C+ - Algebras and the Uncountable: A Systematic Study of the Combinatorics of the Uncountable in the Noncommutative Framework

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Abstract

In this dissertation we investigate nonseparable C*-algebras using methods coming from logic, specifically from set theory. The material is divided into three main parts.

In the first part we study algebras known as counterexamples to Naimark's problem, namely C*-algebras that are not isomorphic to the algebra of compact operators on some Hilbert space, yet still have only one irreducible representation up to unitary equivalence. Such algebras have to be simple, nonseparable and non-type I, and they are known to exist if the diamond principle (a strengthening of the continuum hypothesis) is assumed. With the motivation of finding further characterizations for these counterexamples, we undertake the study of their trace spaces, led by some elementary observations about the unitary action on the state space of these algebras, which seem to suggest that a counterexample to Naimark's problem could have at most one trace. We show that this is not the case and, assuming diamond, we prove that every Choquet simplex with countably many extreme points occurs as the trace space of a counterexample to Naimark's problem and that, moreover, there exists a counterexample whose tracial simplex is nonseparable.

The second part of this dissertation revolves around the Calkin algebra $\mathcal{Q}(H)$ and the general problem of what nonseparable C*-algebras embed into it. We prove that, under Martin's axiom, all C*-algebras of density character less than $2^{\aleph_0}$ embed into the Calkin algebra. Moving to larger C*-algebras, we show that (within ZFC alone) $C^*_{\text{red}}(F_{2^{\aleph_0}})$ and $C^*_{\text{max}}(F_{2^{\aleph_0}})$, where $F_{2^{\aleph_0}}$ is the free group on $2^{\aleph_0}$ generators, and every nonseparable UHF algebra with density character at most $2^{\aleph_0}$, embed into the Calkin algebra. On the other hand, we prove that it is consistent with ZFC + $2^{\aleph_0} \geq \aleph_\alpha$, for every ordinal $\alpha \geq 2$, that the abelian C*-algebra generated by an increasing chain of $\aleph_2$ projections does not embed into $\mathcal{Q}(H)$. Hence, the statement 'Every C*-algebra of density character strictly less than $2^{\aleph_0}$ embeds into the Calkin algebra' is independent from ZFC + $2^{\aleph_0} \geq \aleph_\alpha$, for every ordinal $\alpha > 2$. Finally, we show that the proof of Voiculescu's noncommutative version of the Weyl-von Neumann theorem consists, when looked from the right perspective, of a sequence of applications of the Baire category theorem to certain ccc posets. This allows us, assuming Martin's axiom, to generalize Voiculescu's results to nonseparable C*-algebras of density character less than $2^{\aleph_0}$.

The last part of this manuscript concerns lifting of abelian subalgebras of coronas of non-unital C*-algebras. Given a subset of commuting elements in a corona algebra, we study what could prevent the existence of a commutative lifting of such subset to the multiplier algebra. While for finite and countable families the only issues arising are of K-theoretic nature, for larger families the size itself becomes an obstruction. We prove in fact, for a primitive, non-unital, σ-unital C*-algebra A, that there exists a set of $\aleph_1$ orthogonal positive elements in the corona of A which cannot be lifted to a collection of commuting elements in the multiplier algebra of A.

ancora una volta, a Thias e Sole

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Introduction

> It wasn't a dark and stormy night. It should have been, but that's the weather for you. For every mad scientist who's had a convenient thunderstorm just on the night his Great Work is finished and lying on the slab, there have been dozens who've sat around aimlessly under the peaceful stars while Igor clocks up the overtime.
>
> *Good Omens*, Neil Gaiman and Terry Pratchett

An extremely fruitful interplay between mathematical logic and the theory of algebras of operators on a Hilbert space has been developing over the last 15 years.

Operator algebras were first studied by Murray and von Neumann in the 1930s in response to the birth of quantum mechanics, with the original intention to provide rigorous mathematical foundations to this developing theory. Since Murray and von Neumann's seminal works, this subject has grown into a branch of pure mathematics in its own right, with deep connections with several other areas of mathematics such as algebraic topology, ergodic theory, dynamical systems or geometric group theory.

Mathematical logic, on the other hand, is a discipline straddling mathematics, philosophy and computer science, which came to life in the second half of the 19th century providing the framework for the first systematic study of the foundations of mathematics. This subject recently developed deep connections with operator algebras in the form of model theory and set theory. In this dissertation we focus on some of the interactions between set theory and C*-algebras.

A C*-algebra is an algebra of operators on ℓ_2 which is closed in the norm topology. A recurrent theme in operator algebras (crucial also in this dissertation) is the idea that algebras of operators naturally provide 'quantized' or noncommutative correspondent of well-known mathematical structures. C*-algebras are a textbook example of this. The Gelfand transform establishes in fact an equivalence between the category of unital abelian C*-algebras and the category of compact Hausdorff topological spaces. This brings to the leading principle of this subject, namely that C*-algebras are the noncommutative analogue of topological spaces.

Set theory, on the other hand, is the child of Cantor's investigations on the cardinalities of the subsets of the real line at the end of the 19th century. It grew in a theory with deep metamathematical implications thanks to Gödel incompleteness theorems, and it blossomed after the invention of forcing by Cohen in 1963. With forcing, set theorists finally had the tools to deal with the independence phenomena, discovered thanks to Gödel's results, intrinsic to every first order theory capable of modeling arithmetic.

Unlike model theory, whose applications to von Neumann algebras and C*-algebras have been wide and systematic (see [FHL$^+$16]), the intersections between set theory and operator algebras have been a bit more sparse, albeit extremely significant and deep. Examples are the breakthroughs on Naimark's problem (see [AW04]), on Anderson's conjecture (see [AW08]), and the complete solution of the problem of the existence of an outer automorphism of the Calkin algebra (see [PW07] and [Far11]).

Part of the interactions between set theory and C*-algebras can be roughly orga- nized in four themes: the application of set-theoretic combinatorial statements to produce pathological examples of nonseparable C*-algebras, the translation to the noncommutative context (provided by C*-algebras) of results and techniques concerning boolean algebras and partial orderings (particularly $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$), the study of how set-theoretic axioms determine the properties of a C*-algebra and of its group of automorphisms, and the application of descriptive set theory in classification problems. This dissertation focuses on the first two themes, more specifically the common thread of this thesis is the analysis, by means of combinatorial set-theory, of various examples of nonseparable C*-algebras and of their features. The manuscript is organized into three fairly autonomous chapters. The material in chapter 1 regards Naimark's problem and belongs to the first of the themes listed above. On the other side, chapters 2 and 3 are devoted to different problems about corona algebras (chapter 2 specifically focuses on the Calkin algebra), and the topics discussed there are an example of the second theme.

During the 1940s and 1950s representations of C*-algebras have been extensively studied, and researchers were trying to understand to what extent the representation theory of a C*-algebra determines its isomorphism class. Among all C*-algebras, the algebra of compact operators $K(H)$ carries the simplest possible representation theory, in this case in fact all irreducible representations are unitarily equivalent. In 1951 Naimark asked in [Nai51] whether this strong property characterizes $K(H)$ up to isomorphism. This question is known as Naimark's problem, and in the subsequent years it was settled with a positive answer for the class of type I and the class of separable C*-algebras, but overall it remained unsolved.

About 50 years later, Naimark's problem drew the attention of several researchers in logic, after a major breakthrough towards its solution was made in [AW04]. In this article the authors produced, assuming Jensen's diamond principle (a strengthening of the continuum hypothesis), a *counterexample to Naimark's problem*, namely a C*-algebra with a unique irreducible representation up to unitary equivalence not isomorphic to $K(H)$ for any Hilbert space H. The construction presented in [AW04] is a glaring example of how combinatorial set-theoretic statements can be used to produce nonseparable C*-algebras whose behavior is somewhat irregular, when compared to the separable framework. In fact, while (by Glimm's dichotomy [Gli61]) all non-type I, separable C*-algebras necessarily have continuum many pairwise inequivalent irreducible representations, a counterexample to Naimark's problem is a (nonseparable) non-type I C*-algebra with only one irreducible representation up to unitary equivalence. We remark that it is still not known whether a positive answer to Naimark's problem is relatively consistent with ZFC.

The techniques developed by Akemann and Weaver in [AW04] rely on the results contained in [KOS03], they are very flexible and allow to produce unital counterexamples with various additional properties (e.g. nuclear, UHF, purely infinite, as shown in [FH17]), but little is known about which properties are common to all counterexamples. With the intention of investigating this matter, we look at the trace spaces of unital counterexamples to Naimark's problem. For such C*-algebras, the affine action of the unitary group on the state space is transitive on the extreme points, i.e. the pure states. Since the only states fixed by this action are the traces, it seems conceivable that a counterexample to Naimark's problem could have at most one trace, as happens for affine actions which are transitive on the extreme points of a finite-dimensional simplex. We give a strong negative answer in chapter 1 (whose contents are also presented in [Vac18a]), where we prove, assuming diamond, that every Choquet simplex with countably many extreme points occurs as the tracial simplex of a counterexample to Naimark's problem and that, furthermore, there is a counterexample with a nonseparable trace space.

Chapter 2 is devoted to the Calkin algebra $Q(H)$, the quotient of $B(H)$, the algebra of linear bounded operators on a separable infinite-dimensional Hilbert space, modulo the ideal of compact operators $K(H)$. This C*-algebra has always been object of intense study by the researchers in operator algebras, starting with the work of Weyl and von Neumann on unitary equivalence up to compact perturbation of self-adjoint operators on H (see [Wey09], [VN35]). Their study was the first step in what led to the seminal work [BDF77], which in turn gave life to the theory of extensions, a subject where $Q(H)$ plays a central role, and introduced methods of algebraic topology in the study of C*-algebras.

Over the last 15 years the Calkin algebra has been fertile ground for applications of set theory in operator algebras, due to its structural similarities with the boolean algebra $P(\mathbb{N})/\mathrm{Fin}$, of which it is in fact considered the noncommutative analogue (see [FW12] and [Wea07]). In this framework, what typically happens is that statements and ideas about $P(\mathbb{N})/\mathrm{Fin}$ are translated into noncommutative (or "quantized") correspondents in the context of the Calkin algebra. The problems formulated through this procedure are usually more technical and involved than their commutative counterparts, which nevertheless still provide intuition and ideas for the noncommutative case. Remarkably, it is not rare that this connection between $P(\mathbb{N})/\mathrm{Fin}$ and the Calkin algebra, which is already worth investigating from a set-theoretic perspective, yields results which are related to well-established branches of the theory of C*-algebras, and which are useful also for researchers in those areas. The first example of this phenomenon has been the problem of the existence of outer automorphisms of the Calkin algebra, solved by means of set theory in [PW07] and [Far11], whose original motivation was of K-theoretic nature (see [BDF77]).

The problem of what linear or partial orderings embed into $P(\mathbb{N})/\mathrm{Fin}$ has been widely studied in set theory, for instance because of its connections with the problem of the automatic continuity of Banach algebras homomorphisms (see [DW87]). A systematic study in the nonseparable framework of its noncommutative counterpart, namely investigating what (nonseparable) C*-algebras embed into the Calkin algebra, is, on the other hand, fairly recent (see [FHV17] and [FKV18]).

Chapter 2 focuses on this embedding problem (part of the contents of this chapter are also contained in the joint work [FKV18]). In the first part of the chapter we prove that, given any C*-algebra A, there exists a ccc forcing notion which forces the existence of an embedding of A into $Q(H)$. This theorem is yet another noncommutative version of a known fact about $P(\mathbb{N})/\mathrm{Fin}$: for every partial order $\mathbb{P}$, there is a ccc forcing notion which forces the existence of an embedding of $\mathbb{P}$ into $P(\mathbb{N})/\mathrm{Fin}$. One important consequence of what we prove is that, under Martin's axiom, all C*-algebras of density character less than continuum embed into the Calkin algebra.

Another topic addressed in chapter 2 concerns the class of C*-algebras of density continuum that embed into the Calkin algebra in a given model of ZFC. By the results in [FHV17], the $2^{\aleph_0}$-universality of the Calkin algebra is independent from ZFC. In fact, while the continuum hypothesis implies that all C*-algebras of density continuum embed into the Calkin algebra, there are models of ZFC where some C*-algebras of density $2^{\aleph_0}$ do not embed into $Q(H)$ (this follows for instance from the proper forcing axiom, see [FHV17]). Not much is known about the class of C*-algebras of density continuum that embed into $Q(H)$ for models of ZFC where the continuum hypothesis fails. We prove that $C^*_{\mathrm{red}}(F_{2^{\aleph_0}})$ and $C^*_{\max}(F_{2^{\aleph_0}})$, where $F_{2^{\aleph_0}}$ is the free group on $2^{\aleph_0}$ generators, and all UHF C*-algebras of density at most $2^{\aleph_0}$ embed into the Calkin algebra, regardless of the model of ZFC. On the other hand, we show that the abelian C*-algebra generated by an increasing chain of $\aleph_2$ projections does not embed into $Q(H)$ consistently with ZFC $+\, 2^{\aleph_0} \geq \aleph_\alpha$, for all $\alpha \geq 2$.

Combined with the results exposed in the first part of the chapter, this entails that the statement 'Every C*-algebra of density character strictly less than $2^{\aleph_0}$ embeds into the Calkin algebra' is independent from ZFC + $2^{\aleph_0} \geq \aleph_\alpha$, for every ordinal $\alpha > 2$.

In the last part of chapter 2 we analyze Voiculescu's noncommutative version of the Weyl-von Neumann theorem in [Voi76] from a set-theoretic perspective. More specifically, we show that this theorem can be proved with a sequence of applications of the Baire category theorem to some ccc posets. As a consequence, we obtain that the results in [Voi76] can be generalized to nonspearable C*-algebras of density less than continuum, when Martin's axiom is assumed. This final part of chapter 2, albeit seemingly unrelated to the rest of the material in this chapter, is not a coincidence. The kind of embedding problems for the Calkin algebra we discuss in this dissertation have proven to be way more difficult than their counterparts for $P(\mathbb{N})/\mathrm{Fin}$, both for technical and theoretical reasons (for instance, unlike $P(\mathbb{N})/\mathrm{Fin}$, $Q(H)$ is not countably saturated). Voiculescu's results in [Voi76] (and a deep understanding of them) proved to be invaluable tools when tackling these additional difficulties, as made evident from the proofs contained in chapter 2, [FHV17] and [FKV18].

The last chapter of this dissertation focuses on, given a non-unital C*-algebra A, liftings from the corona algebra $Q(A)$ to the multiplier algebra $M(A)$. By *lifting* of a subset B of $Q(A)$, we mean a collection of elements in $M(A)$ whose image via the quotient map onto $Q(A)$ is B. In chapter 3 we investigate, given a non-unital A, the obstructions that arise when trying to lift a collection of commuting elements in $Q(A)$ to a family in $M(A)$ whose elements still commute.

Although the study of liftings of abelian subalgebras of corona algebras originates from a purely C*-algebraic context, it is not rare to find connections with set theory, even in dated works. It is in fact often the case that the techniques and the combinatorics used in some of the arguments in this framework have a strong set-theoretic flavor (see for instance [AD79], [And79] and, more recently, [CFO14], [Vig15], [SS11], [FW12], [BK17], [Vac16]). Furthermore, the Calkin algebra being the corona of $K(H)$ (as the multiplier algebra of $K(H)$ is $B(H)$), combinatorial arguments and techniques developed in set theory for $P(\mathbb{N})/\mathrm{Fin}$ can be first translated in the context of the Calkin algebra and then, possibly, generalized to a larger class of corona algebras. The previous observation does not apply only to liftings of abelian subalgebras of coronas. An example is, once again, the results on the group of automorphisms of the Calkin algebra, whose generalization to coronas of separable C*-algebras is in progress (see [CF14], [Vig17b], [MV18]).

The main result of the third chapter (also contained in [Vac16]) is a generalization to a wider family of corona algebras of a known theorem about lifting of commuting families of projections in the Calkin algebra. It is known that every countable family of commuting projections in $Q(H)$ can be lifted to a family of projections in $B(H)$ which are diagonalized by the same basis (see [FW12]). On the other hand in [BK17], inspired by some combinatorial arguments which date back to Hausdorff and Luzin concerning the study of uncountable almost disjoint families of subsets of $\mathbb{N}$, it is proved that there is a collection of orthogonal projections of size $\aleph_1$ which cannot be lifted to a commuting family in $B(H)$. Taking inspiration from these results, we undertake a general study of which obstructions arise when trying to lift a commuting subfamily of $Q(A)$ to a commuting subset of $M(A)$, for A primitive non-unital and σ-unital. For such A, while for countable or finite families the only obstacles that arise are of K-theoretic nature, it is always possible to find a collection of orthogonal positive elements of size $\aleph_1$ in $Q(A)$ which cannot be lifted to a commuting family in $M(A)$. Moreover, these positive elements can be chosen to be projections if A has real rank zero, giving a full generalization of the results in [FW12] and [BK17].

Through this dissertation we assume that the reader has some familiarity with C*-algebras and von Neumann algebras, some standard texts we will often refer to are [Mur90], [BO08], [Bla06] and [Dix77]. Even though we will explicitly give most of the definitions concerning set theory, we assume the reader is familiar with cardinal arithmetic and forcing. Standard references for these topics are [Kun11] and [Jec03].

Chapter 1

Trace Spaces of Counterexamples to Naimark's Problem

In 1948 Naimark observed in [Nai48] that the algebra of compact operators $K(H)$ has a unique irreducible representation up to unitary equivalence, the identity representation. A few years later, in [Nai51], he asked whether this property characterizes $K(H)$ up to isomorphism. This question is known as *Naimark's problem*.

Naimark's problem. Let A be a C*-algebra with only one irreducible representation up to unitary equivalence. Is $A \cong K(H)$ for some Hilbert space H?

In the subsequent years an affirmative solution for the problem was proved for the cases of type I C*-algebras and of separably representable C*-algebras (see [Kap51] and [Ros53, Theorem 4] respectively). More recently, an affirmative answer has been found also for certain graph C*-algebras (see [ST17]). Nevertheless, a complete solution is still missing.

Nowadays this problem is considered in a context that has significantly changed since its original formulation. While Naimark's interest basically consisted in understanding to what extent the representation theory of a C*-algebra could define its isomorphism class, Naimark's problem gains a deeper meaning in the light of Glimm's celebrated theorem on type I C*-algebras in [Gli61]. For a separable simple C*-algebra A, Glimm's results imply the equivalence of the following seemingly independent conditions:

1. A is type I,
2. all irreducible representations of A are unitarily equivalent,
3. A has fewer than $2^{\aleph_0}$ inequivalent irreducible representations,
4. A has no type II representation,
5. A has no type III representation.

Most of Glimm's theorem has been extended to nonseparable C*-algebras by Sakai (see [Sak66], [Sak67]), but a negative answer to Naimark's problem would provide an obstruction to a complete generalization of the result in the nonseparable realm. A *counterexample to Naimark's problem* is a C*-algebra with a unique irreducible representation up to unitary equivalence which is not isomorphic to $K(H)$ for any H. Such an algebra would necessarily be nonseparable, simple and non-type I (see proposition 1.1.1), witnessing thus the failure, for nonseparable C*-algebras, of the equivalence of the first two conditions stated above. In this perspective Naimark's problem becomes a preliminary check in the path for a complete generalization of Glimm's theorem to the nonseparable setting.

In 2004 Akemann and Weaver built, assuming the extra set-theoretic axiom known as diamond $\diamondsuit$, the first unital counterexamples to Naimark's problem (see [AW04]). They showed moreover that the existence of a counterexample of density $\aleph_1$ is independent from ZFC. A further refinement of the techniques developed in [AW04] is obtained in [FH17], where the authors build, given $1 \leq n \leq \aleph_0$, a non-type I C*-algebra A not isomorphic to its opposite, with exactly n equivalence classes of irreducible representations, and with no outer automorphisms. It is still not known whether a positive answer to Naimark's problem (and possibly a full generalization of Glimm's theorem to nonseparable C*-algebras) is consistent with ZFC.

Akemann and Weaver's construction (and the one in [FH17]) uses two main ingredients: we already mentioned the first, Jensen's diamond principle $\diamondsuit$. This is a combinatorial statement independent from ZFC which implies the continuum hypothesis (and which will be introduced in the next section). The second ingredient is a deep theorem by Kishimoto, Ozawa and Sakai ([KOS03]) which entails that, for every separable, simple, unital C*-algebra A, the group of automorphisms of A acts transitively on the pure state space of A. We remark that it is not known whether the techniques in [AW04] could be generalized to directly produce counterexamples of densities larger than $\aleph_1$. This is partially due to the homogeneity of the pure state space of separable, simple, unital C*-algebras, implied by the Kishimoto-Ozawa-Sakai transitivity theorem, which is a crucial component of the proofs in [AW04]. Such homogeneity is known to fail for nonseparable C*-algebras. Indeed, using the theory of CCR algebras, it is possible to produce a simple C*-algebra of density $\aleph_1$ with irreducible representations on both separable and nonseparable Hilbert spaces (see [Far10]). Nevertheless, if A is a counterexample to Naimark's problem then the same is true for $A \otimes K(H)$ for any Hilbert space H (see corollary 1.1.5). Therefore $\diamondsuit$ is enough to guarantee the existence of counterexamples of any uncountable density.

As we mentioned before (and will prove later in proposition 1.1.1) a counterexample to Naimark's problem has to be nonseparable, simple and non-type I. The original motivation of the contents of this chapter was to find further characterizations of these algebras and to understand what counterexamples to Naimark's problem should look like. We focus on the study of trace spaces, led by the following general observation regarding group actions on compact convex sets, which initially seemed to suggest some kind of limitation on the size of the tracial simplex of a counterexample to Naimark's problem. Before going any further, we remark that the original construction of the counterexamples given by Akemann and Weaver does not explicitly provide any precise information on the trace space of these algebras (more on this at the beginning of section 1.2).

Let K be a compact convex set and G a group of affine homeomorphisms of K and consider the action

$$\begin{aligned} \Theta : G \times K &\to K \\ (g, x) &\mapsto g(x) \end{aligned}$$

Assume moreover that the action is transitive when restricted to the set of extreme points of K. It is conceivable that the set of the points in K fixed by the action has size no bigger than one, as happens if K is a finite-dimensional simplex. In fact, in this case, if there are at least two points fixed by Θ, we can find a point $y = \sum_{k \leq n} \lambda_k x_k$ such that $g(y) = y$ for all $g \in G$, and $\lambda_i \neq \lambda_j$ for some $i \neq j$, where $\{x_1, \dots, x_n\}$ are affinely independent extremal points of K. However, for any $g \in G$ such that $g(x_i) = x_j$, we get $g(y) \neq y$.

This relates to our context as follows. In a unital counterexample to Naimark's problem A there is a unique irreducible representation modulo unitary equivalence. This implies, by [Mur90, Theorem 5.1.4] and an application of Kadison transitivity theorem ([Mur90, Theorem 5.2.2]), that the action of the unitary group on the state space of A

$$\begin{aligned} \Theta_A : U(A) \times S(A) &\to S(A) \\ (u, \phi) &\mapsto \phi \circ \mathrm{Ad}(u) \end{aligned}$$

is transitive on the pure states of A, namely the extreme points of $S(A)$. Moreover, since the traces are fixed by this action, according to the previous observation it may seem plausible that a counterexample to Naimark's problem could have at most one trace.

Back to an arbitrary action Θ on a compact convex K, we point out that in general, if we do not require K to be finite-dimensional, there is no strict bound on the number of fixed points of Θ even for K separable. This can be proved with an application of the already mentioned Kishimoto-Ozawa-Sakai transitivity theorem from [KOS03] as follows. If A is a separable, simple, unital C*-algebra, then the state space $S(A)$ is a separable compact convex space. Let $\mathrm{AInn}(A)$ be the group of *asymptotically inner* automorphisms of A, i.e., the group of all $\alpha \in \mathrm{Aut}(A)$ such that there exists a continuous path of unitaries $(u_t)_{t\in[0,\infty)} \subseteq U(A)$ such that $\alpha(a) = \lim_{t\to\infty} \mathrm{Ad}(u_t)(a)$ for all $a \in A$. The Kishimoto- Ozawa-Sakai transitivity theorem implies that the action

$$\begin{aligned} \Xi_A : \mathrm{AInn}(A) \times S(A) &\to S(A) \\ (\alpha, \phi) &\mapsto \phi \circ \alpha \end{aligned}$$

is transitive on the extreme points of $S(A)$. On the other hand, since traces are fixed by inner automorphisms, by continuity they are also fixed by the elements of $\mathrm{AInn}(A)$. As every metrizable Choquet simplex occurs as the trace space of some separable simple unital C*-algebra (see [Bla80]), we infer that the set of fixed points in Ξ_A can be considerably large. The same is true for the unitary action Θ_A on the state space of a counterexample to Naimark's problem, as is shown in the main result of this chapter.

Theorem 1.0.1. *Assume* $\diamondsuit$*. Then the following holds:*

1. *For every Choquet simplex with countably many extreme points X, there is a counterexample to Naimark's problem whose trace space $T(A)$ is affinely homeomorphic to X.*

2. *There is a counterexample to Naimark's problem whose trace space $T(A)$ is nonseparable.*

In fact, we obtain the following strengthening of the results in [FH17].

Theorem 1.0.2. *Assume* $\diamondsuit$*. For every Choquet simplex with countably many extreme points X and $1 \leq n \leq \aleph_0$, there is a C*-algebra A such that*

1. *A is simple, unital, nuclear and of density character $\aleph_1$,*

2. *A is not isomorphic to its opposite algebra,*

3. *A has exactly n equivalence classes of pure states,*

4. *all automorphisms of A are inner,*

5. *either of the following conditions can be obtained:*

(a) $T(A)$ *is affinely homeomorphic to* X.

(b) $T(A)$ *is nonseparable.*

Theorem 1.0.2 (in particular its third clause) pushes even further the consistency of the failure of Glimm's dichotomy in [Gli61] in the nonseparable setting, already obtained in [AW04] and [FH17].

Going back to the main motivation of our inquiry, namely understanding what counterexamples to Naimark's problem look like and how they could be characterized, we are still not able to say anything more that such algebras have to be nonseparable, simple and non-type I. On the other hand, theorem 1.0.1 provides a wide variety of counterexamples, and it highlights the flexibility of the techniques in [KOS03] and [AW04]. It would be interesting to know how further this versatility can be pushed, to see for instance if it is possible to obtain any (metrizable or nonseparable) Choquet simplex as the trace space of a counterexample to Naimark's problem, or to investigate the following question.

Question 1.0.3. Is there any K-theoretic or model theoretic obstruction (consistent with being simple) to being a counterexample to Naimark's problem?

This chapter is organized as follows. We start by recalling some necessary background notions on C*-algebras and set theory in section 1.1. In the second part of section 1.1 we quickly sketch the construction of a counterexample to Naimark's problem as in [AW04] and [FH17]. In section 1.2 we show how the study of the trace space of a counterexample to Naimark's problem is reduced to a refinement of the Kishimoto-Ozawa-Sakai theorem in [KOS03]. Such refinement takes place in section 1.3, which is by far the most technical section of the chapter. Finally section 1.4 is devoted to some comments on a possible future direction of research, namely the construction of a counterexample to Naimark's problem with an outer automorphism. We remark that no additional set-theoretic axiom is needed for the proofs of section 1.3.

1.1 Preliminary Notions

1.1.1 Background on C*-algebras and Diamond

If A is a C*-algebra, A_{sa} is the set of its self-adjoint elements, A_+ the set of its positive elements and A^1 the set of its norm one elements. If A is unital, $U(A)$ is the set of all unitaries in A. Denote by $S(A)$ the state space, by $P(A)$ the pure state space, by $T(A)$ the trace space, and by $\partial T(A)$ the set of extremal traces of A, all endowed with the weak* topology. We write F � A when F is a finite subset of A.

Given $\phi \in S(A)$, $(\pi_\phi, H_\phi, \xi_\phi)$ is the GNS cyclic representation associated to ϕ. Two representations (π, H) and (ρ, K) of a C*-algebra A are *unitarily equivalent* if there is a unitary $U : H \to K$ such that $\rho(a) = U\pi(a)U^*$ for all $a \in A$. We recall that if $\phi \in P(A)$, the GNS representation associated to it is irreducible and that, vice versa, every irreducible representation of A is unitarily equivalent to $(\pi_\phi, H_\phi, \xi_\phi)$ for some $\phi \in P(A)$.

A C*-algebra A is *type I* if all irreducible representations (π, H) of A are such that $\pi[A] \supseteq K(H)$.

We denote the group of all automorphisms of A by $\mathrm{Aut}(A)$. Given a unital C*- algebra A and $u \in U(A)$, the *inner automorphism* induced by u on A is $\mathrm{Ad}(u)$ and it sends a to uau^*. An automorphism α is *outer* if it is not induced by a unitary, and we denote the set of all outer automorphisms by $\mathrm{Out}(A)$. An automorphism $\alpha \in \mathrm{Aut}(A)$ is *asymptotically inner* if there exists a continuous path of unitaries $(u_t)_{t\in[0,\infty)}$ in A such that $\alpha(a) = \lim_{t\to\infty} \mathrm{Ad}(u_t)(a)$ for all $a \in A$. We denote the set of all asymptotically inner automorphisms by $\mathrm{AInn}(A)$. For $\alpha \in \mathrm{Aut}(A)$ and $\phi \in S(A)$, the state ϕ is *α-invariant* if $\phi(\alpha(a)) = \phi(a)$ for all $a \in A$.

Given $\phi, \psi \in S(A)$, for A unital, we say that ϕ and ψ are (*unitarily*) *equivalent*, $\phi \sim \psi$ in symbols, if there is $u \in U(A)$ such that $\phi = \psi \circ \mathrm{Ad}(u)$. The states ϕ and ψ are *inequivalent* otherwise. We recall that if A is unital and $\phi, \psi \in P(A)$ then, by Kadison transitivity theorem [Mur90, Theorem 5.2.2], $(\pi_\phi, H_\phi, \xi_\phi)$ and $(\pi_\psi, H_\psi, \xi_\psi)$ are unitarily equivalent if and only if $\phi \sim \psi$.

Given a simple C*-algebra A (i.e. with no non-trivial closed ideals) and $\tau \in T(A)$, we denote the ℓ_2-norm induced by τ on A by $\|\cdot\|_{2,\tau}$ (the subscript τ will be suppressed when there is no risk of confusion). The closure of A in such norm is H_τ, the Hilbert space of the GNS representation associated to τ. Suppose furthermore that τ is α-invariant for some $\alpha \in \mathrm{Aut}(A)$, then the map U_α determined by

$$U_\alpha(\pi_\tau(a)\xi_\tau) = \pi_\tau(\alpha(a))\xi_\tau, \; a \in A$$

extends uniquely to a unitary on H_τ (which we shall denote again by U_α) such that

$$U_\alpha \pi_\tau(a) U_\alpha^* = \pi_\tau(\alpha(a))$$

for all $a \in A$. Thus α can be canonically extended via U_α to an automorphism α_τ of $\pi_\tau[A]''$ (the von Neumann algebra generated by $\pi_\tau[A]$ in $B(H_\tau)$). The automorphism α is *τ-weakly inner* (*τ-strongly outer*) if α_τ is inner (outer).

Given a separably acting type-II$_1$ factor M, let τ be its unique normal tracial state. For a free ultrafilter U on $\mathbb{N}$, the *tracial ultrapower* of M by U is the quotient of the algebra of all sequences in M bounded in norm, denoted by $\ell^\infty(M)$, by its closed ideal

$$c_U = \{\underline{a} \in \ell^\infty(M) : \lim_{n\to U} \|a_n\|_{2,\tau} = 0\}.$$

We denote the tracial ultrapower by M^U. Identifying M with the constant sequences in M^U, we denote the *relative commutant* of M in M^U by $M' \cap M^U$. We say that M has *property Gamma* if $M' \cap M^U$ is non-trivial. We say that M is *full* otherwise. A C*-algebra A has *fiberwise property Gamma* if for all $\tau \in \partial T(A)$ the factor $\pi_\tau[A]''$ has property Gamma.

Given a C*-algebra A and a free ultrafilter U on $\mathbb{N}$, the *ultrapower A_U* is the quotient of the algebra of all sequences in A bounded in norm, denoted by $\ell^\infty(A)$, by its closed ideal

$$c_U = \{\underline{a} \in \ell^\infty(A) : \lim_{n\to U} \|a_n\| = 0\}.$$

Given two vectors ξ and η in a normed vector space, $\xi \approx_\varepsilon \eta$ means $\|\xi - \eta\| < \varepsilon$. For functions ϕ and ψ on a normed vector space, given a finite subset G of the vector space and $\delta > 0$, $\phi \approx_{G,\delta} \psi$ means $\|\phi(\xi) - \psi(\xi)\| < \delta$ for all $\xi \in G$.

The smallest uncountable cardinal is $\aleph_1$, the well-ordered set of all countable ordinals. A *club* in $\aleph_1$ is an unbounded subset $C \subseteq \aleph_1$ such that for every increasing sequence $\{\beta_n\}_{n\in\mathbb{N}} \subseteq C$ the supremum $\sup_{n\in\mathbb{N}}\{\beta_n\}$ belongs to C. A subset of $\aleph_1$ is *stationary* if it meets every club. An increasing transfinite $\aleph_1$-sequence of C*-algebras $\{A_\beta\}_{\beta<\aleph_1}$ is *continuous* if $A_\gamma = \overline{\bigcup_{\beta<\gamma} A_\beta}$ for every limit ordinal $\gamma < \aleph_1$.

The following is Jensen's original formulation of $\diamondsuit$.

The diamond principle ($\diamondsuit$). There exists an $\aleph_1$-sequence of sets $\{X_\beta\}_{\beta<\aleph_1}$ such that

1. $X_\beta \subseteq \beta$ for every $\beta < \aleph_1$,

2. for every $X \subseteq \aleph_1$ the set $\{\beta < \aleph_1 : X \cap \beta = X_\beta\}$ is stationary.

The diamond principle is known to be true in the Gödel constructible universe ([Jec03, Theorem 13.21]) and it implies the continuum hypothesis (CH), thus it is independent from the Zermelo-Fraenkel axiomatization of set theory plus the Axiom of Choice (ZFC).

1.1.2 How to Build a Counterexample to Naimark's Problem

As we mentioned in the introduction, the existence of a counterexample to Naimark's problem is a basic obstruction to a generalization in the nonseparable setting of Glimm's theorem on type I C*-algebras. This is a consequence of the following proposition.

Proposition 1.1.1. *Let A be a counterexample to Naimark's problem. Then A is simple, non-type I and nonseparable.*

Proof. Let I be a closed ideal of A. Since there exists an irreducible representation of A whose kernel contains I, and since all irreducible representations of A are unitarily equivalent (thus have the same kernel), all irreducible representations of A annihilate on I. Therefore $I = \{0\}$. Since all simple type I C*-algebras are elementary, i.e. isomorphic to $K(H)$ for some Hilbert space H, A is necessarily non-type I. Finally A cannot be separable by the results in [Ros53] or by Glimm's theorem on type I C*-algebras in [Gli61]. □

The techniques developed in [AW04] and [FH17] to build counterexamples to Naimark's problem both rely on an application of the Kishimoto-Ozawa-Sakai theorem in [KOS03]. More specifically, such theorem is invoked at the successor steps of a transfinite induction, which eventually produces an increasing continuous $\aleph_1$-sequence of separable infinite-dimensional C*algebras, whose inductive limit is the desired counterexample. The idea to prove theorem 1.0.2 is to mimic this inductive construction and, as we shall see in the next section, the main effort will be to refine the results in [KOS03] in order to have a better control on the trace space of the separable algebras composing the $\aleph_1$-sequence (see theorem 1.2.3 in section 1.2).

We quickly recall the inductive construction presented in [FH17], as it is a fundamental benchmark for the proof of theorem 1.0.2. All omitted details can be found in [FH17], where a continuous model-theoretic equivalent version of $\diamondsuit$, more suitable for working with C*-algebras, is introduced.

The techniques in [FH17] already refine those in [AW04] to produce, given $1 \leq n \leq \aleph_0$, a non-type I C*-algebra A not isomorphic to its opposite, with exactly n equivalence classes of irreducible representations, and with no outer automorphisms. When $n = 1$, this gives a counterexample to Naimark's problem. The algebra A is obtained as an inductive limit of an increasing $\aleph_1$-sequence of infinite-dimensional, separable, simple, unital C*-algebras

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\beta \subseteq \cdots \subseteq A = \bigcup_{\beta<\aleph_1} A_\beta,$$

where each inclusion is unital. For a limit ordinal β define

$$A_\beta = \overline{\bigcup_{\gamma<\beta} A_\gamma}.$$

The crucial part of the construction is the successor step, where the following improvement of the main result of [KOS03] is used.

