GEOMETRY OF POINT-HYPERPLANE AND SPHERICAL FRAMEWORKS

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A DISSERTATION SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN MATHEMATICS AND STATISTICS
YORK UNIVERSITY
TORONTO, ONTARIO

August 2017

Abstract

In this thesis we show that the infinitesimal rigidity of point-hyperplane frameworks in $\mathbb{E}^n$ is equivalent to the infinitesimal rigidity of bar-joint frameworks in $\mathbb{S}^n$ with a set of joints (corresponding to the hyperplanes) located on a hyperplane in $\mathbb{S}^n$. This is done by comparing the rigidity matrix of Euclidean point-hyperplane frameworks and the rigidity matrix of spherical frameworks. This result clearly shows how the first-order rigidity in projective spaces and Euclidean spaces are globally connected. This geometrically significant result is central to the thesis.

This result leads to the equivalence of the first-order rigidity of point-hyperplane frameworks in $\mathbb{E}^n$ with that of bar-joint frameworks with a set of joints in a hyperplane in $\mathbb{E}^n$. This result and some of its important consequences are also presented in the coauthored paper [17]. We also study the rigidity of point-hyperplane frameworks in $\mathbb{E}^n$ and characterize their rigidity.

We next highlight the relationship between point-line frameworks and slider mechanisms in the plane. Point-line frameworks are used to model various types of slider mechanisms. A combinatorial characterization of the rigidity of pinned-slider frameworks in the plane is derived directly as an immediate consequence of the analogous result for pinned bar-joint frameworks in the plane. Using fixed-normal point-line frameworks, we model a second type of slider system in which the slider directions do not change. Also, a third type of slider mechanism is introduced in which the sliders may only rotate around a fixed point but do not translate. This slider mechanism is defined using point-line frameworks with rotatory lines (no translational motion of the lines is allowed). A combinatorial characterization of the generic rigidity of these frameworks is coauthored in [17].

Then we introduce point-hyperplane tensegrity frameworks in $\mathbb{E}^n$. We investigate the rigidity and the infinitesimal rigidity of these frameworks in $\mathbb{E}^n$ using tensegrity frameworks in $\mathbb{S}^n$. We characterize these different types of rigidity for point-hyperplane tensegrity frameworks in $\mathbb{E}^n$ and show how these types of rigidity

are linked together. This leads to a characterization of the rigidity of a broader class of slider mechanisms in which sliders may move under variable distance constraints rather than fixed-distance constraints.

Finally we investigate body-cad constraints in the plane. A combinatorial characterization of their generic infinitesimal rigidity is given. We show how angular constraints are related to non-angular constraints. This leads to a combinatorial result about the rigidity of a specific class of body-bar frameworks with point-point coincidence constraints in $\mathbb{E}^3$.

Acknowledgements

First, I would like to express my deep gratitude and debt to my supervisor, Prof. Walter Whiteley, for his constant support, guidance and encouragement during these years. I am influenced by his geometry-driven mind which was vital to the existence and the shape of this dissertation. He was always available to help me. I learned a lot from him as his student and teaching assistant from which I will benefit in the future.

I am grateful to the members of my Supervisory Committee, Prof. Asia Weiss and Prof. Trueman Machenry for numerous discussions and their help until the completion of my Ph.D.

Many thanks to Prof. Robert Connelly for agreeing to be the external examiner, Prof. Uyen Trang Nguyen as the internal member and Prof. Patrick Ingram as the chair of the Examining Committee of this dissertation who prepared interesting questions and suggested some corrections in the dissertation.

I would also like to thank Primrose Miranda and other staff members of the department of Mathematics and Statistics at York University who were always friendly and helpful.

Last but not least, I sincerely thank Najwan, my wife, the love of my life and my friend for her love and care.

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Chapter 1

Introduction

1.1 Introduction to rigidity

The word *rigidity* is used in a broad sence in mathematics and engineering literature. In geometry, a set S of points equipped with a geometric, topological or differentiable structure is called *rigid* in a space ($\mathbb{E}^n$, for example) if S admits no deformation. A deformation is a transformation that does not globally preserve the structure of the space as opposed to the transformations that do preserve the structure of the entire space, which become the *rigid* transformations of the space. For instance, a famous theorem in differential geometry states that 'the sphere is rigid', in the sense that any compact, connected, smooth surface with constant Gaussian curvature is a sphere in $\mathbb{E}^3$ [6, p. 317]. Physically, this means a sphere made of a flexible but inelastic material is rigid. This theorem globally describes the sphere in the space.

A triangle is a very simple, geometric, discrete structure: three points with pairwise fixed distances between them (side lengths). This defines a rigid geometric structure in Euclidean spaces of any dimension by the well-known Side-Side-Side congruence theorem in Euclidean plane. On the other hand, four distinct points with four distance constraints is not a rigid structure in the plane (Figure 1.1a).

Depending on the nature of the constraints and the geometry of the ambient space, we can give a precise definition of a specific type of rigidity in question. For example, a quadrilateral is defined by four *distance constraints* on four points p_1, p_2, p_3, p_4 in the plane: the distances $\|p_1 - p_2\|$, $\|p_2 - p_3\|$, $\|p_3 - p_4\|$ and $\|p_4 - p_1\|$ are to remain fixed but $\|p_2 - p_4\|$ and $\|p_1 - p_3\|$ may change. If we assign time-dependent coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) to the points p_1, p_2, p_3 and p_4 respectively, then the above distance constraints yield four quadratic equations with eight unknowns $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$ as functions of t as a time variable. There will always be trivial solutions to such a system of equations, which occurs when all the points move under a single 1-parameter rigid motion (all rotate or translate in the same way). Any other type of solution (non-trivial solution) for the system is considered a *deformation* and its existence implies that the framework is *not rigid*. If there is no non-trivial solution then the system is called *rigid*. In the case of the quadrilateral, we intuitively know that there is a non-trivial solution for our system (Figure 1.1a) because the structure is flexible. Now this example may be generalized to any number of points $p_1, \ldots, p_n \in \mathbb{R}^n$ with fixed distance constraints on some pairs of points $p_1, \ldots, p_n$. To express which pairs are constrained, we may use a graph $G = (V, E)$ with $|V|$ vertices corresponding to each point p_i such that the vertex i is connected to the vertex j if the points p_i and p_j are to maintain their distance. This is called a *bar-joint framework* in $\mathbb{E}^n$ and is denoted by $(G, \mathbf{p})$. The constraints define a set of quadratic equations[1] but as the number of points and constraints grow, it becomes more difficult to compute a solution for the system even in the plane (NP-complete). But if we differentiate (with respect to a time variable) these quadratic equations, we obtain a system of linear equations that captures a lot of information about the non-linear system of quadratic equations. The coefficient matrix of the linear system is usually called the *rigidity matrix* of the framework. Rigidity matrices play a crucial role in the study of all types of systems of constraints that we will deal with later on. Of course, the linear system

[1]This is an *affine algebraic variety.*

is not equivalent to the original system of quadratic equations but ‘almost always’, it equivalently determines the rigidity of the framework (See [2]). The derivative of the trivial solutions of the system of quadratic equations (as smooth curves) gives rise to ‘trivial solutions’ for the linear system.

It is key that ‘almost always’ the rigidity or flexibility of a framework $(G, \mathbf{p})$ is not dependent on where the points p_i are placed in $\mathbb{R}^n$ but it depends on the graph G. This makes the rigidity a combinatorial problem (for $n = 2$, see Laman’s Theorem in Chapter 2). It is said that a framework is ‘generically infinitesimally rigid’ if it is rigid almost always.

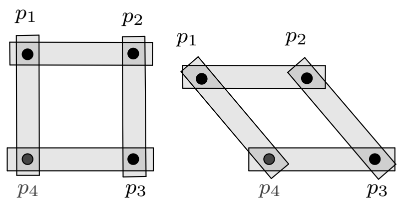
One stream of results in Rigidity dates back to Euler’s Conjecture [18], followed by a result due to Cauchy [7] that states convex polyhedra in three-dimensions with congruent faces connected in the same pattern, are congruent to each other. That means if the faces of a convex polyhedron are made of rigid plates and its edges are replaced by hinges then we have a rigid structure in 3-space (plates and hinges structures). This result was extended by Alexandrov [1] to higher dimensions in 1950. He showed that convex polyhedra are uniquely described by the metric spaces on their surfaces. Pogorelov [47] generalized the result to convex surfaces.

R. Bricard [5] gave an example of a flexible octahedral surface in $\mathbb{E}^3$, which is not embedded because of the self-intersection. H. Gluck [21] showed that Euler’s Conjecture [18] is almost always true for the case of closed, simply connected polyhedral surfaces. R. Connelly [8] found a counterexample to the Euler’s Conjecture.

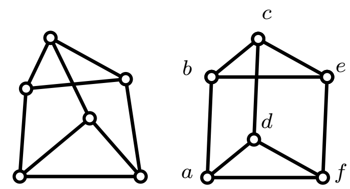
Another stream that helped develop the mathematical theory of rigidity is the *Static Rigidity* that has a rich literature in structural and mechanical engineering. The modern mathematical treatment of structural rigidity is specially influenced by the great physicist and geometer James Clerk Maxwell through the static theory [40],[41] which states:

If $(G, \mathbf{p})$, $|V| \geq n$, is a framework in $\mathbb{E}^n$ then

$$\dim\,(\mathcal{S}) \geq |E| - n|V| + \binom{n+1}{2},$$



(a) 4 distance constraints on 4 distinct points do not uniquely define a convex quadrilateral in plane.



(b) Generic (left) and non-generic (right) configurations on 6 points. In the right figure, ab, cd and ef are parallel with equal length.

Figure 1.1: Planar frameworks.

with equality if and only if the framework is infinitesimally rigid where $\mathcal{S}$ is the stress space of the framework which is the co-kernel of the rigidity matrix of $(G, \mathbf{p})$. This result also connects static rigidity to infinitesimal rigidity of frameworks.

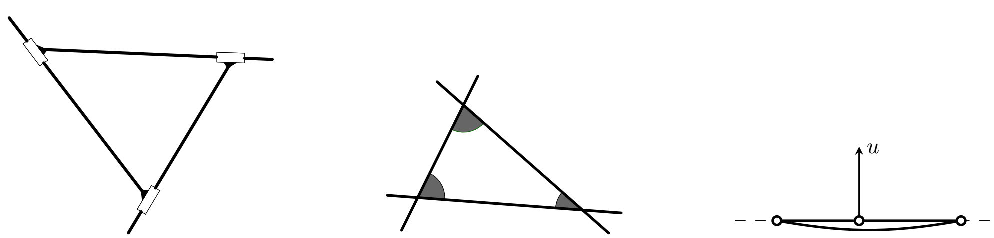
Although static theory provides an equivalent description for infinitesimal rigidity of bar-joint frameworks (see [16]), it is a suitable language for studying *tensegrity frameworks* and also useful in *global rigidity* [9].

1.2 Motivation: sliders, geometric CAD constraints and projective geometry

In mechanical engineering, a *revolute joint* is a type of joint connecting two rigid bodies so that it restricts their relative motion to a rotational motion (one degree of freedom) as opposed to *prismatic joints* that restrict the relative motions of two connected bodies to translational motions.

A variety of mathematical tools have been used to formulate simple mechanical structures (such as open or closed cycles) in order to analyze their motion and their rigidity. There are many standard examples in $\mathbb{E}^2$ and $\mathbb{E}^3$ to which these methods have been applied. Using the theory of Lie Algebras and Lie groups of rigid motions of $\mathbb{E}^n$, a single constraint equation can be written (see [56, p.108]) to describe the configuration space and the incidence geometry of a simple mechanism. Another interesting description of rigid motions of the Euclidean space uses *Clifford Algebra* (see [56], [24]). *Dual quaternions* as a representation of rigid motions in $\mathbb{E}^3$, have been proven useful for analyzing the motions and the incidence geometry of some simple closed cycles such as 6R cycles in the space, see [27] and [26]. Dual quaternions provide a unified representation of rotations and translations in the space. This representation has geometric advantages over the matrix representation of the rigid motions because the center of a rigid motion (and the angle of rotation as the parameter) form the coordinates of the corresponding one-parameter dual quaternion with fixed axis [27]. This representation of rigid motions is in connection with the way 2-extensors represent the infinitesimal rigid motions (Section 2.7.1).

In this context, a prismatic joint is represented by a one-parameter dual quaternion whose axis is at infinity (a translation). For a simple closed cycle (such as a 4-bar linkage, Figure 1.3) of revolute or prismatic joints, the product of a number of dual quaternions (one for each joint) is set to the identity (which is 1 as a number). This equation is called the *closure condition* and its solutions form an algebraic variety whose dimension determines the degrees of freedom of the mechanism. For a triangle, which is the simplest closed cycle of 3 revolute joints, this variety is of dimension zero. In other words, a triangle is rigid. Only when we replace all the



(a) Three prismatic joints with one degree of freedom.

(b) The line model (3 angles fixed) with one degree of freedom.

(c) The non-trivial infinitesimal motion of 3 collinear point.

Figure 1.2: A triangle at infinity and point-line frameworks.

three revolute joints by prismatic joints (Figure 1.2a) the dimension of the corresponding algebraic variety becomes one, which confirms one degree of freedom for the system. On the other hand, the motion of this slider cycle resembles the motion of three lines constrained to maintain the three angles between them, which also has one degree of freedom (Figure 1.2b). Three collinear points connected by bars (a collinear triangle) also has one degree of freedom in terms of *infinitesimal motions* (Figure 1.2c). The bar connecting the two end-points is drawn curved to be visible in the collinear triangle. Computationally, the analysis of the algebraic variety that describes the configuration space of a mechanism could become less effective and very complicated as the number of joints or the number of links increase. This makes the problem NP-hard in general [53].

Sliders frequently occur in different mechanisms, sometimes in combination with revolute joints (see Figure 1.3). A point constrained to move in a specific direction can be viewed as a point constrained to slide along a line. Let's look at the simple bar-joint framework whose underlying graph is $K_{2,2}$. This mechanism (called the *4-bar linkage*) has one degree of freedom. A 4-bar linkage is usually viewed as a closed cycle of 4 bodies (or links) with 4 revolute joints (4R) each of which allows one degree of freedom of rotational relative motion.

Figure 1.3a (in the middle) shows a slider mechanism derived from the 4-bar linkage when the joint p_4 'goes to infinity', meaning the connected links have only a relative translational motion. This slider mechanism has also one degree of freedom as well. We can model this mechanism using a point-line framework in which the joint p_4 is replaced by a line ℓ_1, shown in Figure 1.3a (on the right). Note that joints p_1 and p_3 are sliding along the same line in the slider mechanism. So they are constrained to maintain their distance from a line in the point-line model.

In Figure 1.3b (on the left), the two joints p_2, p_4 go to infinity, that is, they are

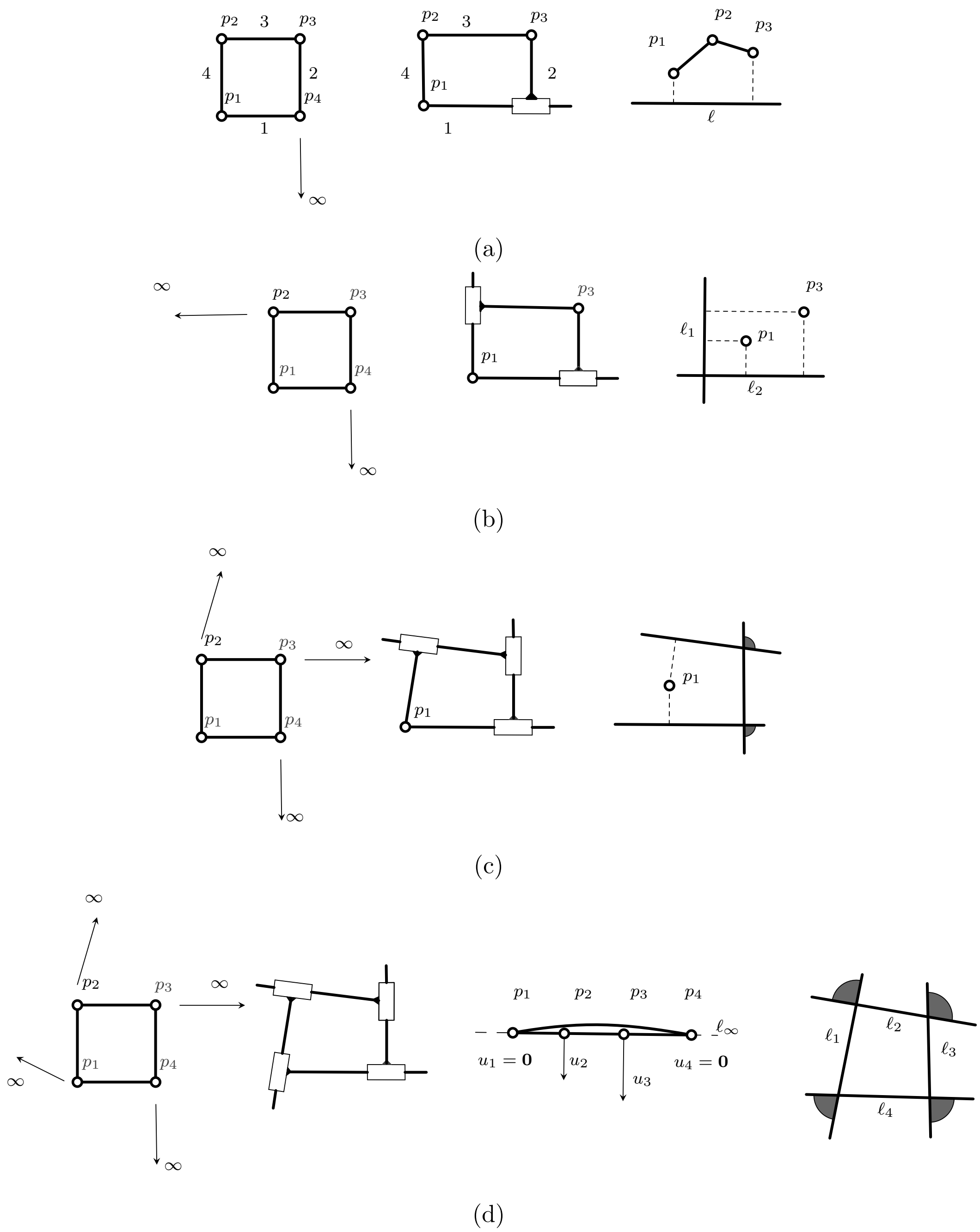


Figure 1.3: Sliders, point-line models and bar-joint frameworks with collinear joints.

replaced by two slider joints in specific directions as shown in Figure 1.3b (in the middle). The corresponding point-line framework is also flexible with one degree of freedom.

In Figure 1.3c, the three joints p_2, p_3, p_4 go to infinity and one revolute joint is left. Again, the corresponding point-line framework shows one degree of freedom.

We can go on and send the fourth joint to infinity to have a cycle of sliders, see Figure 1.3d. The corresponding point-line framework contains no point and 4 lines, with no relative rotational motion between them. The corresponding bar-joint framework is a collinear quadrilateral, which infinitesimally has two degrees of freedom and consequently two degrees of freedom (of finite motion) as a point-line framework and as a slider mechanism. Note that the additional degree of freedom occurred because the joints collapsed on a line (they became collinear). In this thesis, we will explain these connections and shed some light on the link between the rigidity of point-line models and that of bar-joint frameworks that are not generic in Chapter 3.

These types of constraints on the combinations of points and lines (and in higher dimensions, planes or splines [68] and so on) are common in geometric constraints for CAD (Computer Aided Design) software that allows users to put geometric constraints on rigid bodies to design complex systems. The common issue is to detect when a set of arbitrary constraints is independent to avoid redundant constraints. So it is valuable to be able to tell whether a set of constraints is dependent or not by a simple counting criterion. This is the subject of Combinatorial Rigidity. Secondly, it is important to understand the geometry that rules these constraints. This is the subject of Geometric Rigidity.

We study a combination of constraints on points and lines on rigid planar bodies in Chapter 6 that are inspired by CAD constraints and first initiated by A. Lee-St.John et al. in [25] for 3D rigid bodies. The restricted motion of two bodies to sliding along a direction (a line, for example) relative to each other is a translation, which is described as an instantaneous rigid motion whose center (a 2-extensor in $\mathbb{R}^3$) is at infinity in the language of projective geometry.

Projective geometry is known to be the geometry of the first-order motions of bar-joint frameworks because projective transformations preserve the first-order rigidity of bar-joint frameworks and far beyond that. Even in situations that projective transformations do not preserve the constraints (such as tensegrity frameworks) they still help understand different but equivalent types of first-order rigid constraints of those types (Chapter 5). The importance of projective geometry is highlighted in this thesis.

1.3 Outline of thesis

Chapter 2 reviews some fundamental results and techniques in the rigidity theory on which the thesis is relied on. The study of bar-joint and body-bar frameworks covers an extensive part of the literature in Rigidity Theory. We begin to recall some basic definitions and results from infinitesimal rigidity, rigidity, generic rigidity, static rigidity, inductive constructions of bar-joint frameworks in Euclidean spaces. We review the infinitesimal rigidity of spherical bar-joint frameworks and their connection with the infinitesimal rigidity of Euclidean frameworks. The use of the techniques from matroid theory is becoming inevitable to express or achieve more complicated combinatorial results in Rigidity. So we tried to highlight this very briefly in a section. The generic rigidity of point-line frameworks in plane were recently given a combinatorial characterization by B. Jackson and J. Owen [32]. These frameworks became a suitable platform for us to express our rough geometric idea of unifying the rigidity of frameworks with some joints finite and some at infinity as points in the projective plane. Therefore we present the rigidity matrix for point-line frameworks as it is in [32] along with the given combinatorial characterization of the generic rigidity for these frameworks.

We review body-cad constraints introduced in [25] for rigid bodies in $\mathbb{E}^3$. The complexity of the constraints in the space compared to the plane does not allow to include *point-point coincidence* constraints in the combinatorial characterization of the rigidity of these frameworks in 3-space, which was given by A. Lee-St.John and J. Sidman [38] .

