POSITIONAL MOMENTUM AND LIQUIDITY PORTFOLIO MANAGEMENT

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A DISSERTATION SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN ECONOMICS
YORK UNIVERSITY
TORONTO, ONTARIO

APRIL 2020

Abstract

This thesis introduces a new positional momentum management strategy based on the expected future ranks of asset returns and trade volume changes predicted by a bivariate Vector Autoregressive (VAR) model. Chapter one provides some facts about the relationship between return and trade volume changes and the way they have been computed in general. It begins by investigating the simple VAR model to see if we can use the past values of return and trade volume changes to predict their current values. Then recent developments in portfolio management research on momentum portfolios are discussed.

Chapter two introduces a new method to build a positional momentum and liquidity portfolios based on the expected future ranks of asset returns and their trade volume changes. This method is applied to a data set of 1330 stocks traded on the NASDAQ between 2008 and 2016. It is shown that return ranks are correlated with their own past values, and the current and past ranks of trade volume changes. This result leads to a new expected positional momentum strategy providing portfolios of predicted winners, conditional on past ranks of returns and volume changes. This approach further extends to a new expected positional liquid strategy provid-

ing portfolios of predicted liquid stocks. The expected liquid positional strategy selects portfolios of stocks with the strongest realized or predicted increase in trading volume. These new positional management strategies outperform the standard momentum strategies and the equally weighted portfolio in terms of average returns and Sharpe ratio.

Chapter three introduces new positional investment strategies that maximize investors' positional utility from holding assets with high expected future return and liquidity ranks. The optimal allocation vectors provide new investment strategies, such as the optimal positional momentum portfolio, the optimal liquid portfolio and the optimal mixed portfolio that combines high return and liquidity ranks. The future ranks are predicted from a bivariate panel VAR model with time varying autoregressive parameters. We show that there exists a simple linear relationship between the time varying autoregressive parameters of the VAR model and the auto- and cross-correlations at lag one of the return and volume change series of the SPDR. Therefore the autoregressive VAR parameters can be easily updated at each time, which simplifies the implementation of the proposed strategies. The new optimal allocation portfolios are shown to perform well in practice, both in terms of returns and liquidity.

Dedication

This thesis is dedicated to my parents.

Thank you for everything you've done.

Acknowledgements

This thesis would not have been completed without the contributions of many important people. I want to thank my supervisor Joann Jasiak. Despite her many commitments she always made time to talk, help, and offer words of encouragement throughout. Her dedication to her students is incredible and without it this thesis would not have been possible. I also want to thank Christian Gourieroux for his willingness to listen to research ideas, provide feedback, and collaborate. Finally, I want to thank my committee members Paul Rilstone and Augustine Wong for their constructive feedback, thoughtful conversations and positivity.

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Introduction

Predicting the returns on individual securities is the primary objective of research on the predictability of financial markets. What an investor really needs is not a large number of predictions of individual returns, but rather, a ranking of the securities with respect to their returns. The ranking of the individual predicted returns does not guarantee however an optimal decision based on available data.

This thesis intends to improve a momentum strategy by taking into account the auto- and cross-correlations of ranks of returns and trade volume changes instead of just past raw returns and provides the optimal positional portfolio by maximizing the investor's utility function based on the future ranks of return and trade volume changes. The standard positional momentum strategy ranks the asset returns at time t and builds an equally weighted portfolio from the top alpha-percentile of all assets. The value of alpha is fixed at a target top percentile, such as the fifth top percentile, for example. The contrarian positional momentum strategy builds an equally weighted portfolio from the lower alpha percentile.

In finance, the positional theory has many applications, including the job search problems concerning the CEOs, fund managers, or traders in the finance sector [see e.g. Gabaix and Landier (2008), Thanassoulis (2012)]. In terms of positional application, this thesis is focused on the positional portfolio management which maximizes the utility of the expected future position of the portfolio's value. This technique is different than the traditional portfolio momentum management which is based on past portfolio returns and their past ranks. The positional portfolio management proposed in this thesis provides new types of allocations strategies. By comparing the returns on the optimal positional portfolio with the traditional momentum, contrarian (or reversal) strategies and naive equally weighted portfolio, we can measure the gain from implementing the positional portfolio management strategies. In the positional portfolio management all stock returns are ranked cross-sectionally, so that the notion of cross-sectional rank (position) is at the core of the distinction of this management from the standard portfolio management.

Referring to an old Wall Street quote that " It takes volume to make price move", the relationship between assets' returns and their trade volumes has been examined in the literature. Many empirical studies showed that in a dynamic context, information about trading volume improves the forecasts for price changes and return volatility. In Chapter one, the relationship between return and trade volume changes is studied. The motivation for considering both returns and trade volume stems from the empirical evidence documented in financial literature, which suggests that the trade volumes provide additional information and help predict future returns.

In Chapter two the positional momentum strategy is extended in three respects. First, the ranks of asset returns and the ranks of trade volume changes are considered jointly and modelled as a bivariate series. I show that the series of ranks of returns and volume changes are serially correlated and cross-correlated with one another. Second, the positional momentum portfolio based on the observed ranks is replaced by the positional momentum portfolio based on the expected future ranks. The future ranks of return and volume changes are predicted from the past ranks of returns and volume changes. This extends the work by Gagliardini. et al. (2019) who introduced the expected positional momentum strategy involving the predicted future return rank. An investor owning a portfolio of returns with high future ranks (or high Sharpe performance) is not protected from future high liquidity risk. Indeed, a future return winning portfolio may turn out to be an illiquid portfolio. The third contribution in Chapter two is a new expected positional liquid portfolio that contains assets with highest (resp. lowest) future expected changes in trade volumes.

In portfolio management, the feedback and inter-dependence between stocks are of particular interest. Empirical literature [see Demiguel, Nogales and Uppal (2014)] has evidenced that the VAR model can capture serial dependence in stock returns, which is statistically significant. Since the serial dependence and inter-dependence between stocks are of particular interest in portfolio management, multivariate dynamic panel data models provide a useful modeling strategy. The panel VAR model seems to be a powerful tool of analysis. The main advantages of panel VAR models can be summarized as follow: (i) they capture both static and dynamic interdependencies, (ii) they can estimate the co-movements between several variables, (iii) easily incorporate time variations in the coefficients and in the variance of the shocks when estimated by rolling. Panel VARs resemble standard VARs but, because of an additional cross-sectional dimension, they are a much more powerful tool.

In Chapter two a panel VAR model is used to represent the dynamics of ranks of return and volume changes, upon their transformation to bivariate Gaussian ranks. The panel VAR model is considered as a restricted VAR model and estimated by Maximum Likelihood Estimation from monthly returns and trade volumes ranks of 1330 stocks traded on NASDAQ between 2008 and 2016.

Using the utility function as an agent objective function is the foundation of the portfolio selection problem under uncertainty. According to the literature, the utility function measures the investor's relative preference for different levels of wealth. One of the advantages of the utility-based strategy is that it eliminates the arbitrary cut-off point of top 5%, or top 10% of assets to be included in a portfolio. In the portfolio management literature, the investor maximizes his/her expected utility function based on wealth or portfolio return [see Brennan and Torous (1999), Das and Uppal (2004) and Gourieroux and Monfort (2005)][1]. In Chapter three, the investor is assumed to maximize a CARA (Constant Absolute Risk Aversion) utility function of a future position of the assets (ranks of assets).

Chapter three introduces new positional investment strategies that maximize investors' utility from holding assets with high expected future ranks in return and liquidity. This approach allows us to determine the optimal allocations that select assets with respect to their expected future returns and liquidity ranks, where the latter ones are measured by changes in traded volumes. An optimal allocation vec-

[1]Von Neumann and Morgenstern (1994) show that, a rational investor selects the optimal feasible investment by maximising the expected utility of wealth.

tor is also derived for a mixed portfolio of assets with the highest combined ranks of returns and liquidity. The new allocation strategies are called the optimal positional momentum portfolio, the optimal positional liquid portfolio and the optimal positional mixed portfolio, respectively. The new optimal allocations that maximize the positional utility function arise as extensions of a naive equally weighted portfolio that account for serial dependence in the returns and volume change ranks as well as for their co-movements.

Since the parameters of VAR model are shown to be time varying [see Figures 2.7 and 2.8], I propose two methods that allow an investor to update the VAR parameters at each investment time. The first method consists in re-estimating the model at each time by rolling over a fixed window of observations. The second method exploits the relationship between the autoregressive coefficients of the VAR model and the series of auto-and cross-correlations at lag 1 of returns and volume changes of the SPDR (Standard & Poor's Depository Receipts). The SPDR is an Exchange Traded Fund (ETF), i.e. a regularly updated portfolio mimicking the evolution of the $S\&P$ 500 returns. More specifically, we show that the future values of autoregressive VAR coefficients can be predicted from simple linear functions of the current auto- and cross-correlations at lag 1 of SPDR's return and volume changes. These linear functions are easy to compute and simplify the investment procedure as they eliminate the need for re-estimating the panel VAR model by rolling. In the proposed approach, the time varying parameters are considered predetermined. We show heuristically that the approach can be extended to a random coefficient framework, where the autoregressive VAR coefficients are considered as fixed functions of random factors, which are the auto and cross-correlation estimators with their known asymptotic distributions.

In the financial literature the return-to-risk trade-off i.e. the reward-to-risk ratio shows the amount of return gained on an investment correspond to the amount of undertaken risk. The Modern Portfolio Theory (MPT) assumes that investors are risk averse and the literature shows that to get more return we have to take more risk [see Breen, Glosten, and Jagannathan (1989), Nelson (1991), Glosten, Jakannatha and Runkle (1993), Brandt and Kang (2004), etc.]. It means that, given two portfolios with the same expected return, investors will prefer the less risky one. An investor will take more risk only for higher expected returns. On the other hand, this trade-off can vary across investors, as different investors will evaluate the trade-off based on their individual risk aversion characteristics. Computing the level of an individual's risk aversion is the most difficult question since the answer is subjective.[2] In the literature, the risk aversion is considered constant in order to obtain relatively simple formulas for relationships between variables in a model. In Chapter three, I consider the CARA utility function with a constant risk aversion while the investor can adjust the portfolio to the current market conditions by changing the risk aversion coefficient to invest more or less aggressively.

The empirical results in Chapter two show that the expected positional momentum portfolios of winners with high ranked returns predicted from past ranks of returns and trade volume changes (VAR-EPMS), outperform the expected posi-

[2]There exist tests that help determine what is the most appropriate risk for investors. The PASS test by W.G. Droms (1988), the Baillard, Biehl & Kaiser (1986) test, classifies investors in order from "confident" to "anxious" and "careful" to "impetuous", while Barnewal (1987) considered just two types of investors passive and active investors.

tional momentum portfolios of Gagliardini, et al. (2019), the standard momentum strategies [see. e.g. Jegadeesh and Titman (1993, 2001), Hellstrom (2000), Arena. et al., (2008))] and the equally weighted portfolio in terms of average monthly returns and the Sharpe ratio in the long run, and in terms of the cumulative returns over holding periods of 4 and 8 year. Moreover this strategy produces higher average return over 3 and 12 months holding times, as compared to the standard momentum strategies with varying look back periods. This finding is in line with Jagadeesh and Titman (1993) who show that portfolios based on the past 3- to 12-month returns of winners, on average, outperform past losers.

The positional liquid portfolios of stocks introduced in Chapter two provide even better outcomes in terms of the average and cumulative returns than the positional momentum portfolios. The positional portfolios of liquid stocks (LPMS) with recently increased past volumes, outperform the other portfolios in terms of monthly average returns and Sharpe ratio in the long run and in terms of cumulative returns over the horizon of 8 years. It also provides the best investment portfolio over short holding times of 3, 6 and 12 months. The positional portfolio of stocks with an expected liquidity (LEPMS) generates the best cumulative returns over horizons of 2 to 3 years.

The empirical results from Chapter three show that returns on the new optimal positional portfolios are comparable both theoretically and empirically with the naive equally weighted portfolio as well as with the traditional momentum strategies with look-back and holding periods of various length. All positional portfolios provide positive average and cumulative returns. The positional liquid portfolios outperform the positional mixed and momentum portfolios. Also, we observe that for higher risk aversion values, the average and cumulative returns on the positional portfolios decrease. In terms of average returns, the positional portfolios obtained by predicting the future ranks from the VAR(1) model outperform the other portfolios. In terms of cumulative returns, the positional portfolios obtained from fitted values of coefficients based on auto- and cross- correlation of SPDR provide higher returns.

The outline of the First Chapter is as follows: In section 1.2 the literature review is provided. In section 1.3 the relation between returns and trade volume changes is investigated. Section 2.2 introduces the cross-sectional ranks of securities according to their relative returns and trade volume changes in each month. Their transformation to Gaussian ranks is also explained. The panel VAR model of bivariate Gaussian ranks and its estimation are discussed in Section 2.3. Section 2.4 explains how a given portfolio can be positioned among other stocks with respect to either return or changes in trade volume. In Section 2.5, I define the new expected positional momentum and liquidity strategies based on the predicted ranks of returns and volume changes. These strategies are compared among themselves and with the equally weighted portfolio and the standard positional momentum. Section 2.6 concludes the paper. Additional results and proofs are provided in Appendices A, B, C, D, E and F.

Chapter One

Return and Trade Volume

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Return and Trade Volume

1.1 Literature Review

Thomas Hellstrom (2000), introduced a rank measure to the financial literature by ranking a large number of securities according to their relative returns. He predicted the ranks with a linear model and used those ranks in a portfolio selection algorithm. He found that, the optimal portfolio based on those ranks significantly outperforms the benchmark when tested on the Swedish stock market over the period 1993-1997. In the momentum portfolio literature, Gagliardini. et al., (2019) used a different strategy to rank the assets' returns as Gaussian ranks. They studied the positional portfolio management strategies in which the manager maximizes an expected utility function of the cross-sectional rank (position) of the portfolio return. The objective function reflects the manager's goal to be well-ranked among competitors. To implement positional allocation strategies, Gagliardini. et al., (2019) specify a non-linear unobservable factor model for the asset returns which reveals the dynamics of the cross-sectional distribution and the dynamics of the ranks of the individual assets. By using a large data set of stocks returns, they found that the positional strategies outperform the standard momentum and reversal strategies, as well as the equally weighted portfolio. In this study, the future ranks of return and volume changes are predicted from the past ranks of return and volume changes. This extends the work by Gagliardini. et al., (2019). It is also in line withDaniel and Moskowitz (2016) who show that a dynamic momentum strategy based on the forecast of momentum's mean and variance provides higher Sharpe ratio than the static momentum strategy.

The relationship between trade volume and stock returns has been examined in the literature. Several authors studied the contemporaneous relationship between these two variables. Karpoff (1987) uses a bivariate regression and the VAR and VEC models, and examines the IRF and Johansen's Co- integration test to show a bi-directional causality between trading volume and stock return volatility. Gallant, Rossi and Tauchen (1992) study the semi-nonparametric estimation of the joint density of current price change and volume conditional on past price changes and volume. Chordia and Swaminathan (2000) find that, the trade volume is a significant determinant of the lead-lag patterns observed in stock returns. Arena, Haggard and Yan (2008) show the existence of a positive time-series relation between momentum returns and aggregate idiosyncratic volatility.

Bong-Soo Lee and Oliver M.Rui (2000), studied the dynamic relationship between the trade volume and stock return. They examined the causality between these two variables in both domestic and cross-country markets. They found a positive relationship between these two variables and evidenced that this relationship exists even across countries' markets as well. Chandrapala Pathirawasam (2011), examined the relationship between trading volumes and stock returns for 266 stocks traded at the Colombo Stock Exchange (CSE) from 2000-2008. They applied the conventional methodology used by Jagadeesh and Titman (1993), and found that stock returns are positively related to the contemporary change in trading volume, while trading volume change is negatively related to stock returns. They also documented that the investor missperception of future earnings or illiquidity of low volume stocks can be the reason for the negative relationship between trading volume and stock returns. Lee and Swaminathan (2000) show that trading volume helps predict cross-sectional returns for various price momentum portfolios. By introducing the ranks of trade volume changes, I hope to get a more accurate dynamic model to find the optimal momentum positional portfolio.

In Chapter two, the vector autoregressive (VAR) model is used to study the dynamic relationship between the returns' and trade volume changes' ranks. Since both ranks series are normally cross sectionslly distributed, the Maximum Likelihood method is used to estimate a restricted VAR(1) model of both ranks that accommodates the marginal standard Normal density of these variables. Demiguel, Nogales and Uppal (2014) use the vector autoregressive (VAR) model to capture serial dependence in stock returns. In the financial literature, VAR models have been used for strategic asset allocation. For instance, Campbell and Viceira (1999) and (2002), Campbell, Chan, and Viceira (2003), Balduzzi and Lynch (1999), Barberis (2000).

A positive association between trading activity and volume is documented in

Demsetz (1968). Chordia, Roll and Subrahmanyam (2000) used volume as one of the known individual liquidity determinants to uncover suggestive evidence that inventory risks and asymmetric information both affect inter temporal changes in liquidity. Barclay and Hendershott (2004) also used trade activity to determine the liquidity of the stocks during the trading day. Johnson (2008) shows that volume is positively related to the variance of liquidity or liquidity risk. In this thesis I build the liquid positional portfolios from stocks with high ranked trade volume changes.

In Chapter three the time series of auto- and cross-correlations of SPDR are used as common macro factors to predict the parameters of the VAR model. Since SPDR is an Exchange Traded Fund (ETF), i.e. a regularly updated portfolio mimicking the evolution of the $S\&P$ 500 returns, it can be considered as a proxy of the market. Also the relationship between the SPDR and $S\&P$ 500 has been documented in the literature as well. Beaulieu and Morgan (2000) studied the high-frequency relationships between the S&P 500 Index and the SPDR by using minute-by-minute data for November 1997 through February 1998. They showed that the SPDR did not track the index perfectly. Peng Xu (2014) checked the mimicking performance of the SPDR in two ways: first she examined the relation between relative price change of the SPDR and the relative change of the index and second studied the relation between holding period return of the SPDR and the return on the index. She showed that in a linear static analysis, the SPDR mimics the index pretty well, since the historical correlation coefficient between the two return series is 0.98. She also showed that both series will have similar dynamic features, as long as linear dynamics are considered.

In the CARA utility function used in Chapter three, the constant risk aversions are considered. In the literature, constant risk aversion parameters are used to determine the return-to-risk trade-off. Chou (1988) showed that the risk attitude parameter stay stable for correlative periods of time, Safra and Segal (1998) defined the invariant preference relation between outcomes of two distributions as the constant risk aversion and Quiggin and Chambers (2004) show the constancy of the risk aversion since the investor attitude is strongly linked with the family of generalized expected utility preferences.

1.2 Relation Between Return and Trade Volume

Many articles have examined the relationship between stock return and trade volume among different markets. Karpoff (1987) stated four reasons why the relationship between stock price and volume is important to study: (i) it provides a insight into the financial market structure, (ii) this relationship is important in studies which are using the combined data of price and trade volume to draw the conclusion, (iii) it is critical for the debates over the empirical distribution of speculative prices, and finally (iv) the price-trade relations have significant implication for research into future market. Among the articles concerning the relation between price and volume, a few only examined the dynamic relation between trading volume and prices. In a dynamic context, an important point about this relation is that whether information about trad volume improves the forecast of returns. In this section I examine empirically the dynamic relation between the changes in trade volume and stock return.

1.2.1 Data Description

The panel data contains monthly returns and trade volumes changes of 1330 stocks traded on the NASDAQ from October 1999 to October 2016. These stocks have been chosen with respect to the daily average of Turnover/Traded Value of all NASDAQ stocks in 2015. The Turnover/Traded Value is defined as the total amount traded in the security's currency, which is calculated as the sum of numbers of shares times their corresponding prices. Stocks from the highest and the lowest 25th percentiles of Turnover/Traded Value have been selected of Traded Value in October 2016 have been selected [3]. Among those stocks, we selected those observed over the entire sampling period. . After deleting stocks with missing values between October 1999 and October 2016, we end up with 1330 stocks.

The trade volume of a security is defined as the total quantity of shares traded per month. To get the return and the changes in trade volume, the log return and the log volume changes are calculated as follows:

$$\begin{aligned} r_{it} &= ln(\frac{P_{it}}{P_{it-1}}) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \\ tv_{it} &= ln(\frac{TV_{it}}{TV_{it-1}}) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \end{aligned} \tag{1.2.1}$$

where P_{it}, P_{it-1} are the prices at time t and $t-1$, TV_{it}, TV_{it-1} are the trade volume

[3] The Turnover/Traded Value of those stocks was not necessarily in the highest (resp. lowest) 25th percentiles at other times, as it has been changing over the sampling period.

at time t and $t-1$ and r_{it}, tv_{it} are the log return and log changes in trade volume of stock i at time t respectively[4]. The panel contains $n = 1330$ stocks observed over $T = 214$ periods of time (months)[5].

Figures 1.1 and 1.2 present the cross-sectional mean (Figure 1.1) and variance (Figure 1.2) of the returns (r_t) and trade volume changes (tv_t) over time.

In Figure 1.1 we see that the mean returns and mean volume changes do not show any seasonality or trend over time. According to these figures, the mean and variance of trade volume changes are more volatile than those of returns. From October 2000 to October 2004, the mean return varies a lot, and it takes a sharp downturn in July 2001 and May 2002. According to a report by the Cleveland Federal Reserve, this downturn can be viewed as part of a larger bear market or correction that began in 2000. The majority of specialists believe that this downturn could be a reversion to average stock market performance in a longer term context. Indeed from 1998 to 2000, the NASDAQ rose almost 85%, while before that time it had the annual growth of 10% to 15%.

After year 2000, the index dropped to the same level it would have achieved if the annual growth rate followed during 1987-1995 had continued up to 2002. On September 16, 2008, the mean of returns reached its lowest value.

[4]The log changes in trade volume are also referred to as V-ROC (Volume Rate-Of-Change). See Investopedia at https://www.investopedia.com/articles/technical/02/091002.asp for definition and Podobink et.al (2009) for empirical study.

[5]At each date t, the information available is $I_t : \{r_{it}, tv_{it}, i = 1, \cdots, n\} \approx \{P_{it}, TV_{it}, i = 1, \cdots, n\}$.

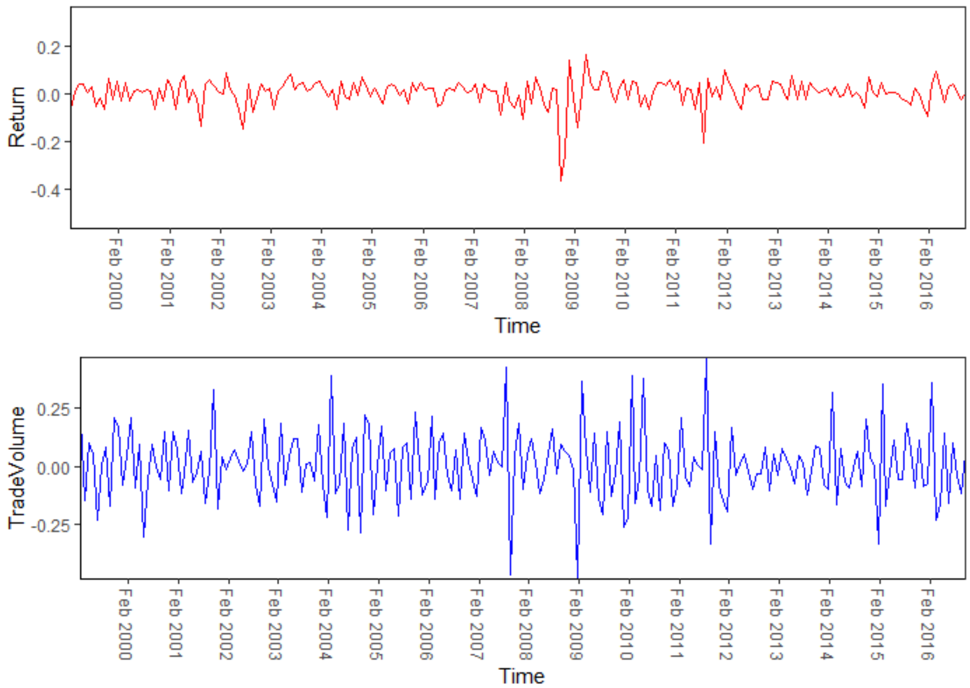

Figure 1.1: Times Series of The Cross-Sectional Mean of Return and Trade Volume
Figure 1.1 displays the cross-sectional mean of return (r_t) and trade volume changes (tv_t) from February 1999 to October 2016. The cross-sectional mean is computed monthly as the average of 1330 stocks' returns and trade volume changes traded on NASDAQ.

The reason was the massive failures of financial institutions in the United States, due primarily to exposure to packaged sub-prime loans and credit default swaps issued to insure these loans and their issuers, which rapidly devolved into a global crisis.

These financial failures resulted in a number of bank failures in Europe and sharp reductions in the value of stocks and commodities worldwide.

Another major fall in stock market was the Black Monday of 2011, which refers to August 8, 2011, when the US and global stock markets crashed following the Friday night credit rating downgrade, by Standard and Poor's of the United States sovereign debt from AAA, or "risk free", to AA. After that in 2014 and 2016, the stock market had experienced the Bull Market. Retail investors, started to put money back in the market in 2013, allowing them to benefit from 2014 in advance.

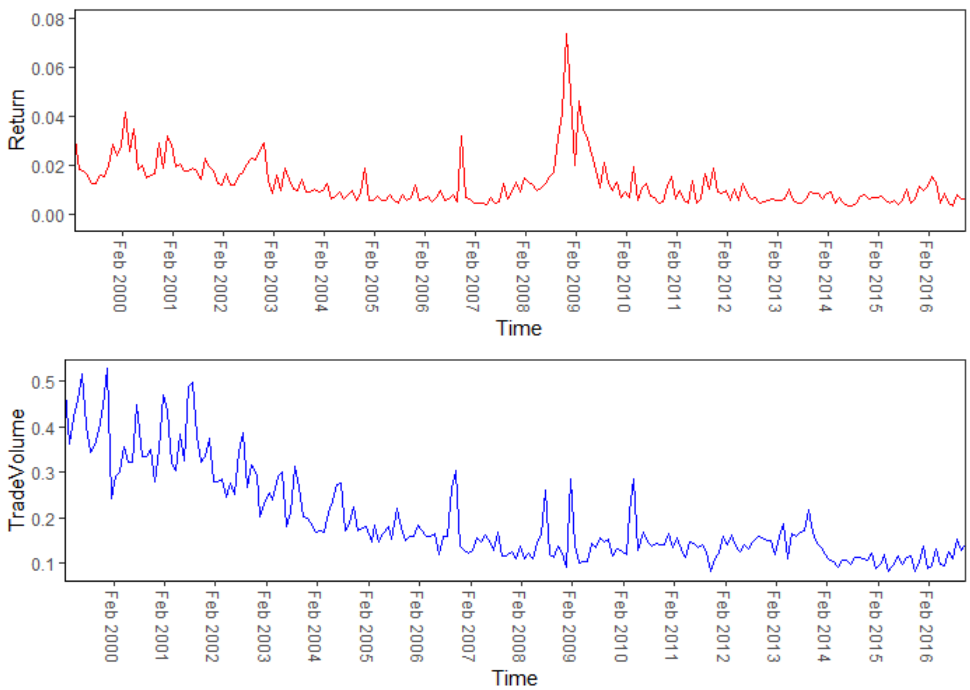

Figure 1.2: Time Series of The Cross-Sectional Variance of Return and Trade Volume
Figure 1.2. displays the cross-sectional variance of return (r_t) and trade volume changes (tv_t) from February 1999 to October 2016. The cross-sectional variance is computed monthly from 1330 stocks' returns and trade volume changes traded on NASDAQ.

In Figure 1.2 we observe that the variances of returns and of volume changes are above 0.01 over the period 1999-2016. After year 2012 both variance series take values between 0.01 and 0.02. There have been periods when the return variance was unusually high or low. From the beginning of 2000 the return volatility was decreasing gradually until 2004, when it reached a more steady pattern (2004-2006). At the end of 2008, the volatility of returns surged to more than 0.07, which is fairly high by historical standards, yet not without precedent. It remained high during the crisis of 2008-2010 but starts to fall from 2009. From 2010 until 2016, it remains in a steady level of lower than 0.01.

The cross-sectional variance of trade volume changes dropped from the average of 0.4 in 1999 to less than 0.2 in 2004 and fluctuated between 0.1 and 0.2 until 2016. In the years 2008, 2009 and 2010 (the crisis), it increased considerably in parallel to the variance of returns. After 2014, both series of cross-sectional variances are less erratic and more smooth.

1.3 Data Stationarity

To check the stationarity of returns and trade volume changes series, the unit root test is applied separately to each panel [6]. In the panel unit root test literature, the null hypothesis is formally stated as H_0: "all of the series have one unit root". While the null hypothesis is common to all the panel unit root tests, the literature considers two different alternative hypotheses, H_1^a : "all of the series do not have unit root" and H_1^b : "at least one of the series has unit root". The alternative H_1^b has been criticized by some authors indicating that if H_0 is rejected we do not know which series have a unit root (Taylor and Sarno 1998). On the other hand, alternative H_1^a implicitly imposes a strong dynamic homogeneity restriction across the panel units (Levin et al. (2002), Im, Pesaran and Shin (2003)) while it may also has power in mixed situations where not all the series are stationary. In practice,

[6] The assumption of cross-sectional independence is common in the literature. It allows for analytically derivation of the asymptotic distributions of the test statistics. In application to financial data, one may argue that some common systematic factors exist. In such a case, the asymptotic distributions change, but the test procedures remain consistent. The statistical adjustment of the tests for common factors is out of the scope of this paper.

those tests that consider the alternative H_1^a are less flexible and may be subject to the same criticism as those considering the alternative H_1^b.

Given these two alternative hypotheses the panel unit root tests, can be obtained in two ways: Approach 1 is based on the t-ratio and approach 2 is based on the p-value. In the first case the alternative hypothesis is H_1^a and in the second one it is H_1^b (Maddala and Wu (1999) and Choi (2001)). The tests based on the t-ratios are panel extensions of the standard Augmented Dickey-Fuller test (ADF) (Said and Dickey (1984)). There are two ways of applying these tests to panel data, either by pooling the units before computing a pooled test statistic (Levin et al.(2002)), or averaging the individual test statistics in order to obtain a group-mean test (Im et al.(2003)). On the other hand, the p-value combination tests are based on the idea that the p-values from N independent ADF tests can easily be combined to obtain a test of the joint hypothesis concerning all the N units. The advantages of the p-value combination approach are its simplicity and flexibility in specifying a different model for each panel unit and the ease in allowing the use of unbalanced panels. Table 1.1 provides the results of the stationarity tests for returns and trade volume changes.

In Table 1.1, Columns 1 and 2 show the outcomes of tests based on the t-ratio which were introduced by Levin, Lin and Chu (Levin et.al (2002)) and Im et.al (2003). Columns 3 to 6 present the outcomes of tests based on the p-value by Maddala and Wu (1999), the modified p-test proposed by Choi (2001), the inverse normal test by Choi (2001) and the logit test by Choi (2001), respectively. All of these tests indicate that the data of monthly returns and trade volume changes are stationary.

Table 1.1: Stationarity Test for Return and Trade Volume Changes

Variables	*Levinlin*	*Ips*	*Madwu*	*Pm*	*Invnormal*	*Logit*
Returns	-565***	-539***	153322***	2065***	-377***	-1159***
Trade Volume Changes	-473***	-561***	154848***	2086***	-380***	-1170***

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: The Table provides the results of six stationarity tests for returns and trade volume changes. Columns 1 and 2 show the results of t-ratio-based stationarity tests of Levin, Lin and Chu (Levin et.al (2002)) and Im et.al (2003), respectively. Columns 3 to 6 present the outcomes of p-value-based stationarity tests of Maddala and Wu (1999), the modified p-test of Choi (2001), the inverse normal test of Choi (2001) and the logit test of Choi (2001), respectively.

1.4 Relation Between Return and Trade Volume Changes

Technical analysts strongly believe that "It takes volume to make price move" (Karpoff, 1987). In fact, they believe that the past security information is not fully incorporated in current security information, and hence, by observing the past security information, the future information can be obtained. The early studies on volume-return relation examined contemporaneous relationships between trading volume and absolute price changes. Hence, they have little relevance on the predictability of future stock price or volume of its trade.

In this section, I examine the volume-return relationships based on monthly data for 1330 stocks which were traded on NASDAQ during the period 1999 to 2016. First, I compute the series of cross-sectional contemporaneous correlation between returns and trade volume changes for every month.

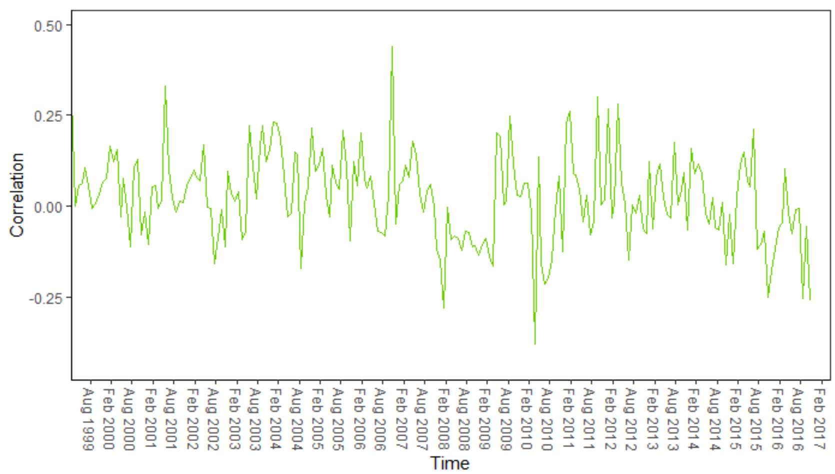

Figure 1.3: Time Series of Cross-Sectional Correlation between Return and Trade Volume

Figure 1.3 shows the montgly time series of the corss-sectional correlation between return and trade volume of 1330 stocks between 1999 and 2016.

Figure 1.3 shows that the relationship between returns and trade volume changes is volatile. Until 2004, the correlation was fluctuating while remaining positive. The equity investors turned more risk-averse in the second and third quarters of 2004 on concerns about the real strength of the global economic recovery. So the investors were very sensitive to the changes in stock return. At the end of the year 2006, the correlation was strongly positive and reached its highest value. On the other hand the correlation turned negative in May 2010. In fact, it seems that investors were not willing to trade when the returns were high, but they where very active when the returns decreased. Over the remaining part of the sampling period, the correlation between stock return and trade volume changes was fluctuating between -0.25 and 0.45. This shows that the return and trade volume changes have been correlated during the period examined, although the sign of correlation was varying. The magnitude of the correlation was higher when significant events occurred in the market.

It is interesting to examine the volatility of this correlation over the time and to determine what factors are affecting it. However, this is out of the scope of this analysis, which is focused on the joint predictability of return and trade volume changes.

The volume-return relationship has two components, which are the contemporary relation between changes in volume and returns and the relationship between the past and current trading volume changes and stock returns. To verify that past information can be used to predict future values of these variables, I estimate a simple Pooled regression and a Fixed Effect (including both time and individual fixed effect) model of current values of these two variables on their past values. First I need to decide how many lags to include in these regressions. To determine the number of lags, I compute the cross-correlation function between the returns and trade volume changes of S&P500, which is a proxy of the market and can provide a rough idea of correlation between these variables in other stocks. Figure 1.4, shows the CCF (Cross-Correlation Function) between the returns and trade volume changes of S&P 500 as a proxy of the market. The CCF shows that the returns and trade volume changes have a negative contemporaneous correlation, while the cross-correlations are only significant at lags 0 and 1.

Another common approach for model order selection in VAR models involves selecting a model order that minimizes one or more information criteria evaluated over a range of model orders. Some of commonly used information criteria are AIC (Akaike information criterion), HQ (Hannan-Quinn criterion), SC (Schwartz information criterion) and FPE (Akaike Final Prediction Error). These criteria evaluate the goodness of fit of the model and indicate the optimal number of lags that ensures a parsimonious fit. A different optimal number of lags was obtained from each criterion, as AIC gave 4, HQ showed 2, SC found 1 and finally FPE presented 4 lags.

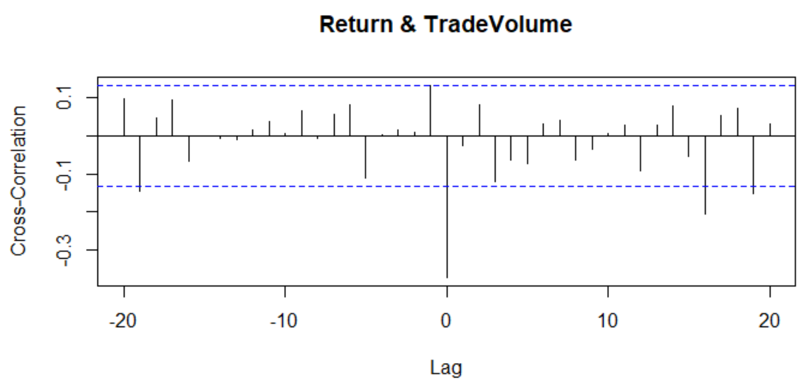

Figure 1.4: Cross-Correlation Function for Return and Trade Volume Changes of S&P500
Figure 1.4 shows the cross-correlation function of return and trade volume changes for S&P500 as a proxy of the market.

Given these different outcomes, the indication of the most stringent SC criterion will be retained. I choose one lag for the dynamic model which was suggested by SC criterion and it also was clear in CCF graph. The dynamic linear models of returns and volume changes are given in equations (1.2-1.3) below.

$$R_{i,t} = \beta_{10} + \beta_{11} R_{i,t-1} + \beta_{12} TV_{i,t-1} + \epsilon_{1,it} \tag{1.4.1}$$

$$TV_{i,t} = \beta_{20} + \beta_{21} R_{i,t-1} + \beta_{22} TV_{i,t-1} + \epsilon_{2,it} \tag{1.4.2}$$

where $R_{i,t}$, $R_{i,t-1}$ and $TV_{i,t}$,$TV_{i,t-1}$ are presenting the current and lagged returns and trade volume changes for stock i respectively. $\epsilon_{1,it}$,$\epsilon_{,it}$ both denote the error term, which are assumed individually identical normally distributed (iid) and mutually independent. These regressions estimated separately by the Least Squares as a Pooled Regression and a Fixed effect model. The results from estimating these simple panel regressions is presented in Table 1.2.

Table 1.2: Return and Trade Volume

Return on Trade Volume	**Intercept**	*LR*	*LTV*
PooledOLS	0.0054***	-0.0132***	0.0074***
FixedEffect		-0.0155***	0.0074***
Trade Volume on Return			
PooledOLS	0.0093***	-0.0443***	-0.3662***
FixedEffect		-0.0441***	-0.3665***

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: The Table provides the results of Pooled OLS and Fixed Effect estimations of Equations (1.2) and (1.3) respectively. The estimated parameters are as follow intercept, coefficient on the first lag of return (LR) and the coefficient on first lag of trade volume (LTV).