Theorem 1.1.2 ([AW04]). *Let A be a separable, simple, unital* C^**-algebra, and let* $\{\phi_h\}_{h\in\mathbb{N}}$ *and* $\{\psi_h\}_{h\in\mathbb{N}}$ *be two sequences of pure states of A such that the ϕ_h's are mutually inequivalent, and similarly the ψ_h's. Then there is an asymptotically inner automorphism α such that $\phi_h \sim \psi_h \circ \alpha$ for all $h \in \mathbb{N}$.*

Theorem 1.1.2 is applied in the proof of the following lemma.

Lemma 1.1.3 ([FH17, Lemma 2.3]). *Let A be a separable, simple, unital* C^**-algebra. Suppose X and Y are disjoint countable sets of inequivalent pure states of A and let E be an equivalence relation on Y. Then there exists a separable, simple, unital* C^**-algebra B such that*

1. *B unitally contains A,*
2. *every $\psi \in X$ has multiple extensions to B,*
3. *every $\phi \in Y$ extends uniquely to a pure state $\tilde{\phi}$ of B,*
4. *if $\phi_0, \phi_1 \in Y$, then $\phi_0 E \phi_1$ if and only if $\tilde{\phi}_0 \sim \tilde{\phi}_1$.*

The algebra B in lemma 1.1.3 is $A \rtimes_\alpha \mathbb{Z}$, where $\alpha \in \mathrm{Aut}(A)$ is provided by theorem 1.1.2 for two sequences of inequivalent pure states which depend on X, Y and E.

Thus, given $\beta < \aleph_1$, the algebra $A_{\beta+1}$ in the $\aleph_1$-sequence introduced above, is obtained by an application of lemma 1.1.3 for A_β, where X, Y and E are chosen accordingly to $\diamondsuit$. Therefore $A_{\beta+1} = A_\beta \rtimes_\alpha \mathbb{Z}$ for some $\alpha \in \mathrm{AInn}(A)$. The diamond principle indicates which X, Y and E we have to choose at each step so that the inductive limit $\cup_{\beta<\aleph_1} A_\beta$ satisfies all the required properties (i.e. having exactly n equivalence classes of irreducible representations and having no outer automorphisms nor antiautomorphisms).

The construction we just sketched allows to produce counterexamples to Naimark's problem of density $\aleph_1$. Starting from those, one can obtain counterexamples of any uncountable density using the following fact.

Proposition 1.1.4. *Let A be a simple* C^**-algebra and B a non-zero hereditary subalgebra of A. A is a counterexample to Naimark's problem if and only if B is.*

Proof. By [Mur90, Theorem 5.5.5], all irreducible representations of A are unitarily equivalent if and only if those of B are. Suppose now that $A \cong K(H)$ for some Hilbert space H. Then B, being a subalgebra of A, is type I, therefore, by proposition 1.1.1, it cannot be a counterexample to Naimark's problem. On the other hand, if $B \cong K(H)$ then, since B is hereditary in A, there is a non-zero $a \in A_+$ such that the hereditary subalgebra generated by a in A is abelian. This, by [Ped79, Lemma 6.1.3] and simplicity of A, implies that $\pi(a)$ has dimension 1 in $B(H_\pi)$ for every irreducible representation (π, H_π) of A. This entails $\pi[A] \supseteq K(H_\pi)$ and therefore, again by simplicity of A, $A \cong \pi[A] \cong K(H_\pi)$. $\square$

Corollary 1.1.5. *Let A be a counterexample to Naimark's problem and H a (not necessarily separable) Hilbert space. Then $A \otimes K(H)$ is a counterexample to Naimark's problem.*

Proof. Let $p \in K(H)$ be a minimal projection. Then $A \otimes p$ is a hereditary subalgebra of $A \otimes K(H)$ isomorphic to A. Use proposition 1.1.4 to conclude. $\square$

1.2 Trace Spaces

The original construction of a counterexample to Naimark's problem by Akemann and Weaver does not explicitly provide any property on the tracial simplex of the algebra itself. However, the following simple proposition allows to infer some useful information.

Proposition 1.2.1. *Let $\{A_\beta\}_{\beta<\aleph_1}$ be an increasing continuous $\aleph_1$-sequence of unital* C*-*algebras such that $A_{\beta+1} = A_\beta \rtimes_{\alpha,r} G_\beta$ for all $\beta < \aleph_1$, G_β being a discrete group. Let A be the inductive limit of the sequence. Suppose furthermore that every $\tau \in T(A_\beta)$ is α_g-invariant for all $g \in G_\beta$. Then for each $\beta < \aleph_1$ there is an embedding*[1] *e_β of $T(A_\beta)$ into $T(A)$.*

Proof. Let B be any unital tracial C*-algebra, $\tau \in T(B)$, and α a homomorphism of a discrete group G (whose identity is e) into $\mathrm{Aut}(B)$ such that τ is α_g-invariant for all $g \in G$. Consider the reduced crossed product $B \rtimes_{\alpha,r} G$ and denote by u_g, for $g \in G$, the unitaries of $B \rtimes_{\alpha,r} G$ corresponding to the elements of the group. The map defined on any finite sum $\sum_{g\in G} a_g u_g$ as

$$\tau\left(\sum_{g\in G} a_g u_g\right) = \tau(a_e)$$

extends uniquely to a trace of $B \rtimes_{\alpha,r} G$. Indeed, τ is $\mathrm{Ad}(u)$-invariant for all $u \in U(B)$ since τ is a trace, and it is $\mathrm{Ad}(u_g)$-invariant for all $g \in G$ since τ is α_g-invariant, hence $\tau(wa) = \tau(aw)$ for all $a \in B \rtimes_{\alpha,r} G$ and $w = w_1 \dots w_k$, where $w_j \in U(B) \cup \{u_g : g \in G\}$ for all $j \le k$. The linear span of the set of products of elements in $U(B) \cup \{u_g : g \in G\}$ is dense in $B \rtimes_{\alpha,r} G$, therefore $\tau(ab) = \tau(ba)$ for all $a, b \in B \rtimes_{\alpha,r} G$. Thus, the embedding e_β can be constructed by induction iterating the extension above at successor steps, and taking the unique extension of previous steps at limit stages. $\square$

In the Akemann-Weaver construction (and in the one from [FH17] we previously recalled) there is no restriction, when starting the induction, on the choice of the first C*-algebra A_0, as long as A_0 is separable simple and unital. Since every metrizable Choquet simplex occurs as the trace space of some separable, simple, unital C*-algebra (see [Bla80]), and since all traces are invariant for asymptotically inner automorphisms (as they are pointwise limits of inner automorphisms), proposition 1.2.1 can be applied to the construction we sketched in the previous section to infer the following.

Corollary 1.2.2. *Assume $\diamondsuit$. For every metrizable Choquet simplex X and $1 \le n \le \aleph_0$, there is a non-type I C*-algebra A not isomorphic to its opposite, with exactly n equivalence classes of irreducible representations, and with no outer automorphisms, such that $T(A)$ contains a homeomorphic copy of X.*

Proposition 1.2.1 implies that the $\aleph_1$-sequence

$$T(A_0) \xleftarrow{r_{1,0}} T(A_1) \xleftarrow{r_{2,1}} \dots T(A_\beta) \xleftarrow{r_{\beta+1,\beta}} \dots \leftarrow T(A)$$

is a projective system whose bonding maps (the restrictions) are surjective. Proposition 1.2.1 also entails that each restriction has a continuous section. Theorem 1.0.2 answers affirmatively the questions whether it is possible to perform the constructions in [AW04] and [FH17] so that the $\aleph_1$-sequence above is forced to be 'strictly increasing' or so that it 'stabilizes' (if $T(A_0)$ has countably many extremal points).

[1] A continuous map which is a homeomorphism onto its image.

Depending on which of the two final clauses of theorem 1.0.2 one wants to obtain, two different strengthenings of lemma 1.1.3 are needed. Clause 5a follows if, when applying lemma 1.1.3 to A_β (hence $B = A_\beta \rtimes_\alpha \mathbb{Z}$), we require in addition that the restriction map $r_{\beta+1,\beta} : T(A_\beta \rtimes_\alpha \mathbb{Z}) \to T(A_\beta)$ is a homeomorphism for all $\beta < \aleph_1$. This would in fact entail that $T(A)$ is affinely homeomorphic to $T(A_0)$. On the other hand, in order to get clause 5b, it is sufficient to require $r_{\beta+1,\beta}$ to be not injective for all $\beta < \aleph_1$, as shown in proposition 1.2.5.

Since α is asymptotically inner, the restriction map $r_{\beta+1,\beta} : T(A_\beta \rtimes_\alpha \mathbb{Z}) \to T(A_\beta)$ is a homeomorphism if and only if all the powers of α are τ-strongly outer for all $\tau \in \partial T(A)$ (see [Tho95, Theorem 4.3]).

Thus, all we need to show is the following variant of theorem 1.1.2.

Theorem 1.2.3. *Let A be an infinite-dimensional, separable, simple, unital C^*-algebra, and let $\{\phi_h\}_{h\in\mathbb{N}}$ and $\{\psi_h\}_{h\in\mathbb{N}}$ be two sequences of pure states of A such that the ϕ_h's are mutually inequivalent, and similarly the ψ_h's.*

1. *Suppose $\partial T(A)$ is countable. There exists an asymptotically inner automorphism α such that $\phi_h \sim \psi_h \circ \alpha$ for all $h \in \mathbb{N}$, and such that α^l is τ-strongly outer for all $\tau \in \partial T(A)$ and all $l \in \mathbb{N}$ if and only if A has fiberwise property Gamma.*

2. *Given a countable $T \subseteq \partial T(A)$, there is an asymptotically inner automorphism α such that $\phi_h \sim \psi_h \circ \alpha$ for all $h \in \mathbb{N}$ and such that α is τ-weakly inner for all $\tau \in T$.*

We remark that in order to prove clause 5b of theorem 1.0.2 it is sufficient to prove item 2 of theorem 1.2.3 for a set T of extremal traces of size 1. It is fairly straightforward to see why fiberwise property Gamma is needed in item 1 of the theorem above. Suppose in fact that there is $\tau \in \partial T(A)$ such that $\pi_\tau[A]$ is full. The automorphism α_τ is approximately inner, since α is. As shown in [Sak74, Theorem 5-6], a way to characterize fullness of type II_1 factors is by saying that all approximately inner automorphisms (with of respect of the norm induced by τ) are inner. This entails that α_τ is inner, hence clause 1 of theorem 1.2.3 cannot be achieved. Property Gamma (which is explicitly used only in proposition 1.3.7) is used to systematically find unitaries with small trace and almost commuting with prescribed finite subsets of A. This allows to keep α_τ and all its powers far (in the norm induced by τ) from inner automorphisms, as shown in lemma 1.3.2.

We assume theorem 1.2.3 (which is proved in section 1.3) for the rest of this section.

Lemma 1.2.4. *Let A be an infinite-dimensional, separable, simple, unital C^*-algebra. Suppose X and Y are disjoint countable sets of inequivalent pure states of A and let E be an equivalence relation on Y. Then there exists a separable simple unital C^*-algebra B such that*

1. *B unitally contains A,*

2. *every $\psi \in X$ has multiple extensions to B,*

3. *every $\phi \in Y$ extends uniquely to a pure state $\tilde{\phi}$ of B,*

4. *if $\phi_0, \phi_1 \in Y$, then $\phi_0 E \phi_1$ if and only if $\tilde{\phi}_0 \sim \tilde{\phi}_1$,*

5. *either of the following conditions can be obtained:*

   (a) *if $\partial T(A)$ is countable and A has fiberwise property Gamma, then B can be chosen so that the restriction map $r : T(B) \to T(A)$ is a homeomorphism,*

(b) *the restriction map $r : T(B) \to T(A)$ is not injective.*

Proof. This lemma can be proved as [FH17, Lemma 2.3] by substituting all the instances of theorem 1.1.2 with theorem 1.2.3. $\square$

Once lemma 1.2.4 is proved, theorem 1.0.2 in the introduction follows from the proof of [FH17, Lemma 2.8] and [FH17, Theorem 1.2], by substituting all instances of [FH17, Lemma 2.3] with our lemma 1.2.4. In order to get item 5a we need to iterate clause 5a of lemma 1.2.4 at each step of the construction. This can be done starting the iteration with a nuclear C*-algebra. Indeed, nuclear C*-algebras have fiberwise property Gamma. Moreover if A in the statement of lemma 1.2.4 is nuclear, the algebra B given by the clause 5a of the lemma can be assumed to be nuclear, thus the fiberwise property Gamma is preserved throughout the construction. Item 5b of theorem 1.0.2 is a consequence of the following fact.

Proposition 1.2.5. *Let $\{A_\beta\}_{\beta<\aleph_1}$ be an increasing continuous $\aleph_1$-sequence as in proposition 1.2.1 and let A be the inductive limit of the $\aleph_1$-sequence. Suppose that the set $\{\beta < \aleph_1 : r_{\beta+1,\beta} : T(A_{\beta+1}) \to T(A_\beta)$ is not injective$\}$ is unbounded in $\aleph_1$. Then $T(A)$ is nonseparable.*

Proof. Suppose $T(A)$ is separable and let $\{\tau_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $T(A)$.

Claim 1.2.5.1. *The set $C = \{\beta < \aleph_1 : \exists n$ s.t. $\tau_n \restriction A_\beta$ has multiple extensions to $A\}$ is unbounded in $\aleph_1$.*

Proof. Suppose the claim is false and let $\gamma < \aleph_1$ be an upper bound for C. Then each $\tau_n \restriction A_\gamma$ has a unique extension to $A_{\gamma+1}$, which, as we already know from the proof of proposition 1.2.1, is defined through the conditional expectation. If γ is big enough there is a trace $\sigma \in T(A_{\gamma+1})$, $a \in A_\gamma$, and $g \in G_\beta$ such that $\sigma(au_g) \neq 0$. If $E > 0$ is small enough, then $\{\tau_n \restriction A_{\gamma+1}\} \cap \{\tau \in T(A_{\gamma+1}) : |\tau(au_g) - \sigma(au_g)| < E\}$ is empty. This is a contradiction since $\{\tau_n \restriction A_{\gamma+1}\}$ is dense in $T(A_{\gamma+1})$. $\square$

The claim entails that there is an $\aleph_1$-sequence of traces (modulo taking a cofinal subsequence of the algebras A_β) $\{\tau_\beta\}_{\beta<\aleph_1}$ such that

1. $\tau_\beta \in T(A_\beta)$ for all $\beta < \aleph_1$,
2. $\tau_\gamma \restriction A_\beta = \tau_\beta$ for all $\gamma > \beta$,
3. the trace τ_β admits two different extensions to $T(A_{\beta+1})$ for every $\beta < \aleph_1$.

This allows to build a discrete set of size $\aleph_1$ in $T(A)$ as follows, which is a contradiction. For any $\beta < \aleph_1$ consider $\tau_{\beta+1}' \in T(A_{\beta+1})$ different from $\tau_{\beta+1}$ and extending τ_β, and pick two open sets in $T(A_{\beta+1})$ dividing them. Their preimage via the restriction map $r_{\beta+1} : T(A) \to T(A_{\beta+1})$ are two open disjoint subsets of $T(A)$ such that only one of them contains all the extensions of $\tau_{\beta+1}$. Hence, any $\aleph_1$-sequence of extensions in $T(A)$ of the elements in $\{\tau_\beta'\}_{\beta<\aleph_1}$ has the required property. $\square$

1.3 A Variant of the Kishimoto-Ozawa-Sakai Theorem

The first part of this section is devoted to the proof of two technical lemmas (lemmas 1.3.1 and 1.3.2). The reader can safely assume these lemmas as blackboxes and go directly to subsection 1.3.2, to see how they are used in the main proofs, before going through part 1.3.1.

1.3.1 Paths of Unitaries

Lemmas 1.3.1 and 1.3.2 are two variants of [KOS03, Lemma 2.2] (for simple C*-algebras).

Lemma 1.3.1. *Let A be an infinite-dimensional, separable, simple, unital C*-algebra, $(\phi_h)_{h\leq m}$ some inequivalent pure states and $\{\tau_1,\dots,\tau_n\} \subseteq \partial T(A)$. For every $F \Subset A$ and $\varepsilon > 0$, there exist $G \Subset A$ and $\delta > 0$ such that, if $(\psi_h)_{h\leq m}$ are pure states which satisfy $\psi_h \approx_{G,\delta} \phi_h$ for all $1 \leq h \leq m$, then for every $K \Subset A$ and every $\varepsilon > 0$ there is a path of unitaries $(u_t)_{t\in[0,1]}$ such that*

1. $u_0 = 1$,
2. $\phi_h \circ Ad(u_1) \approx_{K,\varepsilon} \psi_h$ *for all* $1 \leq h \leq m$,
3. $\|b - Ad(u_t)(b)\| < \varepsilon$ *for all* $b \in F$,
4. $\|u_t - 1\|_{2,k} < \varepsilon$ *for all* $k \leq n$.[2]

Lemma 1.3.2. *Let A be an infinite-dimensional separable, simple, unital C*-algebra with fiberwise property Gamma, $(\phi_h)_{h\leq m}$ some inequivalent pure states and $\tau \in \partial T(A)$. For every $v \in U(A)$, every $F \Subset A$, $l \in \mathbb{N}$ and $\varepsilon > 0$, there exist $G \Subset A$ and $\delta > 0$ such that, if $(\psi_h)_{h\leq m}$ are pure states which satisfy $\psi_h \approx_{G,\delta} \phi_h$ for all $1 \leq h \leq m$, then for every $K \Subset A$ and every $\varepsilon > 0$ there are a path of unitaries $(u_t)_{t\in[0,1]}$ and an $a \in A^1$ such that*

1. $u_0 = 1$,
2. $\phi_h \circ Ad(u_1) \approx_{K,\varepsilon} \psi_h$ *for all* $1 \leq h \leq m$,
3. $\|b - Ad(u_t)(b)\| < \varepsilon$ *for all* $b \in F$,
4. $\|Ad(v)(a) - Ad(u_1^{*l})(a)\|_{2,\tau} > 1/4$.

The reader familiar with the proofs in [KOS03] will notice that the only difference of the two lemmas above with [KOS03, Lemma 2.2] is the additional fourth clause. More specifically, in lemma 1.3.1 we require that the path of unitaries remains close to the identity with respect of the ℓ_2-norm induced by some traces. This is used in the proof of clause 2 of theorem 1.2.3 (in the next subsection) to build, gluing together countably many pieces, a path of unitaries $(u_t)_{t\in[0,\infty)}$ such that $(\mathrm{Ad}(u_t))_{t\in[0,\infty)}$ pointwise converges in norm to an automorphism α, and such that at same time $(\pi_\tau(u_t))_{t\in[0,\infty)}$ strongly converges to a unitary $v \in \pi_\tau[A]$, for some $\tau \in \partial T(A)$. In this situation it is possible to show that $\mathrm{Ad}(v)$ acts like α on $\pi_\tau[A]$, which is therefore τ-weakly inner. On the other hand, the construction in lemma 1.3.2 achieves, in a way, the opposite. In this case we require the path of unitaries to end in a place which is far, with respect of the ℓ_2-norm induced by a trace, from the scalars.

We briefly introduce some notation for the following proposition. Given a state ϕ on a C*-algebra A, we let L_ϕ be the following closed left ideal

$$\{a \in A : \phi(a^*a) = 0\} = \{a \in A : \pi_\phi(a)\xi_\phi = 0\}.$$

We recall that for any state ϕ the intersection $L_\phi \cap L_\phi^*$ is a hereditary subalgebra of A.

[2] We suppress the notation and denote $\|\cdot\|_{2,\tau_k}$ by $\|\cdot\|_{2,k}$.

Proposition 1.3.3. *Let A be an infinite-dimensional, simple, unital* C*-*algebra,* $\tau \in \partial T(A)$ *and* $\phi_1, \ldots, \phi_m$ *some pure states of* A*. Then*

$$M = \{a \in A : \pi_{\phi_j}(a)\xi_{\phi_j} = \pi_{\phi_j}(a^*)\xi_{\phi_j} = 0 \ \forall j \leq m\}$$

is a hereditary subalgebra of A *and* $\pi_\tau[M]$ *is strongly dense in* $\pi_\tau[A]$.

Proof. Since $M = \bigcap_{j\leq m} L_{\phi_j} \cap L^*_{\phi_j}$, the strong closure of $\pi_\tau[M]$ is a hereditary subalgebra of $\pi_\tau[A]''$, therefore it is of the form $p\pi_\tau[A]''p$ for some projection $p \in \pi_\tau[A]''$. Suppose p is not the identity and let $\eta \in H_\tau$ be a unit vector orthogonal to the range of p. Consider the state $\psi(a) = (\pi_\tau(a)\eta, \eta)$. By uniqueness of the GNS representation, $(\pi_\psi, H_\psi, \xi_\psi)$ is unitarily equivalent to $(\pi_\tau, \overline{\pi_\tau[A]\eta}, \eta)$. Since $\pi_\tau[A]''$ is a II$_1$-factor (A is infinite-dimensional and simple), the same is true for $\pi_\psi[A]''$ (see [Dix77, Proposition 5.3.5]). Consider $a \in \bigcap_{j\leq m} L_{\phi_j}$. Then $a^*a \in M$ and this implies

$$\|\pi_\tau(a)p^\perp\|^2 = \|p^\perp\pi_\tau(a^*a)p^\perp\| = 0,$$

hence $\pi_\tau(a)\eta = 0$, which means $\pi_\psi(a)\xi_\psi = 0$, which in turn entails $L_\psi \supseteq \bigcap_{j\leq m} L_{\phi_j}$. Consider the state $\phi = \sum_{j\leq m} \frac{1}{m}\phi_j$, which is such that $L_\phi = \bigcap_{j\leq m} L_{\phi_j}$. By the correspondence between closed left ideals and weak*-closed faces of $S(A)$ (see [Ped79, Theorem 3.10.7][3]) we infer that ψ is contained in the smallest weak*-closed face of $S(A)$ which contains ϕ, which is in fact the set

$$\{\theta \in S(A) : \theta[L_\phi] = 0\}.$$

On the other hand, the smallest face of $S(A)$ containing the state ϕ is

$$F_\phi = \{\theta \in S(A) : \exists\lambda > 0 \ \theta \leq \lambda\phi\}.$$

By the Radon-Nikodym theorem ([Mur90, Theorem 5.1.2]), for every state θ contained in F_ϕ, the GNS representation (π_θ, H_θ) is (unitarily equivalent to) a subrepresentation of (π_ϕ, H_ϕ). Since the latter representation is type I (it is in fact the subrepresentation of a direct sum of irreducible representations), we get to a contradiction if we can prove that F_ϕ is weak*-closed, since this would imply that (π_ψ, H_ψ) is type I. By Radon-Nikodym theorem the map

$$\begin{aligned}\Theta_\phi : \pi_\phi[A]' &\to A^* \\ v &\mapsto (\pi_\phi(\cdot)v\xi_\phi, \xi_\phi)\end{aligned}$$

is a linear map such that $\Theta_\phi[\pi_\phi[A]'_+] \cap S(A) = F_\phi$. Let π denote $\bigoplus_{i\leq m}\pi_{\phi_i}$. We prove that $\pi[A]'$ is finite-dimensional, which entails that also $\pi_\phi[A]'$ is finite-dimensional, since $\pi_\phi[A]' = q\pi[A]'q$ for some projection $q \in \pi[A]'$. This follows from the contents of Chapter 5 of [Dix77]. More specifically, if $\phi_1, \ldots, \phi_n$ are equivalent pure states, given $\pi = \bigoplus_{i\leq n}\pi_{\phi_i}$, then $\pi[A]'$ is a type I$_n$-factor by [Dix77, Proposition 5.4.7], thus it is finite-dimensional. By [Ped79, Theorem 3.8.11], the commutant $\pi[A]'$ is therefore the direct sum of a finite number of finite-dimensional type I factors. □

The previous proposition allows us to prove the following corollary, which can be thought of as an approximate extension to tracial states of the Glimm-Kadison transitivity theorem.

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[3] Here we can consider faces of $S(A)$ instead of $Q(A)$ since A is unital.

Corollary 1.3.4. *Let A be an infinite-dimensional, simple, unital C^*-algebra, $\tau \in \partial T(A)$, $\{(\pi_i, H_i)\}_{i\leq n}$ some inequivalent irreducible representations, $F_i \subset H_i$ finite sets and $T_i \in B(H_i)$. Then the set*

$$\pi_\tau[\{a \in A : \pi_i(a) \restriction_{F_i} = T_i \restriction_{F_i} \ \forall i \leq n\}]$$

is strongly dense in $\pi_\tau[A]$.

Proof. By the Glimm-Kadison transitivity theorem (see [GK60, Corollary 7]) let $a \in A$ be such that, for all $i \leq n$

$$\pi_i(a) \restriction_{F_i} = T_i \restriction_{F_i}.$$

Define for each $i \leq n$ the set

$$L_i = \{a \in A : \pi_i(a)\xi = 0 \ \forall \xi \in F_i\}.$$

Let L be the intersection of all L_i's. By proposition 1.3.3 the set $\pi_\tau[L]$ is strongly dense in $\pi_\tau[A]$, thus the same is true, by linearity, for $\pi_\tau[a + L]$. □

The following proposition is implicitly used in [KOS03, Theorem 3.1]. We give here a full proof of it.

Proposition 1.3.5. *For every $\varepsilon > 0$ and $M \in \mathbb{N}$ there is $\delta > 0$ such that the following holds. Suppose ξ is a norm one vector in an infinite-dimensional Hilbert space H, and that $\{b_j\}_{j\leq M} \subseteq B(H)$ are such that $\sum_j b_j b_j^* \leq 1$ and $\sum_j b_j b_j^* \xi = \xi$. Let moreover $\eta \in H$ be a unit vector orthogonal to the linear span of $\{b_j b_k^* \xi : j, k \leq M\}$ such that, for all $j, k \leq M$*

$$|(b_k^*\xi, b_j^*\xi) - (b_k^*\eta, b_j^*\eta)| < \delta.$$

Then there is a projection $q \in B(H)$ such that

$$\sum_{j\leq M} b_j q b_j^*(\eta + \xi) \approx 0 \quad \text{and} \quad \sum_{j\leq M} b_j q b_j^*(\eta - \xi) \approx \eta - \xi.$$

Proof. By [FKK01, Lemma 3.3], for every $\varepsilon' > 0$ and $M' \in \mathbb{N}$ there is a $\delta' > 0$ such that if $(\xi_1, \ldots, \xi_{M'})$ and $(\eta_1, \ldots, \eta_{M'})$ are two sequences of vectors in a Hilbert space H such that $\sum_i \|\xi_i\|^2 \leq 1$, $\sum_i \|\eta_i\|^2 \leq 1$, and

$$|(\xi_i, \xi_j) - (\eta_i, \eta_j)| < \delta' \ \forall i, j \leq M',$$

then there is a unitary $U \in B(H)$ such that

$$\|U\xi_j - \eta_j\| < \varepsilon' \ \forall j \leq M',$$

Moreover, if H is infinite dimensional and $(\xi_i, \eta_j) = 0$ for all $i, j \leq M'$, then U can be chosen to be self-adjoint. Let $\delta > 0$ be smaller than ε/M and than the δ' given by [FKK01, Lemma 3.3] for $M' = M$ and $\varepsilon' = \varepsilon/M$. Fix ξ, η and b_j for $j \leq M$ as in the statement of the current proposition. Since the linear spans of $\{b_j^*\xi : j \leq M\}$ and $\{b_j^*\eta : j \leq M\}$ are orthogonal, there is a self-adjoint unitary w on H such that, for every $j \leq M$

$$\|w b_j^*\xi - b_j^*\eta\| < \varepsilon/2M,$$

$$\|w b_j^*\eta - b_j^*\xi\| < \varepsilon/2M.$$

This entails, since $\|b_j\| \leq 1$ for all $j \leq M$, $\|b_j w b_j^*\xi - b_j b_j^*\eta\| < \varepsilon/2M$, therefore

$$\Big\|\sum_{j\leq M} b_j w b_j^*\xi - \sum_{j\leq M} b_j b_j^*\eta\Big\| < \varepsilon/2.$$

Similarly we have
$$\Big\| \sum_{j\le M} b_j w b_j^* \eta - \sum_{j\le M} b_j b_j^* \xi \Big\| < \varepsilon/2.$$
Moreover $\sum_j b_j b_j^* \xi = \xi$ and $\delta < \varepsilon/M$ imply $\sum_j b_j b_j^* \eta \approx \eta$. Thus, if q is the projection $(1-w)/2$, it follows that
$$\sum_{j\le M} b_j q b_j^*(\eta+\xi) \approx 0 \quad \text{and} \quad \sum_{j\le M} b_j q b_j^*(\eta-\xi) \approx \eta-\xi.$$
$\square$

Proposition 1.3.6. *For every $\varepsilon > 0$ and $N > 0$ there exists $\delta > 0$ such that for every self-adjoint element a of norm smaller than N on a Hilbert space H, every $r \in [-N,N]$, and all unit vectors $\xi \in H$, we have the following. If $r\xi \approx_\delta a\xi$ then $\exp(i\pi r)\xi \approx \exp(i\pi a)\xi$.*

Proof. Fix $\varepsilon, N > 0$ and let $p(x)$ be a polynomial such that
$$\|(p(x) - \exp(i\pi x))|_{[-N,N]}\|_\infty < \varepsilon/3.$$
It is straightforward to find $\delta > 0$ (depending only on ε, N and $p(x)$) such that $a\xi \approx_\delta r\xi$ implies $p(r)\xi \approx_{\varepsilon/3} p(a)\xi$. Thus we have
$$\exp(i\pi r)\xi \approx_{\varepsilon/3} p(r)\xi \approx_{\varepsilon/3} p(a)\xi \approx_{\varepsilon/3} \exp(i\pi a)\xi.$$
$\square$

Proof of lemma 1.3.1. It is sufficient to show the following claim.

Claim 1.3.6.1. *Let A be an infinite-dimensional, separable, simple, unital C^*-algebra, $(\phi_h)_{h\le m}$ some inequivalent pure states and $\{\tau_1,\dots,\tau_n\} \subseteq \partial T(A)$. For every $F \Subset A$ and $\varepsilon > 0$, there exist $G \Subset A$ and $\delta > 0$ such that the following holds. Suppose $(\psi_h)_{h\le m}$ are pure states such that $\psi_h \sim \phi_h$, and that moreover $\psi_h \approx_{G,\delta} \phi_h$ for all $1 \le h \le m$. Then there exists a path of unitaries $(u_t)_{t\in[0,1]}$ in A satisfying the following*

1. *$u_0 = 1$,*
2. *$\phi_h \circ Ad(u_1) = \psi_h$ for all $1 \le h \le m$,*
3. *$\|b - Ad(u_t)(b)\| < \varepsilon$ for all $b \in F$,*
4. *$\|u_t - 1\|_{2,k} < \varepsilon$ for all $k \le n$.*

In fact the thesis follows from the claim and an application of [FKK01, Lemma 2.3] (see [KOS03, Lemma 2.2] for details).

By an application of the Glimm-Kadison transitivity theorem, there exists $\varepsilon > 0$ such that if $(\theta_h)_{h\le m}$ are inequivalent pure states and $(\chi_h)_{h\le m}$ are pure states such that $\|\theta_h - \chi_h\| < \varepsilon$, then there is a path of unitaries $(v_t)_{t\in[0,1]}$ which satisfies the following, for $K = \max_{b\in F} \|b\|$

1. $v_0 = 1$,
2. $\theta_h \circ \mathrm{Ad}(v_1) = \chi_h$ for all $1 \le h \le m$,
3. $\|v_t - 1\| < \varepsilon/(8K)$ for all $t \in [0,1]$.

In fact for every $h \leq m$, if $\|\theta_h - \chi_h\|$ is small enough, θ_h and χ_h are two vector states on H_{θ_h} induced by two vectors ξ_{θ_h} and ζ_{χ_h} which can be chosen to be very close (depending on $\|\theta_h - \chi_h\|$). Hence there is $u_h \in U(B(H_{\theta_h}))$ which sends ξ_{θ_h} to ζ_{χ_h} and is very close to the identity of $B(H_{\theta_h})$, which in turn implies that $u_h = \exp(ia_h)$ for some $a_h \in B(H_{\theta_h})_{sa}$ whose norm is close to zero. Given the representation $\pi = \bigoplus_{h\leq m} \pi_{\theta_h}$ on $H = \bigoplus_{h\leq m} H_{\theta_h}$, by Glimm-Kadison transitivity theorem there is $b \in B(H)_{sa}$ which behaves like a_h on ξ_{θ_h} for every $h \leq m$, and whose norm is close zero. The required path is $(v_t)_{t\in[0,1]}$, where $v_t = \exp(itb)$. Fix such ϵ.

Let $\epsilon > 0$ be smaller than the δ provided by proposition 1.3.6 for $N = 2^{2n}$ and $\min\{\epsilon/2, \epsilon/4\}$. Let (π_h, H_h, ξ_h) be the GNS representations associated to ϕ_h, let (π, H) be the direct sum of them, and let $p \in B(H)$ be the projection onto the span of the cyclic vectors ξ_h for $h \leq m$. The representation π has an approximate diagonal since it is the direct sum of some inequivalent irreducible representations (see [KOS03, Section 4]), thus there is a positive integer M and some $b_j \in A$ for $j \leq M$ such that

1. $\sum_j b_j b_j^* \leq 1$,
2. $p(1 - \sum_j \pi(b_j b_j^*)) = 0$,
3. $\sup_{c\in A,\ \|c\|\leq 1} \|b \sum_j b_j c b_j^* - \sum_j b_j c b_j^* b\| < \frac{1}{4\, e^{\pi 2^{2n}}\, 2^{2n}}$ for all $b \in F$.