The contribution of the thesis starts in Chapter 3. We introduce point-hyperplane frameworks in $\mathbb{E}^n$ and show that hyperplanes may be treated just as points (at infinity). Of course, this relation between points and hyperplanes is the essence of the *duality* in projective geometry but the fact that the kinematic (at the infinitesimal level) of hyperplanes is also completely explained by kinematic of the corresponding points 'at infinity' is remarkable. This leads to a geometric correspondence between the infinitesimal rigidity of point-hyperplane frameworks and bar-joint frameworks with a set of joints realized on a hyperplane in $\mathbb{E}^n$ (see [17]). This is a remarkable achievement because it makes a clear connection between two worlds that look completely different. It also allows us to transfer the results from the context of bar-joint frameworks to point-hyperplane frameworks. In particular, a combinatorial characterization of the infinitesimal rigidity for bar-joint frameworks in the plane with a set of collinear joints is given as a result transferred from the context of point-line frameworks. We use spherical frameworks to express our geometric correspondence.

The rigidity of point-hyperplane frameworks is studied in $\mathbb{E}^n$. Finally, we provide some examples in the plane and give some remarks on inductive construction of point-line frameworks.

In Chapter 4, we show that point-line frameworks can be used to model different types of slider mechanisms, such as pinned-slider frameworks using pinned point-line frameworks. A combinatorial characterization of the generic rigidity of these frameworks is derived in the plane simply using the results on pinned frameworks in the plane. Next, using fixed-normal point-line frameworks, a type of slider mechanisms is defined for which the sliders move along lines with fixed normals. A combinatorial characterization of the generic rigidity of this type of sliders is given. Our proof describes how an isostatic graph with respect to one type of rigidity (point-line) is transferred to an isostatic graph in another type of rigidity (fixed-normal). Finally we define a type of slider mechanisms for which sliders may only have rotational motions but not translational, using point-line frameworks with rotatory lines whose distance from a reference point is fixed.

In Chapter 5, we introduce and study point-hyperplane tensegrity frameworks in $\mathbb{E}^n$. At first, we establish the static theory and tensegrity frameworks in $\mathbb{S}^n$. Then, we show that the standard results about the tensegrity frameworks also hold in spherical spaces. We need the spherical results to prove the main theorems about the infinitesimal rigidity of point-hyperplane tensegrity frameworks in $\mathbb{E}^n$. Then the rigidity of point-hyperplane tensegrity frameworks are studied and characterized in $\mathbb{E}^n$.

In Chapter 6, we study body-cad constraints in the plane and give a combinatorial characterization of their rigidity in the plane.

Using this combinatorial result in 2D, we describe the infinitesimal rigidity of body-bar frameworks with coincidence constraints in the plane. In turn, this will be used to combinatorially understand the infinitesimal rigidity of a class of spatial rigid bodies with point-point coincidence constraints in 3D.

1.3.1 Statement of authorship

Except for the results that are clearly and directly referenced to the joint work [17] with B. Jackson, A. Nixon, B. Schulze, S. Tanigawa and W. Whiteley in Chapters 3 and 4, this thesis is the original work of the author. The results referenced to the joint work [17] in this thesis are either immediate consequences of the results of this thesis or the author significantly contributed into their formation and towards their final proofs.

Chapter 2

Background: Rigidity of frameworks

In this chapter we review some fundamental concepts of the rigidity of frameworks that form the background of our research and the results or methods that will be used in this thesis.

2.1 Basic definitions of Graph Theory

Graphs are used to record geometric constraints on pairwise geometric objects such as points, lines, planes or rigid bodies and so on.

Definition 2.1.1. A graph is a pair of finite sets $(V, E), V \neq \phi$ where V is called the *set of vertices* and E, disjoint from V, is called the *set of edges* and it consists of unordered pairs of vertices. The set E is a *multiset*, meaning its elements may occur more than once so that every element has a *multiplicity*. We label the vertices with letters indexed with numbers (for example: $v_1, v_2, \ldots, v_n, \ldots$) or simply, with numbers $1, 2, 3, \ldots$ if there is no confusion. Similarly, edges are labeled by indexed letters: an edge e connecting a pair of vertices v_1, v_2 is denoted by $e = \{v_1, v_2\}$ or $e = v_1v_2$ or simply, $e = 12$ if there is no ambiguity. A graph with the set of vertices

V and the set of edges E is denoted by $G = (V, E)$. Also, we use $|V| = v$ and $|E| = e$ for the number of the elements in V and E, respectively.

The following terminologies are useful:

1. For an edge $e = \{v_1, v_2\}$, the two vertices v_1 and v_2 are the *end-points* (or the *ends*) of the edge e.

2. Edges with the same ends are *parallel*.

3. An edge with identical ends, $\{v, v\}$ is a *loop*. We do not allow loops for the graphs in this thesis.

4. A graph is *simple* if it has no parallel edges and no loops. A graph that has parallel edges or loops is called a *multigraph*.

5. Two vertices v_1 and v_2 are *adjacent* if they are connected by an edge, i.e. $e = \{v_1, v_2\} \in E$; the edge e is called *incident* to the vertices v_1 and v_2 and also, v_1 and v_2 are *incident* to e.

6. The *valence*[1] of a vertex v, written as $val(v)$, is the number of edges adjacent to v. By convention, loops are counted twice and parallel edges contribute separately to the valence of a vertex.

7. An *isolated* vertex is a vertex whose valence is 0.

Given a graph $G = (V, E)$ and $E' \subseteq E$, $\nu(E')$ is the number of vertices incident to the edges in E'. We draw a graph with vertices depicted by circles and edges by line segments. A simple graph that contains every possible edge between all the vertices is called a *complete graph*. A complete graph with n vertices is denoted by K_n.

The graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if

[1] It is also called *degree* of v.

1. $V' \subseteq V$, and

2. $E' \subseteq E$.

A subgraph $G' = (V', E')$ of G with $V' = V$ is called a *spanning subgraph*. The *subgraph* $G[E']$ *of* $G = (V, E)$ *induced by the edge set* $E' \subseteq E$, is a graph where V' is the set of the ends of the edges in E'. The subgraph of G with the edge set $E \backslash E'$ is simply written as $G - E'$. Similarly, the graph obtained from G by adding a set of edges E' is denoted by $G + E'$. If $E' = \{e\}$ we write $G - e$ and $G + e$ instead of $G - \{e\}$ and $G + \{e\}$. The *subgraph of* $G = (V, E)$ *induced by the vertex set* $V' \subseteq V$ is a graph where E' is the set of all edges with both ends in V'. This graph is denoted by $G[V']$. The induced graph $G[V \backslash V']$ is denoted by $G - V'$. If $V' = \{v\}$ we write $G - v$ for $G - \{v\}$. A complete subgraph of G is called a *clique* of G. A graph $G = (V, E)$ is said to be *bipartite* if its vertex set has a partition of two sets V_1 and V_2 so that every edge in E has one end in V_1 and another end in V_2. If each vertex of V_1 is connected to each vertex of V_2 the bipartite graph is called a *complete bipartite graph*, denoted by $K_{m,n}$ where $|V_1| = m$, $|V_2| = n$.

By a *path* from vertex u to v, we mean a sequence of vertices $v_0, v_1, \ldots, v_k$ so that :

1. $v = v_0$ and $u = v_k$;

2. $v_0, v_1, \ldots, v_k$ are distinct;

3. v_{i-1} and v_i are adjacent, for $i = 1, \ldots, k$.

A path from vertex u to v is called *closed* if $u = v$. A closed path is called a *circuit*. A graph is *connected* if for every two vertices u and v of G, there is a path from u to v. A connected graph with no circuit is called a *tree*. A *spanning tree* is a spanning subgraph that is a tree.

For two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph whose vertex set is $V_1 \cup V_2$ and its edge set is $E_1 \cup E_2$.

2.2 Bar-joint frameworks in $\mathbb{E}^n$

In this section, we review fundamental concepts of the rigidity of bar-joint frameworks. Throughout this section, by a framework we mean a bar-joint framework.

Definition 2.2.1. An *n-dimensional bar-joint framework*, denoted by $(G, \mathbf{p})$, consists of a simple graph $G = (V, E)$ and an embedding $\mathbf{p}$ of the vertex set V into $\mathbb{E}^n$, $\mathbf{p} : V \to \mathbb{R}^n$ with $\mathbf{p}(i) = p_i$, for every vertex $i \in V$.

Often, the embedding $\mathbf{p}$ is identified by a point in $\mathbb{R}^{nv}$ which in this case, it is called a *configuration* of v points in $\mathbb{E}^n$. It is also said that the graph G is *realized* by $\mathbf{p}$.

First we formalize a definition of a motion of a framework as follows.
Let $(G, \mathbf{p})$ be a framework in $\mathbb{E}^n$. A *motion* of $(G, \mathbf{p})$ is a smooth path $\mathbf{P} : [0, 1] \to \mathbb{R}^{nv}$ with $\mathbf{P}(t) = (P_1(t), \dots, P_v(t)) \in \mathbb{R}^{nv}$ such that $\mathbf{P}(0) = \mathbf{p}$ and

$$\|P_i(t) - P_j(t)\| = \|p_i - p_j\| \qquad \text{for all } t \in [0, 1] \text{ and all } ij \in E,$$

where $\|.\|$ is the Euclidean distance in $\mathbb{R}^n$.

A motion $\mathbf{P} : [0, 1] \to \mathbb{R}^{nv}$ of a framework $(G, \mathbf{p})$ is a *rigid motion* if the distances between all vertices of G are preserved by the motion:

$$\|P_i(t) - P_j(t)\| = \|p_i - p_j\| \qquad \text{for all } t \in [0, 1] \text{ and all } i, j \in V. \tag{2.2.0.1}$$

Explicitly, a rigid motion of points $p_1, \dots, p_v$ in $\mathbb{E}^n$ is of the form $P_i(t) = R(t)p_i + s(t)$, $1 \le i \le v$, where $R(t) \in SO(n)$ is in the Special Orthogonal Group of $\mathbb{E}^n$ for all $t \in [0, 1]$, with $R(0) = I_{n\times n}$, and $s(t) \in \mathbb{R}^n$ for all $t \in [0, 1]$, with $s(0) = 0$. Rigid motions are also called *trivial motions* as they trivially satisfy (2.2.0.1) for any configuration $\mathbf{p}$.

If a motion of a framework $(G, \mathbf{p})$ is not a rigid motion then the distance between at least one pair of vertices is altered by the motion. This motion is called a *deformation* or a *finite flex or finite motion* of $(G, \mathbf{p})$. A non-rigid framework is called *flexible.* A framework is said to be *rigid* if all of its motions are rigid motions, that is, it admits no deformations.

2.2.1 Infinitesimal rigidity

An *infinitesimal motion*[2] of a framework $(G, \mathbf{p})$ in $\mathbb{E}^n$ is a function $\mathbf{p}' : V \to \mathbb{R}^n$, $\mathbf{p}'(i) = p_i' \in \mathbb{R}^n$ so that

$$\langle p_i - p_j, p_i' - p_j' \rangle = 0 \qquad \text{for all } ij \in E, \tag{2.2.1.1}$$

where $\langle \, , \rangle$ is the standard inner product in $\mathbb{R}^n$.

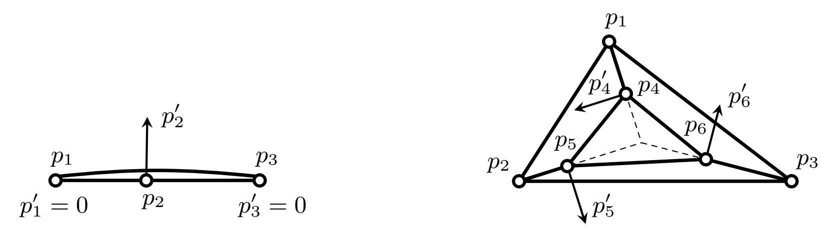
An infinitesimal motion $\mathbf{p}'$ of $(G, \mathbf{p})$ is called an *infinitesimal rigid motion* if $p_i' = Sp_i + t$ for all $i \in V$, some $n \times n$ skew-symmetric matrix S and $t \in \mathbb{R}^n$. In fact, p_i' is the sum of an *infinitesimal rotation* Sp_i and an *infinitesimal translation* t at p_i in $\mathbb{E}^n$. Clearly, infinitesimal rigid motions of a framework $(G, \mathbf{p})$ in $\mathbb{E}^n$ form a linear subspace of $\mathbb{R}^{nv}$. We denote it by $\mathcal{T}(\mathbf{p})$. If the points $p_1, \ldots, p_v$ generate an affine space of dimension at least $n-1$ in $\mathbb{E}^n$, we have $\dim(\mathcal{T}(\mathbf{p})) = \binom{n+1}{2}$. A configuration of points with this property is called *non-degenerate.* We assume the point configurations are non-degenerate unless otherwise is specified.

Remark. Note that $\dim(\mathcal{T}(\mathbf{p}))$ will drop if the above assumption does not hold. For example, for a single bar $\mathbf{p} = (p_1, p_2)$ in $\mathbb{E}^3$, $\dim(\mathcal{T}(\mathbf{p})) = 5$ as the rotation whose axis contains this bar gives velocities $(p_1', p_2') = (0, 0)$ at $\mathbf{p}$, which is null.

Clearly, infinitesimal rigid motions trivially satisfy (2.2.1.1) for any configuration $\mathbf{p}$ because $\langle x, Sx \rangle = 0$, for any $x \in \mathbb{R}^n$ and any $n \times n$ skew-symmetric matrix. Hence they are also called *trivial infinitesimal motions.*

The coefficient matrix of (2.2.1.1), called the *rigidity matrix* of $(G, \mathbf{p})$, is a $e \times nv$ matrix whose rows and each set of n columns are indexed by the edges and the vertices of G, respectively. More clearly, a row ij of this matrix is a vector in $\mathbb{R}^{nv}$

[2] also called an *infinitesimal flex*, *first-order motion* or *first-order flex.*



(a) (b)

Figure 2.1: Singular configurations

that has the coordinates of $p_i - p_j$ for the n components indexed by i (the n columns under p_i in the matrix), the coordinates of $p_j - p_i$ in for the n components indexed by j (the n columns under p_j in the matrix) and zeros for the rest of the components:

$$R(G, \mathbf{p}) = \begin{array}{c c} & \begin{array}{ccccc} & p_i & & p_j & \end{array} \\ \text{edge } ij & \left(\begin{array}{ccccc} & \vdots & & \vdots & \\ \dots & p_i - p_j & \dots & p_j - p_i & \dots \\ & \vdots & & \vdots & \end{array} \right) \end{array}, \qquad (2.2.1.2)$$

By (2.2.1.1) any infinitesimal motion of $(G, \mathbf{p})$ is exactly an element of ker $R(G, \mathbf{p})$.

A framework $(G, \mathbf{p})$ is said to be *infinitesimally rigid* in $\mathbb{E}^n$ if every infinitesimal motion of $(G, \mathbf{p})$ is an infinitesimal rigid motion, i.e. ker $R(G, \mathbf{p}) = \mathcal{T}(\mathbf{p})$. Otherwise, it is called *infinitesimally flexible*. If a framework $(G, \mathbf{p})$ has a non-rigid motion (or non-trivial motion) in $\mathbb{E}^n$, its derivative at $t = 0$ gives rise to a non-trivial infinitesimal motion. So the following result seems plausible (see [10] for a proof):

Theorem 2.2.1. *If a framework $(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^n$ then it is rigid in $\mathbb{E}^n$.*

The converse is not always true. The simplest example is a collinear triangle shown in Figure 2.1a. A more subtle example is *Desargues' configuration* shown in Figure 2.1b in which the dashed lines containing the edges $\{p_2, p_5\}$, $\{p_3, p_6\}$ and $\{p_1, p_4\}$ are concurrent. In both cases, the frameworks are not infinitesimally rigid but they are rigid. The non-trivial infinitesimal motions (up to infinitesimal rigid motions) are illustrated in each case. These infinitesimal motions do not lead to a finite motion but they arise as a result of the *special* configurations. These configurations form the *singular* points of (2.2.1.2) where the rank of (2.2.1.2) increases in an open neighbourhood of such configurations.

More explicitly, the rigidity matrix $R(G, \mathbf{p})$ is closely related to the differential of the *rigidity map* f_G *of* G given by:

$$\begin{aligned} f_G &: \mathbb{R}^{nv} \to \mathbb{R}^e \\ f_G(\mathbf{p}) &= (\ldots, \|p_i - p_j\|^2, \ldots), \end{aligned} \tag{2.2.1.3}$$

It is easy to check that the *differential map* $df_G(\mathbf{p})$ *of* f_G at $\mathbf{p}$ is equal to $2R(G, \mathbf{p})$. A configuration $\mathbf{p}$ is called a *regular*[3] point of f_G if $df_G(\mathbf{p})$ maintains its rank in an open neighbourhood of $\mathbf{p}$ in $\mathbb{R}^{nv}$, or equivalently, the rank of $R(G, \mathbf{q})$ is maximized at $\mathbf{p}$ over all configurations $\mathbf{q} \in \mathbb{R}^{nv}$. In this case, the level set $f_G^{-1}(f_G(\mathbf{p}))$ is a smooth closed embedded manifold in $\mathbb{R}^{nv}$ whose codimension is the rank of $df_G(\mathbf{p})$ (see [37]).

So both the configurations in Figure 2.1 are not regular but they are singular points of f_G. It is proved in [3] that the rigidity of a framework is equivalent to its infinitesimal rigidity for regular configurations.

Theorem 2.2.2. *[3] A framework $(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^n$ if and only if $\mathbf{p}$ is a regular point of f_G and $(G, \mathbf{p})$ is rigid.*

Suppose the rank of $df_G(x)$ is $k = \text{rank } df_G(\mathbf{p})$ for all x in an open neighbourhood around $\mathbf{p}$ in $\mathbb{R}^{nv}$. Let $P(x)$ be the sum of the squares of the determinants of all $k \times k$ minors of $df_G(x)$, for $x \in \mathbb{R}^{nv}$. Obviously $P(x)$ is the sum of a finite number of non-negative polynomials in nv variables. Since $P(x)$ is not zero at $\mathbf{p}$, it is a non-trivial polynomial in nv variables. Therefore the set of all regular points of

[3] In differential geometry, the term *regular* is often used when $df_G(\mathbf{p})$ is surjective, i.e. the rank of $df_G(\mathbf{p})$ is e in our context, but here, it is only required that the rank remains *constant*.

f_G, $\{x \in \mathbb{R}^{nv} | P(x) \neq 0\}$ is a dense open subset of $\mathbb{R}^{nv}$ whose complement set is of Lebesgue measure zero. Hence the above theorem determines the rigidity or flexibility of a graph G in $\mathbb{E}^n$ for *almost all* configurations $\mathbf{p}$ in $\mathbb{E}^n$. In fact, this explains why the rigidity of a framework is sometimes described as the rigidity of its graph. This is the foundation of the *combinatorial* aspect of Rigidity Theory.

In the literature of Rigidity Theory, the term 'generic' is more popular to use than 'regular'. There is a large portion of literature that uses this term in the following meaning:

A configuration $\mathbf{p}$ in $\mathbb{R}^{nv}$ is called *generic* if the set of all the coordinates $\{x_1, \ldots, x_{nv}\}$ of points $p_1, \ldots, p_v$ are *algebraically independent* over $\mathbb{Q}$.

Note that if a configuration is generic then it is regular but the converse is not necessarily true. We say a framework $(G, \mathbf{p})$ is *generically rigid* in $\mathbb{E}^n$ if it is infinitesimally rigid for some, and consequently, for all generic (or regular) configurations $\mathbf{p}$ in $\mathbb{E}^n$.

2.2.2 Static rigidity

In this section, we review key notions and results in the *Static Theory* of rigidity of bar-joint frameworks. This point of view of rigidity has a vast literature in mathematics, physics [40] and structural engineering. The reader can consult [16], [68] and [23] to obtain a general view. Static theory has also been used to derive many results about the rigidity of bar-joint frameworks, such as *inductive constructions* of rigid graphs [61].

Definition 2.2.2. Suppose $(G, \mathbf{p})$ is a framework in $\mathbb{E}^n$. A *self-stress* of $(G, \mathbf{p})$ is a function $\omega : E \to \mathbb{R}$ with $\omega = (\omega_{ij})_{ij \in E}$ such that for every $i \in V$,

$$\sum_{j, ij \in E} \omega_{ij}(p_i - p_j) = 0. \tag{2.2.2.1}$$

It is understood that $\omega_{ij} = \omega_{ji}$ for all $ij \in E$.

The set of stresses of a framework $(G, \mathbf{p})$ forms a linear space in $\mathbb{R}^e$, denoted by $\mathcal{S}(E)$. A self-stress ω could also be identified as member of $\mathbb{R}^E$ so that $\omega R(G, \mathbf{p}) = 0$ by (2.2.2.1), with ω as a row vector in $\mathbb{R}^e$. Linear algebraically, self-stresses are the coefficients of a linear dependence relation among the rows of $R(G, \mathbf{p})$. As (2.2.2.1) indicates, at each vertex p_i, the internal *forces* $\omega_{ij}(p_i - p_j), ij \in E$, add up to zero so that the structure is in equilibrium. Moreover, in structural engineering, self-stresses record internal forces as *tension* (if $\omega_{ij} < 0$) and *compression* (if $\omega_{ij} > 0$) along the bars in the structure.

A non-empty subset E' of the edge set E is called *independent* if there is a realization $\mathbf{p}$ of G for which the corresponding rows in $R(G, \mathbf{p})$ to the edges in E' are linearly independent. The framework $(G, \mathbf{p})$ is called independent if E is independent, i.e. $\mathcal{S}(E)$ is trivial. A framework that is both independent and infinitesimally rigid is called *isostatic*.

Definition 2.2.3. (see [61, p.28],[11]) Let $\mathbf{p}$ be a configuration of points $(p_1, \ldots, p_v)$ in $\mathbb{E}^n$ and $f_1, \ldots, f_v$, $f_i \in \mathbb{R}^n$ is a sequence of forces in $\mathbb{E}^n$ at the points $p_1, \ldots, p_v$, respectively. We say $\mathbf{F} = (f_1, \ldots, f_v)$ is an *equilibrium force*[4] at $\mathbf{p}$ if

$$\begin{aligned} \sum_{i=1}^{v} f_i &= 0, \\ \sum_{i=1}^{v} f_i \wedge p_i &= 0, \end{aligned} \tag{2.2.2.2}$$

where $\wedge$ is the exterior product in $\mathbb{R}^n$.

Using the language of Grassman-Cayley algebra, the equations (2.2.2.2) may be combined into one single equation $\sum_{i=1}^{v} \tilde{f}_i \vee \tilde{p}_i = 0$ where $\tilde{f}_i = (f_i, 0) \in \mathbb{R}^{n+1}$ is a point at infinity (in $\mathbb{P}^n$) and $\tilde{p}_i = (p_i, 1)$ (see [16]).