Table 1.2 shows that in both OLS and Fixed-Effect regressions returns returns show a negative dependence on their own past values and a positive dependence on the past values of trade volume changes and trade volume changes show a negative dependence on past values of return and trade volume changes.

These two simple estimations indicate that the past information on returns and trade volume changes can be used to predict the future return and trade volume changes.

This simple exercise, justifies my approach to the modelling of returns and trade volume changes jointly. I also showed that the past information can be used to predict their future values. Since, in a positional allocation strategy, the goal of managers is to maximize the expected utility function from the cross-sectional ranks or positions, the positional momentum and liquid portfolio based on returns and trade volume changes will be introduced in Chapter two. In Chapter two, the Gaussian ranks are obtained by transforming the empirical cross-sectional distribution functions of returns and trade volume changes. These Gaussian ranks are cross-sectionally Normally distributed and can be modelled by a panel VAR model.

Chapter Two

Positional Momentum and Liquidity Management; A Bivariate Rank Approach

Chapter 2

Positional Momentum and Liquidity Management; A Bivariate Rank Approach

2.1 Introduction

This chapter introduces a new positional momentum management strategy based on the expected future ranks of asset returns and trade volume changes predicted by a bivariate Vector Autoregressive (VAR) model.

The aim of this chapter is to extend the positional momentum strategy in three respects. First, the ranks of asset returns and the ranks of trade volume changes are considered jointly and modelled as a bivariate series. The motivation for this approach stems from the empirical evidence documented in financial literature, which suggests that the trade volumes provide additional information and help predict future returns.

Second, the positional momentum portfolio based on the observed ranks is replaced by the positional momentum portfolio based on the expected future ranks. In this chapter, the future ranks of return and volume changes are predicted from the past ranks of returns and volume changes. This extends the work by Gagliardini. et al., (2019) who introduced the expected positional momentum strategy involving the predicted future return rank. It is also in line with Danial and Moskowitz (2016) who show that a dynamic momentum strategy based on the forecast of momentum's mean and variance provides higher Sharpe ratio than the static momentum strategy.

The search for returns with high future ranks (or high Sharpe performance) does not protect the investor from future high liquidity risk. Indeed, a future return winner portfolio may end up being an illiquid portfolio. The third contribution is a new expected positional liquid portfolio that contains assets that display the highest (resp. lowest) future expected changes in trade volumes. The ranks of return and volume changes are predicted from a bivariate panel Vector Autoregressive model of order one (VAR(1)). In this chapter, a VAR model is used to represent the dynamics of ranks of return and volume changes, upon their transformation to bivariate Gaussian ranks. The main advantage of the panel VAR model is that it has an ability to capture both the autocorrelations of each rank series and the cross-correlations between them. The panel VAR model is estimated from monthly returns and trade volumes ranks of 1330 stocks traded on NASDAQ between 2008 and 2016.

It is shown that return ranks are correlated with their own past values and the current and past ranks of trade volume changes. This result leads to a new expected positional momentum strategy providing portfolios of predicted winners, conditional on past ranks of returns and volume changes. This approach further extends to positional liquidity management. The expected liquid positional strategy selects portfolios of stocks with the strongest realized or predicted increase in trading volume. These new positional management strategies outperform the standard momentum strategies and the equally weighted portfolio in terms of average returns and Sharpe ratio.

This Chapter is organized as follows: Section 2.2 introduces the cross-sectional ranks of securities according to their relative returns and trade volume changes in each month. Their transformation to Gaussian ranks is also explained. The panel VAR model of bivariate Gaussian ranks and its estimation are discussed in Section 2.3. Section 2.4 explains how a given portfolio can be positioned among other stocks with respect to either return or changes in trade volume. In Section 2.5, I define the new expected positional momentum and liquidity strategies based on the predicted ranks of returns and volume changes. These strategies are compared among themselves and with the equally weighted portfolio and the standard positional momentum. Section 2.6 concludes the paper. Additional results and proofs are provided in Appendices A and B.

2.2 Ranks of Returns and Trade Volumes Changes

2.2.1 Ranks

The literature shows that a momentum strategy based on return ranks can outperform the mean-variance strategy based on returns [see Jegadeesh and Titman (1993), Moskowitz, Ooi and Pedersen (2012), Barroso and Santa-Clara (2015)]. This is usually explained by the fact that returns are more volatile than their ranks and the momentum strategy is less sensitive to extreme volatility and more robust. Several empirical studies point out that the rank of stock returns is more predictable than the individual returns. For example, Hellstrom (2000) finds that the ranks can be predicted with a complex linear model, such as a neural network, and his model results show 63% hit rate for the sign of daily threshold-selected 1-day predictions. We show that even more accurate rank predictions can be obtained from dynamic time series model, such as the (Vector) Autoregressive model, applied to rank series transformed into normally distributed variables.

In the literature, the ex-post and ex-ante ranks are distinguished. The ex-post ranks are obtained by ranking all asset returns at time t from the smallest one to the largest one, and next, by dividing their position by the total number of observations. Equivalently the ex-post return rank of asset i can be derived by inverting the empirical cross-sectional (CS) cumulative distribution function (c.d.f) of the returns at date t. In this case the observed ex-post ranks have the discrete empirical uniform distribution on $(1/n,\ 2/n,\ \ldots, 1)$. In the definition of an ex-ante rank, the empirical cross-sectional c.d.f. is replaced by its theoretical distribution function. Hence, the ex-ante ranks have a cross-sectional uniform distribution on the interval $[0, 1]$. The ex-ante ranks can be interpreted as predicted ranks.

Since the ranks are defined up to an increasing transformation, they can be easily transformed into Gaussian ranks as follows [see Gagliardini et al.(2019)]. The Gaussian ranks are obtained from the corresponding uniform ranks by applying the quantile function of the standard Normal distribution. Next, by standardizing the ranks, one ensures the cross-sectional Normal distribution of the rank variables. Let us consider two ex-post Gaussian ranks, one of stock returns and the second of trade volume changes for stock i at time (month) t. These ranks are related to the returns and trade volume changes by the following equations:

$$u_{i,t} = \Phi^{-1}(\hat{F}_t^r(r_{it})) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \tag{2.2.1}$$

$$v_{i,t} = \Phi^{-1}(\hat{F}_t^{tv}(tv_{it})) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \tag{2.2.2}$$

where $u_{i,t}$ is the Gaussian rank of return, $v_{i,t}$ is the Gaussian rank of trade volume changes, Φ is the cumulative distribution function (c.d.f.) of the standard Normal, Φ^{-1} is its inverse which is the quantile function of the standard Normal and $\hat{F}_t^r, \hat{F}_t^{tv}$ are the cross-sectional empirical cumulative distribution functions of return and trade volume changes at date t, respectively.

To compute the Gaussian ranks, I order all returns and trade volume changes from the highest to the lowest in each month, and assign them absolute ranks from 1 to 1330 (since my sample includes 1330 stocks). Next, I divide these ranks by the total number of stocks which gives me the position of each stock in comparison to other stocks in each month. This procedure provides the empirical cross-sectional cumulative distribution functions $\hat{F}_t^r, \hat{F}_t^{tv}$. Next, to transform these ranks into the Gaussian ranks, I find the equivalent quantile of the standard Normal distribution function for each position [See Appendix A.1 for Normal distribution of cross-sectional Gaussian ranks][7].

According to the above transformation, if asset i has return probability equal to $\hat{F}_t^r = 0.90$, then there are 90% of assets in the sample, which have smaller or equal returns at time t while the remaining 10% of assets have larger returns. Equivalently, if asset i has rank 0.90, there is a probability equal to 90% that the return at time t of any other asset is smaller or equal to the return of asset i. For this particular stock, the corresponding Gaussian rank is $u_{it} = 1.28$ which is the 90% quantile of the standard Normal distribution function.

2.2.2 Relation Between Ranks of Returns and Volume Changes

2.2.2.1 Cross-Sectional Correlation of Ranks

The ranks of returns and volume changes are cross-sectionally correlated and their cross-sectional correlation changes over time. Figure 2.1 displays the cross-sectional correlations between the two ranks [8], which are computed for each month from the

[7] Information available at each time t is $J_t : \{u_{it}, v_{it}, i = 1, \cdots, n\}$ and $J_t \subset I_t$.

[8] The sample cross-sectional correlation is $\frac{1}{n}\sum_i(u_{it} - \bar{u_t})(v_{it} - \bar{v_t})/\sqrt{\frac{1}{n}\sum_i(u_{it} - \bar{u_t})^2\frac{1}{n}\sum_i(v_{it} - \bar{v_t})^2}$, where $n = 1330$, $\bar{u_t} = \frac{1}{n}\sum_i u_{it}, t = 1, \cdots, T$ and $\bar{v_t}$ is defined accordingly and $t = 1, .., 214$ denotes the month.

sample of 1330 stocks. The cross-sectional correlation fluctuates over time between -0.2 and 0.2. It reaches its highest value in 2001 (0.31). It turns and stays negative for one year between 2008 and 2009 (the crisis), until it falls to its lowest value in 2014 (-0.19). On average, the cross-sectional correlation is 0.02 over the entire sampling period. As a positive value of cross-correlation indicates that the ranks of returns and trade volume changes increase (decrease) simultaneously, we conclude that the winner stocks tend to be more liquid, on average over the period 2000-2016.

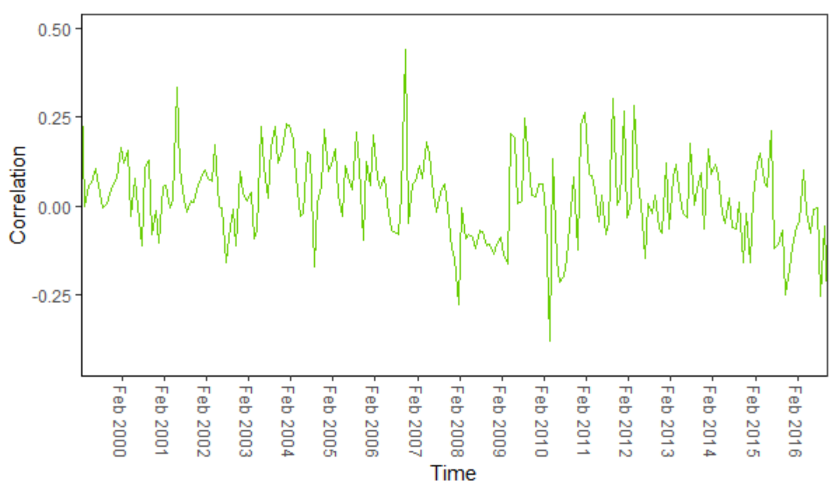

Figure 2.1: Cross-sectional correlation of return and volume change ranks. Figure 2.1 shows the variation in cross-sectional correlations between the return and trade volume change ranks over time. The cross-sectional correlations are computed from the sample of 1330 stocks in each month.

2.2.2.2 Serial Correlation and Cross-Correlation of Ranks

To illustrate the serial correlation and cross-correlation of the two rank series, we consider the example of $S\&P500$. The auto-correlation function (ACF) of each of the rank series u_t^{SP}, v_t^{SP} $t = 1, .., 214$ and the cross-correlation function (CCF) representing the lagged effects of one rank series on another are shown in Figures 2.2, 2.3 and 2.4 respectively.

Figure 2.2 reveals a significant auto-correlation at the first lag in the return rank u_t^{SP}. This finding is consistent with the negative sign of return correlation reported in Jagadeesh (1990). It indicates that the last month's return rank can predict the current rank of return. We also observe the same negative auto-correlation at the first lag in the ranks of trade volume changes (v_t^{SP}) in Figure 2.3.

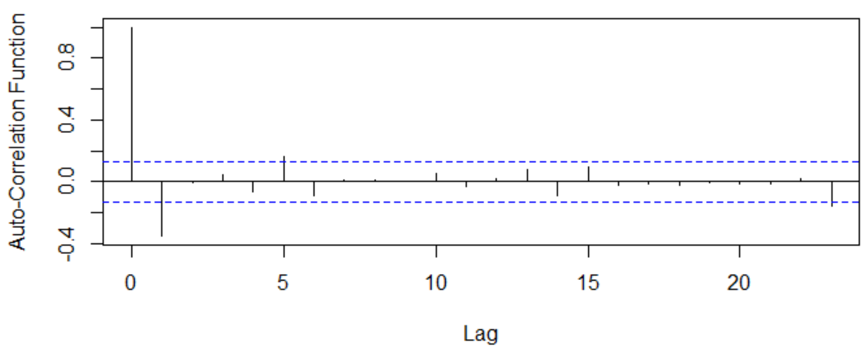

Figure 2.2: Auto-Correlation Function of u_t^{SP} of $S\&P500$

Figure 2.2 shows the auto-correlation function (ACF) of u_t^{SP} return ranks for the $S\&P500$. We observe a significant auto-correlation at the first lag in u_t^{SP}.

Hence, the last month's rank of trade volume changes can also predict the current one. Figure 2.4 shows the cross-correlation function (CCF) of the ranks of returns and trade volume changes of $S\&P500$. We observe a significant negative contemporaneous correlation between the two series of ranks of returns and trade volume changes for $S\&P500$. Hence, a high rank of return on $S\&P500$ tend to occur simultaneously with a low rank of volume change.

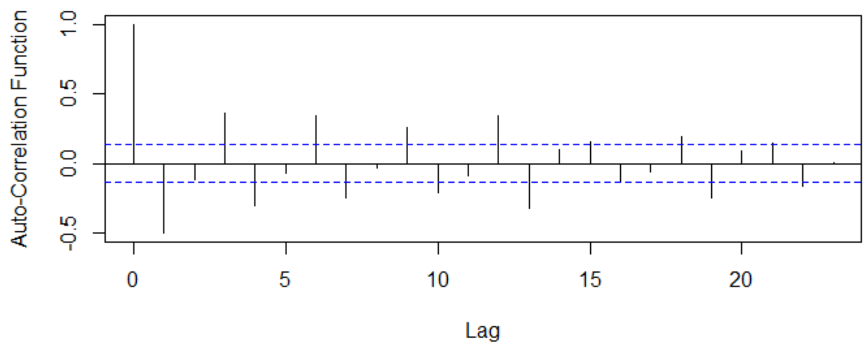

Figure 2.3: Auto-Correlation of v_t^{SP} of $S\&P500$

Figure 2.3 shows the auto-correlation function (ACF) of v_t^{SP} series for the $S\&P500$. We observe a significant auto-correlation at the first lag in v_t^{SP}.

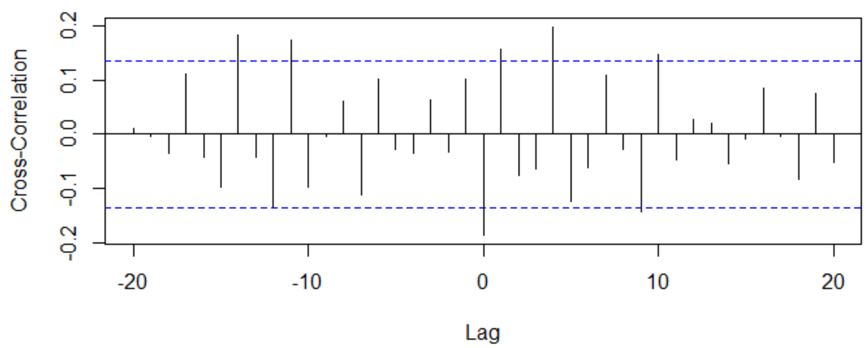

Figure 2.4: Cross-Correlation function (CCF) of u_t^{SP}, v_t^{SP} for $S\&P500$.

Figure 2.4 displays the cross-correlation function (CCF) of the ranks series u_t^{SP}, v_t^{SP} for the $S\&P500$. The CCF shows a significant negative contemporaneous correlation between the two series of ranks of return and trade volume for $S\&P500$. In addition, it shows significant cross-correlations at the first and fourth lags.

In addition, the CCF reveals significant cross-correlations at the first and fourth lags, which suggests that past information on ranks of trade volume changes of $S\&P500$ can help predict the future ranks of returns and vice versa.

So far we have discussed the example of ranks for $S\&P500$. Figures 2.5 and 2.6 illustrate the cross-correlations at lag one in all stocks, by providing the histograms of cross-correlations at lag 1 between u_t, v_{t-1} and v_t, u_{t-1} computed from all stocks in the sample (1330 stocks).

In Figure 2.5, the cross-correlation at lag 1 coefficients take values between -0.13 to 0.18 and the mode of the sample distribution is about 0.04. Hence, on average rank u_t is positively correlated with v_{t-1}. The mode of the distribution in Figure 2.6 is -0.15 which suggests that v_t and u_{t-1} are negatively correlated, on average. The cross-correlation at lag 1 coefficients in Figure 2.6 range between -0.29 and 0.16.

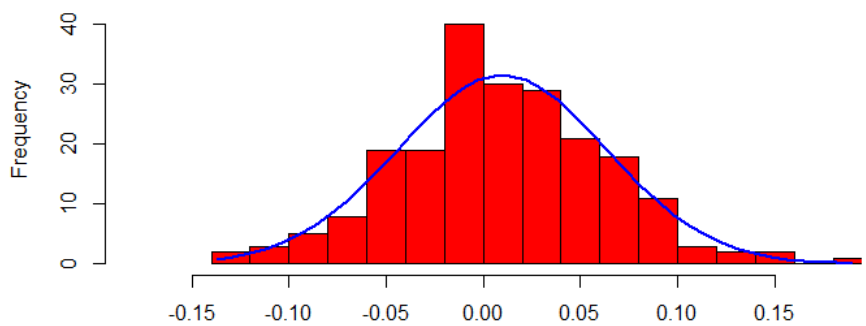

Figure 2.5: Cross-Correlation Between u_t and v_{t-1}
Figure 2.5 display the histograms of cross-correlations between u_t, v_{t-1}. These cross-correlations are computed from 1330 series of ranks for each stock in the sample.

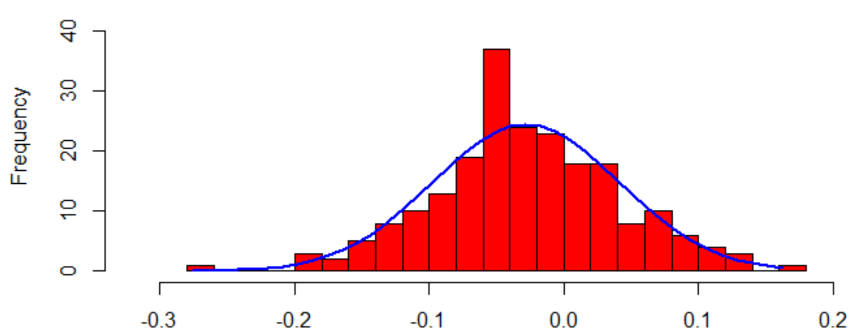

Figure 2.6: Cross-Correlation Between v_t and u_{t-1}
Figure 2.6 display the histograms of cross-correlations between v_t, u_{t-1}. These cross-correlations are computed from 1330 series of ranks for each stock in the sample.

2.3 The Cross-Sectional Gaussian Ranks Model

2.3.1 The Model

Our positional portfolio strategy is about finding the optimal allocation based on the future position of all equities in the portfolio. To predict the future positions, we need to define a joint dynamic model of ranks of return and trade volume changes. According to the results presented in the previous section, the joint dynamics of the two rank series can be represented by a Vector Autoregressive model of order one (VAR(1)).

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \Sigma^{1/2} \begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix}, \qquad t = 2, \cdots, T;\ i = 1, \cdots, n, \tag{2.3.1}$$

where $R = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ is the matrix of autoregressive coefficients, Σ represents the variance matrix of the error terms, and the idiosyncratic disturbance terms $e_{1,it}$ and $e_{2,it}$ are serially independent and identically (i.i.d) standard Normally distributed. The autoregressive coefficients matrix R is assumed to have eigenvalues with modulus less than one to ensure the stationarity of the process. Since the ranks are marginally standard Normally distributed, the marginal variance of the ranks is

$\begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix}$. The diagonal terms are the variances of u_{it} and v_{it} (which are equal to one, since both ranks are cross-sectionally Normally distributed) and η represents the contemporaneous correlation between u_{it} and v_{it}. Thus, we need to impose a constraint on the error variance matrix Σ that follows from the equality below:

$$\begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix} = R \begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix} R' + \Sigma \qquad \textit{where} \;\; |\eta| < 1 \tag{2.3.2}$$

Hence:

$$\Sigma = \begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix} - R \begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix} R' \tag{2.3.3}$$

To estimate the VAR(1) model, let us rewrite equation (4.4) as follows:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = R \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \Sigma^{1/2} \begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix}, \qquad t = 2, \cdots, T; \quad i = 1, \cdots, n. \tag{2.3.4}$$

where $\Sigma = \begin{pmatrix} 1-A & \eta - C \\ \eta - C & 1-B \end{pmatrix}$, with $A = \rho_{11}^2 + \rho_{12}^2 + 2\eta\rho_{11}\rho_{12}$, $B = \rho_{21}^2 + \rho_{22}^2 + 2\eta\rho_{22}\rho_{21}$ and $C = \rho_{11}\rho_{21} + \rho_{12}\rho_{22} + \eta\rho_{12}\rho_{21} + \eta\rho_{11}\rho_{22}$.

The parameters of model (2.7) are estimated by the maximum log likelihood with the following objective function[9] that is maximized with respect to the autoregressive parameters and η as follows:

$$logL(R,\eta) = \sum_{i=1}^{N}\sum_{t=2}^{T}\Big\{ - log(2\pi) - \frac{1}{2}log(|\Sigma|) - \frac{1}{2}\Big(\begin{pmatrix} e_{1,it} \\ \\ e_{2,it} \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} e_{1,it} \\ \\ e_{2,it} \end{pmatrix} \Big)\Big\} \quad (2.3.5)$$

The VAR(1) parameters are estimated from the entire sample of 1330 stocks over the period of 1999 to 2016. Table 2 shows the results of the maximum likelihood estimation. According to the empirical results, all coefficients of the model are strongly significant.

Table 2.1: Estimated VAR(1) Model

$Coefficients$	$Values$
$\hat{\rho}_{11}$	-0.024***
$\hat{\rho}_{12}$	0.010***
$\hat{\rho}_{21}$	-0.024***
$\hat{\rho}_{22}$	-0.354***
$\hat{\eta}$	0.0390***

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: The table provides the parameters of the VAR(1) model, estimated from the entire sample of 1330 stocks over the period 1999 to 2016, by the maximum likelihood (equation 4.8). All coefficients of the model are strongly significant. The estimates of autoregressive coefficients imply that returns' ranks are related negatively with their own past value, while they are related positively with the past value of trade volumes ranks. The ranks of trade volume changes are related negatively with both past ranks of trade volume changes and returns.

[9] After replacing Σ by its expression $\Sigma = \begin{pmatrix} 1-A & \eta - C \\ \eta - C & 1-B \end{pmatrix}$ in the objective function.

The signs of the estimated autoregressive coefficients suggest that:

1) The liquid past losers tend to experience an increase of their current ranks of returns,

2) The non-liquid past losers tend to experience an increase of their current ranks of volume changes.

The contemporaneous correlation $\hat{\eta}$ between the error terms is positive. Hence, a high rank of trade volume change is associated with a high rank of return at the same time t [10]. In other words, the winner stocks are the most liquid ones as well.

An important characteristic of a VAR process is its stationarity. The stationary VAR model contains time series components with time-invariant means, variances, and covariance structure. In practice, the stationarity of an empirical VAR process can be analyzed by considering the companion form and calculating the eigenvalues of the coefficient matrix. The obtained eigenvalues for the VAR(1) model in equation (4.4) are -0.348 and -0.025. Since both eigenvalues are of modulus less than one, we can conclude that the system is stationary.

After estimating the VAR(1) model, we test whether the residuals obey the model's assumption. First, we check for the absence of serial correlation and next, we verify if the error process is normally distributed. The Durbin-Watson (DW) statistic is used to detect the presence of autocorrelation at lag 1 in the residuals (prediction errors) from a regression analysis. The null hypothesis H_0 in the DW test is that the errors are serially uncorrelated and the alternative hypothesis H_1 is

[10] This finding acknowledges the fact that we observed in Section 3.2.1 since the average cross-correlation between ranks of return and trade volume was 0.02.

the existence of a first order autoregressive process in error terms. The DW test applied to the cross-sectional vectors of residuals of model (2.6) indicates that there is no autocorrelation [see Appendix A.2]. The cross-sectional Normal distribution of the residuals and their Normal distribution over time are illustrated in Appendices A.3 and A.4.

Given that the sampling period is long, one can be concerned about the stability of the estimated parameters. To examine the fit of the model over the long sampling period I compute the time series of autoregressive coefficients $\hat{\rho}_{ij}, i, j = 1, 2$ which are obtained by re-estimating the model (equation (2.7)) by rolling, with the window of 109 months ($\simeq$ 9 years).

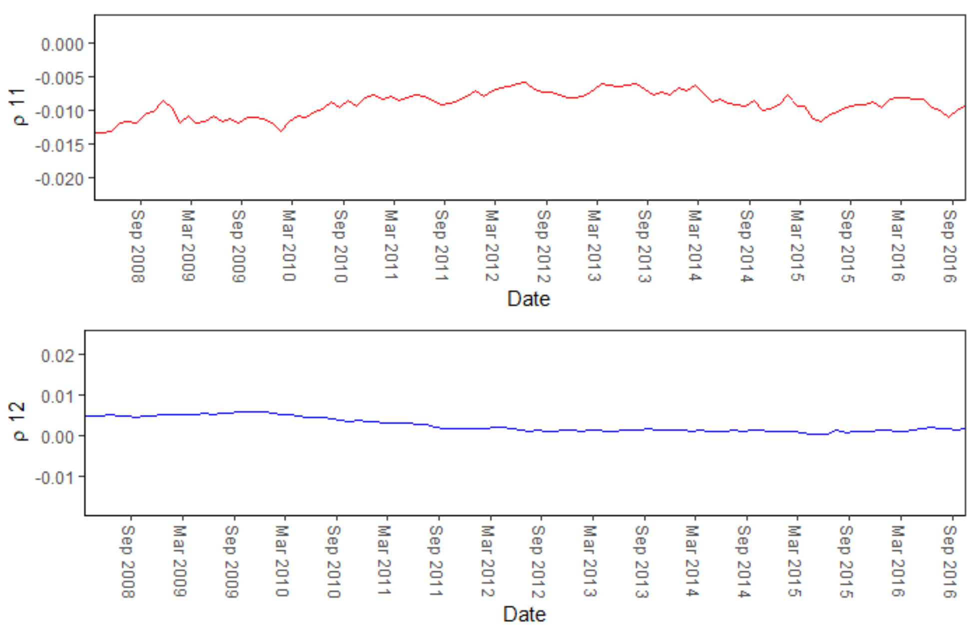

Figure 2.7: Time Series of $\hat{\rho}_{11}, \hat{\rho}_{12}$
Figure 2.7 shows the time series of coefficients $\hat{\rho}_{11}, \hat{\rho}_{12}$, which are obtained by re-estimating the model (equation (2.7)) by rolling with the window of 109 months ($\simeq$ 9 years). Over time, there is little variation and the parameters seem rather stable.

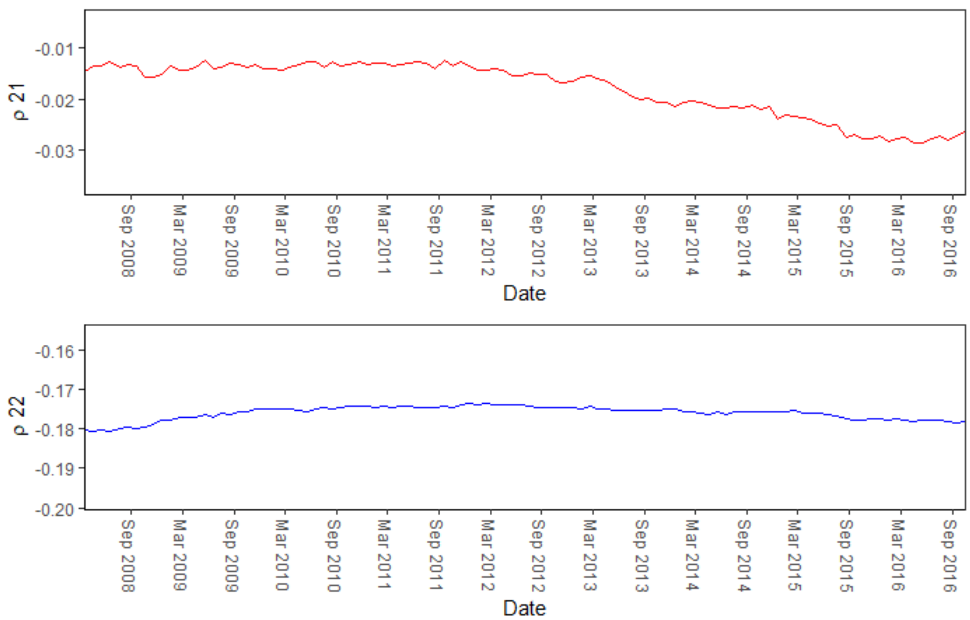

Figure 2.8: Time Series of $\hat{\rho}_{21}, \hat{\rho}_{22}$

Figure 2.8 shows the time series of coefficients $\hat{\rho}_{21}, \hat{\rho}_{22}$, which are obtained by re-estimating the model (equation (2.7)) by rolling with the window of 109 months ($\simeq$ 9 years). Over time, there is little variation and the parameters seem rather stable.

Figures 2.7 and 2.8 show the time series of autoregressive coefficients estimated by rolling over the period: March 2008 - March 2016. We observe slight variation in $\hat{\rho}_{11}$, which is more pronounced than that in $\hat{\rho}_{12}$. Coefficient $\hat{\rho}_{11}$ varies between -0.015 and -0.005, $\hat{\rho}_{12}$ varies between 0 and 0.01, $\hat{\rho}_{21}$ fluctuates between -0.015 and -0.010 and $\hat{\rho}_{22}$ varies around -0.18.

Figure 2.9 shows the time series of $\hat{\eta}$ (contemporaneous correlation between u_{it} and v_{it}) from the rolling estimation over 109 months. We observe that $\hat{\eta}$ displays a downward trend. Over the period 2008 to 2009, it decreases from above 0.2 to slightly less than 0.2 and it continues to decrease at the end of year 2010. Later on, $\hat{\eta}$ remains almost constant until the beginning of 2012. After year 2012, it decreases gradually to zero. Given the slight variation of the parameters and to accommo-

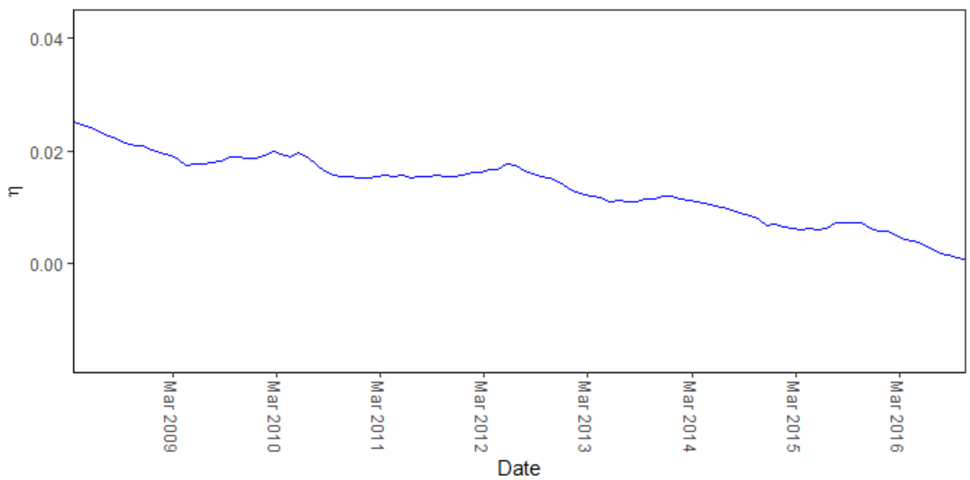

Figure 2.9: Time Series of $\hat{\eta}$

Figure 2.9 shows the time series of $\hat{\eta}$ (contemporaneous correlation between u_{it} and v_{it}) obtained from the rolling estimation with a window of 109 months. The figure shows a downward trend in $\hat{\eta}$.

date the period of crisis 2008 - 2010, I use henceforth the rolling estimation of the VAR(1) model with a window of 109 months. This procedure allows me to update the parameters each month. It produces monthly estimates of model parameters between October 2008 and October 2016, which becomes the new sampling period.

2.4 Positional Portfolio

This Section explains how a given portfolio can be positioned among other stocks on the market with respect to either returns or trade volume changes. It is assumed that there is no short sell, i.e. the numbers of shares of stocks included in the portfolio are non-negative.

2.4.1 Portfolio Return and Portfolio Activity

Let us first consider a portfolio of n stocks. The numbers of shares of these stocks are $\alpha_i,\ i = 1, \cdots, n$. These quantities are non-negative by the no short sell condition. At time t the prices are $P_{i,t},\ i = 1, \cdots, n$ and the dollar values of trade quantities are $TV_{i,t},\ i = 1, \cdots, n$. Then, the total value and total trade value for this portfolio are:

$$\bar{P}_t(\alpha) = \sum_{i=1}^{n} \alpha_i P_{i,t}, \qquad \bar{TV}_t(\alpha) = \sum_{i=1}^{n} \alpha_i TV_{i,t}, \tag{2.4.1}$$

respectively. Thus, the changes between t and $t+1$ are:

$$\frac{\bar{P}_{t+1}(\alpha)}{\bar{P}_t(\alpha)} = \frac{\sum_{i=1}^{n} \alpha_i P_{i,t+1}}{\sum_{i=1}^{n} \alpha_i P_{i,t}} \tag{2.4.2}$$

$$\frac{\bar{TV}_{t+1}(\alpha)}{\bar{TV}_t(\alpha)} = \frac{\sum_{i=1}^{n} \alpha_i TV_{i,t+1}}{\sum_{i=1}^{n} \alpha_i TV_{i,t},}, \tag{2.4.3}$$

These changes can be rewritten as:

$$\frac{\bar{P}_{t+1}(\alpha)}{\bar{P}_t(\alpha)} = \sum_{i=1}^{n} \left(\frac{\alpha_i P_{i,t}}{\sum_{i=1}^{n} \alpha_i P_{i,t}} \frac{P_{i,t+1}}{P_{i,t}} \right) \equiv \sum_{i=1}^{n} \beta^r_{i,t} \frac{P_{i,t+1}}{P_{i,t}} \tag{2.4.4}$$

$$\frac{\bar{TV}_{t+1}(\alpha)}{\bar{TV}_t(\alpha)} = \sum_{i=1}^{n} \left(\frac{\alpha_i TV_{i,t}}{\sum_{i=1}^{n} \alpha_i TV_{i,t}} \frac{TV_{i,t+1}}{TV_{i,t}} \right) \equiv \sum_{i=1}^{n} \beta_{i,t}^{tv} \frac{TV_{i,t+1}}{TV_{i,t}} \tag{2.4.5}$$

Equations (2.11) and (2.12) provide the aggregate formulas of changes in prices and trade volumes, respectively. These allocations are:

a) the allocation in number of shares: $\alpha_i,\ i = 1, \cdots, n$, or

b) the allocation in capitalization: $\beta_{i,t}^{r},\ i = 1, \cdots, n$, or

c) the allocation in dollar weighted activity: $\beta_{i,t}^{tv},\ i = 1, \cdots, n$.

All the above allocations are non-negative, by the no short sell assumption.

2.4.2 How to Position a Portfolio

In order to position a portfolio with no short sell, we use the aggregation formulas of Section 2.4.1. To make the link with the definitions of returns and changes in trade volume defined in the previous section, we also apply the aggregation formulas by substituting the geometric return (change) by their arithmetic counterpart.

This approximation is valid at the first order, as long as the changes are not too large. Thus, we can position a given portfolio among other assets by first defining the return (resp. activity) of that portfolio as:

$$r_t(\beta_t^r) = \beta_t^r r_t = \sum_{i=1}^{n} \beta_{i,t}^r r_{i,t} \tag{2.4.6}$$

$$tv_t(\beta_t^{tv}) = \beta_t^{tv} tv_t = \sum_{i=1}^{n} \beta_{i,t}^{tv} tv_{i,t} \tag{2.4.7}$$

Next, we deduce its position with respect to returns and changes in trade volumes:

$$u_t(\beta_t^r) = \Phi^{-1} F_t^r \Big[\sum_{i=1}^{n} \beta_{i,t}^r (F_t^r)^{-1} \Phi(u_{i,t}) \Big], \tag{2.4.8}$$

$$v_t(\beta_t^{tv}) = \Phi^{-1} F_t^{tv} \Big[\sum_{i=1}^{n} \beta_{i,t}^{tv} (F_t^{tv})^{-1} \Phi(v_{i,t}) \Big], \tag{2.4.9}$$

where the c.d.f. F_t^r and F_t^{tv} are derived with respect to the universe of the n stocks considered.

2.5 Expected Positional Momentum Strategies

This section introduces the positional momentum portfolio strategy based on either the expected ranks of return or expected ranks of volume changes conditional on their past, and examines the comparative performance of the proposed portfolios. The standard positional momentum strategy consists of adjusting the portfolio by buying stocks or other securities with high observed past returns and selling stocks with poor observed past returns. The expected positional momentum strategy introduced by Gagliardini et. al. (2019) extends the momentum portfolio methodology by providing at time t portfolios of stocks with high expected return ranks at each time $t+1$. The future ranks are forecasted out-of-sample from a univariate autoregressive AR(1) model of return ranks [Appendix B].

The proposed approach in this section, the momentum positional portfolio is based on the expected return ranks and is adjusted at each time t, conditional on the past return and volume change ranks. In our study, the expected ranks at $t+1$ are forecast out-of-sample from the bivariate VAR(1) model (equation 2.7) of return and volume change ranks at each time t. In Section 2.5.1, the momentum portfolios based on the expected return ranks are presented. The liquid portfolios based on the expected trade volume change ranks are presented in Section 2.5.2.

The literature has documented that stocks reverse in returns at short monthly horizons (see e.g. Jegadesh (1990), Avramov et.al. (2006)) and long horizon (Hong and Stein (1999)) likely due to overreaction of some investors to news (De Bondt and Thaler (1985), Hong and Stein (1999)). Therefore, we also consider the positional reverse momentum portfolios and positional reverse liquid portfolios which contain stocks with low expected return ranks and low expected volume change ranks, respectively.