Fix $\delta = \delta'/2$, δ' being the value given by proposition 1.3.5 for M and ϵ. Fix moreover

$$G = \{b_j b_k^* : j, k \leq M\}.$$

Suppose $\psi_h \sim \phi_h$ and $\psi_h \approx_{G,\delta} \phi_h$ for all $h \leq m$. For every $h \leq m$ pick $w_h \in U(A)$ such that $\phi_h \circ \mathrm{Ad}(w_h) = \psi_h$, and let η_h denote the vector $w_h \xi_h$. By Glimm's lemma (see [BO08, Lemma 1.4.11]) there are, for every $h \leq m$, $\zeta_h \in H_h$ unit vectors orthogonal to $\{\pi(b_j b_k^*)\xi_h, \pi(b_j b_k^*)\eta_h : j, k \leq M\}$ such that, if $\theta_h = \omega_{\zeta_h} \circ \pi_h$, we have $\theta_h \approx_{G,\delta} \psi_h$ for every $h \leq m$. As a consequence $\theta_h \approx_{G,\delta'} \phi_h$ for all $h \leq m$, which implies, for $j, k \leq M$

$$|\langle \pi(b_k)^*\xi_h, \pi(b_j)^*\xi_h\rangle - \langle \pi(b_k)^*\zeta_h, \pi(b_j)^*\zeta_h\rangle| < \delta'.$$

By an application of propositions 1.3.5 and 1.3.6 for $\xi = \xi_h$, $\eta = \zeta_h$ and $b_j = \pi_h(b_j)$, we obtain a projection $q_h \in B(H_h)$ such that $v_h = \exp(i\pi \sum_j b_j q_h b_j^*)$ satisfies $\zeta_h \approx_{\epsilon/2} v_h\xi_h$. By Glimm-Kadison transitivity theorem there is $a \in A^1_{sa}$ which agrees with q_h on $S_h = \mathrm{span}\{\pi(b_j^*)\xi_h, \pi(b_j^*)\zeta_h, \pi(q_h)\pi(b_j^*)\xi_h, \pi(q_h)\pi(b_j^*)\zeta_h : j \leq M\}$ for every $h \leq m$. For each $k \leq n$ corollary 1.3.4 provides one $a_k \in A_{sa}$ such that $\|a_k\|_{2,k} \leq \epsilon^2/(2^{4n}M)$, which moreover agrees with q_h on S_h for all $h \leq m$. From the proof of corollary 1.3.4 and Kaplansky density theorem, it is possible to see that each a_k can be chosen of norm smaller than 2. Define $\bar{a}$ to be the sum $\sum_j b_j a_1 \ldots a_n a^2 a_n \ldots a_1 b_j^*$. This is a positive element whose norm is smaller than 2^{2n}. Define u_t for $t \in [0, 1]$ to be $\exp(it\pi\bar{a})$. Thus, combining proposition 1.3.6 with the previous construction, we get $\|\pi(u_1)\xi_h - \zeta_h\| < \epsilon/2$ for all $h \leq m$. This implies $\|\phi_h \circ \mathrm{Ad}(u_1) - \theta_h\| < \epsilon$. Moreover for all $b \in F$ we have

$$\|[u_t, b]\| \leq e^{\pi\|\bar{a}\|}\|[\bar{a}, b]\| \leq \epsilon/4.$$

Finally, let $\tilde{a}_k$ be $a_k/\|a_k\|$. Then for each $k \leq n$ we can show that

$$\tau_k(\bar{a}^2) \leq 2^{4n} \sum_{j\leq M} \tau_k(b_j \tilde{a}_1 \ldots \tilde{a}_n a^2 \tilde{a}_n \ldots \tilde{a}_1 b_j^*) =$$

$$= 2^{4n} \sum_{j\leq M} \tau_k(\tilde{a}_k \dots \tilde{a}_n a^2 \tilde{a}_n \dots \tilde{a}_1 b_j^* b_j \tilde{a}_1 \dots \tilde{a}_{k-1}) \leq$$

$$\leq 2^{4n} \sum_{j\leq M} [\tau_k((\tilde{a}_k \dots \tilde{a}_n a^2 \tilde{a}_n \dots \tilde{a}_k)^2]^{1/2} [\tau_k((\tilde{a}_{k-1} \dots \tilde{a}_1 b_j^* b_j \tilde{a}_1 \dots \tilde{a}_{k-1})^2]^{1/2} \leq$$

$$\leq 2^{4n} \sum_{j\leq M} \tau_k(\tilde{a}_k \dots \tilde{a}_n a^2 \tilde{a}_n \dots \tilde{a}_k)^{1/2} \leq 2^{4n} \sum_{j\leq M} \tau_k(\tilde{a}_k^2)^{1/2} \leq \varepsilon^2.$$

Therefore $\|a\|_{2,k} \leq \varepsilon$, thus $\|u_t - 1\|_{2,k} \leq \varepsilon/4$. Perform the same construction between $(\theta_h)_{h \leq m}$ and $(\psi_h)_{h\leq m}$ to find a path of unitaries $(v_t)_{t\in[0,1]}$ such that $\|\psi_h \circ \mathrm{Ad}(v_1) - \theta_h\| < \varepsilon$ for all $h \leq m$, $\|[v_t, b]\| \leq \varepsilon/4$ for all $t \in [0,1]$ and $b \in F$ and finally such that $\|v_t - 1\|_{2,k} \leq \varepsilon/4$ for all $t \in [0,1]$ and $k \leq n$. By what we said at the beginning of the proof, we can find two paths of untaires $(u'_t)_{t\in[0,1]}$, $(v'_t)_{t\in[0,1]}$ such that $\phi_h \circ \mathrm{Ad}(u_1 u'_1) = \theta_h$ and $\psi_h \circ \mathrm{Ad}(v_1 v'_1) = \theta_h$ for all $h \leq m$, and such that $\|u'_t - 1\| < \varepsilon/(8K)$, $\|v'_t - 1\| < \varepsilon/(8K)$ for all $t \in [0,1]$. Then $(u_t u'_t {v'_t}^* v_t^*)_{t\in[0,1]}$ is the required path. □

The following proposition is the only place where fiberwise property Gamma is required. We refer to [KR14] for all the omitted details concerning central sequence C*-algebras in the next proposition.

Proposition 1.3.7. *Let A be an infinite-dimensional, separable, simple, unital C*-algebra with fiberwise property Gamma, $\tau \in \partial T(A)$, and $l \in \mathbb{N}$. Given any $F \Subset A$ and $\varepsilon > 0$, there is a unitary $v = e^{ia}$ for some $a \in A_{sa}$, such that $\|\mathrm{Ad}(v)(c) - c\| < \varepsilon$ for all $c \in F$ and $|\tau(v^l)| < 1/8$.*

Proof. By assumption, $\pi_\tau[A]$ is a type-II$_1$ factor M with property Gamma, hence there is a unitary $u = \exp\{ilb\}$ for some $b \in (M \cap M^{\mathcal{U}})_{sa}$, such that the trace (which is the $\mathcal{U}$-limit of τ along $M^{\mathbb{N}}$) of u is zero (see [Con76, Theorem 2.1-Lemma 2.4]). By [KR14, Theorem 3.3] (see also [AK16]) there is an $\bar{a} \in A \cap A_{\mathcal{U}}$ such that $\pi_\tau(\bar{a}) =_{\mathcal{U}} b$.[4] Thus, given any $F \Subset A$ and $\varepsilon > 0$, by strong continuity of the exponential map (see [Mur90, Theorem 4.3.2]), there is $a \in A_{sa}$ such that $v = \exp(ia)$ is a unitary which satisfies $\|\mathrm{Ad}(v)(c) - c\| < \varepsilon$ for all $c \in F$, and $|\tau(v^l)| < 1/8$. □

Proof of lemma 1.3.2. Similarly to lemma 1.3.1, it is sufficient to prove the following claim and then apply [FKK01, Lemma 2.3]

Claim 1.3.7.1. *Let A be an infinite-dimensional, separable, simple, unital C*-algebra with fiberwise property Gamma, $(\phi_h)_{h\leq m}$ some inequivalent pure states and $\tau \in \partial T(A)$. For every $v \in U(A)$, every $F \Subset A$, $l \in \mathbb{N}$ and $\varepsilon > 0$, there exist $G \Subset A$ and $\delta > 0$ such that the following holds. Suppose $(\psi_h)_{h\leq m}$ are pure states such that $\psi_h \sim \phi_h$, and that moreover $\psi_h \approx_{G,\delta} \phi_h$ for all $1 \leq h \leq m$. Then there exist a path of unitaries $(u_t)_{t\in[0,1]}$ in A and $a \in A^1$ satisfying the following*

1. *$u_0 = 1$,*
2. *$\phi_h \circ \mathrm{Ad}(u_1) = \psi_h$ for all $1 \leq h \leq m$,*
3. *$\|b - \mathrm{Ad}(u_t)(b)\| < \varepsilon$ for all $b \in F$,*
4. *$\|\mathrm{Ad}(v)(a) - \mathrm{Ad}(u_1^{*l})(a)\|_{2,\tau} > 1/4$.*

[4] For $\bar{a} = (a_n)$, $\pi_\tau(\bar{a})$ denotes the sequence $(\pi_\tau(a_n))$.

We shall denote $\| \cdot \|_{2,\tau}$ simply by $\| \cdot \|_2$. The proof splits in two cases. First, assume there is some $a \in A^1$ such that

$$\|\mathrm{Ad}(v)(a) - a\|_2 > 1/4.$$

Then the proof can be carried on as in lemma 1.3.1 (with an empty set of traces) by adding a to F and picking E small enough.

Let's therefore assume that for all $a \in A^1$ the following holds

$$\|\mathrm{Ad}(v)(a) - a\|_2 \leq 1/4.$$

Our aim is to produce a path of unitaries $(u_t)_{t\in[0,1]}$ which satisfies the first three clauses of the lemma plus $|\tau(u^l_1)| < 1/4$. In fact, this implies $\|u^l_1 - \tau(u^l_1)\|_2 \geq 3/4$, which, by [FHS13, Lemma 4.2], is enough to find an $a \in A^1$ such that

$$\|\mathrm{Ad}(u_1^l)(a) - a\|_2 > 1/2.$$

To do this, fix G and δ given by lemma 1.3.1 for F, $\min\{1/(8l), E/2\}$ and $\{\tau\}$. Now pick $s \in U(A)$ given by proposition 1.3.7 for $F \cup G$, l and $\min\{\delta/2, E/2\}$. This implies that if $(\psi_h)_{h\leq m}$ are pure states such that $\psi_h \approx_{G,\delta/2} \phi_h$, then $\psi_h \circ \mathrm{Ad}(s^*) \approx_{G,\delta} \phi_h$ for all $h \leq m$. Thus we get from lemma 1.3.1 a path of unitaries $(w_t)_{t\in[0,1]}$ such that

1. $w_0 = 1$,
2. $\phi_h \circ \mathrm{Ad}(w_1) \circ \mathrm{Ad}(s) = \psi_h$ for all $1 \leq h \leq m$,
3. $\|b - \mathrm{Ad}(w_t)(b)\| < E/2$ for all $b \in F$,
4. $\|w_1^l - 1\|_2 < 1/8$.

Since $s = e^{ia}$ for some $a \in A_{sa}$, let s_t be equal to e^{ita} for $t \in [0, 1]$. Hence the path defined by $u_t = w_t s_t$ for $t \in [0, 1]$ gives the thesis. $\square$

1.3.2 Gluing Paths

We are ready to prove theorem 1.2.3. We split the proof in two parts, the first for clause 1, the second for clause 2.

Proof of theorem 1.2.3 - part 1. We first show that if A has fiberwise property Gamma, then there is an asymptotically inner automorphism α such that $\phi_h \sim \psi_h \circ \alpha$ for all $h \in \mathbb{N}$, and such that α^l is τ-strongly outer for all $\tau \in \partial T(A)$ and all $l \in \mathbb{N}$. Fix a dense $\{a_i\}_{i\in\mathbb{N}}$ in A, a dense $\{\sigma_j\}_{j\in\mathbb{N}}$ in $U(A)$ and let $\{\tau_k\}_{k\in\mathbb{N}}$ be an enumeration of $\partial T(A)$. The construction proceeds by induction on the triples $(l, j, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. These three indices keep track of the fact that we want to build an automorphism α such that, for all $(l, j, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, the l-th power of its extension $\alpha^l_{\tau_k}$ to $\pi_{\tau_k}[A]$ is far away from all $\mathrm{Ad}(\sigma_j)$ in the ℓ_2-nom induced by τ_k Let $=_S$ be any well-ordering of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and assume that the three smallest elements of such ordering are $(1, 1, 1) < (1, 1, 2) < (1, 2, 1)$ (this is needed to introduce step 1 and 2 of the construction, as will be clarified later). We will present in detail step 1 and 2 of the construction, then the generic n-th step.

Step 1: *a1)* Apply lemma 1.3.2 to ϕ_1 for $F_1 = \{a_1\}$, $l = 1$, $E_1 = 2^{-6}$, $v = \sigma_1$, $\tau = \tau_1$, to find a G_1 � A and $\delta_1 > 0$ which satisfy the thesis of the lemma.

b1) Fix $\tilde{\psi}_1 \sim \psi_1$ such that $\tilde{\psi}_1 \approx_{G_1,\delta_1} \phi_1$.

a2) Apply lemma 1.3.1 to $\tilde{\psi}_1$ for $F_1 = F_1$, ε_1, $\{\tau_1, \tau_2\}$, to find a $G_1 \Subset A$ and $\delta_1 > 0$ which satisfy the thesis of the lemma.

b2) Fix $K = G_1 \cup F_1$ and $\varepsilon = \min\{\delta_1, 1/2\}$, and let $(v_{1,t})_{t\in[0,1]}$ be a path of unitaries in A and $b_{1,1,1} \in A^1$ given by the application of lemma 1.3.2 in part *a1* such that (we will denote $v_{1,1}$ simply by v_1):

- $v_{1,0} = 1$,
- $\phi_1 \circ \mathrm{Ad}(v_1) \approx_{K,\varepsilon} \tilde{\psi}_1$,
- $\|b - \mathrm{Ad}(v_{1,t})(b)\| < \varepsilon_1$ for all $b \in F_1$,
- $\|\mathrm{Ad}(\sigma_1)(b_{1,1,1}) - \mathrm{Ad}(v_1^*)(b_{1,1,1})\|_{2,1} > 1/4$.

Step 2: *a1)* Apply lemma 1.3.2 to $\phi_1 \circ \mathrm{Ad}(v_1)$ for $F_2 = F_1 \cup \{a_i, \mathrm{Ad}(v_1^*)(a_i) : i \leq 2\} \cup \{b_{1,1,1}\}$, $l = 1$, $\varepsilon_2 = 2^{-7}$, $v = v_1\sigma_1$, $\tau = \tau_2$ to find a $G_2 \Subset A$ and $\delta_2 > 0$ which satisfy the thesis of the lemma.

b1) Fix $K = G_2 \cup F_2$ and $\varepsilon = \min\{\delta_2, 1/4\}$, and let $(w_{1,t})_{t\in[0,1]}$ be a path of unitaries in A given by the application of lemma 1.3.1 in part *a2* of the previous step such that (we will denote $w_{1,1}$ simply by w_1):

- $w_{1,0} = 1$,
- $\phi_1 \circ \mathrm{Ad}(v_1) \approx_{K,\varepsilon} \tilde{\psi}_1 \circ \mathrm{Ad}(w_1)$,
- $\|b - \mathrm{Ad}(w_{1,t})(b)\| < \varepsilon_1$ for all $b \in F_1$,
- $\|w_1 - 1\|_{2,k} < \varepsilon_1$ for all $k \leq 2$.

Let u_1 be equal to $w_1 v_1^*$. We have that

$$\|\mathrm{Ad}(\sigma_1)(b_{1,1,1}) - \mathrm{Ad}(u_1)(b_{1,1,1})\|_{2,1} \geq \|\mathrm{Ad}(\sigma_1)(b_{1,1,1}) - \mathrm{Ad}(v_1^*)(b_{1,1,1})\|_{2,1} - 2^{-5} > 1/8.$$

Conclude by fixing $\tilde{\psi}_2 \sim \psi_2$ such that $\phi_2 \circ \mathrm{Ad}(v_1) \approx_{K,\varepsilon} \tilde{\psi}_2 \circ \mathrm{Ad}(w_1)$.

a2) Apply lemma 1.3.1 to $(\tilde{\psi}_1 \circ \mathrm{Ad}(w_1), \tilde{\psi}_2 \circ \mathrm{Ad}(w_1))$ for $F_2 = F_2 \cup \{\mathrm{Ad}(w_1^*)(a_i) : i \leq 2\}$, ε_2, $\{\tau_1, \tau_2\}$ to find a $G_2 \Subset A$ and $\delta_2 > 0$ which satisfy the thesis of the lemma.

b2) Fix $K = G_2 \cup F_2$ and $\varepsilon = \min\{\delta_2, 1/4\}$, and let $(v_{2,t})_{t\in[0,1]}$ be a path of unitaries in A and $b_{1,1,2} \in A^1$ given by the application of lemma 1.3.2 in part *a1* such that (we will denote $v_{2,1}$ simply by v_2)

- $v_{2,0} = 1$,
- $\phi_h \circ \mathrm{Ad}(v_1 v_2) \approx_{K,\varepsilon} \tilde{\psi}_h \circ \mathrm{Ad}(w_1)$ for $h \leq 2$,
- $\|b - \mathrm{Ad}(v_{2,t})(b)\| < \varepsilon_2$ for all $b \in F_2$,
- $\|\mathrm{Ad}(v_1\sigma_1)(b_{1,1,2}) - \mathrm{Ad}(v_2^*)(b_{1,1,2})\|_{2,2} > 1/4$.

Assume (l, j, k) is the n-th element of the ordering induced on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ by $\prec$. Assume moreover that in part *a2* of step $n-1$ lemma 1.3.1 is applied to a set of traces $\{\tau_k : k \leq K\}$ such that $K \geq k$. Assuming (l', j', k') is the immediate successor of (l, j, k), we define K' to be equal to $\max\{K, k'\}$ and $L = \max\{l : (l, j, k) \preceq (l', j', k')\}$.[5]

Step n: *a1)* Apply lemma 1.3.2 to $(\phi_h \circ \mathrm{Ad}(v_1 \dots v_{n-1}))_{h\leq n}$ for $F_n = F_{n-1} \cup \{a_i, \mathrm{Ad}(v_{n-1}^* \dots v_1^*)(a_i) : i \leq n\} \cup \{b_{l,j,k} : (l, j, k) \prec (l', j', k')\} \cup \{u_{n-2} v_{n-1}^*\}$, $l = l'$, $\varepsilon_n = 2^{-(5+n)}/2L^2$, $\tau = \tau_{k'}$, $v = (v_{n-1}u_{n-2}^*)^{l'}\sigma_{j'}$ to find a $G_n \Subset A$ and $\delta_n > 0$ which satisfy the thesis of the lemma.

---

[5] This is the reason we had to specify the first elements of the ordering $\prec$ and why we had to apply lemma 1.3.1 in part *a2* of step 1 to $\{\tau_1, \tau_2\}$, since for step 1 we have $K = 2$.

b1) Fix $K = G_n \cup F_n$ and $E = \min\{\delta_n, 2^{-n}\}$, and let $(w_{n-1,t})_{t\in[0,1]}$ be a path of unitaries in A given by the application of lemma 1.3.1 in part *a2* of the previous step such that (we will denote $w_{n-1,1}$ simply by w_{n-1}):

- $w_{n-1,0} = 1$,
- $\phi_h \circ \mathrm{Ad}(v_1 \dots v_{n-1}) \approx_{K,t} \tilde{\psi}_h \circ \mathrm{Ad}(w_1 \dots w_{n-1})$ for $h \leq n-1$,
- $\|b - \mathrm{Ad}(w_{n-1})(b)\| < E_{n-1}$ for all $b \in F_{n-1}$,
- $\|w_{n-1} - 1\|_{2,k} < E_{n-1}$ for all $k \leq K$.

Let u_{n-1} be equal to $u_{n-2}w_{n-1}v^*_{n-1}$. For every $(l,j,k) < (l,j,k)$ we have, assuming that (l,j,k) corresponds to the N-th element of the well-ordering $<$:

$$\|\mathrm{Ad}(\sigma_j)(b_{l,j,k}) - \mathrm{Ad}(u^l_{n-1})(b_{l,j,k})\|_{2,k} \geq$$

$$\geq \|\mathrm{Ad}(\sigma_j)(b_{l,j,k}) - \mathrm{Ad}((w_1v_1^* \dots v^*_{N-2}v^*_{N-1}v^*_N)^l)(b_{l,j,k})\|_{2,k} - 2^{-4} \geq$$

$$\geq \|\mathrm{Ad}((v_{N-1}u^*_{N-2})^l\sigma_j)(b_{l,j,k}) - \mathrm{Ad}(v^{*l}_N)(b_{l,j,k})\|_{2,k} - 2^{-3} > 1/8.$$

Conclude by fixing $\tilde{\psi}_n \sim \psi_n$ such that $\phi_n \circ \mathrm{Ad}(v_1 \dots v_{n-1}) \approx_{K,t} \tilde{\psi}_n \circ \mathrm{Ad}(w_1 \dots w_{n-1})$.

a2) Apply lemma 1.3.1 to $(\tilde{\psi}_h \circ \mathrm{Ad}(w_1 \dots w_{n-1}))_{h\leq n}$ for $F_n = F_n \cup \{\mathrm{Ad}(w^*_{n-1} \dots w^*_1)(a_i) : i \leq n\}$, E_n, $\{\tau_k : k \leq K\}$ to find a $G_n \Subset A$ and $\delta_n > 0$ which satisfy the thesis of the lemma.

b2) Fix $K = G_n \cup F_n$ and $E = \min\{\delta_n, 2^{-n}\}$, and let $(v_{n,t})_{t\in[0,1]}$ be a path of unitaries in A and $b_{l^t,j^t,k^t} \in A^1$ given by the application of lemma 1.3.2 in part *a1* such that (we will denote $v_{n,1}$ simply by v_n):

- $v_{n,0} = 1$,
- $\phi_h \circ \mathrm{Ad}(v_1 \dots v_n) \approx_{K,t} \tilde{\psi}_h \circ \mathrm{Ad}(w_1 \dots w_{n-1})$ for $h \leq n$,
- $\|b - \mathrm{Ad}(v_{n,t})(b)\| < E_n$ for all $b \in F_n$,
- $\|\mathrm{Ad}((v_{n-1}u^*_{n-2})^{l^t}\sigma_{j^t})(b_{l^t,j^t,k^t}) - \mathrm{Ad}(v^{*l^t}_n)(b_{l^t,j^t,k^t})\|_{2,k^t} > 1/4$.

The proof that Φ and Ψ, defined respectively as the pointwise limits of $\{\mathrm{Ad}(v_n)\}_{n\in\mathbb{N}}$ and $\{\mathrm{Ad}(w_n)\}_{n\in\mathbb{N}}$, are two automorphisms of A such that $\phi_h \circ \Phi \sim \psi_h \circ \Psi$ for all $h \in \mathbb{N}$ is as in [KOS03, Theorem 2.1]. Suppose now that $\alpha = \Psi \circ \Phi^{-1}$, and that α^l is a τ_k-weakly inner automorphism for some $k, l \in \mathbb{N}$. Thus, there is a σ_j such that, for all $a \in A^1$

$$\|\mathrm{Ad}(\sigma_j)(a) - \alpha^l(a)\|_{2,k} \leq 1/16.$$

Let $n \in \mathbb{N}$ be bigger than N and such that $\|\mathrm{Ad}(u^l_n)(b_{l,j,k}) - \alpha^l(b_{l,j,k})\|_{2,k} < 1/16$, N being the number corresponding to (l,j,k) with respect to $\leq$. Hence by construction it follows that

$$\|\mathrm{Ad}(\sigma_j)(b_{l,j,k}) - \mathrm{Ad}(u^l_n)(b_{l,j,k})\|_{2,k} > 1/8,$$

which is a contradiction.

For the other direction, suppose that there is $\tau \in \partial T(A)$ such that $\pi_\tau[A]$ is full. By [Sak74, Theorem 5-6] this is equivalent to say that all approximately inner automorphisms (in the norm induced by τ) on $\pi_\tau[A]$ are inner. Since α is approximately inner, it follows that α_τ is approximately inner in the norm induced by τ. The automorphism α_τ is therefore inner. □

Proof of theorem 1.2.3 - part 2. Fix a dense $\{a_i\}_{i\in\mathbb{N}}$ in A.

Step 1: *a1)* Apply lemma 1.3.1 to ϕ_1 for $F_1 = \{a_1\}$, $E_1 = 2^{-1}$, $\{\tau_1\}$, to find a G_1 � A and $\delta_1 > 0$ which satisfy the thesis of the lemma.

b1) Fix $\tilde{\psi}_1 \sim \psi_1$ such that $\tilde{\psi}_1 \approx_{G,\delta} \phi_1$.

a2) Apply lemma 1.3.1 to $\tilde{\psi}_1$ for $F_1 = F_1$, E_1, $\{\tau_1\}$, to find a G_1 � A and $\delta_1 > 0$ which satisfy the thesis of the lemma.

b2) Fix $K = G_1 \cup F_1$ and $E = \min\{\delta_1, 1/2\}$, and let $(v_{1,t})_{t\in[0,1]}$ be a path of unitaries in A given by the application of lemma 1.3.1 in part *a1* such that (we will denote $v_{1,1}$ simply by v_1):

- $v_{1,0} = 1$,
- $\phi_1 \circ \mathrm{Ad}(v_1) \approx_{K,t} \tilde{\psi}_1$,
- $\|b - \mathrm{Ad}(v_{1,t})(b)\| < E_1$ for all $b \in F_1$,
- $\|v_1 - 1\|_{2,1} < E_1$.

Step n: *a1)* Apply lemma 1.3.1 to $(\phi_h \circ \mathrm{Ad}(v_1 \dots v_{n-1}))_{h\leq n}$ for $F_n = F_{n-1} \cup \{a_i, \mathrm{Ad}(v^*_{n-1} \dots v^*_1)(a_i) : i \leq n\}$, $E_n = 2^{-n}$, $\{\tau_1, \dots, \tau_n\}$ to find a G_n � A and $\delta_n > 0$ which satisfy the thesis of the lemma.

b1) Fix $K = G_n \cup F_n$ and $E = \min\{\delta_n, 2^{-n}\}$, and let $(w_{n-1,t})_{t\in[0,1]}$ be a path of unitaries in A given by the application of lemma 1.3.1 in part *a2* of the previous step such that (we will denote $w_{n-1,1}$ simply by w_{n-1}):

- $w_{n-1,0} = 1$,
- $\phi_h \circ \mathrm{Ad}(v_1 \dots v_{n-1}) \approx_{K,t} \tilde{\psi}_h \circ \mathrm{Ad}(w_1 \dots w_{n-1})$ for $h \leq n-1$,
- $\|b - \mathrm{Ad}(w_{n-1})(b)\| < E_{n-1}$ for all $b \in F_{n-1}$,
- $\|w_{n-1} - 1\|_{2,k} < E_{n-1}$ for all $k \leq n-1$.

a2) Apply lemma 1.3.1 to $(\tilde{\psi}_h \circ \mathrm{Ad}(w_1 \dots w_{n-1}))_{h\leq n}$ for $F_n w_1 = F_n \cup \{\mathrm{Ad}(w^*_{n-1} \dots {}^*)(a_i) : i \leq n\}$, E_n, $\{\tau_1, \dots, \tau_n\}$ to find a G_n � A and $\delta_n > 0$ which satisfy the thesis of the lemma.

b2) Fix $K = G_n \cup F_n$ and $E = \min\{\delta_n, 2^{-n}\}$, and let $(v_{n,t})_{t\in[0,1]}$ be a path of unitaries in A given by the application of lemma 1.3.1 in part *a1* such that (we will denote $v_{n,1}$ simply by v_n):

- $v_{n,0} = 1$,
- $\phi_h \circ \mathrm{Ad}(v_1 \dots v_n) \approx_{K,t} \tilde{\psi}_h \circ \mathrm{Ad}(w_1 \dots w_{n-1})$ for $h \leq n$,
- $\|b - \mathrm{Ad}(v_{n,t})(b)\| < E_n$ for all $b \in F_n$,
- $\|v_n - 1\|_{2,k} < E_n$ for all $k \leq n$.

The proof that Φ and Ψ, defined respectively as the pointwise limits of $\{\mathrm{Ad}(v_n)\}_{n\in\mathbb{N}}$ and $\{\mathrm{Ad}(w_n)\}_{n\in\mathbb{N}}$, are two automorphisms of A such that $\phi_h \circ \Phi \sim \psi_h \circ \Psi$ for all $h \in \mathbb{N}$ is as in [KOS03, Theorem 2.1]. If $u_t = w_t v_t^*$, then the path of unitaries $(u_t)_{t\in[0,\infty)}$ is such that $\alpha(a) = \lim_{t\to\infty} \mathrm{Ad}(u_t)(a)$ for all $a \in A$ is the required automorphism. By construction, for each $n \in \mathbb{N}$ and all $k \leq n$ we have that

$$\|u_{n+1} - u_n\|_{2,k} = \|u_{n+1}u_n^* - 1\|_{2,k} = \|w_{n+1}v^*_{n+1} - 1\|_{2,k} < 2^{-(n-1)}.$$

Thus, given any $\tau \in \{\tau_k\}_{k\in\mathbb{N}}$, the sequence $\{\pi_\tau(u_n)\}_{n\in\mathbb{N}}$ is strongly convergent on $B(H_\tau)$ (recall that the strong convergence of $\{\pi_\tau(u_n)\}_{n\in\mathbb{N}}$ is equivalent to the convergence of $\{u_n\}_{n\in\mathbb{N}}$ in the ℓ_2-norm induced by τ). Let v be its strong limit. Then $\mathrm{Ad}(v)$ extends α, in fact for every $a, x, y \in A$ and $E > 0$, for $n \in \mathbb{N}$ big enough the following holds

$$(v\pi_\tau(a)v^*x, y)_\tau = (\pi_\tau(a)v^*x, v^*y)_\tau \approx (\pi_\tau(au^*_n)x, \pi_\tau(u^*_n)y)_\tau =$$
$$= (\pi_\tau(u_n a u^*_n)x, y)_\tau \approx (\pi_\tau(\alpha(a))x, y)_\tau.$$

The argument extends by density to all $x, y \in H_\tau$ and all $a \in \pi_\tau[A]$. $\square$

1.4 Outer Automorphisms

An interesting question related to this topic (see also the introduction of [FH17]) is the existence of a counterexample to Naimark's problem with an outer automorphism. This problem is related to the following freeness result.

Theorem 1.4.1 ([Kis81, Theorem 2.1]). *Let A be a separable, simple, unital C*-algebra and $\alpha \in Out(A)$. Then there exist two inequivalent pure states $\phi, \psi \in P(A)$ such that $\phi = \psi \circ \alpha$*

This result is linked in turn to the following question on inner automorphisms which, to our knowledge, is open.

Question 1.4.2. Let A be a unital C*-algebra and let α be an automorphism of A. Suppose that, whenever A is embedded in a C*-algebra B, α extends to an automorphism of B. Is α inner?

The analogous question has a positive answer for the category of groups (see [Sch87]), and an application of theorem 1.4.1 shows that this is also the case for separable, simple, unital C*-algebras. In fact, let A be a separable, simple, unital C*-algebra and $\alpha \in \mathrm{Out}(A)$. Suppose that $\phi, \psi \in P(A)$ are two inequivalent pure states such that $\phi = \psi \circ \alpha$. Since A is simple, the GNS representation associated to ϕ provides a map $\pi_\phi : A \to B(H_\phi)$ which is an embedding of A into $B(H_\phi)$. Identify A with $\pi_\phi[A]$ and suppose α can be extended to an automorphism of $B(H_\phi)$, which means that there is $u \in U(B(H_\phi))$ such that $\mathrm{Ad}(u) \restriction_A = \alpha$. The pure state ψ is thus equal to the vector state induced by $u\xi_\phi$, therefore an application of the Kadison transitivity theorem entails that ϕ and ψ are unitarily equivalent, which is a contradiction. A generalization of theorem 1.4.1 to nonseparable C*-algebras would settle the question also in the nonseparable simple case. A positive answer to the following question would show the impossibility of such generalization.

Question 1.4.3. Does a counterexample to Naimark's problem with an outer automorphism consistently exist?

Chapter 2

Embedding C*-algebras into the Calkin Algebra

The Calkin algebra $Q(H)$ is the quotient of the algebra of bounded linear operators $B(H)$ on a separable infinite-dimensional Hilbert space H, modulo the ideal of the compact operators $\mathcal{K}(H)$. Its first formal definition by Calkin dates back to 1941 [Cal41], making it the first example of an abstract C*-algebra which is not a von Neumann algebra[1]. Nevertheless, the implicit presence of the Calkin algebra can be tracked back already in the early works on operator algebras by Weyl and von Neumann [Wey09] and [VN35]. Here the authors fully characterize when two self-adjoint operators in $B(H)$ are unitarily equivalent up to a *compact difference* in terms of their spectra. The Calkin algebra became predominant after the research by Weyl and von Neumann was extended to normal operators and later, in the seminal paper [BDF77], to the classification of essentially normal operators, which led in turn to a fruitful interaction between C*-algebras and algebraic topology.

From a set-theoretic perspective, the Calkin algebra is an important point of con- tact with operator algebras, due to its structural similarities with the boolean algebra $P(\mathbb{N})/\mathrm{Fin}$, of which it is in fact considered the noncommutative analogue. The bond between these two objects is formally motivated by the Stone and the Gelfand-Naimark dualities. The Stone duality theorem links boolean algebras with compact, Hausdorff, zero-dimensional topological spaces, while the Gelfand-Naimark duality yields an equivalence between the category of compact Hausdorff spaces and the category of abelian unital C*-algebras. In this framework, the abelian C*-algebra associated to $P(\mathbb{N})/\mathrm{Fin}$ is $\ell^\infty(\mathbb{N})/c_0$, which diagonally embeds into the Calkin algebra. As a consequence, results about $P(\mathbb{N})/\mathrm{Fin}$ translate into (frequently nontrivial) questions about $Q(H)$.

In this chapter we study the analogue of the question “Which linear orderings embed into $P(\mathbb{N})/\mathrm{Fin}$?”. This topic has been extensively studied in set theory, one of the motivations being, for instance, the deep connections with the problem of the automatic continuity of Banach algebras homomorphisms. More in detail, Woodin’s condition for the automatic continuity of Banach algebras homomorphisms from $C([0,1])$ asserts that if there exists a discontinuous homomorphism from $C([0,1])$ into a Banach algebra, then a nontrivial initial segment of an ultrapower $\mathbb{N}^\mathbb{N}/\mathcal{U}$ embeds into $P(\mathbb{N})/\mathrm{Fin}$ (see [DW87]). This is usually stated in terms of embedding into the directed set $(\mathbb{N}^\mathbb{N}, \leq^*)$, but a linear order embeds into $(\mathbb{N}^\mathbb{N}, \leq^*)$ if and only if it embeds into $P(\mathbb{N})/\mathrm{Fin}$ (see for instance [Far96, Proposition 0.1] or [Woo84, Lemma 3.2]).

[1] In [Cal41] Calkin provided a faithful (hence isometric) representation of $Q(H)$ on a Hilbert space spanned by an orthonormal basis of size continuum.

In order to put our study into the proper context, we start by reviewing some known results about the topic of embeddings of linear orderings into $P(\mathbb{N})/\mathrm{Fin}$. To begin, $P(\mathbb{N})$ embeds as a boolean algebra into $P(\mathbb{N})/\mathrm{Fin}$. To define an embedding, send for instance $A \subseteq \mathbb{N}$ to the equivalence class of the set $\{(2n+1)2^m : n \in \mathbb{N}, m \in A\}$. Every countable linear ordering $\mathbb{L}$ embeds into $P(\mathbb{N})$, and therefore into $P(\mathbb{N})/\mathrm{Fin}$. One way to see this is to enumerate the elements of $\mathbb{L}$ as a_n, for $n \in \mathbb{N}$, and define $\Phi : \mathbb{L} \to P(\mathbb{N})$ by $\Phi(a_m) = \{n : a_n \leq a_m\}$.

Since $P(\mathbb{N})/\mathrm{Fin}$ is a countably saturated atomless boolean algebra, all linear orderings of cardinality $\aleph_1$ embed into it. Thus the continuum hypothesis, CH, implies that a linear order embeds into $P(\mathbb{N})/\mathrm{Fin}$ if and only if its cardinality is at most $2^{\aleph_0}$. The assertion that all linear orderings of cardinality at most $2^{\aleph_0}$ embed into $P(\mathbb{N})/\mathrm{Fin}$ is also relatively consistent with ZFC plus the negation of CH, as shown by Laver in [Lav79]. Laver's model is however an exception, in the absence of CH it is often possible to find linear orders of size $2^{\aleph_0}$ which do not embed into $P(\mathbb{N})/\mathrm{Fin}$. It is well-known for instance that $2^{\aleph_0}$ can be arbitrarily large and $\aleph_2$ does not embed into $P(\mathbb{N})/\mathrm{Fin}$ (see proposition 2.5.2).

The main question we investigate in this chapter is the noncommutative analogue of what we have exposed so far.

Question 2.0.1. What C*-algebras embed into the Calkin algebra?

This is also a noncommutative analogue of the question "What abelian C*-algebras embed into ℓ_∞/c_0?". By the Gelfand-Naimark duality, this translates to ask what compact Hausdorff spaces are continuous images of $\beta\mathbb{N} \setminus \mathbb{N}$, the Čech-Stone remainder of $\mathbb{N}$. By Parovičenko's theorem having weight at most $\aleph_1$ is a sufficient condition (alternatively, this can be proved by elementary model theory, see the discussion in [DH01, p. 1820]). However, the situation in ZFC is quite nontrivial ([DH99], [DH00]).

The analogue of the cardinality of a C*-algebra A (or of a topological space) is the density character. It is defined as the least cardinality of a dense subset of A. Thus the C*-algebras of density character $\aleph_0$ are exactly the separable C*-algebras. The density character of a nonseparable C*-algebra is equal to the minimal cardinality of a generating subset and also to the minimal cardinality of a dense $(\mathbb{Q} + i\mathbb{Q})$-subalgebra.

Every separable C*-algebra embeds into $B(H)$ and therefore, by a standard amplification argument, into $Q(H)$. In addition, all C*-algebras of density character $\aleph_1$ embed into $Q(H)$, but the proof is surprisingly nontrivial ([FHV17]) due to the failure of countable saturation in the Calkin algebra ([FH13, Section 4]; the Calkin algebra is not even countably homogeneous, see [FH16]).

Since the density character of $Q(H)$ is $2^{\aleph_0}$, C*-algebras with larger density character do not embed into $Q(H)$ and once again CH gives the simplest possible characterization of the class of C*-algebras that embed into $Q(H)$. In the first part of this chapter we make the next step and we investigate what happens when CH fails, focusing on C*-algebras of density character strictly less than $2^{\aleph_0}$.