The set of equilibrium loads at $\mathbf{p}$ is a linear space by (2.2.2.2). We denote it by $\mathcal{E}(\mathbf{p})$. The space of equilibrium loads at $\mathbf{p} = (p_1, \ldots, p_v)$ is the orthogonal complement of $\mathcal{T}(\mathbf{p})$ in $\mathbb{R}^{nv}$, see [68], [11].

[4] also called *equilibrium load.*

A framework $(G, \mathbf{p})$ is *statically rigid* in $\mathbb{E}^n$ if every equilibrium force $\mathbf{F} = (f_1, \dots, f_v)$ is in the row space of the rigidity matrix $R(G, \mathbf{p})$. Namely,

$$\sum_{j, ij \in E} \omega_{ij}(p_i - p_j) = f_i, \tag{2.2.2.3}$$

for every $i \in V$ and some stress $\omega = (\omega_{ij}) \in \mathbb{R}^e$. If (2.2.2.3) holds for some force $\mathbf{F} = (f_1, \dots, f_v)$, the stress ω is called a *resolution* of $\mathbf{F}$ or $\mathbf{F}$ is resolved by stress ω. Note that if a force $\mathbf{F}$ is resolved then it is an equilibrium force. A framework $(G, \mathbf{p})$ is statically rigid if rank $R(G, \mathbf{p}) = nv - \binom{n+1}{2}$. Indeed, static rigidity and infinitesimal rigidity are dual concepts.

Theorem 2.2.3. *([16], [61], [11]) A framework $(G, \mathbf{p})$ in $\mathbb{E}^n$ is statically rigid if and only if it is infinitesimally rigid.*

The language of static rigidity has been used to tackle some problems in rigidity [68]. The term 'stress', defined above, comes from the vocabulary of the static theory of rigidity but its use extends further to the study of important geometric problems such as *global rigidity*, see [9]. In addition, the static theory of rigidity is used to study the rigidity of *tensegrity frameworks* [50]. Here we describe these types of frameworks.

Suppose $G = (V, E)$ is a simple graph. Assign signs $+$ or $-$ to some edges, not necessarily all. This partitions the edges into three sets: E_+, the edges with $+$ sign; E_-, the edges with minus sign and $E_\circ$ are the edges with no sign assigned to them. The graph G is now called a *signed graph*, which we denote by $G^\pm = (V, E_\circ, E_-, E_+)$.

Suppose G is realized in $\mathbb{E}^n$ by an embedding $\mathbf{p} = (p_1, \dots, p_v) \in \mathbb{R}^{nv}$. Consider the following system:

$$\langle p_i - p_j, p_i' - p_j' \rangle = 0 \qquad \text{for all } ij \in E_\circ, \tag{2.2.2.4}$$

$$\langle p_i - p_j, p_i' - p_j' \rangle \leq 0 \qquad \text{for all } ij \in E_-, \tag{2.2.2.5}$$

$$\langle p_i - p_j, p_i' - p_j' \rangle \geq 0 \qquad \text{for all } ij \in E_+, \tag{2.2.2.6}$$

where p_i' are unknowns for every $i = 1, \ldots, v$. The signed graph G along with a configuration $\mathbf{p}$ in $\mathbb{E}^n$ subject to the constraints (2.2.2.4)–(2.2.2.6), is called a *tensegrity framework* in $\mathbb{E}^n$ and we denote it by $(G^{\pm}, \mathbf{p})$.

The tensegrity framework $(G^{\pm}, \mathbf{p})$ is called *infinitesimally rigid* if the only possible solutions $p_i' \in \mathbb{R}^n$, $i = 1, \ldots, v$ of the system (2.2.2.4)–(2.2.2.6) are the trivial infinitesimal motions of the framework $(G, \mathbf{p})$. It turns out that the rigidity of a tensegrity framework $(G^{\pm}, \mathbf{p})$ is closely related to that of the corresponding framework $(G, \mathbf{p})$. For a definition of the *rigidity* of a tensegrity framework, we refer the reader to [50], [69].

A *proper stress* $\omega = (\omega_{ij})_{ij \in E}$ of a tensegrity framework $(G^{\pm}, \mathbf{p})$ is a self-stress of $(G, \mathbf{p})$ that *respects* the signs of all the corresponding edges in $G^{\pm}$, i.e. $\omega_{ij} \leq 0$ if $ij \in E_-$, $\omega_{ij} \geq 0$ if $ij \in E_+$ and no condition on ω_{ij} if $ij \in E_\circ$.
A proper stress $\omega = (\omega_{ij})_{ij \in E}$ is called *strict* if $\omega_{ij} \neq 0$, for all $ij \in E_- \cup E_+$. A tensegrity framework $(G^{\pm}, \mathbf{p})$ is called *statically rigid* if every equilibrium force at $\mathbf{p}$ is resolved by a proper stress, as described in (2.2.2.3).

Theorem 2.2.1 has an analogue for tensegrity frameworks. See [10], [12] of Connelly and [50] of Roth and Whiteley for a proof.

Theorem 2.2.4. *If a tensegrity framework $(G^{\pm}, \mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^n$ then it is rigid in $\mathbb{E}^n$.*

We next state the main result that characterizes the rigidity of tensegrity frameworks in $\mathbb{E}^n$. It also establishes connections between the infinitesimal rigidity and the rigidity for tensegrity frameworks.

Theorem 2.2.5. *([50, Theorem 5.2.]) Suppose $(G^{\pm}, \mathbf{p})$ is a tensegrity framework in $\mathbb{E}^n$. Then the following are equivalent:*

(a) *$(G^{\pm}, \mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^n$.*

(b) *$(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^n$ and there exists a strict stress of $(G^{\pm}, \mathbf{p})$.*

(c) $(G, \mathbf{p})$ *is rigid in* $\mathbb{E}^n$, $\mathbf{p}$ *is a regular point of* G *and there exists a strict stress of* $(G^{\pm}, \mathbf{p})$.

(d) $(G^{\pm}, \mathbf{p})$ *is rigid in* $\mathbb{E}^n$, $\mathbf{p}$ *is a regular point of* G *and there exists a strict stress of* $(G^{\pm}, \mathbf{p})$.

A configuration $\mathbf{p}$ is *fully regular*[5] if rank $df_A(\mathbf{p})$ is maximized in an open neighbourhood of $\mathbf{p}$, for every subgraph A of G.

Theorem 2.2.6. *([50, Theorem 5.8.]) If* $\mathbf{p}$ *is fully regular for a graph* G *then* $(G^{\pm}, \mathbf{p})$ *is rigid in* $\mathbb{E}^n$ *if and only if it is infinitesimally rigid in* $\mathbb{E}^n$.

2.3 Equivalence of infinitesimal rigidity of spherical and Euclidean bar-joint frameworks

It is well-known that the infinitesimal rigidity of spherical bar-joint frameworks on the upper (or resp. lower) hemisphere $\mathbb{S}^n_+$ (or resp. $\mathbb{S}^n_-$) is equivalent to the infinitesimal rigidity of bar-joint frameworks in $\mathbb{E}^n$ under the central projection (see [52], [54] and [47, Chapter V]). See Figure 2.2.

The central projection provides a geometric correspondence between the infinitesimal rigidity class of spherical frameworks and their projection into $\mathbb{E}^n$. But this correspondence naturally excludes a hyperplane in $\mathbb{S}^n$ (the 'equator') whose points have no image in $\mathbb{E}^n$ under the central projection. This thesis extends this geometric correspondence to include the equator in $\mathbb{S}^n$!

2.3.1 Spherical bar-joint frameworks

A n-sphere $\mathbb{S}^n$ is defined as the subset,

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\},$$

[5]The term *general position* is used in [50] instead of fully regular.

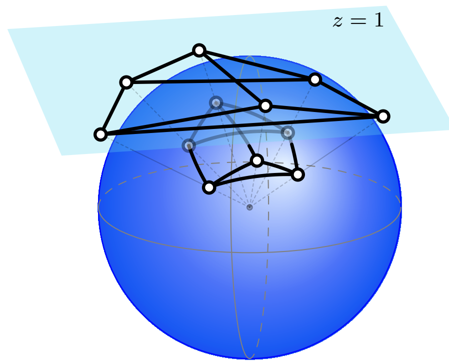


Figure 2.2: Infinitesimal rigidity of a framework on upper hemisphere is equivalent to that of its projection in the Euclidean plane.

of $\mathbb{R}^{n+1}$. The upper hemisphere $\mathbb{S}^n_+$ is $\mathbb{S}^n_+ = \{x \in \mathbb{S}^n | \langle x, e_{n+1} \rangle > 0\}$, where $e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ is of length 1. Similarly, $\mathbb{S}^n_- = \{x \in \mathbb{S}^n | \langle x, e_{n+1} \rangle < 0\}$. The equator of $\mathbb{S}^n$ consists of $x \in \mathbb{S}^n$ such that $\langle x, e_{n+1} \rangle = 0$.

Let $G = (V, E)$ be a simple, finite, undirected graph with $|V| = v$ vertices and $|E| = e$ edges. A *bar-joint framework in* $\mathbb{S}^n$, denoted by $(G, \mathbf{p})$, is composed of a simple graph $G = (V, E)$ and an embedding $\mathbf{p}$ of vertices V into $\mathbb{S}^n$, i.e. $\mathbf{p} = (p_1, p_2, \dots, p_v)$ where $p_i \in \mathbb{S}^n$ for all $i \in V$. A *motion* of a framework $(G, \mathbf{p})$ is a smooth path $\mathbf{p}(t) = (p_1(t), \dots, p_v(t)) : [0, 1] \to \mathbb{R}^{(n+1)v}$ with $\mathbf{p}(0) = \mathbf{p}$ such that

$$\begin{aligned} \langle p_i(t), p_j(t) \rangle &= \langle p_i, p_j \rangle, \\ \langle p_i(t), p_i(t) \rangle &= 1, \end{aligned} \tag{2.3.1.1}$$

for all $t \in [0, 1]$ and all $ij \in E$, $i \in V$. A framework $(G, \mathbf{p})$ is *rigid* if for every motion $\mathbf{p}(t)$ we have $p_i(t) = R(t)p_i$, for all $i \in V$ and all t close enough to 0 where $R(t)$ is the one-parameter group generated by a rotation $R_0 = R \in SO(n + 1)$. An *infinitesimal motion* of $(G, \mathbf{p})$ is an assignment of vectors $p'_i \in \mathbb{R}^{n+1}$, for each $i \in V$, such that

$$\begin{aligned} \langle p_j, p'_i \rangle + \langle p_i, p'_j \rangle &= 0 \qquad \text{for all } ij \in E, \\ \langle p_i, p'_i \rangle &= 0 \qquad \text{for all } i \in V. \end{aligned} \tag{2.3.1.2}$$

A framework $(G, \mathbf{p})$ is called *infinitesimally rigid* if for any infinitesimal motion

$(p'_1, \ldots, p'_v)$ of $(G, \mathbf{p})$, $p'_i = Sp_i$ for every $i \in V$ and for some skew symmetric $(n+1) \times (n+1)$ matrix S. Otherwise, it is called *infinitesimally flexible*. The infinitesimal motions of $(G, \mathbf{p})$ form a linear subspace of $\mathbb{R}^{(n+1)v}$, which is also, the kernel of a $(e+v) \times (n+1)v$ matrix called the *rigidity matrix* $R_{\mathbb{S}}(G, \mathbf{p})$ of the framework $(G, \mathbf{p})$ in $\mathbb{S}^n$:

$$R_{\mathbb{S}}(G, \mathbf{p}) = \begin{array}{c} \\ \\ \text{edge } \{i,j\} \\ \\ \text{vertex } i \\ \\ \text{vertex } j \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & \\ \ldots & p_j & \ldots & p_i & \ldots \\ & \vdots & \vdots & \vdots & \\ \ldots & p_i & \cdots & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \ldots & 0 & \cdots & p_j & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}, \tag{2.3.1.3}$$

where $p_i \in \mathbb{S}^n$, $i \in V$. The rows are indexed by the finite set $E \cup V$ and corresponding to each vertex i in V, there are $n+1$ columns indexed by i. One can observe that the equations in (2.3.1.2) are actually *homogeneous* with respect to the pairs (p_i, p'_i) for any $i \in V$. This means p_i may be scaled to have any length other than 1 so that points p_i are on different concentric spheres while the infinitesimal rigidity or infinitesimal flexibility remains invariant. Geometrically, this is where the *projective geometry* begins to appear. The matrix $R_{\mathbb{S}}(G, \mathbf{p})$ captures a lot of information about the infinitesimal rigidity of different types of frameworks, as we will see in Chapter 3.

It is important to remember that, not all infinitesimal motions arise as the derivatives of real motions but all non-trivial finite motions do generate non-trivial infinitesimal motions. This again suggests that infinitesimal rigidity implies rigidity. Similar to the argument in Section 2.2.2 we can see that if $\mathbf{p}$ is a regular point of $R_{\mathbb{S}}(G, \mathbf{p})$ then infinitesimal rigidity is equivalent to rigidity in $\mathbb{S}^n$.

The equations in (2.3.1.2) can also be equivalently rewritten as

$$\begin{aligned} \langle p_j - p_j, p_i' - p_j' \rangle &= 0 \qquad \text{for all } ij \in E, \\ \langle p_i, p_i' \rangle &= 0 \qquad \text{for all } i \in V. \end{aligned} \tag{2.3.1.4}$$

This correspondence appears in the rigidity matrix through simple row reductions on the matrix $R_{\mathbb{S}}(G, \mathbf{p})$ to obtain the matrix $R_{\mathbb{S}^*}(G, \mathbf{p})$

$$R_{\mathbb{S}^*}(G,\mathbf{p}) = \begin{array}{c} \\ \\ \text{edge } \{i,j\} \\ \\ \text{vertex } i \\ \\ \text{vertex } j \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & p_i & & p_j & \end{array} \\ \left(\begin{array}{ccccc} & \vdots & & \vdots & \\ \dots & p_i - p_j & \dots & p_j - p_i & \dots \\ & \vdots & \vdots & \vdots & \\ \dots & p_i & \cdots & \mathbf{0} & \cdots \\ & \vdots & \vdots & \vdots & \\ \dots & \mathbf{0} & \cdots & p_j & \cdots \\ & \vdots & & \vdots & \end{array} \right) \end{array}. \tag{2.3.1.5}$$

We summarize these observations with a proposition.

Proposition 2.3.1. *There is an invertible* $(e + v) \times (e + v)$ *matrix* T *such that* $T \times R_{\mathbb{S}}(G, \mathbf{p}) = R_{\mathbb{S}^*}(G, \mathbf{p})$. *As a result, they have isomorphic kernels.*

There are two geometric perspectives related to the representation (2.3.1.4):

- It describes the rigidity (matrix) of a bar-joint framework in $\mathbb{E}^{n+1}$ obtained from $(G, \mathbf{p})$ by connecting each of the v vertices of $(G, \mathbf{p})$ with a new point o (the *origin*) (see Figure 2.3) and then, restricting the infinitesimal motions to those which fix the origin. In matrix notation, this is achieved by deleting the columns corresponding to the origin from the rigidity matrix. This process is called *coning* and the resulting framework is called the *cone framework* denoted by $(G * o, \mathbf{p})$. The above matrix is also called the *cone rigidity matrix* of the cone framework $(G * o, \mathbf{p})$. For more details on coning see [66], [54].

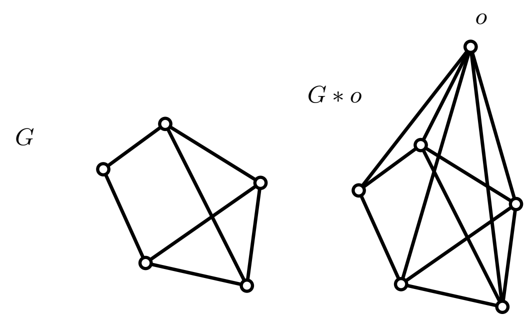


Figure 2.3: A graph G is coned to $G * o$ by a vertex o.

- It describes the rigidity matrix of a bar-joint framework $(G, \mathbf{p})$ in $\mathbb{E}^{n+1}$ *supported* on $\mathbb{S}^n$ (see [44]). Namely, the vertices are realized in $\mathbb{S}^n$ and restricted to move in $\mathbb{S}^n$ but the bars connecting the vertices are not in $\mathbb{S}^n$.

Let $(G, \mathbf{p})$ be a framework in $\mathbb{S}^n$. We may assume that no point p_i, $i \in V$ is in the equator. We then move all the points $p_i = (x_{i,1}, \ldots, x_{i,n+1})$, $x_{i,n+1} \neq 0$, along their radial ray so that they lie in the hyperplane $x_{n+1} = 1$. In other words, we replace p_i by

$$\tilde{p}_i = (x_{i,1}/x_{i,n+1}, \ldots, x_{i,n}/x_{i,n+1}, 1),$$

in the rigidity matrix $R_{\mathbb{S}^*}(G, \mathbf{p})$. So we obtain the following new system which is equivalent to (2.3.1.2) and (2.3.1.4):

$$\begin{aligned} \langle \tilde{p}_j - \tilde{p}_j, \tilde{p}'_i - \tilde{p}'_j \rangle &= 0 \qquad \text{for all } ij \in E, \\ \langle \tilde{p}_i, \tilde{p}'_i \rangle &= 0 \qquad \text{for all } i \in V. \end{aligned} \tag{2.3.1.6}$$

Note that $\tilde{p}'_i = \frac{1}{\langle p_i, e_{n+1} \rangle} p'_i$ where $p'_i \in T_{p_i}\mathbb{S}^n, i \in V$ are motions of $(G, \mathbf{p})$ in $\mathbb{S}^n$. In fact, the point p_i was scaled by the factor $1/\langle p_i, e_{n+1} \rangle$ and therefore, its velocity p'_i is scaled by the same factor.

The resulting matrix, denoted by $R_{\mathbb{A}}(G, \tilde{\mathbf{p}})$, is equivalent to (2.3.1.3) and (2.3.1.5):

$$R_{\mathbb{A}}(G,\tilde{\mathbf{p}}) = \begin{array}{rc} & \begin{array}{ccccc} & \tilde{p}_i & & \tilde{p}_j & \end{array} \\ \begin{array}{r} \\ \text{edge } ij \\ \\ \text{vertex } i \\ \\ \text{vertex } j \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \dots & \tilde{p}_i - \tilde{p}_j & \dots & \tilde{p}_j - \tilde{p}_i & \dots \\ & \vdots & \vdots & \vdots & \\ \dots & \tilde{p}_i & \dots & \mathbf{0} & \dots \\ & \vdots & \vdots & \vdots & \\ \dots & \mathbf{0} & \dots & \tilde{p}_j & \dots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}. \tag{2.3.1.7}$$

In particular, (2.3.1.3), (2.3.1.5) and (2.3.1.7) have isomorphic kernels (see also [54]).

We can now go a step further and completely embed the framework $(G, \tilde{\mathbf{p}})$ into $\mathbb{E}^n$ while the infinitesimal rigidity or the infinitesimal flexibility of the framework is maintained.

To see this, let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$ be the natural projection from $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$. Let

$$\pi(\tilde{p}_i) = \hat{p}_i,$$

and

$$\pi(\tilde{p}_i') = \hat{p}_i',$$

for every $i \in V$. This describes a one-one correspondence, simultaneously between both the points $\tilde{p}_i \leftrightarrow \hat{p}_i$ and their infinitesimal motions $\tilde{p}_i' \leftrightarrow \hat{p}_i'$ for every $i \in V$, considering $\langle \tilde{p}_i, \tilde{p}_i' \rangle = 0$ and $\langle \tilde{p}_i, e_{n+1} \rangle = 1$ for every $i \in V$. If $\hat{p}_i'$ is a motion at $\hat{p}_i$ in $\mathbb{E}^n$ then we can recover $\tilde{p}_i'$ by augmenting a new last component to $\hat{p}_i'$, i.e., $\tilde{p}_i' = (\hat{p}_i', -\langle \hat{p}_i', \hat{p}_i \rangle) \in \mathbb{R}^{n+1}$.

In addition, this correspondence preserves trivial infinitesimal motions. Under this correspondence, (2.3.1.6) in $\mathbb{R}^{n+1}$ is equivalent to the following system in $\mathbb{E}^n$:

$$\langle \hat{p}_j - \hat{p}_j, \hat{p}_i' - \hat{p}_j' \rangle = 0 \qquad \text{for all } ij \in E. \tag{2.3.1.8}$$

The coefficient matrix of (2.3.1.8) looks like:

$$
R(G,\hat{\mathbf{p}}) = \begin{array}{c c} & \begin{array}{ccccc} & \hat{p}_i & & \hat{p}_j & \end{array} \\ \text{edge } ij & \begin{pmatrix} & \vdots & & \vdots & \\ \dots & \hat{p}_i - \hat{p}_j & \dots & \hat{p}_j - \hat{p}_i & \dots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}, \tag{2.3.1.9}
$$

which is the *rigidity matrix* of a bar-joint framework in $\mathbb{E}^n$. Now the matrix $R(G, \hat{\mathbf{p}})$ is of size $e \times nv$. In fact, one column under each point and the v rows corresponding to the vertices in (2.3.1.7), one for each point have been removed from (2.3.1.3). The framework $(G, \hat{\mathbf{p}})$ in $\mathbb{E}^n$ is the *projective image* of $(G, \mathbf{p})$ in $\mathbb{S}^n$ under the central projection.

The transfer from (2.3.1.2) to (2.3.1.8) can be directly described by the following bijections:

$$
\begin{aligned}
\phi &: \mathbb{S}^n \to \mathbb{E}^n \\
\phi(p_i) &= \pi\Big(\frac{1}{\langle p_i, e_{n+1}\rangle} p_i\Big),
\end{aligned} \tag{2.3.1.10}
$$

for every $i \in V$ and

$$
\begin{aligned}
\psi_{p_i} &: T_{p_i}\mathbb{S}^n \to \mathbb{R}^n \\
\psi_{p_i}(p_i') &= \pi\Big(\frac{1}{\langle p_i, e_{n+1}\rangle} p_i'\Big),
\end{aligned} \tag{2.3.1.11}
$$

for every $i \in V$[6]. Figure 2.4 shows this correspondence for a single bar. It shows how trivial infinitesimal motions are projected from sphere to those in the plane and vice versa.

We, therefore, have the following important theorem.

Theorem 2.3.2. *[52] Suppose $(G, \mathbf{p})$ is a bar-joint framework in the upper hemisphere $\mathbb{S}^n_+$ whose central projection into $\mathbb{E}^n$ is $(G, \hat{\mathbf{p}})$. Then $(G, \mathbf{p})$ is infinitesimally rigid in $\mathbb{S}^n$ if and only if $(G, \hat{\mathbf{p}})$ is infinitesimally rigid in $\mathbb{E}^n$.*

[6] The bijection ψ is *not* the differential of the smooth mapping ϕ in (2.3.1.10).