For the positional strategy, it is important to define the investment universe which can be different than the positional universe. The investment universe is a set of assets potentially introduced in the portfolio, while the positional universe is a set of assets that has been used to define the ranks. For instance, for a fund manager, the investment universe may be a fraction of the stocks, whereas the positioning universe can be the set of all stocks which are trading in the stock market.

This section, considers the investment universe equivalent to the positional universe (as in Section 5), which contains 1330 stocks returns and trade volumes in the balanced panel from NASDAQ market from October 2008 to October 2016.

2.5.1 Positional Momentum Strategies

This section examines the performance of the momentum strategy based on the expected return ranks. It also compares the proposed methodology based on the VAR(1) model in equation (2.7) with the positional momentum strategy introduced by Gagliardini et. al. (2019) which is based on predicted return ranks from the AR(1) model [see Appendix B]. All coefficients in both the VAR(1) and AR(1) models are estimated and updated monthly by using a rolling window of 109 months that allows us to accommodate the crisis period.

2.5.1.1 Definition of The Strategies

The following strategies are considered to compute portfolios with monthly adjustments of asset allocations and equal look-back periods of one month over the period 2008 to 2016:

1) **The Expected Positional Momentum Strategy (EPMS)**

This strategy selects at time t an equally weighted portfolio of stocks with the 5% highest expected return ranks[11]. The expected ranks at time t are forecast one-step

[11] The timing of the forecast is moved to t instead to $t+1$ in order to compare the prediction-based strategies with the standard momentum.

ahead at time $t-1$ from,

a) the bivariate VAR(1) model (equation (2.7)),

b) the univariate AR(1) model (Appendix B).

2) **The Expected Positional Reverse Strategy (EPRS)**

This strategy is similar to the EPMS except for including in the portfolio stocks with the 5% lowest expected return ranks at time t, which are forecast one-step ahead at time $t-1$ from

a) the bivariate VAR(1) model (equation (2.7)),

b) the univariate AR(1) model (in Appendix B).

3) **Equally Weighted Portfolio (EW)**

This portfolio is an Equally Weighted (EW) portfolio of all 1330 stocks. It is used in the performance study as the benchmark and a market portfolio proxy.

4) **The Positional Momentum Strategy (PMS)**

This is the standard strategy that selects at time t an equally weighted portfolio including all stocks whose observed returns at time $t-1$ are in the upper 5% quantile of the CS (cross-sectional) distribution respectively. The Gaussian ranks of return of these stocks are such that $u_{i,t-1} \geq 1.64$.

5) **The Positional Reversal Strategies (PRS)**

This standard strategy selects at time t an equally weighted portfolio including all stocks with the observed return ranks at time $t-1$ in the lowest 5% quantile of the CS distribution.

2.5.1.2 Return Performance of the Strategies

The time series of monthly portfolio returns generated by the above five strategies are illustrated in Figure 2.10. We observe a period of high volatility at the beginning of the sample due to the crisis. During that period, the EPMS provided positive returns while avoiding strong negative returns displayed by the PRS over the entire sampling period.

During the crisis, the VAR-EPMS shows surprisingly high returns in 2009. After 2008, the volatility decreases and the monthly returns on the EPMS and the PRS portfolios continue to dominate the other strategies, except for March 2016 when the EPRS strategy is more efficient. As shown in Table 2.2, the VAR-based EPMS strategy outperforms all other strategies in terms of average monthly returns.

Table 2.2 presents the statistics summarizing the monthly returns on the five positional portfolios over the sampling period 2008-2016, including both the expected positional strategies based on VAR(1) and AR(1) models. On average, the VAR-based EPMS strategy provides the highest monthly return and outperforms all other strategies.

Moreover, the average monthly return on the equally weighted portfolio (EW) is slightly lower than on the PRS portfolio while the PRS has a higher average monthly return then the PMS portfolio. This implies that the reversal portfolios based on the lowest 5% past ranks of returns (PRS) provide higher average returns than the PMS which is based on the highest 5% past ranks of returns. Hence, the portfolio of past losers (stocks with low return ranks), provides a higher average monthly return than the portfolio of past winners (stocks with high past return ranks).

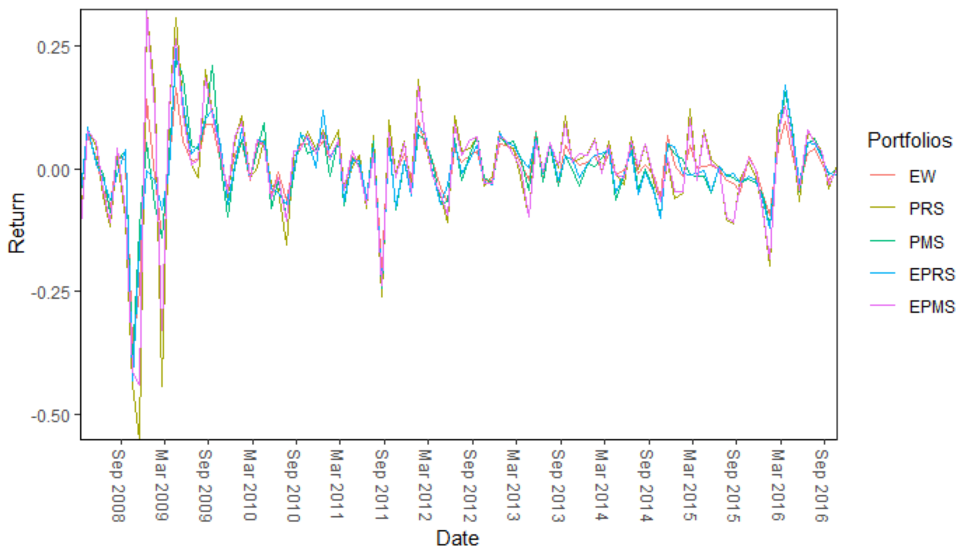

Figure 2.10: Monthly Portfolio Returns on Momentum Portfolios.
Figure 2.10 shows the time series of monthly portfolio returns generated by the five strategies, from September 2008 to October 2016. The returns on portfolios are color-coded as follows: EW-red, PRS-olive, PMS-green, EPRS-blue, EPMS-purple.

The EPRS portfolios based on forecasts from either the AR and VAR models have lower average returns than the EW portfolio, which are positive. Moreover, the reversal portfolio based on the expected ranks of returns EPRS obtained from the bivariate VAR model provides a higher average return than the EPRS based on the univariate AR model.

The last row of Table 2.2 provides the positional Sharpe ratio on the momentum portfolios. The Sharpe ratio is obtained from the following formula:

$$SR = \frac{\bar{r}_p - r_f}{\sigma_p} \tag{2.5.1}$$

Table 2.2: Monthly Returns on Positional Momentum Portfolios

	EW	*PMS*	*PRS*	EPMS		EPRS	
				VAR	*AR*	*VAR*	*AR*
Mean	0.0042	0.0020	0.0043	**0.0064**	0.0037	0.0034	0.0022
Standard Deviation	0.0676	0.0798	0.1202	0.1054	0.1090	0.0770	0.0743
Skew	-2.1586	-1.0888	-1.654	-1.3467	-1.5703	-1.6173	-1.1998
Kurt	12.168	9.222	9.6130	8.444	9.6056	12.485	9.7066
Sharpe Ratio	0.0416	0.0067	0.0240	**0.0471**	0.0206	0.0258	0.0103

Note: Table 2.2 presents the summary statistics of monthly returns on the positional portfolios over the sampling period September 2008 to October 2016. On average, the VAR-EPMS strategy outperforms all other strategies and has the highest Sharpe ratio. The average return of the VAR- EPRS is higher than the AR- EPRS but lower than the EW.

where $\bar{r}_p$ is the mean of the portfolio return, r_f is the risk free returns on the last date of portfolio holding (October 2016) and σ_p is the standard deviation of portfolio returns. In this paper, the time series of 10-year US Generic Government Treasury Bond is considered as a risk-free return. Table 2.2 shows that the highest average return and Sharpe ratio are on the EPMS obtained from the bivariate VAR model. Both the bivariate VAR-EPMS and VAR-EPRS portfolios have higher Sharpe ratios than the AR-EPMS and AR-EPRS.

Figures 2.11 and 2.12 show the cumulative portfolio returns over time on the PMS, PRS and EW portfolios with the inception date of January 2008, compared with the the VAR-EPMS and VAR-EPRS in Figure 2.11 and the AR-EPMS and AR-EPRS portfolios in Figure 2.12.

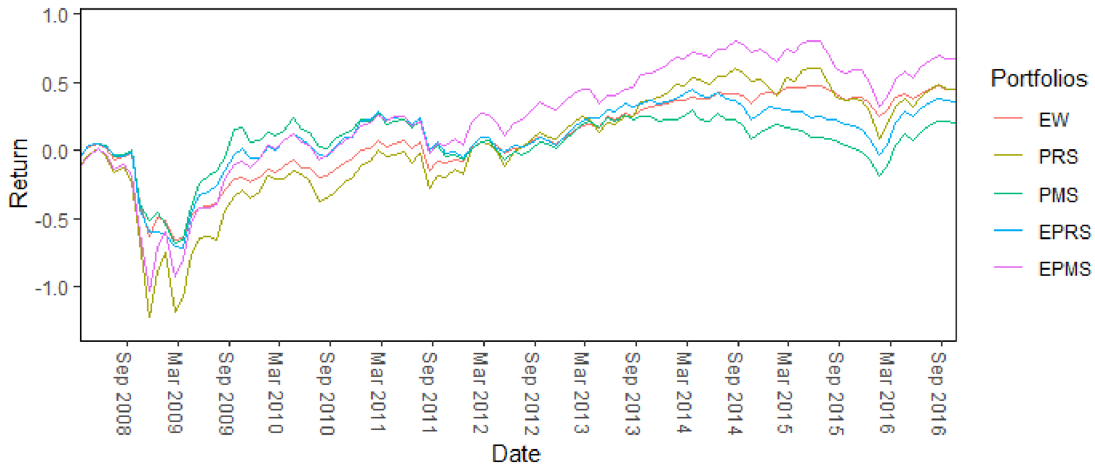

Figure 2.11: Cumulative Returns on Positional Portfolios for VAR(1) Model

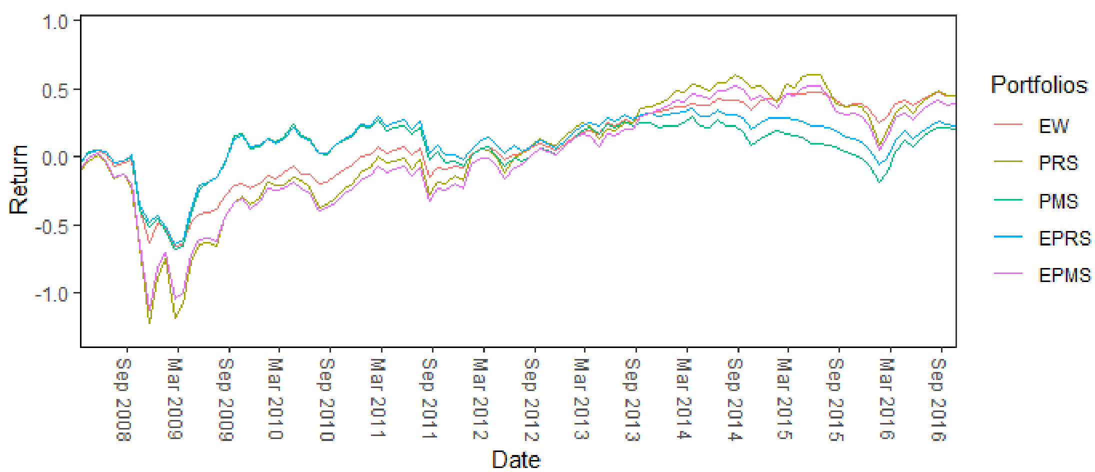

Figure 2.12: Cumulative Returns on Positional Portfolios for AR(1) Model

Figures 2.11 and 2.12 compare the time series of cumulative portfolio returns on the positional momentum portfolios PMS, PRS and EW with the VAR-EPMS, VAR-EPRS (Figure 2.11) and AR-EPMS, AR-EPRS (Figure 2.12) with the inception date of January 2008. The returns on portfolios are color-coded as follows: EW-red, PRS-olive, PMS-green, EPRS-blue, EPMS-purple.

In Figure 2.11, we observe that in early 2008, the cumulative returns on the PMS and VAR-EPRS are close and higher than on the other portfolios whose cumulative returns are more volatile. The PRS and the VAR-EPMS portfolios provide close cumulative returns, although during the crisis period the EW outperforms both VAR-EPMS and VAR-EPRS portfolios. Over the period 2009 - 2010, the PMS portfolio provides the highest cumulative return. From 2010 to 2011, the cumulative returns on the EPMS, EPRS and PMS are very close. After year 2011, the VAR-EPMS portfolio outperforms all other portfolios. Figure 2.12 shows a similar pattern to that observed in Figure 2.11 during the crisis period 2008 to 2009. Over the period 2009 to 2011, the AR-EPRS and PMS portfolios provide the highest cumulative returns, while the AR-based EPMS and PRS portfolios provide the lowest cumulative returns.

From 2011 to 2012, the AR-EPRS has the highest cumulative return and the AR-EPMS has the lowest cumulative return.

From the beginning of year 2012 until the end of 2013, the cumulative returns on all portfolios are close, although for most of that time, the AR-EPRS outperforms the other portfolios. After September 2013 until September 2015, the AR-EPRS is the leading portfolio. Between September 2015 until mid 2016, the EW portfolio has the highest cumulative return and the PMS portfolio has the lowest cumulative return. Between mid 2016 until September 2016, the AR-EPRS provides the highest cumulative return. By comparing Figures 2.11 and 2.12, we find that after year 2011, the VAR-EPMS, i.e. the positional momentum portfolio which is based on the VAR model predictions, outperforms other positional momentum portfolios in terms of the cumulative return.

Table 2.3 shows the cumulative return on October 2016 on the positional momentum portfolios with different inception dates. If an investor holds the positional momentum portfolios since January 1st, 2008 (for 8 years, i.e. the longest holding period), then the best investment is the VAR-EPMS that provides the highest cumulative return, six times higher than the cumulative return on the EW with inception date of 2014 (which is the highest cumulative return over the shortest holding period).

Among the portfolios with inception date of January 1st, 2010, the highest return is on the PRS which slightly exceeds the return on the VAR- EPMS. Over the 4-year holding period, the PRS and the VAR-EPMS have the highest cumulative return. Over the shortest holding period of 2 years, the EW provides the highest cumulative return.

We also observe that when the holding period decreases from 8 to 6 years, the cumulative returns on the VAR-EPMS and AR-EPM increase, although they decrease when the holding period is reduced from 6 to 4-years. The cumulative returns on the VAR-EPMS and AR-EPMS portfolios decrease further over the shortest 2-year holding period. On the contrary, the cumulative returns on the VAR-EPRS and AR-EPRS grow when the holding period is extended from 8 to 4 years, but also decrease significantly over the shortest 2-year holding period.

The results in Table 2.3 also show that the positional portfolios with expected ranks obtained from the bivariate VAR model provide significantly higher cumulative returns than those obtained from the univariate AR model.

Table 2.3: Cumulative Returns on Positional Momentum Portfolios

	EW	*PMS*	*PRS*	EPMS		EPRS	
				VAR	*AR*	*VAR*	*AR*
From 2008	0.444	0.206	0.451	**0.667**	0.384	0.358	0.230
From 2010	0.646	0.136	**0.750**	**0.729**	0.716	0.412	0.151
From 2012	0.528	0.280	**0.630**	**0.630**	0.613	0.419	0.249
From 2014	**0.102**	-0.015	0.034	**0.047**	0.019	0.003	-0.073

Note: Table 2.3 shows the cumulative returns on October 19th 2016 on holding the positional momentum portfolios from different inception dates. If one holds the positional momentum portfolios from January 1st, 2008, the best investment is the VAR-EPMS, which provides the highest cumulative return, six times higher than the cumulative return on the EW with inception date of 2014 (which is the highest cumulative return in the shortest holding period). Among the portfolios with inception date of January 1st, 2010 the highest cumulative return is on the PRS, followed by the VAR-EPMS. Over the 4-year holding period, the PRS and VAR-EPMS provide the highest cumulative return. Over the shortest 2-year holding period the EW provides the highest cumulative return.

Table 2.4 illustrates the returns over short holding periods. Four holding periods of 3,6,9 and 12 months are considered . The first seven columns of Table 2.4 show the cumulative returns on all five positional portfolios obtained by re-investing every month [12], with one month look-back period, over the entire sampling period. The last two columns report the results with 3, 6, 9 and 12 month look-back periods. When the rolling investment with 3-month holding period is considered, the VAR-EPMS outperforms all other portfolios in terms of the Sharpe ratio and the average cumulative return. Over the 6 and 9-month holding periods, the PRS based on 6 and 9 months look-back periods have the highest returns. Over the 12-month holding period, the VAR-EPMS outperforms the other portfolios again in terms of the average cumulative return.

[12]We refer to this strategy as a rolling investment.

We also report that the PRS outperforms the PMS at holding periods of 3, 6 and 9 months. This is consistent with the finding of Lehmann (1990) and Jegadeesh (1990) who find that stock returns exhibit strong reversals for short look-back periods (one to six months) [see also, Moskowitz, Ooi and Pedersen (2012)].

Table 2.4: Rolling Cumulative Returns on Positional Momentum Portfolios

Holding Period	Look-Back=1 Month							Look-Back=Holding	
	EW	*PMS*	*PRS*	EPMS		EPRS		*PMS*	*PRS*
3 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.013	0.006	0.014	**0.021**	0.012	0.011	0.007	0.003	0.020
Standard Deviation	0.118	0.148	0.200	0.178	0.178	0.151	0.151	0.135	0.182
Sharpe Ratio	0.101	0.033	0.067	0.112	0.061	0.062	0.039	0.009	0.092
6 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.026	0.010	0.029	0.042	0.024	0.020	0.012	**0.044**	**0.056**
Standard Deviation	0.184	0.229	0.307	0.271	0.271	0.236	0.236	0.164	0.223
Sharpe Ratio	0.136	0.038	0.089	0.149	0.084	0.078	0.047	0.216	0.214
9 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.043	0.015	0.050	0.068	0.043	0.031	0.020	**0.077**	**0.125**
Standard Deviation	0.213	0.264	0.352	0.311	0.311	0.272	0.272	0.148	0.453
Sharpe Ratio	0.197	0.052	0.138	0.213	0.133	0.109	0.069	0.431	0.248
12 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.073	0.029	0.090	**0.110**	0.079	0.054	0.037	0.086	0.055
Standard Deviation	0.206	0.260	0.336	0.302	0.302	0.267	0.267	0.100	0.188
Sharpe Ratio	0.347	0.106	0.264	0.361	0.257	0.197	0.134	0.690	0.120

Note: The first seven columns of Table 2.4 display the average cumulative returns on all five positional momentum portfolios obtained by re-investing every month, with a one month look-back period. The last two columns show the cumulative returns on positional momentum portfolios based on the top and bottom 5% stocks, with 3,6,9 and 12 months look-back periods and holding periods of 3,6,9 and 12 months respectively. The VAR-EPMS outperforms the other portfolios in terms of the Sharpe ratio and average cumulative return over the 3 month. In the 6 and 9 month holding periods, the PRS based on 6 and 9 months look-back periods have the highest return. In the 12 month holding period, the VAR-based EPMS beats the other portfolios.

2.5.2 Expected Liquidity Positional Momentum Strategy

2.5.2.1 Definition of the Strategies

This section introduces the positional liquid portfolio management strategies that are defined below and named according to the terminology introduced in the previous section, with the letter "L" for liquidity added to the acronyms [13]:

1) **The Liquid Expected Positional Momentum Strategy (LEPMS)**

This strategy selects at time t an equally weighted portfolio of stocks with the 5% highest expected ranks of trade volume changes at each month. The expected ranks at time t are forecast one-step ahead at time $t-1$ from a) the bivariate VAR(1) model (equation. (2.7)), b) the univariate AR(1) model for liquidity, as estimated by the ranks of volume changes.

2) **The Liquid Expected Positional Reverse Strategy (LEPRS)**

It selects at time t the 5% lowest expected ranks of trade volume changes of time t, which are forecast one-step ahead at time $t-1$ from a) the bivariate VAR(1) model (equation (2.7), b) the univariate AR(1) model for liquidity.

The LPMS and LPRS are the momentum portfolios directly defined from the past ranks of trade volume changes at time $t-1$. The EW portfolio remains an equally weighted portfolio of all stocks, as in the previous Section.

[13] Many micro-structure models suggest, it is easier to trade when the market is active, therefore linking the trade volume and liquidity.

2.5.2.2 Return Performance

Let us now compare the performance of the liquid expected positional portfolios in terms of returns. Figure 2.13 shows the time series of monthly returns on the liquid positional portfolios LPMS, LPRS, VAR-LEPMS, VAR-LEPRS and EW. We observe a period of high volatility at the beginning of the sample due to the crisis.

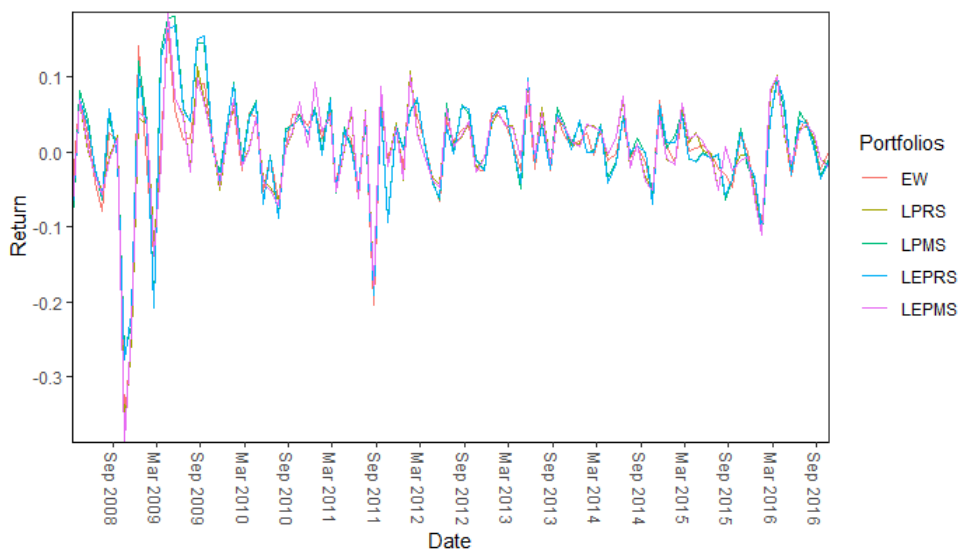

Figure 2.13: Monthly Returns on Liquid Positional Portfolios.
Figure 2.13 shows the time series of monthly returns on liquid positional momentum portfolios generated by the five strategies from September 2008 to October 2016. The returns on portfolios are color-coded as follows: EW-red, LPRS-olive, LPMS-green, VAR-LEPRS-blue, VAR-LEPMS-purple.

During that period, the VAR-LEPRS provides the highest returns for most of the time (this is in line with Daniel and Moskowitz (2016)) while at the end of 2008 all liquid portfolios report negative returns. Moreover, the volatility of the liquid positional portfolios is lower than the volatility of momentum positional portfolios in the previous Section (see Figure 2.10). After 2008, the volatility decreases and the monthly returns on the VAR-LEPRS portfolio along with the LPMS continue to dominate the other strategies, until March 2010. Between March 2010 and March 2012 the VAR-LEPMS has the highest monthly return. Next, until September 2013 the VAR-LEPRS and LPMS are the leading portfolios again. From September 2013 until the end of the sampling period, the returns on all portfolios are close and vary between -0.1 and 0.1.

Table 2.5, shows the summary statistics on the returns on liquid (expected) positional portfolios over the entire sampling period. According to these results, all liquid expected positional portfolio strategies produce higher average return than the equally weighed portfolio (EW). The LPMS portfolio which is based on the highest volume change ranks at time $(t-1)$ has the highest average return.

Table 2.5: Returns on Positional Liquid Portfolios

	EW	*LPMS*	*LPRS*	LEPMS		LEPRS	
				VAR	*AR*	*VAR*	*AR*
Mean	0.0042	**0.0101**	0.0048	0.0045	0.0042	**0.0085**	0.0077
Standard Deviation	0.0676	0.0723	0.06862	0.0692	0.0698	0.0710	0.0717
Skew	-2.1586	-1.0298	-2.1103	-2.1448	-2.2811	-1.0657	-1.4560
Kurt	12.168	6.1592	12.742	12.940	13.539	6.2801	8.2578
Sharpe Ratio	0.0416	**0.1198**	0.0491	0.0442	0.0401	**0.1004**	0.0879

Note: Table 2.5 shows the summary statistics for returns on the liquid positional portfolios over the entire sampling period. All liquid expected positional portfolio strategies produce higher average returns than the equally weighed portfolio (EW). The LPMS portfolio which is based on the highest trade volume ranks at time $(t-1)$ has the highest average return. The average return on the VAR- LEPRS, which is based on the expected lowest volume changes ranks is very close to the average LPMS return.

The VAR-LEPRS, which is based on the expected lowest volume ranks produces an average return slightly lower than the average LPMS return. By comparing the results in Table 2.5 to those in Table 2.2, we find that the highest average returns are obtained from the positional liquid LPMS, and the VAR- LEPRS, followed by the VAR-EPMS strategy. Therefore, the LPMS along with these two VAR-based portfolios outperform the other positional portfolios in terms of average monthly returns and the Sharpe ratio. On average the positional portfolios based on very liquid assets with either past strong increase in trading activity or a predicted strong decline in trading volume provide higher average returns than portfolios of stocks with the highest return ranks.

As in the previous section, we can compare the positional liquid portfolios in terms of their cumulative returns. Figures 2.14 and 2.15 show the time series of cumulative returns on the positional liquid portfolios LPMS, LPRS and EW held since 2008 compared to VAR-LEPMS and VAR-LEPRS and AR-LEPMS and AR-LEPRS respectively.

Both Figures reveal parallel patterns of cumulative returns on all liquid positional portfolios. In 2008, all portfolios provide close cumulative returns, and these cumulative returns decline rapidly at the end of the year. From 2009 until the end of the sampling period, the LPMS has the highest cumulative returns, followed by the LEPRS. From 2009 until the beginning of 2014, the equally weighted portfolio has higher cumulative returns than the LPRS and the LEPMS.

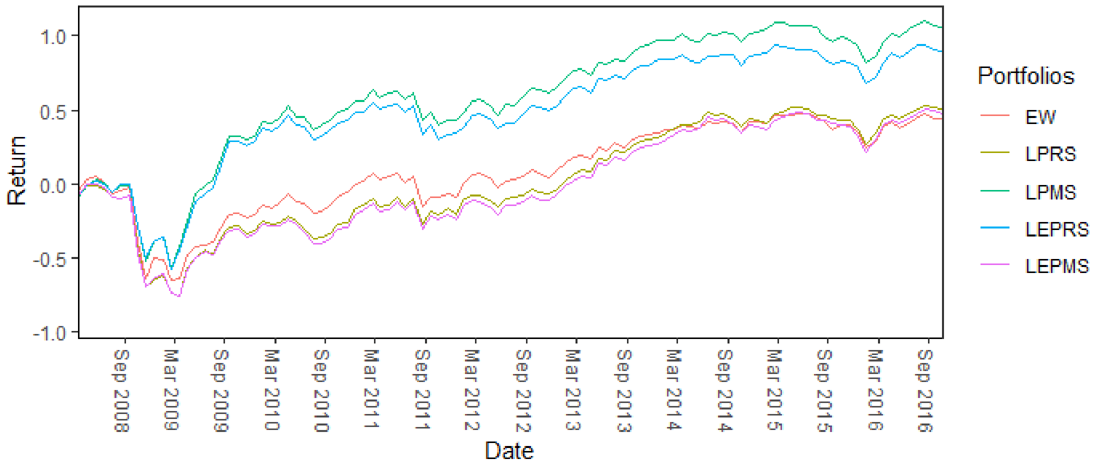

Figure 2.14: Cumulative Returns on Positional Liquid Portfolios for VAR(1) Model

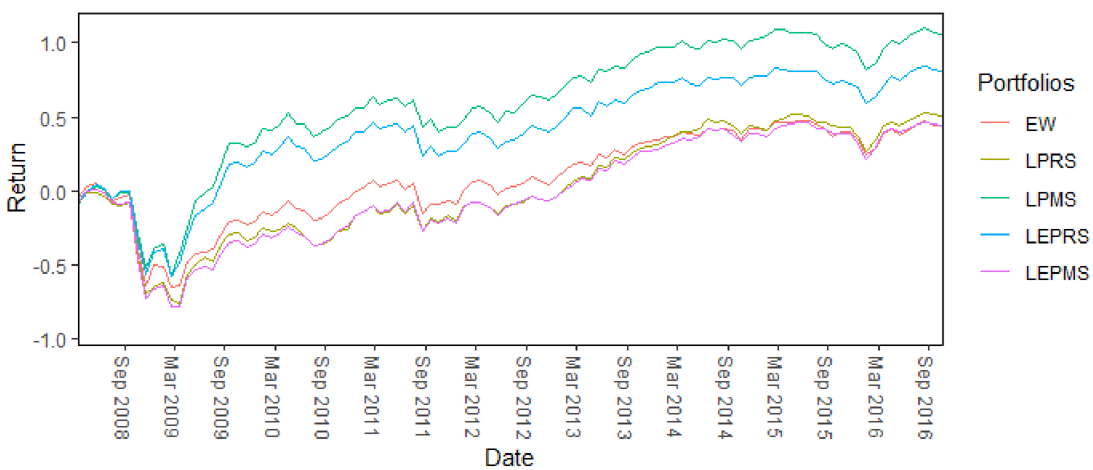

Figure 2.15: Cumulative Returns on Positional Liquid Portfolios for AR(1) Model
Figures 2.14 and 2.15 show the time series of cumulative returns on positional liquid portfolios held since 2008. The returns on portfolios are color-coded as follow: EW-red, LPRS-olive, LPMS-green, LEPRS-blue, LEPMS-purple. In both Figures the LPMS outperforms other strategies and is followed by the LEPRS.

Table 2.6 shows the cumulative returns on the positional liquid portfolios with different inception dates. These results show that all positional liquid portfolios provide positive cumulative returns. Over the holding period 2008-2016, the highest cumulative return is provided by the portfolio with a past strong increase in trade volume (LPMS) followed by the portfolio with a predicted strong decline in trade volume ,VAR-LEPRS. Among the portfolios held from January 2010, the highest cumulative return is obtained from the portfolio with a past strong decline in trade volume (LPRS) which slightly exceeds the return on the VAR-LEPMS. Over shorter holding periods (from 2012 to 2016 and from 2014 to 2016), the VAR-LEPMS which is based on the highest expected trade volume increases, outperforms other portfolios in terms of the cumulative return.

Table 2.6 also reveals that in the long-run (8-year holding period), the LPMS provides the highest cumulative return while in short-run (2-year holding period) the VAR-LEPMS provides the highest cumulative return. Hence, in the long-run, an investment in a portfolio of stocks with the past highest increases in trade volume provides the highest cumulative return. In the short-run, an investment in a portfolio of stocks with the highest VAR-predicted increase in trade volume provides the highest cumulative return.

By comparing the results from Table 2.6 with Table 2.3, we find that the positional liquid portfolios LPMS of stocks with the past highest increases in trade volume, outperforms the other positional momentum portfolios in the long run (8 years holding period). Therefore, if an investor is planning to invest in long run, the

LPMS is the best choice for that investment.

Table 2.6: Cumulative Returns on Positional Liquid Portfolios

	EW	*LPMS*	*LPRS*	LEPMS		LEPRS	
				VAR	*AR*	*VAR*	*AR*
From 2008	0.444	**1.051**	0.501	0.469	0.442	**0.893**	0.807
From 2010	0.646	0.724	**0.819**	**0.809**	0.800	0.605	0.611
From 2012	0.528	0.617	0.707	**0.709**	0.658	0.553	0.545
From 2014	0.102	0.079	0.195	**0.204**	0.166	0.051	0.079

Note: Table 2.6 shows the cumulative returns on October 19th 2016 on positional liquid portfolios from different inception dates. If one holds the positional liquid portfolio from January 1st, 2008, the highest cumulative return would be provided from the LPMS, followed by the VAR-LEPRS. For a 6-year holding period, the highest cumulative return is on the LPRS, which slightly exceeds the return on the VAR-LEPMS. Over shorter holding periods of 4 and 2 years, the VAR-LEPMS outperforms the other portfolios.

Over the 6-year holding period (2010-2016) the LPRS followed by the VAR-LEPMS outperform the PRS and the VAR-EPMS, respectively. Over the shorter period of investment of 4 and 2 years, the VAR-LEPMS which is the portfolio of liquid stocks predicted from the VAR model outperforms all positional momentum portfolios in Table 2.6. These results show that if one intends to invest over shorter horizons, that investor will benefit the most by holding the VAR-LEPMS. Also by comparing these two tables we find that, in the post-crisis period, the VAR-LEPMS provides the higher cumulative return than the VAR-EPMS.

Let us now discuss the results on rolling investment with four holding periods of 3,6,9 and 12 months respectively. In Table 2.7 the first seven columns display the average cumulative returns on all five positional liquid portfolios obtained by investing every month with one month look-back period. The last two columns show the cumulative returns on positional liquid portfolios based on the observed past top and bottom 5% stocks, with 3,6,9 and 12 months look-back periods. Among the portfolios with 1-month look-back period, the LPMS outperforms the other strategies, while the second best portfolio is the VAR-LEPRS. Among the portfolios with equal look-back holding period the LPMS provides the highest return too.

By comparing Table 2.7 with Table2.7, we find that the LPMS provides higher cumulative return over all holding periods and for all look-back periods considered. Therefore an investor who invests in a portfolio of liquid assets obtains the highest cumulative returns. These two tables show that, in general, the positional liquid portfolios which are based on liquidity, provide higher cumulative returns than the positional momentum portfolios which are based on winner stocks.

Table 2.7: Rolling Cumulative Returns on Positional Liquid Portfolios

Holding Period	Look-Back=1 Month							Look-Back=Holding	
	EW	*LPMS*	*LPRS*	LEPMS		LEPRS		*LPMS*	*LPRS*
3 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.013	**0.032**	0.016	0.015	0.014	**0.027**	0.024	0.019	0.017
Standard Deviation	0.118	0.131	0.127	0.128	0.129	0.129	0.127	0.117	0.125
Sharpe Ratio	0.101	0.235	0.114	0.106	0.098	0.202	0.183	0.129	0.107
6 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.026	**0.065**	0.032	0.030	0.028	**0.056**	0.049	**0.056**	0.053
Standard Deviation	0.184	0.209	0.199	0.199	0.201	0.205	0.199	0.169	0.133
Sharpe Ratio	0.136	0.304	0.155	0.143	0.133	0.266	0.241	0.283	0.335
9 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.043	**0.099**	0.052	0.049	0.047	0.086	0.076	**0.130**	**0.101**
Standard Deviation	0.213	0.246	0.231	0.231	0.235	0.241	0.231	0.276	0.157
Sharpe Ratio	0.197	0.399	0.222	0.207	0.193	0.350	0.325	0.423	0.563
12 Months				*VAR*	*AR*	*VAR*	*AR*		
Mean	0.0731	**0.146**	0.087	0.081	0.080	**0.126**	0.115	0.115	0.108
Standard Deviation	0.206	0.253	0.222	0.224	0.229	0.247	0.229	0.115	0.130
Sharpe Ratio	0.347	0.571	0.385	0.359	0.344	0.507	0.496	0.845	0.698

Note: The first seven columns of Table 8 display the average cumulative returns on all five positional liquid portfolios obtained by re-investing every month with holding periods of 3,6,9 and 12 months and one month look-back period. The last two columns show the cumulative returns for positional liquid portfolios based on the top and bottom 5% stocks, for 3,6,9 and 12 months look-back periods. Over the 3- and 6-month holding periods, the LPMS outperforms all other portfolios in terms of the Sharpe ratio and the average cumulative return. Over the 9-month holding period the LPMS with the same holding period is the best portfolio. Over the 12-month holding period, the LPMS and VAR-LEPRS provide the highest returns, respectively.

2.6 Conclusion

This paper introduced positional momentum and liquid portfolio management strategies which are based on the expected bivariate ranks of returns and trade volume changes. The ranks of returns and of trade volume changes are transformed to Gaussian ranks by the quantile function, i.e. the inverse of the cumulative Normal distribution function. The expected ranks are predicted from a conditionally Gaussian vector autoregressive model of order one VAR(1), which represents their joint dynamics. The predicted ranks are used to build the Expected Positional Momentum/Reversal portfolios (EPMS and EPRS) of stocks with high/low ranked expected returns. For portfolio liquidity management, I introduce the Liquid (Expected) Positional Momentum and Reversal portfolios (LPMS, LEPMS, LPRS and LEPRS) of stocks with high and low ranked (Expected) trade volume changes.

These allocation strategies were applied to an investment universe consisting of 1330 stocks which are traded on the NASDAQ market between 2008-2016. The empirical results show that the VAR-based positional momentum EPMS portfolios outperform the equally weighted portfolio, the traditional momentum portfolios and the EPMS portfolios with return ranks predicted from a univariate AR model. This finding suggests that the trade volume ranks help predict the return ranks and improve the positional portfolio performance. Also, a dynamic momentum strategy based on predictions provides higher portfolio returns than the traditional static strategies, such as the EW, PMS and PRS. Moreover, the predictions enhanced by past volume ranks improve that performance even more, which is consistent with Lee and Swaminathan (2000) who demonstrate that past volume changes help predict the momentum magnitude.

The analysis of the positional liquid portfolios shows that the static LPMS strategy produces higher cumulative returns than the VAR or AR prediction-based LEPMS at short horizons of 3,6,9, and 12 months or long horizon of 8 years. This shows that volume rank predictions are not optimal in this context and the static LPMS strategies based on past observed volume ranks can be used instead. It seems that portfolios based on previously liquid stocks, generate higher current returns than portfolios of stocks with predicted liquid stocks. This can be explained by a delayed effect of price (return) response to a past volume increase. Given the limited short-term supply, prices increase due to the liquidity constraint in the past term, generate high returns in the current term. Accordingly, the stocks with high predicted current volume increases will generate high returns over the next period rather than instantaneously. Therefore, the LPMS strategies based on past observed high volume changes tend to outperform the LEPMS strategies based on the predicted current high volume changes.

Regarding the monthly returns, the positional portfolios based on very liquid assets with either past strong increase in trading activity (LPMS) or a predicted strong decline in trading volume (VAR-LEPRS) outperforms the EPMS which contains stocks with the highest expected returns and the equally weighted portfolio. This interesting result shows that a positional portfolio of liquid stocks outperforms a portfolio of expected winners stocks. In the long term, the cumulative return over the period 2008-2016 including the crisis indicate that the positional liquid portfolio with past strong increase in trading volume (LPMS) outperforms all positional momentum portfolios. Over the 4 and 2 years holding periods, the positional liquid portfolio with the highest predicted trade volume based on the VAR model (VAR-LEPMS) provides the highest cumulative return. In terms of cumulative returns over shorter holding periods of 3,6,9 and 12 months, the positional liquid portfolio based on past strong increase in trading volume (LPMS) outperforms all positional momentum portfolios.