Theorem 2.0.2. *The assertion 'Every* C**-algebra of density character strictly less than* $2^{\aleph_0}$ *embeds into the Calkin algebra' is independent from ZFC. More precisely, it is independent from ZFC +* $2^{\aleph_0} = \aleph_\alpha$ *for every* $\alpha > 2$.

The most involved part in the proof of theorem 2.0.2 is showing that the statement 'All C*-algebras of density character strictly less than $2^{\aleph_0}$ embed into $Q(H)$' is consistent with ZFC+$2^{\aleph_0} > \aleph_2$. This will be achieved via theorem 2.0.3 (which is proved in section 2.3) using forcing.

The method of forcing was introduced by Cohen to prove the independence of CH from ZFC, and later developed to deal with more general independence phenomena (see section 2.1.2). The *countable chain condition* (or ccc) is a property of forcing notions that ensures no cardinals or cofinalities are collapsed, and all stationary sets are preserved, in the forcing extension (see the beginning of section 2.1.2).

Theorem 2.0.3. *For every* C^**-algebra* A *there exists a ccc forcing notion* E_A *which forces the existence of an embedding of* A *into* $Q(H)$.[2]

Rephrasing the statement of theorem 2.0.3, every C^*-algebra, regardless of its density character, can be embedded into the Calkin algebra in a forcing extension of the universe obtained without collapsing any cardinals or cofinalities.

Theorem 2.0.3 (whose proof is given in section 2.3) was inspired by an analogous fact holding for partial orders and $P(\mathbb{N})/\mathrm{Fin}$: for every partial order P there is a ccc forcing notion which forces the existence of an embedding of P into $P(\mathbb{N})/\mathrm{Fin}$. While the proof of this latter fact is an elementary exercise, the proof of theorem 2.0.3 is fairly sophisticated. At a critical place it makes use of some variations of Voiculescu's results in [Voi76] (see corollaries 2.1.3 and 2.1.4).

The following corollary is the consistency result needed to prove one part of theorem 2.0.2 and follows from the proof of theorem 2.0.3.

Corollary 2.0.4. *Assume Martin's axiom, MA. Then every* C^**-algebra with density character strictly less than* $2^{\aleph_0}$ *embeds into the Calkin algebra.*

In the case when the continuum is not greater than $\aleph_2$, the conclusion of corollary 2.0.4 follows from [FHV17].

In section 2.5 we investigate the embedding problem in $Q(H)$ for some C^*-algebras of density character $2^{\aleph_0}$. The continuum hypothesis implies that all C^*-algebras of density $2^{\aleph_0}$ embed into $Q(H)$, but there are models of ZFC where this does not happen (see [FHV17] and corollary 2.5.5). Identifying the class of C^*-algebras of density character $2^{\aleph_0}$ that embed in $Q(H)$ in a given model of ZFC is generally a task out of reach (the analogous problem for $P(\mathbb{N})/\mathrm{Fin}$ and linear orders is already extremely challenging). In section 2.5 we prove that the C^*-algebra generated by an increasing chain of $\aleph_2$ projections does not embed into $Q(H)$ consistently with $\mathrm{ZFC} + 2^{\aleph_0} \geq \aleph_\alpha$, for every $\alpha \geq 2$. On the other hand, we show that $C^*_{\mathrm{red}}(F_{2^{\aleph_0}})$ and $C^*_{\max}(F_{2^{\aleph_0}})$, where $F_{2^{\aleph_0}}$ is the free group on $2^{\aleph_0}$ generators, embed into the Calkin algebra in every model of ZFC. The proof of the first fact is based on an argument on isomorphic names for real numbers by Kunen ([Kun68]). The proof of the latter is a simple application of the fact that $C^*_{\max}(F_{2^{\aleph_0}})$ is residually finite-dimensional and, for $C^*_{\mathrm{red}}(F_{2^{\aleph_0}})$, of a deep result by Haagerup and Thorbjørnsen ([HT05]). It is possible to generalize the notion of UHF algebra to nonseparable C^*-algebras by saying that a C^*-algebra is UHF if it is isomorphic to a tensor product of full matrix algebras (more on this in [FK10], [FK15]). We conclude section 2.5 by showing that all UHF algebras of density character at most $2^{\aleph_0}$ embed into $Q(H)$.

Question 2.0.5. Does $\bigotimes_{\alpha<2^{\aleph_0}} O_2$ consistently fail to embed into the Calkin algebra?

The results exposed in section 2.5 combined with theorem 2.0.3 allow us to prove theorem 2.0.2

[2] Given a C^*-algebra A in a model M of ZFC, it is often the case that the set A is not a C^*-algebra in a forcing extension of M, since it might not be closed anymore. Through this chapter we will implicitly identify A with its completion when passing to forcing extensions.

Proof of theorem 2.0.2. As pointed out above, if the cardinality of the continuum is not greater than $\aleph_2$ then all C*-algebras of density character strictly less than $2^{\aleph_0}$ embed into the Calkin algebra. We prove the statement of the corollary for $\alpha = 3$, as the proof for the other cases is analogous. Martin's axiom is relatively consistent with the continuum being equal to $\aleph_3$ ([Jec03, Theorem 16.13]) and by Corollary 2.0.4 in this case all C*-algebras of density character not greater than $\aleph_2$ embed into the Calkin algebra. On the other hand, in a model obtained by adding $\aleph_3$ Cohen reals to a model of CH we get that $2^{\aleph_0} = \aleph_3$, and that the Calkin algebra has no chains of projections of order type $\aleph_2$, as shown in proposition 2.5.5. Therefore in this model the abelian C*-algebra $C(\aleph_2 + 1)$ (where the ordinal $\aleph_2 + 1$ is endowed with the order topology) does not embed into $Q(H)$. $\square$

Finally, the last section of this chapter revolves around Voiculescu's theorem in [Voi76] (theorem 2.1.2). The contents of [Voi76] played a key role in the development of the theory of extensions of separable C*-algebras. An *extension* of a unital C*-algebra A (or rather its *Busby invariant*) is a unital embedding of A into $Q(H)$. Given a unital C*-algebra A, let Ext(A) be the set of all the extensions of A modulo unitary transformation of H, i.e. we identify two embeddings τ_1 and τ_2 for which there is a unitary transformation U of H such that $\tau_1 = \mathrm{Ad}(U) \circ \tau_2$.[3] Exploiting the fact that $H \oplus H \cong H$, it is possible to define the sum of two (classes of) extensions via the direct sum, and endow Ext(A) with a semigroup structure. One of the main consequences of [Voi76] is that, for a unital separable C*-algebra A, the semigroup Ext(A) always has an identity element, namely the class of all *trivial* extensions (an extension is trivial if it admits a multiplicative lift to $B(H)$). This, along with the results in [CE76], entails for instance that Ext(A) is a group for every nuclear separable unital C*-algebra A. The behavior of Ext(A) is much wilder when A is not in the above class, and for nonseparable C*-algebras Ext(A) could be empty (see [HR00, Section 2.6-2.7] for an introduction to the basic properties of the functor Ext). We remark that, by corollary 2.0.4, Martin's axiom entails that for all C*-algebras A of density less than continuum Ext(A) is non-empty. In section 2.6 we introduce a new perspective on the proof of Voiculescu's theorem (as given by Arveson in [Arv77]) which emerged during the work on the proof of theorem 2.0.3. More in detail, we prove that most of the arguments in [Arv77] used to prove Voiculescu's theorem (theorem 2.1.2) are diagonalization arguments which are equivalent to applications of the Baire category theorem (lemma 2.1.7) to some appropriate ccc posets. This allows us, assuming Martin's axiom, to generalize the contents of [Voi76] also to nonseparable C*-algebras of density less than continuum (see theorem 2.6.1).

2.1 Preliminary results

2.1.1 C*-algebras

Some definitions were already given in chapter 1, but we recall them here for the reader's convenience. In this chapter H always denotes the separable Hilbert space $\ell_2(\mathbb{N})$ and $B(H)$ is the space of linear bounded operators on H. $F(H)$ is the space of all finite-rank operators on H and its norm-closure, $K(H)$, is the ideal of compact operators. The notation $U(H)$ is reserved for the group of unitary operators on H. The Calkin algebra $Q(H)$ is the quotient of $B(H)$ by the compact operators and, through this chapter, $\pi : B(H) \to Q(H)$ is the quotient map.

[3] It is not uncommon to study the set of all extensions of A also modulo other equivalence relations, more on this in [Bla98, Chapter VII, Section 15.4].

Consistently with the notation of the previous chapter, we write $F(H)^{\leq 1}_{+}$ for the collection of all finite-rank positive contractions on H. For $h \in F(H)$, h^{+} is the orthogonal projection onto $h[H]$, the range of h, and h is the projection onto the 1-eigenspace of h (i.e. the space of all vectors ξ such that $h\xi = \xi$).

An operator $T \in B(H)$ is *way above* S, $T \gg S$ in symbols, if $TS = S$. We write $T \sim_{K(H)} S$, and say that T and S *agree modulo the compact operators*, to indicate that $T - S \in K(H)$. Similarly, given a C*-algebra A, two maps $\phi, \psi : A \to B(H)$ *agree modulo the compact operators* if $\phi(a) \sim_{K(H)} \psi(a)$ for every $a \in A$.

A net of operators $\{T_i\}_{i\in I}$ *strongly converges* to an operator T if for each $\xi \in H$ the net $\{T_i\xi\}_{i\in I}$ converges to $T\xi$. We remark that to verify the strong convergence of a norm-bounded net it suffices to check it on a dense subset of H. Given two vectors ξ and η of a normed vector space and $E > 0$, the notation $\xi \approx \eta$ stands for $\|\xi - \eta\| < E$.

If A is a C*-algebra, we write $F \Subset A$ to mean that F is a finite subset of A and $C^*(F)$ denotes the C*-subalgebra of A generated by F. If A is unital and $u \in A$ is a unitary element, then $\mathrm{Ad}(u)$ denotes the automorphism of A which sends a to uau^*.

A representation $\phi : A \to B(H)$ is called *essential* if $\phi(a) \in K(H)$ implies $\phi(a) = 0$ for all $a \in A$. Note that all (non-zero) representations of simple, infinite-dimensional C*-algebras on H are faithful (i.e. injective) and essential. A unital, injective $*$-homomorphism $\Theta : A \to Q(H)$ is *trivial* if there exists a unital (and necessarily essential) representation $\phi : A \to B(H)$ such that $\pi \circ \phi = \Theta$ and ϕ is the (*multiplicative*) *lift* of Θ. Moreover, Θ is called *locally trivial* if its restriction to any unital separable C*-subalgebra of A is trivial.

A bounded linear map $\sigma : A \to B$ between unital C*-algebras is *unital completely positive* (abbreviated as *u.c.p.*) if $\sigma(1) = 1$ and it is *completely positive*, namely is such that

$$\sum_{i,j<n} b_i^* \sigma(a_i^* a_j) b_j \geq 0$$

for all $n \in \mathbb{N}$ and all $a_0, \dots, a_{n-1} \in A$, $b_0, \dots, b_{n-1} \in B$. U.c.p. maps are always contractive and $*$-preserving.

Given a C*-algebra $A \subseteq B(H)$, an approximate unit $(h_\lambda)_{\lambda\in\Lambda}$ of $K(H)$ is *quasicentral* for A if $\lim_\lambda \|ah_\lambda - h_\lambda a\| = 0$ for every $a \in A$.

Given a cardinal λ, a C*-algebra A is *(injectively) λ-universal* if it has density character λ and all C*-algebras of density character λ embed into A.

Mainly for convenience, for the proof of theorem 2.0.3, we shall exclusively be concerned with embeddings of unital and simple C*-algebras into the Calkin algebra, as any unital $*$-homomorphism from a unital simple C*-algebra into $Q(H)$ is automatically injective. This causes no loss of generality, thanks to the following proposition.

Proposition 2.1.1 ([FHV17, Lemma 2.1]). *Every C*-algebra A embeds into a unital and simple C*-algebra of the same density character of A.*

The label 'Voiculescu's theorem' often refers to a not well-defined collection of results and corollaries from [Voi76], for us it always refers to the following specific theorem. Throughout the following statements (and the rest of the chapter), as mentioned at the beginning of this section, Hilbert spaces denoted by H are always assumed to be separable and infinite-dimensional.

Theorem 2.1.2 ([Arv77, Theorem 4]). *Let H, L be two separable Hilbert spaces, $A \subseteq B(H)$ a separable unital C*-algebra and $\sigma : A \to B(L)$ a unital completely positive map such that $\sigma(a) = 0$ for all $a \in A \cap K(H)$. Then there is a sequence of isometries $V_n : L \to H$ such that $\sigma(a) - V_n^* a V_n \in K(L)$ and $\lim_{n\to\infty} \|\sigma(a) - V_n^* a V_n\| = 0$ for all $a \in A$.*

The following two corollaries of theorem 2.1.2 are needed in the proof of theorem 2.0.3.

Corollary 2.1.3 ([BO08, Corollary 1.7.5]). *Let A be a unital separable C^*-algebra and let $\phi, \psi : A \to B(H)$ be two essential faithful unital representations. Then, for every F � A and $E > 0$, there exists a unitary $u \in U(H)$ such that*

1. *$Ad(u) \circ \phi \sim_{K(H)} \psi$,*

2. *$Ad(u) \circ \phi(a) \approx \psi(a)$ for all $a \in F$.*

Corollary 2.1.4. *Let A be a unital, separable C^*-algebra and let ϕ, $\psi : A \to B(H)$ be two essential faithful unital representations. Then, for every F � A and every finite-dimensional subspace $K \subseteq H$, there exists a unitary $w \in U(H)$ such that*

1. *$Ad(w) \circ \phi \sim_{K(H)} \psi$,*

2. *$Ad(w) \circ \phi(a)(\xi) = \phi(a)(\xi)$ for every $a \in F$ and $\xi \in K$.*

In particular, the set

$$\{Ad(w) \circ \phi : w \in U(H), Ad(w) \circ \phi(a) \sim_{K(H)} \psi(a) \text{ for all } a \in A\}$$

has ϕ in its closure with respect to strong convergence.

Proof. Let F � A, $K \subseteq H$ a finite-dimensional subspace and let $P \in B(H)$ be the orthogonal projection onto K. By corollary 2.1.3 we can find a unitary $v \in U(H)$ such that $\mathrm{Ad}(v) \circ \phi$ and ψ agree modulo the compact operators. Let Q be the finite-rank projection onto the space spanned by the set $K \cup \{\phi(a)K : a \in F\}$ and let $w \in U(H)$ be a finite-rank modification of v such that $wQ = Qw = Q$. Then $\mathrm{Ad}(w) \circ \phi$ and $\mathrm{Ad}(v) \circ \phi$ agree modulo the compact operators and $\mathrm{Ad}(w) \circ \phi(a)P = \phi(a)P$ for all $a \in F$. $\square$

See also [Arv77] and [HR00, Section 3] for a detailed proof of corollary 2.1.3, which is a standard consequence of the results in [Voi76]. Another result needed in the proof of theorem 2.0.3 (whose proof heavily relies on corollary 2.1.3) is the following.

Theorem 2.1.5 ([FHV17, Theorem A]). *All C^*-algebras of density $\aleph_1$ embed into the Calkin algebra. Moreover, the embedding can be chosen to be locally trivial.*

The following lemma is invoked multiple times in section 2.3 to take care of some technical details.

Lemma 2.1.6. *Let $T \in B(H)$ be a finite-rank projection. For every $E > 0$ there exists $\delta > 0$ such that if $S \in B(H)$ and $IT - SI < \delta$, then there is a unitary $u \in U(H)$ satisfying the following.*

1. *$uT[H] \subseteq S[H]$, namely the image space of uT is contained in the image space of S,*

2. *$uT \approx T$,*

3. *$u - Id_H \in F(H)$,*

4. *for every orthogonal projection P onto a subspace of $T[H]$ such that $SP = P$, $uP = P$ holds.*

Proof. Let $\{\xi_1, \dots, \xi_k\}$ be an orthonormal basis of the space of all eigenvectors of S whose eigenvalue is 1 and which are moreover contained in $T[H]$. Fix $\{\xi_1, \dots, \xi_n\}$ an orthonormal basis of $T[H]$ extending $\{\xi_1, \dots, \xi_k\}$. If $\delta < 1$, the set $\{S\xi_1, \dots, S\xi_n\}$ (which linearly spans $ST[H]$) is linearly independent. In fact, if $\xi \in T[H]$ has norm one and is such that $S\xi = 0$, then $\|T\xi\| = \|\xi\| < \delta$, which is a contradiction. Applying the Gram-Schmidt process to $\{S\xi_1, \dots, S\xi_n\}$ we obtain an orthonormal basis $\{\eta_1, \dots, \eta_n\}$ for $ST[H]$ which, for a sufficiently small choice of δ, is such that

$$\|\xi_i - \eta_i\| < \frac{E}{n}, \; i = 1, \dots, n.$$

Denote by V the finite-dimensional space spanned by $T[H]$ and $ST[H]$. Let $\{\xi_1, \dots, \xi_m\}$ be an orthonormal basis of V that extends $\{\xi_1, \dots, \xi_n\}$ and, similarly, $\{\eta_1, \dots, \eta_m\}$ an orthonormal basis of V extending $\{\eta_1, \dots, \eta_n\}$. This naturally defines a unitary $w : V \to V$ by sending the vector ξ_i to η_i for every $i = 1, \dots, m$. Finally, define $u \in U(H)$ to be equal to w on V and equal to the identity on the orthogonal complement of V. The unitary u satisfies the desired properties. □

2.1.2 Set Theory and Forcing

As stated in the introduction, theorem 2.0.3 is an application of the method of forcing. For a standard introduction to this topic see [Kun11]; see also [DW87] and [Wea14].

We recall some technical definitions. A *partially ordered set* (or simply *poset*) $(\mathsf{P}, \leq)$ is a set equipped with a binary transitive antisymmetric reflexive relation $\leq$. Two elements p, q of a poset $(\mathsf{P}, \leq)$ are *compatible* if there exists $s \in P$ such that $s \leq p$ and $s \leq q$. Otherwise, p and q are *incompatible*. A subset $\Delta \subseteq \mathsf{P}$ is *dense* if for every $p \in \mathsf{P}$ there is $q \in \Delta$ such that $q \leq p$. A subset Δ of P is *open* if it is close downwards, i.e. $p \in \Delta$ and $q \leq p$ implies $q \in \Delta$. A subset $A \subseteq \mathsf{P}$ is an *antichain* if its elements are pairwise incompatible. The poset $(\mathsf{P}, \leq)$ satisfies the *countable chain condition* (henceforth abbreviated as *ccc*) if every antichain is at most countable. $(\mathsf{P}, \leq)$ has *property K* if every uncountable subset of P contains a further uncountable subset in which any two elements are compatible. Given a cardinal λ, a *λ-chain* is a subset $\{p_\alpha : \alpha < \lambda\}$ of P such that $p_\alpha < p_\beta$ for all $\alpha < \beta < \lambda$. A non-empty subset G of P is a *filter* if $q \in G$ and $q \leq p$ implies $p \in G$, and if for any $p, q \in G$ there exists $r \in G$ such that $r \leq p$, $r \leq q$. Given a family D of dense subsets of P, a filter G is *D-generic* if it meets every dense of D.

A *forcing notion* (or *forcing*) is a partially ordered set (poset), whose elements are called *conditions*. Naively, the forcing method produces, starting from a poset P, an extension of von Neumann's universe V. The extension is obtained by adding to V a filter G of P which intersects *all* dense open subsets of P. This generic extension, usually denoted by $V[G]$, is a model of ZFC, and its theory depends on combinatorial properties of P and (to some extent) on the choice of G. A condition $p \in \mathsf{P}$ *forces* a sentence ϕ in the language of ZFC if ϕ is true in $V[G]$ whenever G is a generic filter containing p. If ϕ is true in every generic extension $V[G]$, we say that P *forces* ϕ.

Unless P is trivial, no filter intersects every dense open subset of P. For this reason, the forcing method is combined with a Löwenheim-Skolem reflection argument and applied to countable models of ZFC. If M is a countable model of ZFC and $\mathsf{P} \in M$, then the existence of an M-generic filter G (i.e. intersecting every open dense subset of P in M) of P is guaranteed by the Baire category theorem ([Kun11, Lemma III.3.14])[4].

[4] For metamathematical reasons related to Gödel's incompleteness theorem, one usually considers models of a large enough finite fragment of ZFC. By other metamathematical considerations, for all practical purposes this issue can be safely ignored; see [Kun11, Section IV.5.1].

An obvious method for embedding a given C*-algebra A into the Calkin algebra is to generically add a bijection between a dense subset of A and $\aleph_0$ (i.e. to 'collapse' the density character of A to $\aleph_0$). The completion of A in the forcing extension (routinely identified with A) is then separable and therefore embeds into the Calkin algebra of the extension. However, if the density character of A is collapsed, then this results in a C*-algebra that has little to do with the original algebra A. We shall give two examples.

Fix an uncountable cardinal κ. If A is $C^*_{red}(F_\kappa)$, the reduced group algebra of the free group with κ generators, then collapsing κ to $\aleph_0$ makes A isomorphic to $C^*_{red}(F_{\aleph_0})$ (better known as $C^*_{red}(F_\infty)$). It is not difficult to prove that, if a cardinal κ is not collapsed, then the completion of $C^*_{red}(F_\kappa)$ in the extension is isomorphic to $C^*_{red}(F_\kappa)$ as computed in the extension. This is not automatic as, for example, the completion of the ground model Calkin algebra in a forcing extension will rarely be isomorphic to the Calkin algebra in the extension.

A more drastic example is provided by the 2^κ nonisomorphic C*-algebras each of which is an inductive limit of full matrix algebras of the form $M_{2^n}(\mathbb{C})$ for $n \in \mathbb{N}$ constructed in [FK15, Theorem 1.2]. After collapsing κ to $\aleph_0$, all of these C*-algebras become isomorphic to the CAR algebra. This is because it can be proved that the K-groups of A are invariant under forcing and, by Glimm's classification result, unital and separable inductive limits of full matrix algebras are isomorphic (e.g. [Bla06]). A similar effect can be produced even with a forcing that preserves cardinals if it collapses a stationary set ([FK15, Proposition 6.6]).

Instead of 'collapsing' the cardinality of A, our approach is to 'inflate' the Calkin algebra. More precisely, we prove that Martin's axiom implies that the Calkin algebra has already been 'inflated'.

The following lemma is an equivalent version of the more common topological formulation of the Baire category theorem.

Lemma 2.1.7 (Baire category theorem, [Jec03, Lemma 14.4])**.** *If* $(\mathbb{P}, <)$ *is a partially ordered set and* D *is a countable collection of dense subsets of* $\mathbb{P}$*, then there exists a* D*-generic filter on* $\mathbb{P}$*. Moreover, for any* $p \in \mathbb{P}$*, there is a* D*-generic filter* G *such that* $p \in G$*.*

Forcing axioms are far-reaching extensions of the Baire category theorem that enable one to apply forcing without worrying about metamathematical issues. Martin's axiom is the simplest (and most popular) forcing axiom.

Martin's axiom (MA). If $(\mathbb{P}, <)$ is a poset that satisfies the countable chain condition, and D is a collection of fewer than $2^{\aleph_0}$ dense subsets of $\mathbb{P}$, then there exists a D-generic filter on $\mathbb{P}$.

Martin's axiom is a combinatorial statement which is independent from ZFC. It is a vacuous consequence of CH (by lemma 2.1.7), but it is also consistent that, given any regular $\kappa > \aleph_1$, $2^{\aleph_0} = \kappa$ and MA holds (see [Jec03, Theorem 16.13]).

The proof strategy in section 2.3 is as follows. Given a C*-algebra A, we start by defining a forcing notion $\mathbb{E}_A$ (definition 2.3.2) whose generic filters (if any) allow to build an embedding of A into $Q(H)$ (proposition 2.3.5). We then proceed to show that $\mathbb{E}_A$ is ccc (proposition 2.3.7), and that the existence of sufficiently generic filters inducing the existence of an embedding of A into $Q(H)$ is guaranteed in models of ZFC + MA (corollary 2.0.4).

The following lemma will be used when proving that a given forcing notion is ccc. A family C of sets forms a Δ-*system* with *root* R if $X \cap Y = R$ for any two distinct sets X

and Y in C. When the sets in C are pairwise disjoint, one obtains the special case with $R = \emptyset$.

Lemma 2.1.8 (Δ-system lemma, [Kun11, Lemma III.2.6]). *Every uncountable family of finite sets contains an uncountable Δ-system.* $\square$

2.2 Boolean Algebras and Quasidiagonal C*-algebras

In this section we discuss two special cases of theorem 2.0.3, those corresponding to the classes of abelian and quasidiagonal C*-algebras. Their proofs (the first of which is standard) are intended to provide intuition and demonstrate the increase in complexity regarding the corresponding forcing notions that are implemented. It also displays the natural progression behind theorem 2.0.3. We will omit most of the technical details in this section, as the results discussed here can be easily inferred by the proofs of the subsequent parts of the chapter. The reader eager to transition right away to the proof of theorem 2.0.3 may safely skip to section 2.3.

2.2.1 Embedding Abelian C*-algebras into ℓ_∞/c_0

The main focus in this part will be on obtaining the abelian version of theorem 2.0.3.

Proposition 2.2.1. *For every abelian C*-algebra A there exists a ccc forcing notion which forces that A embeds into ℓ_∞/c_0.*

Exploiting the fact that the categories of Boolean algebras, Stone spaces (i.e. zero-dimensional, compact, Hausdorff spaces) and C*-algebras of continuous functions on Stone spaces are all equivalent (by a combination of the Stone duality [GH09, Theorem 31-32] and the Gelfand-Naimark duality [Mur90, Theorem 2.1.10]), one can translate the statement of the proposition above to a statement regarding Boolean algebras. In particular, it is enough to show that for any Boolean algebra B there exists a ccc forcing notion which forces that B embeds into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. If B is a Boolean algebra, we denote by $\mathrm{St}(\mathsf{B})$ its *Stone space*, the space of all ultrafilters on B equipped with the Stone topology.

To see the aforementioned translation, first of all note that it suffices to prove the assertion of proposition 2.2.1 for C*-algebras of the form $C(Y)$ with Y being a Stone space, as every abelian C*-algebra embeds into such an algebra. Indeed, any abelian C*-algebra $C(X)$ naturally embeds into the von Neumann algebra $L^\infty(X)$ which, being a real rank zero unital C*-algebra, is of the form $C(Y)$ with Y zero-dimensional, compact and Hausdorff. We provide an alternative proof for the reader who is not familiar with the theory of von Neumann algebras. Every non-unital, abelian C*-algebra embeds into its unitization, which is a C*-algebra of continuous functions on a compact, Hausdorff space X. For any compact, Hausdorff space X, let X_d consist of the underlying set of X equipped with the discrete topology. Then, the identity map from X_d to X uniquely extends to a continuous map from βX_d onto X and this, in turn, implies the existence of an embedding of $C(X)$ into $C(\beta X_d)$. The Čech-Stone compactification of a discrete space is always zero-dimensional and this establishes the previous claim.

Let X be a Stone space and consider the Boolean algebra B of all clopen subsets of X. By the Stone duality, the existence of a ccc forcing notion that forces the embedding of B into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ yields (in any generic extension of the universe) a continuous surjection from $\mathrm{St}(\mathcal{P}(\mathbb{N})/\mathrm{Fin}) \cong \beta\mathbb{N} \setminus \mathbb{N}$ onto $\mathrm{St}(\mathsf{B}) \cong X$. By contravariance due to the Gelfand-Naimark duality, one obtains an injective $*$-homomorphism from $C(X)$ into $C(\beta\mathbb{N} \setminus \mathbb{N})$, with the latter being isomorphic to ℓ_∞/c_0.

Thus, we turn our attention to providing the forcing notion guaranteed by the following folklore proposition.

Proposition 2.2.2. *For every boolean algebra* $\mathbb{B}$*, there exists a ccc forcing notion* $\mathbb{E}_{\mathbb{B}}$ *which forces that* $\mathbb{B}$ *embeds into* $P(\mathbb{N})/\mathrm{Fin}$.

We view $P(\mathbb{N})/\mathrm{Fin}$ as the space of all binary sequences $2^{\mathbb{N}}$ modulo the equivalence relation

$$x \sim y \text{ if and only if } |\{n \in \mathbb{N} : x(n) \neq y(n)\}| < \aleph_0$$

for all $x, y \in 2^{\mathbb{N}}$.

Definition 2.2.3. Fix a boolean algebra $\mathbb{B}$ and let $\mathbb{E}_{\mathbb{B}}$ be the set of all triples

$$p = (B_p, n_p, \psi_p)$$

where

1. B_p is a finite boolean subalgebra of $\mathbb{B}$,
2. $n_p \in \mathbb{N}$,
3. $\psi_p : B_p \to 2^{n_p}$ is an arbitrary map.

Given $p, q \in \mathbb{E}_{\mathbb{B}}$, we say that $p < q$ if and only if

4. $B_q \subseteq B_p$,
5. $n_q < n_p$,
6. $\psi_p(a)(i) = \psi_q(a)(i)$ for all $a \in B_q$ and $i < n_q$,
7. the map

$$\begin{aligned} B_q &\to 2^{n_p - n_q} \\ a &\mapsto \psi_p(a)_{\restriction [n_q, n_p)} \end{aligned}$$

is an injective homomorphism of boolean algebras.

This defines a partial order on $\mathbb{E}_{\mathbb{B}}$. Conditions in $\mathbb{E}_{\mathbb{B}}$ represent partial maps from a finite subset of $\mathbb{B}$ to an initial segment of a characteristic function corresponding to a subset of $\mathbb{N}$.

Any finite Boolean subalgebra of $\mathbb{B}$ is isomorphic to the Boolean algebra given by the powerset of a finite set and hence can be embedded into 2^m for $m \in \mathbb{N}$ large enough. Therefore it is always possible to extend a given condition $p \in \mathbb{E}_{\mathbb{B}}$ to a $q < p$ such that B_q contains any arbitrary finite subset of $\mathbb{B}$ and $n_q > n_p$, while making sure that in the added segment the map is an injective homomorphism. For this reason, a generic filter G in $\mathbb{E}_{\mathbb{B}}$ provides a pool of maps which can be 'glued' together in a coherent way, inducing thus a function Ψ_G which, by genericity, is defined everywhere on $\mathbb{B}$:

$$\begin{aligned} \Psi_G : \mathbb{B} &\to P(\mathbb{N}) \\ b &\mapsto \bigcup_{\{p \in G : b \in B_p\}} \psi_p(b). \end{aligned}$$

Here we identify $\psi_p(b) \in 2^{n_p}$ with the corresponding subset of n_p. Moreover, by definition of the order relation on $\mathbb{E}_{\mathbb{B}}$, the map Ψ_G is, modulo the ideal of finite sets, injective and preserves all Boolean operations.

By using a standard uniformization argument and an application of the Δ-system lemma (lemma 2.1.8), when given an uncountable set of conditions $U \subseteq \mathbb{E}_{\mathbb{B}}$, it is possible to find an uncountable $W \subseteq U$, $n \in \mathbb{N}$ and $Z \Subset \mathbb{B}$ such that $n_p = n$, $B_p \cap B_q = Z$ and $\psi_p(b) = \psi_q(b)$ for all $p, q \in W$ and $b \in Z$. Thus the problem of whether $\mathbb{E}_{\mathbb{B}}$ is ccc is reduced to the following:

Lemma 2.2.4. *Let $p, q \in \mathbb{E}_{\mathbb{B}}$ be two conditions such that $n_p = n_q$ and the maps ψ_p, ψ_q agree on $B_p \cap B_q$. Then, p and q are compatible.*

To see that this holds, define B_s to be the (finite) Boolean subalgebra of $\mathbb{B}$ that is generated by $B_p \cup B_q$ and choose a Boolean algebra isomorphism

$$f : B_s \to 2^m$$

for some $m \in \mathbb{N}$. Set $n_s = n_p + m$ and define the map ψ_s to be equal to ψ_p concatenated with f on B_p, equal to ψ_q concatenated with f on $B_q \setminus B_p$ and equal to zero elsewhere. Then, the condition $s = (B_s, n_s, \psi_s)$ extends both p and q.

2.2.2 Embedding Quasidiagonal C^*-algebras into the Calkin Algebra

Quasidiagonal C^*-algebras possess strong local properties that remarkably simplify the proof of theorem 2.0.3. In this case, in fact, the 'natural' analogue of the poset introduced in the previous subsection does the job without too much additional effort.

A unital C^*-algebra A is *quasidiagonal* if for every finite set $F \Subset A$ and $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and a u.c.p. map $\sigma : A \to M_n(\mathbb{C})$ such that

$$\|\sigma(ab) - \sigma(a)\sigma(b)\| < \varepsilon \text{ for all } a, b \in F$$

and

$$\|\sigma(a)\| > \|a\| - \varepsilon \text{ for all } a \in F.$$

In this section we prove the following proposition.

Proposition 2.2.5. *For every quasidiagonal C^*-algebra A there exists a ccc poset $\mathbb{QD}_A$ which forces an embedding of A into $Q(H)$.*

As opposed to the proof of theorem 2.0.3 in section 2.3, where we can apply proposition 2.1.1, we will not assume that A is simple in the proof of proposition 2.2.5. Such assumption would have made definition 2.2.6 slightly simpler, but, to our knowledge, it is not known whether it is possible to embed a given quasidiagonal C^*-algebra into a simple quasidiagonal one (an application of the Downward Löwenheim-Skolem theorem ([FHL+ar, Theorem 2.6.2]) would then provide a quasidiagonal simple C^*-algebra with the same density character as the one we started with). We may assume though that A is unital. To begin, fix $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis of H and for every $n \in \mathbb{N}$ let R_n be the orthogonal projection onto the linear span of the set $\{e_k : k \leq n\}$. Since for every $n \in \mathbb{N}$ the space $R_n B(H) R_n$ is finite-dimensional, choose D_n a countable dense subset that contains R_n. For $n < m \in \mathbb{N}$, we also require that $D_n \subseteq R_n D_m R_n$.

Similar to the case of Boolean algebras, we define a forcing notion for a quasidiagonal C^*-algebra whose conditions represent partial maps from a finite subset of A to an 'initial segment' in $B(H)$, which in this case is a *corner* $R_n B(H) R_n$ for some $n \in \mathbb{N}$. Extensions of conditions are defined as to yield better approximations, maps are defined on a bigger domain and take values on a larger corner in $B(H)$. It is only on a sufficient part of the larger corner that we shall request that the new maps preserve the norm of elements and all algebraic operations, modulo a small error (which disappears once one passes to the Calkin algebra).

Definition 2.2.6. Let A be a unital, quasidiagonal C*-algebra and define QD_A to be the set of all tuples

$$p = (F_p, n_p, E_p, \psi_p)$$

such that

1. $F_p \Subset A$ and $1 \in F_p$,
2. $n_p \in \mathbb{N}$,
3. $E_p \in \mathbb{Q}^+$,
4. $\psi_p : F_p \to D_{n_p}$ is a map such that $\psi_p(1) = 1$ and

$$\|\phi_p(a)\| \leq \|a\| \text{ for all } a \in F_p.$$

For $p, q \in \mathrm{QD}_A$, we write $p < q$ if the following hold

5. $F_q \subseteq F_p$,
6. $n_q \leq n_p$,
7. $E_p < E_q$,
8. $\psi_p(a)R_{n_q} = \psi_q(a)$ and $R_{n_q}\psi_p(a) = \psi_q(a)$ for all $a \in F_q$,
9. $\|\psi_p(a)(R_{n_p} - R_{n_q})\| > \|a\| - E_q$ for all $a \in F_q$,
10. for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$ define

$$\Delta^{p,+}_{a,b,\lambda,\mu} := \psi_p(\lambda a + \mu b) - \lambda\psi_p(a) - \mu\psi_p(b),$$

$$\Delta^{p,*}_{a} := \psi_p(a^*) - \psi_p(a)^*,$$
$$\Delta^{p,}_{a,b} := \psi_p(ab) - \psi_p(a)\psi_p(b),$$

Then we require

(a) $\|\Delta^{p,+}_{a,b,\lambda,\mu}(R_{n_p} - R_{n_q})\| < E_q - E_p$ if $a, b, \lambda a + \mu b \in F_q$,

(b) $\|\Delta^{p,*}_{a}(R_{n_p} - R_{n_q})\| < E_q - E_p$ if $a, a^* \in F_q$,

(c) $\|\Delta^{p,}_{a,b}(R_{n_p} - R_{n_q})\| < E_q - E_p$ if $a, b, ab \in F_q$.