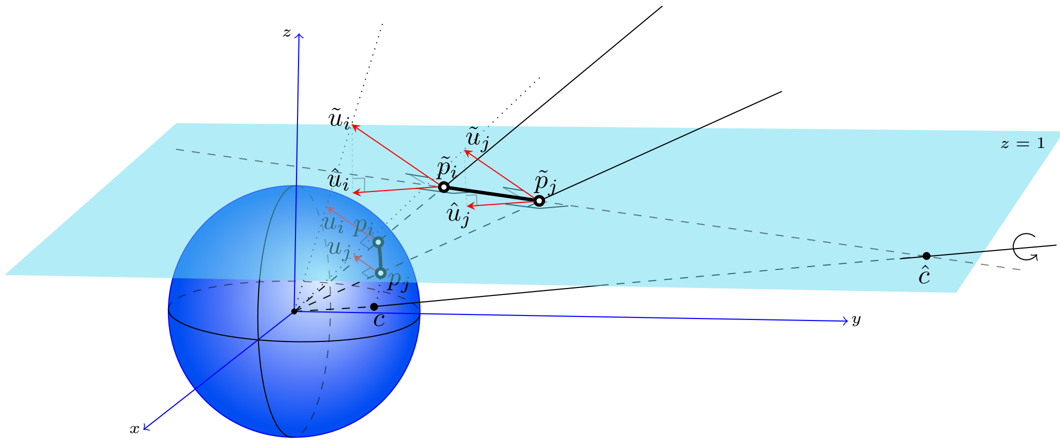


Figure 2.4: One-to-one correspondence of infinitesimal rigid motions on the upper hemisphere $\mathbb{S}^2_+$ and the plane.

2.4 Matroid Theory and Rigidity Matroids

Matroid theory is originated as a generalization of the concept of dependence and independence in linear algebra and graph theory. The reader is referred to [20], [46] for definition(s) of a matroid, basic properties and associated important concepts such as *submodular functions*.

As was briefly explained before, generic rigidity has a combinatorial nature which is explored through the rigidity matrix of the related frameworks at generic configurations. So matroid theory naturally arises from rigidity matrices of frameworks. More specifically, when a graph is realized by a generic embedding $\mathbf{p}$ then the independent rows of the rigidity matrix define independent sets of a matroid called the *rigidity matroid*, see [23], [68]. So we can shift our focus from the framework to its associated graph when $\mathbf{p}$ is generic. For example, we say some edges of a graph are *independent* if the corresponding rows in the rigidity matrix are independent.

An important function associated to a matroid on a ground set S is the *rank function*: for any $T \subseteq S$, $r(T)$ is the size of a maximal independent subset of T, which is well-defined. A matroid is uniquely determined by its rank function [46].

A set of edges is called a *basis* of the rigidity matroid on the vertex set V of G if it is a maximal independent set of the rigidity matroid on V. A graph whose edges are the basis for the rigidity matroid is called an *isostatic* graph.

Now let's consider some well-known rigidity matroids from bar-joint frameworks. Suppose G is a graph and $\mathbf{p}$ embeds the vertices of G collinear as distinct points on a line. This is a generic 1-dimensional framework in $\mathbb{R}$. The rigidity matrix $R(G, \mathbf{p})$ is the *matrix representation* of the *cycle matroid* of the graph G, which is also the *1-rigidity matroid.*

Proposition 2.4.1. *[68] The 1-rigidity matroid of a graph G is the cycle matroid of G. In particular:*

(a) *A set E' of edges is independent if and only if it is a forest.*

(b) *A set E' of edges is independent if and only if $|F'| \leq \nu(F') - 1$ for every non-empty $F' \subseteq E'$.*

(c) *A set E' of edges is a basis for the 1-rigidity matroid on the vertex set of G if and only if it is a spanning tree on the vertex set of G.*

(d) *A set E' of edges is a basis for the 1-rigidity matroid on the vertex set of G if and only if $|E'| = v - 1$ and $|F'| \leq \nu(F') - 1$ for every non-empty $F' \subseteq E'$.*

Given a graph G and a generic embedding $\mathbf{p}$ of the vertices of G in $\mathbb{E}^2$. The *2-rigidity matroid* is characterized by *Laman's Theorem*; a landmark theorem in the theory of rigidity that gives a characterization of the generic rigidity of bar-joint frameworks in $\mathbb{E}^2$.

Theorem 2.4.2 (Laman's Theorem)**.** *[35] Let G be a graph $G = (V, E)$ with $v \geq 2$ vertices. Then G is a basis for the* 2-rigidity matroid *on v vertices if and only if $e = 2v - 3$ and $|E'| \leq 2\nu(E') - 3$ for every non-empty $E' \subseteq E$.*

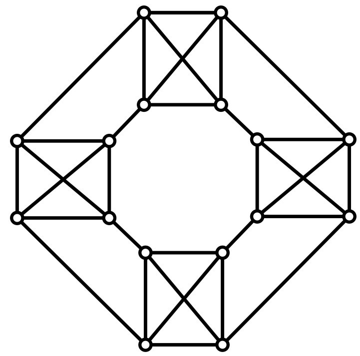


Figure 2.5: A graph with 4 maximal rigid components.

Crapo's theorem [15] and Recski's theorem [48] are another two equivalent combinatorial characterizations of isostatic graphs in the plane.

Each of these characterizations has an associated polynomial time algorithm to verify whether a given graph is isostatic. The *pebble game* [36] is widely used for this purpose.

The following result describes the rank function r of the 2-rigidity matroid.

Theorem 2.4.3. *([39], [23]) For a graph $G = (V, E)$, the rank function of 2-rigidity matroid is given by*

$$r(E) = min \sum_{i=1}^{k} (2\nu(E_i) - 3),$$

where the minimum is taken over all partitions $\{E_i\}_{i=1}^{k}$ of E. The minimum is achieved when E_i's are the maximal rigid components of G.

Example 2.4.1. Figure 2.5 shows a graph whose maximal rigid components consists of 4 quadrilaterals with their diagonals R_1, R_2, R_3 and R_4 along with 8 single bars B_i, $1 \leq i \leq 8$. Since $r(R_i) = 5$, $i = 1, 2, 3, 4$ and $r(B_i) = 1, 1 \leq i \leq 8$, by Theorem 2.4.3, we have

$$r(E) = 4r(R_1) + 8r(B_1) = 20 + 8 = 28.$$

So the graph has one degree of freedom:

$$(2v - 3) - r(E) = 2 \times 16 - 3 - 28 = 1.$$

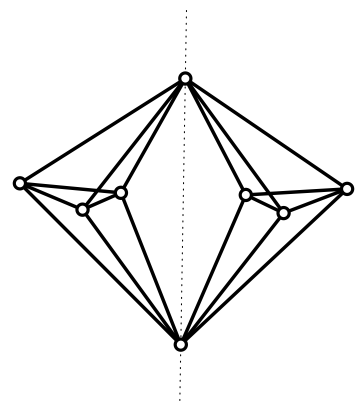


Figure 2.6: Double banana: a flexible framework in 3-space which is a circuit of the 3-rigidity matroid with $e = 3v - 6$.

The following theorem gives a necessary condition for n-dimensional rigidity matroid, which is just a consequence of the fact that the space of infinitesimal rigid motions is of dimension $\binom{n+1}{2}$ for (non-degenerate) configurations in $\mathbb{E}^n$.

Theorem 2.4.4. *([68, p.237]) Let G be a graph with $v \geq n$ vertices. If E is an independent set of edges on v vertices in the n-dimensional rigidity matroid then*

$$|F| \leq n\nu(F) - \binom{n+1}{2} \qquad \text{for all } F \subseteq E \text{ with } \nu(F) \geq n. \tag{2.4.0.1}$$

An edge set F with $|F| > n\nu(F) - \binom{d+1}{2}$ is dependent in the n-rigidity matroid.

Figure 6.14a shows an example of a graph with the set E of 18 edges and 8 vertices for which $|E| = 3|V| - 6 = 18$ but E is generically dependent. The framework is flexible under a rotation with the axis illustrated as a dotted line. This means that the counting condition in the above theorem is not sufficient to characterize independent (or dependent) sets in 3-rigidity matroid. At the moment, there is no known characterization of isostatic graphs in 3-space or higher dimensions.

2.5 Inductive constructions of rigid graphs

As we saw, a system of geometric constraints may define a matroid that describes the generic rigidity of those constraints and therefore, defines the combinatorial conditions on the associated graph that characterizes the rigidity of the system for almost all realizations. One of the active topics of possible research in the Rigidity Theory aims to develop inductive methods based on which the rigid graphs (in the related matroids) can be constructed from simple ones such as a single bar, and conversely.

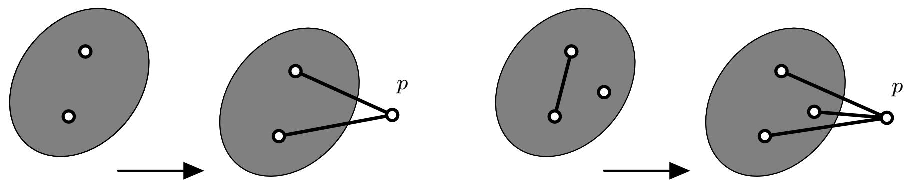
Perhaps the *Henneberg methods* [29] is the most famous example of inductive methods that were introduced to characterize 2-isostatic graphs; the graphs associated to isostatic bar-joint frameworks in the plane. Henneberg methods give a recipe for adding or removing a 2-valent or a 3-valent vertex from a given 2-isostatic graph so that it yields a new 2-isostatic graph. Because the existence of 2-valent or 3-valent vertices for an isostatic graph G (unless it is single bar) is guaranteed by the count $e = 2v - 3$, these methods are all that is needed:

1. We can simply remove (add) a 2-valent vertex from (to) an 2-isostatic graph to obtain a new one (Figure 2.7a). This is called a 2-addition or 2-removal move.

2. We can replace any edge $e = \{v_1, v_2\}$ by a new 3-valent vertex v^* that is connected to v_1 and v_3 and any third vertex v_3 in the graph. This operation is called *edge-split*. Conversely, suppose G is a 2-isostatic graph with a 3-valent vertex v^* adjacent to three vertices v_1, v_2, v_3 then for some edge $e \in \{v_1v_2, v_1v_3, v_2v_3\}$, the graph $(G - v^*) + e$ is 2-isostatic (Figure 2.7b).

See [61] for a proof of the above claims and extensions. The operations 1 and 2 form the *Henneberg moves*. Therefore,

Proposition 2.5.1. *[61] A graph with at least two vertices is 2-isostatic if and only if it is obtained from a single bar by a sequence of Henneberg moves.*



(a) Adding or removing a 2-valent vertex preserves isostatic graphs in the plane.

(b) An edge-split move and its reverse preserve isostatic graphs in the plane.

Figure 2.7: Henneberg methods

Henneberg moves also preserve isostatic graphs in $\mathbb{E}^n$ (n-isostatic graphs). But not all isostatic graphs are obtained from these moves only for $n > 2$. The reader can see [45] for more inductive techniques of various types of graphs.

2.6 Point-line frameworks in the plane and their rigidity matroid

A combinatorial characterization of the generic rigidity of point-line frameworks in the plane is given in [32]. We review the following preliminary concepts from [32] needed to state that result which will be used in the next chapter.

Definition 2.6.1. A *point-line graph* $G = (V_P \cup V_L, E)$ is a simple graph whose vertex-set is partitioned into two subsets V_P and V_L called *point-vertices* and *line-vertices*, respectively. This will naturally partition the set E into three subsets E_{PP}, E_{PL}, E_{LL}. The sets E_{PP}, E_{PL} and E_{LL} are respectively, the set of the edges of G with both their end-points in V_P, the set of the edges of G with one end in V_P and the other in V_L and the set of the edges of G with both their end-points in V_L.

A point-line graph records the pairwise distance constraints between a pair of points, a pair of a point and a line, and angle constraints between a pair of lines. We use v_p for $|V_P|$ and v_l for $|V_L|$. Point-vertices are labeled as $p_1, \dots, p_{v_p}$ and line-vertices are labeled as $\ell_1, \dots, \ell_{v_l}$ when we draw point-line graphs. We often refer to a point p_i by its index i as $i \in V_P$ and similarly, to a line ℓ_i as $i \in V_L$. For simplicity, we will denote a point-line graph by $G = (V, E)$, if point-vertices and line-vertices are understood.

Definition 2.6.2. A *point-line framework* $(G, \mathbf{p}, \ell)$ in the plane consists of a point-line graph $G = (V, E)$, an embedding $\mathbf{p} : V_P \to \mathbb{R}^2$ of point-vertices into $\mathbb{R}^2$ and an embedding $\ell : V_L \to \mathbb{E}^2$ of line-vertices in V_L into $\mathbb{E}^2$ such that $\ell_i = (a_i, b_i)$ if the line ℓ_i has the equation $x = a_i y + b_i$, for every $i \in V_L$.

If $V_L = \phi$, then $(G, \mathbf{p}, \ell)$ is just a bar-joint framework. In general, just like bar-joint frameworks, a point-line framework $(G, \mathbf{p}, \ell)$ with $v \geq 2$ could have a 3-dimensional trivial infinitesimal motions in the plane. A line in the plane has two degrees of freedom just like a point: a rotational motion and a translation perpendicular to the line. If $(\mathbf{p}, \ell)$ is viewed as a point of $\mathbb{R}^{2v}$, it is called a *point-line configuration* in $\mathbb{E}^2$. A point-line configuration is called *generic* if the collection of the coordinates $\{x_i, y_i\}_{i \in V_P} \bigcup \{a_i\}_{i \in V_L}$ of all points and lines are algebraically independent over $\mathbb{Q}$. Soon we will see why there is no concern about b_i's for the lines $i \in V_L$ in a generic point-line configuration.

Given a point-line graph G, the *rigidity map* $f_G : \mathbb{R}^{2v} \to \mathbb{R}^e$ with $f_G(\mathbf{p}, \ell) = (\dots, f_{ij}(\mathbf{p}, \ell), \dots), ij \in E$, is defined as follows:

$$f_{ij}(\mathbf{p}, \ell) = \begin{cases} (x_i - x_j)^2 + (y_i - y_j)^2 & ij \in E_{PP}, \\ (x_i - a_j y_i - b_j)(1 + a_j^2)^{-1/2} & ij \in E_{PL}, \\ \tan^{-1} a_i - \tan^{-1} a_j & ij \in E_{LL}, i < j. \end{cases}$$

Using some elementary row-column operations (see [32]), the Jacobian $df_G(\mathbf{p}, \ell)$ can be simplified to the *point-line rigidity matrix* $R(G, \mathbf{p}, \ell)$:

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} & \multicolumn{2}{c}{p_h} & & \multicolumn{2}{c}{p_i} & & \multicolumn{2}{c}{\ell_j} & & \multicolumn{2}{c}{\ell_k} \end{array} \\
\begin{array}{l} \text{Point-point} \\ \\ \text{Line-line} \\ \\ \text{Point-line} \end{array} &
\begin{pmatrix}
\cdots & x_h - x_i & y_h - y_i & \cdots & x_i - x_h & y_i - y_h & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & -1 & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
\cdots & 1 & -a_j & \cdots & 0 & 0 & \cdots & -x_i a_j - y_i & -1 & \cdots & 0 & 0
\end{pmatrix}
\end{array}.
$$

A point-line framework is called *infinitesimally rigid* if the rank of $R(G, \mathbf{p}, \ell)$ is $2v - 3$. In other words, only the trivial infinitesimal motions (infinitesimal rigid motions of the plane) are in the kernel of $R(G, \mathbf{p}, \ell)$. A framework that is not infinitesimally rigid is called *infinitesimally flexible*. A set of edges of a point-line graph is called *independent* if the corresponding rows in the rigidity matrix $R(G, \mathbf{p}, \ell)$ are independent for some generic configuration $(\mathbf{p}, \ell) \in \mathbb{R}^{2v}$. A point-line graph G is called *isostatic* if $e = 2v - 3$ and E is independent.

Example 2.6.1. Figure 2.8a shows a point-line graph G_1 with no point-vertex. Lines ℓ_1, ℓ_2 and ℓ_3 are to maintain their mutual angles. But we know that this geometrically is a dependent set of constraints because the angle sum in a triangle is fixed, of course. Also, any generic realization of G as a point-line framework results in an infinitesimally flexible and flexible framework as each line can independently move parallel to itself, which is not a trivial motion. This gives rise to one degree of freedom for the system. Obviously, G_1 is rigid as a bar-joint graph though.

Figure 2.8b shows another point-line graph $G_2 = (V_2, E_2)$ with 3 line-vertices ℓ_1, ℓ_2, ℓ_3 and 6 point-vertices $p_1, \ldots, p_6$. A realization of G is also demonstrated in Figure 2.8b. Point-line distance constraints are illustrated by dashed-lines. The generic framework $(G_2, \mathbf{p}, \ell)$ in Figure 2.8b is flexible because the lines ℓ_1, ℓ_2, ℓ_3 can move parallel to themselves while all the constraints are maintained. The graph G_2 as a bar-joint graph is rigid because, by Theorem 2.4.3, $r(E_2) = 5+5+5 = 15 = |E|$ and also, $2|V_2| - 3 = 15$. So G_2 is isostatic as a bar-joint framework.

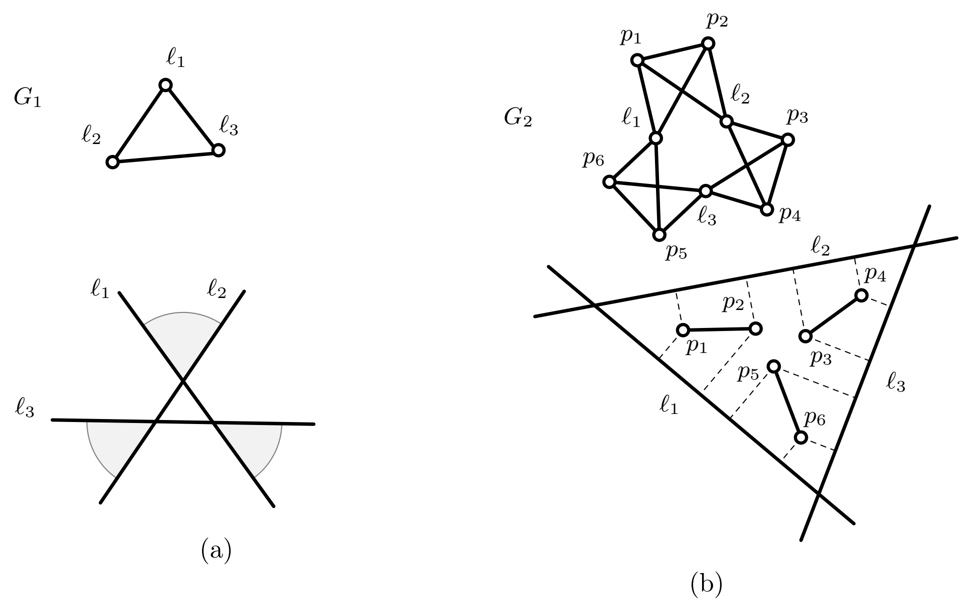


Figure 2.8: Examples of point-line graphs and their realizations.

For a set of edges E, $\nu_P(E)$ and $\nu_L(E)$ denote the number of point-vertices and the number of line-vertices incident to the edges in E, respectively.

The following theorem gives a combinatorial characterization of isostatic point-line graphs in the plane.

Theorem 2.6.1. *[32] Given a point-line graph $G = (V, E)$ on v vertices in the plane. G is isostatic if and only if $e = 2v - 3$ and for every non-empty subset E' of E, we have*

$$|E'| \leq \sum_{i=1}^{s} (2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(E') - 1,$$

for every partition $\{A_i\}_{i=1}^{s}$ of E'.

It is shown in [32] that the point-line matroid is induced by the submodular function $\rho + \nu_L - 1$ where

$$\rho(F) = \min \sum_{i=1}^{s} (2\nu_P(A_i) + \nu_L(A_i) - 2),$$

for every partition $\{A_i\}_{i=1}^s$ of $\phi \neq F \subseteq E$. The authors also give a polynomial algorithm to decide whether a given point-line graph is independent using pebble game.

Theorem 2.6.1 can now detect that the graphs G_1 and G_2 are dependent. Consider $G_1 = (V_1, E_1)$ and partition its edge-set into three single bars $A_1 = \{\ell_1, \ell_2\}$, $A_2 = \{\ell_2, \ell_3\}$ and $A_3 = \{\ell_1, \ell_3\}$. We have $\nu_P(A_i) = 0$ and $\nu_L(A_i) = 2$ for every $i = 1, 2, 3$ but

$$3 = |E_1| > \nu_L(E_1) - 1 = 2,$$

for this particular partition. Therefore E_1 is not independent. It is easy to see that the rank of E is 2 in the point-line rigidity matroid because two angle constraints are independent.

For graph $G_2 = (V_2, E_2)$, partition its edge set into three subsets A_1, A_2, A_3 where each A_i is one of the three quadrilaterals in the graph G. Considering $\nu_P(A_i) = \nu_L(A_i) = 2$ for $i = 1, 2, 3$, we obtain

$$15 = |E_2| > 3(4 + 2 - 2) + 2 = 14,$$

for this partition. Therefore E_2 is dependent.

2.7 Body-bar frameworks and their rigidity matroid

It often happens that a bar-joint framework consists of several rigid blocks in $\mathbb{E}^n$ that are connected together in a certain way. These rigid blocks become *rigid bodies* in $\mathbb{E}^n$, which their size or their shape is not of our concern but the bodies must span an affine space of dimension at least $n-1$ in $\mathbb{E}^n$. All points attached to a rigid body now move under a single rigid motion. This section reviews the methods and results derived to study this type of frameworks.

As an example, the bar-joint framework with the graph shown in Figure 2.5 can be considered a body-bar framework with 4 bodies cyclically connected with pairs

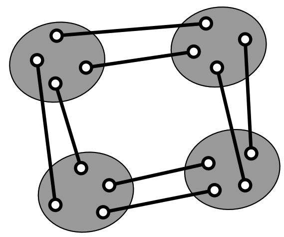


Figure 2.9: A body-bar framework in plane.

of bars, see Figure 2.9. The shaded ellipses are symbols of rigid bodies in this thesis (Chapter 6).