It seems that stocks which were liquid in the past generate higher returns than the portfolios of past winner stocks. As pointed out in Section 3.2.2 , Figure 2.4, past volume ranks are positively correlated with the current return ranks. High ranked returns tend to follow past high volume increases and enhance the performance of the LPMS and LEPMS portfolios.

I also find that over the period $2008 - 2010$ (the crisis) the LPMS portfolio provides higher average cumulative returns than the equally weighted and the EPMS portfolios. After the crisis, the LPRS followed by the VAR-LEPMS portfolio provide higher cumulative returns than the other positional momentum portfolios.

Chapter Three

Optimal Positional Momentum and Liquidity Management

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Optimal Positional Momentum and Liquidity Management

3.1 Introduction

This chapter introduces new positional investment strategies that maximize investors' positional utility from holding assets with high expected future return and liquidity ranks. In this chapter the investor is assumed to maximize a CARA (Constant Absolute Risk Aversion) utility function of future position of the assets (ranks of assets). In this respect, I follow the approach of Gagliardini, Gourieroux, Rubin (2019) who introduce a positional utility, which is an increasing function of future asset return positions rather than of future portfolio returns. The optimal allocation vectors provide new investment strategies, such as the optimal positional momentum portfolio, the optimal liquid portfolio and the optimal mixed portfolio that combines high return and liquidity ranks. The future ranks are predicted from a bivariate panel VAR model with time varying autoregressive parameters.

It has been shown that returns on the new optimal portfolios are comparable both theoretically and empirically with the naive equally weighted portfolio as well as with the traditional momentum strategies with look-back and holding periods of various length. The autoregressive parameters of the VAR model displayed variation over time. To accommodate that variation, a time varying parameter VAR model is considered and two methods that allow an investor to update the VAR parameters at each investment time are proposed. The first method consists in re-estimating the model at each time by rolling over a fixed window of observations. The second method exploits the relationship between the autoregressive coefficients of the VAR model and the series of auto-and cross-correlations at lag 1 of returns and volume changes of the SPDR (Standard Poor's Depositary Receipts). The SPDR is an Exchange Traded Fund (ETF), i.e. a regularly updated portfolio mimicking the evolution of the $S\&P$ 500 returns.

More specifically, I show that the future values of autoregressive VAR coefficients can be predicted from simple linear functions of the current auto- and cross-correlations at lag 1 of SPDR return and volume changes. These linear functions are easy to compute and simplify the investment procedure as they eliminate the need for re-estimating the panel VAR model by rolling. In the proposed approach, the time varying parameters are considered predetermined. I show heuristically that the approach can be extended to a random coefficient framework, where the autoregressive VAR coefficients are considered as fixed functions of random factors, which are the auto and cross-correlation estimators with their known asymptotic distributions.

This chapter is organized as follows. Section 3.2 introduces the panel VAR model and its parameter estimates based on the entire sample. It also provides the evidence of time variation of the autoregressive coefficients and extends the model to a time varying parameter VAR model. Section 3.3 documents empirically and establishes the linear relationship between the auto- and cross-correlations of the return and volume change series of SPDR and the series of autoregressive coefficients of the VAR model. Section 3.4 derives the optimal allocation vectors from maximizing the positional CARA utility functions of expected ranks of return and volume changes that lead to the optimal momentum, liquid and mixed portfolios. Section 3.5 presents the empirical results. Section 3.6 concludes the paper. Additional results are gathered in Appendices C, D, E and F.

3.1.1 The Ranks

This chapter examines the dynamics of Gaussian ranks of return and trade volume changes computed from 1330 stocks observed monthly over the period of April 1999 to October 2016. The ranks are defined in Chapter two, Section 2 as follows:

$$u_{i,t} = \Phi^{-1}(\hat{F}_t^r(r_{it})) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \tag{3.1.1}$$

$$v_{i,t} = \Phi^{-1}(\hat{F}_t^{tv}(tv_{it})) \qquad t = 1, \cdots, T; \quad i = 1, \cdots, n, \tag{3.1.2}$$

where $u_{i,t}$ is the Gaussian rank of return $(r_{i,t})$, $v_{i,t}$ is the Gaussian rank of trade volume change $(tv_{i,t})$, Φ is the cumulative distribution function (c.d.f) of the standard Normal, Φ^{-1} is its inverse, i.e. the quantile function of the standard Normal and $\hat{F}_t^r, \hat{F}_t^{tv}$ are the cross-sectional empirical cumulative distribution functions of return and trade volume changes at date t, respectively.

3.1.2 The Model

The positional portfolio strategy is about finding the optimal allocation based on the future position of all equities in the portfolio. To predict the future positions, we define a joint dynamic model of ranks of return and trade volume changes $(u_{it}, v_{it} : i = 1, \cdots, n, t = 1, \cdots, T)$. The joint dynamics of the two rank series can be represented by a Vector Autoregressive model of order one (VAR(1)) as follow:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \Sigma^{1/2} \begin{pmatrix} e_{1,it} \\ e_{2,it} \end{pmatrix} \qquad t = 2, \cdots, T;\ i = 1, \cdots, n, \tag{3.1.3}$$

where $R = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ is the matrix of autoregressive coefficients , Σ represents the conditional variance matrix and the idiosyncratic disturbance terms $(e_{1it}, e_{2,it})$

are serially independent and identically (i.i.d.) standard Normal distributed. The autoregressive matrix R is assumed to have eigenvalues with modulus less than one to ensure the stationarity of the process. The ranks are marginally standard Normally distributed with the marginal variance of the ranks $\begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix}$. Let us introduce an additional assumption as follow:

Assumption 1, The marginal variance of ranks is an identity matrix.

The above assumption implies that $\eta = 0$ [14]. Moreover, it constraints the error variance matrix Σ as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R' + \Sigma. \tag{3.1.4}$$

From equation (3.4) we can compute the matrix Σ as follows:

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - R \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R' = Id - RR', \tag{3.1.5}$$

where Id is a 2×2 identity matrix. Matrix Σ depends on the autoregressive coefficients of the VAR(1) model:

[14] This assumption is not very stringent. In Chapter 2, we have empirically documented that $\hat{\eta}$ is small and tends to 0 at the end of the sampling period.

$$\Sigma = \begin{pmatrix} 1-\rho_{11}^2-\rho_{12}^2 & -\rho_{11}\rho_{21}-\rho_{12}\rho_{22} \\ -\rho_{11}\rho_{21}-\rho_{12}\rho_{22} & 1-\rho_{21}^2-\rho_{22}^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2. \end{pmatrix} \tag{3.1.6}$$

The VAR(1) model (3.3) can be rewritten as follows:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,it} \\ \epsilon_{2,it} \end{pmatrix}, \qquad t = 2, \cdots, T;\ i = 1, \cdots, n, \tag{3.1.7}$$

where error vectors $(\epsilon_{1,it}, \epsilon_{2,it})$ are jointly normally distributed with mean 0 and variance Σ. The marginal densities of the error terms are:

$$\begin{aligned} \epsilon_{1,it} &\sim N(0, \sigma_1^2), \\ \epsilon_{2,it} &\sim N(0, \sigma_2^2), \end{aligned} \tag{3.1.8}$$

where $\sigma_1^2 = 1 - \rho_{11}^2 - \rho_{12}^2$, $\sigma_2^2 = 1 - \rho_{21}^2 - \rho_{22}^2$ and $cov(\epsilon_{1,it}, \epsilon_{2,it}) = \sigma_{12} = -\rho_{11}\rho_{21} - \rho_{12}\rho_{22}$. The parameters of model (3.7) are estimated by the maximum likelihood method with the following objective function that is maximized with respect to the autoregressive parameters ($\rho_{11}, \rho_{12}, \rho_{21}$ and ρ_{22},):

$$logL = \sum_{i=1}^{N} \sum_{t=2}^{T} \Big\{ - log(2\pi) - \frac{1}{2} log(|Id - RR'|) - \frac{1}{2} \Big[\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} - R \begin{pmatrix} u_{it-1} \\ v_{it-1} \end{pmatrix} \Big]'$$
$$\cdot (Id - RR')^{-1} \Big[\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} - R \begin{pmatrix} u_{it-1} \\ v_{it-1} \end{pmatrix} \Big] \Big\} \tag{3.1.9}$$

Table 3.1 shows the results of the maximum likelihood estimation from ranks of all 1330 stocks over the entire sampling period 1999-2016.

Table 3.1: Estimated VAR(1) Model for 1330 Stocks

Coefficients	**Values**	**S-D**	**Confidence Interval**
ρ_{11}	-0.024***	0.002	(-0.029 , -0.020)
ρ_{12}	0.012***	0.002	(0.008 , 0.016)
ρ_{21}	-0.010***	0.002	(-0.014 , -0.004)
ρ_{22}	-0.354***	0.001	(-0.357 , -0.351)

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: The table provides the parameters of the VAR(1) model, estimated from the entire sample of 1330 stocks over the period 1999 to 2016, by the maximum likelihood (equation 3.9). All coefficients of the model are strongly significant.

The empirical results show that all coefficients of the model are statistically significant. The estimated signs of the autoregressive coefficients suggest that:

1) low ranks of past returns and high ranks of past volume changes tend to increase the current ranks of returns,

2) low ranks of past returns and low ranks of past volume changes tend to increase

the current ranks of volume changes.

An important characteristic of a VAR process is its stationarity. A stationary VAR model has time-invariant mean, variance, and covariance structure. In practice, the stationarity of an empirical VAR process can be analyzed by calculating the eigenvalues of the autoregressive coefficient matrix $(\hat{R})$. The computed eigenvalues of $(\hat{R})$ are -0.358 and -0.025. Since both eigenvalues are of modulus less than one, we can conclude that the VAR(1) model is stationary.

Given that the sampling period is long, one can be concerned about the stability of the estimated parameters. Therefore, we re-estimate the equation (3.7) by rolling with the window of 108 months ($\simeq$ 9 years). The rolling estimation yields the estimates of the following VAR(1) with time varying coefficients:

$$\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} = \begin{pmatrix} \rho_{11,t} & \rho_{12,t} \\ \rho_{21,t} & \rho_{22,t} \end{pmatrix} \begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,it} \\ \epsilon_{2,it} \end{pmatrix}, \qquad t = 2, \cdots, T;\ i = 1, \cdots, n, \tag{3.1.10}$$

where the error variances are time varying as well: $\sigma_{1t}^2 = 1 - \rho_{11,t}^2 - \rho_{12,t}^2$, $\sigma_{2t}^2 = 1 - \rho_{21,t}^2 - \rho_{22t}^2$ and $cov(\epsilon_{1,it}, \epsilon_{2,it}) = \sigma_{12t} = -\rho_{11,t}\rho_{21,t} - \rho_{12,t}\rho_{22,t}$.

Figures 3.1 and 3.2 show the time series of autoregressive coefficients of model (3.10) estimated by rolling over the period: March 2008 - September 2016. We observe that there is some variation in $\hat{\rho}_{11,t}$, which is more pronounced than in $\hat{\rho}_{12,t}$. Coefficient $\hat{\rho}_{11,t}$ varies between -0.015 and -0.005 and coefficient $\hat{\rho}_{12,t}$ varies between 0 and 0.01. Coefficient $\hat{\rho}_{21,t}$ takes lower values and fluctuates between -0.015 and

-0.025. Coefficient $\hat{\rho}_{22,t}$ varies around -0.18.

Let $\hat{R}_t,\ t = 1, ..., T$ denote the time series of matrices of time varying autoregressive coefficients from model (3.10). The eigenvalues of matrices $\hat{R}_t,\ t = 1, ..., T$ computed over $t = 1, ..., T$ are of modulus less than one, indicating that the time varying coefficient VAR(1) model is stationary. We also compute the eigenvalues of the constrained matrices $\hat{\Sigma}_t = Id - R_t R_t',\ t = 1, ..., T$ that are positive at all times $t = 1, ..., T$.

In practice, the rolling estimation of a panel VAR(1) model can be difficult. Therefore, in next Section we explore an alternative approach, where the autoregressive coefficients can be modelled as simple linear functions of time varying factors that are easy to compute.

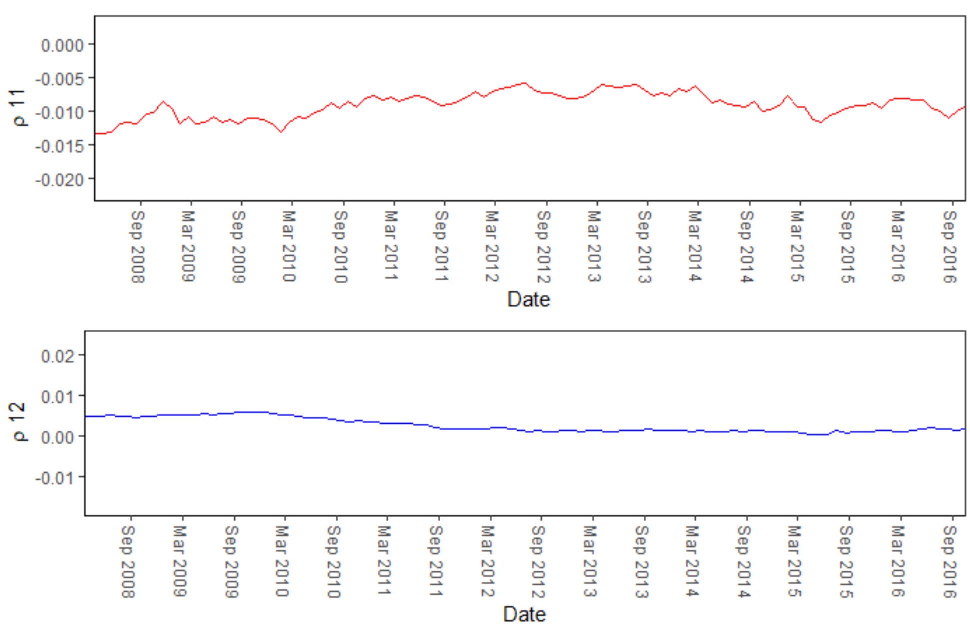

Figure 3.1: Time Series of $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}$

Figure 3.1 shows the time series of coefficients $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}$, which are obtained by re-estimating model (3.10) by rolling with the window of 108 months ($\simeq$ 9 years).

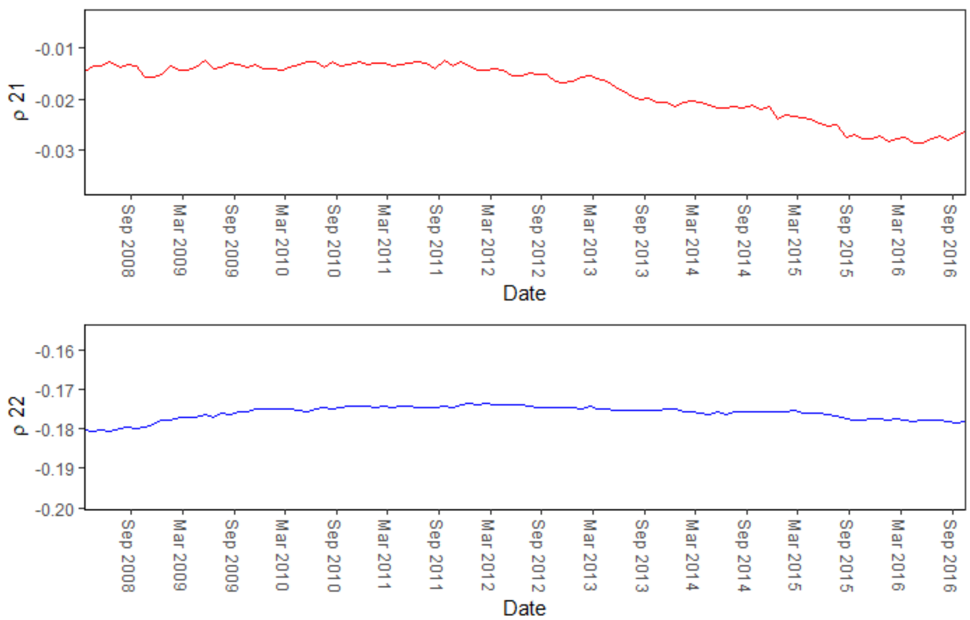

Figure 3.2: Time Series of $\hat{\rho}_{21,t}, \hat{\rho}_{22,t}$

Figure 3.2 shows the time series of coefficients $\hat{\rho}_{21,t}, \hat{\rho}_{22,t}$, which are obtained by re-estimating the model (equation (3.10)) by rolling with the window of 108 months ($\simeq$ 9 years).

3.2 Dynamic Autoregressive Coefficient Model

The stock prices behavior is reflected by the dynamics of stock market indexes such as the $S\&P500$ and by the prices of its mimicking portfolio, called the $SPDR$ (Standard & Poor's Depository Receipts) quoted on NYSE with ticker SPY.

3.2.1 Standard & Poor's Depository Receipts (SPDR) as Market Factor

The Standard and Poor's Depository Receipt (SPDR),[15] is an exchange traded fund which holds all of the S&P 500 Index stocks and is designed to reflect the price and yield performance of the S&P 500 Index. The SPDR, first issued by the State Street Global Advisors' investment management group (SSGA) and is traded on the American Stock Exchange (AMEX) since 1993. The SPDR index fund is designed to track the S&P 500 stock market index.

The aim behind this ETF is to provide an investment vehicle that at least roughly produces returns in line with the S&P 500 Index. Unlike mutual funds, the SPDR's trust shares are not created for investors at the time of their investment. In fact, they have a fixed number of shares that are bought and sold on the open market to align their holdings with the S&P 500 index. The S&P 500 index itself is composed of U.S. big companies across all Global Industry Classification Standard (GICS) sectors with a market capitalization of $5 billion or greater. Some literature showed that the SPDR is not mimicking S&P 500 perfectly [see Beaulieu and Morgan (2000)]. while some studies show that the SPDR is mimicking S&P 500 in a linear analysis. For instance, Peng Xu (2014) showed that in a linear dynamics analysis the SPDR and S&P 500 has similar dynamic features while. Since the SPDR is designed to reflect the price and yield performance of the SP 500 Index, it can be considered as the pulse of the U.S. equity market or a common factor that encompasses the effects

[15] Often referred to as the âĂIJspiderâĂİ, and its symbol in the market is SPY

of all news and events on the stock market.

The SPDR is consistently one of the high volume trading vehicles in the U.S. exchanges[16]. Many investors and hedge funds use this fund because it represents the S&P 500 index and by a single purchase, they will have exposure to a wide range of large U.S. companies. Not only the volume but also its good price movement make the SPDR attractive to traders.

Figure 3.3 shows the relationship between the monthly returns on $SPDR$ and $S\&P500$ recorded over the period April 1999 to October 2016[17].

Figure 3.3: SPDR and S&P500 Returns

Figure 3.3 shows the time series of S&P 500 and SPDR's returns from April 1999 to October 2016.

[16]Peng Xu (2014) showed that, the average daily trading volume from Jan, 2001 to Dec, 2005 is over 38 million shares and the average trading value per day is over 4 billion

[17]The returns of SPDR and S&P 500 are computed as log return and the dividends haven't been considered in the return.

We observe that these returns are moving in parallel and are both fluctuating roughly between -0.1 to 0.1. There are periods when the volatility of SPDR's returns is higher than the volatility of the return on S&P 500. For instance, on February, 2000, September, 2001, August, 2002, October, 2008 or September 2011 the returns of SPDR declined more than the returns on S&P 500. Also at the beginning of years 2009 and 2012, July 2013 and at the end of 2014 the returns of SPDR increased more than the returns on S&P 500.

The historical correlation between the returns on SPDR and on S&P500 is 0.66 and the regression coefficient between the squared returns of SPY and S&P 500 is 1.34.[18], suggesting that the SPDR mimics the index rather well as far as a linear static analysis is concerned. Applying a simple linear regression[19], also showed that, these two historical correlations are both statistically significant. From what We observe in Figure 3.3 and also from the linear regressions' results, we can conclude that the returns on SPDR approximate the S&P 500 returns very closely. Therefore, the returns on the SPDR can be considered as a proxy for the market portfolio return.

[18] It corresponds to Peng Xu (2014) who showed the positive historical correlation coefficient between the two return series.

[19] A simple linear regression model between SPDR and S&P 500 returns has been estimated as $r_{SPDR} = a_0 + a_1 r_{S\&P500} + e$, and between the squared returns as $r^2_{SPDR} = a_0 + a_1 r^2_{S\&P500} + e$. where r_{SPDR}, r_{SP} are the return of SPDR and S&P500 respectively, r^2_{SPDR}, r^2_{SP} are the squared return of SPDR and S&P 500, a_0, a_1 are the constant and the coefficient respectively and e is the error term.

3.2.2 Relation Between The Returns and Trade Volume Changes of SPDR

Let us now consider the series of $SPDR$ returns and trade volume changes recorded monthly between April 1999 and October 2016. The trade volume is defined as the total quantity of shares traded per month. The log return and the log volume changes are calculated as follows:

$$
\begin{aligned}
r_t^S &= ln(\frac{P_t^S}{P_{t-1}^S}), \qquad t = 1, \cdots, T \\
tv_t^S &= ln(\frac{TV_t^S}{TV_{t-1}^S}), \qquad t = 1, \cdots, T,
\end{aligned}
\tag{3.2.1}
$$

where P_t^S, P_{t-1}^S are the prices of SPDR at times t and $t-1$, TV_t^S, TV_{t-1}^S are the trade volume changes of SPDR at times t and $t-1$.

A simple way to determine whether there exists a relationship between the series of SPDR returns and trade volume changes, is to examine the cross-correlation function. Figure 3.4 shows the cross-correlation function of returns and trade volume changes of SPDR. We observe that the cross-correlation at lag one is significant. Hence, past trade volume changes can help predict the current returns. We also detect a significant negative contemporaneous correlation between the returns and trade volume changes of SPDR.

Figure 3.5 illustrates the contemporaneous correlation in a regression of SPDR trade volume changes on the returns i.e. r_t^S on tv_t^S. The regression line has a negative

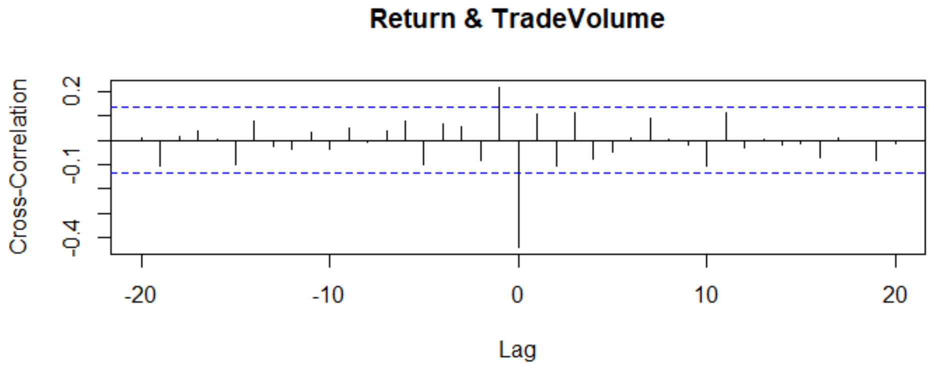

Figure 3.4: SPDR: Cross-Correlation Function of r_t^S and tv_t^S
Figure 3.4 shows the cross-correlation function of returns and trade volume changes of SPDR. There is significant correlation at lags 0 and one.

slope which is consistent with the negative contemporaneous correlation in Figure 3.4. Hence, a high positive return on SPDR is associated with a high negative trade volume change at time t.

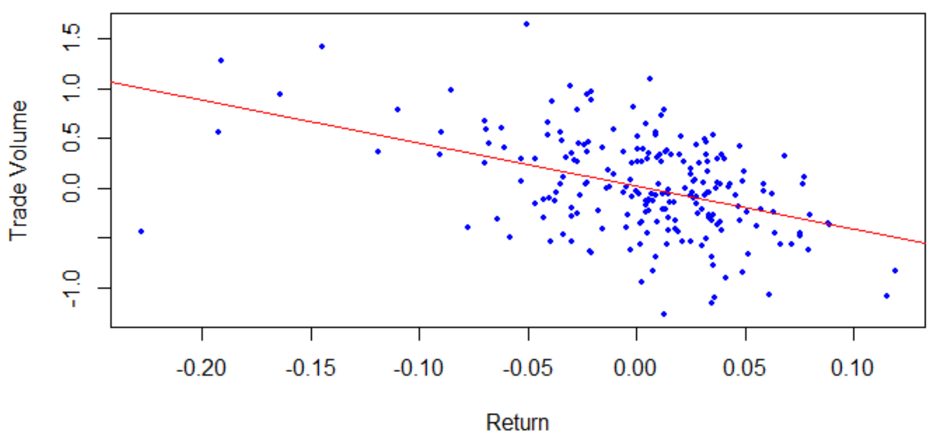

Figure 3.5: SPDR: Regression line of r_t^S on tv_t^S
Figure 3.5 shows the regression line of returns as a linear function of trade volume trade volume changes of SPDR at time t. The regression line has a negative slope, which shows a negative contemporaneous correlation between return and trade volume changes of SPDR.

Table 3.2: Linear Regression of SPDR's Trade Volume Changes on Returns

Coefficients	**Values**	**S-D**	**Confidence Interval**
Intercept	0.023	0.030	(-0.037 , 0.083)
Correlation	-4.335***	0.613	(-5.562 ,-3.108)

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: The Table shows the coefficients of a linear regression of trade volume changes on the returns of SPDR. The regression coefficient is negative and significant.

By applying a simple linear regression, one can also evidence a significant negative relation between trade volume and return. Table 3.2, shows the result of the linear regression of SPDR's trade volume over its return. The regression coefficient is Statistically significant and negative. It means that if the return increase the contemporaneous trade volume would decrease.

3.2.3 Comparing Return and Liquidity Persistence: SPDR and Stock Ranks

We have shown that the dynamics of returns on SPDR mimic the dynamics of market returns and the SPDR returns are correlated with the SPDR's trade volume changes. Moreover, the liquidity of SPDR is the liquidity of an asset with a return equal to the market return.

Let us now explore whether the persistence and cross-correlation of returns and trade volume changes of SPDR is similar to rank persistence in all stocks in our sample. That persistence on average over the entire sampling period is approximated by the estimated autoregressive coefficients of stock return and liquidity ranks $\hat{\rho}_{ij}, i, j = 1, 2$ of model (3.10) reported in Table 3.1. The time varying stock persistence at each time t is approximated by the series of time-varying autoregressive coefficients $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}, \hat{\rho}_{21,t}, \hat{\rho}_{22,t}$ estimated by rolling and displayed in Figures 3.1 and 3.2. We proceed with a dynamic analysis and compare these four series with the series of sample auto- and cross-correlations at lag one of r_t^S and tv_t^S (return and trade volume of SPDR), both estimated by rolling with a window of 108 months ($\sim$ 9 years).

Let the dynamic sample autocorrelations at lag 1 be denoted by $AC(r^S)_t$ and $AC(tv^S)_t$ for returns and trade volume changes, respectively. The dynamic sample cross-correlations at lag 1 between r_t^S and tv_{t-1}^S are denoted by $CC(r^S, tv^S)_t$. The sample cross-correlations between $tv_t^S$and r_{t-1}^S are denoted by $CC(tv^S, r^S)_t$. The distributional properties of these time series are examined and compared in Appendix C, which displays their histograms and non-parametric normal density estimates.

Table 3.3 below shows the means, modes and standard deviations (S.D.) of the time series of $AC(r^S)_t$, $AC(tv^S)_t$, $CC(r^S, tv^S)_t$ and $CC(tv^S, r^S)_t$ in comparison with the autoregressive coefficient series ($\hat{\rho}_{jk,t},\ i, j = 1, 2,\ t = 1, ..., T$).

The mean of the sample auto-correlations of SPDR returns and the mean and mode of $\hat{\rho}_{11t}$ are negative. The mean and mode of cross-correlations of r_t^S, tv_{t-1}^S and $\hat{\rho}_{12t}$ have the same sign and are positive while they are bigger for the cross-correlations of r_t^S, tv_{t-1}^S. The mean and mode of cross-correlations of tv_t^S, r_{t-1^S} are positive while the are negative for $\hat{\rho}_{21t}$. Both sample auto-correlation at lag one of SPDR's trade volume changes and $\hat{\rho}_{22t}$ have negative mean and mode.

Table 3.3: Summary Statistics for Cross- and Auto- Correlation of SPDR and Autoregressive Coefficients $\hat{\rho}_{jk,t}$

Coefficients	**Mean**	**Mode**	**S.D.**
$AC(r^S)_t$	-0.0001	0.0501	0.0846
$\hat{\rho}_{11,t}$	-0.0090	-0.0084	0.0000
$CC(r^S, tv^S)_t$	0.0647	0.0037	0.1020
$\hat{\rho}_{12,t}$	0.0026	0.0013	0.0016
$CC(tv^S, r^S)_t$	0.2494	0.2991	0.0691
$\hat{\rho}_{21,t}$	-0.0179	-0.0138	0.005
$AC(tv^S)_t$	-0.5089	-0.5176	0.0182
$\hat{\rho}_{22,t}$	-0.1758	-0.1748	0.0016

Note: Table 3.3, shows Summary Statistics for Cross-Correlation and Auto-Correlation of return and trade volume changes of SPDR and the Autoregressive Coefficients $\hat{\rho}_{jk,t}$

Table 3.4 shows the results of the t-test of equality of means of these time series[20]. The t-test of the equality of means of sample auto- and cross-correlation of SPDR and the autoregressive coefficients of the VAR(1) model reject the null hypothesis except for the auto-correlation at lag one of r_t^S and $\hat{\rho}_{11t}$.

Table 3.4: T-Test of Equality of the Means

Null Hypothesis	**P-Value**
Mean(AC(r^S)$_t$)=Mean($\hat{\rho}_{11t}$)	0.155
Mean(CC(r^S, tv^S)$_t$)=Mean($\hat{\rho}_{12t}$)	0.000
Mean(CC(tv^S, r^S)$_t$)=Mean($\hat{\rho}_{21t}$)	0.000
Mean(AC(tv^S)$_t$)=Mean($\hat{\rho}_{22t}$)	0.000

Note: Table 3.4, shows the t-test results of equality of the mean of Auto- and Cross-Correlation of SPDR and the Autoregressive Coefficients $\hat{\rho}_{jk,t}$

Figures 3.6-3.9 below illustrate and compare the dynamics of the series of sample auto-and cross-correlations of SPDR with the autoregressive coefficient dynam-

[20]The test is asymptotically valid, due to the non-normality of the computed coefficient series

ics. The right panels show the estimated time varying autoregressive coefficients $\hat{\rho}_{jk,t}, (\ j, k = 1, 2,\ \ t = 1, ..., T)$ plotted with the red line. The left panels show the sample auto- or cross-correlations of r_t^S and tv_t^S at lag one. In all panels, the green and blue lines are indicating the upper and the lower bounds of confidence intervals, respectively.

Figure 3.6 compares the dynamics of $AC(r^S)_t$ and the time series $\hat{\rho}_{11,t}$. We observe that the auto-correlations of SPDR returns increase over time and have two major troughs in October 2008 and October 2011. After year 2012, the auto-correlations remain steady and positive. We observe similar dynamics, although at a different level in $\hat{\rho}_{11,t}$ in the right hand side of Figure 3.6.

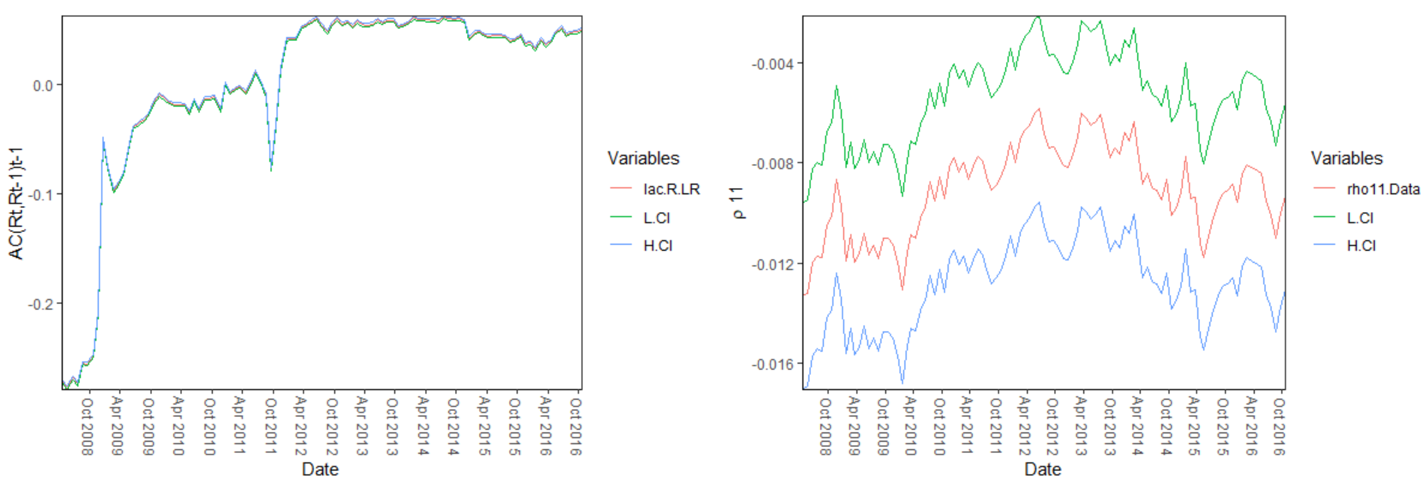

Figure 3.6: Time Series of SPDR Autocorrelations $AC(r^S)_t$ and Coefficients $\hat{\rho}_{11t}$
Figure 3.6 compares the sample auto-correlations at lag one of SPDR's returns with the time series of autoregressive coefficients $\hat{\rho}_{11t}$ from model (3.10). The red line in the left plot shows the sample auto-correlations of (r_t^S, r_{t-1}^S) and the coefficients $\hat{\rho}_{11t}$ in the right plot. In both plots the green and blue lines show the upper and lower bounds of confidence intervals.

The series $\hat{\rho}_{11,t}$ is always negative, and it is growing from December 2008 until August 2012. After August 2012, it starts to decrease. Before August 2012, it has two major peaks on January 2009 and August 2012 and two major drops on March 2010 and October 2011. After August 2012 the series $\hat{\rho}_{11,t}$ reaches its lowest value on July 2015.

Figure 3.7 shows the dynamics of $\mathrm{CC}(r^S, tv^S)_t$ compared to the time series $\hat{\rho}_{12t}$ in the right plot. Both the cross-correlations and $\hat{\rho}_{12t}$ are decreasing over time. In the left plot, the cross-correlations between r_t and tv_{t-1} have two peaks on October 2011 and 2015. In the right plot, we observe that the series $\hat{\rho}_{12t}$ has three major peaks on February 2010, June 2012 and July 2016.

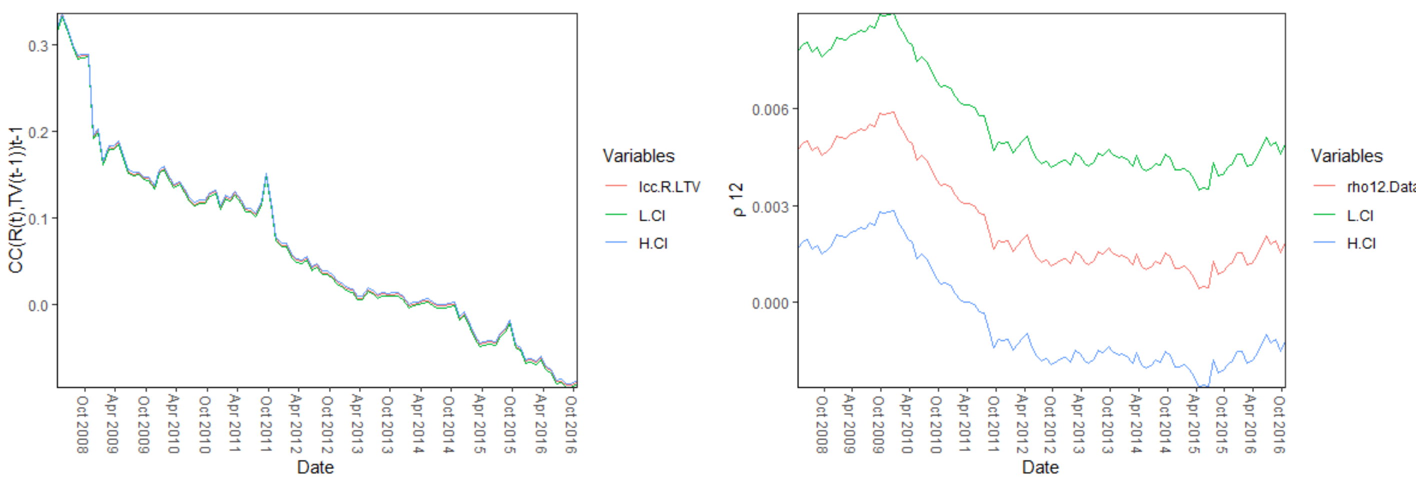

Figure 3.7: Time Series of SPDR Cross-correlations $CC(r^S, tv^S)_t$ and Coefficients $\hat{\rho}_{12t}$

Figure 3.7 compares the sample cross-correlations of (r_t^S, tv_{t-1}^S) of SPDR and the time series of coefficients $\hat{\rho}_{12t}$ from model (3.10). The red line in the left plot shows the sample cross-correlations of (r_t^S, tv_{t-1}^S) and the coefficients $\hat{\rho}_{12t}$ in the right plot. In both plots the green and blue lines show the upper and lower bounds of confidence intervals.

Figure 3.8 compares the dynamics of sample cross-correlations $CC(tv^S, r^S)_t$ of SPRD with the time series of coefficients $\hat{\rho}_{21t}$. The cross-correlations decrease until February 2010 and increase afterwards. They reach the minimum value on December 2008 while always remaining positive. The series of coefficients, $\hat{\rho}_{21t}$ always takes negative values. Coefficients $\hat{\rho}_{21t}$ stay at a constant level until December 2011, and decrease afterwards.