Item 8 entails, for $a \in F_q$

$$R_{n_q}\psi_p(a)R_{n_q} = \psi_q(a)$$

and

$$R_{n_q}\psi_p(a)(1 - R_{n_q}) = (1 - R_{n_q})\psi_p(a)R_{n_q} = 0.$$

This property displays the block-diagonal fashion of the extension of conditions and plays a crucial role in ascertaining that the relation $<$ is transitive. To demonstrate it, by considering multiplication as an example, for conditions $p < q < s$ in QD_A we have that

$$\|\Delta^{p,\cdot}_{a,b}(R_{n_p} - R_{n_s})\| \leq \|\Delta^{p,\cdot}_{a,b}(R_{n_p} - R_{n_q})\| + \|\Delta^{p,\cdot}_{a,b}(R_{n_q} - R_{n_s})\| < E_q - E_p + \|\Delta^{p,\cdot}_{a,b}(R_{n_q} - R_{n_s})\|.$$

Item 8 implies that

$$\psi_p(c)(R_{n_q} - R_{n_s}) = \psi_q(c)(R_{n_q} - R_{n_s}) = (R_{n_q} - R_{n_s})\psi_q(c)(R_{n_q} - R_{n_s})$$

for all $c \in F_s$. Thus

$$\psi_p(a)\psi_p(b)(R_{n_q} - R_{n_s}) = \psi_p(a)(R_{n_q} - R_{n_s})\psi_q(b)(R_{n_q} - R_{n_s}) = \psi_q(a)\psi_q(b)(R_{n_q} - R_{n_s})$$

which in turn yields

$$\|\Delta^{p,\cdot}_{a,b}(R_{n_q} - R_{n_s})\| < E_s - E_q.$$

Note that for any finite set F � A and $n \in \mathbb{N}$ there are only countably many maps $\psi : F \to D_n$ as in condition 4 of the previous definition. This, along with a standard uniformization argument and an application of the Δ-system lemma (lemma 2.1.8), reduces the problem of whether the poset QD_A is ccc to the following lemma.

Lemma 2.2.7. *Let $p, q \in \mathrm{QD}_A$ be two conditions such that $n_p = n_q$, $E_p = E_q$ and the maps ψ_p, ψ_q agree on $F_p \cap F_q$. Then, p and q are compatible.*

Proof. For $E_s = E_p/8$, let $m \in \mathbb{N}$ and $\phi : F_s = F_p \cup F_q \to M_m(\mathbb{C})$ be given as in the definition of quasidiagonality. By setting $n_s = n_p + m$, identifying $M_m(\mathbb{C})$ with the corner $(R_{n_s} - R_{n_p})B(H)(R_{n_s} - R_{n_p})$ and approximating ϕ via the dense sets up to E_s, define a map ψ_s which block-diagonally extends both ψ_p and ψ_q via this approximation of ϕ. In this manner, the resulting condition $s = (F_s, n_s, E_s, \psi_s) \in \mathrm{QD}_A$ extends both p and q. □

The previously described argument also gives the basic idea of how to extend a given condition by diagonally adjoining a finite-dimensional block in which, modulo a small error, all algebraic operations and the norm of all elements are preserved. This hints that a generic filter induces (analogously to the case of Boolean algebras in the previous subsection; see also proposition 2.3.5) a map from A into $Q(H)$ which is an isometric (and thus injective) $*$-homomorphism.

2.3 The General Case

In this section we proceed to define the forcing notion $\mathbb{E}_A$ and give the proof of theorem 2.0.3.

2.3.1 The Poset

For what follows in this section, A is a simple unital C*-algebra. Fix $P \subseteq B(H)$ an increasing countable sequence of finite-rank projections converging strongly to the identity and C a countable dense subset of $F(H)^{\leq 1}_+$. For $R \in P$ and $h \in C$ let $S_{R,h}$ be the orthogonal projection onto the span of $h^+[H] \cup R[H]$. Fix a countable dense subset

$$D_{R,h} \subseteq \{S_{R,h}Th^+ : T \in B(H)\}$$

that contains h^+. We need the dense sets $D_{R,h}$ and C to satisfy certain closure properties in order to carry out the arguments below. We describe these properties in detail here, but the reader can safely ignore them for now and come back to them when reading the proof of proposition 2.3.4.

Definition 2.3.1. The countable sets C and $D_{R,h}$ previously defined are required to have the following closure properties.

1. For all $c_1, \ldots, c_k \in C$ and $R \in P$, the intersection of C with the set (recall that $h \gg c$ stands for $hc = c$)
$$\{h \in \mathcal{F}(H)_+^{\leq 1} : h \gg c_1, \ldots, h \gg c_k, h \geq R\}$$
is dense in the latter.

2. Given $R \in P$ and $h, k \in C$, the intersection of $D_{R,h}$ with the set
$$\{T \in S_{R,h}\mathcal{B}(H)h^+ : Tk^-[H] \subseteq h^-[H], Th^-[H] \subseteq h^+[H]\}$$
is dense in the latter.

3. Given $R, R \in P$, $h_1, h_2, k \in C$, and $T \in D_{R',h_2}$, the intersection of D_{R,h_1} with the set
$$\{T \in S_{R,h_1}\mathcal{B}(H)h_1^+ : Th_1^+ = T', h_1^-T = h_2^-T', Tk^-[H] \subseteq h_1^-[H], Th_1^-[H] \subseteq h_1^+[H]\}$$
is dense in the latter.

It is straightforward to build countable dense sets with such properties by countable iteration.[5]

Before proceeding to the definition of the poset, we pause to give some insight and justify the considerably higher complexity it possesses when compared with the abelian or quasidiagonal case. The rough idea is, again, to define a poset where each condition represents a partial map from a finite subset of A into some finite-dimensional corner of $\mathcal{B}(H)$. The ordering guarantees that stronger conditions behave like $*$-homomorphisms on larger and larger subspaces of H up to an error which tends to zero. The countable, dense sets $D_{R,h}$ considered in the beginning of this section serve as the codomains of these partial maps and, as a result, for any finite subset of A there are only countable many possible maps into any given corner. The main difference with the quasidiagonal case is that we cannot expect conditions to look like block-diagonal matrices anymore. This has troublesome consequences, mostly caused by the multiplication (and to a minor extent by the adjoint operation). The main issue is that, given $p < q$, one cannot expect that a property similar to the consequence of item 8 of definition 2.2.6, that is
$$R_{n_q}\psi_p(a)(1 - R_{n_q}) = (1 - R_{n_q})\psi_p(a)R_{n_q} = 0$$
can hold in general. Therefore (and with the comments succeeding definition 2.2.6 in mind), even defining a partial order that is transitive proves to be non-trivial. An even bigger issue that comes up is the extension of a condition to a stronger one with larger domain. While in the quasidiagonal case it is sufficient to add a finite-dimensional block with some prescribed properties, completely ignoring how ψ_p is defined, in the general case one has to explicitly require for ψ_p to allow at least one extension in order to avoid $\mathbb{E}_A$ having atomic conditions[6]. These and other technical reasons lead to the following definition.

Definition 2.3.2. Let $\mathbb{E}_A$ be the set of the tuples
$$p = (F_p, E_p, h_p, R_p, \psi_p)$$
where

[5] A logician can use a large enough countable elementary submodel of a sufficiently large hereditary set containing all the relevant objects as a parameter to outright define these sets.

[6] Given a poset $(\mathbb{P}, <)$, $p \in P$ is atomic if $q \leq p$ implies $q = p$.

1. $F_p \subset A$, $1 \in F_p$ and if $a \in F_p$ then $a^* \in F_p$,

2. $E_p \in \mathbb{Q}^+$,

3. $h_p \in C$,

4. $R_p \in P$,

5. $\psi_p : F_p \to D_{R_p,h_p}$, is a map and there exist a faithful, essential, unital $*$-homomorphism $\Phi_p : C^*(F_p) \to B(H)$ and a projection $k_p \leq h_p^-$ such that for all $a \in F_p$

   (a) $k_p = k^-$ for some $k \in C$,
   (b) $\psi_p(1) = h_p^+$,
   (c) $\|(\psi_p(a) - \Phi_p(a))(h_p^+ - k_p)\| < \frac{E_p}{3M_p}$, where

   $$L_{F_p} = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \exists \mu \in \mathbb{C}, \exists a, b \in F_p \text{ s.t. } a \neq 0 \text{ and } \lambda a + \mu b \in F_p\}$$

   and

   $$M_p = \max\{3\|a\|, 3\|\psi_p(a)\|, L_{F_p} : a \in F_p\},$$

   (d) $\|\psi_p(a) + \Phi_p(a)(1 - h_p^+)\| < \frac{3}{2}\|a\|$,
   (e) $\psi_p(a)k_p[H] \subseteq h_p^-[H]$ and $\psi_p(a)h_p^-[H] \subseteq h_p^+[H]$,
   (f) $\Phi_p(a)k_p[H] \subseteq h_p^-[H]$ and $\Phi_p(a)h_p^-[H] \subseteq h_p^+[H]$.

We refer to the pair (k_p, Φ_p) as the *promise* for the condition p. Given $p, q \in \mathbb{E}_A$, we write $p < q$ if and only if

6. $F_p \supseteq F_q$,

7. $E_p < E_q$,

8. $h_p \gg h_q$,

9. $R_p \geq R_q$,

10. $\psi_p(a)h_q^+ = \psi_q(a)$ for all $a \in F_q$,

11. $h_q^- \psi_p(a) = h_q^- \psi_q(a)$ for all $a \in F_q$,

12. (a) $\|\Delta^{p,+}_{a,b,\lambda,\mu}(h_p^- - h_q^-)\| < E_q - E_p$ for $a, b, \lambda a + \mu b \in F_q$,
    (b) $\|\Delta^{p,*}_a(h_p^- - h_q^-)\| < E_q - E_p$ for $a \in F_q$,
    (c) $\|\Delta^{p,\cdot}_{a,b}(h_p^- - h_q^-)\| < E_q - E_p$ for $a, b, ab \in F_q$,

   where the quantities $\Delta^{p,+}_{a,b,\lambda,\mu}$, $\Delta^{p,*}_a$ and $\Delta^{p,\cdot}_{a,b}$ are as in definition 2.2.6.

Item 5e above is an example of how the problem of transitivity is addressed and this becomes clear in the next proposition. The promise in item 5 is witnessing that there is at least one way to extend p (via Φ_p) to conditions with arbitrarily large (finite-dimensional) domain. It will become clear later (see propositions 2.3.4, 2.3.6, 2.3.7) how corollary 2.1.4 implies that the choice of a specific Φ_p is not a real constraint to how extensions of p are going to look like.

Proposition 2.3.3. *The relation $<$ defined on E_A is transitive.*

Proof. Let $p, q, s \in \mathbb{E}_A$ be such that $p < q < s$. It is straightforward to check that conditions 6-9 hold between p and s. Clauses 10 and 11 follow since $h_q \gg h_s$ implies $h_q^- \geq h_s^+$. We recall that for two projections p, q the relation $p \leq q$ is equivalent to $pq = qp = p$. We divide the proof of condition 12 in three claims, one for each item.

Claim 2.3.3.1. *If $a, b, \lambda a + \mu b \in F_s$ then $I\Delta^{p,+}_{a,b,\lambda,\mu}(h_p^- - h_s^-)I < E_s - E_p$.*

Proof. We have

$$I\Delta^{p,+}_{a,b,\lambda,\mu}(h_p^- - h_s^-)I \leq I\Delta^{p,+}_{a,b,\lambda,\mu}(h_p^- - h_q^-)I + I\Delta^{p,+}_{a,b,\lambda,\mu}(h_q^- - h_s^-)I.$$

Since $p < q < s$, we know that $\psi_p(c)h_q^+ = \psi_q(c)$ for all $c \in F_q$, hence we can conclude

$$I\Delta^{p,+}_{a,b,\lambda,\mu}(h_p^- - h_q^-)I + I\Delta^{p,+}_{a,b,\lambda,\mu}(h_q^- - h_s^-)I = I\Delta^{p,+}_{a,b,\lambda,\mu}(h_p^- - h_q^-)I + I\Delta^{q,+}_{a,b,\lambda,\mu}(h_q^- - h_s^-)I$$

$$< E_q - E_p + E_s - E_q = E_s - E_p$$

as required. ∎

Claim 2.3.3.2. *If $a \in F_s$ then $I\Delta^{p,*}_a(h_p^- - h_s^-)I < E_s - E_p$.*

Proof. We have

$$I\Delta^{p,*}_a(h_p^- - h_s^-)I \leq I\Delta^{p,*}_a(h_p^- - h_q^-)I + I\Delta^{p,*}_a(h_q^- - h_s^-)I.$$

Since $p < q < s$, for all $c \in F_q$ we have $\psi_p(c)h_q^+ = \psi_q(c)$ and $h_q^-\psi_p(c) = h_q^-\psi_q(c)$, which entails $\psi_p(c)^* h_q^- = \psi_q(c)^* h_q^-$. Thus we conclude that

$$I\Delta^{p,*}_a(h_p^- - h_q^-)I + I\Delta^{p,*}_a(h_q^- - h_s^-)I = I\Delta^{p,*}_a(h_p^- - h_q^-)I + I\Delta^{q,*}_a(h_q^- - h_s^-)I < E_s - E_p,$$

as required. ∎

Claim 2.3.3.3. *If $a, b, ab \in F_s$ then $I\Delta^{p,\cdot}_{a,b}(h_p^- - h_s^-)I < E_s - E_p$.*

Proof. We have

$$I\Delta^{p,\cdot}_{a,b}(h_p^- - h_s^-)I \leq I\Delta^{p,\cdot}_{a,b}(h_p^- - h_q^-)I + I\Delta^{p,\cdot}_{a,b}(h_q^- - h_s^-)I < E_q - E_p + I\Delta^{p,\cdot}_{a,b}(h_q^- - h_s^-)I$$

Since $\psi_p(c)h_q^+ = \psi_q(c)$ for all $c \in F_q$ we get

$$(\psi_p(ab) - \psi_p(a)\psi_p(b))(h_q^- - h_s^-) = (\psi_q(ab) - \psi_p(a)\psi_q(b))(h_q^- - h_s^-)$$

and therefore

$$(\psi_p(ab) - \psi_p(a)\psi_p(b))(h_q^- - h_s^-) = \Delta^{q,\cdot}_{a,b}(h_q^- - h_s^-) + (\psi_q(a) - \psi_p(a))\psi_q(b)(h_q^- - h_s^-).$$

The rightmost term is zero since $\psi_q(b)\xi \in h_q^+[H]$ for all $\xi \in h_q^-[H]$ and $\psi_p(a)h_q = \psi_q(a)h_q$. This ultimately leads to the thesis since $I\Delta^{q,\cdot}_{a,b}(h_q^- - h_s^-)I < E_s - E_q$. ∎

This completes the proof. $\square$

2.3.2 Density and Countable Chain Condition

As in definition 2.3.2, for $F \Subset A$, let

$$L_F = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \exists \mu \in \mathbb{C}, \exists a, b \in F \text{ such that } \lambda a + \mu b \in F\}$$

and

$$J_F = \max\{\|a\| : a \in F\}.$$

For $p \in \mathbb{E}_A$, let

$$M_p = \max\{3\|a\|, 3\|\psi_p(a)\|, L_{F_p} : a \in F_p\}.$$

For $F \Subset A$ and $p \in \mathbb{E}_A$ let

$$M(p, F) = 3 \max\{3M_p + 1, L_F, 2J_F + 1\}.$$

Finally, for $p \in \mathbb{E}_A$ and a fixed promise (k_p, Φ_p) for the condition p, define the constants

$$N(p, \Phi_p) = \max\{\|(\psi_p(a) - \Phi_p(a))(h_p^+ - h_p^-)\| : a \in F_p\}$$

and

$$D(p, \Phi_p) = \min\{3\|a\|/2 - \|\psi_p(a) + \Phi_p(a)(1 - h_p^+)\| : a \in F_p\}.$$

Proposition 2.3.4. *Given $F \Subset A$, $E \in \mathbb{Q}^+$, $h \in \mathcal{C}$ and $R \in \mathcal{P}$, the set*

$$\Delta_{F,,h,R} = \{p \in \mathbb{E}_A : F_p \supseteq F, E_p \leq E, h_p \gg h, R_p \geq R\}$$

is open dense in $\mathbb{E}_A$.

Proof. It is straightforward to check that $\Delta_{F,,h,R}$ is open. Fix a condition

$$q = (F_q, E_q, h_q, R_q, \psi_q)$$

and let (k_q, Φ_q) be a promise for the condition q. By item 5c of definition 2.3.2 there is a δ such that

$$N(q, \Phi_q) < \delta < \frac{E_q}{3M_q}.$$

Fix moreover a *small enough* γ, more precisely such that

$$\gamma \leq \min\{E, E_q - 3M_q\delta, D(q, \Phi_q)\}.$$

Let $F_p = F_q \cup F \cup F^*$. Applying corollary 2.1.3, let Φ be a faithful essential unital representation of $C^*(F_p)$ such that

$$\|\Phi_{|F_q} - \Phi_{q|F_q}\| < \frac{\gamma}{36M}$$

with $M = M(q, F_p)$. Consider, by condition 1 of definition 2.3.1, an operator $k \in \mathcal{C}$ be such that $k \gg h$, $k \gg h_q$, $k \gg R_q$ and denote k^- by k_p. Let T be the finite-rank projection onto the space spanned by the set $\{\Phi(a)k[H] : a \in F_p\}$. By item 1 of definition 2.3.1, since $T \gg k$, we can pick $l \in \mathcal{C}$ such that $l \gg k$ and $l \approx_{\frac{\gamma}{18M}} T$. Moreover, by lemma 2.1.6, picking l closer to T if needed, there is a unitary $u \in \mathcal{U}(H)$ such that:

1. u is a compact perturbation of the identity,
2. $uT[H] \subseteq l[H]$,

3. u is the identity on $k_p[H]$ (since $l \gg k_p$),

4. $\|(\mathrm{Ad}(u)\Phi(a) - \Phi(a))k_p\| < \frac{\gamma}{36M}$ for all $a \in F_p$.

This entails that $\Phi' = \mathrm{Ad}(u) \circ \Phi$ is such that $\Phi'(a)k_p[H] \subseteq l[H]$ and

$$\|(\Phi'(a) - \Phi_q(a))k_p\| < \frac{\gamma}{18M}$$

for all $a \in F_q$. Let Q be the finite-rank projection onto the space spanned by the set $\{\Phi'(a)l[H] : a \in F_p\}$ and let K be the finite-rank operator equal to the identity on $l[H]$, equal to $\frac{1}{2}\mathrm{Id}$ on $Q(H) \cap l[H]^{\perp}$ (remember that $Q \geq l^+$ since $1 \in F_p$) and equal to zero on $Q[H]^{\perp}$. By item 1 of definition 2.3.1 there is $h_p \in C$ such that $h_p \gg l$ and $h_p \approx_{\frac{\gamma}{15M}} K$. Moreover, picking h_p closer to K if necessary we may assume that $\dim(h_pQ[H]) = \dim(Q[H])$ and that $h_p^- = l^+$. The first equality can be obtained with the argument exposed at the beginning of the proof of lemma 2.1.6, while the second is as follows. Suppose $\xi \in l[H]^{\perp}$ is a norm one vector, then $\xi = \xi_1 + \xi_2$, where ξ_1 and ξ_2 are orthogonal vectors of norm smaller than 1 such that $K\xi_1 = \frac{1}{2}\xi_1$ and $K\xi_2 = 0$. Hence, if h_p is close enough to K it follows that $\|h_p\xi\| < 1$. The equality $\dim(h_pQ[H]) = \dim(Q[H])$ allows us to find a unitary v such that

5. v is a compact perturbation of the identity,

6. v sends $Q[H]$ in $h_p[H]$,

7. v is the identity on $l[H]$.

The representation $\Phi_p = \mathrm{Ad}(v) \circ \Phi'$ is such that

8. $\Phi_p(a)k_p[H] \subseteq h_p^-[H]$ for all $a \in F_p$,

9. $\Phi_p(a)h_p^-[H] \subseteq h_p^+[H]$ for all $a \in F_p$,

10. $\|(\Phi_p(a) - \Phi_q(a))k_p\| < \frac{\gamma}{18M}$ for all $a \in F_q$.

Let $R_p \in P$ be such that $R_p \geq R, R_q$ and

$$\|(1 - R_p)\Phi_p(a)h_p^+\| < \frac{\gamma}{18M}$$

for all $a \in F_p$. Consider now, given $a \in F_q$, the operator

$$\phi(a) = \psi_q(a) + (1 - h_q^-)\Phi_p(a)(h_p^- - h_q^+) + (1 - h_q^-)R_p\Phi_p(a)(h_p^+ - h_p^-)$$

and for $a \in F_p \setminus F_q$ the operator

$$\phi(a) = \Phi_p(a)h_p^- + R_p\Phi_p(a)(h_p^+ - h_p^-).$$

For all $a \in F_p$ we have

$$\phi(a)k_p[H] \subseteq h_p^-[H]$$

and

$$\phi(a)h_p^-[H] \subseteq h_p^+[H],$$

moreover for $a \in F_q$ we also have

$$\phi(a)h_q^+ = \psi_q(a)$$

and

$$h_q^-\phi(a) = h_q^-\psi_q(a).$$

Let $\psi_p : F_p \to D_{R_p,h_p}$ be a function such that:

11. $\psi_p(1) = h^+_p$,

12. for all $a \in F_p$, $\psi_p(a) \approx_{\frac{\gamma}{18M}} \phi(a)$ and we also require that

    (a) $\psi_p(a)k_p[H] \subseteq h^-_p[H]$ for all $a \in F_p$,

    (b) $\psi_p(a)h^-_p[H] \subseteq h^+_p[H]$ for all $a \in F_p$,

    (c) $\psi_p(a)h^+_q = \psi_q(a)$ for all $a \in F_q$,

    (d) $h^-_q \psi_p(a) = h^-_q \psi_q(a)$ for all $a \in F_q$.

Such a function ψ_p exists because of the requirements on D_{R_p,h_p} we asked in items 2 and 3 of definition 2.3.1.

Claim 2.3.4.1. *For all $a \in F_p$ we have $\|(\psi_p(a) - \Phi_p(a))(h^+_p - k_p)\| < \frac{\gamma}{6M}$*

Proof. The inequality is trivially true for $a = 1$. For $a \in F_p \setminus F_q$ we have

$$\psi_p(a)(h^+_p - k_p) \approx_{\frac{\gamma}{18M}} \Phi_p(a)(h^-_p - k_p) + R_p\Phi_p(a)(h^+_p - h^-_p) \approx_{\frac{\gamma}{18M}} \Phi_p(a)(h^+_p - k_p),$$

where the last approximation is a consequence of

$$\|(1 - R_p)\Phi_p(a)h^+_p\| < \frac{\gamma}{18M}.$$

Now let $a \in F_q \setminus \{1\}$. Similarly to the previous case we get

$$\psi_p(a)(h^+_p - k_p) \approx_{\frac{\gamma}{9M}} (1 - h^-_q)\Phi_p(a)(h^+_p - k_p).$$

By definition we have $(h^+_p - h^+_q)\Phi_q(a)h^-_q = 0$. We use

$$\|(\Phi_p(a) - \Phi_q(a))k_p\| < \frac{\gamma}{18M}$$

and $k_p \geq h^-_q$ to infer that $(h^+_p - h^+_q)\Phi_p(a)h^-_q \approx_{\frac{\gamma}{18M}} 0$. Since F_q is self-adjoint, we also obtain that

$$h^-_q\Phi_p(a)(h^+_p - h^+_q) \approx_{\frac{\gamma}{18M}} 0.$$

This allows us to conclude that $\psi_p(a)(h^+_p - k_p) \approx_{\frac{\gamma}{6M}} \Phi_p(a)(h^+_p - k_p)$. $\square$

Claim 2.3.4.2. *For all $a \in F_p$ we have $\|\psi_p(a) + \Phi_p(a)(1 - h^+_p)\| < \frac{3}{2}\|a\|$.*

Proof. Let $a \in F_p \setminus F_q$. Then we have

$$\psi_p(a) + \Phi_p(a)(1 - h^+_p) \approx_{\frac{\gamma}{18M}} \Phi_p(a)h^-_p + R_p\Phi_p(a)(h^+_p - h^-_p) + \Phi_p(a)(1 - h^+_p) \approx_{\frac{\gamma}{18M}} \Phi_p(a),$$

hence the thesis follows since $\|\Phi_p(a)\| \leq \|a\|$ and we can assume $\gamma \leq \|a\|$. Consider now $a \in F_q$. Since in the previous claim we showed that

$$h^-_q\Phi_p(a)(h^+_p - h^+_q) \approx_{\frac{\gamma}{18M}} 0,$$

we have

$$\psi_p(a) + \Phi_p(a)(1 - h^+_p) \approx_{\frac{\gamma}{18M}} \phi(a) + \Phi_p(a)(1 - h^+_p) \approx_{\frac{\gamma}{9M}} \psi_q(a) + \Phi_p(a)(1 - h^+_q).$$

Recall that $\Phi_p = \mathrm{Ad}(w) \circ \Phi$, where w is a unitary which behaves like the identity on k_p (hence on h^+_q and R_q as well), thus $w(1 - h^+_q) = (1 - h^+_q)w$ and $\psi_q(a) = \mathrm{Ad}(w)(\psi_q(a))$ for all $a \in F_q$. Moreover Φ was defined so that

$$\|\Phi_{\restriction F_q} - \Phi_{q\restriction F_q}\| < \frac{\gamma}{36M}.$$

Therefore the following holds

$$\|\psi_q(a) + \Phi_p(a)(1 - h_q^+)\| = \|\psi_q(a) + \Phi(a)(1 - h_q^+)\| \approx_{\frac{\gamma}{36M}} \|\psi_q(a) + \Phi_q(a)(1 - h_q^+)\| < \frac{3}{2}\|a\|,$$

which implies the thesis since $\gamma \leq \|a\|$. □

This finally entails that, letting $E_p = \frac{\gamma}{6}$

$$p = (F_p, E_p, h_p, R_p, \psi_p)$$

is an element of $\Delta_{F,,h,R}$. It is in fact straightforward to check that $M_p \leq M = M(q, F_p)$ if γ is small enough. We are left with checking that $p < q$. Conditions 6-11 follow from the definition of p.

Claim 2.3.4.3. *For all $a, b, \lambda a + \mu b \in F_q$ we have $\|(\Delta^{p,+}_{a,b,\lambda,\mu})(h_p^- - h_q^-)\| < E_q - E_p$.*

Proof. Given $c \in F_q$ we have, by definition of δ (see the beginning of the proof), $\|(\psi_q(c) - \Phi_q(c))(h_q^+ - k_q)\| < \delta$, and the same is true if we replace $(h_q^+ - k_q)$ with $(h_p^- - h_q^-)$, since $(h_q^+ - k_q) \geq (h_p^- - h_q^-)$. Moreover, by definition of Φ_p, $\|(\Phi_p(c) - \Phi_q(c))k_p\| < \frac{\gamma}{18M}$ holds. This, along with the fact that F_q is self-adjoint, $\Phi_q(c)h_q^-[H] \subseteq h_q^+[H]$ (item 5f of definition 2.3.2) and $k_p \geq h_q^+$, entails that $\|h_q^-\Phi_p(c)(h_p - k_p)\| < \frac{\gamma}{18M}$. Therefore

$$(\Delta^{p,+}_{a,b,\lambda,\mu})(h_p^- - h_q^-) \approx_{\frac{\gamma}{6}} (\phi(\lambda a + \mu b) - \lambda\phi(a) - \mu\phi(b))(h_p^- - h_q^-) \approx_{3M\delta + \frac{\gamma}{3}} 0,$$

as required. □

Claim 2.3.4.4. *For all $a \in F_q$ we have $\|(\Delta^{p,*}_a)(h_p^- - h_q^-)\| < E_q - E_p$.*

Proof. Using approximations analogous to previous claim, we have that

$$\begin{aligned}(\Delta^{p,*}_a)(h_p^- - h_q^-) &\approx_{\frac{\gamma}{9}} (\phi(a^*) - \phi(a)^*)(h_p^- - h_q^-) \\ &\approx_{\delta + \frac{\gamma}{9}} (\Phi_p(a^*) - \psi_q(a)^* - (h_p^- - h_q^+)\Phi_p(a^*)(1 - h_q^-) \\ &\quad - (h_p^+ - h_p^-)\Phi_p(a^*)R_p(1 - h_q^-))(h_p^- - h_q^-)\end{aligned}$$

Since F_p is self-adjoint and by definition of R_p

$$\|h_p^+\Phi_p(c)(1 - R_p)\| < \frac{\gamma}{18M}$$

for all $c \in F_q$, thus $(h_p^+ - h_p^-)\Phi_p(a^*)R_p(1 - h_q^-) \approx_{\frac{\gamma}{18M}} (h_p^+ - h_p^-)\Phi_p(a^*)(1 - h_q^-)$. Hence we obtain

$$(\Delta^{p,*}_a)(h_p^- - h_q^-) \approx_{\delta + 5\frac{\gamma}{18}} (\Phi_p(a^*) - \psi_q(a)^* - (h_p^+ - h_q^+)\Phi_p(a^*)(1 - h_q^-))(h_p^- - h_q^-).$$

Furthermore we have

$$
\begin{aligned}
\psi_q(a)^*(h_p^- - h_q^-) = ((h_p^- - h_q^-)\psi_q(a))^* &= ((h_p^- - h_q^-)\psi_q(a)h_q^+)^* \\
&= ((h_p^- - h_q^-)\psi_q(a)(h_q^+ - k_q))^*,
\end{aligned}
$$

where the last equality is a consequence of $\psi_q(c)k_qH \subseteq h_q^- H$ for all $c \in F_q$ (item 5e of definition 2.3.2). Since

$$\|(\psi_q(c) - \Phi_q(c))(h_q^+ - k_q)\| < \delta, \ \|(\Phi_p(c) - \Phi_q(c))k_p\| < \frac{\gamma}{18M},$$

we get that

$$(\Delta_a^{p,*})(h_p^- - h_q^-) \approx_{2\delta+\frac{\gamma}{3}} \Phi_p(a^*)(h_p^- - h_q^-) - (h_p^+ - k_q)\Phi_p(a^*)(h_p^- - h_q^-).$$

Moreover, by how we defined Φ_p we have

$$\Phi_p(a^*)(h_p^- - h_q^-) = h_p^+\Phi_p(a^*)(h_p^- - h_q^-)$$

and

$$(1 - h_q^-)\Phi_p(c)k_q \approx_{\frac{\gamma}{18M}} (1 - h_q^-)\Phi_q(c)k_q = 0$$

for all $c \in F_q$. This last approximation entails, since F_q is self-adjoint, that

$$\|k_q\Phi_p(c)(1 - h_q^-)\| < \frac{\gamma}{18M}$$

for all $c \in F_q$. $\square$

Claim 2.3.4.5. *For all $a, b, ab \in F_q$ we have $\|(\Delta_{a,b}^{p,\cdot})(h_p^- - h_q^-)\| < \varepsilon_q - \varepsilon_p$.*

Proof. Similarly to the previous claims, we have the following approximations

$$(\Delta_{a,b}^{p,\cdot})(h_p^- - h_q^-) \approx_{\frac{\gamma}{6}} (\phi(ab) - \phi(a)\phi(b))(h_p^- - h_q^-) \approx_{2M_q\delta+\frac{2\gamma}{9}} \|(\Phi_p(ab) - \phi(a)\Phi_p(b))(h_p^- - h_q^-)\|.$$

As noted in the previous claim, for all $c \in F_q$ we have

$$\|k_q\Phi_p(c)(1 - h_q^-)\| < \frac{\gamma}{18M},$$

hence the same is true with $(h_p^- - h_q^-)$ in place of $(1 - h_q^-)$. Thus

$$\begin{aligned}\phi(a)\Phi_p(b)(h_p^- - h_q^-) &\approx_{\frac{\gamma}{18M}} \phi(a)(1 - k_q)\Phi_p(b)(h_p^- - h_q^-)\\ &\approx_{M_q\delta+\frac{\gamma}{6}} \Phi_p(a)(1 - k_q)\Phi_p(b)(h_p^- - h_q^-)\\ &\approx_{\frac{\gamma}{18M}} \Phi_p(a)\Phi_p(b)(h_p^- - h_q^-),\end{aligned}$$

as required. $\square$

This completes the proof. $\square$

Fix B a dense unital $(\mathbb{Q} + i\mathbb{Q})$-$*$-subalgebra of A with cardinality equal to the density character[7] of A. We define the family D as follows

$$D = \{\Delta_{F,\varepsilon,h,R} : F \Subset B, \varepsilon \in \mathbb{Q}^+, h \in C, R \in P\}.$$

Proposition 2.3.5. *Suppose there exists a $\mathcal{D}$-generic filter G for E_A. Then there exists a unital embedding Φ_G of A into the Calkin algebra.*

[7] The density character of a topological space X is defined as $\chi(X) = \min\{|D| : D \subseteq X \text{ dense}\}$

Proof. Let G be a D-generic filter and fix $a \in B$. The net $\{\psi_p(a)\}_{\{p\in G: a\in F_p\}}$ (indexed according to $(G, >)$, which is directed since G is a filter) is strongly convergent in $B(H)$. Indeed, given $q \in G$, $E > 0$ and $\xi_1, \ldots \xi_k$ norm one vectors in H, let $p \in G$ be such that $p < q$ and $h^+_p\xi_j \approx_{\frac{E}{3\|a\|/2}} \xi_j$ for $1 \leq j \leq k$ (which exists by genericity of G). Then, for all $s < p$ in G and $1 \leq j \leq k$ we have

$$\psi_s(a)\xi_j \approx \psi_s(a)h^+_p\xi_j = \psi_p(a)\xi_j.$$

Thus the net $\{\psi_p(a)\}_{\{p\in G: a\in F_p\}}$ strongly converges to a linear map from H to H, which is bounded since $\|\psi_p(a)\| < 3\|a\|/2$ for all $p \in G$. Let $\Phi_G : B \to Q(H)$ be the map $\pi \circ \Psi$.

Claim 2.3.5.1. *The map $\Phi_G : B \to Q(H)$ is a unital $*$-homomorphism of* $(\mathbb{Q} + i\mathbb{Q})$-*algebras. .*

Proof. For $a, b \in B$, we prove that $\Psi(ab) - \Psi(a)\Psi(b)$ is compact. Let $E > 0$ and pick $p \in G$ such that $a, b, ab \in F_p$ and $E_p < E$. We claim that

$$\|(\Psi(ab) - \Psi(a)\Psi(b))(1 - h^-_p)\| < E.$$

Suppose this fails, and let $\xi \in (1 - h^-_p)[H]$ be a norm one vector such that

$$\|(\Psi(ab) - \Psi(a)\Psi(b))\xi\| > E.$$

By genericity of G we can find $q \in G$ such that $q < p$ and

$$\|(\Psi(ab) - \Psi(a)\Psi(b))\eta\| > E,$$

where $\eta = h_q\xi$. Now let $s < q$ in G such that $\Psi(b)\eta$ is close enough to $h_s\Psi(b)\eta$ to obtain

$$\|(\psi_s(ab) - \psi_s(a)\psi_s(b))\eta\| > E.$$

This is a contradiction since $s < p$ implies

$$\|(\psi_s(ab) - \psi_s(a)\psi_s(b))(h^-_s - h^-_p)\| < E_p < E.$$

Similarly it can be checked that Φ_G is $(\mathbb{Q} + i\mathbb{Q})$-linear and self-adjoint. Moreover, Φ_G is bounded since Ψ is. The claim follows since Ψ maps the unit of A to the identity on H. $\square$

Extending Φ_G to the complex linear span of B, we obtain a unital, bounded $*$-homomorphism into the Calkin algebra. This is a dense (complex) $*$-subalgebra of A, hence we can uniquely extend to obtain a unital $*$-homomorphism from A into $Q(H)$, which is injective, since A is simple.

$\square$

Note that the fact that Φ_G above is bounded is crucial in allowing to extend it and obtain a $*$-homomorphism defined on all of the algebra A. To see how this can fail, the identity map on the (algebraic) group algebra of any non-amenable discrete group cannot be extended to a $*$-homomorphism from the reduced group C*-algebra to the universal one (see [BO08, Theorem 2.6.8]).