2.7.1 Center of infinitesimal rigid motions in $\mathbb{E}^n$

In the context of body-bar frameworks, the set of points attached to a body move under a single rigid motion which is the motion that the entire body undergoes. Instead of a matrix representation of the rigid motions, we want a representation of infinitesimal rigid motions in $\mathbb{E}^n$ that allows us to write the constraints linearly so that it produces a rigidity matrix for the constraints. This representation is provided through the *exterior algebra* and it also has geometric benefits specially in terms of projective geometry and the incidence geometry of the constraints.

In the following we use the notation of *Grassmann-Cayley* algebra: the *join operator* $\vee$ on points in $\mathbb{R}^{n+1}$ instead of the exterior product $\wedge$ in the exterior algebra as *Grassmann-Cayley* algebra has another operation called *meet* $\wedge$ to represent the intersection of two spaces whose output is a smaller space (see [62]).

In general every r-extensor $p_1 \vee \cdots \vee p_r,\ p_i \in \mathbb{R}^{n+1}$ uniquely defines a linear subspace N generated by p_i's in $\mathbb{R}^{n+1}$ of dimension r, if p_i's are independent. This subspace can be given 'coordinates' by putting the row vectors p_i's in a $r \times (n+1)$ matrix and listing all $r \times r$ minors in some order. There are $\nu = \binom{n+1}{r}$ of these numbers, not simultaneously zero, that are called the *Plücker coordinates* of the subspace N. They are unique to each subspace up to a non-zero scalar. So they are considered points of the projective space $\mathbb{P}^{\nu-1}$ but not any point. They have to satisfy some equations called *Plücker relations*. Therefore they belong to a projective algebraic variety in $\mathbb{P}^{\nu-1}$ called *Grassmannian* $G(r, n+1)$ (see [58]).

Here, from [64] by N. White and W. Whiteley, we recall key tools, notations and techniques we will need later for our problems in the plane.

Any rotation (including a translation) in $\mathbb{E}^n$ has a center (or axis) of dimension $n-2$ in $\mathbb{P}^n$, which is a subspace of dimension $n-1$ in $\mathbb{R}^{n+1}$. In fact, for $n-1$ vectors $a_1, \dots, a_{n-1}$ in $\mathbb{R}^{n+1}$, the $(n-1)$-*extensor* $Z = a_1 \vee a_2 \vee \dots \vee a_{n-1}$ represents an infinitesimal rigid motion in $\mathbb{E}^n$ whose action at a point $p = (p_1, \dots, p_n, 1)$ is $Z \vee p$. This is called the *motion* at p induced by Z. The $(n-1)$-extensor $Z = a_1 \vee a_2 \vee \dots \vee a_{n-1}$ is referred to as the *center* of the motion[7]. In fact, the hyperplane $Z \vee p$ in $\mathbb{R}^{n+1}$ has the Plücker coordinates in the form of a $(n+1)$-vector $(v_1, \dots, v_n, v_{n+1})$ where $(v_1, \dots, v_n)$ is the instantaneous velocity (or instantaneous motion) at the point $(p_1, \dots, p_n)$ in $\mathbb{E}^n$ and $v_{n+1} = -\langle (v_1, \dots, v_n), (p_1, \dots, p_n) \rangle$. This is because

$$\det[a_1, \dots, a_{n-1}, p, p] = \langle (v_1, \dots, v_n, v_{n+1}), p \rangle = 0.$$

Note that the above equation also explains by which order the Plücker coordinates $(v_1, \dots, v_n, v_{n+1})$ could be ordered.

If Z is in the *hyperplane at infinity* in $\mathbb{P}^n$ then Z represents the center of an *instantaneous translation* in $\mathbb{E}^n$. It determines a $(n-1)$-dimensional subspace in the hyperplane at infinity that, in turn, dually determines a unique vector $v \in \mathbb{R}^n$, which is an instantaneous translation in the direction of v in $\mathbb{E}^n$. In fact, it can be easily checked that the velocity induced by Z at any two point p and q in $\mathbb{E}^n$ are identical to v.

In $\mathbb{E}^2$, the center of an infinitesimal rigid motion is a 3-vector $c = p \vee q$ for two points $p, q \in \mathbb{P}^2$. In $\mathbb{E}^3$, the center of a typical infinitesimal rigid motion is a 6-vector $c = p \vee q$ for two points $p, q \in \mathbb{P}^3$. Also, this is a *screw* in the space with the axis

[7] The motion $Z' = \alpha Z$ where α is a scalar has the same center as Z but different *angular velocity* if Z is a rotation. This can be accurately determined after an appropriate normalization.

$p \vee q$. For example, a screw whose axis is the z-axis is $s = (0,0,0,1) \vee (0,0,1,0) = (0,0,0,0,0,-1)$ in the standard order $12, 13, 14, 23, 24, 34$ where ij means the minor of columns ij. The motion of s at a point $p = (p_1, p_2, p_3, 1)$ is $s \vee p = (p_2, -p_1, 0, 0)$ in the order $234, -134, 124, -123$.

2.7.2 Infinitesimal rigidity of body-bar frameworks in $\mathbb{E}^n$

Let B_1 and B_2 are two rigid bodies in $\mathbb{E}^n$ and p, q are points on B_1 and B_2, respectively. Suppose p and q are connected by a bar, namely they are constrained to maintain their distance in $\mathbb{E}^n$. Let u and v are infinitesimal motions in $\mathbb{E}^n$ that respect this constraint, namely, vectors in $\mathbb{R}^n$ such that $\langle u - v, p - q\rangle = 0$. Since points p and q are part of rigid bodies, their motions are induced by Z_1 and Z_2, the centers of infinitesimal motions of B_1 and B_2 , respectively. Considering p and q in their homogeneous coordinates, we then have $Z_1 \vee p = (u, -\langle u, p\rangle)$ and similarly, $Z_2 \vee q = (u, -\langle u, q\rangle)$ as explained above. Therefore $Z_1 \vee p \vee q = \det[Z_1, p, q] = \langle Z_1 \vee p, q\rangle$. Thus,

$$0 = \langle u - v, p - q\rangle = -Z_1 \vee p \vee q - Z_2 \vee q \vee p = -Z_1 \vee (p \vee q) + Z_2 \vee (p \vee q), \tag{2.7.2.1}$$

Note that $Z_1 \vee (p \vee q)$ or $Z_2 \vee (p \vee q)$ are determinants that two of their rows are p and q. We expand these determinants with respect to two rows p and q we can write $Z_1 \vee p \vee q = \langle Z_1^*, p \vee q\rangle$ where Z_1^* is viewed as a $\binom{n+1}{n-1}$-vector with such an order of coordinates that when we take its dot-product to $p \vee q$ we obtain $\det[Z_1, p, q]$. In fact, if $Z = p_1 \vee \cdots \vee p_{n-1}$ is an $(n-1)$-extensor then $Z^* = (P_{12}, -P_{13}, \dots, (-1)^{i+j-1} P_{ij}, \dots)$ where P_{ij} is the $(n-1) \times (n-1)$ minor obtained by omitting columns i and j from the matrix whose rows are $p_1, \dots, p_{n-1}$. The operator * as described above is well defined, bijective and linear. Therefore (2.7.2.2) can written as

$$0 = \langle u - v, p - q\rangle = \langle Z_1^* - Z_2^*, p \vee q\rangle. \tag{2.7.2.2}$$

Note that $p \vee q$ also determines the line through the points p and q in $\mathbb{E}^n$.

Definition 2.7.1. [64] A *body-bar framework* in $\mathbb{E}^n$ is a finite collection of rigid bodies $B_1, \dots, B_v$ and some rigid bars $\{p_{i,e}, p_{j,e}\}_{i=1}^{e}$ for some $e \in \mathbb{N}$, connecting pairs of distinct points $p_{i,e}, p_{j,e}$ on two bodies B_i and B_j for some $1 \leq i, j \leq v$. Associated to a body-bar framework, there is a finite multigraph $G = (V, E)$, with no loops, and with vertices corresponding to each body and edges to the bars. This multigraph G is called a *body-bar graph.* Any body-bar graph may be realized as a body-bar framework by assigning an ordered pair of distinct points in $\mathbb{R}^{n+1}$ to each edge $e \in E$. A body-bar framework with multigraph G and an embedding of edges $\mathbf{p}$ as just described, is denoted by $(G, \mathbf{p})$.

Note that each bar $\{p_{i,e}, p_{j,e}\}$ determines a 2-extensor or a line $p_{i,e} \vee p_{j,e}$ in $\mathbb{E}^n$. As (2.7.2.2) indicates, the constraint corresponding to the bar $\{p_{i,e}, p_{j,e}\}$ only depends on the line $p_{i,e} \vee p_{j,e}$ so that the pair $p_{i,e}, p_{j,e}$ could be replaced by any other pairs of distinct points on the line $p_{i,e} \vee p_{j,e}$ without changing the constraint.

An *infinitesimal motion* of a body-bar framework $(G, \mathbf{p})$ is an assignment of centers Z_i as $(n-1)$-extensors to each body B_i so that the length of each bar $e = \{p_{i,e}, p_{j,e}\}$ with $p_{i,e} \in B_i$ and $p_{j,e} \in B_j$ for two bodies B_i, B_j is infinitesimally preserved, i.e. $\langle Z_i^* - Z_j^*, p_{i,e} \vee p_{j,e} \rangle = 0$. This defines a system of linear equations of $|E| = e$ equations and $v\binom{n+1}{2}$ unknown, that is $Z_i, 1 \leq i \leq v$. The corresponding *body-bar rigidity matrix* has the following format:

$$R_b(G, \mathbf{p}) = \begin{array}{c} \\ \\ \text{edge } e \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & B_i & & B_j & \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & \\ \dots & p_{i,e} \vee p_{j,e} & \dots & -p_{i,e} \vee p_{j,e} & \dots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}. \tag{2.7.2.3}$$

The rigidity matrix $R_b(G, \mathbf{p})$ has one row for each bar (or edge of G) and $\binom{n+1}{2}$ columns per body, bodies are indexed by some order. The kernel of $R_b(G, \mathbf{p})$ is of the form $(Z_1^*, \dots, Z_v^*)$ where Z_i is the center of instantaneous motion of the body B_i.

A body-bar framework $(G, \mathbf{p})$ is called *infinitesimally rigid* if $Z_1 = \cdots = Z_v = Z$, where Z is the center of an infinitesimal rigid motion of $\mathbb{R}^n$.

Therefore $\dim(\ker(R_b(G, \mathbf{p}))) = \binom{n+1}{2}$. A set of edges of G are called *independent* if the corresponding rows in the rigidity matrix are independent for almost all realizations $\mathbf{p}$ of G. A body-bar graph G is called *isostatic* if it has $e = \binom{n+1}{2}(v-1)$ independent edges.

To analyze the combinatorics of the independence of the rows of $R_b(G, \mathbf{p})$, a more general matrix pattern is considered in [64] that includes that of body-bar frameworks as a special case.

Definition 2.7.2. [64] Let G be a multigraph with no loops. A *k-frame matrix* for G consists of one row for each edge and k columns for each vertex, where if $\{u, v\}$ is an edge of G then the row for e has a k-tuple x_e in the columns for u and $-x_e$ in the columns for v, and 0 in all other columns. For a particular choice of x_e, $e \in E$ via an embedding $\mathbf{p} : E \to \mathbb{R}^k$, $\mathbf{p}(e) = x_e$, the k-frame matrix is denoted by $M(G, \mathbf{p})$ and is called a *k-frame* of $(G, \mathbf{p})$. If $(G, \mathbf{p})$ has distinct algebraically independent coordinates for all entries in the x_e's, we call $(G, \mathbf{p})$ a *generic k-frame* for G.

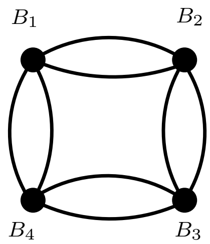
Abstractly, a *motion* of a k-frame $(G, \mathbf{p})$ is defined to be a vector $Z = (Z_1, \ldots, Z_v) \in \mathbb{R}^{kv}$ of length kv which is orthogonal to the row space of the matrix $M(G, \mathbf{p})$. A *trivial motion* of a k-frame is a motion for which $Z_1 = \cdots = Z_v$. A k-frame is called *rigid* if it has only trivial motions. A k-frame is *k-isostatic* if every motion of it is trivial and deleting any edge results in a non-trivial motion for the k-frame.

Theorem 2.7.1. *[64] Let G be a multigraph. The following are equivalent:*

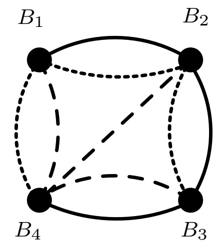
(a) *A k-frame of G is k-isostatic.*

(b) *There is a set of k edge-disjoint spanning trees which covers G.*

(c) *The rigidity matrix is the matroid union of k cycle matroids of G.*



(a) A flexible body-bar graph with one degree of freedom.



(b) A generically rigid body-bar graph.

Figure 2.10: A body-bar graph in the plane (a) and a rigid graph derived from it (b) with 3 spanning trees on the vertices.

(d) $e = k(v-1)$ *and for any non-empty subset* $E' \subseteq E$,

$$|E'| \leq k(|V(E')| - 1).$$

We note that a body-bar framework is a special case of a k-frame when $k = \binom{n+1}{2}$ and x_e is a 2-extensor for every $e \in E$. The combinatorial characterization of the rigidity of body-bar frameworks in $\mathbb{E}^n$ was first given by Tay [60] using inductive techniques. The equivalence of (b) and (d) are due to Tutte and Nash-Williams [43].

Example 2.7.1. Consider the body-bar framework in Figure 2.9. The associated body-bar graph G is shown in Figure 2.10. In the plane, we have $k = 3$. For the multigraph G, $v = 4$. So we need $3 \times (4-1) = 9$ edges (constraints) to possibly have a rigid framework. Therefore, by Theorem 2.7.1, the framework is not rigid as it is underconstrained. The framework has one degree of freedom as predicted using Theorem 2.4.3 in Example 2.4.1. Figure 2.10b shows that adding an extra edge will result in a rigid framework. An edge-disjoint union of 3 spanning trees is shown in the graph as dotted edges, dashed edges and ordinary edges. The extra edge could be placed anywhere. The rigidity of this new framework can be verified as a bar-joint framework, as well.

2.7.3 Body-cad constraints in $\mathbb{E}^3$

Motivated by constraint-based CAD software, a broader set of geometric constraints on rigid bodies, other than only the distance constraints on distinct points, has been introduced in [25] by A. Lee-St.John et al. and the corresponding rigidity matrix was developed for 3-dimensional structures. These constraints include, not only points but more diverse geometric objects in 3-space such as lines and planes that are affixed to different rigid bodies:

1. *Point-point constraints*, which include *distance constraints* on distinct points of different bodies (as it is for body-bar frameworks) and *coincidence constraints* on two coincident points in 3-space that is, the two coincident points each attached to two different bodies are to remain coincident.

2. *Point-line constraints* including distance or coincidence constraints on pairs of point-lines attached to a pair of bodies in 3-space.

3. *Line-line constraints* including parallel, perpendicular, fixed angular, coincidence and distance constraints on pairs of lines attached to pair of bodies. These constraints are not identical in 3-space.

4. *Point-plane constraints* including distance or coincidence constraints on pairs of point-planes attached to pairs of bodies in 3-space.

5. *Line-plane constraints* including parallel, perpendicular, fixed angular, coincidence and distance constraints on pairs of lines-planes attached to pairs of bodies in 3-space.

6. *Plane-plane constraints* including parallel, perpendicular, fixed angular, coincidence and distance constraints on pairs of planes attached to pairs of bodies in 3-space.

These constraints are named *body-cad constraints*. We refer the reader to [25] for the details on the algebraic expression of the constraints and examples.

For $n = 3$, the centers of infinitesimal motions are 2-extensors, called *screws* in $\mathbb{E}^3$ indicating the axis of the instantaneous motion of a rigid body. The dimension of the linear space of infinitesimal rigid motions in $\mathbb{E}^3$ is 6. So, with v bodies, the rank of the rigidity matrix of a body-cad system is at most $6(v-1)$. As it is for body-bar frameworks, a body-cad framework is infinitesimally rigid if all the infinitesimal motions of the system are trivial. Namely, the only solution (infinitesimal motions) of the system are trivial motions i.e., all bodies have the same center of motion. A body-cad system is infinitesimally rigid if the rank of its rigidity matrix is $6(v-1)$; it is called *isostatic (or minimally rigid)* if it is infinitesimally rigid and its associated graph has $6(v-1)$ edges.

Angular constraints may incluse parallel, perpendicular or arbitrary fixed angular constraints such as *line-line non-parallel fixed angular*, *line-line parallel*, *line-line perpendicular line-plane parallel*, *line-plane perpendicular*, *line-plane fixed angular*, *plane-plane parallel*, *plane-plane perpendicular*, *plane-plane fixed angular* although all of these reduce to line-line non-parallel fixed angular or line-line parallel using the normals to the planes.

It is important to know that angular constraints exhibit a rather special behaviour than distance constraints. We shall shed some light on the geometric reasons of this in this thesis. At the first place, this difference is visible from the rigidity matrix of the constraints. In fact, a pattern of 'generic zeros' in the rows involving angular constraints in the rigidity matrix explicitly reveals the difference. These rows in the matrix and the corresponding edges in the body-cad graph G are coloured *red* and the rest of edges are *black* to emphasize the difference. So a body-cad graph generally has two types of edges: red R and black B. This is denoted by $G = (V, R \cup B)$.

We should also note that some of the above constraints need more than one equation to be expressed algebraically. So they may occupy more than one row in the rigidity matrix or correspond to more than one edge in the graph. The constraint corresponding to one row in the rigidity matrix or equivalently, one edge in G are called *primitive*. For example, a line-line coincidence constraint in $\mathbb{E}^3$ corresponds to 4 primitive constraints: 2 red (angular) constraints to preserve parallel lines and 2 black constraints to maintain coincidence of a point of one line on the other line (see [25]). More explicitly, a line-line coincidence constraint is composed of a line-line parallel and a point-line coincidence constraint each of which corresponds to two algebraic constraints or two rows in the rigidity matrix.

The following theorem by A. Lee-St.John and J. Sidman gives a combinatorial characterization of generic rigidity for a body-cad system in $\mathbb{E}^3$. The proof is based on the technique employed in [64].

Theorem 2.7.2. *[38] A 3-dimensional body-cad framework with no point-point coincidence constraints is generically minimally rigid if and only if in its associated primitive cad graph $G = (V, R \cup B)$ there is some set of black edges $B' \subseteq B$ such that*

1. *$B \backslash B'$ is the edge-disjoint union of 3 spanning trees, and*
2. *$R \cup B'$ is the edge-disjoint union of 3 spanning trees.*

A point-point coincidence constraint between two bodies in 3-space imposes 3 black constraints on two bodies. Double banana in Figure 6.14a is an example of two bodies with two point-point coincidence constraints, which is generically flexible in $\mathbb{E}^3$ while its graph satisfies the criterion in the above theorem. So point-point coincidence constraints in 3D are excluded in the combinatorial criterion of the generic rigidity.

In Chapter 6, we will develop cad constraints in the plane and will see that the analogous 2D combinatorial characterization is true in the plane even when point-point coincidences are included.

Chapter 3

Rigidity of point-hyperplane frameworks

In this chapter we define point-hyperplane frameworks and study their geometric rigidity. We will show that the first-order rigidity of spherical frameworks in $\mathbb{S}^n$ with some joints on the 'equator' are equivalent to the first-order rigidity of point-hyperplane frameworks in $\mathbb{E}^n$. In terms of projective geometry, this provides a one-to-one geometric correspondence between the infinitesimal rigidity of a framework in $\mathbb{P}^n$ with some joints at infinity and the infinitesimal rigidity of a class of point-hyperplane frameworks in the Euclidean space $\mathbb{E}^n$. In other words, this correspondence gives a Euclidean interpretation of the bar-joint frameworks in $\mathbb{P}^n$ with some joints at infinity.

Using projective transformations, it will be shown that the first-order rigidity of a point-hyperplane framework in $\mathbb{E}^n$ is equivalent to the first-order rigidity of a bar-joint framework with a set of joints (corresponding to the hyperplanes) in a hyperplane in $\mathbb{E}^n$ (see also [17]). In particular, this establishes the equivalence of the first-order rigidity of point-line frameworks in the plane and that of bar-joint frameworks in $\mathbb{E}^2$ with some collinear joints. Since a combinatorial characterization of the 'generic' rigidity of point-line frameworks in $\mathbb{E}^2$ has been recently given, the mentioned correspondence also provides a combinatorial characterization of the 'generic' rigidity of bar-joint frameworks with some collinear joints in the plane.

As indicated, some results of this chapter are coauthored in [17].

3.1 Bar-joint frameworks with some joints on the equator in $\mathbb{S}^n$

In Chapter 2 we described bar-joint frameworks in the upper hemispherical space $\mathbb{S}^n_+$ (or equivalently $\mathbb{S}^n_-$) and Theorem 2.3.2 showed that their infinitesimal rigidity is equivalent to that of the bar-joint framework in $\mathbb{E}^n$ obtained by the central projection of the spherical configuration into the affine chart containing the Euclidean space $\mathbb{E}^n$; the affine chart of 'finite points' in the language of projective geometry. However, the correspondence (2.3.1.10)–(2.3.1.11) in Section 2.3 implicitly excludes the points in the 'hyperplane at infinity' in $\mathbb{P}^n$ (or in the equator in $\mathbb{S}^n$) because obviously, there is no point in $\mathbb{E}^n$ corresponding to a point in the hyperplane at infinity under the central projection. But what if some vertices fall into the chart of the 'points at infinity'? what does this situation correspond to in a Euclidean space? We will answer this question in this chapter.

We denote the affine hyperplane $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$ by $\mathbb{A}^n$. This hyperplane is the chart of *finite points* in the projective space $\mathbb{P}^n$, which may be taken identical to $\mathbb{E}^n$. The point $N = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ is the *north pole* of $\mathbb{S}^n$. The *equator* V_{eq} of $\mathbb{S}^n$ is the intersection of the hyperplane

$$H_\infty = \{x \in \mathbb{R}^{n+1} : \langle x, N \rangle = 0\},$$

in $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$. The hyperplane H_∞, the *hyperplane at infinity* in $\mathbb{P}^n$, is the chart of the *points at infinity* in $\mathbb{P}^n$.