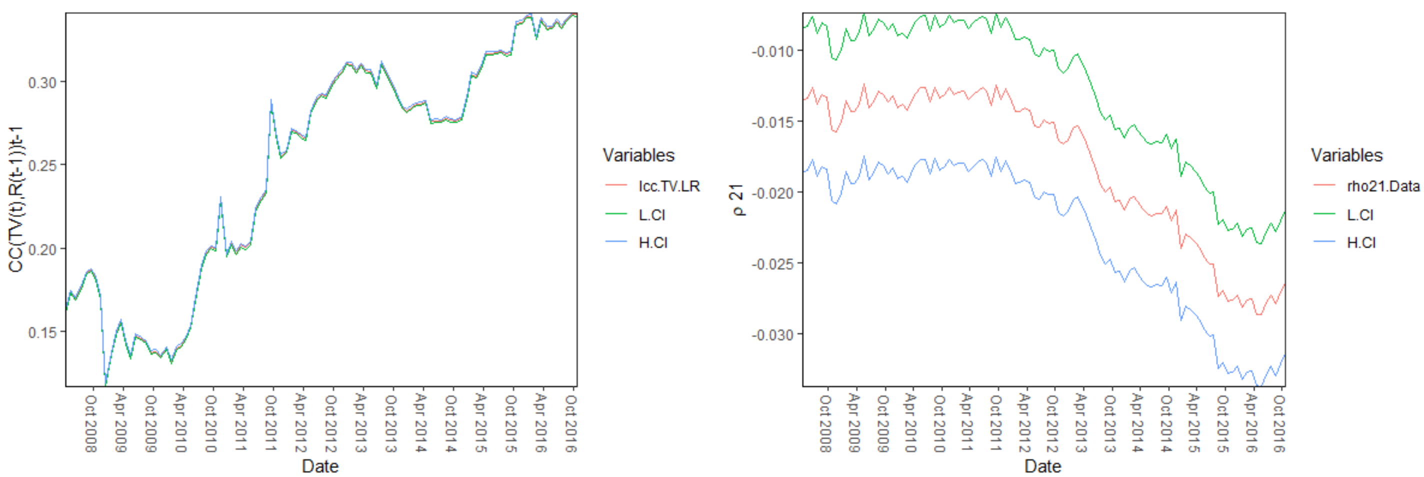

Figure 3.8: Time Series of SPDR Cross-correlations $CC(tv^S, r^S)_t$ and Coefficients $\hat{\rho}_{21t}$
Figure 3.8 compares the sample cross-correlations of (tv_t^S, r_{t-1}^S) of SPDR with coefficients $\hat{\rho}_{21t}$ from model (3.10). The red line in the left plot shows the sample cross-correlations of (tv_t^S, r_{t-1}^S) and the coefficients $\hat{\rho}_{21t}$ in the right plot. In both plots the green and the blue lines show the upper and lower bounds of confidence intervals.

Figure 3.9, shows the sample auto-correlations $\mathrm{AC}(tv^S)_t$ and the time series of coefficients $\hat{\rho}_{22t}$. The dynamics of these two series are different, but they both always take negative values. The auto-correlations at lag one of trade volume changes of SPDR reach their first peak on January 2009 and drop to their minimum value on September 2011. Next, that series grows until November 2014 and then drops to its second minimum value on October 2015. In the right plot, we observe that series $\hat{\rho}_{22t}$ increases until April 2012, and decreases afterwards.

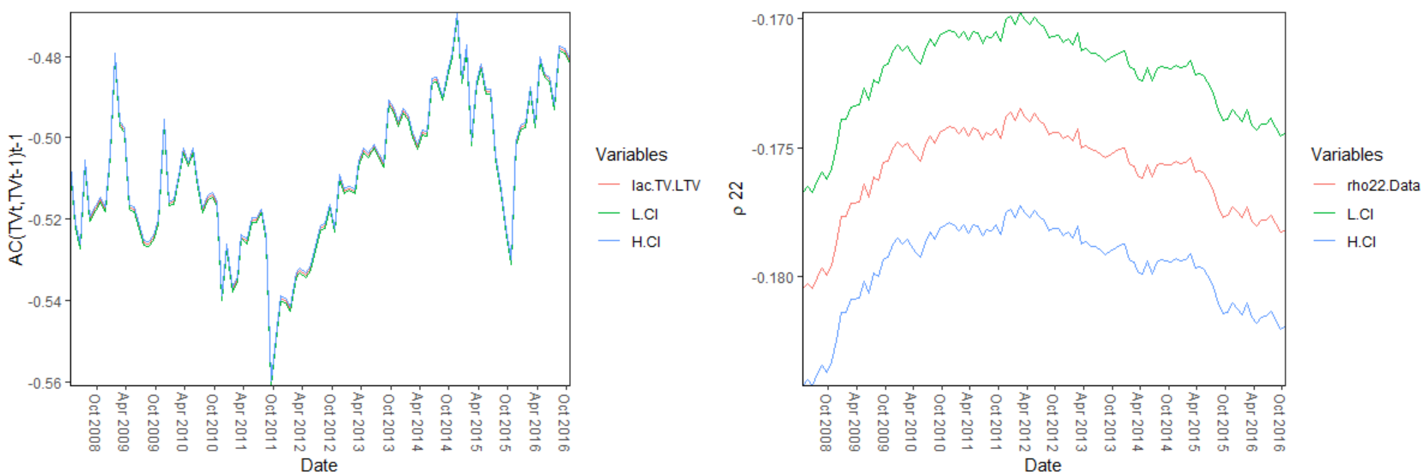

Figure 3.9: Time Series of Auto-correlations $AC(tv^S)_t$ and Coefficients $\hat{\rho}_{22t}$
Figure 3.9 compares the sample auto-correlations at lag one of SPDR's trade volume changes with the time series of coefficients $\hat{\rho}_{22t}$ from model (3.10). The red line in the left plot shows the sample auto-correlations of (tv^S_t, tv^S_{t-1}) and the coefficients $\hat{\rho}_{22t}$ in the right plot. In both plots the green and the blue lines show the upper and lower bounds of confidence intervals.

The empirical analysis of the dynamics and distributional properties of autoregressive coefficients $\hat{\rho}_{jk,t}$ and sample auto- and cross-correlations of SPDR returns and trade volume changes leads to the modelling of autoregressive coefficients as functions of the sample correlation functions of SPDR.

3.2.4 Dynamic Factor Models of $\rho_{jk,t}$

The following regressions reveal the existence of statistically significant linear relationship between the series of autoregressive coefficients $\rho_{jkt},\ (jk = 1, 2,\ t = 1, ..., T)$ and the auto- and cross-correlations of SPDR's return and trade volume changes.

$$\hat{\rho}_{11,t} = a_{110} + a_{11}AC(r^S)_{t-1} + d_{1,t}, \tag{3.2.2}$$

$$\hat{\rho}_{12,t} = a_{120} + a_{12}CC(r^S, tv^S)_{t-1} + d_{2,t}, \tag{3.2.3}$$

$$\hat{\rho}_{21,t} = a_{210} + a_{21}CC(tv^S r^S)_{t-1} + d_{3,t}, \tag{3.2.4}$$

$$\hat{\rho}_{22,t} = a_{220} + a_{22}AC(tv^S)_{t-1} + d_{4,t}, \tag{3.2.5}$$

where $\hat{\rho}_{11,t}, \hat{\rho}_{12,t}, \hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ are the series of autoregressive coefficients of VAR(1) model (3.10) displayed in Figures 3.1 and 3.2, and $AC(r^S)_{t-1}$ and $AC(tv^S)_{t-1}$ are the lagged values of auto-correlations of SPDR return and trade volume changes, $CC(r^S tv^S)_{t-1}$ is the lagged value of the cross-correlation between r^S_t and tv^S_{t-1}, $CC(tv^S r^S)_{t-1}$ is the lagged value of the cross-correlation between tv^S_t and r^S_{t-1}. Parameters $a_{110}, a_{120}, a_{210}$ and a_{220} are the intercepts, a_{11}, a_{12}, a_{21} and a_{22} are the regression coefficients and $d_{1,t}, d_{2,t}, d_{3,t}$ and $d_{4,t}$ are the disturbance terms which are assumed to have mean zero, fixed variances and are orthogonal to the regresses.

Table 3.5 shows the results of estimating the above linear regressions. All regression coefficients are statistically significant [21].

[21] The regression lines are provided in Appendix B.

Table 3.5: Linear Regression Coefficients

Dependent Variable	a_{jk0}	a_{jk}	R^2	RSE
ρ_{11}	-0.009***	0.012***	0.35	0.001
ρ_{12}	0.001***	0.014***	0.68	0.001
ρ_{21}	-0.003*	-0.058***	0.58	0.003
ρ_{22}	-0.192***	-0.031***	0.12	0.002

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Note: Table 3.5 shows the results of estimating linear equations (3.12)-(3.15). a_{jk0} are the intercepts, a_{jk} are the regression coefficients, R^2 shows the multiple R-squared and RSE shows the residual standard error.

This result implies that by using the lagged values of auto- and cross-correlations of SPDR's return and trade volume changes, we can predict the parameters of the VAR(1) model as follows:

$$\hat{\hat{\rho}}_{11,t} = \hat{a}_{110} + \hat{a}_{11} AC(r^S)_{t-1}, \tag{3.2.6}$$

$$\hat{\hat{\rho}}_{12,t} = \hat{a}_{120} + \hat{a}_{12} CC(r^S, tv^S)_{t-1}, \tag{3.2.7}$$

$$\hat{\hat{\rho}}_{21,t} = \hat{a}_{210} + \hat{a}_{21} CC(tv^S r^S)_{t-1}, \tag{3.2.8}$$

$$\hat{\hat{\rho}}_{22,t} = \hat{a}_{220} + \hat{a}_{22} AC(tv^S)_{t-1}. \tag{3.2.9}$$

Next, the fitted values of $\hat{\rho}_{11}, \hat{\rho}_{12}, \hat{\rho}_{21}$ and $\hat{\rho}_{22}$ are computed from equations (3.16) to (3-19). The following figures show the fitted series $\hat{\hat{\rho}}_{11t}, \hat{\hat{\rho}}_{12t}, \hat{\hat{\rho}}_{21t}$ and $\hat{\hat{\rho}}_{22t}$ and compare them to the dependent variables $\hat{\rho}_{11t}, \hat{\rho}_{12t}, \hat{\rho}_{21t}$ and $\hat{\rho}_{22t}$.

In all four plots, the fitted values of autoregressive coefficients show less fluctuation then the estimates. However, their patterns are close to those of the the estimated autoregressive parameters and they remain inside the confidence intervals of the estimated autoregressive parameters. As we can see in all these Figures, at the end of the sample period there is a gap between the fitted value and the time series of the coefficients.

To reduce the gap between the estimated ($\hat{\rho}_{jk}$) and fitted coefficients ($\hat{\hat{\rho}}_{jk}$) at the end of the sampling period, for out-of-sample forecasts, the fit can be adjusted locally, by calibrating the regression coefficients.

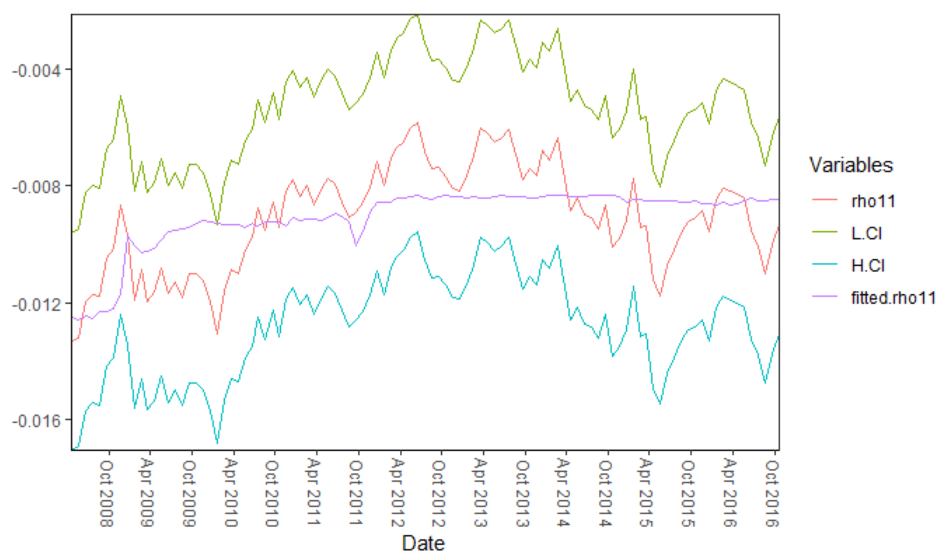

Figure 3.10: Time Series of $\hat{\rho}_{11t}$ and Fitted Values $\hat{\hat{\rho}}_{11t}$

Figure 3.10 compares the time series of estimated $\hat{\rho}_{11t}$ and the fitted values of $\hat{\hat{\rho}}_{11t}$. The red line shows the estimated $\hat{\rho}_{11t}$ from VAR(1) model (2.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{11t}$.

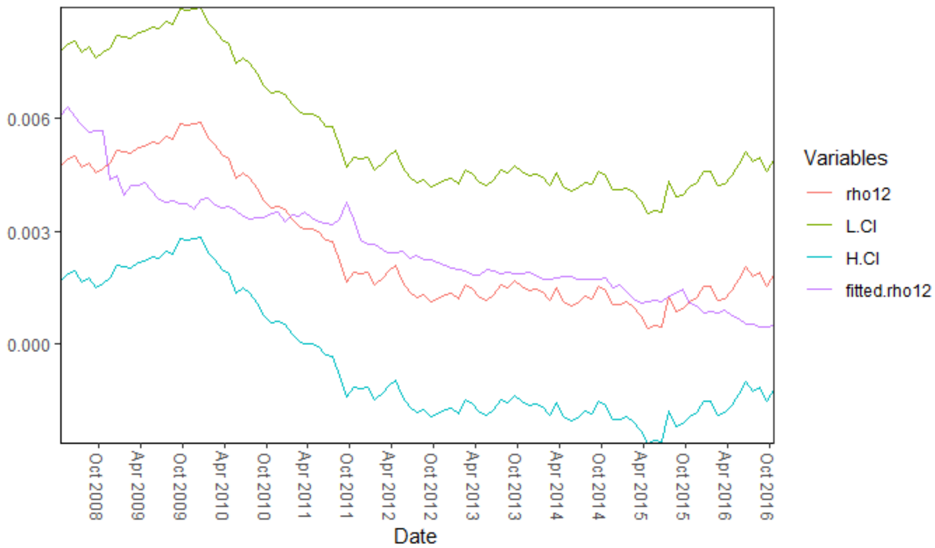

Figure 3.11: Time Series of $\hat{\rho}_{12t}$ and Fitted Values $\hat{\hat{\rho}}_{12t}$

Figure 3.11 compares the time series of estimated $\hat{\rho}_{12t}$ and the fitted values $\hat{\hat{\rho}}_{12t}$. The red line shows the estimated $\hat{\rho}_{12t}$ from VAR(1) model (3.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{12t}$.

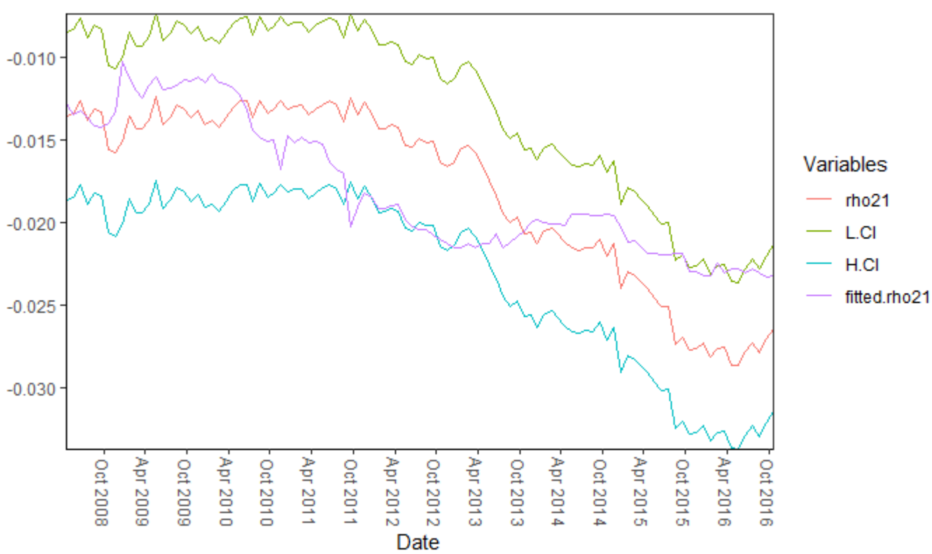

Figure 3.12: Time Series of $\hat{\rho}_{21t}$ and Fitted Values $\hat{\hat{\rho}}_{21t}$

Figure 3.12 compares the time series of estimated $\hat{\rho}_{21t}$ and the fitted values $\hat{\hat{\rho}}_{21t}$. The red line shows the estimated $\hat{\rho}_{21t}$ from VAR(1) model (3.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{21t}$.

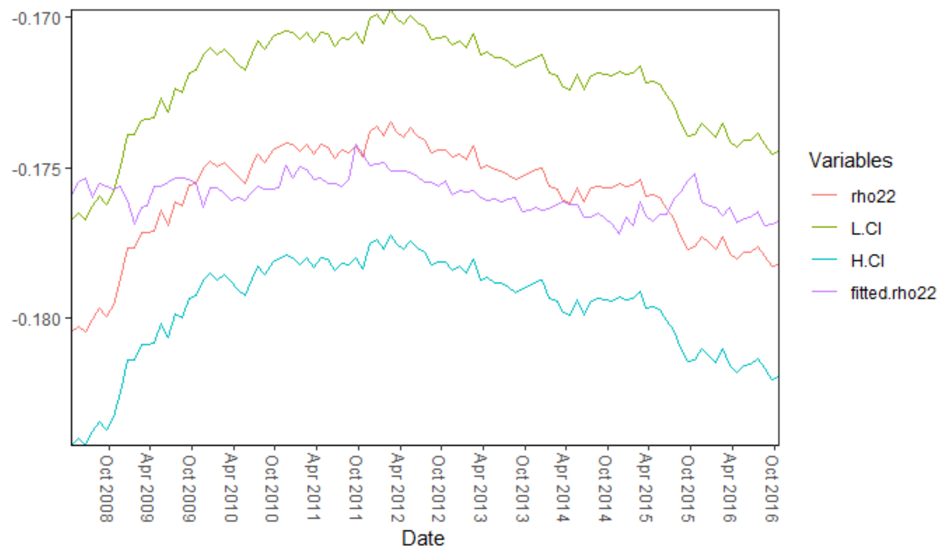

Figure 3.13: Time Series of $\hat{\rho}_{22t}$ and Fitted Values $\hat{\hat{\rho}}_{22t}$

Figure 3.13 compares the time series of estimated $\hat{\rho}_{22t}$ and the fitted values $\hat{\hat{\rho}}_{22t}$. The red line shows the estimated $\hat{\rho}_{22t}$ from VAR(1) model (3.10), green and blue lines show it's upper and lower confidence intervals. The purple line shows the fitted values of $\hat{\hat{\rho}}_{22t}$.

3.2.5 Rank Forecasts

The previous Section showed that the $SPDR$ approximates the behavior of the market portfolio as it returns are close to those of $S\&P500$. Therefore, it can be considered as an observable factor. It follows that the four explanatory variables in equations (3.16) to (3.19) can be considered as fixed functions of factor returns and trade volume changes, determining the autoregressive coefficients of the VAR(1) model and the persistence of stock return and liquidity ranks.

This result provides an alternative approach to forecasting out of sample the future ranks of stock returns and volume changes from the VAR(1) model (3-10). At time $T+1$, the future true rank is:

$$\begin{pmatrix} u_{iT+1} \\ v_{iT+1} \end{pmatrix} = \begin{pmatrix} \rho_{11,T+1} & \rho_{12,T+1} \\ \rho_{21,T+1} & \rho_{22,T+1} \end{pmatrix} \begin{pmatrix} u_{i,T} \\ v_{i,T} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,iT+1} \\ \epsilon_{2,iT+1} \end{pmatrix}, \qquad i = 1, \cdots, n. \quad (3.2.10)$$

It can be forecast using the last values of coefficients $\rho_{jkT},\ j,k = 1,2$ estimated by rolling and displayed in Figures 3.1 and 3.2. This approach assumes implicitly that the autoregressive coefficients remain constant between times T and $T+1$. Then the estimated ranks are as follows:

$$\begin{pmatrix} \hat{u}_{iT+1} \\ \hat{v}_{iT+1} \end{pmatrix} = \begin{pmatrix} \hat{\rho}_{11,T} & \hat{\rho}_{12,T} \\ \hat{\rho}_{21,T} & \hat{\rho}_{22,T} \end{pmatrix} \begin{pmatrix} u_{i,T} \\ v_{i,T} \end{pmatrix}, \qquad i = 1, \cdots, n, \qquad (3.2.11)$$

Instead of re-estimating the VAR(1) by rolling equation (3.1), one can predict the autoregressive coefficients $\hat{\hat{\rho}}_{jk,T+1}$ from equations (3.16) to (3.19) for fixed values of linear regression coefficients given in Table 3.5. Next, the bivariate ranks can be predicted as follows:

$$\begin{pmatrix} \hat{u}_{iT+1} \\ \hat{v}_{iT+1} \end{pmatrix} = \begin{pmatrix} \hat{\hat{\rho}}_{11,T+1} & \hat{\hat{\rho}}_{12,T+1} \\ \hat{\hat{\rho}}_{21,T+1} & \hat{\hat{\rho}}_{22,T+1} \end{pmatrix} \begin{pmatrix} u_{i,T} \\ v_{i,T} \end{pmatrix}, \qquad i = 1, \cdots, n, \qquad (3.2.12)$$

The relative performance of the two forecast methods is assessed empirically in

Section 3.5. In the next Section, the predicted ranks of returns and trade volumes are used as the approximations of the expected future ranks to build optimal portfolio allocations.

3.3 Optimal Positional Management

In this Section we determine the optimal portfolio allocations for an investor with a CARA utility function.

3.3.1 Optimal Positional Allocations

The empirical analysis presented in the previous Section concerned the empirical ranks of returns and trade volume changes transformed to Gaussian variables. The portfolio management, however is based on their theoretical counterparts. Therefore, we distinguish and define the theoretical ex-ante ranks from the assumed theoretical c.d.f of each of these two series, denoted by F_t^r and F_t^{tv}. Then, the ex-ante ranks are defined as follows:

$$u_{it}^* = F_t^r(r_{it}), \tag{3.3.1}$$

$$v_{it}^* = F_t^{tv}(tv_{it}), \tag{3.3.2}$$

The theoretical Gaussian ranks are given by $u_{it} = \Phi^{-1}(u_{it}^*) = Q_t^r(r_{it})$ and $v_{it} = \Phi^{-1}(v_{it}^*) = Q_t^{tv}(tv_{it})$, where $Q_t^r = \Phi^{-1} \circ F_t^r$ and $Q_t^{tv} = \Phi^{-1} \circ F_t^{tv}$. As the Gaussian ranks of returns and volume changes are Normally cross-sectionally distributed, at each time t the relationship between asset i returns and trade volume changes and their respective ranks can be defined by the following stochastic transformations:

$$r_{i,t} = \sigma_{r,t} u_{it} + \mu_{r,t} \qquad t = 1, \cdots, T \ \ , i = 1, \cdots, n, \tag{3.3.3}$$

$$tv_{i,t} = \sigma_{tv,t} v_{it} + \mu_{tv,t} \qquad t = 1, \cdots, T \ \ , i = 1, \cdots, n, \tag{3.3.4}$$

where $\mu_{r,t}, \mu_{tv,t}$ are the cross-sectional means of returns and trade volume changes and $\sigma_{r,t}, \sigma_{tv,t}$, represent the cross-sectional standard deviations of the marginal Normal distributions of return and trade volume changes at time t. This transformation, implies that the cross-sectional marginal distributions of assets' returns and trade volume changes at date t are Gaussian as well ($N(\mu_{r,t}, \sigma_{r,t})$ and $N(\mu_{tv,t}, \sigma_{tv,t})$ respectively).

Let us consider two types of investors; investor 1 is looking for a portfolio that provides the highest possible future return rank and investor 2 is looking for a portfolio with the highest possible future liquidity rank.

The quantile functions are time varying and are given below for the return and trade volume changes, respectively:

$$Q_t^r(r_{it}) = \frac{r_{it} - \mu_{r,t}}{\sigma_{r,t}} \tag{3.3.5}$$

$$Q_t^{tv}(tv_{it}) = \frac{tv_{it} - \mu_{tv,t}}{\sigma_{tv,t}} \tag{3.3.6}$$

The investor maximizes a CARA utility function in either return or trade volume changes subject to a constraint $\beta' h = 1$, where h is a unit vector of length n. This implies that the sum of all portfolio allocations is equal to one, and the optimal portfolio contains risky assets only. For investors 1 and 2, the future ranks of portfolio returns and traded volumes are $Q^{r}_{t+1}(\beta'_r r_{t+1})$ and $Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1})$, respectively. These investors maximize the conditional expected utilities $E_t U[Q^{r}_{t+1}(\beta'_r r_{t+1})]$ and $E_t U[Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1})]$, where $U(u) = -exp(-A_r u)$ or $U(v) = -exp(-A_{tv} v)$ and E_t denotes the expectation conditional on the current and past returns, volumes and the predetermined current values of the autoregressive coefficients. Therefore, the optimal positional momentum strategy consists in selecting assets with the optimal relative allocation vector $\hat{\beta}_{r,t}$, where:

$$\begin{aligned} \beta^{*}_{r,t} &= A_{\beta_r:\beta'_r h=1}\ E_t\big[U(Q^{r}_{t+1}(\beta'_r r_{t+1}\big] \\ &= A_{\beta_r:\beta'_r h=1}\ E_t\big[U\big(Q^{r}_{t+1}\Big(\sum_{i=1}^{n}\beta_{r,i} {Q^{r}_{t+1}}^{-1}(u_{it+1})\Big)\big)\big], \end{aligned} \tag{3.3.7}$$

The optimal positional allocation vector based on the liquidity ranks is:

$$\begin{aligned} \beta^{*}_{tv,t} &= A_{\beta_{tv}:\beta'_{tv} h=1}\ E_t\big[U(Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1}\big] \\ &= A_{\beta_{tv}:\beta'_{tv} h=1}\ E_t\big[U\big(Q^{tv}_{t+1}\Big(\sum_{i=1}^{n}\beta_{tv,i} {Q^{tv}_{t+1}}^{-1}(v_{it+1})\Big)\big)\big] \end{aligned} \tag{3.3.8}$$

Let us consider the positional momentum and liquid portfolios which each contains relative risky allocation vectors β'_r and β'_{tv}, respectively. The future return and trade volume change of these portfolios are given by:

$$\beta'_r r_{t+1} = \sigma_{r,t+1}\beta'_r u_{t+1} + \mu_{r,t+1}\beta'_r h \tag{3.3.9}$$

$$\beta'_{tv} tv_{t+1} = \sigma_{tv,t+1}\beta'_{tv} v_{t+1} + \mu_{tv,t+1}\beta'_{tv} h \tag{3.3.10}$$

since $\beta_r' h = \beta_{tv}' h = 1$. By substituting the future positions of return and volume change ranks into (3.27) and (3.28) respectively, the future positions of the portfolios become:

$$Q_{t+1}^{r}(\beta_r' r_{t+1}) = \frac{\sigma_{r,t+1}\beta_r' u_{t+1} + \mu_{r,t+1}\beta_r' h - \mu_{r,t+1}}{\sigma_{r,t+1}} = \beta_r' u_{t+1}, \qquad \forall\ \beta_r, \quad (3.3.11)$$

$$Q_{t+1}^{tv}(\beta_{tv}' tv_{t+1}) = \frac{\sigma_{tv,t+1}\beta_{tv}' v_{t+1} + \mu_{tv,t+1}\beta_{tv}' h - \mu_{tv,t+1}}{\sigma_{tv,t+1}} = \beta_{tv}' v_{t+1}, \qquad \forall\ \beta_{tv}, \quad (3.3.12)$$

Equations (3.33) and (3.34) show that, the position of the future return and trade volume change of the momentum and liquid positional portfolios is a linear combination of the future Gaussian ranks of return and trade volume changes of the individual risky asset $(u_{i,t+1}, v_{i,t+1})$, with weights equal to the elements of the relative risky allocations β_r and β_{tv}. The future positions of the return and trade volume of the portfolios are equal to the shares of each asset in the portfolio multiplied by its future rank. Therefore, in order to predict the future positions of returns and trade volumes of the portfolio, we can use their future Gaussian ranks weighted by their respective shares in each portfolio. This result is a consequence of the linearity of the transformed quantile function (Q_{t+1}) under the Normality assumption on the cross-sectional distributions [see equations (3.27)-(3.28)], and holds for any dynamics of the ranks.

More specifically, by considering the dynamics of ranks introduced in the panel VAR model (equation 3.10), the future positions of returns and trade volume changes can be written as functions of their current ranks as follows:

$$\begin{aligned} Q^r_{t+1}(\beta_r' r_{t+1}) &= \beta_r' u_{t+1} \\ &= \sum_{i=1}^{n} \beta_{r,i}\rho_{11,t+1}u_{i,t} + \sum_{i=1}^{n} \beta_{r,i}\rho_{12,t+1}v_{i,t} + \sum_{i=1}^{n} \beta_{r,i}\epsilon_{1,it+1} \end{aligned} \tag{3.3.13}$$

$$\begin{aligned} Q^{tv}_{t+1}(\beta_{tv}' tv_{t+1}) &= \beta_{tv}' v_{t+1} \\ &= \sum_{i=1}^{n} \beta_{tv,i}\rho_{21,t+1}u_{i,t} + \sum_{i=1}^{n} \beta_{tv,i}\rho_{22,t+1}v_{i,t} + \sum_{i=1}^{n} \beta_{tv,i}\epsilon_{2,it+1}, \end{aligned} \tag{3.3.14}$$

where coefficients $\rho_{11,t+1}, \rho_{12,t+1}, \rho_{21,t+1}$ and $\rho_{22,t+1}$, in the autoregressive matrix R_t and the conditional variance matrix Σ_t are assumed to be predetermined and are known to the investor at time t [22]. These equations show that the future positions of returns and trade volume changes can be easily computed from the current ranks of returns and trade volume changes.

In the optimizations (3.29) and (3.30), the risk aversion coefficients A depends on the investor. We assume that the risk aversion of Investor 1 is A_r and that of investor 2 is A_{tv}. After substituting the quantile functions in the utility function and given that errors $\epsilon_{1,it}, \epsilon_{2,it}$ in equation (3.10), are independent Gaussian white noise processes, the expected positional utilities to be maximized are as follows:

$$\begin{gathered} -E[exp(-A_r Q^r_{t+1}(\beta_r' r_{t+1})) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}] = \\ -\Big[exp\Big(-A_r \sum_{i=1}^{n} \beta_{r,i}\rho_{11,t+1}u_{i,t} - A_r \sum_{i=1}^{n} \beta_{r,i}\rho_{12,t+1}v_{i,t} + \frac{1}{2}A_r^2 \sum_{i=1}^{n} \beta_{r,i}^2\sigma_{1,t+1}^2\Big)\Big] \end{gathered} \tag{3.3.15}$$

[22]See Appendix C for a heuristic demonstration of case of stochastic autoregressive coefficients.

where $\sigma^2_{1t+1} = 1 - \rho^2_{11,t+1} - \rho^2_{12,t+1}$, subject to $\beta_r' h = 1$ and,

$$
\begin{gathered}
-E[exp(-A_{tv}Q^{tv}_{t+1}(\beta_{tv}' tv_{t+1})) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}] = \\
-\Big[exp\Big(-A_{tv}\sum_{i=1}^{n}\beta_{tv,i}\rho_{21,t+1}u_{i,t} - A_{tv}\sum_{i=1}^{n}\beta_{tv,i}\rho_{22,t+1}v_{i,t} + \frac{1}{2}A^2_{tv}\sum_{i=1}^{n}\beta^2_{tv,i}\sigma^2_{2,t+1}\Big)\Big]
\end{gathered}
\tag{3.3.16}
$$

where $\sigma^2_{2t+1} = 1 - \rho^2_{21,t+1} - \rho^2_{22,t+1}$, subject to $\beta_{tv}' h = 1$, for investor 2. In each of the above equations ((3.37) and (3.38)), the expected positional utility is independent of the cross-sectional mean and standard deviation of returns and trade volumes ($\mu_{r,t}, \mu_{tv,t}$ and $\sigma_{r,t}, \sigma_{tv,t}$) at time t and depends on the current position of asset i return and trade volume change (u_{it} and v_{it}).

The Lagrangian functions for the maximization of the expected positional utility with respect to the portfolio allocation vectors β_r' and β_{tv}', subject to the constraints $\beta_r' h = 1$ and $\beta_{tv}' h = 1$ are:

$$
\begin{aligned}
L_r = -\Big[exp\Big(&-A_r\sum_{i=1}^{n}\beta_{r,i}\rho_{11,t+1}u_{i,t} - A_r\sum_{i=1}^{n}\beta_{r,i}\rho_{12,t+1}v_{i,t} + \\
&\frac{1}{2}A^2_r\sum_{i=1}^{n}\beta^2_{r,i}\sigma^2_{1,t+1}\Big)\Big] + \lambda_r(1 - \beta_r' h)
\end{aligned}
\tag{3.3.17}
$$

$$
\begin{aligned}
L_{tv} = -\Big[exp\Big(&-A_{tv}\sum_{i=1}^{n}\beta_{tv,i}\rho_{21,t+1}u_{i,t} - A_{tv}\sum_{i=1}^{n}\beta_{tv,i}\rho_{22,t+1}v_{i,t} + \\
&\frac{1}{2}A^2_{tv}\sum_{i=1}^{n}\beta^2_{tv,i}\sigma^2_{2,t+1}\Big)\Big] + \lambda_{tv}(1 - \beta_{tv}' h)
\end{aligned}
\tag{3.3.18}
$$

where λ_r and λ_{tv} are the Lagrange multipliers. The first-order condition for $\beta_{r,t}, \beta_{tv,t}$

are:

$$-A_r\Big[(\rho_{11,t+1}u_t+\rho_{12,t+1}v_t-A_r\sigma^2_{1,t+1}\beta_{r,t})$$
$$exp\Big[-A_r(\rho_{11,t+1}\beta'_{r,t}u_t-\rho_{12,t+1}\beta_{r,t}v_t)+\frac{1}{2}A_r^2\beta'_{r,t}\beta_{r,t}\sigma^2_{1,t+1}\Big)\Big]-\lambda_{r,t}h=0 \quad (3.3.19)$$

$$-A_{tv}\Big[(\rho_{21,t+1}u_t+\rho_{22,t+1}v_t-A_{tv}\sigma^2_{2,t+1}\beta_{tv})$$
$$exp\Big[-A_{tv}(\rho_{21,t+1}\beta'_{tv,t}u_t-\rho_{22,t+1}\beta_{tv,t}v_t)+\frac{1}{2}A_{tv}^2\beta'_{tv,t}\beta_{tv,t}\sigma^2_{2,t+1}\Big)\Big]-\lambda_{tv,t}h=0 \quad (3.3.20)$$

By solving the above equations with respect to $\beta_{r,t},\lambda_{r,t}$ and $\beta_{tv,t},\lambda_{tv,t}$ the optimal portfolio shares at time t are as follows:

$$\beta^*_{r,t}=\frac{1}{A_r}\frac{\rho_{11,t+1}u_t+\rho_{12,t+1}v_t}{\sigma^2_{1,t+1}}-\frac{1}{A_r^2}\frac{\lambda_{r,t}h}{\sigma^2_{1,t+1}} \quad (3.3.21)$$

$$\beta^*_{tv,t}=\frac{1}{A_{tv}}\frac{\rho_{21,t+1}u_t+\rho_{22,t+1}v_t}{\sigma^2_{2,t+1}}-\frac{1}{A_{tv}^2}\frac{\lambda_{tv,t}h}{\sigma^2_{2,t+1}} \quad (3.3.22)$$

In terms of vector we get:

$$\beta^*_{r,t}{}'h=\frac{(\rho_{11t+1}u_{.t}+\rho_{12t+1}v_{.t})}{A_r\sigma^2_{1,t+1}}-n\frac{\lambda_{r,t}}{A_r^2\sigma^2_{1,t+1}}=1, \quad (3.3.23)$$

$$\beta^*_{tv,t}{}'h=\frac{(\rho_{21t+1}u_{.t}+\rho_{22t+1}v_{.t})}{A_{tv}\sigma^2_{2,t+1}}-n\frac{\lambda_{tv,t}}{A_{tv}^2\sigma^2_{2,t+1}}=1, \quad (3.3.24)$$

where $u_{.t}$ and $v_{.t}$ are the vectors of ranks of asset $i=1,\ldots,n$ at time t. Then we have:

$$\frac{\lambda_{r,t}}{A_r^2\sigma^2_{1,t+1}}=\frac{\rho_{11t+1}\overline{u_t}+\rho_{12t+1}\overline{v_t}}{A_r\sigma^2_{1,t+1}}-\frac{1}{n}, \quad (3.3.25)$$

$$\frac{\lambda_{tv,t}}{A_{tv}^2\sigma_{2,t+1}^2} = \frac{\rho_{21t+1}\overline{u_t} + \rho_{22t+1}\overline{v_t}}{A_{tv}\sigma_{2,t+1}^2} - \frac{1}{n}, \tag{3.3.26}$$

where $\overline{u_t} = \frac{1}{n}\sum_{i=1}^{n} u_{it}$ and $\overline{v_t} = \frac{1}{n}\sum_{i=1}^{n} v_{it}$. By substituting the above expressions into (3.45) and (3.46) we get the vectors of optimal allocations as follow:

$$\beta_{r,t}^* = \frac{1}{n}h + \frac{\rho_{11t+1}(u_t - \overline{u_t}h) + \rho_{12t+1}(v_t - \overline{v_t}h)}{A_r\sigma_{1,t+1}^2} \tag{3.3.27}$$

$$\beta_{tv,t}^* = \frac{1}{n}h + \frac{\rho_{21t+1}(u_t - \overline{u_t}h) + \rho_{22t+1}(v_t - \overline{v_t}h)}{A_{tv}\sigma_{2,t+1}^2} \tag{3.3.28}$$

The optimal relative positional allocations $\beta_{r,t}^{\prime *}, \beta_{tv,t}^{\prime *}$ (equations (3.49) and (3.50)) are linear combinations of two well-known portfolios. The first one is the equally weighted portfolio with weight 1/n for each asset and the second portfolio is an arbitrage portfolio (i.e. zero-cost portfolio) with dynamic allocations proportional to the deviations of the current ranks from their cross-sectional averages. Since these arbitrage portfolios contain the vector of expected future ranks in deviation from their cross-sectional averages $\Big((\rho_{11,t+1}(u_{it} - \overline{u_t}) + \rho_{12,t+1}(v_{it} - \overline{v_t})$ in equation (3.49) and $(\rho_{21,t+1}(u_{it} - \overline{u_t}) + \rho_{22,t+1}(v_{it} - \overline{v_t}))$ in equation (3.50) $\Big)$, it can be interpreted as a momentum portfolio in equation (3.49) and liquid portfolio in equation (3.50). When the sign of the sum of persistence coefficients $\rho_{jk,t} + \rho_{jj,t}$ (where j,k=1,2) is positive, the arbitrage portfolio will be long in assets with large expected deviation of their future ranks from their cross-sectional average, and when the sum of persistence coefficients is negative then, it will be short in assets with small expected deviation of their future ranks from their cross-sectional average.