With the only part of theorem 2.0.3 remaining unproven being the fact that the poset is ccc, we begin with the following lemma yielding sufficient conditions for the compatibility of elements of $\mathbb{E}_A$.

Lemma 2.3.6. *Let $p, q \in \mathbb{E}_A$ be two conditions such that:*

1. *$h_p = h_q$ and $R_p = R_q$,*

2. *$\psi_p(a) = \psi_q(a)$ for all $a \in F_p \cap F_q$,*

3. *there exist unital $*$-homomorphisms $\Phi_p : C^*(F_p) \to B(H)$ and $\Phi_q : C^*(F_q) \to B(H)$ which are faithful and essential, and a projection k satisfying the following.*

   (a) The pairs (k, Φ_p) and (k, Φ_q) are promises for p and q, respectively.

   (b) There are δ_p and δ_q such that $N(p, \Phi_p) < \delta_p < \frac{E_p}{3M_p}$ and $N(q, \Phi_q) < \delta_q < \frac{E_q}{3M_q}$, and if

   $$\gamma \leq \min\{E_p - 3M_p\delta_p, D(p, \Phi_p), E_q - 3M_q\delta_q, D(q, \Phi_q)\}$$

   and

   $$M = \max\{M(p, F_p \cup F_q), M(q, F_p \cup F_q)\},$$

   then every $a \in F_p \cap F_q$ satisfies $\|\Phi_p(a) - \Phi_q(a)\| < \frac{\gamma}{18M}$.

   (c) There is a trivial embedding $\Theta : C^*(F_p \cup F_q) \to Q(H)$ such that $\pi \circ \Phi_p = \Theta_{|C^*(F_p)}$ and $\pi \circ \Phi_q = \Theta_{|C^*(F_q)}$.

Then p and q are compatible.

Proof. We suppress the notation and denote h_p by h, R_p by R and k_p by k. Let Φ be a faithful essential unital representation lifting Θ to $B(H)$. Since Φ_p and $\Phi_{|F_p}$ agree modulo the compacts, and Φ_q and $\Phi_{|F_q}$ agree modulo the compacts, there exists (by condition 1 of definition 2.3.1) $k \in C$ such that $k \gg h$, $k \gg R$, and in addition the following holds. For all $a \in F_p$ we have

$$\|(\Phi_p(a) - \Phi(a))(1 - k^-)\| < \frac{\gamma}{36M},$$

and for all $a \in F_q$ we have

$$\|(\Phi_q(a) - \Phi(a))(1 - k^-)\| < \frac{\gamma}{36M}.$$

We shall denote k^- by k_s. Arguing as in the first part of the proof of proposition 2.3.4 we can find $h_s \gg k_s$ in C and a unitary w such that:

1. w is a compact perturbation of the identity,

2. $wk_s = k_sw = k_s$,

and by letting $\Phi_p = (\mathrm{Ad}w) \circ \Phi_p$, $\Phi_q = (\mathrm{Ad}w) \circ \Phi_q$ and $\Phi = (\mathrm{Ad}w) \circ \Phi$, we also have that

3. $\|(\Phi_p(a) - \Phi_p(a))k_s\| < \frac{\gamma}{36M}$ for all $a \in F_p$,

4. $\|(\Phi_q(a) - \Phi_q(a))k_s\| < \frac{\gamma}{36M}$ for all $a \in F_q$,

5. $\|(\Phi(a) - \Phi(a))k_s\| < \frac{\gamma}{36M}$ for all $a \in F_p \cup F_q$,

6. $\Phi_p(a)k_s[H] \subseteq h_s^-[H]$ and $\Phi_p(a)h_s^-[H] \subseteq h_s^+[H]$ for all $a \in F_p$,

7. $\Phi_q(a)k_s[H] \subseteq h_s^-[H]$ and $\Phi_q(a)h_s^-[H] \subseteq h_s^+[H]$ for all $a \in F_q$,

8. $\Phi(a)k_s[H] \subseteq h_s^-[H]$ and $\Phi(a)h_s^-[H] \subseteq h_s^+[H]$ for all $a \in F_p \cup F_q$.

Let $R_s \in P$ be such that $R_s \geq R$ and for all $a \in F_p$ and all $b \in F_q$ we have

$$\|(1 - R_s)\Phi_p(a)h_s^+\| < \frac{\gamma}{18M},$$
$$\|(1 - R_s)\Phi_q(b)h_s^+\| < \frac{\gamma}{18M}.$$

Given $a \in F_p$, consider the operator

$$\phi(a) = \psi_p(a) + (1 - h^-)\Phi_p(a)(h_s^- - h^+) + (1 - h^-)R_s\Phi_p(a)(h_s^+ - h_s^-)$$

and for $a \in F_q \setminus F_p$

$$\phi(a) = \psi_q(a) + (1 - h^-)\Phi_q(a)(h_s^- - h^+) + (1 - h^-)R_s\Phi_q(a)(h_s^+ - h_s^-).$$

Define now the function $\psi_s : F_p \cup F_q \to D_s$ as an approximation of ϕ in the same way it was done in the proof of proposition 2.3.4. Suitably adapting the arguments in such proof to the present situation allows to show that

$$s = (F_p \cup F_q, \gamma/6, h_s, R_s, \psi_s)$$

is an element of E_A with promise (k_s, Φ). We follow the proof of claim 2.3.4.1 in order to check that the quantity $\|(\psi_s(a) - \Phi(a))(h_s^+ - k_s)\|$ is small enough for $a \in F_p \cup F_q$, using in addition that for all $a \in F_p$

$$\|(\Phi_p(a) - \Phi(a))(1 - k_s)\| < \frac{\gamma}{36M}$$

and that for all $a \in F_q$

$$\|(\Phi_q(a) - \Phi(a))(1 - k_s)\| < \frac{\gamma}{36M}.$$

This entails the same inequality between Φ_p and Φ (and between Φ_q and Φ) since the unitary w fixes k_s. The proofs of $s < p$ and $s < q$ go along the lines of those in claim 2.3.4.3, 2.3.4.4 and 2.3.4.5, keeping the following caveat in mind. It might happen, for instance, that p and q are such that $a \in F_p \cap F_q$ and $b, ab \in F_q \setminus F_p$. In this case $\Delta^{q,\cdot}_{a,b}(h_s^- - h_q^-)$ can be approximated (following the proof of claim 2.3.4.5) as $(\Phi_q(ab) - \Phi_p(a)\Phi_q(b))(h_s^- - h_q^-)$. This is where the condition $\Phi_p(a) \approx_{\frac{\gamma}{18M}} \Phi_q(a)$, required in item 3b of the statement of the present lemma, plays a key role, showing that the latter term is close to zero. The same argument applies for the analogous situations where Φ_p and Φ_q appear in the same formulas for the addition and the adjoint operation. ∎

Property K is a strengthening of the countable chain condition (see definition the beginning of section 2.1).

Proposition 2.3.7. *The poset E_A has property K and hence satisfies the countable chain condition.*

Proof. We prove that the poset E_A has property K, namely that any uncountable family of conditions has an uncountable subset of compatible conditions. Let $\{p_\alpha : \alpha < \aleph_1\}$ be a set of conditions[8] in E_A and for each $\alpha < \aleph_1$ fix a promise (k_α, Φ_α) for the condition p_α. By passing to an uncountable subset if necessary, we may assume $E_\alpha = E$, $h_\alpha = h$, $R_\alpha = R$, $k_\alpha = k$ for all $\alpha < \aleph_1$. An application of the Δ-system lemma yields a finite set $Z \subset A$

[8] We suppress the notation and denote F_{p_α} by F_a, $\mathbb{E}_{p_\alpha}$ by $\mathbb{E}_a$, etc.

such that $F_\alpha \cap F_\beta = Z$ for all $\alpha, \beta < \aleph_1$. Since Z is finite and $D_{R,h}$ is countable, we can furthermore assume that for all $\alpha, \beta < \aleph_1$ if $a \in F_\alpha \cap F_\beta$ then $\psi_\alpha(a) = \psi_\beta(a)$. Consider

$$F = \bigcup_{\alpha<\aleph_1} F_\alpha.$$

By [FHV17] there is a locally trivial embedding $\Theta : C^*(F) \to Q(H)$. For each $\alpha < \aleph_1$ fix a lift $\Theta_\alpha : C^*(F_\alpha) \to B(H)$ of $\Theta_{|C^*(F_\alpha)}$. Corollary 2.1.4 applied to Φ_α and Θ_α provides a faithful essential unital $\Phi_\alpha : C^*(F_\alpha) \to B(H)$ such that

1. $\Phi_\alpha(a) - \Theta_\alpha(a) \in K(H)$ for all $a \in F_\alpha$, hence $\pi \circ \Phi_\alpha = \Theta_{|C^*(F_\alpha)}$,
2. $\Phi_\alpha(a)h_\alpha^+ = \Phi_\alpha(a)h_\alpha^+$ for all $a \in F_\alpha$.

This entails that the pair (k_α, Φ_α) is still a promise for p_α. Hence, with no loss of generality, we can assume $\pi \circ \Phi_\alpha = \Theta_{|C^*(F_\alpha)}$ for every $\alpha < \aleph_1$. This in particular implies that

$$\Phi_\alpha(a) \sim_{K(H)} \Phi_\beta(a), \text{ for all } a \in Z$$

Fix an arbitrary $\gamma > 0$. We can assume that for all $\alpha, \beta \in \aleph_1$ and all $a \in F_\alpha \cap F_\beta$

$$\|\Phi_\alpha(a) - \Phi_\beta(a)\| < \gamma.$$

Indeed, start by fixing $\delta < \aleph_1$. Then for each $\alpha < \aleph_1$ there is $P_\alpha \in P$ such that

$$\|(\Phi_\alpha - \Phi_\delta)_{|Z}(1 - P_\alpha)\| < \gamma/5$$

and $R_\alpha \in P$ such that

$$\|(1 - R_\alpha)\Phi_{\alpha|Z}P_\alpha\| < \gamma/5.$$

By the pigeonhole principle there is an uncountable $U \subseteq \aleph_1$ such that $R_\alpha = R$ and $P_\alpha = P$ for all $\alpha \in U$. Since $RB(H)P$ is finite-dimensional we can also require that

$$\|R(\Phi_\alpha - \Phi_\beta)_{|Z}P\| < \gamma/5$$

for all $\alpha, \beta \in U$. Thus, for $a \in Z$, we have that:

$$\|\Phi_\alpha(a) - \Phi_\beta(a)\| \leq \|(\Phi_\alpha - \Phi_\beta)_{|Z}P\| + \|(\Phi_\alpha - \Phi_\delta)_{|Z}(1-P)\| + \|(\Phi_\beta - \Phi_\delta)_{|Z}(1-P)\| < \gamma.$$

Since the choice of γ in the claim is arbitrary, lemma 2.3.6 implies that we can pass to an uncountable subset in which any two conditions p_α and p_β are compatible. $\square$

Proof of Corollary 2.0.4. By proposition 2.1.1 it suffices to prove the statement for unital and simple C*-algebras. For any unital and simple C*-algebra A, the collection D of open, dense subsets of E_A (as defined prior to proposition 2.3.5) has cardinality equal to the density character of A. Since the poset E_A is ccc, this implies that if the density character of A is strictly less than $2^{\aleph_0}$, then Martin's axiom ensures the existence of a D-generic filter for E_A and the corollary follows by proposition 2.3.5. $\square$

2.4 Concluding Remarks on Theorem 2.0.3

It would be desirable to have a simpler forcing notion in place of E_A defined in the course of the proof of theorem 2.0.3. This would allow for an analysis of the names for C*-subalgebras of $Q(H)$ and better control of the structure of $Q(H)$ in the extension. In particular, it would be a step towards proving that a given C*-algebra can be 'gently placed' into $Q(H)$ (cf. [Woo84, p. 17-18]). In this regard, we conjecture the following.

Conjecture 2.4.1. Let A be an abelian and nonseparable C*-algebra. If the density character of A is greater than $2^{\aleph_0}$, then E_A forces that A does not embed into ℓ_∞/c_0.

We now propose related directions of study, taking inspiration from the commutative setting.

2.4.1 The Question of Minimality of Generic Embeddings

From the very beginnings of forcing, it has been known that a given partial ordering E can be embedded into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ by a ccc forcing. The simplest such forcing notion was denoted H_{E} and studied in [Far96] where it was proved that H_{E} embeds E into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ in a minimal way. If a cardinal $\kappa > 2^{\aleph_0}$ is such that E does not have a chain of order type κ or κ^*, then in the forcing extension $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ does not have chains of order type κ or κ^* (this is a consequence of [Far96, Theorem 9.1]). In addition, if $\min(\kappa, \lambda) > 2^{\aleph_0}$ and E does not have (κ, λ)-gaps[9] then in the forcing extension by H_{E} there are no (κ, λ)-gaps ([Far96, Theorem 9.2]) in V. We do not know whether analogous results apply to E_A or some variant thereof. In the noncommutative setting, the following question is even more natural.

Question 2.4.2. Consider the class $E = E(Q(H))$ of all C*-algebras that embed into the Calkin algebra. Can any notrivial closure properties of E be proved in ZFC? For example:

1. Do $A \in E$ and $B \in E$ together imply $A \otimes B$ in E (take the spatial tensor product, or even the algebraic tensor product)?

2. If $A_n \in E$ for $n \in \mathbb{N}$ and $A = \lim_n A_n$, is $A \in E$?

We conjecture that the answers to both 1 and 2 are negative. The analogous class E_{Fin} of all linear orderings that embed into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ does not seem to have any nontrivial closure properties provable in ZFC. For example, it is relatively consistent with ZFC that there exists a linear ordering L and a partition $\mathsf{L} = \mathsf{L}_1 \sqcup \mathsf{L}_2$ such that $\mathsf{L}_1 \in E_{\mathrm{Fin}}$ and $\mathsf{L}_2 \in E_{\mathrm{Fin}}$ but $\mathsf{L} \notin E_{\mathrm{Fin}}$ ([Far96, Proposition 1.4]).

2.4.2 Complete embeddings

Given a forcing notion P, its subordering P_0 is a *complete subordering* of P if for every generic filter $G \subseteq \mathsf{P}_0$ one can define a forcing notion P/G such that P is forcing equivalent to the two-step iteration $\mathsf{P}_0 * \mathsf{P}/G$ (for an intrinsic characterization of this relation see [Kun11, Definition III.3.65]).

A salient property of the forcing notion H_{E} (section 2.4.1) is that $\mathsf{E} \mapsto H_{\mathsf{E}}$ is a covariant functor from the category of partial orderings and order-isomorphic embeddings

[9]Given two cardinals κ and λ, a (κ,λ)-*gap* in a poset P is composed by a strictly increasing sequence $\{f_\alpha : \alpha < \kappa\} \subseteq \mathsf{P}$ and a strictly decreasing sequence $\{g_\beta : \beta < \lambda\} \subseteq \mathsf{P}$ such that $f_\alpha < g_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$, and moreover such that there is no $h \in \mathsf{P}$ greater than all f_α's and smaller than all g_β's.

as maps into the category of forcing notions with complete embeddings as morphisms. This is a consequence of [Far96, Proposition 4.2], where the compatibility relation in H_{E} has been shown to be 'local' in the sense that the conditions p and q are compatible in $H_{\mathrm{supp}(p)\cup\mathrm{supp}(q)}$ if and only if they are compatible in H_{E}.

Analogous arguments show that the mapping $\mathsf{B} \mapsto \mathsf{E}_{\mathsf{B}}$ defined on section 2.2.1 is a covariant functor from the category of Boolean algebras and injective homomorphisms into the category of ccc forcing notions with complete embeddings as morphisms. As a result, if D is a Boolean subalgebra of B and G is E_{D}-generic, then forcing with the poset E_{B} is equivalent to first forcing with E_{D} and then with $\mathsf{E}_{\mathsf{B}/G}$.

It is not difficult to prove that the association $A \mapsto \mathsf{QD}_A$ as in proposition 2.2.6 does not have this property, as QD_{C}, naturally considered as a subordering of $\mathsf{QD}_{M_2(\mathsf{C})}$, is not a complete subordering. More generally, if m is a proper divisor of n then the poset $\mathsf{QD}_{M_m(\mathsf{C})}$ is not a complete subordering of $\mathsf{QD}_{M_n(\mathsf{C})}$. We do not know whether there is an alternative definition of a functor $A \mapsto \mathsf{QD}_A$ that satisfies the conclusion of proposition 2.2.6. The latter remark also applies to the poset E_A given in theorem 2.0.3.

2.4.3 $2^{\aleph_0}$-universality

One line of research following the path opened with theorem 2.0.2, would be to understand which C*-algebras of density character $2^{\aleph_0}$ embed into the Calkin algebra. We recall from the beginning of section 2.1 that for a cardinal λ, a C*-algebra A is *(injectively) λ-universal* if it has density character λ and all C*-algebras of density character λ embed into A. The results in [FHV17] entail that the $2^{\aleph_0}$-universality of the Calkin algebra is independent from ZFC. On the one hand CH implies that $\mathcal{Q}(H)$ is $2^{\aleph_0}$-universal. Conversely, the proper forcing axiom implies that $\mathcal{Q}(H)$ is not $2^{\aleph_0}$-universal because some abelian C*-algebras of density $2^{\aleph_0}$ do not embed into it (see [Vig17a, Corollary 5.3.14 and theorem 5.3.15]; see also corollary 2.5.5). Can the Calkin algebra be $2^{\aleph_0}$-universal even when CH fails? The analogous fact for $P(\mathsf{N})/\mathrm{Fin}$ and linear orders, namely that there is a model of ZFC where CH fails and all linear orders of size $2^{\aleph_0}$ embed into $P(\mathsf{N})/\mathrm{Fin}$, has been proved in [Lav79] (see also [BFZ90] for the generalization to boolean algebras). We do not know whether these techniques can be generalized to provide a model in which CH fails and the Calkin algebra is a $2^{\aleph_0}$-universal C*-algebra, but the fact that E_A has property K is a step (possibly small) towards such a model. A poset with property K is *productively ccc*, in the sense that its product with any ccc poset is still ccc. A salient feature of the forcing iterations used in both [Lav79] and [BFZ90] is that they are not 'freezing' any gaps in $\mathsf{N}^{\mathsf{N}}/\mathrm{Fin}$ and $P(\mathsf{N})/\mathrm{Fin}$.[10]

Lemma 2.4.3. *For any C*-algebra A, the poset E_A cannot freeze any gaps in $P(\mathsf{N})/\mathrm{Fin}$.*

Proof. Every gap in $P(\mathsf{N})/\mathrm{Fin}$ or $\mathsf{N}^{\mathsf{N}}/\mathrm{Fin}$ that can be split without collapsing $\aleph_1$ can be split by a ccc forcing. This is well-known result of Kunen ([Kun76]) not so easy to find in the literature.[11] Therefore if a gap can be split by a ccc forcing P, then a poset which freezes it destroys the ccc-ness of P. But E_A has property K, and is therefore productively ccc. $\square$

While the gap spectra of $P(\mathsf{N})/\mathrm{Fin}$ and $\mathsf{N}^{\mathsf{N}}/\mathrm{Fin}$ are closely related, the gap spectrum of the poset of projections in the Calkin algebra is more complicated. The following

[10] A gap is 'frozen' if it cannot be split in a further forcing extension without collapsing $\aleph_1$.

[11] See e.g., [TF95, Fact on p. 76]. It is not difficult to see that a 'Suslin gap' as in [TF95, Definition 9.4] can be split by a natural ccc forcing whose conditions are finite K_0-homogeneous sets.

proposition was proved, but not stated, in [ZA14], and we include a proof for reader's convenience.

Theorem 2.4.4. *Martin's axiom implies that the poset of projections in the Calkin algebra contains a $(2^{\aleph_0}, 2^{\aleph_0})$-gap which cannot be frozen.*

Proof. By [ZA14, Theorem 4], there exists (in ZFC) a gap in this poset whose sides are analytic and σ-directed. This gap cannot be frozen, and Martin's axiom is used only to 'linearize' it. By the discussion following [ZA14, Corollary 2], each of the sides of this gap is Tukey equivalent to the ideal of Lebesgue measure zero sets ordered by the inclusion. Since the additivity of the Lebesgue measure can be increased by a ccc poset ([Kun11, Lemma III.3.28]), Martin's axiom implies that this gap contains an $(2^{\aleph_0}, 2^{\aleph_0})$-gap and that any further ccc forcing that increases the additivity of the Lebesgue measure will split the gap. $\square$

2.5 C*-algebras of Density Continuum

Given a model M of ZFC, it is generally extremely hard to identify the class of the C*-algebras of density continuum of M that embed into $Q(H)$. A preliminary and more reasonable task could be to focus on simple examples of C*-algebras of density $2^{\aleph_0}$ (e.g. group C*-algebras of groups of size $2^{\aleph_0}$, nonseparable UHF algebras, etc.), and see whether they consistently fail to embed into $Q(H)$ or not.

In this section we address this matter for some of specific example. In the first part, using a trick derived from Kunen's PhD thesis [Kun68], we show that, after adding any number of Cohen reals, there are no well-ordered increasing chains of projections in $Q(H)$ of size larger than the ground model continuum. This also allows us to present a simple model of ZFC where the Calkin algebra is not $\aleph_2$-universal and $2^{\aleph_0} \geq \aleph_2$ (see also [FHV17, Corollary 3.1]). In the second part of this section, with a simple application of the results in [HT05], we show that the reduced group C*-algebra generated by the free group on $2^{\aleph_0}$ generators embeds into $Q(H)$. Similarly, we use the fact that the full group C*-algebra generated by the free group F_r, for $r \in \mathbb{N}$, is residually finite-dimensional to show that $C^*_{\max}(F_{2^{\aleph_0}})$ also embeds into $Q(H)$. Finally, in the last subsection, we prove that $\bigotimes_{\alpha<2^{\aleph_0}} M_{n_\alpha}(\mathbb{C})$, as n_α varies in $\mathbb{N}$ and $M_{n_\alpha}(\mathbb{C})$ is the algebra of $n_\alpha \times n_\alpha$ complex matrices, embeds into $Q(H)$, regardless of the model of ZFC.

2.5.1 Isomorphic Names

Definition 2.5.1. Given a set of ordinals S, $(\mathbb{C}_S, \leq)$ is the set of all partial functions with finite domain from S to 2 with the order relation given by the extension.

When S is a cardinal κ, the previous definition gives the Cohen forcing adding a generic subset of κ. It is straightforward to check that the forcing notion adding κ Cohen reals can be identified with $\mathbb{C}_\kappa$.

The following fact about the poset $(\mathbb{N}^\mathbb{N}, \lesssim^*)$ is a well-known consequence of the contents of [Kun68, Section 12].[12]

Proposition 2.5.2. *In the generic extension given by $\mathbb{C}_\kappa$, there are no chains in $(\mathbb{N}^\mathbb{N}, \leq^*)$ of size bigger than the ground model continuum.*

[12]For $f, g \in \mathbb{N}^\mathbb{N}$, we write $f \leq^* g$ iff $f(n) \leq g(n)$ for all but finitely many n.

Proof. Let $\lambda = (2^{\aleph_0})^+$, let $\{f_\alpha : \alpha < \lambda\}$ be a set of names for reals and $p \in \mathbb{C}_\kappa$ forcing it to be a λ-chain in $\mathbb{N}^\mathbb{N}$. By [Kun11, IV.3.10] we can assume that for every $\alpha < \lambda$

$$f_\alpha = \bigcup \{\{(n,m)\} \times A^\alpha_{n,m} : n,m \in \mathbb{N}\},$$

where $A^\alpha_{n,m}$ is a maximal antichain of conditions q such that $q \Vdash f_\alpha(n) = m$. For each

$\alpha < \lambda$ we define the support of f_α as the subset of κ

$$\operatorname{supp}(f_\alpha) := \bigcup \{\operatorname{dom}(q) : q \in \cup_{n,m\in\mathbb{N}} A^\alpha_{n,m}\}.$$

Each of these supports is countable, hence by [Kun68, Lemma 12.6] we can assume there is a countable $S \subseteq \kappa$ containing the domain of p such that $\operatorname{supp}(f_\alpha) \cap \operatorname{supp}(f_\beta) \subseteq S$ for all $\alpha, \beta < \lambda$. In order to add a single f_α to the generic model we only need a countable iteration of Cohen's forcing, in particular f_α is added by a forcing which is isomorphic to $\mathbb{C}_S * \mathbb{C}_\alpha$, where $\mathbb{C}_S$ is the poset of all conditions in $\mathbb{C}_\kappa$ whose domain is contained in S, and $\mathbb{C}_\alpha$ is the poset of all conditions in $\mathbb{C}_\kappa$ whose domain is contained in $\operatorname{supp}(f_\alpha) \setminus S$. We remark that, modulo taking a subset of λ of cardinality λ, $\operatorname{supp}(f_\alpha) \setminus S$ is non-empty for every $\alpha < \lambda$. If that were not the case $\cup_{\alpha<\lambda}\operatorname{supp}(f_\alpha)$ would be countable, and there could be at most $2^{\aleph_0}$ different names in $\{f_\alpha : \alpha < \lambda\}$, which is a contradiction. Without loss of generality we assume that $\operatorname{supp}(f_\alpha) \setminus S$ has the same order type for all $\alpha < \lambda$, and therefore that all $\mathbb{C}_\alpha$'s are isomorphic. We can moreover assume that all f_α's correspond to the same name in $\mathbb{C}_S * \mathbb{C}_\alpha$, as there are at most $2^{\aleph_0}$ different names for reals in a countable iteration of Cohen's forcing. Given $\alpha < \beta < \lambda$, the bijection from κ to κ swapping $\operatorname{supp}(f_\alpha) \setminus S$ with $\operatorname{supp}(f_\beta) \setminus S$ induces an automorphism θ on $\mathbb{C}_\kappa$ and on the $\mathbb{C}_\kappa$-names (see [Jec03, Lemma 14.36]) which fixes p and switches f_α with f_β, thus on the one hand we have

$$p \Vdash f_\alpha <^* f_\beta,$$

on the other

$$\theta(p) \Vdash \theta(f_\alpha) <^* \theta(f_\beta) \Leftrightarrow p \Vdash f_\beta <^* f_\alpha,$$

which is a contradiction. $\square$

The proof we just exposed is rather flexible, in fact it can be used also to prove the following corollary.

Corollary 2.5.3. *Identify* $\mathbb{N}^\mathbb{N}$ *with the real numbers with their standard Borel structure, and let* $\preceq$ *be a Borel order on* $\mathbb{N}^\mathbb{N}$. *Then, in the generic extension given by* $\mathbb{C}_\kappa$, *there are no chains in* $(\mathbb{N}^\mathbb{N}, \preceq)$ *of size bigger than the ground model continuum.*

Proof. Any Borel subset of $\mathbb{N}^\mathbb{N}$ can be coded by a real $r \subseteq \mathbb{N}$. Repeat verbatim the proof of proposition 2.5.2 adding the support of the standard $\mathbb{C}_\kappa$-name of r to S (such support is countable). Because of this, r, and therefore the order $\preceq$, is fixed by the automorphism θ introduced in proposition 2.5.2. $\square$

The corollary above allows to generalize proposition 2.5.2 to chains of projections of the Calkin algebra as follows. First observe that all projections of $Q(H)$ lift to projections of $B(H)$ (see [FW12, Lemma 5.3]). Thus, in order to check that there are no λ-chains of projections in $Q(H)$, it is sufficient to prove that there are no λ-chains of projections in $B(H)$ for the order $\leq^*$ defined as

$$P =_{s^*} Q \Leftrightarrow P\,(1 - Q) \in K(H).$$

Second, the unit ball of $B(H)$ is an uncountable Polish space (i.e. a separable and completely metrizable topological space) when equipped with the strong topology, and the set of its projections is Borel, hence Borel isomorphic to the real numbers. Therefore, in order to show that after forcing with $\mathbb{C}_\kappa$ there are no well-ordered increasing chains of projections in $Q(H)$ of size larger than the ground model continuum, it is sufficient to show (thanks to corollary 2.5.3) that the order $\leq^*$ on the projections of $B(H)$ is Borel.

Proposition 2.5.4. *The order relation $\leq^*$ on the projections of $B(H)$ is Borel with respect of the strong operator topology.*

Proof. Fix a projection R and $\xi \in H$. Fix moreover an orthonormal basis $\{\xi_j\}_{j\in\mathbb{N}}$ of H and $\{\eta_k\}_{k\in\mathbb{N}}$ countable dense in the unit sphere of H. The set of all pairs of projections (P, Q) such that

$$\|PQ(1-R)\xi\| < \varepsilon$$

is open in $B(H) \times B(H)$ by continuity (in the strong topology) of the multiplication on bounded sets. We have that $P \leq^* Q$ if and only if $P(1-Q) \in K(H)$ if and only if for all $n \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\|P(1-Q)(1-R_N)\xi_k\| < 1/n,$$

where R_N is the projection onto the space spanned by $\{\xi_j : j \leq N\}$. The relation $\leq^*$ is therefore Borel. $\square$

Corollary 2.5.5. *In the generic extension given by $\mathbb{C}_\kappa$ there are no increasing chains of projections in $Q(H)$ of size bigger than the ground model continuum. In particular it is consistent with the failure of CH that the Calkin algebra is not $\aleph_2$-universal.*

Proof. A model of ZFC witnessing the second assertion can be obtained adding $\aleph_2$ Cohen reals to a model of CH. $\square$

2.5.2 Embedding $C^*_{\mathrm{red}}(F_{2^{\aleph_0}})$ into the Calkin Algebra

In the paper [HT05] the authors show that for $r \in \mathbb{N} \cup \{\aleph_0\}$ the C*-algebra $C^*_{\mathrm{red}}(F_r)$, i.e. the reduced C*-algebra generated by the free group with r generators, embeds into $\prod_{n\in\mathbb{N}} M_n(\mathbb{C})/\bigoplus_{n\in\mathbb{N}} M_n(\mathbb{C})$,[13] thus into the Calkin algebra. They prove in fact the following theorem.

Theorem 2.5.6 ([HT05, Theorem B]). *Let $\lambda : F_r \to B(\ell^2(F_r))$ be the left regular representation of the free group on r generators, with $r \in \mathbb{N} \cup \{\aleph_0\}$. Then there exists a sequence of unitary representations $\pi_n : F_r \to M_n$ such that for all $a_1, \ldots, a_k \in F_r$ and $c_1, \ldots, c_k \in \mathbb{C}$ the following holds*

$$\lim_{n\to\infty} \left\| \sum c_j \pi_n(a_j) \right\| = \left\| \sum c_j \lambda(a_j) \right\| .$$

The algebra $C^*_{\max}(F_2)$ (and similarly $C^*_{\max}(F_r)$ for every $r \in \mathbb{N}$ and $C^*_{\max}(F_\infty)$) is residually finite-dimensional, namely it has a faithful representation which is direct sum of finite-dimensional representation (see [Cho80, Theorem 7]). We have thus the analogous version of the theorem above.

[13] $\prod_{n\in\mathbb{N}} M_n(\mathbb{C})$ is the C*-algebra of all uniformly bounded sequence of matrix algebras, while

$_{n\in\mathbb{N}} M_n(\mathbb{C})$ is the ideal of $_{n\in\mathbb{N}} M_n(\mathbb{C})$ of the sequences converging to zero.

Lemma 2.5.7. *Let $\sigma : F_r \to B(H_u)$ be the universal representation of the free group on r generators, with $r \in \mathbb{N} \cup \{\aleph_0\}$. Then there exists a sequence of unitary representations $\pi^\sigma_n : F_r \to M_n$ such that for all $a_1, \dots, a_k \in F_r$ and $c_1, \dots, c_k \in \mathbb{C}$ the following holds*

$$\lim_{n\to\infty} \sum c_j \pi^\sigma_n(a_j) = \sum c_j \sigma(a_j) .$$

We remark that a crucial difference between the π^λ_n's and the π^σ_n's is that the latter can always be extended to representations of $C^*_{\max}(F_r)$, while the former do not extend to $C^*_{\mathrm{red}}(F_r)$ (this is the key point to show that $\mathrm{Ext}(C^*_{\mathrm{red}}(F_r))$ is not a group, see [HT05, Remark 8.6]).

We have therefore the following result.

Theorem 2.5.8. *Let $\lambda : F_{2^{\aleph_0}} \to B(\ell^2(F_{2^{\aleph_0}}))$ and $\sigma : F_{2^{\aleph_0}} \to B(H_u)$ be the left regular representation and the universal representation of the free group on $2^{\aleph_0}$ generators, respectively. For $\theta \in \{\lambda, \sigma\}$, there exists a sequence of unitary representations $\tau^\theta_n : F_{2^{\aleph_0}} \to M_{k(n)}$ such that for all $a_1, \dots, a_k \in F_{2^{\aleph_0}}$ and $c_1, \dots, c_k \in \mathbb{C}$ the following holds*

$$\lim_{n\to\infty} \sum c_j \tau^\theta_n(a_j) = \sum c_j \theta(a_j) .$$

Proof. For each $r \in \mathbb{N} \cup \{\aleph_0\}$, index the generators of F_{2^r} with the set of strings of 0's and 1's of length r. Fix $D_0 \subseteq D_1 \subseteq \dots$ an increasing countable sequence of finite subsets of $\mathbb{C}$ such that $\bigcup_{n\in\mathbb{N}} D_n$ is dense in $\mathbb{C}$. Given an element s in a certain free group F_r, we think it as a finite word whose letters are taken from the set of the generators of F_r and their inverses. The *length of* s is the length of its reduced form, i.e. the word representing s where no non-trivial simplifications are possible. For every $n \in \mathbb{N}$ fix a unitary representation $\pi^\theta_n : F_{2^n} \to M_{k(n)}$ given by theorem 2.5.6 and lemma 2.5.7 such that for all $a_1, \dots, a_n \in F_{2^n}$ of length at most n and $c_1, \dots, c_n \in D_n$ the following holds

$$\sum c_j \pi^\theta_n(a_j) \approx_{1/n} \sum c_j \theta_n(a_j) ,$$

where $\lambda_n : F_{2^n} \to B(\ell^2(F_{2^n}))$ and $\sigma_n : F_{2^n} \to B(H_n)$ are the left regular representation and the universal representation of F_{2^n} respectively, and $\theta_n \in \{\lambda_n, \sigma_n\}$. Given $n \in \mathbb{N}$, let moreover $\zeta_n : F_{2^{\aleph_0}} \to F_n$ be the group homomorphism which sends a_s to $a_{s|n}$. Define $\tau^\theta_n : F_{2^{\aleph_0}} \to M_{k(n)}$ as $\pi^\theta_n \circ \zeta_n$ for every $n \in \mathbb{N}$. Fix $\varepsilon > 0$, $a_1 \dots a_k \in F_{2^{\aleph_0}}$ and $c_1, \dots, c_k \in \mathbb{C}$. Pick $n \in \mathbb{N}$ big enough so that $n \geq k$, $1/n < \varepsilon/2$, c_j is approximated up to $\varepsilon/2k$ by some $c'_j \in D_n$ for all $j \leq k$, a_j has length smaller than n and ζ_n is injective when restricted to the set of all generators of $F_{2^{\aleph_0}}$ which (or whose inverses) appear in some a_j for $j \leq k$. By enlarging, if necessary, such set it is possible to define an injective group homomorphism $\eta_n : F_{2^n} \to F_{2^{\aleph_0}}$ which is a section of ζ_n. Thus by [BO08, Propositions 2.5.8-2.5.9] and the previous definitions we get

$$\sum c_j \theta(a_j) = \sum c_j \theta_n(\zeta_n(a_j)) \approx_\varepsilon \sum c_j \pi^\theta_n(\zeta_n(a_j)) = \sum c_j \tau^\theta_n(a_j) .$$

$\square$

We remark that when $\theta = \sigma$ in the proof above, all the maps τ^σ_n extend to representations of $C^*_{\max}(F_{2^{\aleph_0}})$, hence we also get the following corollary.

Corollary 2.5.9. *The* C**-algebra* $C^*_{max}(F_{2^{\aleph_0}})$ *is residually finite dimensional. In particular it embeds into* $B(H)$.

2.5.3 Embedding $\bigotimes_{\alpha<2^{\aleph_0}} M_{n_\alpha}(\mathbb{C})$ into the Calkin algebra

A nonseparable C*-algebra A is UHF if it is isomorphic to a tensor product of full matrix algebras (see [FK10], [FK15]). This subsection is devoted to prove the following proposition.