Suppose $(G, \mathbf{p})$ is a bar-joint framework in $\mathbb{S}^n$ with some points on the equator V_{eq}. As explained in Chapter 2, a first-order motion of the framework $(G, \mathbf{p})$ is $\mathbf{p}' = (p'_i)_{i\in V} \in \mathbb{R}^{(n+1)v}$ such that $p'_i \in T_{p_i}\mathbb{S}^n$ for every $i \in V$ and

$$\begin{aligned} \langle p_j, p'_i\rangle + \langle p_i, p'_j\rangle &= 0 \qquad \text{for all } ij \in E, \\ \langle p_i, p'_i\rangle &= 0 \qquad \text{for all } i \in V. \end{aligned} \tag{3.1.0.1}$$

The framework $(G, \mathbf{p})$ in $\mathbb{S}^n$ is called *infinitesimally rigid* if (3.1.0.1) only has trivial solutions, namely, all the solutions of (3.1.0.1) are of the form $p'_i = Sp_i$ for every $i \in V$ and some skew-symmetric $(n+1)\times(n+1)$ matrix. In fact, the trivial motions are the *infinitesimal rigid motions* of the framework $(G, \mathbf{p})$.

The set of all first-order motions of a framework $(G, \mathbf{p})$ forms a linear space in $\mathbb{R}^{(n+1)v}$, a subspace of which is the space of trivial motions denoted by $\mathcal{T}(\mathbf{p})$:

$$\mathcal{T}(\mathbf{p}) = \{(p'_1, \dots, p'_v) \in \prod_{i=1}^{v} T_{p_i}\mathbb{S}^n \mid p'_i = Sp_i, S^t = -S, S \in M_{(n+1)\times(n+1)}\}.$$

If the points $p_1, \dots, p_v$ of the configuration $\mathbf{p}$ in $\mathbb{S}^n$ generate a linear subspace of $\mathbb{R}^{n+1}$ of dimension at least n then $\dim \mathcal{T}(\mathbf{p}) = \binom{n+1}{2}$, which is the dimension of the linear space of *infinitesimal rotations* in $\mathbb{R}^{n+1}$ or equivalently, the dimension of the space of $(n+1)\times(n+1)$ skew-symmetric matrices. The coefficient matrix of (3.1.0.1) is called the *rigidity matrix* of the framework $(G, \mathbf{p})$ and we denoted it by $R_{\mathbb{S}}(G, \mathbf{p})$ in Chapter 2.

As one can easily see, (3.1.0.1) is invariant under scaling the pairs (p_i, p'_i), for every $i \in V$. In particular, if we replace a point p_i with its *antipodal* $-p_i$ then the infinitesimal rigidity will be preserved.

The system (3.1.0.1) associated to a framework $(G, \mathbf{p})$ in $\mathbb{S}^n$ can be equivalently expressed as:

$$\begin{aligned} \langle \tilde{p}_i - \tilde{p}_j, \tilde{p}'_i - \tilde{p}'_j\rangle &= 0 \qquad ij \in E, \\ \langle \tilde{p}_i, \tilde{p}'_i\rangle &= 0 \qquad i \in V. \end{aligned} \tag{3.1.0.2}$$

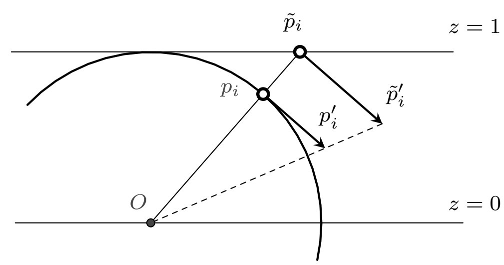


Figure 3.1: The transfer of velocities along the radial rays.

where

$$\tilde{p}_i = \frac{1}{\langle e_{n+1}, p_i \rangle} p_i \qquad \text{for every } p_i \notin V_{\text{eq}}, \tag{3.1.0.3}$$

$$\tilde{p}_i = p_i \qquad \text{for every } p_i \in V_{\text{eq}}. \tag{3.1.0.4}$$

The natural bijection between the solution space of (3.1.0.1) and (3.1.0.2) is the following:

$$\tilde{p}'_i = \frac{1}{\langle e_{n+1}, p_i \rangle} p'_i \qquad \text{for every } p_i \notin V_{\text{eq}}, \tag{3.1.0.5}$$

$$\tilde{p}'_i = p'_i \qquad \text{for every } p_i \in V_{\text{eq}}, \tag{3.1.0.6}$$

if p'_i is a motion at p_i. That is, the motions will be rescaled by the same factor as the points were rescaled. Note that (3.1.0.3)–(3.1.0.6) define invertible mappings so that we can recover the spherical system (3.1.0.1) from (3.1.0.2) using the following inverses:

$$p_i = \frac{1}{\|\tilde{p}_i\|} \tilde{p}_i, \tag{3.1.0.7}$$

$$p'_i = \frac{1}{\|\tilde{p}_i\|} \tilde{p}'_i, \tag{3.1.0.8}$$

for all points $i \in V$. Intuitively, the points that are not in the equator V_{eq} have been projected to into the affine hyperplane $x_{n+1} = 1$ (or $\mathbb{E}^n$) but the points in V_{eq} and their motions simply remain unchanged under the above transformation (see Figure 3.1). In addition, the points in V_{eq} (or points at infinity) could also be arbitrarily rescaled. In fact, $\tilde{\mathbf{p}}$ may also be viewed as a configuration in the projective space $\mathbb{P}^n$ for which the points $\tilde{p}_i, p_i \notin V_{\text{eq}}$ are finite points as their last coordinate is non-zero while $\tilde{p}_i, p_i \in V_{\text{eq}}$ are the points at infinity as their last coordinate is zero, in their *homogeneous coordinates*. This makes sense if we recall that the *metric* of the projective space is defined the same as that of a spherical space except that the points need not to have a unit length [51]. We denote the coefficient matrix of (3.1.0.2) by $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$:

$$
\begin{array}{r c}
 & \begin{array}{ccccc|ccccc} & \multicolumn{3}{c}{\text{finite points}} & & & \multicolumn{3}{c}{\text{points at infinity}} & \\ & \tilde{p}_h & & \tilde{p}_i & & & \tilde{p}_j & & \tilde{p}_k & \end{array} \\
\begin{array}{r} \\ \text{finite-finite} \\ \\ \hline \\ \text{infinity-infinity} \\ \\ \hline \\ \text{finite-infinity} \\ \\ \hline \\ \text{finite vertices} \\ \vdots \\ \text{vertices at infinity} \\ \vdots \end{array} &
\left(\begin{array}{ccccc|ccccc}
 & \vdots & & \vdots & & & & & & \\
\cdots & \tilde{p}_h - \tilde{p}_i & \cdots & \tilde{p}_i - \tilde{p}_h & \cdots & & & 0 & & \\
 & \vdots & & \vdots & & & & & & \\ \hline
 & & & & & & \vdots & & \vdots & \\
 & & 0 & & & \cdots & \tilde{p}_j - \tilde{p}_k & \cdots & \tilde{p}_k - \tilde{p}_j & \cdots \\
 & & & & & & \vdots & & \vdots & \\ \hline
 & & & \vdots & & & \vdots & & & \\
 & 0 & & \tilde{p}_i - \tilde{p}_j & & & \tilde{p}_j - \tilde{p}_i & & 0 & \\
 & & & \vdots & & & \vdots & & & \\ \hline
 & \vdots & & \vdots & & & \vdots & & \vdots & \\
\cdots & \tilde{p}_h & \cdots & 0 & & \cdots & 0 & \cdots & 0 & \cdots \\
 & \vdots & & \vdots & & & \vdots & & \vdots & \\
\cdots & 0 & \cdots & 0 & & \cdots & 0 & \cdots & \tilde{p}_k & \cdots \\
 & \vdots & & \vdots & & & \vdots & & \vdots &
\end{array}\right)
\end{array}. \tag{3.1.0.9}
$$

From (3.1.0.5)–(3.1.0.8), we have the following proposition.

Proposition 3.1.1. *The systems (3.1.0.1) and (3.1.0.2) are equivalent with isomorphic solution spaces and isomorphic spaces of trivial solutions.*

According to the column operations we did to obtain $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ from $R_{\mathbb{S}}(G, \mathbf{p})$, the bijective transformation that maps the kernel of $R_{\mathbb{S}}(G, \mathbf{p})$ to the kernel of $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$

is the $(n+1)v \times (n+1)v$ matrix with the block form

$$Q = \begin{array}{c} \begin{array}{cc} \text{Finite Joints} & \qquad\qquad \text{Infinite Joints} \end{array} \\ \left(\begin{array}{c|c} \begin{matrix} z_1^{-1} I_{(n+1)\times(n+1)} & & 0 \\ & \ddots & \\ 0 & & z_{v_f}^{-1} I_{(n+1)\times(n+1)} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} I_{(n+1)\times(n+1)} & & 0 \\ & \ddots & \\ 0 & & I_{(n+1)\times(n+1)} \end{matrix} \end{array}\right) \end{array}.$$

The blocks of the identity matrices $I_{(n+1)\times(n+1)}$ in Q corresponding to the joints at infinity emphasize that the velocities assigned to the joints at infinity ("velocities at infinity") are the same as that of the joints of the associated spherical framework located on the equator.

The spherical rigidity matrix $R_{\mathbb{S}}(G, \mathbf{p})$ and the matrix $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ equivalently describe the infinitesimal rigidity of a spherical framework $(G, \mathbf{p})$, however the special similarity in the appearance of the matrix $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ to the rigidity matrix of a bar-joint framework in $\mathbb{E}^n$ makes it more suitable for our purposes in the next section.

3.2 Point-hyperplane frameworks in $\mathbb{E}^n$

Informally, a point-hyperplane framework is a collection of points and hyperplanes in $\mathbb{E}^n$ with pairwise distance constraints on some pairs of points, points and hyperplanes and angle constraints on some pairs of hyperplanes. Namely, the distance or the angle between the appropriate pairs are to remain fixed. A graph can be employed to show which pairs of points or hyperplanes are constrained.

A *point-hyperplane graph* $G = (V, E)$ is a simple, finite, undirected graph whose vertex set V is composed of an ordered subset V_P of vertices corresponding to the points (point-vertices) and an ordered subset V_L of vertices corresponding to the hyperplanes (hyperplane-vertices); its edge set E is the disjoint union of the subsets E_{PP}, E_{PL} and E_{LL} where

- E_{PP} is the set of the edges incident to pairs of point-vertices. These edges represent the distance constraints on those pairs of points.
- E_{PL} is the set of the edges with one end-point in V_P (a point-vertex) and the other end-point in V_L (a hyperplane-vertex). These edges represent the distance constraints on point-hyperplane pairs.
- E_{LL} is the set of the edges incident to pairs of hyperplane-vertices. These edges represent the angle constraint on those pairs of hyperplanes.

In the case $V_L = \phi$, a point-hyperplane graph becomes a bar-joint graph.

Notation 1. Given a point-hyperplane graph $G = (V, E)$, we use $|V| = v$ for the total number of vertices, $|V_P| = v_p$ for the number of point-vertices, $|V_L| = v_l$ for the number of hyperplane-vertices and $|E| = e$ for the number of the edges of G. The ordering on vertices and edges is arbitrary. The vertices are indexed by natural numbers $V_P = \{1, \ldots, v_p\}$ for points and $V_L = \{1, \ldots, v_l\}$ for hyperplanes but we also use p_i for the ith point in V_P and ℓ_i for the ith hyperplane in V_L. The edge $\{i, j\}$ in E is also denoted by $ij \in E$.

A *point-hyperplane framework* $(G, \mathbf{p}, \ell)$ consists of a point-hyperplane graph G, an embedding of the point-vertices $\mathbf{p} : V_P \to \mathbb{R}^n$ with $\mathbf{p}(i) = p_i = (x_{i,1}, \ldots, x_{i,n})$, $i \in V_P$ into $\mathbb{R}^n$, along with a parametrization of hyperplanes $\ell : V_L \to \mathbb{R}^n$ with $\ell(i) = \ell_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{R}^n$ for every $i \in V_L$ such that the hyperplane ℓ_i has the equation $x_1 + a_{i,1}x_2 + \cdots + a_{i,n-1}x_n + a_{i,n} = 0$, in $\mathbb{E}^n$. This representation of the hyperplanes assumes that no hyperplane has a normal vector perpendicular to the x_1-axis in $\mathbb{R}^n$. This is no loss of generality because we may always rotate the entire framework so that the normals to the hyperplanes are not perpendicular to the x_1-axis. The pair $(\mathbf{p}, \ell)$ just described, is called a *point-hyperplane configuration* in $\mathbb{E}^n$.

To start studying the rigidity of point-hyperplane constraints, we first introduce the algebraic expressions of the constraints. Given a point-hyperplane graph $G = (V, E)$, the *rigidity map* of G is a smooth function $f_G : \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l} \to \mathbb{R}^e$ with $f_G(\mathbf{p}, \ell) = (\ldots, f_{ij}(\mathbf{p}, \ell), \ldots)$ where

$$\begin{aligned}
f_{ij}(\mathbf{p},\ell) &= \|p_i - p_j\|^2 \qquad ij \in E_{PP},\\
f_{ij}(\mathbf{p},\ell) &= \frac{x_{i,1} + a_{j,1}x_{i,2} + \cdots + a_{j,n-1}x_{i,n} + a_{j,n}}{(1 + a_{j,1}^2 + \cdots + a_{j,n-1}^2)^{1/2}} \qquad ij \in E_{PL},\\
f_{ij}(\mathbf{p},\ell) &= \cos^{-1}\frac{1 + a_{i,1}a_{j,1} + \cdots + a_{i,n-1}a_{j,n-1}}{(1 + a_{i,1}^2 + \cdots + a_{i,n-1}^2)^{1/2}(1 + a_{j,1}^2 + \ldots + a_{j,n-1}^2)^{1/2}} \qquad ij \in E_{LL}.
\end{aligned} \tag{3.2.0.1}$$

for every $(\mathbf{p}, \ell) \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$. For every $ij \in E$, the expression for $f_{ij}(\mathbf{p}, \ell)$ at $(\mathbf{p}, \ell)$ is:

- the squared distance between points p_i and p_j if $ij \in E_{PP}$,
- the signed distance between point p_i and hyperplane ℓ_j if $ij \in E_{PL}$,
- the angle[1] between hyperplanes ℓ_i and ℓ_j if $ij \in E_{LL}$.

Given a point-hyperplane framework $(G, \mathbf{p}, \ell)$, we can calculate the differential $df_G(\mathbf{p}, \ell)$ of the rigidity map f_G at the configuration $(\mathbf{p}, \ell)$. After some calculations

[1] *The* angle between two hyperplanes is defined as the angle between their normal vectors.

and simplifications (see below), we obtain the following $e \times nv$ matrix:

$$\begin{array}{l} \\ \text{point-point} \\ \\ \text{hyperplane-hyperplane} \\ \\ \text{point-hyperplane} \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} & p_h & & p_i & & \ell_j & & \ell_k & \end{array} \\
\begin{pmatrix}
\dots & p_h - p_i & \dots & p_i - p_h & \dots & 0 & \dots & 0 & \dots \\
 & \vdots & & \vdots & & \vdots & & \vdots & \\
\dots & 0 & \dots & 0 & \dots & \Theta_{jk} & & \Theta_{kj} & \dots \\
 & \vdots & & \vdots & & \vdots & & \vdots & \\
\dots & H_j & \dots & 0 & \dots & D_{hj} & \dots & 0 & \dots
\end{pmatrix}
\end{array}, \quad (3.2.0.2)$$

where

$$\begin{aligned}
H_j =&(1, a_{j,2}, \dots, a_{j,n-1}) \qquad j \in V_L, \\
\|H_j\| =&(1 + a_{j,2}^2 + \dots + a_{j,n-1}^2)^{1/2} \qquad j \in V_L, \\
D_{hj} =&(x_{h,2}\|H_j\|^2 - a_{j,1}\alpha_{hj}, \dots, x_{h,n}\|H_j\|^2 - a_{j,n-1}\alpha_{hj}, 1) \qquad h \in V_P, j \in V_L, \\
\Theta_{jk} =&-(a_{k,1}\|H_j\|^2 - a_{j,1}\theta_{jk}, \dots, a_{k,n-1}\|H_j\|^2 - a_{j,n-1}\theta_{jk}, 0) \qquad j, k \in V_L, \\
\alpha_{hj} =&x_{h,1} + a_{j,1}x_{h,2} + \dots + a_{j,n-1}x_{h,n} \qquad h \in V_P, j \in V_L, \\
\theta_{jk} =&1 + a_{j,1}a_{k,1} + \dots + a_{j,n-1}a_{k,n-1} \qquad j, k \in V_L.
\end{aligned}$$

We denote the matrix (3.2.0.2) by $R_{\mathbb{E}}(G, \mathbf{p}, \ell)$. This matrix is obtained from the Jacobian matrix as follows. Multiply a point-hyperplane row hj, $h \in V_p, j \in V_L$ by $\|H_j\|$. Multiply a hyperplane-hyperplane row jk by

$$\|H_j\|\|H_k\|\sqrt{1 - \frac{\theta_{jk}^2}{\|H_j\|^2\|H_k\|^2}}.$$

For every $i \in V_L$, multiply the first $n-1$ columns under the hyperplane ℓ_i by $\|H_i\|^2$. Finally, for every column l, $l = 1, \dots, n-1$ under a hyperplane $\ell_j, j \in V_L$, the scalar multiple $a_{j,l}a_{j,n}$ of the last column is added to the lth column. Consequently, the Jacobian matrix of the rigidity map f_H defined above, and the matrix (3.2.0.2) are *equi-rank* for any subgraph H of G.

It is vital to note that the latter column operation makes the last coordinate $a_{i,n}$ of each hyperplane $\ell_i = (a_{i,1}, \ldots, a_{i,n})$ disappear from the Jacobian matrix. Therefore $a_{i,n}, i \in V_L$ does not appear in (3.2.0.2). As we will see, geometrically, this means that the rigidity of a point-hyperplane configuration in $\mathbb{E}^n$ is *invariant* under any translation of any individual hyperplane in the configuration.

Given a point-hyperplane framework $(G, \mathbf{p}, \ell)$ in $\mathbb{E}^n$, we say a point-hyperplane framework is *equivalent* to $(G, \mathbf{p}, \ell)$ if it is obtained from $(G, \mathbf{p}, \ell)$ by translating some hyperplanes ℓ_i, $i \in V_L$. So every framework $(G, \mathbf{p}, \ell)$ is equivalent to a family of point-hyperplane frameworks. In particular, any framework $(G, \mathbf{p}, \ell)$ in $\mathbb{E}^n$ is equivalent to the framework $(G, \mathbf{p}, \ell^\circ)$ where $\ell_i^\circ = (a_{i,1}, \ldots, a_{i,n}, 0)$ for every $i \in V_L$. ℓ_i° is the hyperplane parallel to ℓ_i through the origin.

Given a point-hyperplane configuration $(\mathbf{p}, \ell)$ in $\mathbb{E}^n$, let $\tilde{p}_i = (x_{i,1}, \ldots, x_{n,1}, 1)$ where $p_i = (x_{i,1}, \ldots, x_{n,1})$ for every $i \in V_P$ and $\tilde{\ell}_i = (\vec{\ell_i}, 0) \in \mathbb{R}^{n+1}$ for every $i \in V_L$ where $\vec{\ell_i} = (1, a_{i,1}, \ldots, a_{i,n-1})$ is the *orientation* of the hyperplane ℓ_i whose equation is $x_1 + a_{i,1}x_2 + \cdots + a_{i,n-1}x_n + a_{i,n} = 0$. We say a configuration $(\mathbf{p}, \ell)$ is *non-degenerate* if $\tilde{p}_1, \ldots, \tilde{p}_{v_p}$ and $\tilde{\ell}_1, \ldots, \tilde{\ell}_{v_l}$ generate a vector space of dimension at least n in $\mathbb{R}^{n+1}$.

We define dim $(\mathbf{p}, \ell) =$ dim $\langle \tilde{p}_1, \ldots, \tilde{p}_{v_p}, \tilde{\ell}_1, \ldots, \tilde{\ell}_{v_l} \rangle - 1$. A non-degenerate point-hyperplane configuration in $\mathbb{E}^n$ has a full dimension of infinitesimal rigid motions that are not all identity on the configuration. If the vector space spanned by $\tilde{p}_1, \ldots, \tilde{p}_{v_p}$ and $\tilde{\ell}_1, \ldots, \tilde{\ell}_{v_l}$ is of dimension $m < n$ then the configuration is called *degenerate*. A single point or any number of parallel lines in the plane are each degenerate configurations in $\mathbb{E}^2$. A point and a set of parallel planes in $\mathbb{E}^3$ form a degenerate point-plane configuration. For degenerate configurations, the dimension of rigid motions of the configuration (that are not identity when restricted to the configuration) is not full; it is less than $n(n+1)/2$. We always assume that point-hyperplane configurations are non-degenerate unless otherwise is specified.

We say that a point-hyperplane framework $(G, \mathbf{p}, \ell)$ is *infinitesimally rigid* if the rank of (3.2.0.2) is $nv - \binom{n+1}{2}$; it is *independent* if the rank of (3.2.0.2) is e; it is *generic* if the set of the coordinates

$$\cup_{1\le i\le v_p}\{x_{i,1},\dots,x_{i,n}\}\bigcup\cup_{1\le j\le v_l}\{a_{j,1},\dots,a_{j,n-1}\}$$

are algebraically independent.

As one might have already noticed, the matrix (3.2.0.2) is complicated and not intuitive enough to work with. In the next section, we will give a geometric view of (3.2.0.2) by showing it is equivalent to (3.1.0.9) as far as the infinitesimal rigidity of point-hyperplane frameworks is concerned.

3.3 Point-hyperplane frameworks vs bar-joint frameworks in $\mathbb{E}^n$

Suppose $(G, \mathbf{p}, \ell)$ is a point-hyperplane framework in $\mathbb{E}^n$ where $\mathbf{p} = (p_i)_{i\in V_P}$, $p_i = (x_{i,1}, ..., x_{i,n})$ and $\ell_i = (a_{i,1}, \dots, a_{i,n})$ for hyperplanes $i \in V_L$ with equations $x_1 + a_{i,1}x_2 + \dots +, a_{i,n-1}x_n + a_{i,n} = 0$.