This interpretation of the arbitrage part of the positional portfolio implies that the optimal positional allocation deviates from the equally weighted portfolio by over-weighting the assets with larger current ranks, when the sum of persistence coefficients is positive and deviates from the equally weighted portfolio by over-weighting the assets with small current ranks, when the sum of persistence coefficients is negative. The weight of the arbitrage portfolio in the optimal risky allocations $\beta^*_{r,it}$ and $\beta^*_{tv,it}$ are positively correlated with the persistence of ranks coefficients ($\rho_{11,t+1}, \rho_{12,t+1}$ in equation (3.49) and $\rho_{21,t+1}, \rho_{22,t+1}$ in equation (3.50)) and negatively correlated with the risk aversion coefficients (A_r in equation (3.49) and A_{tv} in equation (4.50)) of the investors.

The optimal allocation vectors $\beta^*_{r,t}, \beta^*_{tv,t}$ that determine the positional portfolio strategies depend on the choice of the positional utility function and on the positional universe of stocks which is used to compute the ranks. Moreover, these optimal allocations of the positional investor are defined by considering functions Q_{t+1} as the exogenous functions, which in this paper, are the quantile functions.

3.3.2 Optimal Mixed Positional Allocations

Let us consider investment strategy that select assets with the highest return and liquidity ranks. The optimal allocation vector β^* is obtained by maximizing the positional CARA utility function as follows:

$$\begin{aligned} &- E[exp(-(A_r Q^r_{t+1}(\beta'_r r_{t+1}) + A_{tv} Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1}))) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}] \\ &\qquad = -E\Big[exp\big(- (A_r \beta' u_{t+1} + A_{tv} \beta' v_{t+1}) \mid \underline{r_t}, \underline{tv_t}, R_{t+1}\big)\Big] \qquad (3.3.29) \end{aligned}$$

subject to $\beta' h = 1$. By analogy to the previous section, we predict the future ranks u_{t+1} and v_{t+1} from the bivariate VAR(1) model (equation 3.10) with time varying coefficients, which are considered predetermined at time t. Next we maximize:

$$-\Big[exp\Big(- A_r \sum_{i=1}^{n} \beta_i(\rho_{11,t+1}u_{it} + \rho_{12,t+1}v_{it}) - A_{tv} \sum_{i=1}^{n} \beta_i(\rho_{21,t+1}u_{it} + \rho_{22,t+1}v_{it}) + \frac{1}{2}\sum_{i=1}^{n} \beta_i^2 (A_r^2 \sigma_{1,t+1}^2 + A_{tv}^2 \sigma_{2,t+1}^2 + 2A_r A_{tv} \sigma_{12,t+1}^2)\Big)\Big] \tag{3.3.30}$$

where $\sigma_{12,t+1} = -\rho_{11,t+1}\rho_{21,t+1} - \rho_{12,t+1}\rho_{22,t+1}$. To simplify the exposition, let us use the vector notation:

$$-\Big[exp\Big(- A_r(\rho_{11,t+1}\beta' u_t + \rho_{12,t+1}\beta' v_t) - A_{tv}(\rho_{21,t+1}\beta' u_t + \rho_{22,t+1}\beta' v_t) + \frac{1}{2}\beta'\beta(A_r^2 \sigma_{1,t+1}^2 + A_{tv}^2 \sigma_{2,t+1}^2 + 2A_r A_{tv} \sigma_{12,t+1})\Big)\Big] \tag{3.3.31}$$

subject to $\beta' h = 1$. The Lagrangian of the constrained maximization is:

$$L_M = -\Big[exp\Big(- A_r(\rho_{11,t+1}\beta' u_t + \rho_{12,t+1}\beta' v_t) - A_{tv}\beta'(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t) + \frac{1}{2}\beta\beta'(A_r^2 \sigma_{1,t+1}^2 + A_{tv}^2 \sigma_{2,t+1}^2 + 2A_r A_{tv} \sigma_{12,t+1}^2)\Big)\Big] + \lambda(1 - \beta' h) \tag{3.3.32}$$

where λ is the Lagrange multiplier. The first-order condition for β_t, λ_t is:

$$[A_r(\rho_{11,t+1}u_t + \rho_{12,t+1}v_t) + A_{tv}(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t) - (A_r^2 \sigma_{1,t+1}^2 + A_{tv}^2 \sigma_{2,t+1}^2 + 2A_r A_{tv}\sigma_{12,t+1})\beta_t] exp\Big(- A_r(\rho_{11,t+1}\beta' u_t + \rho_{12,t+1}\beta' v_t) - A_{tv}(\rho_{21,t+1}\beta' u_t + \rho_{22,t+1}\beta' v_t) + \frac{1}{2}\beta_t'\beta_t(A_r^2 \sigma_{1,t+1}^2 + A_{tv}^2 \sigma_{2,t+1}^2 + 2A_r A_{tv} \sigma_{12,t+1}^2)\Big) - \lambda_t h = 0 \tag{3.3.33}$$

which yields:

$$\beta_t^* = \frac{A_r(\rho_{11,t+1}u_t + \rho_{12,t+1}v_t) + A_{tv}(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t)}{A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2 + 2A_rA_{tv}\sigma_{12,t+1}} - \\ \frac{\lambda_t h}{A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2 + 2A_rA_{tv}\sigma_{12,t+1}} \quad (3.3.34)$$

Let m_t denotes the nominator of the first term: $m_t = A_r(\rho_{11,t+1}u_t + \rho_{12,t+1}v_t) + A_{tv}(\rho_{21,t+1}u_t + \rho_{22,t+1}v_t)$, and Δ_t denotes the common denominator: $\Delta_t = A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2 + 2A_rA_{tv}\sigma_{12,t+1}$. We can rewrite equation (4.55) as follows:

$$\beta_t^* = \frac{m_t}{\Delta_t} - \frac{\lambda_t h}{\Delta_t}. \quad (3.3.35)$$

By taking into account the constraint $\beta_t^{*\prime}h = 1$, we get equation (4.57) in terms of vector as below:

$$\begin{aligned} \beta_t^{*\prime}h &= \frac{m_{.t}}{\Delta_t} - \frac{\lambda_t h'h}{\Delta_t} = 1 \\ &= \frac{m_{.t}}{\Delta_t} - \frac{\lambda n}{\Delta_t} = 1, \end{aligned} \quad (3.3.36)$$

By solving for $\frac{\lambda_t}{\Delta_t}$ we get:

$$\frac{\lambda}{\Delta_t} = \frac{1}{n}\frac{m_{.t}}{\Delta_t} - \frac{1}{n} \quad (3.3.37)$$

By substituting equation (4.59) into the expression of β^* (equation (4.57)), we get the optimal allocation vector as follows:

$$\beta^* = \frac{m_t}{\Delta_t} - \frac{\frac{1}{n} m_t}{\Delta_t} + \frac{1}{n}$$
$$\beta^* = \frac{1}{n}h + \frac{1}{\Delta_t}(m_t - \overline{m_t}h) \tag{3.3.38}$$

which is the optimal allocation vector:

$$\beta_t^* = \frac{1}{n}h + \frac{A_r\big(\rho_{11,t+1}(u_t - \overline{u}_t) + \rho_{12,t+1}(v_t - \overline{v}_t)\big)}{A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2 + 2A_rA_{tv}\sigma_{12,t+1}} + \frac{A_{tv}\big(\rho_{21,t+1}(u_t - \overline{u}_t) + \rho_{22,t+1}(v_t - \overline{v}_t)\big)}{A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2 + 2A_rA_{tv}\sigma_{12,t+1}} \tag{3.3.39}$$

It is easy to see that the above formula simplifies when $\sigma_{12,t} = 0$:

$$\beta_t^* = \frac{1}{n} + \frac{A_r(m_{r,t} - \overline{m}_{r,t}h) + A_{tv}(m_{tv,t} - \overline{m}_{tv,t}h)}{A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2}, \tag{3.3.40}$$

where $(m_{r,t} - \overline{m}_{r,t}h) = \rho_{11,t+1}(u_t - \overline{u}_t h) + \rho_{12,t+1}(v_t - \overline{v}_t h)$ and $(m_{tv,t} - \overline{m}_{tv,t}h) = \rho_{21,t+1}(u_t - \overline{u}_t h) + \rho_{22,t+1}(v_t - \overline{v}_t h)$. We see that:

$$\beta_t^* = \frac{1}{n}h + \frac{A_r^2\sigma_{1t+1}^2 \frac{(m_{r,t} - \overline{m}_{r,t}h)}{A_r\sigma_{1t+1}^2} + A_{tv}^2\sigma_{2t+1}^2 \frac{(m_{tv,t} - \overline{m}_{tv,t}h)}{A_{tv}\sigma_{2t+1}^2}}{A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2}$$

$$= \frac{1}{n}h + \pi_{r,t}\Big[\frac{(m_{r,t} - \overline{m}_{r,t}h)}{A_r\sigma_{1t+1}^2}\Big] + \pi_{tv,t}\Big[\frac{(m_{tv,t} - \overline{m}_{tv,t}h)}{A_{tv}\sigma_{2t+1}^2}\Big], \tag{3.3.41}$$

where $\pi_{r,t} = \frac{A_r^2 \sigma_{1t+1}^2}{A_r^2 \sigma_{1,t+1}^2 + A_{tv}^2 \sigma_{2,t+1}^2}$ and $\pi_{tv,t} = \frac{A_{tv}^2 \sigma_{2t+1}^2}{A_r^2 \sigma_{1,t+1}^2 + A_{tv}^2 \sigma_{2,t+1}^2}$. It follows from equation (3.62), that when $\sigma_{12,t+1} = 0$, the optimal mixed positional allocation contains two portfolios. The first one is the equally weighted portfolio with weights $1/n$ and the second one is a weighted average of the positional momentum and positional liquidity allocations.

3.3.3 Optimal Positional Portfolios

From the optimal positional allocation vectors we define the following three types of optimal positional portfolios:

Definition 1: The efficient positional momentum portfolio is based on the optimal positional allocation $\beta_{r,t}^*$ which maximizes the $CARA$ positional utility function under condition $\beta_r' h = 1$ for positional risk aversion parameters A_r and a bivariate VAR model component of returns ranks dynamics.

As the liquidity ensures uninterrupted availability of funds, we extend this approach further and introduce a new positional liquid portfolio which is efficient in terms of liquidity as follows;

Definition 2: The efficient positional liquid portfolio is based on the optimal positional allocations $\beta_{tv,t}^*$ which maximizes the $CARA$ positional utility function under constraint $\beta_{tv}' h = 1$ for positional risk aversion parameters A_{tv} and a bivariate VAR model component of trade volumes' ranks dynamics.

Some investors are interested in maximizing the returns while also looking for quick access to funds as well. The third approach introduced as a new mixed positional portfolio, which is efficient in terms of both return and liquidity.

Definition 3: The efficient positional mixed portfolio is based on the optimal positional allocations β_t^* which maximizes the $CARA$ positional utility function under constraint $\beta' h = 1$ for risk aversion parameters A_r, A_{tv} and a bivariate VAR model of return and trade volumes' ranks dynamics.

3.4 Optimal Positional Strategies

In this Section, we implement the optimal positional strategies defined in Section 3.3. The positional strategies are applied to an investment universe corresponding to the $n = 1330$ stocks traded in NASDAQ market from 1999 to 2016. The positional risk aversion parameters are considered constant and take values $0.5, 1, 3, 5$. The expected ranks of returns are predicted from the bivariate VAR(1) model (equation 3.7) of ranks of returns and trade volume changes using either the autoregressive parameters $\hat{\rho}_{jk,t}$, $jk = 1, 2$ estimated by rolling (equation), or $\hat{\hat{\rho}}_{jk,t+1}$, $jk = 1, 2$ predicted from the factor model (equations 3.16-3.19). This strategy compute optimal portfolios with monthly adjustments of asset allocations and equal look-back periods of one month over the period 2008 to 2016. The returns of the positional portfolios are compared with the returns on the equal weighted portfolio (EW), that are obtained from rolling with a window of 108 months.

3.4.1 Optimal Positional Momentum Portfolios

The optimal positional momentum portfolios contain stocks with allocations β_t^r, defined as follows:

$$\beta_{r,t}^* = \frac{1}{n}h + \frac{\rho_{11,t+1}(u_t - \overline{u}_t) + \rho_{12,t+1}(v_t - \overline{v}_t)}{A_r \sigma_{1,t+1}^2} \tag{3.4.1}$$

Table 3.6, shows the average of the time series of the optimal positional portfolios' returns, their standard deviations and Sharpe ratios and compares those returns with the equally weighted portfolio's return. Two types of positional momentum portfolios are considered: one with the future ranks predicted with estimated $\hat{\rho}_{11,t},\ \hat{\rho}_{12,t}$ from VAR(1) model (equation 3.7), and the other one with fitted value of estimated coefficients from equations (3.16) and (3.17) $\hat{\hat{\rho}}_{11,t+1},\ \hat{\hat{\rho}}_{12,t+1}$. The positional momentum portfolios are computed for four different values of risk aversions ($A_r = 0.5, 1, 3, 5$). We observe that all portfolios are providing positive returns and higher than equally weighed portfolio's return. By increasing the risk aversion value, the return of the optimal positional momentum portfolios decreased which is consistent with the risk-return trade-off in financial literature [23]. In other word, lower risk aversion tends to higher returns due to higher undertaken risk. However the Sharpe ratio for the risk aversion equal to 3 is higher than other values.

In all values of risk aversions, the positional momentum portfolios based on estimated $\hat{\rho}_{11,t},\ \hat{\rho}_{12,t}$ are providing higher return than those based on fitted values

[23]Many literature show that the more return sought, the more risk that must be undertaken (Breen, Glosten,and Jagannathan (1989), Nelson (1991), Glosten, Jakannatha and Runkle (1993), Brandtand Kang (2004), etc).

of $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$. But the portfolios based on the fitted ρ's are providing higher Sharpe ratios than those based on estimated ρ's.

Table 3.6: Summary of Positional Momentum Portfolios' Returns

	Estimated ρ's			Fitted ρ's		
Risk Aversion	**Mean**	**S-D**	**Sh-R**	**Mean**	**S-D**	**Sh-R**
$A_r = 0.5$	**2.198**	1.176	1.867	2.192	1.089	2.011
$A_r = 1$	**1.101**	0.581	1.890	1.098	0.534	2.052
$A_r = 3$	**0.370**	0.191	**1.922**	0.369	0.171	**2.144**
$A_r = 5$	**0.223**	0.120	1.851	**0.223**	0.105	2.093

	Mean	S-D	Sh-R
EW	0.004	0.067	0.042

Note: Table 3.6 shows the average of the time series return of the optimal positional momentum portfolios with the future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the with fitted value of estimated coefficients from equations (3.16) and (3.17) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$ (Fitted ρ's).

Figure 3.14 shows the time series of returns on the positional momentum portfolios for different values of risk aversion. Both positional momentum portfolios (based on Estimated ρ's and the Fitted $\rho's$) with risk aversion equal to 0.5 are outperforming all other portfolios. With 0.5 risk aversion, the positional momentum portfolio based on Fitted ρ's outperform the other one until January 2009. After that the positional momentum portfolio based on Estimated ρ's provides the highest return until June 2010. From July 2010 to March 2014, the portfolio based on Fitted ρ's has the highest return. While from July 2014 and so on, the portfolio based on Estimated ρ's gives us the highest return.

As we can see, by increasing the risk aversion value, the return on the positional momentum portfolios decrease. The higher the risk aversion value, the lower the return. The equally weighted portfolio has the lowest return, as compared to the other portfolios.

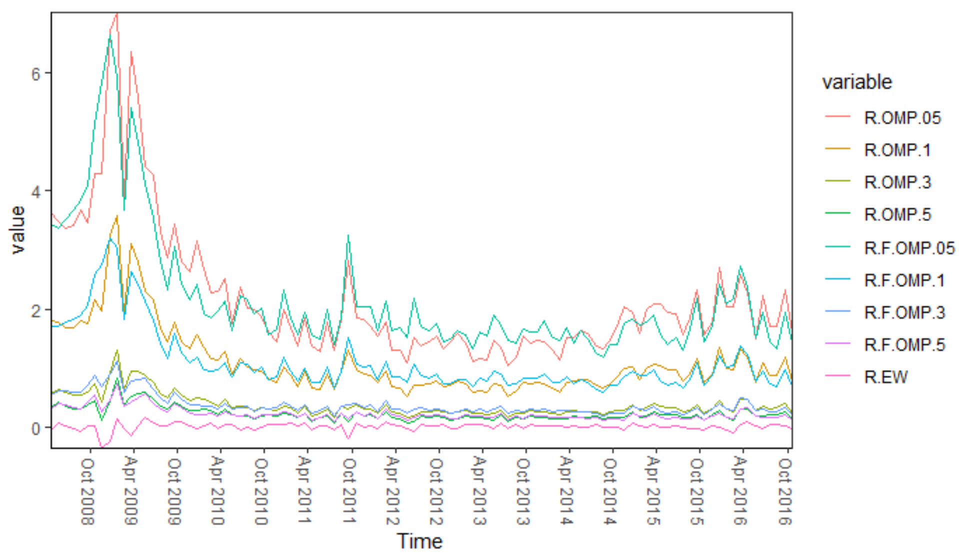

Figure 3.14: Time Series of Positional Momentum Strategies' Returns

Figure 3.14 compares the time series of returns of positional momentum portfolios. The red, orange, olive and green line show the returns of optimal positional momentum portfolios computed from estimated parameters of VAR model for different A_r. The light green, light blue, blue and purple line show the returns of optimal positional momentum portfolios computed from fitted values of parameters from equations (3.16) and (3.17) for different A_r. The pink line shows the mean of the equally weighted portfolio.

Table 3.7 shows the cumulative return of these optimal positional momentum portfolios with the inception date of April 2008 until October 2016. As we can see by increasing the risk aversion the cumulative returns on positional momentum portfolios increase. In all values of risk aversions, the positional momentum portfolios based on estimated ρ's provide higher cumulative return than the portfolios based on fitted ρ's, however these cumulative returns are very close.

Table 3.7: Cumulative Return of Positional Momentum Portfolios Until October 2016

Risk Aversion	Estimated ρ's	Fitted ρ's
$A_r = 0.5$	**134.18**	133.92
$A_r = 1$	**67.137**	67.004
$A_r = 3$	**22.437**	22.392
$A_r = 5$	**13.497**	13.470
EW	0.087	

Note: Table 3.7 shows the cumulative return of optimal positional momentum portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) and (3.17) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$ (Fitted ρ's).

Figure 3.15 shows the time series of cumulative returns on all these positional momentum portfolios. Both positional momentum portfolios (based on Estimated ρ's and the Fitted $\rho's$) with risk aversion equal to 0.5 are outperforming all other portfolios. With 0.5 risk aversion, the positional momentum portfolio biased on Fitted ρ's outperform the other one until MAY 2009. After that the positional momentum portfolio based on Estimated ρ's provides the highest return until June 2012. From January 2013 to February 2016, the portfolio based on Fitted ρ's has the highest return. As we can see, by increasing the risk aversion value, the return on the positional momentum portfolios decrease. The higher the risk aversion value, the lower the return. The equally weighted portfolio has the lowest return, as compared to the other portfolios.

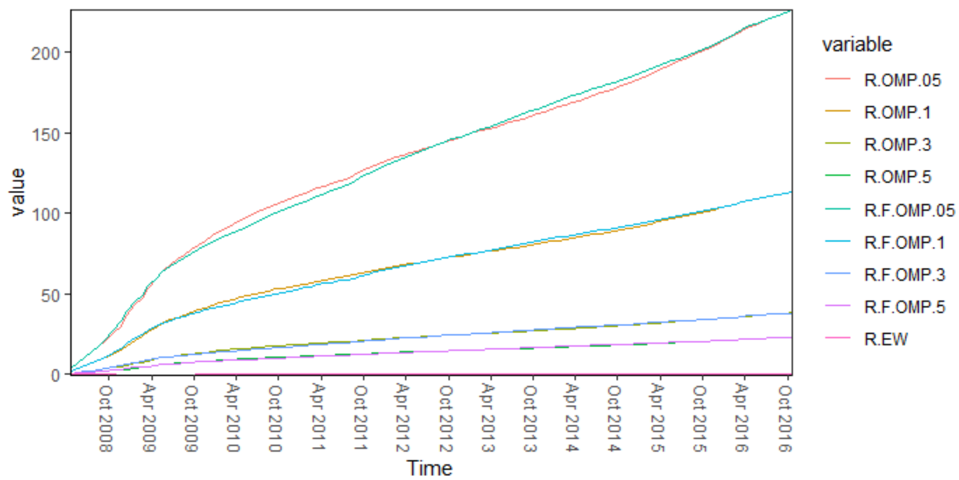

Figure 3.15: Time Series of Cumulative Returns of Positional Momentum Strategies
Figure 3.15 compares the time series of cumulative returns of positional momentum portfolios if one hold the portfolio until October 2016. The red, orange, olive and green line show the cumulative returns of optimal positional momentum portfolios computed from estimated parameters of VAR model for different A_r. The light green, light blue, blue and purple line show the cumulative returns of optimal positional momentum portfolios computed from fitted values of parameters from equations (3.16) and (3.17) for different A_r. The pink line shows the mean of the equally weighted portfolio.

3.4.2 Optimal Positional Liquid Portfolios

The optimal positional liquid portfolios contain stocks with allocations $\beta^*_{tv,t}$, defined as follows:

$$\beta^*_{tv,t} = \frac{1}{n}h + \frac{\rho_{21,t+1}(u_t - \overline{u}_t) + \rho_{22,t+1}(v_t - \overline{v}_t)}{A_{tv}\sigma^2_{2,t+1}} \tag{3.4.2}$$

Table 3.8 shows the average of the time series of the optimal positional liquid portfolios returns, their standard deviations and Sharpe ratios and compares those returns with the equally weighted portfolio's return.

Table 3.8: Summary of Positional Liquid Portfolios' Returns

	Estimated ρ's			Fitted ρ's		
Risk Aversion	**Mean**	**S-D**	**Sh-R**	**Mean**	**S-D**	**Sh-R**
$A_{tv} = 0.5$	**3.795**	2.198	**0.778**	3.742	2.211	**0.753**
$A_{tv} = 1$	**1.899**	1.100	0.773	1.873	1.106	0.748
$A_{tv} = 3$	**0.636**	0.367	0.753	0.627	0.369	0.729
$A_{tv} = 5$	**0.383**	0.221	0.733	0.378	0.222	0.710

	Mean	S-D	Sh-R
EW	0.004	0.067	0.042

Note: Table 3.8 shows the average of the time series return of the optimal positional liquid portfolios with the future ranks predicted with estimated $\hat{\rho}_{21,t}$, $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the with fitted value of estimated coefficients from equations (3.18) and (3.19) $\hat{\hat{\rho}}_{21,t+1}$, $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

Two types of positional liquid portfolios are considered, one with the future ranks predicted with the estimated $\hat{\rho}_{21,t}, \hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) and another one with the fitted values $\hat{\hat{\rho}}_{21,t+1}, \hat{\hat{\rho}}_{22,t+1}$ obtained from factor model (equations (3.18) and (3.19)). The positional liquid portfolios are computed for four different values of risk aversion ($A_{tv} = 0.5, 1, 3, 5$). The returns on both types of positional liquid portfolios with the estimated and fitted autoregressive coefficients are positive and higher than equally weighted portfolio. Also we see that, the higher the risk aversion value, the lower the average return and Sharpe ratio on the positional liquid portfolio, which is again in consistent with the risk-return trade-off. In both types of positional liquid portfolios, the portfolios based on estimated ρ's from VAR(1) model provide higher average return than the one based on fitted ρ's from equation (3.18) and (3.19), however their values are very close. By comparing Tables 3.8 with 3.6, we find that the positional liquid portfolios are providing higher average return than the positional momentum portfolios. But the positional momentum portfolios are providing higher Sharpe ratios than the positional liquid portfolios. Hence, the positional portfolios of liquid assets give higher average returns and lower Sharpe ratios than the positional portfolios of winners.

Figure 3.16 shows the time series of returns on the positional liquid portfolios for different values of risk aversions. The positional liquid portfolios with risk aversion equal to 0.5 are outperforming all other portfolios. The higher the risk aversion, the lower the return on the positional liquid portfolio. The equally weighted portfolio has the lowest return compare to other considered portfolios. But in some periods of time the equally weighted portfolio provides higher return than the others. For instance, from November 2008 to February 2009, when all positional liquid portfolios reached to their lowest value, the equally weighted portfolio outperforms them.

Table 3.9: Cumulative Return of Positional Liquid Portfolios Until October 2016

Risk Aversion	Estimated ρ's	Fitted ρ's
$A_{tv} = 0.5$	185.00	**187.07**
$A_{tv} = 1$	92.544	**93.581**
$A_{tv} = 3$	30.906	**31.251**
$A_{tv} = 5$	18.578	**18.786**
EW	0.087	

Note: Table 3.9 shows the cumulative return of optimal positional liquid portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{21,t}$, $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.18) and (3.19) $\hat{\hat{\rho}}_{21,t+1}$, $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

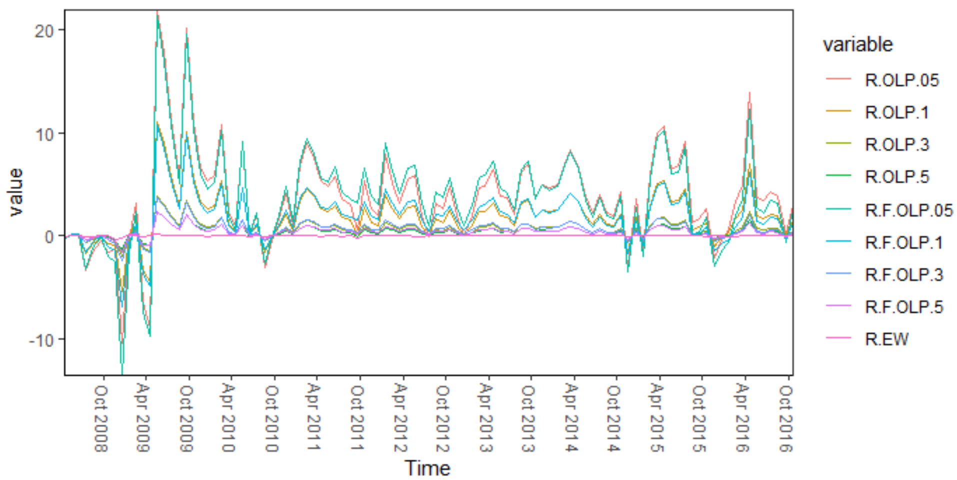

Figure 3.16: Time Series of Positional Liquid Strategies' Returns
Figure 3.16 compares the time series of positional liquid portfolios' returns. The red, orange, olive and green line show the returns of optimal positional liquid portfolios computed from estimated parameters of VAR model for different A_{tv}. The light green, light blue, blue and purple line show the returns of optimal positional liquid portfolios computed from fitted values of parameters from equations (3.18) and (3.19) for different A_{tv}. The pink line shows the mean of the equally weighted portfolio.

Table 3.9 shows the cumulative returns on positional liquid portfolios with the inception date of April 2008. We observe that, the positional liquid portfolios based on fitted ρ's outperforms all other portfolios for all values of risk aversions. Also by increasing the risk aversions' values, the cumulative returns decrease. By comparing Table 3.9 with Table 3.7, we see that a positional portfolio which is based on liquid assets provided higher cumulative return than the positional portfolios based on winners.

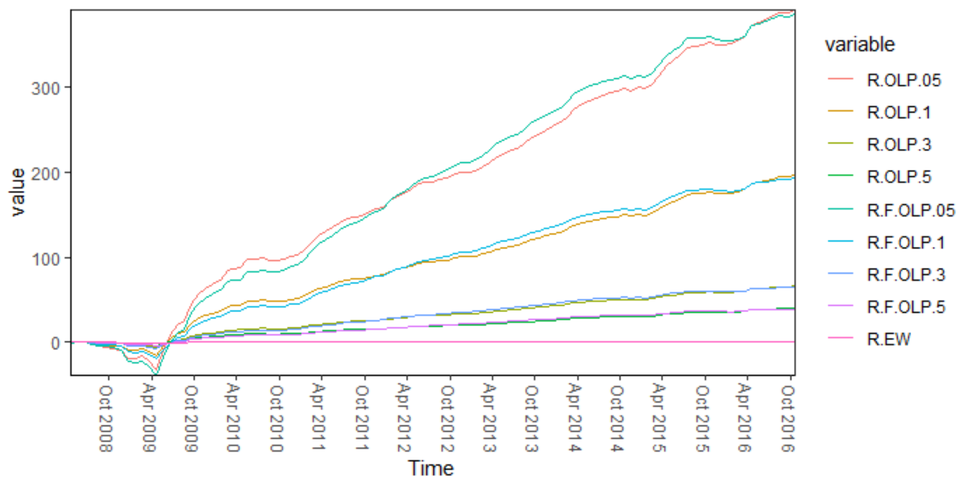

Figure 3.17: Time Series of Cumulative Returns of Positional Liquid Strategies
Figure 3.17 compares the time series of cumulative returns of positional liquid portfolios if one hold the portfolio until October 2016. The red, orange, olive and green line show the cumulative returns of optimal positional liquid portfolios computed from estimated parameters of VAR model for different A_{tv}. The light green, light blue, blue and purple line show the cumulative returns of optimal positional liquid portfolios computed from fitted values of parameters from equations (3.18) and (3.19) for different A_{tv}. The pink line shows the mean of the equally weighted portfolio.

Figure 3.17 shows the time series of cumulative return of positional liquid portfolios from April 2008 to October 2016. Again by increasing the risk aversions' values the cumulative returns decreased. Both positional liquid portfolios (based on estimated ρ's and fitted ρ's) with risk aversion 0.5 outperform all other portfolios. While the equally weighted portfolio provides the lowest cumulative returns compare to other portfolios. At the beginning of the sample period (April 2008 to July 2009), the crisis period, all portfolios' cumulative returns are below equally weighted. After that the cumulative returns on the positional liquid portfolios increase.

3.4.3 Optimal Mixed Positional Portfolios

The optimal mixed positional portfolios contain assets with allocations β_t^* defined as follows:

$$\beta_t^* = \frac{1}{n}h + \frac{A_r(\rho_{11,t+1}(u_t - \overline{u}_t) + \rho_{12,t+1}(v_t - \overline{v}_t))}{\Delta_t} + \frac{A_{tv}(\rho_{21,t+1}(u_t - \overline{u}_t) + \rho_{22,t+1}(v_t - \overline{v}_t))}{\Delta_t} \tag{3.4.3}$$

where $\Delta_t = A_r^2\sigma_{1,t+1}^2 + A_{tv}^2\sigma_{2,t+1}^2 + 2A_rA_{tv}\sigma_{12,t+1}$. Table 3.10 shows the average return, the standard deviations and the Sharpe ratios of positional mixed portfolios and of the equally weighted portfolio. Again, two types of positional mixed portfolios are considered, one with future ranks predicted with $\hat{\rho}_{jk,t}, j, k = 1, 2$ estimated by rolling (equation) and another one with $\hat{\hat{\rho}}_{jk,t+1}, j, k = 1, 2$ predicted from the factor model (equation).

Table 3.10, shows the positional mixed portfolios computed for different values of risk aversions $A_r = A_{tv} = 0.5, 1, 3, 5$. Both types of positional mixed portfolios are providing positive return and higher than equally weighted portfolio. Same as positional momentum portfolios, the positional mixed portfolios which is based on estimated ρ's have higher return than the ones based on fitted ρ's. But the Sharp ratios of the Fitted ρ's are higher than the estimated ρ's.

Also we observe that the higher the risk aversion value, the higher the average return and the Sharpe ratio on the positional mixed portfolios.

By comparing Tables 3.10,3.8 and 3.6, we find that the positional liquid portfolios are providing an average return higher than the positional mixed portfolios. However, the average returns on the positional mixed portfolios are higher than the positional momentum portfolios. Hence, the positional portfolio which is based on liquid winners provides higher average return than the positional portfolio based on just winners. In fact, by considering the liquidity along with the winners, we can improve the return of the positional portfolios. Also the positional portfolios based on just liquid assets would provide even higher average return than the positional portfolio based on liquid winners.

Table 3.10: Summary of Positional Mixed Portfolios' returns

	Estimated ρ's			Fitted ρ's		
Risk Aversion	**Mean**	**S-D**	**Sh-R**	**Mean**	**S-D**	**Sh-R**
$A_r = A_{tv} = 0.5$	**2.983**	2.323	**1.283**	2.954	2.295	**1.286**
$A_r = A_{tv} = 1$	**1.493**	1.177	1.267	1.479	1.164	1.269
$A_r = A_{tv} = 3$	**0.500**	0.416	1.199	0.496	0.412	1.200
$A_r = A_{tv} = 5$	**0.302**	0.266	1.129	0.299	0.263	1.129

	Mean	S-D	Sh-R
EW	0.004	0.067	0.042

Note: Table 3.10 shows the average of the time series return of the optimal positional mixed portfolios with the future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{21,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the with fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

In terms of Sharpe ratios, the positional momentum portfolios based on fitted ρ's are providing higher values than the positional liquid and mixed portfolios.

Figure 3.18 shows the time series of the returns of the positional mixed portfolios. We observe very similar patterns as Figure 3.16. By increasing the risk aversions' values, the return on the positional mixed portfolios decrease. When the risk aversion is 0.5, both portfolios based on estimated ρ's and fitted ρ's are providing the highest return, while the equally weighted provides the lowest return.

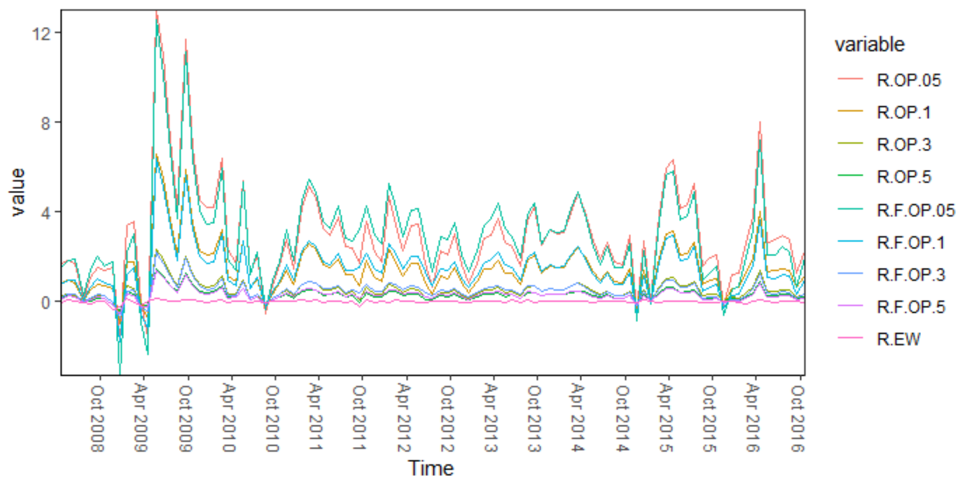

Figure 3.18: Time Series of Mixed Positional Strategies' Returns

Figure 3.18 compares the time series of the positional mixed portfolios' returns. The red, orange, olive and green line show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_r and A_{tv}. The light green, light blue, blue and purple line show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_r and A_{tv}. The pink line shows the mean of the equally weighted portfolio.

Table 3.11: Cumulative Return of Positional Mixed Portfolios Until October 2016

Risk Aversion	Estimated ρ's	Fitted ρ's
$A_r = A_{tv} = 0.5$	159.13	**160.02**
$A_r = A_{tv} = 1$	79.610	**80.05**
$A_r = A_{tv} = 3$	26.595	**26.743**
$A_r = A_{tv} = 5$	15.991	**16.081**
EW	0.087	

Note: Table 3.11 shows the cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's).

Table 3.11 shows the cumulative returns on positional mixed portfolios with inception date of April 2008 until October 2016. We observe that cumulative return on positional mixed portfolios based on fitted ρ's are higher than those based on estimated ρ's. Again, higher risk aversion value provides lower cumulative return. In both Table 3.11 and 3.9 the positional portfolios based on fitted ρ's provide higher return than those based on estimated ρ's. While in Table 3.7, the positional portfolios based on estimated ρ's yield in higher returns. By comparing Table 3.11,3.9 and 3.7, we observe that a positional portfolio based on liquid assets outperforms other positional portfolios. However, a positional portfolios based on liquid winners has higher cumulative return than the positional portfolio based on just winner stocks.

Figure 3.19 shows the cumulative returns on the positional mixed portfolios from April 2008 until October 2016. When risk aversion is 0.5, the positional portfolio based on liquid winners obtained from fitted ρ's provides the highest return until January 2009. From January 2009 to March 2012 the positional mixed portfolio based on estimated ρ's has the higher return. After that until April 2016 the positional portfolio based on fitted ρ's outperforms other portfolios, while from May 2016, the positional portfolio based on estimated ρ' outperforms the others.

Let us now assume that $A_r \neq A_{tv}$. Below, we examine the positional mixed portfolios with different values of risk aversion. First, we consider A_r fixed and compute the positional mixed portfolios for different values of A_{tv}. Next, we consider A_{tv} fixed and compute the positional mixed portfolios for different values of A_r.

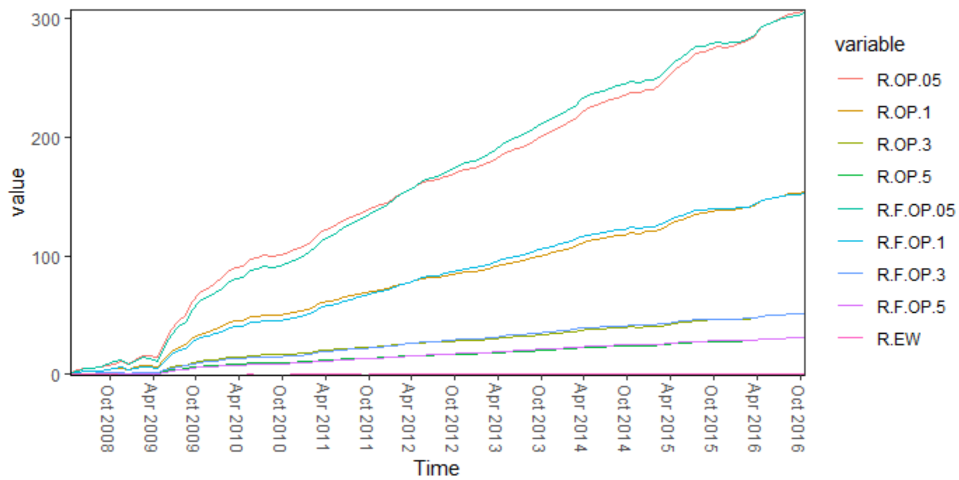

Figure 3.19: Time Series of Cumulative Returns of Positional Mixed Strategies
Figure 3.19 compares the time series of cumulative returns of positional mixed portfolios if one hold the portfolio until October 2016. The red, orange, olive and green line show the cumulative returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_r and A_{tv}. The light green, light blue, blue and purple line show the cumulative returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when for different A_r and A_{tv}. The pink line shows the mean of the equally weighted portfolio.