Proposition 2.5.10. *Let $(n_\alpha)_{\alpha<2^{\aleph_0}}$ be a $2^{\aleph_0}$-sequence of natural numbers. The C*-algebra $\bigotimes_{\alpha<2^{\aleph_0}} M_{n_\alpha}(\mathbb{C})$, where $M_{n_\alpha}(\mathbb{C})$ is the C*-algebra of $n_\alpha \times n_\alpha$ matrices with complex entries, embeds into $Q(H)$.*

Proof. Identify the elements of $2^{\aleph_0}$ with infinite sequences with entries in $\{0, 1\}$ and 2^n with the set of finite strings with entries in $\{0, 1\}$ of length n. For every $n \in \mathbb{N}$ define

$$H_n = \underbrace{\mathbb{C}^{n!} \otimes \cdots \otimes \mathbb{C}^{n!}}_{2^n \text{ times}}.$$

Decompose H as follows.

$$H = \bigoplus_{n\in\mathbb{N}} H_n.$$

Fix $\alpha < 2^{\aleph_0}$ and, for $a \in M_{n_\alpha}(\mathbb{C})$, let $\sigma_\alpha(a) \in B(H)$ be the operator acting as the identity on H_n if $n < n_\alpha$, and otherwise as:

$$\mathrm{Id}_{n!} \otimes \cdots \otimes \mathrm{Id}_{n!} \otimes (a \otimes \mathrm{Id}_{n!/n_\alpha}) \otimes \mathrm{Id}_{n!} \otimes \cdots \otimes \mathrm{Id}_{n!},$$

where $(a \otimes \mathrm{Id}_{n!/n_\alpha})$ appears in the position corresponding to $\alpha\restriction n$, the restriction of α to the first n entries.

The composition of σ_α with the quotient map $\pi : B(H) \to Q(H)$ is a unital embedding of $M_{n_\alpha}(\mathbb{C})$ into $Q(H)$. Moreover, given two different $\alpha, \beta < 2^{\aleph_0}$, the images of σ_α and σ_β commute on $\bigoplus_{n\geq k} H_n$, being k is the first coordinate where α and β differ. Thus the images of $\pi \circ \sigma_\alpha$ and $\pi \circ \sigma_\beta$ commute. Therefore there exists a $*$-homomorphism σ of $\bigotimes_{\alpha<2^{\aleph_0}} M_{n_\alpha}(\mathbb{C})$ into $Q(H)$ such that $\sigma\restriction M_{n_\alpha}(\mathbb{C}) = \sigma_\alpha$. Finally, σ is injective since it is unital and $\bigotimes_{\alpha<2^{\aleph_0}} M_{n_\alpha}(\mathbb{C})$ is simple. □

The next step in this setting would be to investigate whether the argument used in this proof can be adapted to $\bigotimes_{\alpha<2^{\aleph_0}} O_2$.

2.6 Voiculescu's Theorem for Nonseparable C*-algebras

In [Arv77], the author gave a proof of theorem 2.1.2 which (is different from the original one and) relies on the use of quasicentral approximate units of the compact operators. We recall, from the beginning of section 2.1, that, given a C*-algebra $A \subseteq B(H)$, an approximate unit $\{h_\lambda\}_{\lambda\in\Lambda}$ of $K(H)$ is *quasicentral* for A if $\lim_\lambda \|h_\lambda a - a h_\lambda\| = 0$ for all $a \in A$. The main point we want to make in this section is that the arguments used in the first two sections of [Arv77] to prove Voiculescu's theorem, are diagonalization arguments equivalent to applications of the Baire category theorem (lemma 2.1.7) to some appropriate ccc posets. This allows us to generalize Voiculescu's theorem as follows.

Theorem 2.6.1. *Assume MA. Let H, L be two separable Hilbert spaces, $A \subseteq B(H)$ a unital C*-algebra of density less than $2^{\aleph_0}$ and $\sigma : A \to B(L)$ a unital completely positive map such that $\sigma(a) = 0$ for all $a \in A \cap K(H)$. Then there is a sequence of isometries $V_n : L \to H$ such that $\sigma(a) - V_n^* a V_n \in K(L)$ and $\lim_{n\to\infty} \|\sigma(a) - V_n^* a V_n\| = 0$ for all $a \in A$.*

We recall that MA is consistent with $2^{\aleph_0}$ being as big as desired. On the other hand the spaces H and L are still assumed to be separable, hence the theorem applies only to separably representable C*-algebras.

It is known that for every C*-algebra $A \subseteq B(H)$ there is an approximate unit of the compact operators which is quasicentral for A (see [Arv77, Theorem 1 p.330]). Moreover, if A is separable, the quasicentral approximate unit can be chosen to be sequential. We start by showing how MA pushes this property to C*-algebras of density less than continuum. This is a simple fact, nevertheless it should give an idea of the flavor of this section and it should clarify, at least to the reader familiar with the proof of Voiculescu's theorem given in [Arv77], how to get to the proof of theorem 2.6.1.

Proposition 2.6.2. *Assume MA. Let $A \subseteq B(H)$ be a C*-algebra of density less than $2^{\aleph_0}$. Then there exists a sequential approximate unit $\{h_n\}_{n\in\mathbb{N}}$ of $K(H)$ which is quasicentral for A.*

Proof. Fix a countable dense K in $K(H)^1_+$ and B dense in A of size smaller than continuum. Let $\mathbb{P}$ be the set of tuples

$$p = (F_p, J_p, n_p, (h^p_j)_{j\leq n_p})$$

where $F_p \Subset A$, $J_p \Subset K(H)$, $n_p \in \mathbb{N}$ and $h^p_j \in K$ for all $j \leq n_p$. For $p, q \in \mathbb{P}$ we say $p < q$ if and only if

1. $F_q \subseteq F_p$,
2. $J_q \subseteq J_p$,
3. $n_q \leq n_p$,
4. $h^p_j = h^q_j$ for all $j \leq n_q$,
5. if $n_q < n_p$ then, for all $n_q < j \leq n_p$, all $k \in J_q$ and all $a \in F_q$, the following holds

$$\|[a, h^p_j]\| < 1/j, \ \|h^p_j k - k\| < 1/j.$$

The relation $<$ makes $\mathbb{P}$ a partial order which satisfies the ccc, since any two conditions p, q such that $n_p = n_q$ and $(h^p_j)_{j\leq n_p} = (h^q_j)_{j\leq n_q}$ are compatible (since there always exists a sequential approximate unit of $K(H)$ which is quasicentral for the C*-algebra generated by a finite subset of A). Let D be the collection of the sets

$$\Delta_{F,J,n} = \{p \in \mathbb{P} : F_p \supseteq F, J_p \supseteq J, n_p \geq n\},$$

where $F \Subset B$, $J \Subset K$ and $n \in \mathbb{N}$. The sets $\Delta_{F,J,n}$ are open dense because for every separable subalgebra of $B(H)$ there there is a sequential approximate unit of $K(H)$ which is quasicentral for it. A generic D-filter produces a sequential approximate unit of $K(H)$ which is quasicentral for A. Since D has size smaller than $2^{\aleph_0}$, MA guarantees the existence of such a filter. □

2.6.1 Finite Dimension

The following lemma is a preliminary step in the proof of Voiculescu's theorem in [Arv77], and it can be thought as a finite-dimensional version of Voiculescu's theorem.

Lemma 2.6.3 ([Arv77, Lemma p. 335])**.** *Let H be a separable, infinite-dimensional Hilbert space, $A \subseteq B(H)$ a separable unital C^*-algebra and $\sigma : A \to B(\mathbb{C}^m)$ a unital completely positive map such that $\sigma(a) = 0$ for all $a \in A \cap K(H)$. Then there is a sequence of isometries $V_n : \mathbb{C}^m \to H$ such that $\lim_{n\to\infty}\|\sigma(a) - V_n^* a V_n\| = 0$ for all $a \in A$. Moreover, given $L \subseteq H$ a finite-dimensional subspace, the isometries V_n can be chosen to have range orthogonal to L.*

This lemma is used in [Arv77] to carry on the argument in the infinite dimensional case, passing through block-diagonal maps. We follow the same path.

2.6.2 Block-Diagonal Maps

A completely positive map $\sigma : A \to B(L)$ is *block-diagonal* if there is a decomposition $L = \bigoplus_{n\in\mathbb{N}} L_n$, where L_n is finite-dimensional for all $n \in \mathbb{N}$, which induces a decomposition $\sigma = \bigoplus_{n\in\mathbb{N}} \sigma_n$ into completely positive maps $\sigma_n : A \to B(L_n)$. We use lemma 2.6.3 to prove theorem 2.6.1 in the case where σ is block-diagonal.

Lemma 2.6.4. *Assume MA. Let H, L be two separable Hilbert spaces, $A \subseteq B(H)$ a unital C^*-algebra of density less than $2^{\aleph_0}$ and $\sigma : A \to B(L)$ a block-diagonal unital completely positive map such that $\sigma(a) = 0$ for all $a \in A \cap K(H)$. Then there is a sequence of isometries $V_n : L \to H$ such that $\sigma(a) - V_n^* a V_n \in K(L)$ and $\lim_{n\to\infty}\|\sigma(a) - V_n^* a V_n\| = 0$ for all $a \in A$.*

Proof. By hypo thesis $L = \bigoplus_{n\in\mathbb{N}} L_n$, where L_n is finite-dimensional for all $n \in \mathbb{N}$, and σ decomposes as $\bigoplus_{n\in\mathbb{N}} \sigma_n$, where $\sigma_n(a) = 0$ whenever $a \in A \cap K(H)$ for all $n \in \mathbb{N}$. Let K be a countable dense subset of the unit ball of H such that, for every $\xi \in K$ the set $\{\eta \in K : \eta \perp \xi\}$ is dense in $\{\eta \in H : \|\eta\| = 1, \eta \perp \xi\}$. Let B be a dense subset of A of size smaller than $2^{\aleph_0}$ and fix an orthonormal basis $\{\xi_j^n\}_{j\le k_n}$ for each L_n. Consider the set $\mathbb{P}$ composed by tuples

$$p = (F_p, n_p, (W_i^p)_{i\le n_p}),$$

where F_p is a finite subset of A, $n_p \in \mathbb{N}$ and W_i^p is an isometry of L_i into H such that $W_i^p \xi_j^i \in K$ for every $j \le k_i$ and $i \le n_p$. We say $p \le q$ for two elements in $\mathbb{P}$ if and only if

1. $F_q \subseteq F_p$,
2. $n_q \le n_p$,
3. $W_i^p = W_i^q$ for all $i \le n_q$,
4. for $n_q < i \le n_p$ (if any) we require $W_i L_i$ to be orthogonal to $\{W_j L_j, aW_j L_j, a^* W_j L_j : j \le i, a \in F_q\}$ and
$$\|\sigma_i(a) - W_i^* a W_i\| < E/2^{i+1}$$
for all $a \in F_q$.

By lemma 2.6.3 two conditions p, q such that $n_p = n_q$ and $(W_i^p)_{i\le n_p} = (W_i^q)_{i\le n_q}$ are compatible, thus a standard uniformization argument entails that the poset $(\mathbb{P}, <)$ is ccc. Let D be the collection of the sets

$$\Delta_{F,n} = \{p \in \mathbb{P} : F_p \supseteq F, n_p \ge n\}$$

as F varies among the finite subsets of B and $n \in \mathbb{N}$. Again by lemma 2.6.3, $\Delta_{F,n}$ is open dense in $\mathbb{P}$. By MA, let G be a D-generic filter. Let V be the isometry from $\bigoplus L_n$

into H defined as $\bigoplus_{n\in\mathbb{N}} W_n$ where $W_n = W^p_n$ for some $p \in G$ such that $n_p \geq n$. The isometry is well defined since G is a filter. The proof that $\sigma(a) - V^* aV \in K(L)$ and that $\|\sigma(a) - V^* aV\| < \varepsilon$ for all $a \in A$ is the same as the first part of the proof of [Arv77, Theorem 4]. $\square$

Lemma 2.6.5. *Assume MA. Let H be a separable Hilbert space, A a unital C^*-algebra of density less than $2^{\aleph_0}$ and $\sigma : A \to B(H)$ a unital completely positive map. Then there is a block-diagonal completely positive map $\sigma : A \to B(L)$, where L is separable, and a sequence of isometries $V_n : H \to L$ such that $\sigma(a) - V_n^*\sigma(a)V_n \in K(H)$ and $\lim_{n\to\infty}\|\sigma(a) - V_n^*\sigma(a)V_n\| = 0$ for all $a \in A$.*

Proof. We use the same poset (and notation) defined in proposition 2.6.2 to generate an approximate unit of $K(H)$ which is quasicentral for $\sigma[A]$. Adjusting suitably the inequality in item 5 of the definition of the poset (see [Arv77, Lemma p.332]), by MA there is a generic filter of $\mathbb{P}$ which generates a quasicentral unit $(h_n)_{n\in\mathbb{N}}$ such that if $a \in F_p$ for some $p \in G$, then for all $n > n_p$ we have

$$\|[(h_{n+1} - h_n)^{1/2}, \sigma(a)]\| < \varepsilon/2^n .$$

From here, the proof is the same as in [HR00, Theorem 3.5.5]. $\square$

The proof of theorem 2.6.1 follows composing the isometries coming from lemmas 2.6.4 and 2.6.5.

Similarly to how is done in [HR00, Theorem 3.4.6], it is possible to obtain that the sequence $(V_n)_{n\in\mathbb{N}}$ in theorem 2.6.1 is composed of unitaries if σ is a $*$-homomorphism. We get therefore the following strengthening of corollaries 2.1.3 and 2.1.4.

Corollary 2.6.6. *Assume MA. Let A be a unital C^*-algebra of density less than $2^{\aleph_0}$ and let $\phi, \psi : A \to B(H)$ be two essential faithful unital representations. Then, for every $F \Subset A$ and $\varepsilon > 0$, there exists a unitary $u \in U(H)$ such that*

1. *$Ad(u) \circ \phi \sim_{K(H)} \psi$,*
2. *$Ad(u) \circ \phi(a) \approx_\varepsilon \psi(a)$ for all $a \in F$.*

Corollary 2.6.7. *Assume MA. Let A be a unital C^*-algebra of density less than $2^{\aleph_0}$ and let $\phi, \psi : A \to B(H)$ be two essential faithful unital representations. Then, for every $F \Subset A$ and every finite-dimensional subspace $K \subseteq H$, there exists a unitary $w \in U(H)$ such that*

1. *$Ad(w) \circ \phi \sim_{K(H)} \psi$,*
2. *$Ad(w) \circ \phi(a)(\xi) = \phi(a)(\xi)$ for every $a \in F$ and $\xi \in K$.*

In particular, the set

$$\{Ad(w) \circ \phi : w \in U(H), Ad(w) \circ \phi(a) \sim_{K(H)} \psi(a) \text{ for all } a \in A\}$$

has ϕ in its closure with respect to strong convergence.

2.6.3 Independece

Consider the following question.

Question 2.6.8. Is the thesis of theorem 2.6.1 (and corollaries 2.6.6 and 2.6.7) independent from ZFC, or is it true even without assuming MA?

A possible strategy to show that theorem 2.6.1 consistently fails without MA could revolve around the following proposition.

Proposition 2.6.9. *There exists a* C^**-algebra* M *of density character* $2^{\aleph_0}$ *which admits two essential faithful unital representations* ϕ, ψ *on a separable, infinite-dimensional Hilbert space* H*, such that there is no unitary* u *of* H *that satisfies* $Ad(u) \circ \phi \sim_{K(H)} \psi$.

Proof. Consider the diagonal embedding ϕ of $M = L^\infty([0,1])$ into $B(L^2([0,1]))$ mapping f to the operator M_f, which sends each $g \in L^2([0,1])$ to fg. Consider moreover the amplification of the diagonal embedding

$$\psi : M \to B(L^2([0,1]) \oplus L^2([0,1])) \cong B(L^2([0,1]))$$
$$f \mapsto (M_f, M_f)$$

Denote by Φ the composition of ϕ with π (the quotient map from $B(L^2([0,1]))$ onto the Calkin algebra), and by Ψ the composition of ψ with π. Although, by corollary 2.1.3, for every countable subset F of M there is a unitary transformation u of $L^2([0,1])$ such that $\pi(u^*)\Phi(f)\pi(u) = \Psi(f)$ for all $f \in C^*(F)$, there is no unitary transformation sending globally Φ to Ψ. The reason for this is that $\Phi[M]$ is a masa of the Calkin algebra (and so is every unitary transformation of it) while $\Psi[M]$ is not. $\square$

Starting from the algebra M given by the previous proposition, suppose there is a forcing extension of the universe where $2^{\aleph_0}$ is bigger than the ground model continuum, but no unitary transformation of $L^2([0,1])$ that sends Φ to Ψ is added. This would provide a model of ZFC answering question 2.6.8.[14]

[14] After the submission of this dissertation, it was shown in [Vac18b, Section 4] that indeed there exists of a model of ZFC where MA does not hold and the thesis of theorem 2.6.1 fails. The model is not obtained following the remark about proposition 2.6.9 given above, but via an application of Cohen's forcing and a simple cardinality argument.

Chapter 3

Obstructions to Lifting Abelian Subalgebras of Corona Algebras

Given a C*-algebra A, its *multiplier algebra* $M(A)$ is the largest unital C*-algebra containing A as an essential ideal (see [Bla06, Section II.7.3]). In the abelian case the multiplier algebra corresponds, via the Gelfand transform, to the Čech-Stone compactification of a locally compact Hausdorff space. The *corona algebra* $Q(A)$ of a C*-algebra A is the quotient $M(A)/A$. In this chapter we denote by π the canonical projection from $M(A)$ onto $Q(A)$. A *lifting* in $M(A)$ of a set $B \subseteq Q(A)$ is a set $A \subseteq M(A)$ (possibly of the same size of B) such that $\pi[A] = B$. The study of which properties of $B \subseteq Q(A)$ can be preserved in a lifting, and the analysis of the relations between B and its preimage $\pi^{-1}[B]$, have developed into a theory in its own right with strong connections with the study of stable relations in C*-algebras. A general introduction to this subject can be found in [Lor97].

This chapter focuses on liftings of abelian subalgebras of $Q(A)$, a topic which has been widely studied, for instance, as a mean to produce interesting examples (or counterexamples) of $*$-algebras and in the investigation of masas in the Calkin algebra. Before starting, we give a short list of references for the reader interested in some applications. Remarkably, a lot of these works (even the older ones) rely on combinatorial or diagonalization arguments of set-theoretic nature.

In [AD79] the authors, assuming the continuum hypothesis, produce a nonseparable C*-algebra A whose abelian subalgebras are all separable. The algebra A is a lifting in $\ell_\infty(M_2(\mathbb{C}))$ of an abelian subalgebra of $\ell_\infty(M_2(\mathbb{C}))/c_0(M_2(\mathbb{C}))$ generated by $\aleph_1$ orthogonal projections. Here $\ell_\infty(M_2(\mathbb{C}))$ is the C*-algebra of all countable bounded (in norm) sequences of elements of $M_2(\mathbb{C})$ and $c_0(M_2(\mathbb{C})) \subseteq \ell_\infty(M_2(\mathbb{C}))$ is the C*-algebra of those sequences which converge to zero. It was later shown that the continuum hypothesis is not necessary to prove the existence of nonseparable C*-algebra whose abelian subalgebras are all separable (see [BK17]; see also [Pop83, Corollary 6.7]).

Another example of a lifting result which was proven assuming the continuum hypothesis is due to Anderson in [And79]. The paper shows the existence of a masa (i.e. a maximal abelian subalgebra) of the Calkin algebra which is generated by its projections and which does not lift to a masa in $B(H)$. It is not known whether the continuum hypothesis is necessary to prove this fact (see also [SS11]).

More recently, the study of liftings led to the first example in [CFO14] (and its refinement in [Vig15]) of an amenable nonseparable Banach algebra which is not isomorphic to a C*-algebra. Once again, this algebra is the lift in $\ell_\infty(M_2(\mathbb{C}))$ of an abelian C*-algebra in $\ell_\infty(M_2(\mathbb{C}))/c_0(M_2(\mathbb{C}))$ of density $\aleph_1$. The problem of the existence of a separable Banach algebra which is not isomorphic to a C*-algebra is still open.

In this chapter we focus on the following problem. Let A be a noncommutative non-unital C*-algebra, and let B be a commutative family in $Q(A)$.What kind of obstructions could prevent the existence of a commutative lifting of B in $M(A)$? We consider collections with various properties, but our main concern and focus is the role played by the cardinality of the set we want to lift. The following table summarizes all the cases we are going to analyze. The symbols "✓" and "×" indicate whether it is possible or not to have a lifting for collections on the left column whose size is the cardinal in the top line. $Q(H)$ is, as in the previous chapter, the Calkin algebra on a separable Hilbert space H.

$Q(A) \to M(A)$	$< \aleph_0$	$\aleph_0$	$\aleph_1$
Commuting self-adjoint → Commuting self-adjoint	×	×	×
Commuting projections →Commuting projections	✓ in $Q(H)$	✓ in $Q(H)$	×
Commuting projections → Commuting positive	✓	✓	×
Orthogonal positive → Orthogonal positive	✓	✓	×
Orthogonal positive → Commuting positive	✓	✓	×

It is clear from the table that starting with an uncountable collection is a fatal obstruction. We also remark that the two columns in the middle, representing the lifting problem for finite and countable collections, have the same values. One reason for this phenomenon is that the obstructions in this scenario are all of K-theoretic nature and involve only a finite number of elements, as we shall see in the next paragraph (see also [Dav85]). This situation also relates to other compactness phenomena (at least at the countable level) that corona algebras of σ-unital algebras satisfy, due to their partial countable saturation (see [FH13]). Most of the results in the table about finite and countable families are already known ([Lor97], [FW12, Lemma 5.34], [Lor97, Lemma 10.1.12]). The main contribution of this paper concerns the right column, for which some theorems about projections in the Calkin algebra have already been proved ([FW12, Theorem 5.35], [BK17]).

Let A be $K(H)$, the algebra of the compact operators on a separable Hilbert space H, so that $M(A) = B(H)$ and $Q(A) = Q(H)$. By a well-known K-theoretic obstruction, the unilateral shift is a normal element in $Q(H)$ which does not lift to a normal element in $B(H)$ (more on this in [BDF77] and [Dav10]). An element is normal if and only if its real and imaginary part commute. This proves that it is not always possible to lift a couple of commuting self-adjoint elements in a corona algebra to commuting self-adjoint elements in the multiplier algebra.

One possible way to bypass this obstruction is to strengthen the hypotheses on the collection we start with. In [FW12, Lemma 5.34] it is proved that any countable family of commuting projections in the Calkin algebra can be lifted to a family of commuting projections in $B(H)$. Moreover, the authors provide a lifting of simultaneously diagonalizable projections. Proving a more general statement about liftings, in section 3.1 we show that any countable collection of commuting projections in a corona algebra can be lifted to a commutative family of positive elements in the multiplier algebra. We remark that it is not always possible to lift projections in a corona algebra to projections in the multiplier algebra. This occurs when $Q(A)$ has real rank zero but $M(A)$ has not, which is the case for instance if $A = Q(H) \otimes K(H)$ (see [Zha92, Example 2.7(iii)]) or $A = Z \otimes K(H)$, where Z is the Jiang-Su algebra (see [LN16]).

Two elements in a C*-algebra are *orthogonal* if their product is zero. Any countable family of orthogonal positive elements in a corona algebra admits a commutative lifting. This is a consequence of the more general result [Lor97, Lemma 10.1.12], which is relaid in this paper as proposition 3.1.2.

We cannot expect to be able to generalize verbatim the above result for uncountable families of orthogonal positive elements. This is the case since, by a cardinality obstruction, a multiplier algebra $M(A)$ which can be faithfully represented on a separable Hilbert space H, cannot contain an uncountable collection of orthogonal positive elements. The existence of such a collection in $M(A)$ (and thus in $B(H)$) would in fact imply the existence of an uncountable set of orthogonal vectors in H, contradicting the separability of H.

We could still ask whether it is possible to lift an uncountable family of orthogonal positive elements to a family of commuting positive elements. This leads to an obstruction of set-theoretic nature. In [FW12, Theorem 5.35], it is shown that there exists an $\aleph_1$-sized collection of orthogonal projections in the Calkin algebra whose uncountable subsets cannot be lifted to families of simultaneously diagonalizable projections in $B(H)$. This result is refined in [BK17, Theorem 7], where the authors provide an $\aleph_1$-sized set of orthogonal projections in $Q(H)$ which contains no uncountable subset that lifts to a collection of commuting operators in $B(H)$. The main result of this paper is a generalization of [BK17, Theorem 7].

Theorem 3.0.1. *Assume A is a primitive, non-unital, σ-unital C^*-algebra. Then there is a collection of $\aleph_1$ pairwise orthogonal positive elements of $Q(A)$ containing no uncountable subset that simultaneously lifts to commuting elements in $M(A)$.*

Corollary 3.0.2. *Assume A is a primitive, real rank zero, non-unital, σ-unital C^*-algebra. Then there is a collection of $\aleph_1$ pairwise orthogonal projections of $Q(A)$ containing no uncountable subset that simultaneously lifts to commuting elements in $M(A)$.*

The proof of theorem 3.0.1 is inspired by the combinatorics used in [BK17] and [FW12], which goes back to Luzin and Hausdorff and the study of uncountable almost disjoint families of subsets of $\mathbb{N}$ and Luzin's families (see [Luz47]). We remark that no extra set theoretic assumption (such as the continuum hypothesis) is required in our proof.

The chapter is structured as follows: in section 3.1 we report the results needed to settle the problem for liftings of countable families of commuting projections and of orthogonal positive elements. Section 3.2 is devoted to the proof of theorem 3.0.1. In section 3.3 we introduce a reflection (in the set-theoretic sense) problem related to this topic and a partial solution to it.

3.1 Countable Collections

In [FW12, Lemma 5.34] Farah and Wofsey prove that any countable set of commuting projections in the Calkin algebra can be lifted to a set of simultaneously diagonalizable projections in $B(H)$. The thesis in the following proposition is weaker, but it holds in a more general context.

Proposition 3.1.1. *Let $\phi : A \to B$ be a surjective $*$-homomorphism between two C^*-algebras and let $\{p_n\}_{n\in\mathbb{N}}$ be a collection of commuting projections of B. Then there exists a set $\{q_n\}_{n\in\mathbb{N}}$ of commuting positive elements of A such that $\phi(q_n) = p_n$.*

Proof. We can assume that both A and B are unital, that $\phi(1_A) = 1_B$ and that $1_B \in \{p_n\}_{n\in\mathbb{N}}$. Let $C \subseteq B$ be the abelian C^*-algebra generated by the set $\{p_n\}_{n\in\mathbb{N}}$. Consider the element

$$b = \sum_{n\in\mathbb{N}} \frac{2p_n - 1}{3^n}.$$

Let X be the spectrum of b in A. The algebra C is generated by b (see [Ric60, p. 293] for a proof), thus $C \cong C(X)$. Fix $a \in A$ such that $\phi(a) = b$. The element $(a+a^*)/2$ is still in the preimage of b since b is self-adjoint, thus we can assume $a \in A_{sa}$. If Y is the spectrum of a, we have in general that $X \subseteq Y$. Fix $f_n \in C(X)_+$ such that $f_n(b) = p_n$. Since the range of f_n is contained in $[0, 1]$ and the spaces Y and X are compact and Hausdorff, by the Tietze extension theorem ([Wil70, Theorem 15.8]), for every $n \in \mathbb{N}$, there is a continuous $F_n : Y \to [0, 1]$ such that $F_n \restriction_X = f_n$. Set $q_n = F_n(a)$. The map ϕ acts on $C(Y)$ as the restriction on X (here we identify $C^*(a)$ and $C^*(b)$ with $C(Y)$ and $C(X)$ respectively), therefore $\phi(q_n) = p_n$ for every $n \in \mathbb{N}$. $\square$

The q_n's can be chosen to be projections if there is a self-adjoint a in the preimage of b whose spectrum is X. By the Weyl-von Nuemann theorem, this is the case when ϕ is the quotient map from $B(H)$ onto the Calkin algebra (see [Dav96, Theorem II.4.4]).

We focus now on lifting sets of orthogonal positive elements, starting with a set of size two. Let therefore $\phi : A \to B$ be a surjective $*$-homomorphism of C*-algebras, and let b_1, $b_2 \in B_+$ be such that $b_1 b_2 = 0$. Consider the self-adjoint $b = b_1 - b_2$ and let $a \in A$ be a self-adjoint such that $\phi(a) = b$. Then the positive and negative part of a are two orthogonal positive elements such that $\phi(a_+) = b_1$, $\phi(a_-) = b_2$. The situation is not much different when dealing with countable collections, as shown in [Lor97, Lemma 10.1.12]. We report here the full proof.

Proposition 3.1.2 ([Lor97, Lemma 10.1.12]). *Assume $\phi : A \to B$ is a surjective $*$-homomorphism between two* C**-algebras. Let $\{b_n\}_{n\in\mathbb{N}}$ be a collection of orthogonal elements in $B_+^{\leq 1}$. Then there exists a set $\{a_n\}_{n\in\mathbb{N}}$ of orthogonal elements in $A_+^{\leq 1}$ such that $\phi(a_n) = b_n$.*

Proof. Define for $j \in \mathbb{N}$

$$c_j = \sum_{i\geq j} 2^{-i} b_i$$

For each $j \in \mathbb{N}$, let C_j be the hereditary C*-algebra $c_j B c_j$. We have therefore that

1. $b_i \in C_j$ for $i \geq j$;
2. $b_i c_j = 0$ for $i < j$.

Start lifting b_1 and c_2 to two orthogonal positive elements in A, call them a_1 and d_2 respectively. Let D_2 be the hereditary subalgebra generated by d_2 in A. Notice that a_1 is orthogonal to every element in D_2 and that $\pi[D_2] = C_2$. Consider now b_2 and c_3, which belong to C_2. Lift them to two orthogonal positive elements in D_2. Call these lifts a_2 and d_3 respectively. The elements a_1, a_2 and d_3 are orthogonal. Let D_3 be the hereditary subalgebra generated by d_3 in A and iterate this procedure. $\square$

3.2 Uncountable Collections

Throughout this section, let A be a σ-unital non-unital primitive C*-algebra. A C*-algebra is *σ-unital* if it admits a countable approximate unit, and it is *primitive* if it admits a faithful irreducible representation. We can thus assume that A is a noncommutative strongly dense C*-subalgebra of $B(H)$ for a certain Hilbert space H. A sequence of operators $\{x_n\}_{n\in\mathbb{N}}$ *strictly converges* to $x \in B(H)$ if and only if $x_n a \to xa$ and $a x_n \to ax$ in norm for all $a \in A$. In this scenario $M(A)$ can be identified with the idealizer

$$\{x \in B(H) : xA \subseteq A, Ax \subseteq A\}$$

or with the strict closure of A in $B(H)$ ([Bla06, II.7.3.5]). Given two elements a, b in a C*-algebra A, we denote the commutator $ab - ba$ by $[a, b]$. Moreover, from now on denote by $(e_n)_{n\in\mathbb{N}}$ an approximate unit of A such that:

1. $e_0 = 0$,
2. $\|e_i - e_j\| = 1$ for $i \neq j$,
3. $e_ie_j = e_i$ (i.e. $e_j \gg e_i$) for every $i < j$.

Such an approximate unit exists since A is σ-unital, as proved in [Ped90, Section 2].

The proof of theorem 3.0.1 follows closely the one given by Bice and Koszmider for [BK17, Theorem 7], and a lemma similar to [BK17, Lemma 6] is required.

Lemma 3.2.1. *Let A be a primitive, non-unital, σ-unital* C**-algebra. There exists a family $(a_\beta)_{\beta\in\aleph_1} \subseteq M(A)_+ \setminus A$ such that:*

1. *$\|a_\beta\| = 1$ for all $\beta \in \aleph_1$;*
2. *$a_\alpha a_\beta \in A$ for all distinct $\alpha, \beta \in \aleph_1$;*
3. *given $d_1, d_2 \in M(A)$, for all $\beta \in \aleph_1$, all $n \in \mathbb{N}$, and all but finitely many $\alpha < \beta$:*

$$\|[(a_\alpha + d_1e_n), (a_\beta + d_2e_n)]\| \geq \frac{1}{8}.$$

The rough idea to prove this lemma is to build, for every $\beta < \aleph_1$, a strictly increasing function $f_\beta : \mathbb{N} \to \mathbb{N}$ and a norm-bounded sequence $\{c^\beta_k\}_{k\in\mathbb{N}} \subseteq A_+$ to define

$$a_\beta = \sum_{k\in\mathbb{N}} (e_{f_\beta(2k+1)} - e_{f_\beta(2k)})^{\frac{1}{2}} c^\beta_k (e_{f_\beta(2k+1)} - e_{f_\beta(2k)})^{\frac{1}{2}}.$$

Note that this series belongs to $M(A)$ by [Ped90, Theorem 4.1] (see also [FH13, item(10) p.48]). In order to satisfy the thesis of the lemma, we build each c^β_k so that, for some $\alpha < \beta$ and some $n \in \mathbb{N}$, the following holds

$$\|[(a_\alpha + e_n), (c^\beta_k + e_n)]\| \geq \frac{1}{8}.$$

The choice of f_β will guarantee orthogonality in $Q(A)$ exploiting, for $n_2 < n_1 < m_2 < m_1$, the following fact:

$$(e_{m_1} - e_{m_2})(e_{n_1} - e_{n_2}) = 0.$$

The main ingredient used to build c^β_k is Kadison's transitivity theorem, which we are allowed to use since A is primitive.

Proof of lemma 3.2.1. Since the C*-algebra A is primitive, we can assume that there is a Hilbert space H such that $A \subseteq B(H)$ and A acts irreducibly on H. For each $n < m$, denote the space $(e_m - e_n)H$ by $S_{n,m}$. We start by building a_0. Let $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(n) = \begin{cases} 2^{n+1} - 1 & \text{if } n \text{ is even} \\ 2^n & \text{if } n \text{ is odd.} \end{cases}$$

For every $k \in \mathbb{N}$ there is a unit vector ξ in the range of $e_{f(2k+1)} - e_{f(2k)}$. By the definition of the approximate unit $(e_n)_{n\in\mathbb{N}}$, the vector ξ is a 1-eigenvector of $e_{f(2k+2)}$. This, along with the (algebraic) irreducibility of $A \subseteq B(H)$, entails that

$$AS_{f(2k+1),f(2k)} = H.$$

Denote the algebra $(e_{f(2k+1)} - e_{f(2k)})A(e_{f(2k+1)} - e_{f(2k)})$ by A_k. We have that

$$A_kH \supseteq S_{f(2k),f(2k+1)}.$$

Let $\xi_k^0, \eta_k^0 \in S_{f(2k),f(2k+1)}$ be two orthogonal[1] norm one vectors. Since A acts irreducibly on H and A_k is a hereditary subalgebra of A, it follows that A_k acts irreducibly on $B(A_kH)$ (see [Mur90, Theorem 5.5.2]). Therefore, by Kadison's transitivity theorem, we can find a self-adjoint $c_k^0 \in A_k$ such that

$$\begin{aligned} c_k^0(\xi_k^0) &= \xi_k^0, \\ c_k^0(\eta_k^0) &= 0, \end{aligned}$$

and $\|c_k^0\| = 1$. We can suppose that c_k^0 is positive by taking its square, doing so will not change its norm nor the image of ξ_k^0 and η_k^0. Consider the function

$$f_0(n) = \begin{cases} f(n) - 1 & \text{if } n \text{ is even} \\ f(n) + 1 & \text{if } n \text{ is odd.} \end{cases}$$

We have that

$$\begin{aligned} e_{f_0(2k+1)}c_k^0 &= c_k^0 e_{f_0(2k+1)} = c_k^0, \\ e_{f_0(2k)}c_k^0 &= c_k^0 e_{f_0(2k)} = 0. \end{aligned}$$

This entails

$$(e_{f_0(2k+1)} - e_{f_0(2k)})c_k^0 = c_k^0 = c_k^0(e_{f_0(2k+1)} - e_{f_0(2k)})$$

and therefore also

$$c_k^0 = (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2} c_k^0 (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2}.$$

The norm $\|c_k^0\|$ is bounded by 1 for every $k \in \mathbb{N}$, therefore the sum

$$a_0 = \sum_{k\in\mathbb{N}} c_k^0 = \sum_{k\in\mathbb{N}} (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2} c_k^0 (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2}$$

is strictly convergent (see [Ped90, Theorem 4.1] or [FH13, Item (10) p.48]), hence $a_0 \in M(A)_+$. Furthermore:

$$\begin{aligned} \|a_0\| &= \Big\| \sum_{k\in\mathbb{N}} (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2} c_k^0 (e_{f_0(2k+1)} - e_{f_0(2k)})^{1/2} \Big\| \leq \\ &\leq \Big\| \sum_{k\in\mathbb{N}} e_{f_0(2k+1)} - e_{f_0(2k)} \Big\| \leq 1. \end{aligned}$$

In order to show that $a_0 \notin A$, first observe that

$$a_0(\xi_k^0) = \sum_{m<k} c_m^0(\xi_k^0) + c_k^0(\xi_k^0) + \sum_{m>k} c_m^0(\xi_k^0) = c_k^0(\xi_k^0) = \xi_k^0.$$

The first sum annihilates since $\xi_k^0 \in S_{f(2k),f(2k+1)}$ implies $\xi_k^0 = (e_{f_0(2k+1)} - e_{f_0(2k)})(\xi_k^0)$, and for $m < k$

$$c_m^0(e_{f_0(2k+1)} - e_{f_0(2k)})(\xi_k^0) = c_m^0 e_{f_0(2m+1)}(e_{f_0(2k+1)} - e_{f_0(2k)})(\xi_k^0) = 0,$$

[1] We can always assume $S_{n,n+1}$ has at least 2 linearly independent vectors for each $n \in \mathbb{N}$ by taking, if

necessary, a subsequence $(e_{k_j})_{j\in\mathbb{N}}$ from the original approximate unit.

which follows by $f_0(2m+1) < f_0(2k) < f_0(2k+1)$. The second series also annihilates, indeed for $m > k$ we have $c^0_n e_{f_0(2k+1)} = c^0_n e_{f_0(2m)} e_{f_0(2k+1)} = 0$ (the same equation also holds for $e_{f_0(2k)}$). Using the same argument, it can be proved that

$$a_0(\xi) = c^0_n(\xi)$$

for every $\xi \in S_{f_0(2n), f_0(2n+1)}$. Observe that $\|(a_0 - e_{f_0(2m+1)} a_0)(\xi^0_k)\| = 1$ for $k > m$, thus $a_0 \notin A$.