Let

$$\begin{aligned}
\tilde{p}_i &= (x_{i,1}, ..., x_{i,n}, 1) && \text{for all points } p_i, i \in V_P,\\
\tilde{p}_i &= (1, a_{i,1}, ..., a_{i,n-1}, 0) && \text{for all hyperplanes } \ell_i = (a_{i,1}, \dots, a_{i,n}),\ i \in V_L.
\end{aligned} \tag{3.3.0.1}$$

More clearly, $\tilde{p}_i$ gives the affine coordinates of points p_i in the affine hyperplane $x_{i,n+1} = 1$ for every $i \in V_P$ while $\tilde{p}_i$ is the coordinates of a 'point at infinity', which is determined by a normal vector to each hyperplane $i \in V_L$.

From Proposition 3.1.1, the matrix $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ detects the infinitesimal rigidity (or flexibility) of the spherical framework $(G, \hat{\mathbf{p}})$ as well as $R_{\mathbb{S}}(G, \hat{\mathbf{p}})$ where $\hat{\mathbf{p}}$ is given by the following:

$$\begin{aligned}
\hat{p}_i &= \tilde{p}_i/\|\tilde{p}_i\| && \text{for every } i \in V,\\
\hat{p}'_i &= \tilde{p}'_i/\|\tilde{p}_i\| && \text{for every } i \in V,
\end{aligned} \tag{3.3.0.2}$$

that rescale $(\tilde{p}_i)_{i\in V}$ and their motions $(\tilde{p}'_i)_{i\in V}$ by the same factor $\frac{1}{\|\tilde{p}_i\|}$ for each $i \in V$.

Under (3.3.0.2), the points $\tilde{p}_i = (1, a_{i,1}, \ldots, a_{i,n-1}, 0), i \in V_L$ corresponding to the hyperplane ℓ_i, are being projected to points $\hat{p}_i$ in the equator V_{eq} of $\mathbb{S}^n$. However, points $\hat{p}_i$'s do not belong to V_{eq} if $i \in V_P$.

The next theorem relates the infinitesimal rigidity of a point-hyperplane framework $(G, \mathbf{p}, \ell)$ in $\mathbb{E}^n$ to that of a bar-joint framework $(G, \hat{\mathbf{p}})$ in $\mathbb{S}^n$.

Theorem 3.3.1. *A point-hyperplane framework $(G, \mathbf{p}, \ell)$ is infinitesimally rigid (resp. independent) in $\mathbb{E}^n$ if and only if the spherical bar-joint framework $(G, \hat{\mathbf{p}})$ with points $\hat{p}_i, i \in V_L$ on the equator V_{eq}, is infinitesimally rigid (resp. independent) in $\mathbb{S}^n$.*

Proof. Consider the $(v+e) \times (n+1)v$ matrix $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ in (3.1.0.9) at $\tilde{\mathbf{p}}$ in $\mathbb{R}^{(n+1)v}$ where $\tilde{\mathbf{p}}$ is given in (3.3.0.1). Using elementary row-column operations, we will show that $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ and the matrix in (3.2.0.2) are rank-equivalent. Then the theorem will follow from (3.3.0.2) and Proposition 3.1.1.
The 'finite-finite' rows in $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ are already in the desired form with respect to their corresponding point-point rows in (3.2.0.2). We then operate the elementary row operations $r_j - r_{j,k} \hookrightarrow r_{j,k}$ for all $j, k \in V_L$ and $r_h - r_{h,j} \hookrightarrow r_{h,j}$ for all $h \in V_P, j \in V_L$ to have the following form of that part of $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$:

$$\begin{array}{lc}
 & \begin{array}{ccccccc} & \tilde{p}_h & & \tilde{p}_j & & \tilde{p}_k & \end{array} \\
\begin{array}{l} \\ \text{finite, infinite} \\ \\ \text{infinite, infinite} \\ \\ \text{vertex } \tilde{p}_h \\ \\ \text{vertex } \tilde{p}_j \\ \\ \text{vertex } \tilde{p}_k \\ \\ \end{array} &
\left(\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
\ldots & (1, a_{j,1}, \ldots, a_{j,n-1}, 0) & \ldots & (x_{h,1}, x_{h,2}, \ldots, x_{h,n}, 1) & \ldots & 0 & \ldots \\
 & \vdots & & \vdots & & \vdots & \\
\ldots & 0 & \ldots & (1, a_{k,1}, \ldots, a_{k,n-1}, 0) & \ldots & (1, a_{j,1}, \ldots, a_{j,n-1}, 0) & \ldots \\
 & \vdots & & \vdots & & \vdots & \\
\ldots & (x_{h,1}, x_{h,2}, \ldots, x_{h,n}, 1) & \ldots & 0 & \ldots & 0 & \ldots \\
 & \vdots & & \vdots & & \vdots & \\
\ldots & 0 & \ldots & (1, a_{j,1}, \ldots, a_{j,n-1}, 0) & \ldots & 0 & \ldots \\
 & \vdots & & \vdots & & \vdots & \\
\ldots & 0 & \ldots & 0 & \ldots & (1, a_{k,1}, \ldots, a_{k,n-1}, 0) & \ldots \\
 & \vdots & & \vdots & & \vdots &
\end{array}\right)
\end{array},$$

Now the columns under the finite points $i \in V_P$ are already in the appropriate form with respect to the corresponding columns in (3.2.0.2). Next we need to do some column operations. For all finite-infinite rows hj with $h \in V_P, j \in V_L$, we apply the row operation $-\frac{\alpha_{hj}}{\|H_j\|^2} r_j + r_{h,j} \hookrightarrow r_{h,j}$. Also, for the rows jk with $j, k \in V_L$, we apply the row operation $\frac{\theta_{jk}}{\|H_j\|^2} r_j - r_{j,k} \hookrightarrow r_{j,k}$. Then, for every $j \in V_L$, we apply the column operation $a_{j,l} c_{j,l} + c_{j,1} \hookrightarrow c_{j,1}$ for every $2 \leq l \leq n$. Divide the first column under every $j \in V_L$ by $\|H_j\|^2$. Finally, multiply columns 1 to $n-1$ by $\|H_j\|^2$ and divide row j by $\|H_j\|^2$, for every $j \in V_L$. Now the unique pivot 1 has been created in the first column under every point $\tilde{p}_j, j \in V_L$ located in the vertex-row j. The pivot 1 for every point p_i, $i \in V_P$ is located in the last column under p_i, in the vertex-row i. If we remove all the columns and rows containing the pivot 1's from $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$ we obtain (3.2.0.2). This completes the proof. □

See [17] for a different exposition of Theorem 3.3.1.

Definition 3.3.1. Given a point-hyperplane framework $(G, \mathbf{p}, \ell)$ in $\mathbb{E}^n$, we call the spherical framework $(G, \hat{\mathbf{p}})$ described above, the *projection* of $(G, \mathbf{p}, \ell)$ into $\mathbb{S}^n$. Conversely, a bar-joint framework $(G, \hat{\mathbf{p}})$ in $\mathbb{S}^n$ with some vertices realized in the equator will correspond to a family of point-hyperplane frameworks that are equivalent to $(G, \mathbf{p}, \ell^\circ)$ where G is regarded as a point-hyperplane graph such that $i \in V_P$ if $\hat{p}_i \notin V_{\text{eq}}$ and $i \in V_L$ if $\hat{p}_i \in V_{\text{eq}}$, with concurrent hyperplanes so that the hyperplane ℓ_i has the equation $a_{i,1}x_1 + \cdots + a_{i,n}x_n = 0$ where $\hat{p}_i = (a_{i,1}, \ldots, a_{i,n}, 0) \in V_{\text{eq}}$ and $p_i = \pi(\frac{\hat{p}_i}{\langle \hat{p}_i, e_{n+1} \rangle})$ for all points $p_i \notin V_{\text{eq}}$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the natural projection.

With the notation developed above, we have the following corollary:

Corollary 3.3.2. *Suppose $(G, \mathbf{p}, \ell)$ is a point-hyperplane framework in $\mathbb{E}^n$. Then the following are equivalent:*

(a) $(G, \mathbf{p}, \ell)$ *is infinitesimally rigid.*

(b) rank $R_{\mathbb{E}}(G, \mathbf{p}, \ell) = nv - \binom{n+1}{2}$.

(c) rank $R_{\mathbb{P}}(G, \tilde{\mathbf{p}}) = nv + v - \binom{n+1}{2}$.

(d) rank $R_{\mathbb{S}}(G, \hat{\mathbf{p}}) = nv + v - \binom{n+1}{2}$.

Even though the parametrization of hyperplanes as introduced in the previous section used a minimal number of parameters and as a result, provided a smaller size of matrix (3.2.0.2), it is rather complicated to work with. As pointed out, we could rescale the points at infinity $\tilde{p}_i$ in (3.1.0.4) as desired without affecting the infinitesimal rigidity but it helps to normalize the normal vectors $\tilde{p}_i = (1, a_{i,1}, \ldots, a_{i,n-1}, 0)$, $1 \leq i \leq v_l$, of hyperplanes ℓ_i to have the unit length when they are being parameterized. Namely, we can parameterize a hyperplane ℓ_i as $(a'_{i,1}, \ldots, a'_{i,n+1})$ so that $a'^2_{i,1} + \ldots + a'^2_{i,n} = 1$ and $a'_{i,n+1}$ is arbitrary. This will not change the rank of $R_{\mathbb{P}}(G, \tilde{\mathbf{p}})$. It also removes the implicit restriction of parametrization that forces the first coordinate in $\tilde{p}_i = (1, a_{i,1}, \ldots, a_{i,n-1}, 0)$ to be 1 while this is not necessary because, geometrically, this dictates a specific *orientation* to the hyperplanes so that they have the normals with positive first coordinates. By normalizing the vectors normal to hyperplanes ℓ_i, the expressions in (3.2.0.1) will be neater. This shortcut transition from point-hyperplane frameworks in $\mathbb{E}^n$ to spherical frameworks in $\mathbb{S}^n$ might reduce the projective geometric perspective of the problem. This slightly different, but equivalent approach is adopted in [17], which we explain at the end of this chapter and will use it in Chapter 5.

It is well-known that projective transformations preserve regularities (as well as singularities) in the literature of the rigidity of frameworks [16], [30]. In the view of the fact that the infinitesimal rigidity of point-hyperplane frameworks can be equivalently described using the matrix (3.1.0.9), which resembles the rigidity matrix of a bar-joint framework, one might expect that projective transformations would preserve the rigidity of point-hyperplane frameworks as well. Indeed, we will see below that this is true. We now begin to explain what a 'projective image' of a point-hyperplane framework is.

Suppose $(G, \mathbf{p}, \ell)$ is a point-hyperplane framework in $\mathbb{E}^n$ and $(G, \hat{\mathbf{p}})$ is its projection in $\mathbb{S}^n$. Let T be a *projective transformation* in $\mathbb{E}^n$ represented by an invertible linear transformation $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Applying T to the configuration $\hat{\mathbf{p}}$ results in a new configuration $\hat{\mathbf{q}} = (\hat{q}_i)_{i \in V}$ in $\mathbb{S}^n$ given by $\hat{q}_i = \frac{T(\hat{p}_i)}{\|T(\hat{p}_i)\|}$. Using (3.3.0.2), we project back $\hat{\mathbf{q}}$ into the Euclidean space $\mathbb{E}^n$ to obtain a point-hyperplane framework $(G, \mathbf{q}, \jmath^\circ)$ of concurrent hyperplanes at the origin, possibly after relabeling of the vertices of G as points or hyperplanes depending on whether q_i is now on the equator V_{eq} of $\mathbb{S}^n$ or not. We denote this derived graph of G under a projective transformation T by G_T and we call $(G_T, \mathbf{q}, \jmath^\circ)$ and any equivalent framework in its class, a *projective image* of the point-hyperplane framework $(G, \mathbf{p}, \ell)$ in $\mathbb{E}^n$ under T.

Theorem 3.3.3. *Suppose $(G, \mathbf{p}, \ell)$ is a point-hyperplane framework in $\mathbb{E}^n$. Then $(G, \mathbf{p}, \ell)$ is infinitesimally rigid (resp. independent) if and only if any projective image of $(G, \mathbf{p}, \ell)$ is infinitesimally rigid (resp. independent).*

Proof. Suppose $(G_T, \mathbf{q}, \jmath)$ is any projective image of $(G, \mathbf{p}, \ell)$ under a projective transformation T. Let $(G, \hat{\mathbf{p}})$ and $(G, \hat{\mathbf{q}})$ be the projections of $(G, \mathbf{p}, \ell)$ and $(G_T, \mathbf{q}, \jmath)$, respectively into $\mathbb{S}^n$. It is enough to show that $(G, \hat{\mathbf{p}})$ and $(G_T, \hat{\mathbf{q}})$ are both infinitesimally rigid or infinitesimally flexible in $\mathbb{S}^n$. Since $(G_T, \mathbf{q}, \jmath)$ is a projective image of $(G, \mathbf{p}, \ell)$, $\hat{q}_i = T(\hat{p}_i)/\|T(\hat{p}_i)\|$ for every $i \in V$. Then $\hat{\mathbf{p}}' = (\hat{p}'_i)_{i \in V}$ is a motion of the framework $(G, \hat{\mathbf{p}})$ if and only if $\hat{\mathbf{q}}' = (\hat{q}'_i)_{i \in V}$ where $\hat{q}'_i = T^{-t}(\hat{p}'_i)/\|T(\hat{p}_i)\|$, $i \in V$ is a motion of $(G, \hat{\mathbf{q}})$. This is because $x^t y = 0$, is equivalent to $(Tx)^t (T^{-t} y) = 0$, for every $x, y \in \mathbb{R}^{n+1}$. Now the theorem follows from Theorem 3.3.1. $\square$

A projective image of a point-hyperplane framework $(G, \mathbf{p}, \ell)$ could be a bar-joint framework if no vertex of G is projected to the hyperplane at infinity. This special type of projective images of a point-hyperplane framework yields bar-joint frameworks with all points $i \in V_L$ located on a hyperplane in $\mathbb{E}^n$. The invariance of the infinitesimal rigidity for point-hyperplane frameworks under projective transformations was observed and used in [17] to drive the following theorem and some other results.

Theorem 3.3.4. *[17] Suppose* $(G, \mathbf{q})$ *is a bar-joint framework in* $\mathbb{E}^n$ *with some points* q_i *realized in an affine hyperplane in* $\mathbb{E}^n$ *for all* $i \in V_0 \subseteq V$*. There exists a point-hyperplane framework* $(G', \mathbf{p}, \ell)$ *in* $\mathbb{E}^n$ *whose infinitesimal rigidity is equivalent to that of* $(G, \mathbf{q})$ *in* $\mathbb{E}^n$*.*

Proof. Let H be the affine hyperplane on which the points q_i's are realized for all $i \in V_0$ and G' be the point-hyperplane graph where $V_L = V_0$. Suppose T is a projective transformation which maps H to the hyperplane at infinity H_∞ in $\mathbb{P}^n$. Define $p_i = T(q_i) \in \mathbb{E}^n$ for all $i \in V \backslash V_0$, and $\ell_i = T(q_i) \in H_\infty$, which is a hyperplane in $\mathbb{E}^n$ through the origin with normal vector is $T(q_i)$. The point-hyperplane framework $(G, \mathbf{p}, \ell)$ is a projective image of $(G, \mathbf{q})$ so their infinitesimal rigidity are equivalent by Theorem 3.3.3. $\square$

Theorem 3.3.4 gives a geometric insight into the rigidity of an important class of non-generic bar-joint frameworks in Euclidean spaces. Consequently, all the known results and techniques in theory of the rigidity of bar-joint frameworks could be applied to point-hyperplane frameworks and vice versa. In particular, we will use point-line frameworks to understand the combinatorial and geometric rigidity of bar-joint framework with some collinear joints in the plane in Section 3.5.

3.4 Rigidity of point-hyperplane frameworks in $\mathbb{E}^n$

In order to understand motions in any geometric space it is essential to first understand the *isometries* of the space and how they act on the geometric objects in the space. Recall that an isometry of $\mathbb{E}^n$ is a bijective map $T : \mathbb{E}^n \to \mathbb{E}^n$ such that $\|T(x)-T(y)\| = \|x-y\|$ for all $x, y \in \mathbb{E}^n$. To obtain the image of a hyperplane under an isometry, we may find the action of the isometry on a point on the hyperplane and its action on a normal vector of the hyperplane as a point (see Section 5.3.2). Then, if needed, we rescale the resulting equation of the new hyperplane to be in the form described in Section 3.2 with the first coordinate 1.

Given a point-hyperplane graph G with v vertices, consider the rigidity function $f_G : \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ defined by (3.2.0.1). Let K be the complete graph on the vertex set of G. Note that $f_K(\mathbf{p}, \ell) = f_K(\mathbf{q}, \jmath)$ for all $(\mathbf{p}, \ell), (\mathbf{q}, \jmath) \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ if and only if the mapping $p_i \leftrightarrow q_i$, $\ell_i \leftrightarrow \jmath_i$ is the restriction of an *isometry* of $\mathbb{E}^n$ to the configuration $(\mathbf{p}, \ell)$.

In fact, $f_K^{-1}(f_K(\mathbf{p}, \ell))$ is the set of all point-hyperplane configurations $(\mathbf{q}, \jmath) \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ that are *congruent* to $(\mathbf{p}, \ell)$. Because the set of all isometries of $\mathbb{E}^n$ forms a smooth manifold (of dimension $n(n+1)/2$) so $f_K^{-1}(f_K(\mathbf{p}, \ell))$ is also smooth manifold that may be parameterized by the set of isometries of $\mathbb{E}^n$ that are not the identity on $(\mathbf{p}, \ell)$. Exactly similar to [2, p. 283], it can be shown that $f_K^{-1}(f_K(\mathbf{p}, \ell))$ is of dimension $n(n+1)/2$ if $(\mathbf{p}, \ell)$ is non-degenerate. If $(\mathbf{p}, \ell)$ is degenerate of dimension $m < n - 1$ then the dimension of $f_K^{-1}(f_K(\mathbf{p}, \ell))$ is $(m+1)(2n-m)/2$.

It is clear that $f_K^{-1}(f_K(\mathbf{p}, \ell)) \subseteq f_G^{-1}(f_G(\mathbf{p}, \ell))$ for any graph G on v vertices.

Definition 3.4.1. Let G be a point-hyperplane graph on v vertices, K is the complete graph on v vertices and $(\mathbf{p}, \ell) \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ is a point-hyperplane configuration in $\mathbb{E}^n$. The point-hyperplane framework $(G, \mathbf{p}, \ell)$ is *rigid* in $\mathbb{E}^n$ if there exists a neighbourhood U of $(\mathbf{p}, \ell)$ in $\mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ such that

$$f_K^{-1}(f_K(\mathbf{p}, \ell)) \cap U = f_G^{-1}(f_G(\mathbf{p}, \ell)) \cap U.$$

The framework $(G, \mathbf{p}, \ell)$ is called *flexible* in $\mathbb{E}^n$ if there exists a continuous path $\gamma : [0,1] \to \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ such that $\gamma(0) = (\mathbf{p}, \ell)$ and $\gamma(t) \in f_G^{-1}(f_G(\mathbf{p}, \ell)) - f_K^{-1}(f_K(\mathbf{p}, \ell))$ for all $t \in (0, 1]$.

Therefore, a framework $(G, \mathbf{p}, \ell)$ is rigid in $\mathbb{E}^n$ if for every $(\mathbf{q}, \jmath) \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ near $(\mathbf{p}, \ell)$ with $f_G(\mathbf{p}, \ell) = f_G(\mathbf{q}, \jmath)$, there exists an isometry of $\mathbb{E}^n$ that maps the configuration $(\mathbf{p}, \ell)$ to the configuration $(\mathbf{q}, \jmath)$.

The following proposition demonstrates the equivalence of different notions of flexibility.

Proposition 3.4.1. *Let G be a point-hyperplane graph with v vertices, K the complete graph with v vertices, and $(\mathbf{p}, \ell) \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$. The following are equivalent:*

(a) *$(G, \mathbf{p}, \ell)$ is not rigid in $\mathbb{E}^n$.*

(b) *$(G, \mathbf{p}, \ell)$ is flexible in $\mathbb{E}^n$.*

(c) *There exists a analytic path γ in $f_G^{-1}(f_G(\mathbf{p}, \ell))$ with $\gamma(0) = (\mathbf{p}, \ell)$ and $\gamma(t) \notin f_K^{-1}(f_K(\mathbf{p}, \ell))$.*

Proof. The proof is similar to Proposition 1 in [2]. □

It is a standard result in differential geometry (see [37], for example) that if $(\mathbf{p}, \ell)$ is a regular point of f_G in $\mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ then $f_G^{-1}(f_G(\mathbf{p}, \ell))$ is a smooth closed embedded submanifold in $\mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$.

For any given point-hyperplane graph G, a point-hyperplane configurations $(\mathbf{p}, \ell)$ is called *regular* if $(\mathbf{p}, \ell)$ is a regular point of f_G given by (3.2.0.1). Regular point-hyperplane configurations of a point-hyperplane graph form an open dense subset of $\mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$.

Theorem 3.4.2. *Let G be a point-hyperplane graph on v vertices, with e edges and the edge function $f_G : \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l} \to \mathbb{R}^e$. Suppose $(\mathbf{p}, \ell) \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ is a regular point of f_G and let $m = \dim(\mathbf{p}, \ell)$. Then the framework $(G, \mathbf{p}, \ell)$ is rigid in $\mathbb{E}^n$ if and only if*

$$\operatorname{rank} df_G(\mathbf{p}, \ell) = nv - (m+1)(2n-m)/2. \tag{3.4.0.1}$$

Consequently, $(G, \mathbf{p}, \ell)$ is flexible if and only if

$$\operatorname{rank} df_G(\mathbf{p}, \ell) < nv - (m+1)(2n-m)/2. \tag{3.4.0.2}$$

Proof. We omit the proof since it is similar to that of the main result in [2, p.282] for bar-joint frameworks. $\square$

Since we usually assume that $(\mathbf{p}, \ell)$ is non-degenerate so m is equal to $n-1$ or n. Then (3.4.0.1) and (3.4.0.2) simplify to

$$\operatorname{rank} df_G(\mathbf{p}, \ell) = nv - n(n+1)/2, \tag{3.4.0.3}$$

and

$$\operatorname{rank} df_G(\mathbf{p}, \ell) < nv - n(n+1)/2. \tag{3.4.0.4}$$

Note that $f_K^{-1}(f_K(\mathbf{p}, \ell))$ is a smooth manifold whose tangent space $T_{(\mathbf{p},\ell)}$ at point $(\mathbf{p}, \ell)$ lies in ker $df_G(\mathbf{p}, \ell)$ since if $x' \in \mathbb{R}^{nv_p} \times \mathbb{R}^{nv_l}$ is in the tangent space of $f_K^{-1}(f_K(\mathbf{p}, \ell))$ at $(\mathbf{p}, \ell)$ then there exists a smooth path $x(t) : [0, 1] \to f_K^{-1}(f_K(\mathbf{p}, \ell))$ such that $x(0) = (\mathbf{p}, \ell)$ and $x'(0) = x'$. Therefore $f_G(x(t)) = f_G(\mathbf{p}, \ell)$ for all $t \in [0, 1]$. This implies $df_G(\mathbf{p}, \ell)x' = 0$, i.e., $x' \in$ ker $df_G(\mathbf{p}, \ell)$. Therefore $T_{(\mathbf{p},\ell)} \subseteq$ ker $df_G(\mathbf{p}, \ell)$. By definition, $(G, \mathbf{p}, \ell)$ is infinitesimally rigid in $\mathbb{E}^n$ if $T_{(\mathbf{p},\ell)} =$ ker $df_G(\mathbf{p}, \ell)$. Theorem 3.4.2 implies if $(G, \mathbf{p}, \ell)$ is rigid for a regular configuration $(\mathbf{p}, \ell)$ then it is infinitesimally rigid. The converse is also true: Infinitesimal rigidity of a point-hyperplane framework in $\mathbb{E}^n$ implies its rigidity. See [10] for a proof in the context of bar-joint frameworks. We will also give the proof of a version of this in Chapter 6. Because the proofs are similar we omit them here.