Table 3.12, shows the average returns, their Sharp ratios and the cumulative returns of the positional mixed portfolios when $A_r = 0.5$. Again, we build two different type of portfolios, one based on estimated parameters (Estimated ρ's) and the other from the fitted values (Fitted ρ's). Same as before by increasing the value of risk aversion, the average and cumulative returns decreased.

In terms of average return the mixed portfolios based on Estimated ρ's are providing higher return, while in terms of cumulative returns the positional mixed portfolios based on fitted ρ's are providing higher returns. But in general the cumulative returns of these two strategies are very close. When the risk aversion is 0.5 the Fitted ρ's has higher Sharpe ratio but for other values of risk aversions the estimated ρ's are providing higher Sharpe ratios, which are decreasing by increasing the value of risk aversion.

Table 3.12: Summary of Mixed Positional Portfolios' Returns, $A_r = 0.5$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_{tv} = 0.5$	**2.983**	1.283	2.954	**1.286**	159.13	**160.02**
$A_{tv} = 1$	**1.960**	**1.028**	1.938	1.011	101.05	**101.82**
$A_{tv} = 3$	**0.679**	**0.836**	0.671	0.813	33.780	**34.109**
$A_{tv} = 5$	**0.401**	**0.781**	0.396	0.758	19.758	**19.960**
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.12 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_r constant and equal to 0.5.

Figure 3.20 shows the time series of returns on the positional mixed portfolios when $A_r = 0.5$. The positional mixed portfolios with risk aversion equal to 0.5 are outperforming all other portfolios. The higher the risk aversion value, the lower the return of the positional liquid portfolios. The equally weighted portfolio has the lowest return.

Table 3.13 shows the average return, their Sharpe rations and cumulative returns of positional mixed portfolios when $A_r = 1$. We observe that in terms of average returns the positional mixed portfolios which is based on estimated ρ's outperforms the other, while in terms of cumulative returns the positional mixed portfolios based on fitted ρ's provide higher returns. Again we see the risk-return trade-off, higher risk aversion with lower returns.

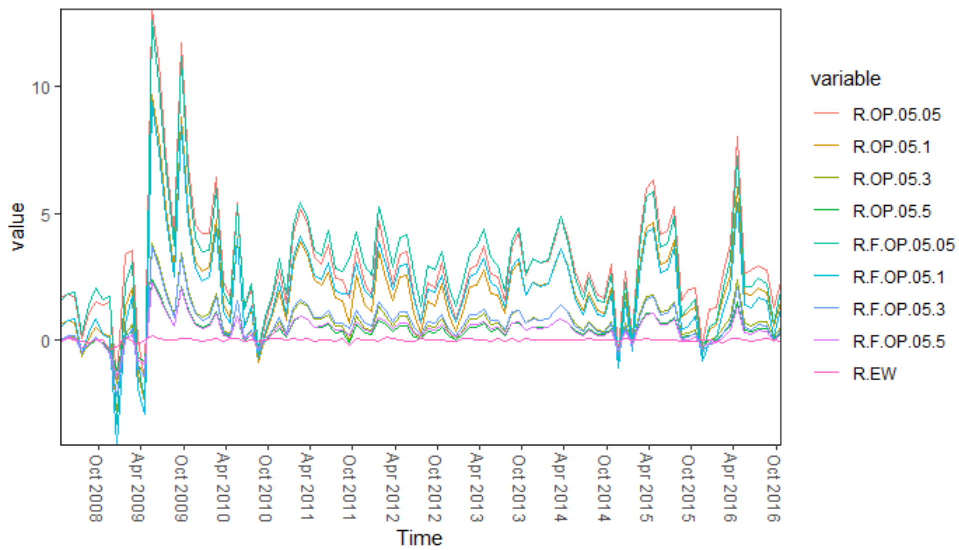

Figure 3.20: Time Series of Mixed Positional Strategies' Returns With $A_r = 0.5$

Figure 3.20 compares the time series of positional mixed portfolios' returns for $A_r = 0.5$. The red, orange, olive and green line show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_{tv}. The light green, light blue, blue and purple line show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_{tv}. The pink line shows the mean of the equally weighted portfolio.

Table 3.13: Summary of Mixed Positional Portfolios' Returns, $A_r = 1$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_{tv} = 0.5$	**1.626**	1.687	2.954	**1.764**	90.094	**90.395**
$A_{tv} = 1$	**1.960**	1.267	1.479	**1.269**	79.610	**80.057**
$A_{tv} = 3$	**0.683**	**0.916**	0.675	0.896	34.625	**34.922**
$A_{tv} = 5$	**0.411**	**0.828**	0.406	0.806	20.500	**20.694**
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.13 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_r constant and equal to 1.

For the risk aversions equal to 0.5 and 1 the fitted ρ's are providing higher Sharpe ratios, while for the risk aversions equal to 3 and 5 the estimated ρ's are providing higher Sharpe ratios.

Figure 3.21 shows the time series of returns on the positional mixed portfolios for $A_r = 1$. From April 2008 to January 2010 the positional mixed portfolios in both ways by $A_{tv} = 0.5$ outperforms others. Between February 2010 to June 2014 the positional mixed portfolios based on fitted ρ's with $A_{tv} = 0.5,\ 1$ are providing the highest returns. After that the positional mixed portfolios based on Estimated ρ's with $A_{tv} = 0.5,\ 1$ have the highest returns.

Table 3.14, shows the average returns, their Sharpe ratios and the cumulative returns of positional mixed portfolios with $A_r = 3$. Same as Table 3.13, in terms of average returns, the positional mixed portfolios based on estimated ρ's are providing higher returns. In terms of cumulative returns the positional mixed portfolios obtained from fitted ρ's have higher returns than the other portfolios.

Unlike before, when A_{tv} is equal to 1 and 3, the positional mixed portfolios obtained from both ways (estimated and fitted ρ's) are providing higher average and cumulative return than when $A_{tv} = 0.5$. The fitted ρ's are providing higher Sharpe ratios and decreasing for higher values of risk aversion except when the risk aversion is equal to 5. For the risk aversion= 5 the estimated ρ's is providing higher Sharpe ratio.

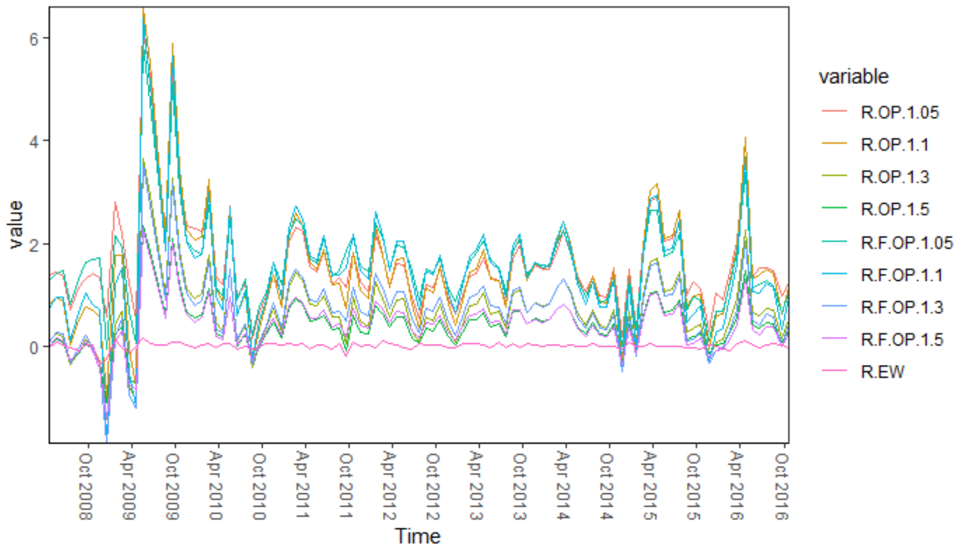

Figure 3.21: Time Series of Mixed Positional Strategies' Returns With $A_r = 1$
Figure 3.21 compares the time series of positional mixed portfolios' returns for $A_r = 1$. The red, orange, olive and green line show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_{tv}. The light green, light blue, blue and purple line show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_{tv}. The pink line shows the mean of the equally weighted portfolio.

Table 3.14: Summary of Mixed Positional Portfolios' Returns, $A_r = 3$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_{tv} = 0.5$	**0.459**	2.130	0.457	**2.430**	26.694	**26.706**
$A_{tv} = 1$	**0.518**	1.845	0.515	**1.993**	29.245	**29.306**
$A_{tv} = 3$	**0.500**	1.199	0.496	**1.200**	26.595	**26.743**
$A_{tv} = 5$	**0.379**	**0.999**	0.375	0.985	19.617	**19.757**
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.14 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_r constant and equal to 3.

Figure 3.22 shows the time series of the returns on the positional mixed portfolios when $A_r = 3$. The returns on these portfolios are very close, while the equally weighted portfolio provides the lowest return. In most of the time the positional mixed portfolios obtained from fitted ρ's are providing higher return specially with $A_{tv} = 1,\ 3$.

Table 3.15, shows the average returns, their Sharpe ratios and the cumulative returns on positional mixed portfolios when $A_r = 5$. In terms of average return the positional mixed portfolios obtained from estimated ρ' outperform other portfolios. The risk-return trade-off is reveres in these portfolios, since the one obtained for $A_r = 0.5$ has the lowest average return than the others.

The positional mixed portfolios obtained from fitted ρ's have higher cumulative returns than the other ones except for $A_{tv} = 0.5$, where the portfolio obtained from estimated ρ's provides the higher return.

The fitted ρ's are providing higher Sharpe ratios and they are increasing by higher values of risk aversions. By comparing Tables 3.15, 3.14, 3.13 and 3.12, we observe that by increasing the value of A_r the average and cumulative returns decreased. Also the optimal mixed positional portfolios based on fitted ρ's are providing higher Sharpe ratios when we fixed $A_r = 5$. In terms of average returns the positional mixed portfolios obtained from estimate ρ's outperforms those obtained from fitted ρ's. However, the positional mixed portfolios obtained from fitted ρ's provide higher cumulative returns for all values of A_r. The only exception is when $A_r = 5$ and $A_{tv} = 0.5$ where the portfolio obtained from estimated ρ's provides higher cumulative return than the one obtained from fitted ρ's.

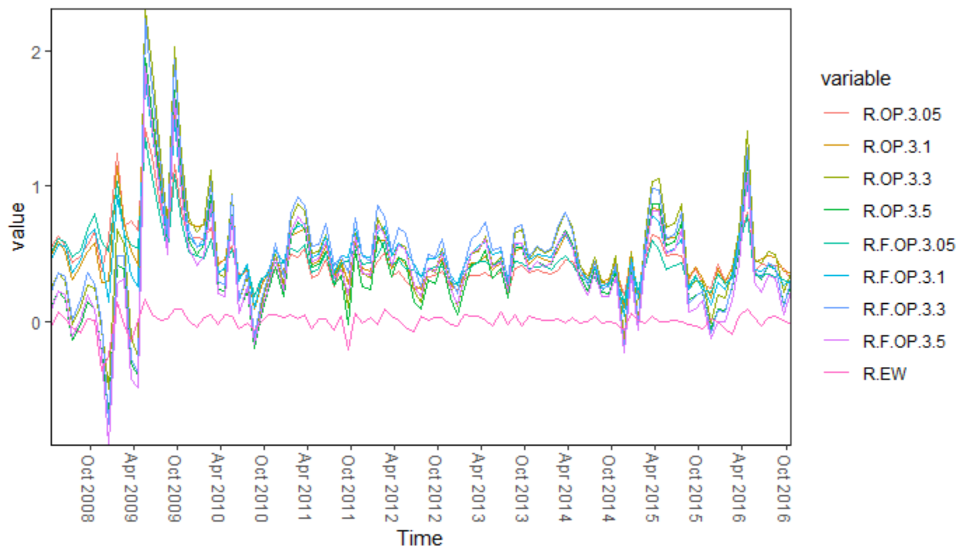

Figure 3.22: Time Series of Mixed Positional Strategies' Returns With $A_r = 3$

Figure 3.22 compares the time series of positional mixed portfolios' returns for $A_r = 3$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR for different A_{tv}. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) when for different A_{tv} respectively. The pink line shows the mean of the equally weighted portfolio.

Table 3.15: Summary of Mixed Positional Portfolios' Returns, $A_r = 5$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_{tv} = 0.5$	**0.257**	1.997	0.256	**2.290**	**15.142**	15.135
$A_{tv} = 1$	**0.286**	1.913	0.284	**2.135**	16.444	**16.457**
$A_{tv} = 3$	**0.330**	1.381	0.327	**1.415**	17.993	**18.064**
$A_{tv} = 5$	**0.302**	1.129	0.299	**1.129**	15.991	**16.081**
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.15 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_r constant and equal to 5.

Figure 3.23 shows the time series of the positional mixed portfolios' return when $A_r = 5$. Unlike the previous Figures, the positional mixed portfolios obtained from Now let us consider A_{tv} constant and build the positional mixed portfolios with different values of A_r.

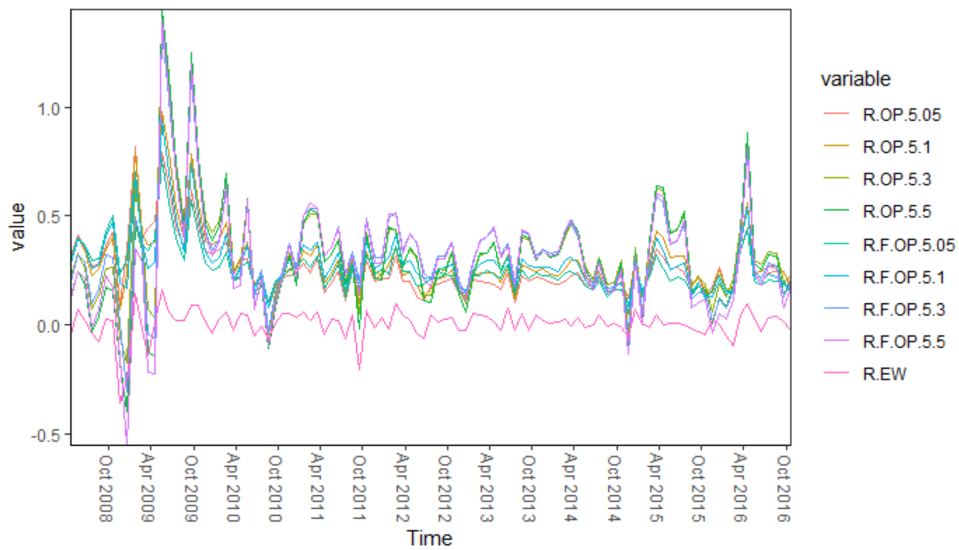

Figure 3.23: Time Series of Mixed Positional Strategies' Returns, $A_r = 5$
Figure 3.23 compares the time series of positional mixed portfolios' returns for $A_r = 5$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_{tv}. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_{tv}. The pink line shows the mean of the equally weighted portfolio.

Table 3.16, shows the average returns, their Sharpe ratios and the cumulative returns on positional mixed portfolios when $A_{tv} = 0.5$. Same as before, the positional mixed portfolios obtained from estimate ρ's are providing the higher average returns. While in terms of cumulative returns the portfolios obtained from fitted ρ's are outperforming the others except for $A_r = 5$.

For risk aversion equal to 1 the portfolio based on estimated ρ's is providing higher Sharpe ratio, while for other values of risk aversions, the fitted ρ's are providing higher Sharpe ratios. Both estimated and fitted ρ's when $A_t v = 5$ are providing the higher returns than the others.

Table 3.16: Summary of Mixed Positional Portfolios' Returns, $A_{tv} = 0.5$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_r = 0.5$	**2.983**	1.283	2.954	**1.286**	159.134	**160.027**
$A_r = 1$	**1.626**	**1.687**	1.764	1.613	90.094	**90.395**
$A_r = 3$	**0.459**	2.130	0.457	**2.430**	26.694	**26.706**
$A_r = 5$	**0.257**	1.997	0.256	**2.290**	**15.142**	15.135
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.16 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_{tv} constant and equal to 0.5.

Figure 3.24 shows the time series of the positional mixed portfolios when $A_{tv} = 0.5$. In most of the time positional mixed portfolios obtained from both ways by $A_r = 0.5$ have higher returns than the others. we can see the risk-return trade-off relations in this Figure. In other word by increasing the value of risk aversion (A_r) the return on the portfolios fall.

Table 3.17 shows the average returns, their Sharpe ratios and the cumulative returns of positional mixed portfolios when $A_{tv} = 1$.

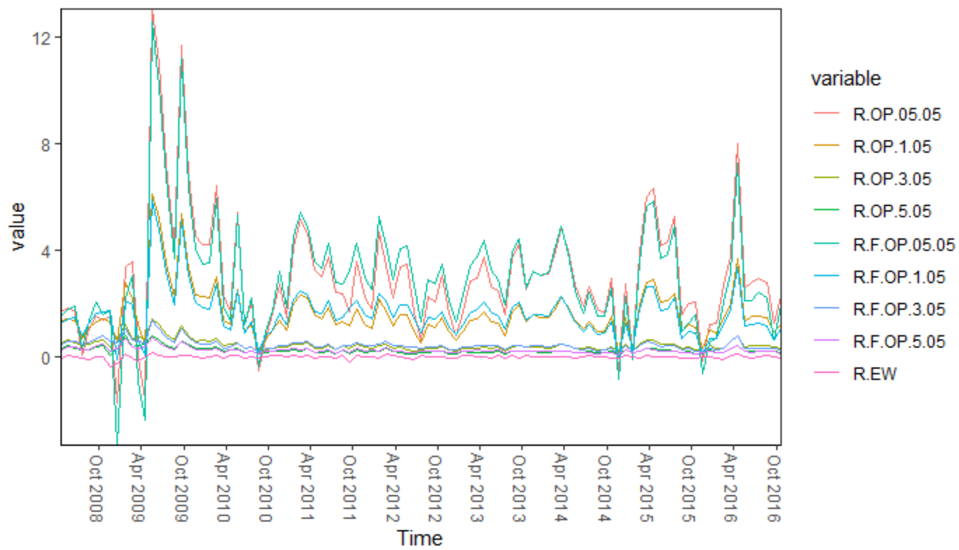

Figure 3.24: Time Series of Mixed Positional Strategies' Returns, $A_{tv} = 0.5$

Figure 3.24 compares the time series of positional mixed portfolios' returns for $A_{tv} = 0.5$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_r. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_r. The pink line shows the mean of the equally weighted portfolio.

Table 3.17: Summary of Mixed Positional Portfolios' Returns, $A_{tv} = 1$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_r = 0.5$	**1.960**	**1.028**	1.938	1.011	101.054	**101.827**
$A_r = 1$	**1.493**	1.267	1.479	**1.269**	79.610	**80.057**
$A_r = 3$	**0.518**	1.845	0.515	**1.993**	29.245	**29.306**
$A_r = 5$	**0.286**	1.913	0.284	**2.135**	16.444	**16.457**
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.17 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 2.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_{tv} constant and equal to 1.

We observe that by increasing the value of risk aversion (A_{tv}) the average and the cumulative returns of positional mixed portfolios are decreasing. In terms of average return, the positional portfolios obtained from estimated ρ's are providing higher returns. But in terms of cumulative returns the positional mixed portfolios based on fitted ρ's are outperforming the others. For risk aversion equal to 0.5, the estimated ρ's has higher Sharpe ratio, while for other values for risk aversion, the fitted ρ's have higher Sharpe ratios.

Figure 3.25 shows the time series of the positional mixed portfolios when $A_{tv} = 1$. Same as Figure 3.24, the portfolios with the lowest risk aversion provide the highest returns and the equally weighted portfolio provides the lowest return. By increasing the risk aversion value, the return of the positional mixed portfolios increase.

Table 3.18 shows the returns and Sharpe ratios on the positional mixed portfolios for $A_{tv} = 3$. In Table 3.18 we see that by decreasing the risk aversion the average and cumulative returns increased. The positional mixed portfolios obtained from estimated ρ's outperform other portfolios. In terms of cumulative returns the positional mixed portfolios obtained from fitted ρ's are providing higher returns.

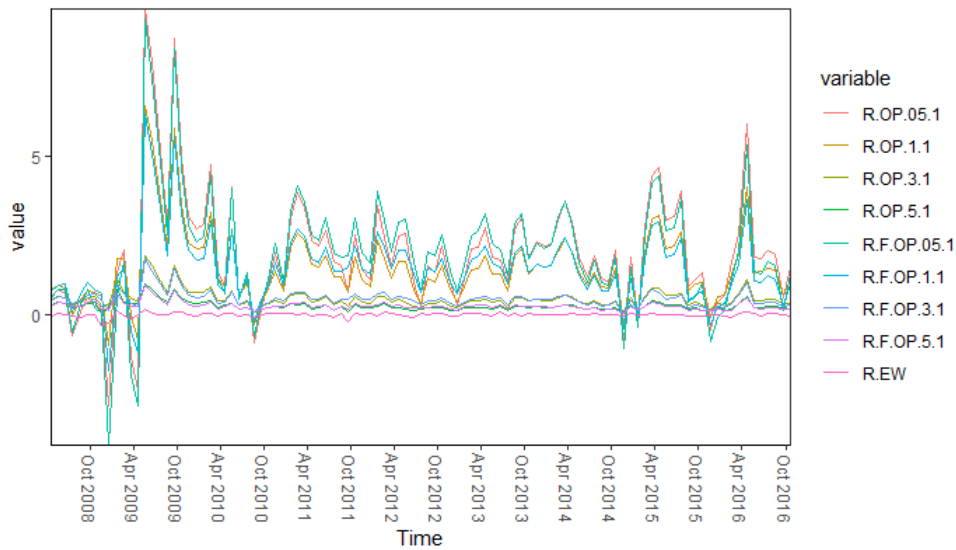

Figure 3.25: Time Series of Mixed Positional Strategies' Returns, $A_{tv} = 1$

Figure 3.25 compares the time series of positional mixed portfolios' returns for $A_{tv} = 1$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_r. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_r. The pink line shows the mean of the equally weighted portfolio.

Table 3.18: Summary of Mixed Positional Portfolios' Returns, $A_{tv} = 3$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_r = 0.5$	**0.679**	**0.836**	0.671	0.813	33.780	**34.109**
$A_r = 1$	**0.683**	**0.916**	0.675	0.896	34.625	**34.922**
$A_r = 3$	**0.500**	1.199	0.496	**1.200**	26.595	**26.743**
$A_r = 5$	**0.330**	1.381	0.327	**1.415**	17.993	**18.0641**
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.18 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_{tv} constant and equal to 3.

When risk aversion is equal to 0.5 and 1, the estimated ρ's are providing higher Sharpe ratios, but for the values of risk aversions equal to 3 and 5 the fitted ρ's are providing higher Sharpe ratios.

Figure 3.26 shows the time series f positions mixed portfolios when $A_{tv} = 3$. These time series returns are showing same patterns as Figure 3.25. We also see the risk-return trade-off as well, higher risk yield in higher returns.

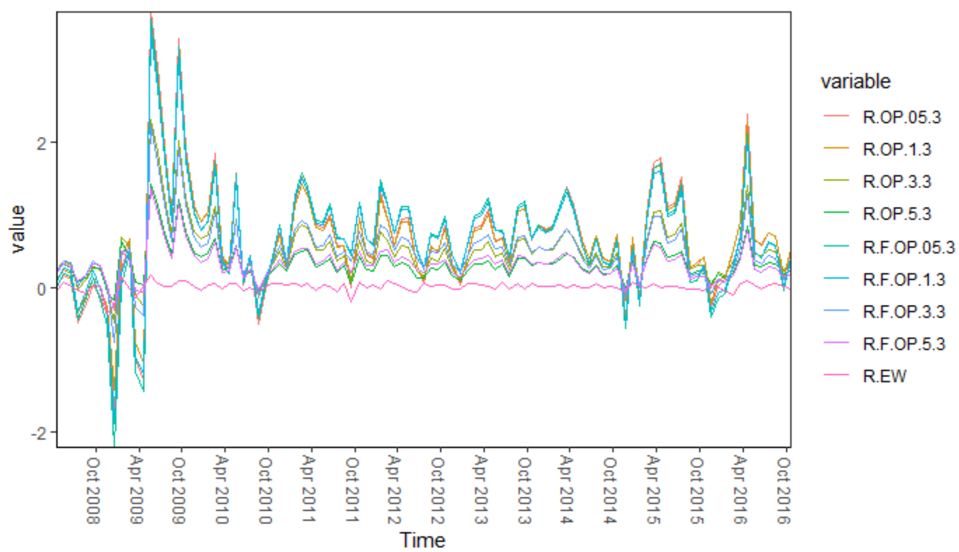

Figure 3.26: Time Series of Mixed Positional Strategies' Returns, $A_{tv} = 3$
Figure 3.26 compares the time series of positional mixed portfolios' returns for $A_{tv} = 3$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_r. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_r. The pink line shows the mean of the equally weighted portfolio.

Table 3.19 provides the average return, their Sharpe ratios and the cumulative returns on positional mixed portfolios when $A_{tv} = 5$. The positional mixed portfolios obtained from estimated ρ's provide higher average returns. In terms of cumulative returns positional portfolios based on fitted ρ's are providing higher returns. Higher

Table 3.19: Summary of Mixed Positional Portfolios' Returns, $A_{tv} = 5$

	Average Return				Cumulative Return	
Risk Aversion	**Eρ's**	**Eρ-SR**	**Fρ's**	**Fρ-SR**	**Estimated ρ's**	**Fitted ρ's**
$A_r = 0.5$	**0.401**	**0.781**	0.396	0.758	19.758	**19.960**
$A_r = 1$	**0.411**	**0.828**	0.406	0.806	20.500	**20.694**
$A_r = 3$	**0.379**	**0.999**	0.375	0.985	19.617	**19.757**
$A_r = 5$	**0.302**	**1.129**	0.299	1.129	15.991	**16.081**
	Mean		S-D		Sh-R	
EW	0.004		0.067		0.042	

Note: Table 3.19 shows the average and cumulative return of optimal positional mixed portfolios with the inception date of April 2008 based on future ranks predicted with estimated $\hat{\rho}_{11,t}$, $\hat{\rho}_{12,t}$, $\hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) (Estimated ρ's) and the fitted value of estimated coefficients from equations (3.16) to (3.19) $\hat{\hat{\rho}}_{11,t+1}$, $\hat{\hat{\rho}}_{12,t+1}$, $\hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$ (Fitted ρ's), considering A_{tv} constant and equal to 5.

risk yield in higher average and cumulative returns. The estimated ρ's are providing higher Sharpe ratios than the fitted ones and by increasing the value of risk aversions they increase as well.

Figure 3.27 provided the time series of the positional mixed portfolios when $A_{tv} = 5$. Similar to the previous Figures, we can observe the risk-return trade-off. Higher risk aversion, lower returns. The returns of all these portfolios are very close while the equally weighted still provides the lowest return.

Comparing Table 3.19, 3.18, 3.17 and 3.16 shows that, by increasing the value of A_{tv} the average and cumulative returns decreased. In terms of average return the positional mixed portfolios obtained from estimated ρ's are providing higher returns, while in terms of cumulative returns the positional mixed portfolios obtained from fitted ρ's. By comparing the results in all Tables provided in Section 5, we see that the positional liquid portfolios provides the highest average and cumulative returns compare to other strategies.

It means that a positional portfolio based on liquid assets provides higher return than the positional portfolio based on winners. Also we found that the positional mixed portfolios provide higher average and cumulative returns than positional momentum portfolios. In other words a positional portfolio based on liquid winners provides higher average and cumulative returns than a positional portfolio based on just winner stocks.

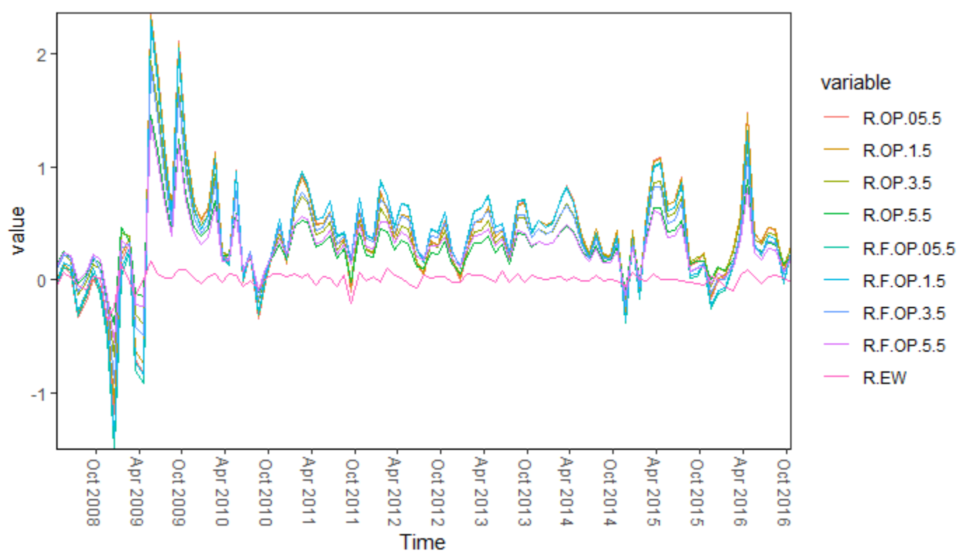

Figure 3.27: Time Series of Mixed Positional Strategies' Returns, $A_{tv} = 5$
Figure 3.27 compares the time series of positional mixed portfolios' returns for $A_{tv} = 5$. The red, orange, olive and green lines show the returns of optimal positional mixed portfolios computed from estimated parameters of VAR model for different A_r. The light green, light blue, blue and purple lines show the returns of optimal positional mixed portfolios computed from fitted values of parameters from equations (3.16) to (3.19) for different A_r. The pink line shows the mean of the equally weighted portfolio.

3.5 Conclusion

This paper introduced new positional investment strategies that maximize investors positional utility from portfolios of assets with expected high return ranks, high liquidity ranks and high combined return-liquidity ranks. The optimal allocation vectors are computed from return and volume change ranks modelled as a panel VAR with time varying coefficients. We show that the autoregressive VAR parameters can be well approximated by linear functions of auto- and cross- correlations of the returns and volume change series of the SPDR tracking portfolio.

The empirical results indicate that all positional portfolios provide positive average and cumulative return. The positional liquid portfolios outperform the positional mixed and momentum portfolios respectively. Also, we observe that for higher risk aversion values, the average and cumulative returns on the positional portfolios decrease. In terms of average returns the positional portfolios obtained from estimated coefficients $\hat{\rho}_{11,t},\ \hat{\rho}_{12,t},\ \hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) outperforms the other portfolios. While in terms of cumulative returns the positional portfolios obtained from fitted values of coefficients based on auto- and cross- correlation of SPDR ($\hat{\hat{\rho}}_{11,t+1},\ \hat{\hat{\rho}}_{12,t+1},\ \hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$) provide higher returns.

Conclusion

This thesis intends to improve a momentum strategy by taking into account the auto- and cross-correlations of ranks of returns and trade volume changes instead of just past raw returns and provides the optimal positional portfolio by maximizing the investor's utility function based on the future ranks of return and trade volume changes. The positional portfolio management proposed in this thesis provides new types of allocations strategies. By comparing the returns on the optimal positional portfolio with the traditional momentum, contrarian (or reversal) strategies and naive equally weighted portfolio, we can measure the gain from implementing the positional portfolio management strategies. In the positional portfolio management all stock returns are ranked cross-sectionally, so that the notion of cross-sectional rank (position) is at the core of the distinction of this management from the standard portfolio management.

Many empirical studies showed that in a dynamic context, information about trading volume improves the forecasts for price changes and return volatility. In Chapter one, the relationship between return and trade volume changes is studied. The motivation for considering both returns and trade volume stems from the empirical evidence documented in financial literature, which suggests that the trade volumes provide additional information and help predict future returns.

Chapter Two extends the positional momentum strategy in three respects. First, the ranks of asset returns and the ranks of trade volume changes are considered jointly and modelled as a bivariate series. Second, the positional momentum portfolio based on the observed ranks is replaced by the positional momentum portfolio based on the expected future ranks. In this chapter, the future ranks of return and volume changes are predicted from the past ranks of returns and volume changes. The third contribution is a new expected positional liquid portfolio that contains assets that display the highest (resp. lowest) future expected changes in trade volumes. The ranks of return and volume changes are predicted from a bivariate panel Vector Autoregressive model of order one (VAR(1)). It is shown that return ranks are correlated with their own past values and the current and past ranks of trade volume changes. This results leads to a new expected positional momentum strategy providing portfolios of predicted winners, conditional on past ranks of returns and volume changes. This approach further extends to positional liquidity management. The expected liquid positional strategy selects portfolios of stocks with the strongest realized or predicted increase in trading volume. These new positional management strategies outperform the standard momentum strategies and the equally weighted portfolio in terms of average returns and Sharpe ratio.

Chapter Three introduces new optimal positional investment strategies that maximize investors' positional utility from holding assets with high expected future return and liquidity ranks. The investor is assumed to maximize a CARA (Constant

Absolute Risk Aversion) utility function of future position of the assets (ranks of assets). The optimal allocation vectors provide new investment strategies, such as the optimal positional momentum portfolio, the optimal liquid portfolio and the optimal mixed portfolio that combines high return and liquidity ranks. The future ranks are predicted from a bivariate panel VAR model with time varying autoregressive parameters.

It has been shown that returns on the new optimal portfolios are comparable both theoretically and empirically with the naive equally weighted portfolio as well as with the traditional momentum strategies with look-back and holding periods of various length. To accommodate that variation, a time varying parameter VAR model is considered and two methods that allow an investor to update the VAR parameters at each investment time are proposed. The first method consists in re-estimating the model at each time by rolling over a fixed window of observations. The second method exploits the relationship between the autoregressive coefficients of the VAR model and the series of auto-and cross-correlations at lag 1 of returns and volume changes of the SPDR (Standard Poor's Depositary Receipts). These linear functions are easy to compute and simplify the investment procedure as they eliminate the need for re-estimating the panel VAR model by rolling. In the proposed approach, the time varying parameters are considered predetermined. I show that the approach can be extended to a random coefficient framework, where the autoregressive VAR coefficients are considered as fixed functions of random factors, which are the auto and cross-correlation estimators with their known asymptotic distributions. The empirical results indicate that all positional portfolios provide positive average and cumulative return. The positional liquid portfolios outperform the positional mixed and momentum portfolios respectively. Also, we observe that for higher risk aversion values, the average and cumulative returns on the positional portfolios decrease. In terms of average returns the positional portfolios obtained from estimated coefficients $\hat{\rho}_{11,t},\ \hat{\rho}_{12,t},\ \hat{\rho}_{21,t}$ and $\hat{\rho}_{22,t}$ from VAR(1) model (equation 3.7) outperforms the other portfolios. While in terms of cumulative returns the positional portfolios obtained from fitted values of coefficients based on auto- and cross-correlation of SPDR ($\hat{\hat{\rho}}_{11,t+1},\ \hat{\hat{\rho}}_{12,t+1},\ \hat{\hat{\rho}}_{21,t+1}$ and $\hat{\hat{\rho}}_{22,t+1}$) provide higher returns.

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Appendices

A Diagnostic Tests For Error Terms

A.1 Normality of Cross-Sectional Gaussian Ranks

Figures A.1 and A.2 display the Q-Q plots for the two transformed observed ranks vectors u and v in October 2016.

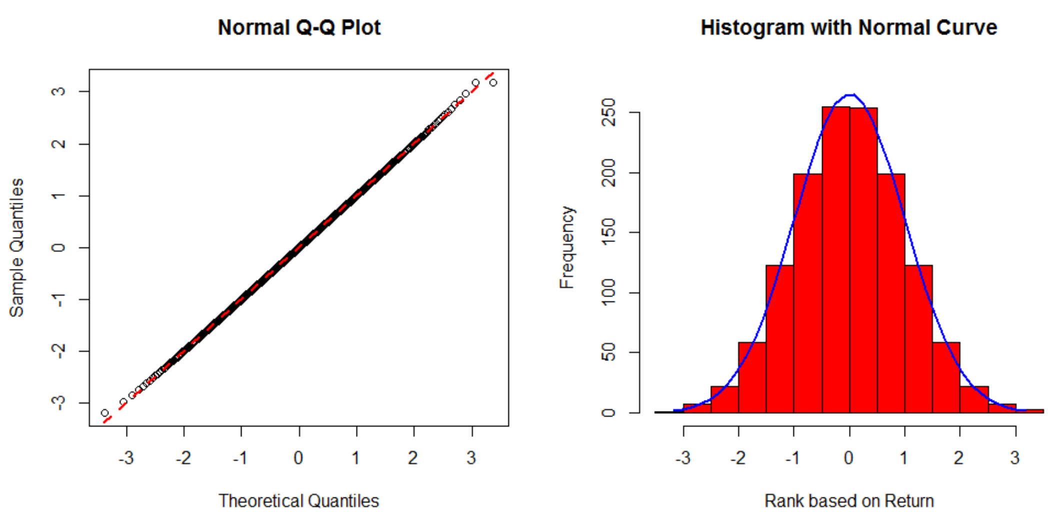

Figure A.1: Q-Q Plot of Transformed Return Ranks u_i
Figure A.1 displays the Q-Q plot of the transformed return ranks u_i in October 2016. The figures confirm the cross-sectional Gaussian distribution of transformed return ranks.

The figures confirm the cross-sectional Gaussian distribution of return ranks and trade volume change ranks. As expected, both ranks are cross-sectionally Normally distributed. In addition, the Shapiro normality tests applied to ranks u and v in each month, indicate that both ranks are Normally distributed cross-sectionally at each period of time.