The construction proceeds by transfinite induction on $\aleph_1$, the first uncountable cardinal. At step $\beta < \aleph_1$ we assume to have a sequence of elements $(a_\alpha)_{\alpha<\beta}$ in $M(A)_+$ and functions $(f_\alpha)_{\alpha<\beta}$ such that:

i. For all $\alpha < \beta$ the function $f_\alpha : \mathbb{N} \to \mathbb{N}$ is strictly increasing and, given any other $\gamma < \alpha$, for all $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $j > N$ and all $i \in \mathbb{N}$ the following holds

$$|f_\alpha(j) - f_\gamma(i)| > 2^k.$$

Furthermore, we ask that for all $\alpha < \beta$ and all $k \in \mathbb{N}$:

$$f_\alpha(2(k+1)) - f_\alpha(2k+1) > 2^{2k+1}.$$

ii. For each $\alpha < \beta$ there exists a sequence $(c^\alpha_k)_{k\in\mathbb{N}}$ of positive norm 1 elements in A such that

$$a_\alpha = \sum_{k\in\mathbb{N}} c^\alpha_k.$$

Moreover we require that

$$e_{f_\alpha(2k+1)} c^\alpha_k = c^\alpha_k e_{f_\alpha(2k+1)} = c^\alpha_k,$$
$$e_{f_\alpha(2k)} c^\alpha_k = c^\alpha_k e_{f_\alpha(2k)} = 0,$$

and that there exist $\xi^\alpha_k, \eta^\alpha_k \in S_{f_\alpha(2k), f_\alpha(2k+1)}$, two norm one orthogonal vectors, such that $c^\alpha_k(\xi^\alpha_k) = \xi^\alpha_k$ and $c^\alpha_k(\eta^\alpha_k) = 0$.

iii. Given $\alpha < \beta$ and $d_1, d_2 \in M(A)$, for all $l \in \mathbb{N}$, and for all but possibly l many $\gamma < \alpha$ the following holds:

$$\|[(a_\alpha + d_1 e_l), (a_\gamma + d_2 e_l)]\| \geq \frac{1}{2}.$$

It can be shown, as we already did for a_0, that for all $\alpha < \beta$:

a. $a_\alpha \in M(A)_+ \setminus A$;

b. $\|a_\alpha\| = 1$;

c. $a_\alpha(\xi) = c^\alpha_k(\xi) \in S_{f_\alpha(2k), f_\alpha(2k+1)}$ for every $\xi \in S_{f_\alpha(2k), f_\alpha(2k+1)}$.

Moreover, by items (i)-(ii), along with the fact that for $n_2 < n_1 < m_2 < m_1$

$$(e_{m_1} - e_{m_2})(e_{n_1} - e_{n_2}) = 0,$$

we have that $a_\alpha a_\gamma \in A$ for all $\alpha, \gamma < \beta$.

We want to find f_β and a_β such that the families $\{a_\alpha\}_{\alpha<\beta+1}$ and $\{f_\alpha\}_{\alpha<\beta+1}$ satisfy the three inductive hypotheses. This will be sufficient to continue the induction and to obtain the thesis of the lemma. Since β is a countable ordinal, the sequence $(a_\alpha)_{\alpha<\beta}$ is either finite or can be written as $(a_{\alpha_n})_{n<\mathbb{N}}$, where $n \mapsto \alpha_n$ is a bijection between $\mathbb{N}$ and β. We assume that β is infinite, since the finite case is easier. In order to ease the notation, we shall denote a_{α_n} by a_n (and similarly f_{α_n} by f_n, $c^k_{\alpha_n}$ by c^k_n, etc.).

The construction of a_β proceeds inductively on the set $\{(i,j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}$ ordered along with any well-ordering of type ω such that $(i, j) \leq (i', j')$ implies $j \leq j'$, like for example

$$(i, j) \leq (i', j') \iff j \leq j' \text{ or } j = j', \, i \leq i'.$$

Suppose we are at step M, which corresponds to a certain couple (i, j). At step M we provide a $c^\beta_M \in A_+$ such that, for every $d_1, d_2 \in M(A)$

$$\|[(a_j + d_1 e_i), (c^\beta_M + d_2 e_i)]\| \geq \frac{1}{2}$$

and we define two values of f_β. Assume that $f_\beta(n)$ has been defined for $n \leq 2M - 1$. Let $m \in \mathbb{N}$ be the smallest natural number such that

$$f_j(2m) > \max\{i + 2, f_\beta(2M - 1) + 2^{2M-1} + 1\}$$

and such that, for $l \geq 2m$, the inequality $|f_j(l) - f_k(n)| > 2^M + 1$ holds for all $k \in \mathbb{N}$ such that $\alpha_k < \alpha_j$, and all $n \in \mathbb{N}$. By inductive hypothesis there are two norm one orthogonal vectors $\xi^j_m, \eta^j_m \in S_{f_j(2m), f_j(2m+1)}$ such that $a_j(\xi^j_m) = \xi^j_m$ and $a_j(\eta^j_m) = 0$. Set $\xi^\beta_M = \frac{1}{\sqrt{2}}(\xi^m_j + \eta^m_j)$ and $\eta^\beta_M = \frac{1}{\sqrt{2}}(\xi^m_j - \eta^m_j)$. Using Kadison's transitivity theorem, fix a positive, norm one element

$$c^\beta_M \in (e_{f_j(2m+1)} - e_{f_j(2m)})A(e_{f_j(2m+1)} - e_{f_j(2m)})$$

such that

$$c^\beta_M(\xi^\beta_M) = \xi^\beta_M,$$
$$c^\beta_M(\eta^\beta_M) = 0.$$

Let $f_\beta(2M) = f_j(2m) - 1$ and $f_\beta(2M + 1) = f_j(2m + 1) + 1$. We have therefore that

$$e_{f_\beta(2M+1)} c^\beta_M e_{f_\beta(2M+1)} = c^\beta_M,$$
$$e_{f_\beta(2M)} c^\beta_M = c^\beta_M e_{f_\beta(2M)} = 0.$$

Moreover:

$$\|(a_j + d_1 e_i)(c^\beta_M + d_2 e_i)(\xi^\beta_M) - (c^\beta_M + d_2 e_i)(a_j + d_1 e_i)(\xi^\beta_M)\| = \tag{$*$}$$
$$\|a_j c^\beta_M(\xi^\beta_M) - c^\beta_M a_j(\xi^\beta_M)\| = \frac{1}{2\sqrt{2}}\|\xi^m_j - \eta^m_j\| = \frac{1}{2}.$$

This is the case since $e_i(\xi) = 0$ for every $\xi \in S_{f_j(2m), f_j(2m+1)}$ (we chose m so that $f_j(2m) > i + 2$) and $c^\beta_M(\xi^\beta_M), a_j(\xi^\beta_M) = c^j_m(\xi^\beta_M) \in S_{f_j(2m), f_j(2m+1)}$. Define

$$a_\beta = \sum_{n \in \mathbb{N}} c^\beta_n = \sum_{n \in \mathbb{N}} (e_{f_\beta(2n+1)} - e_{f_\beta(2n)})^{\frac{1}{2}} c^\beta_n (e_{f_\beta(2n+1)} - e_{f_\beta(2n)})^{\frac{1}{2}}.$$

This series is strictly convergent since all c^β_n's have norm 1. The families $\{f_n\}_{n<\mathbb{N}} \cup \{f_\beta\}$ and $\{a_n\}_{n<\mathbb{N}} \cup \{a_\beta\}$ satisfy items (i)-(ii) of the inductive hypothesis[2].

[2]The induction to define a_β and f_β is on the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}$ ordered with a well-ordering of type ω such that $(i, j) \leq (i', j')$ implies $j \leq j'$. This is used to show that f_β satisfies clause i of the inductive hypothesis.

Finally we verify clause (iii). Notice that, by construction, for every $k \in \mathbb{N}$, given $\xi \in S_{f_\beta(2k), f_\beta(2k+1)}$ we have

$$a_\beta(\xi) = c^\beta_k(\xi).$$

Let $i \leq j \in \mathbb{N}$, denote the step corresponding to the couple (i, j) by M, and let $m \in \mathbb{N}$ be such that $f_\beta(2M) = f_j(2m) - 1$ (by construction we can find such m). Remember that $\xi^\beta_M = \frac{1}{\sqrt{2}}(\xi^m_j + \eta^m_j) \in S_{f_\beta(2M), f_\beta(2M+1)}$. Given $d_1, d_2 \in \mathcal{M}(A)$, we have that

$$\|(a_j + d_1 e_i)(a_\beta + d_2 e_i)(\xi^\beta_M) - (a_\beta + d_2 e_i)(a_j + d_1 e_i)(\xi^\beta_M)\| =$$
$$\|a_j a_\beta(\xi^\beta_M) - a_\beta a_j(\xi^\beta_M)\| = \frac{1}{2\sqrt{2}}\|\xi^m_j - \eta^m_j\| = \frac{1}{2}.$$

This equation can be shown using the same arguments used to prove $(*)$.

Notice that if β is finite, we only obtain a finite number of c^β_n therefore their sum (which is finite) does not belong to $\mathcal{M}(A) \setminus A$. In this case it is sufficient to add an infinite number of addends, as we did for a_0. Suppose that β is (the ordinal corresponding to) $N \in \mathbb{N}$, then the previous construction defines f_N only up until $2N + 1$. Let $f_N(2(N + 1))$ be the smallest integer such that

- $f_N(2(N + 1)) - f_N(2N + 1) > 2^{2N+1}$;
- $|f_N(2(N + 1)) - f_j(n)| > 2^{2(N+1)}$ for all $j < N$; and for all $n \in \mathbb{N}$.

Define

$$f_N(2(N + 1) + 1) = f_N(2(N + 1)) + 3$$

and continue inductively the definition of f_N. For each $n > N$ we can therefore, as we did for a_0 using Kadison's transitivity theorem, find a positive element

$$c^N_n \in \overline{(e_{f_N(2n+1)-1} - e_{f_N(2n)+1})A(e_{f_N(2n+1)-1} - e_{f_N(2n)+1})}$$

which moves a norm one vector $\xi^N_n \in S_{f_N(2n), f_N(2n+1)}$ into itself, and another orthogonal norm one vector η^N_n to zero. If we define a_N to be the sum of such c^N_n's, it is possible to show, using the same arguments exposed when β was assumed to be infinite, that the families $\{f_n\}_{n<N} \cup \{f_\beta\}$ and $\{a_n\}_{n<N+1}$ satisfy items i-iii of the inductive hypothesis. $\square$

The proof of theorem 3.0.1 is analogous to the one given in [BK17, Theorem 7], but it uses our lemma 3.2.1 instead of [BK17, Lemma 6].

Proof of theorem 3.0.1. Let $(e_n)_{n\in\mathbb{N}} \subseteq A$ be the approximate unit defined at the beginning of the current section, and let $(a_\beta)_{\beta\in\aleph_1}$ be the $\aleph_1$-sized collection obtained from lemma 3.2.1. Suppose there is an uncountable $U \subseteq \aleph_1$ and $(d_\beta)_{\beta\in U} \subseteq A$ such that

$$[(a_\alpha + d_\alpha), (a_\beta + d_\beta)] = 0$$

for all $\alpha, \beta \in U$. By using the pigeonhole principle, we can suppose that $\|d_\beta\| \leq M$ for some $M \in \mathbb{R}$, and that there is a unique $n \in \mathbb{N}$ such that

$$\|d_\beta - d_\beta e_n\| \leq \frac{1}{64(M + 1)}$$

for all $\beta \in U$.

Therefore, for every $\beta \in U$ and all but finitely many $\alpha \in U$ such that $\alpha < \beta$, we have

$$0 = \|[(a_\alpha + d_\alpha), (a_\beta + d_\beta)]\| \geq \|[(a_\alpha + d_\alpha e_n), (a_\beta + d_\beta e_n)]\| - \frac{1}{16} \geq \frac{1}{16}.$$

This is a contradiction when $\{\alpha \in U : \alpha < \beta\}$ is infinite. Indeed, in this case there exists at least one (in fact infinitely many!) $\alpha < \beta$ for which the inequality that we displayed above holds. $\square$

Proof of corollary 3.0.2. The proof follows verbatim the one given for lemma 3.2.1 plus theorem 3.0.1. The only difference is that, each time Kadison's transitivity theorem is invoked in lemma 3.2.1, it is possible to use a stronger version of Kadison's transitivity theorem for C*-algebras of real rank zero (see for instance [Bic13, Theorem 6.5]) which allows to chose at each step a projection. This stronger version of Kadison's transitivity theorem can be used throughout the whole iteration since hereditary subalgebras of real rank zero C*-algebras have real rank zero. $\square$

If A is a commutative non-unital C*-algebra, then the problem of lifting commuting elements from $Q(A)$ to $M(A)$ is trivial, as both $Q(A)$ and $M(A)$ are abelian. In section 3.2 we ruled out this possibility by asking for A to be primitive. From this perspective, primitivity can be thought as a strong negation of commutativity.

The other important feature we required to prove theorem 3.0.1 is σ-unitality. We do not know whether this assumption could be weakened, but it certainly cannot be removed tout-court. Indeed, there are extreme examples of primitive, non-σ-unital C*-algebras whose corona is finite-dimensional (see [Sak71] and [GK18]), for which theorem 3.0.1 is trivially false. Our conjecture is that there might be a condition on the order structure of the approximate unit of A which is weaker than σ-unitality, but still makes theorem 3.0.1 true. For instance, it would be interesting to know whether the techniques used in theorem 3.0.1 could be applied to the algebra of the compact operators on a nonseparable Hilbert space, or more in general to a C*-algebra A with a projection $p \in M(A)$ such that pAp is primitive, non-unital and σ-unital.

We remark that the proof of theorem 3.0.1 we gave can be adapted to any primitive C*-algebra A which admits an increasing approximate unit $\{e_\alpha\}_{\alpha\in\kappa}$, for κ regular cardinal, to produce a κ^+-sized family of orthogonal positive elements in $Q(A)$ which cannot be lifted to a set of commuting elements in $M(A)$.

3.3 A Reflection Problem

Question 3.3.1. Assume $F \subseteq Q(A)_{sa}$ is a commutative family such that any smaller (in the sense of cardinality) subset can be lifted to a set of commuting elements in $M(A)_{sa}$. Can F be lifted to a collection of commuting elements in $M(A)_{sa}$?

Theorem 3.0.1 and proposition 3.1.2 entail that this is not true in general for primitive, non-unital, σ-unital C*-algebras if $|F| = \aleph_1$, pointing out the set theoretic incompactness of $\aleph_1$ for this property.

If the family F is infinite and countable, then question 3.3.1 has a positive answer in the Calkin algebra.

Proposition 3.3.2. *Suppose that A is a separable abelian* C**-subalgebra of $Q(H)$ such that every finitely-generated subalgebra of A has an abelian lift. Then A has an abelian lift.*

The proof of this proposition relies on Voiculescu's theorem [Arv77, Theorem 4] (see also theorem 2.1.2), starting from the following lemma. We recall that an embedding of a given C*-algebra A into the Calkin algebra is trivial if it admits a multiplicative lift to $B(H)$.

Lemma 3.3.3. *Let A be a separable unital abelian C*-subalgebra of $Q(H)$. If there exists a unital abelian C*-algebra $B \subseteq B(H)$ lifting A, then the identity map on A, saw as an embedding into $Q(H)$, is trivial.*

Proof. Since B is abelian, there exists a masa (maximal abelian subalgebra) of $B(H)$ containing B. Masas in $B(H)$ are von Neumann algebras and, as such, they are generated by their projections. This entails that A is contained in a separable unital abelian subalgebra $C(Y)$ of $Q(H)$ which is generated by its projections. By [BDF77, Theorem 1.15] there exists a unital $*$-homomorphism $\Psi : C(Y) \to B(H)$ lifting the identity on $C(Y)$. Let Φ be the restriction of Ψ to $C(X)$. □

Proof of proposition 3.3.2. Suppose that $F = \{a_n\}_{n\in\mathbb{N}} \subseteq Q(H)_{sa}$ is an abelian family such that every finite subset of F has a commutative lift. Without loss of generality, we can assume that $a_0 = 1$. By lemma 3.3.3 we can assume that, for every $k \in \mathbb{N}$, there is a unital $*$-homomorphism $\Phi_k : \mathrm{C}^*(\{a_n\}_{n\le k}) \to B(H)$ lifting the identity map on $\mathrm{C}^*(\{a_n\}_{n\le k})$. By Voiculescu's theorem [Arv77, Theorem 4] (theorem 2.1.2) we can moreover assume that, for every $n \in \mathbb{N}$, the sequence $\{\Phi_k(a_n)\}_{k\ge n}$ converges to some self-adjoint operator A_n in $B(H)$ such that $A_n - \Phi_k(a_n)$ is compact for every $k \in \mathbb{N}$. The family $\{A_n\}_{n\in\mathbb{N}}$ is a commutative lifting of $\{a_n\}_{n\in\mathbb{N}}$. □

More general forms of Voiculescu's theorem are known to hold for extensions of various separable C*-algebras other than $K(H)$ (see [EK01], [Gab16], [Sch18, Section 2.2]). Such generalizations could potentially be used to carry out the arguments exposed above for coronas of other separable nuclear stable C*-algebras. We remark however the importance of being able to lift separable abelian subalgebras of $Q(H)$ to abelian algebras in $B(H)$ with the same spectrum, as guaranteed by lemma 3.3.3. This is false in general in other coronas, as it happens for instance when $A = Z \otimes K(H)$. In this case, projections in $Q(A)$ do not necessarily lift to projections in $M(A)$, since the former has real rank zero but the latter has not (see [LN16]).

The following example proves that question 3.3.1 has negative answer for finite families with an even number of elements.

Example 3.3.4. Let S^n be the n-dimensional sphere. The algebra $C(S^n)$ is generated by $n + 1$ self-adjoint elements $\{h_i\}_{0\le i\le n}$ satisfying the relation

$$h_0^2 + \cdots + h_n^2 = 1.$$

Let $F = \{h_i\}_{0\le i\le n}$. The relation above implies that the joint spectrum of a subset of F of size $m \le n$ is the m-dimensional ball B^m. The space B^m is contractible, therefore the group $\mathrm{Ext}(B^m)$ is trivial (see [HR00, Section 2.6-2.7] for the definition of the functor Ext and its basic properties). As a consequence, for any $[\tau] \in \mathrm{Ext}(S^n)$, any proper subset of $\tau[F]$ can be lifted to a set of commuting self-adjoint operators in $B(H)$. On the other hand $\mathrm{Ext}(S^{2k+1}) = \mathbb{Z}$ for every $k \in \mathbb{N}$. We conclude that any non-trivial extension τ of $C(S^{2k+1})$ produces, by lemma 3.3.3, a family $\tau[F]$ of size $2k + 2$ in the Calkin algebra for which Question 3.3.1 has negative answer.

The argument above does not apply to families of odd cardinality, since $\mathrm{Ext}(S^{2k}) = \{0\}$ for every $k \in \mathbb{N}$. However, in [Dav85] (see also [Voi81], [Lor88]), the author builds a set of three commuting self-adjoint elements in the corona algebra of $\prod_{n \in \mathbb{N}} M_n(\mathbb{C})$ with no commutative lifting to the multiplier algebra, whose proper subsets of size two all admit a commutative lifting. The answer to question 3.3.1 for larger finite families with an odd number of elements is, to the best of our knowledge, unknown.

Bibliography

[AD79] C. A. Akemann and J. E. Doner, *A nonseparable C^*-algebra with only separable abelian C^*-subalgebras*, Bull. London Math. Soc. **11** (1979), no. 3, 279–284. MR 554393

[AK16] H. Ando and E. Kirchberg, *Non-commutativity of the central sequence algebra for separable non-type I C^*-algebras*, J. Lond. Math. Soc. (2) **94** (2016), no. 1, 280–294. MR 3532173

[And79] J. Anderson, *Pathology in the Calkin algebra*, J. Operator Theory **2** (1979), no. 2, 159–167. MR 559603

[Arv77] W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), no. 2, 329–355. MR 0438137

[AW04] C. Akemann and N. Weaver, *Consistency of a counterexample to Naimark's problem*, Proc. Natl. Acad. Sci. USA **101** (2004), no. 20, 7522–7525. MR 2057719

[AW08] ______, *$B(H)$ has a pure state that is not multiplicative on any masa*, Proc. Natl. Acad. Sci. USA **105** (2008), no. 14, 5313–5314. MR 2403097

[BDF77] L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. of Math. (2) **105** (1977), no. 2, 265–324. MR 0458196

[BFZ90] J. Baumgartner, R. Frankiewicz, and P. Zbierski, *Embedding of Boolean algebras in $P(\omega)/\text{fin}$*, Fund. Math. **136** (1990), no. 3, 187–192.

[Bic13] T. Bice, *The projection calculus*, Münster J. Math. **6** (2013), no. 2, 557–581. MR 3148223

[BK17] T. Bice and P. Koszmider, *A note on the Akemann-Doner and Farah-Wofsey constructions*, Proc. Amer. Math. Soc. **145** (2017), no. 2, 681–687. MR 3577870

[Bla80] B. E. Blackadar, *Traces on simple AF C^*-algebras*, J. Funct. Anal. **38** (1980), no. 2, 156–168. MR 587906

[Bla98] B. Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR 1656031

[Bla06] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. MR 2188261

[BO08] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008. MR 2391387

[Cal41] J. W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. (2) **42** (1941), 839–873. MR 0005790

[CE76] M. D. Choi and E. G. Effros, *The completely positive lifting problem for C^*-algebras*, Ann. of Math. (2) **104** (1976), no. 3, 585–609. MR 0417795

[CF14] S. Coskey and I. Farah, *Automorphisms of corona algebras, and group cohomology*, Trans. Amer. Math. Soc. **366** (2014), no. 7, 3611–3630. MR 3192609

[CFO14] Y. Choi, I. Farah, and N. Ozawa, *A nonseparable amenable operator algebra which is not isomorphic to a C^*-algebra*, Forum Math. Sigma **2** (2014), e2, 12. MR 3177805

[Cho80] M. D. Choi, *The full C^*-algebra of the free group on two generators*, Pacific J. Math. **87** (1980), no. 1, 41–48. MR 590864

[Con76] A. Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$*, Ann. of Math. (2) **104** (1976), no. 1, 73–115. MR 0454659

[Dav85] K. R. Davidson, *Almost commuting Hermitian matrices*, Math. Scand. **56** (1985), no. 2, 222–240. MR 813638

[Dav96] K. R. Davidson, *C*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. MR 1402012

[Dav10] K. R. Davidson, *Essentially normal operators*, A glimpse at Hilbert space operators, Oper. Theory Adv. Appl., vol. 207, Birkhäuser Verlag, Basel, 2010, pp. 209–222. MR 2743421

[DH99] A. Dow and K. P. Hart, *ω^* has (almost) no continuous images*, Israel J. Math. **109** (1999), 29–39. MR 1679586

[DH00] A. Dow and K. P. Hart, *The measure algebra does not always embed*, Fund. Math. **163** (2000), no. 2, 163–176. MR 1752102

[DH01] A. Dow and K. P. Hart, *A universal continuum of weight $\aleph$*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1819–1838. MR 1707489

[Dix77] J. Dixmier, *C^*-algebras*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15. MR 0458185

[DW87] H. G. Dales and W. H. Woodin, *An introduction to independence for analysts*, London Mathematical Society Lecture Note Series, vol. 115, Cambridge University Press, Cambridge, 1987. MR 942216

[EK01] G. A. Elliott and D. Kucerovsky, *An abstract Voiculescu-Brown-Douglas-Fillmore absorption theorem*, Pacific J. Math. **198** (2001), no. 2, 385–409. MR 1835515

[Far96] I. Farah, *Embedding partially ordered sets into ω_ω*, Fund. Math. **151** (1996), no. 1, 53–95. MR 1405521

[Far10] _____, *Graphs and CCR algebras*, Indiana Univ. Math. J. **59** (2010), no. 3, 1041–1056. MR 2779071

[Far11] _____, *All automorphisms of the Calkin algebra are inner*, Ann. of Math. (2) **173** (2011), no. 2, 619–661. MR 2776359

[FH13] I. Farah and B. Hart, *Countable saturation of corona algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **35** (2013), no. 2, 35–56. MR 3114457

[FH16] I. Farah and I. Hirshberg, *The Calkin algebra is not countably homogeneous*, Proc. Amer. Math. Soc. **144** (2016), no. 12, 5351–5357. MR 3556277

[FH17] _____, *Simple nuclear C^*-algebras not isomorphic to their opposites*, Proc. Natl. Acad. Sci. USA **114** (2017), no. 24, 6244–6249. MR 3667529

[FHL+16] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter, *Model theory of C^*-algebras*, arXiv preprint arXiv:1602.08072 (2016).

[FHL+ar] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter, *Model theory of* C^**-algebras*, Memoirs AMS (to appear).

[FHS13] I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras I: stability*, Bull. Lond. Math. Soc. **45** (2013), no. 4, 825–838. MR 3081550

[FHV17] I. Farah, I. Hirshberg, and A. Vignati, *The Calkin algebra is $\aleph_1$-universal*, arXiv preprint arXiv:1707.01782 (2017).

[FK10] I. Farah and T. Katsura, *Nonseparable UHF algebras I: Dixmier's problem*, Adv. Math. **225** (2010), no. 3, 1399–1430. MR 2673735

[FK15] _____, *Nonseparable UHF algebras II: Classification*, Math. Scand. **117** (2015), no. 1, 105–125. MR 3403788

[FKK01] H. Futamura, N. Kataoka, and A. Kishimoto, *Homogeneity of the pure state space for separable C^*-algebras*, Internat. J. Math. **12** (2001), no. 7, 813–845. MR 1850673

[FKV18] I. Farah, G. Katsimpas, and A. Vaccaro, *Embedding C^*-algebras into the Calkin algebra*, arXiv preprint arXiv:1810.00255 (2018).

[FW12] I. Farah and E. Wofsey, *Set theory and operator algebras*, London Math. Soc. Lecture Note Series **406** (2012), 63–120.

[Gab16] J. Gabe, *A note on nonunital absorbing extensions*, Pacific J. Math. **284** (2016), no. 2, 383–393. MR 3544306

[GH09] S. Givant and P. Halmos, *Introduction to Boolean algebras*, Undergraduate Texts in Mathematics, Springer, New York, 2009. MR 2466574

[GK60] J. G. Glimm and R. V. Kadison, *Unitary operators in C^*-algebras*, Pacific J. Math. **10** (1960), 547–556. MR 0115104

[GK18] S. Ghasemi and P. Koszmider, *An extension of compact operators by compact operators with no nontrivial multipliers*, J. Noncommut. Geom. **12** (2018), no. 4, 1503–1529. MR 3896233

[Gli61] J. Glimm, *Type I C^*-algebras*, Ann. of Math. (2) **73** (1961), 572–612. MR 0124756

[HR00] N. Higson and J. Roe, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000, Oxford Science Publications. MR 1817560

[HT05] U. Haagerup and S. Thorbjørnsen, *A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group*, Ann. of Math. (2) **162** (2005), no. 2, 711–775. MR 2183281

[Jec03] T. Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513

[Kap51] I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255. MR 0042066

[Kis81] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Comm. Math. Phys. **81** (1981), no. 3, 429–435. MR 634163

[KOS03] A. Kishimoto, N. Ozawa, and S. Sakai, *Homogeneity of the pure state space of a separable C^*-algebra*, Canad. Math. Bull. **46** (2003), no. 3, 365–372. MR 1994863

[KR14] E. Kirchberg and M. Rørdam, *Central sequence C^*-algebras and tensorial absorption of the Jiang-Su algebra*, J. Reine Angew. Math. **695** (2014), 175–214. MR 3276157

[Kun68] K. Kunen, *Inaccessibility properties of cardinals*, ProQuest LLC, Ann Arbor, MI, 1968, Thesis (Ph.D.)–Stanford University. MR 2617841

[Kun76] K. Kunen, *(κ, λ^*)-gaps under MA*, preprint, 1976.

[Kun11] _____, *Set theory*, Studies in Logic (London), vol. 34, College Publications, London, 2011.

[Lav79] R. Laver, *Linear orders in $(\omega)^\omega$ under eventual dominance*, Logic Colloquium '78 (Mons, 1978), Stud. Logic Foundations Math., vol. 97, North-Holland, Amsterdam-New York, 1979, pp. 299–302. MR 567675

[LN16] H. Lin and P. W. Ng, *The corona algebra of the stabilized Jiang-Su algebra*, J. Funct. Anal. **270** (2016), no. 3, 1220–1267. MR 3438334

[Lor88] T. A. Loring, *K-theory and asymptotically commuting matrices*, Canad. J. Math. **40** (1988), no. 1, 197–216. MR 928219

[Lor97] _____, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs, vol. 8, American Mathematical Society, Providence, RI, 1997. MR 1420863

[Luz47] N. N. Luzin, *On subsets of the series of natural numbers*, Izvestiya Akad. Nauk SSSR. Ser. Mat. **11** (1947), 403–410. MR 0021576

[Mur90] G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990. MR 1074574

[MV18] P. McKenney and A. Vignati, *Forcing axioms and coronas of nuclear C^*-algebras*, arXiv preprint arXiv:1806.09676 (2018).

[Nai48] M. A. Naimark, *Rings with involutions*, Uspehi Matem. Nauk (N.S.) **3** (1948), no. 5(27), 52–145. MR 0027448

[Nai51] _____, *On a problem of the theory of rings with involution*, Uspehi Matem. Nauk (N.S.) **6** (1951), no. 6(46), 160–164. MR 0046574

[Ped79] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979. MR 548006

[Ped90] ___, *The corona construction*, Operator Theory: Proceedings of the 1988 GPOTS-Wabash Conference (Indianapolis, IN, 1988), Pitman Res. Notes Math. Ser., vol. 225, Longman Sci. Tech., Harlow, 1990, pp. 49–92. MR 1075635

[Pop83] S. Popa, *Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras*, J. Operator Theory **9** (1983), no. 2, 253–268. MR 703810

[PW07] N. C. Phillips and N. Weaver, *The Calkin algebra has outer automorphisms*, Duke Math. J. **139** (2007), no. 1, 185–202. MR 2322680

[Ric60] C. E. Rickart, *General theory of Banach algebras*, The University Series in Higher Mathematics, D. van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. MR 0115101

[Ros53] A. Rosenberg, *The number of irreducible representations of simple rings with no minimal ideals*, Amer. J. Math. **75** (1953), 523–530. MR 0057477

[Sak66] S. Sakai, *On a characterization of type I C^*-algebras*, Bull. Amer. Math. Soc. **72** (1966), 508–512. MR 0192363

[Sak67] _____, *On type I C^*-algebras*, Proc. Amer. Math. Soc. **18** (1967), 861–863. MR 0213887

[Sak71] S. Sakai, *Derivations of simple C*-algebras. III*, Tôhoku Math. J. (2) **23** (1971), 559–564. MR 0318902

[Sak74] _____, *On automorphism groups of II_1-factors*, Tôhoku Math. J. (2) **26** (1974), 423–430. MR 0380443

[Sch87] P. E. Schupp, *A characterization of inner automorphisms*, Proc. Amer. Math. Soc. **101** (1987), no. 2, 226–228. MR 902532

[Sch18] C. Schafhauser, *Subalgebras of simple AF-algebras*, arXiv preprint arXiv:1807.07381 (2018).

[SS11] S. Shelah and J. Steprans, *Masas in the Calkin algebra without the continuum hypothesis*, J. Appl. Anal. **17** (2011), no. 1, 69–89. MR 2805847

[ST17] N. Suri and M. Tomforde, *Naimark's problem for graph C^*-algebras*, Illinois J. Math. **61** (2017), no. 3-4, 479–495. MR 3845730

[TF95] S. Todorcevic and I. Farah, *Some applications of the method of forcing*, Yenisei, Moscow, 1995.

[Tho95] K. Thomsen, *Traces, unitary characters and crossed products by* **Z**, Publ. Res. Inst. Math. Sci. **31** (1995), no. 6, 1011–1029. MR 1382564

[Vac16] A. Vaccaro, *Obstructions to lifting abelian subalgebras of corona algebras*, arXiv preprint arXiv:1611.06272 (2016).

[Vac18a] ______, *Trace spaces of counterexamples to Naimark's problem*, J. Funct. Anal. **275** (2018), no. 10, 2794–2816. MR 3853080

[Vac18b] ______, *Voiculescu's theorem for nonseparable C^*-algebras*, arXiv preprint arXiv:1811.09352 (2018).

[Vig15] A. Vignati, *An algebra whose subalgebras are characterized by density*, J. Symb. Log. **80** (2015), no. 3, 1066–1074. MR 3395360

[Vig17a] A. Vignati, *Logic and C^*-algebras: Set theoretical dichotomies in the theory of continuous quotients*, Ph.D. thesis, York University, Toronto, 2017.

[Vig17b] A. Vignati, *Nontrivial homeomorphisms of Čech-Stone remainders*, Münster J. Math. **10** (2017), no. 1, 189–200. MR 3624107

[VN35] J. Von Neumann, *Charakterisierung des spektrums eines integraloperators.*

[Voi76] D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113. MR 0415338

[Voi81] ______, *Remarks on the singular extension in the C^*-algebra of the Heisenberg group*, J. Operator Theory **5** (1981), no. 2, 147–170. MR 617972

[Wea07] N. Weaver, *Set theory and C^*-algebras*, Bull. Symbolic Logic **13** (2007), no. 1, 1–20. MR 2300900

[Wea14] N. Weaver, *Forcing for mathematicians*, World Scientific, 2014.

[Wey09] H. Weyl, *Über beschränkte quadratische formen, deren differenz vollstetig ist*, Rendiconti del Circolo Matematico di Palermo (1884-1940) **27** (1909), no. 1, 373–392.

[Wil70] S. Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970. MR 0264581

[Woo84] W. H. Woodin, *Discontinuous Homomorphisms of $C(\Omega)$ and Set Theory*, ProQuest LLC, Ann Arbor, MI, 1984, Thesis (Ph.D.)–University of California, Berkeley. MR 2634119

[ZA14] B. Zamora-Aviles, *Gaps in the poset of projections in the Calkin algebra*, Isr. J. Math. **202** (2014), no. 1, 105–115.

[Zha92] S. Zhang, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras. I*, Pacific J. Math. **155** (1992), no. 1, 169–197. MR 1174483