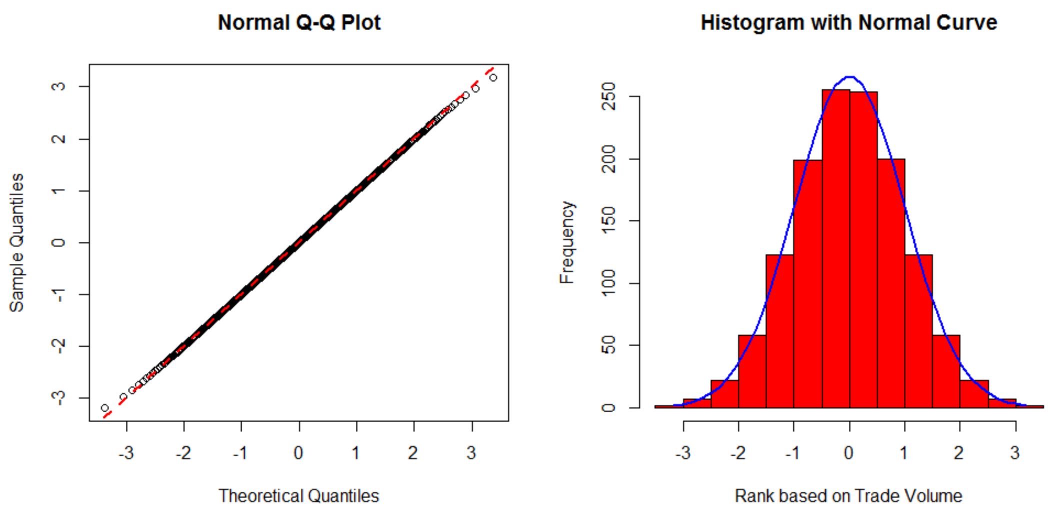

Figure A.2: Q-Q Plot of Transformed Volume Change Ranks v_i

Figure A.2 Displays the Q-Q plot of the transformed trade volume changes ranks v_i in October 2016. The figures confirm the cross-sectional Gaussian distribution of transformed trade volume changes ranks.

A.2 Autocorrelation Test

The Durbin-Watson (DW) statistic is used to test for the presence of autocorrelation at lag 1 in regression residuals when the classical assumptions are satisfied). The null hypothesis is that the errors are serially uncorrelated and the alternative is that they follow a first order autoregressive process. If $\hat{e}_t$ is the residual associated with the observation at time t, then the test statistic is:

$$d = \frac{\sum_{i=2}^{T}(\hat{e}_t - \hat{e}_{t-1})^2}{\sum_{i=1}^{T}\hat{e}_t^2} \tag{A.2.1}$$

where T is the number of observations. Since d is approximately equal to $2(1-r)$, where r is the sample autocorrelation of the residuals, $d = 2$ indicates no autocorrelation.

From model (2.6) it follows that bivariate residual is $\hat{e}_t = \begin{pmatrix} \hat{e}_{1t} \\ \hat{e}_{2t} \end{pmatrix} = \hat{\Sigma}^{-\frac{1}{2}}\left[\begin{pmatrix} u_{it} \\ v_{it} \end{pmatrix} - \hat{R}\begin{pmatrix} u_{i,t-1} \\ v_{i,t-1} \end{pmatrix}\right]$, where $\hat{\Sigma}$ and $\hat{R}$ are the estimation of Σ and R. Then the DW statistics can be computed separately from $\hat{e}_{1t}$ and $\hat{e}_{2t}$. The values of d for the VAR(1) model (equation(2.7)) are as follows:

$$\begin{aligned} \hat{d}_{e_1} &= 1.99 \\ \hat{d}_{e_2} &= 2.15 \end{aligned} \tag{A.2.2}$$

Since the obtained $d's$ are very close to 2, we conclude that there is no autocorrelation in error terms.

A.3 Cross-Sectional Normality of Residuals

Figures A.3 and A.4 show the Q-Q plot and the cross-sectional histogram of residuals in equation (2.7).

As we can see both residual terms are cross-secionally normally distributed. We also

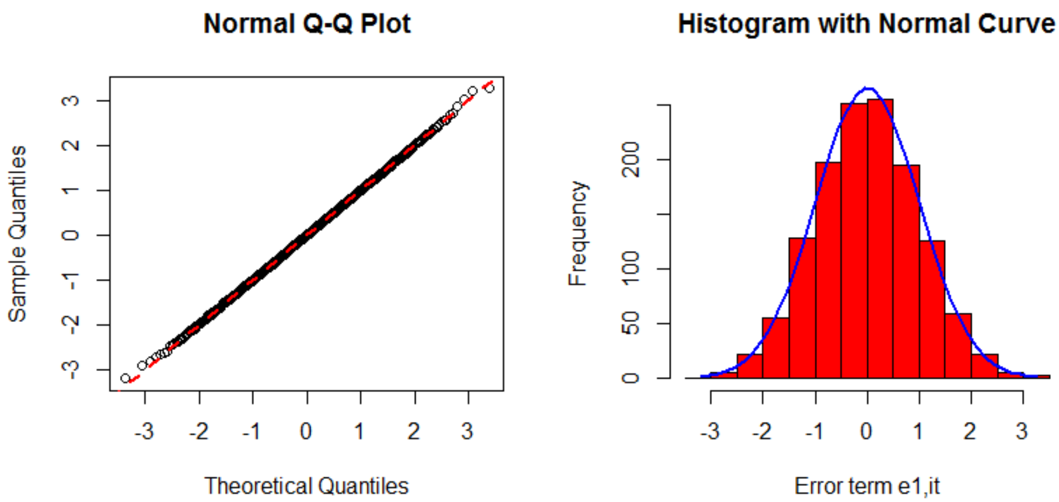

Figure A.3: Q-Q plot and Cross-Sectional Histogram of $\hat{e}_1$

Figure A.3 displays the Q-Q plot and cross-sectional histogram of $\hat{e}_1$ estimated from equation (2.7) on October, 2016.

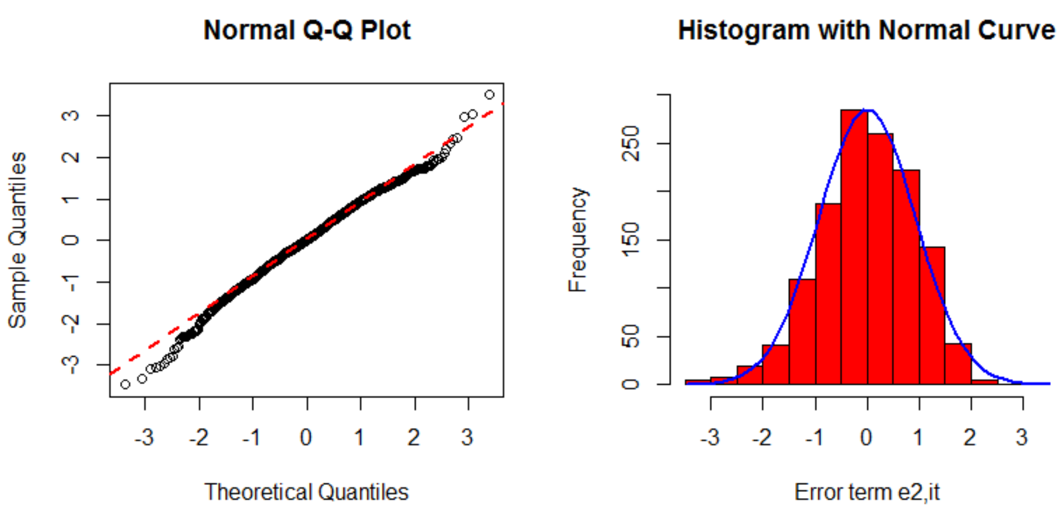

Figure A.4: Q-Q Plot and Cross-Sectional Histogram of $\hat{e}_2$

Figure A.4 displays the Q-Q plot and cross-sectional histogram of $\hat{e}_2$ estimated from equation (2.7) on October, 2016.

perform the Shapiro Normality test for the cross-sectional residuals obtained from VAR(1) model in equation (2.7). The Shapiro test never rejects normality cross-sectionally for $\hat{e}_{1,it}$ and 42% of times Shapiro test rejects normality cross-sectionally for $\hat{e}_{2,it}$. The following Figures show the Q-Q plot and the distribution function of the cross-sectional residuals in October, 2016.

B Normality of Serial Residuals for $S\&P500$

The Shapiro test of Normality is applied to time series of residuals obtained from the VAR(1) model estimated for $S\&P500$. The test did not rejects the normality for $\hat{e}_1$ ($p-value = 0.42$), but it rejected the normality of $\hat{e}_2$ since the p-value is 0.02. Therefore, for $S\&P500$ the distribution of residuals from the return rank equation of the VAR(1) model is normal while it is not normal for the residuals from the volume ranks equation of the VAR(1) model.

Figures B.1 and B.2 display the Q-Q plots and the historical distribution function of the series of residuals from VAR(1) model for $S\&P500$. Figure B.1 shows that for $S\&P500$, the historical error term $\hat{e}_1$ estimated from the VAR(1) model is Normally distributed while Figure B.2 shows that the historical error term $\hat{e}_2$ estimated from the VAR(1) model is not Normally distributed.

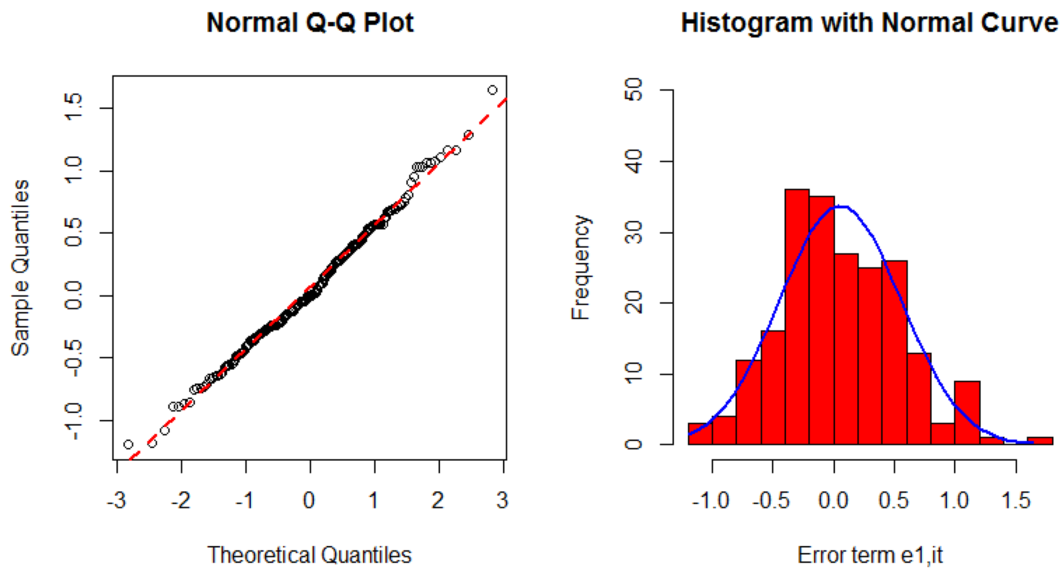

Figure B.1: Q-Q Plot and Histogram of Residual $\hat{e}_1$ for S&P.500.

Figure B.1 displays the Q-Q plot and histogram of residual $\hat{e}_1$ for S&P.500. The histogram and the Q-Q plot both show that $\hat{e}_1$ is Normally distributed.

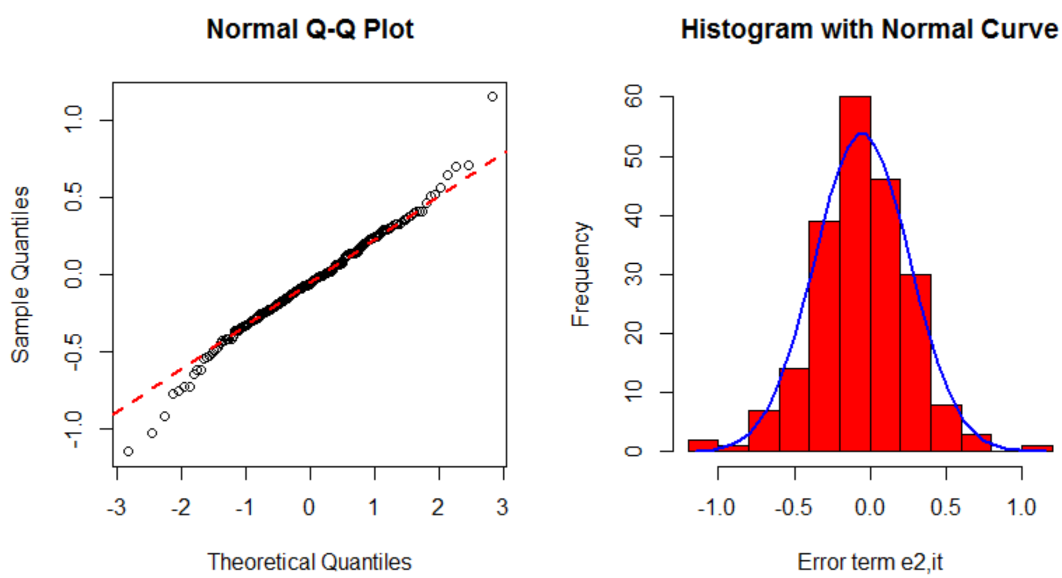

Figure B.2: Q-Q plot and Histogram of $\hat{e}_2$ for $S\&P500$.

Figure B.2 displays the Q-Q plot and histogram of $\hat{e}_2$ for $S\&P500$. The histogram and the Q-Q plot both show that $\hat{e}_2$ is slightly different than Normal distribution.

C Histograms of Auto and Cross-Correlations of SPDR Return and Trade volume Changes and of the Series of Estimated Autoregressive Coefficients

The following figures show the histograms of the time series of sample auto- cross-correlations of SPDR return and trade volume changes and of the time series of autoregressive coefficients $\hat{\rho}_{jk,t},\ j=1,2,\ k=1,2,\ t=1,...,T$ of the VAR(1) model (equation 3.10). All series are estimated by rolling with a window of 9 years over the sampling period.

Figure C.1, shows that sample auto-correlations of r_t^S take values mostly between -0.1 and 0.05 and their density is asymmetric with a long left tail. The series $\hat{\rho}_{11t}$ takes smaller values between -0.01 and 0.006, and has a symmetric density. Figure C.2 shows that the cross-correlations of r_t^S,tv_{t-1}^S take values mostly between -0.1 and 0.2 and their density displays asymmetry in the right tail. The series $\hat{\rho}_{21t}$ takes only positive values, with the most frequently observed values in the interval (0.001,0.002). The density of cross-correlations of tv_t^S,r_{t-1}^S given in Figure C.3 is almost bimodal. These cross-correlations take positive values only. The density of $\hat{\rho}_{21t}$ is similar in shape but its support includes small positive and negative values. Figure C.4 shows the density of sample auto-correlations of tv_t^S, which take negative values. Their density is symmetric and bell-shaped. The density of $r\hat{h}o_{22t}$, which also take negative values only, is asymmetric with a long left tail.

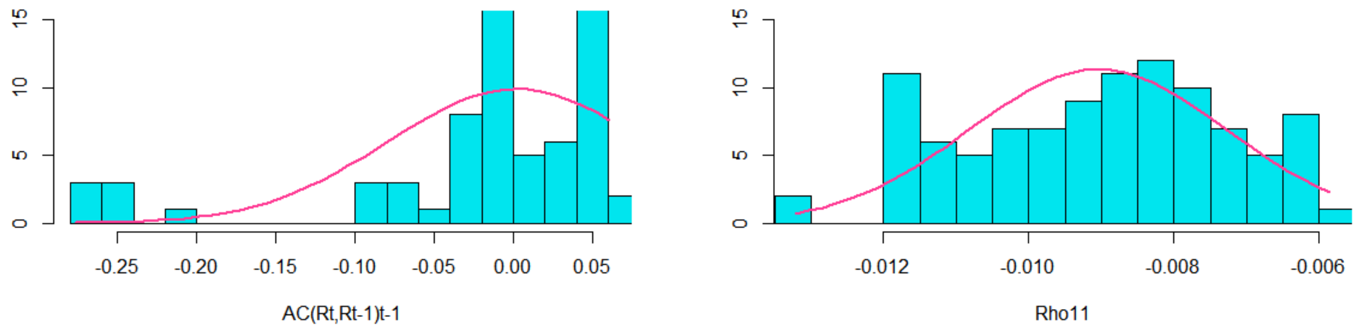

Figure C.1: Histograms of autocorrelations at lag one of r_t^S and of $\hat{\rho}_{11t}$

Figure C.1 compares the histograms of autocorrelations at lag one of SPDR's returns and the estimated $\hat{\rho}_{11t}$ from equation (3.10). In both plots the red line shows the kernel density estimates.

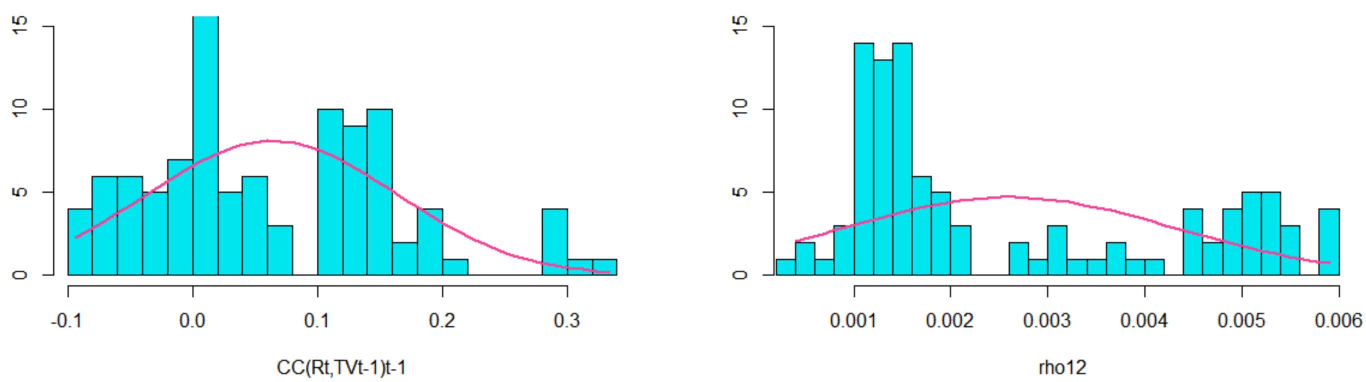

Figure C.2: Histograms of cross-correlation of r_t^S, tv_{t-1}^S and of $\hat{\rho}_{12t}$

Figure C.2 compares the histograms of cross-correlations of SPDR's r_t^S, tv_{t-1}^S and the estimated $\hat{\rho}_{12t}$ from equation (3.10). In both plots the red line shows the kernel density estimates.

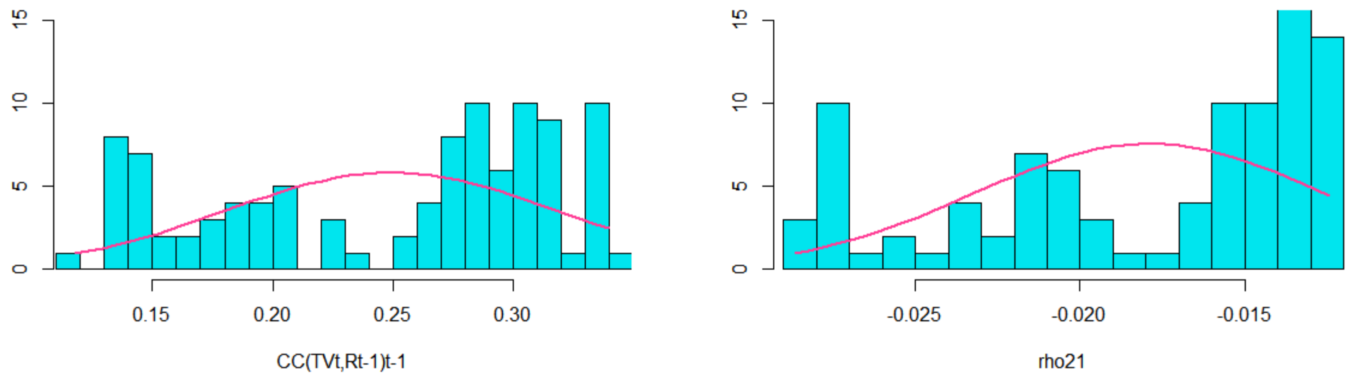

Figure C.3: Histograms of cross-correlations of tv_t^S, r_{t-1}^S and of $\hat{\rho}_{21t}$

Figure C.3 compares the histograms of cross-correlations of tv_t^S, r_{t-1}^S and the estimated $\hat{\rho}_{21t}$ from equation (3.10). In both plots the red line shows the kernel density estimates.

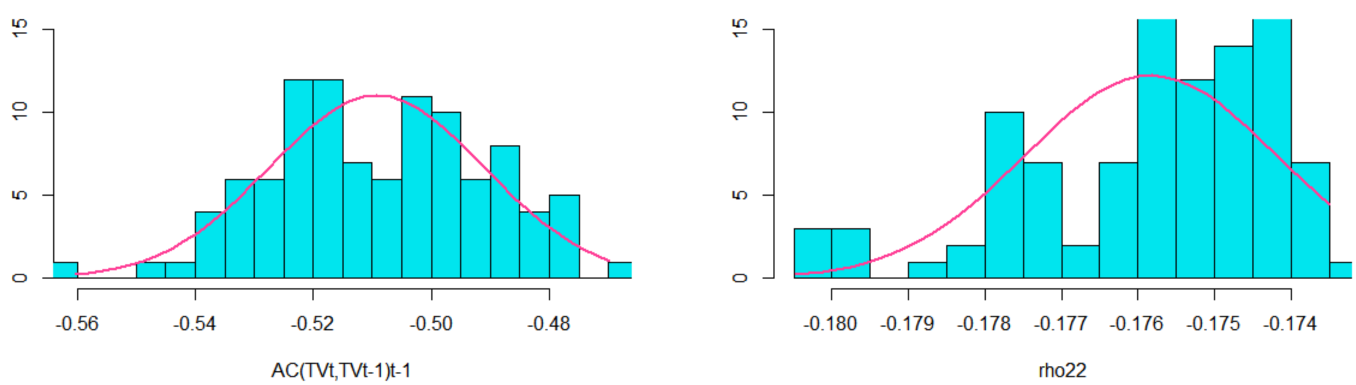

Figure C.4: Histograms of autocorrelations at lag one of tv_t^S and of $\hat{\rho}_{22t}$

Figure C.4 compares the histograms of autocorrelation at lag one of SPDR's trade volume changes and the estimated $\hat{\rho}_{22t}$ from equation (3.10). In both plots the red line shows the kernel density estimates.

D Factor Models-Scatters and Regression Lines

The following Figures illustrate the regressions of $\hat{\rho}_{jk,t}$ on auto- and cross-correlations of SPDR returns and volume changes (equations 3.16 to 3.19). We observe that the scatters are irregular and the linear models provide fairly good approximations.

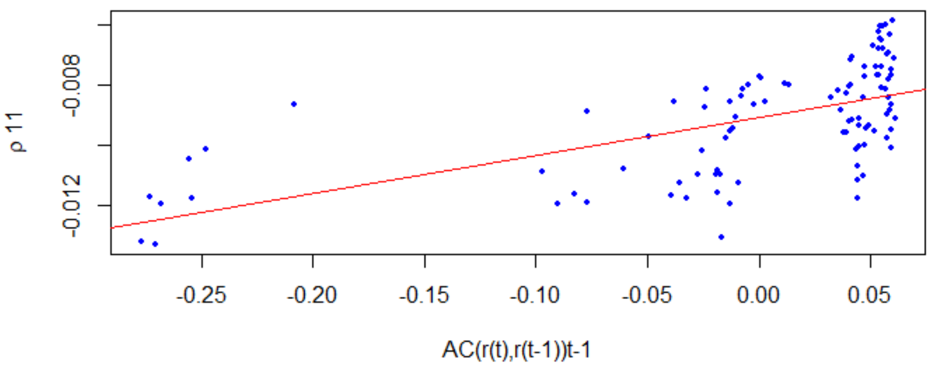

Figure D.1: Regression of $\hat{\rho}_{11t}$ on $AC(r^S)_{t-1}$

Figure D.1 shows the simple linear regression of $\hat{\rho}_{11t}$ on $AC(r^S)_{t-1}$. This linear regression shows a positive relation between $\hat{\rho}_{11t}$ on $AC(r^S)_{t-1}$.

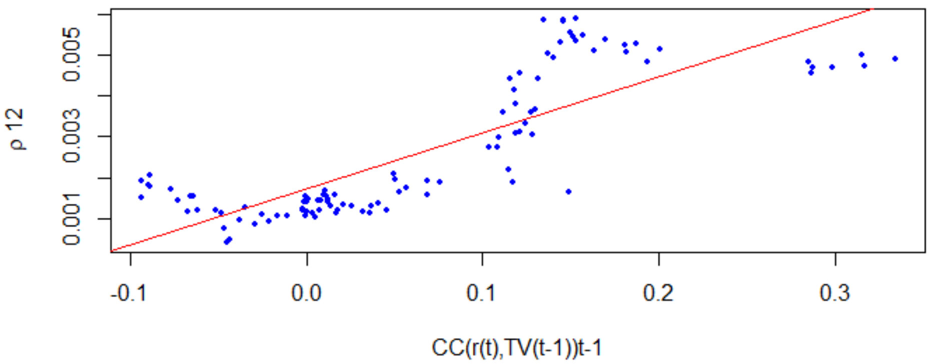

Figure D.2: Regression of $\hat{\rho}_{12t}$ on $CC(r^s tv^S)_{t-1}$

Figure D.2 shows the simple linear regression of $\hat{\rho}_{12t}$ on $CC(r^s tv^S)_{t-1}$. This linear regression shows a positive relation between $\hat{\rho}_{12t}$ on $CC(r^s tv^S)_{t-1}$.

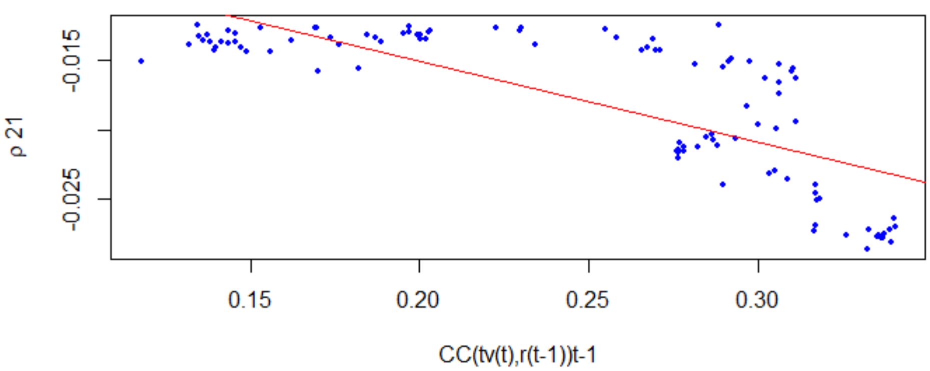

Figure D.3: Regression of $\hat{\rho}_{21t}$ on $CC(tv^S r^s)_{t-1}$

Figure D.3 shows the linear regression of $\hat{\rho}_{21t}$ on $CC(tv^S r^s)_{t-1}$. This regression line shows a negative relation between $\hat{\rho}_{21t}$ on $CC(tv^S r^s)_{t-1}$

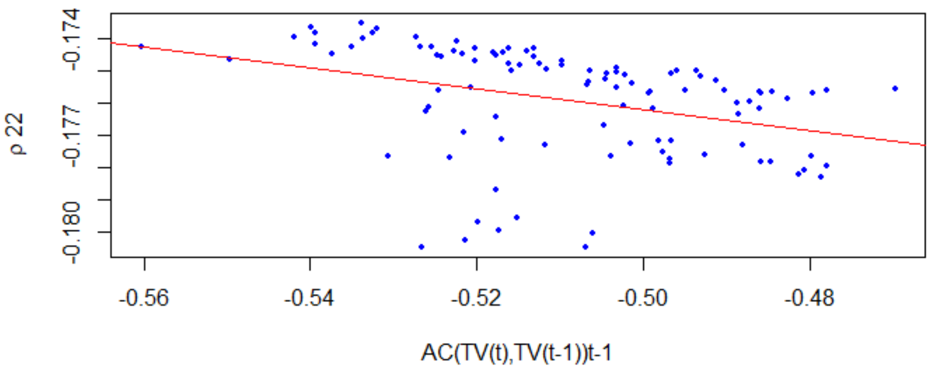

Figure D.4: Regression of $\hat{\rho}_{22t}$ on $AC(tv^S)_{t-1}$

Figure D.4 shows linear regression of $\hat{\rho}_{22t}$ on $AC(tv^S)_{t-1}$. This linear regression shows the negative relation between $\hat{\rho}_{22t}$ on $AC(tv^S)_{t-1}$.

E Stochastic Autoregressive Coefficients

This section illustrates the changes to the optimal allocation vectors when the autoregressive coefficients $\rho_{ij,t+1}$ are considered as random functions of factor F_t. The factor F_t represents jointly the returns r_t^S and trade volume changes tv_t^S of SPDR at time t that determine the autoregressive coefficients $\rho_{ij,t+1}$.

The expected positional utilities to be maximized are as follows:

$$-E[exp(-A_r Q^r_{t+1}(\beta_r' r_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_t}]$$

$$= -E\{E[exp(-A_r Q^r_{t+1}(\beta_r' r_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_{t+1}}] | \underline{r_t}, \underline{tv_t}, \underline{F_t}\}$$

$$= -E\Big[exp\Big(-A_r\rho_{11,t+1}\sum_{i=1}^{n}\beta_{r,i}u_{i,t} - A_r\rho_{12,t+1}\sum_{i=1}^{n}\beta_{r,i}v_{i,t} + \frac{1}{2}A_r^2\sum_{i=1}^{n}\beta_{r,i}^2\sigma_{1,t+1}^2\Big) | \underline{r_t}, \underline{tv_t}, \underline{F_t}\Big]$$

$$= -E_t(exp[-A_r\rho_{11,t+1}\beta_r' u_t - A_r\rho_{12,t+1}\beta_r' v_t] + \frac{1}{2}A_r^2\beta_r'\beta_r\sigma_{1,t+1}^2) \quad (3.3)$$

subject to $\beta_r' h = 1$ and,

$$
\begin{aligned}
&- E[exp(-A_{tv}Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_t}] \\
&= -E\{E[exp(-A_{tv}Q^{tv}_{t+1}(\beta'_{tv} tv_{t+1})) \mid \underline{r_t}, \underline{tv_t}, \underline{F_t}] | \underline{r_t}, \underline{tv_t}, \underline{F_t}\} \\
&= -E\Big[exp\Big(-A_{tv}\rho_{21,t+1}\sum_{i=1}^{n}\beta_{tv,i}u_{i,t} - A_{tv}\rho_{22,t+1}\sum_{i=1}^{n}\beta_{tv,i}v_{i,t} + \frac{1}{2}A_{tv}^2\sum_{i=1}^{n}\beta^2_{tv,i}\sigma^2_{2,t+1}\Big)|\underline{r_t}, \underline{tv_t}, \underline{F_t}\Big] \\
&= -E_t(exp[-A_{tv}\rho_{21,t+1}\beta'_{tv}u_t - A_{tv}\rho_{22,t+1}\beta'_{tv}v_t] + \frac{1}{2}A_r^2\beta'_{tv}\beta_{tv}\sigma^2_{2,t+1}) \quad (3.4)
\end{aligned}
$$

subject to $\beta'_{tv} h = 1$. The above optimization problems are difficult to solve. In order to simplify the optimal allocation vectors, we can consider their first-order expansion with respect to $\rho_{jk,t+1}$ (where $j, k = 1, 2$) for small $\rho_{jk,t+1}$. At first-order approximation with respect to the persistence parameters, we have $E_t(\sigma^2_{1,t+1}) = E_t(\sigma^2_{2,t+1}) \simeq 1$ [24].

$$
\begin{aligned}
&- E_t[(1 - A_r\rho_{11,t+1}\beta_r' u_t - A_r\rho_{12,t+1}\beta_r' v_t)\exp\frac{1}{2}A_r^2\beta_r'\beta_r] \\
&\simeq -(1 - A_r E_t\rho_{11,t+1}\beta_r' u_t - A_r E_t\rho_{12,t+1}\beta_r' v_t)\exp\frac{1}{2}A_r^2\beta_r'\beta_r \\
&\simeq -\exp[-A_r E_t\rho_{11,t+1}\beta_r' u_t - A_r E_t\rho_{12,t+1}\beta_r' v_t - \frac{1}{2}A_r^2\beta_r'\beta_r \quad (3.5)
\end{aligned}
$$

[24] In Section 2 we showed that in practice, the positional persistence values at different dates can be rather small (see Figures 2.7,2.8). Therefore the assumption that $E_t\sigma^2_{1,t+1} = E_t\sigma^2_{2,t+1} \simeq 1$ is plausible

$$
\begin{aligned}
&- E_t[(1 - A_{tv}\rho_{21,t+1}\beta_{tv}' u_t - A_{tv}\rho_{22,t+1}\beta_{tv}' v_t) \exp \frac{1}{2} A_{tv}^2 \beta_{tv}' \beta_{tv}] \\
&\simeq -(1 - A_{tv} E_t\rho_{21,t+1}\beta_{tv}' u_t - A_{tv} E_t\rho_{22,t+1}\beta_{tv}' v_t) \exp \frac{1}{2} A_{tv}^2 \beta_{tv}' \beta_{tv} \\
&\simeq - \exp[-A_{tv} E_t\rho_{21,t+1}\beta_{tv}' u_t - A_{tv} E_t\rho_{22,t+1}\beta_{tv}' v_t - \frac{1}{2} A_{tv}^2 \beta_{tv}' \beta_{tv} \quad (3.6)
\end{aligned}
$$

which are objective functions similar to those in Section (3.3) with the autoregressive coefficients $\rho_{jk,t+1},\ j,k = 1,2$ replaced by their expectations $E_t\rho_{jk,t+1},\ j,k = 1,2$. Hence, the approximate optimal positional allocations are as follows:

$$
\beta_{r,it}^* = \frac{1}{n} + \frac{1}{A_r}\Big[E_t\rho_{11,t+1}u_{it} + E_t\rho_{12,t+1}v_{it} - \frac{1}{n}\sum_{i=1}^{n}(E_t\rho_{11,t+1}u_{it} + E_t\rho_{12,t+1}v_{it})\Big] \quad (3.7)
$$

$$
\beta_{tv,it}^* = \frac{1}{n} + \frac{1}{A_{tv}}\Big[E_t\rho_{21,t+1}u_{it} + E_t\rho_{22,t+1}v_{it} - \frac{1}{n}\sum_{i=1}^{n}(E_t\rho_{21,t+1}u_{it} + E_t\rho_{22,t+1}v_{it})\Big] \quad (3.8)
$$

By simplifying the above expressions we get:

$$
\beta_{r,it}^* = \frac{1}{n} + \frac{1}{A_r}\Big(E_t\rho_{11,t+1}(u_{it} - \overline{u_t}) + E_t\rho_{12,t+1}(v_{it} - \overline{v_t})\Big) \quad (3.9)
$$

$$
\beta_{tv,it}^* = \frac{1}{n} + \frac{1}{A_{tv}}\Big(E_t\rho_{21,t+1}(u_{it} - \overline{u_t}) + E_t\rho_{22,t+1}(v_{it} - \overline{v_t})\Big) \quad (3.10)
$$

where $\overline{u_t} = 1/n\sum_{i=1}^{n} u_{it}$ and $\overline{v_t} = 1/n\sum_{i=1}^{n} v_{it}$ are the cross-sectional averages of the Gaussian ranks at time t. When the number of assets (n) tends to infinity, these cross-sectional averages tend to zero, which is the mean of the standard Normal distribution.

The above optimal allocations are linear combination of two portfolios. The first one has positive weights $\frac{1}{n}$. The second portfolio on the right hand side of each solution is an arbitrage portfolio (zero-cost portfolio), with weights involving the ranks ($E_t\rho_{11,t}u_{it} + E_t\rho_{12,t}v_{it}$ and $E_t\rho_{21,t}u_{it} + E_t\rho_{22,t}v_{it}$, respectively.

F Square Root of Matrix Σ

To find the $\Sigma^{1/2}$ let us consider:

$$\Sigma = \begin{pmatrix} 1-\rho_{11}^2-\rho_{12}^2 & 1-\rho_{11}\rho_{21}-\rho_{12}\rho_{22} \\ 1-\rho_{21}\rho_{11}-\rho_{22}\rho_{12} & 1-\rho_{21}^2-\rho_{22}^2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.11}$$

Since $\Sigma^{1/2}$ is symmetric I can consider B=C. The square root of variance matrix is:

$$\Sigma^{1/2} = \pm\Big(\frac{1}{R}\Big)\begin{pmatrix} A+T & B \\ B & D+T \end{pmatrix} \tag{3.12}$$

where $T = |Det|^{1/2}$ and $R^2 = A + D + 2T$. We get:

$$\begin{aligned} T &= \sqrt{AD - B^2} \\ R &= \sqrt{A + D + 2\sqrt{AD - B^2}} \end{aligned} \tag{3.13}$$

By substituting $F.3$ into $F.2$ we get:

$$\Sigma^{1/2} = \pm\left(\frac{1}{\sqrt{A + D + 2\sqrt{AD - B^2}}}\right)\begin{pmatrix} A + \sqrt{AD - B^2} & B \\ B & D + \sqrt{AD - B^2} \end{pmatrix}$$
$$\Sigma^{1/2} = \pm\begin{pmatrix} \frac{A+\sqrt{AD-B^2}}{\sqrt{A+D+2\sqrt{AD-B^2}}} & \frac{B}{\sqrt{A+D+2\sqrt{AD-B^2}}} \\ \frac{C}{\sqrt{A+D+2\sqrt{AD-B^2}}} & \frac{D+\sqrt{AD-B^2}}{\sqrt{A+D+2\sqrt{AD-B^2}}} \end{pmatrix} \tag{3.14}$$

By substituting A, B and D from equation $F.1$ into equation $F.4$ we get the following:

$$T = \sqrt{\rho_{21}^2(-1+\rho_{12}^2) + \rho_{22}^2(-1+\rho_{11}^2) - \rho_{11}^2 - \rho_{12}^2 + 2\rho_{11}\rho_{21}(1-\rho_{22}\rho_{12}) + 2\rho_{22}\rho_{12}}$$
$$R = \sqrt{2 - (\rho_{11}^2 + \rho_{12}^2 + \rho_{21}^2 + \rho_{22}^2) + 2\sqrt{T}} \tag{3.15}$$

$$\Sigma^{1/2} = \pm\begin{pmatrix} \frac{1-\rho_{11}^2-\rho_{12}^2+\sqrt{T}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} & \frac{1-\rho_{11}\rho_{21}-\rho_{12}\rho_{22}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} \\ \frac{1-\rho_{21}\rho_{11}-\rho_{22}\rho_{12}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} & \frac{1-\rho_{21}^2-\rho_{22}^2+\sqrt{T}}{\sqrt{2-(\rho_{11}^2+\rho_{12}^2+\rho_{21}^2+\rho_{22}^2)+2\sqrt{T}}} \end{pmatrix} \tag{3.16}$$