We therefore have the following important result which is analogous to the result for bar-joint frameworks in [3, p. 173] by Asimow and Roth.

Theorem 3.4.3. *A point-hyperplane framework $(G, \mathbf{p}, \ell)$ in infinitesimally rigid in $\mathbb{E}^n$ if and only if $(\mathbf{p}, \ell)$ is a regular point of f_G and $(G, \mathbf{p}, \ell)$ is rigid.*

Proof. Suppose $(G, \mathbf{p}, \ell)$ is infinitesimally rigid in $\mathbb{E}^n$. Let $(\mathbf{q}, \jmath)$ be a point-hyperplane configuration in an open neighbourhood of $(\mathbf{p}, \ell)$ such that $l \geq m$ where dim $(\mathbf{q}, \jmath) = l$

and dim $(\mathbf{p}, \ell) = m$. Then

$$\begin{aligned}\text{rank } df_G(\mathbf{q}, \jmath) &\geq \text{rank } df_G(\mathbf{p}, \ell) = nv - (m+1)(2n-m)/2 \\ &\geq nv - (l+1)(2n-l)/2 = \text{rank } df_G(\mathbf{q}, \jmath).\end{aligned}$$

Therefore rank $df_G(\mathbf{q}, \jmath) =$ rank $df_G(\mathbf{p}, \ell)$. Namely $df_G(\mathbf{p}, \ell)$ maintains its rank in an open neighbourhood of $(\mathbf{p}, \ell)$ in $\mathbb{R}^{nv}$. So $(\mathbf{p}, \ell)$ is regular and by Theorem 3.4.2 $(G, \mathbf{p}, \ell)$ is rigid. The converse also follows from Theorem 3.4.2. $\square$

This theorem shows that the rigidity and infinitesimal rigidity of point-hyperplane frameworks are equivalent for regular configurations. The difference may occur only at singular configurations. See Chapter 7, Figure 7.2a for an example of a singular point-line configuration as a result of symmetry. The point-line framework in $\mathbb{E}^2$ is the projection of a symmetric Desargues' configuration on $\mathbb{S}^2$ into the plane. The point-line framework is infinitesimally flexible but it is rigid.

Understanding the correspondence between finite and infinitesimal flexes of spherical bar-joint and Euclidean point-line frameworks was one of our main motivations for the development of this thesis. Translational finite motions of lines in a point-line framework are not necessarily captured on the sphere as finite flexes of the spherical framework but they are infinitesimally recorded on the sphere at even singular bar-joint configurations by Theorem 3.3.1. In the next section, we will see examples of singular configurations of bar-joint frameworks in the plane whose infinitesimal flex becomes a finite flex in the corresponding class of point-line frameworks.

3.5 Point-line frameworks in the plane

In this section, we investigate the consequences of the results of the previous sections to the plane.

Suppose $(G, \mathbf{p}, \ell)$ is a point-line framework in $\mathbb{E}^2$. In [32], a combinatorial characterization of the generic rigidity of point-line frameworks was given. In that paper, a line ℓ_i is parameterized by a pair (a_i, b_i) such that its equation is $\ell_i : x = a_i y + b_i$. This slightly different parametrization from the one in Section 3.2 just results in a minus sign in the columns under each line in the rigidity matrix. As a combinatorial reason, this choice creates a desired patterns of 1's and -1's in the rigidity matrix (see below). With this form of parametrization, the following matrix is obtained from (3.2.0.2) for $n = 2$:

$$
\begin{array}{rc}
 & \begin{array}{ccccccccccc} \quad & p_h & & \quad & & p_i & \quad & & \ell_j & & \ell_k \end{array} \\
\begin{array}{r} \text{Point-point} \\ \\ \text{Line-line} \\ \\ \text{Point-line} \end{array} &
\begin{pmatrix}
\cdots & x_h - x_i & y_h - y_i & \cdots & x_i - x_h & y_i - y_h & \cdots & 0 & 0 & 0 & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -a_j + a_k & 0 & -(-a_j + a_k) & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 1 & -a_j & \cdots & 0 & 0 & \cdots & -x_i a_j - y_i & -1 & 0 & 0
\end{pmatrix}
\end{array}.
$$

Part of the assumption of genericity of a point-line configuration is that there is *no* parallel lines in the configuration. Therefore, $a_i \neq a_j$, if $i \neq j$ for all $i, j \in V_L$. Under this assumption, the previous matrix can be simplified more by dividing every 'line-line' row $jk \in E_{LL}$ by $-a_j + a_k$ to obtain the final form of the matrix as the following:

$$
\begin{array}{rc}
 & \begin{array}{cccccccccccc} \quad & p_h & & \quad & & p_i & \quad & & \ell_j & & \quad & \ell_k \end{array} \\
\begin{array}{r} \text{Point-point} \\ \\ \text{Line-line} \\ \\ \text{Point-line} \end{array} &
\begin{pmatrix}
\cdots & x_h - x_i & y_h - y_i & \cdots & x_i - x_h & y_i - y_h & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & -1 & 0 \\
 & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
\cdots & 1 & -a_j & \cdots & 0 & 0 & \cdots & -x_i a_j - y_i & -1 & \cdots & 0 & 0
\end{pmatrix}
\end{array},
\tag{3.5.0.1}
$$

The pattern of 1, -1 in all the rows involving the lines (all the edges adjacent to a line-vertex) is used to understand the matroid represented by this matrix.

Let $(G, \mathbf{p}, \ell)$ be a point-line framework in the plane with $p_i = (x_i, y_i)$ for all points $i \in V_P$ and each line ℓ_i has the coordinates (a_i, b_i), for all lines $i \in V_L$. By our conventional notation, $\tilde{p}_i = (x_i, y_i, 1)$ for all points $i \in V_P$ and $\tilde{p}_i = (1, -a_i, 0)$, for every lines ℓ_i, $i \in V_L$. Substituting this into (3.1.0.9), we obtain a 3-dimensional representation of $(G, \mathbf{p}, \ell)$ that unifies the geometry of points and lines. By Theorem 3.3.1, the rigidity of a point-line framework may be described by this representation.

It is easy to check that (3.5.0.1) is obtained from (3.1.0.9) using only row operations[2] followed by deleting the last column under each finite point (for each $i \in V_p$ in (3.5.0.1)) and the first column under each point at infinity (corresponding to each line $i \in V_L$ in (3.5.0.1)) and finally, the deletion of all vertex-rows in (3.1.0.9). Therefore, if $u = (u_{i,1}, u_{i,2})_{i \in V} \in \mathbb{R}^{2v}$ is in the kernel of (3.5.0.1) then $\tilde{u} = (\tilde{u}_i)_{i \in V} \in \mathbb{R}^{3v}$ will be in the kernel of (3.1.0.9) where

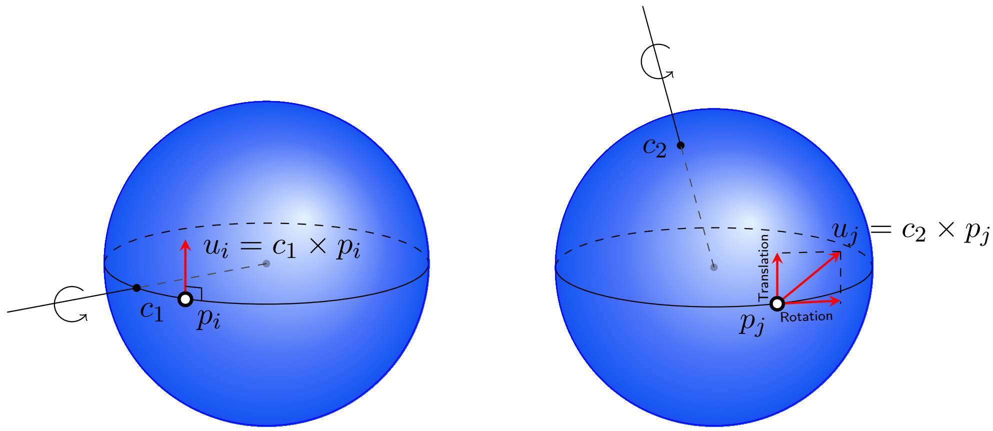
$$\begin{aligned} \tilde{u}_i &= (u_{i,1}, u_{i,2}, -\langle (x_{i,1}, x_{i,2}), (u_{i,1}, u_{i,2}) \rangle) && \text{for all } i \in V_P, \\ \tilde{u}_j &= (-a_j u_{j,1}, -u_{j,1}, -u_{j,2}) && \text{for all } j \in V_L. \end{aligned} \tag{3.5.0.2}$$

Note that this correspondence is one-to-one because $\langle \tilde{p}_i, \tilde{u}_i \rangle = 0$, for all $i \in V$.

According to (3.3.0.2), after scaling each point (finite or infinite) $\tilde{p}_i, i \in V$ by the reciprocal of its length $\frac{1}{\|\tilde{p}_i\|}$, we obtain the corresponding motions of the projected spherical bar-joint framework $(G, \hat{\mathbf{p}})$. This gives us an intuitive sense of the motions at a line ℓ_i in the Euclidean plane using its corresponding point $\tilde{p}_i / \|\tilde{p}_i\| = (1, -a_i, 0)(1 + a_i^2)^{-1/2}$ on the equator of $\mathbb{S}^2$. To have a better view, let's consider the correspondence between trivial motions on $\mathbb{S}^2$ and $\mathbb{E}^2$ under the central projection that includes the points on the equator. To obtain the motion at a point $p = (x, y)$ undergoing an instantaneous rotation with center (c_1, c_2, c_3) around the point $(c_1/c_3, c_2/c_3, 1)$ in the plane, we calculate $(c_1, c_2, c_3) \times (x, y, 1)$ and then read the first two coordinates, by (3.5.0.2). If p represents a line $x = ay + b$, we calculate $(c_1, c_2, c_3) \times (1, -a, 0)$ and read the last two coordinates by (3.5.0.2).

In the plane, the pure infinitesimal rotation has a center $(0, 0, c_3)$ around the origin for some $c_3 \in \mathbb{R}$ (the *angular velocity*) and a trivial infinitesimal pure translation in the direction of $(c_2, -c_1)$ has the center $(c_1, c_2, 0)$. The effect of a trivial

[2] for higher dimensions, we needed column operations as well.



(a) A pure translation of a line in the plane whose normal is determined by a point $p_i \in V_{\text{eq}}$ can be obtained by the action of a rotation c_1 at p_i whose axis is through the equator.

(b) A general rotation c_2 of a line whose normal is determined by $p_j \in V_{\text{eq}}$ can be decomposed into a pure rotational and pure translational components

Figure 3.2: Infinitesimal motions at lines in the plane with normals determined by p_i and p_j can be visualized using the infinitesimal motions at the point p_i and p_j on the equator of $\mathbb{S}^2$.

motion (general rotation) with center (c_1, c_2, c_3) at a point or line p is

$$(c_1, c_2, c_3) \times p = (c_1, c_2, 0) \times p + (0, 0, c_3) \times p,$$

which is the sum of a pure rotational motion and a pure translational motion (see Figure 3.2b). The motion of pure rotations with center $(0, 0, c_3)$ at line $(1, -a, 0)$ is $(ac_3, -c_3, 0)$, which reads as $(-c_3, 0)$ in the plane. The motion of a trivial translation in direction $(c_2, -c_1)$ at a line $(1, -a, 0)$ is $(0, 0, -ac_1 - c_2)$ (see Figure 3.2a) which reads as $(0, -ac_1 - c_2)$ in the plane. Figure 3.2 visualizes motions at two lines in the plane whose normals are determined by points p_i and p_j shown on the equator. The motion with center c_1 on the equator (which is a translation in the plane) at p_i is u_i. This is a pure translation on a line ℓ_i with normal p_i in the plane. The axis through c_2 is the center of a general rotation motion whose motion at p_j is u_j and is decomposed to a pure translation and a pure rotation respectively perpendicular and parallel to the plane $z = 1$.

The following theorem is the main result in the plane:

Theorem 3.5.1. *[17] The following are equivalent in the plane:*

(a) *A point-line framework $(G, \mathbf{p}, \ell)$ is generically infinitesimally rigid.*

(b) *A projection image of $(G, \mathbf{p}, \ell)$ as a bar-joint framework $(G, \mathbf{q})$ with all the joints q_i, $i \in V_L$ collinear, is infinitesimally rigid.*

(c) *The graph G has a spanning subgraph $G' = (V, E')$ with $|E'| = 2|V| - 3$ such that*

$$|F'| \leq \sum_{i=1}^{s}(2\nu_P(A_i) + \nu_L(A_i) - 2) + \nu_L(F') - 1,$$

for all partitions $\{A_1, \ldots, A_s\}$ of any non-empty subset F' of E'.

Proof. The equivalence of (a) and (b) is a consequence of Theorem 3.3.4 for $n = 2$. The equivalence of (a) and (c) is the main result in [32]. □

This theorem shows that at the infinitesimal level, a point-line framework is essentially a bar-joint framework whose vertices corresponding exactly to the line-vertices in the associated point-line graph, are realized collinear. Namely, a point-line framework is a bar-joint framework with a special geometry in the world of infinitesimal rigidity. We now apply the above result to some important examples below.

Example 3.5.1. Consider the bar-joint framework $(G, \hat{\mathbf{p}})$ on $\mathbb{S}^2$ shown in Figure 3.3a. It has 6 vertices and 9 edges. Three of its joints are placed collinear on the equator of $\mathbb{S}^2$. Therefore, $(G, \hat{\mathbf{p}})$ is not infinitesimally rigid on the sphere because a projective image of it is a bar-joint framework with three collinear points and a cycle on the collinear vertices that has wasted one edge (see Figure 3.4b). When

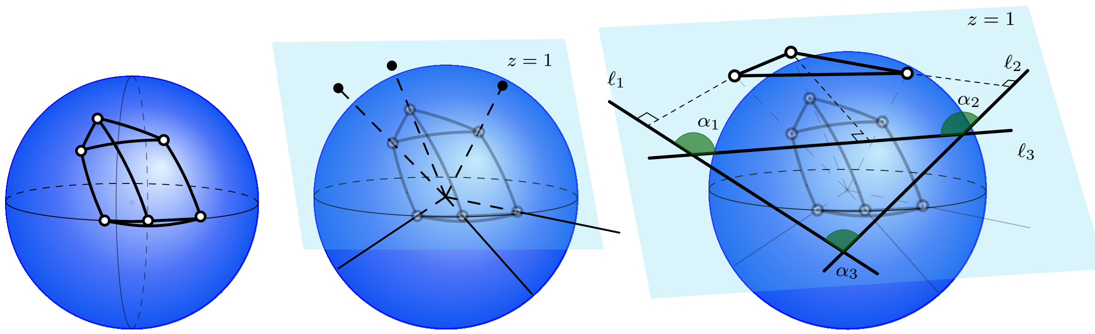


Figure 3.3: (a) shows a spherical framework with 3 joints on the equator projected to the plane in (b). Its infinitesimal rigidity is equivalent to that of a class of point-line frameworks in the plane (c).

$(G, \hat{\mathbf{p}})$ is projected under the central projection the joints on the equator map to infinity (Figure 3.3b), each of which may be replaced by a line from the associated pencil of parallel lines (Figure 3.3c). The point-line configuration $(G, \mathbf{p}.\ell)$ shown in the plane $z = 1$ in Figure 3.3c is a representative of a class of equivalent point-line frameworks whose rigidity is equivalent to the spherical framework $(G, \hat{\mathbf{p}})$. The associated point-line graph G is illustrated in Figure 3.4a.

By Theorem 3.3.1, $(G, \mathbf{p}, \ell)$ is not generically infinitesimally rigid because the spherical framework $(G, \hat{\mathbf{p}})$ is not. By Theorem 3.5.1, the infinitesimal rigidity of $(G, \mathbf{p}, \ell)$ is equivalent to a bar-joint framework $(G, \mathbf{p})$ in the plane with collinear points corresponding to the line-vertices of G, which is shown in Figure 3.4a. $(G, \mathbf{p})$ is dependent because of the cycle on points p_1, p_2 and p_3 (corresponding to the lines ℓ_1, ℓ_2 and ℓ_3). Note that collinear edges correspond to angle constraints on the line ℓ_1, ℓ_2 and ℓ_3, which is also a dependent set of geometric constraints on the lines in the plane since if any two of the angles α_1, α_2 or α_3 are fixed, the third one will be determined and fixed automatically. Also note that the framework with collinear points in Figure 3.4b is not infinitesimally rigid but is rigid as a bar-joint framework. On the other hand, the corresponding point-line framework is infinitesimally

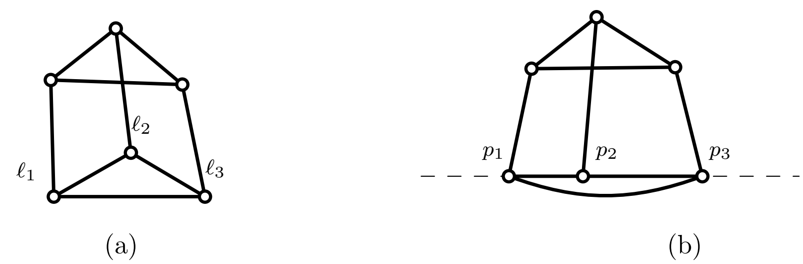


Figure 3.4: Point-line graph (a) and its corresponding bar-joint framework (b).

flexible, and flexible as a point-line configuration by Theorem 3.4.2 because such a configuration is regular as a point-line configuration.

We can use the count in Theorem 3.5.1 part (c) to detect the dependence in the point-line graph. The graph G itself is the only spanning graph with $|E| = 2|V| - 3$. Let F' is the edge set of the cycle on vertices ℓ_1, ℓ_2 and ℓ_3. Therefore $|F'| = 3$, $\nu_L(F') = 3$ and $\nu_P(F') = 0$. For the partition $\{A_1, A_2, A_3\}$ of F' where $A_1 = \{\ell_1, \ell_2\}$, $A_2 = \{\ell_1, \ell_3\}$ and $A_3 = \{\ell_2, \ell_3\}$, we have

$$3 > (0 + 2 - 2) + (0 + 2 - 2) + (0 + 2 - 2) + 3 - 1 = 2.$$

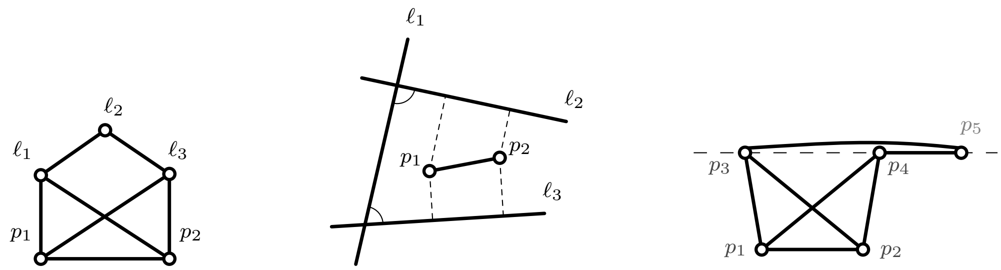
Thus F' is dependent.

In general, if F' is a non-empty subset of edges incident to line-vertices in a point-graph G then by part (c) in Theorem 3.5.1 we must have

$$|F'| \leq \nu(F') - 1,$$

which is the independence condition of the cycle matroid of a graph. As a result, any cycle on line-vertices or equivalently, collinear points in G create dependence. Recall that cycle matroid characterizes the rigidity of 1-dimensional frameworks.

Example 3.5.2. Consider the point-line graph in Figure 3.5a of which a generic point-line realization in the plane is shown in Figure 3.5b. The infinitesimal rigidity of this point-line framework in the plane is equivalent to the infinitesimal rigidity



(a) A point-line graph G (b) A point-line realization of G (c) A bar-joint realization of G

Figure 3.5: Point-line vs. bar-joint frameworks

of the bar-joint framework (corresponding to G) shown in Figure 3.5c where the collinear points p_1, p_2 and p_3 are the vertices corresponding to the line-vertices in G.

This bar-joint framework is not infinitesimally rigid in the plane because the edge $\{p_3, p_4\}$ is induced by both the edges with end-points $p_1, \ldots, p_4$ and the edges with end points p_3, p_4, p_5. This makes the framework dependent.

To see the dependence of G via the combinatorial criterion in Theorem 3.5.1, first note that G is the only spanning graph with $|E| = 2|V| - 3 = 2.5 - 3 = 7$. With lines regarded as collinear point we have, $V_P = \{p_1, p_2\}$, $V_L = \{p_3, p_4, p_5\}$. Consider the partition $\{A_1, A_2, A_3\}$ of E where A_1 is the set of edges with end-points p_1, p_2, p_3, p_4, A_2 is the singleton of the edge $\{p_3, p_5\}$ and A_3 is the singleton of the edge $\{p_4, p_5\}$. So $\nu_P(A_2) = \nu_P(A_3) = 0$, $\nu_P(A_1) = \nu_P(E) = 2$ and also $\nu_L(A_1) = 2$, $\nu_L(A_2) = \nu_L(A_2) = 2$. Therefore

$$7 > \big((2 \times 2 + 2 - 2) + (0 + 2 - 2) + (0 + 2 - 2)\big) + 3 - 1 = 6.$$

for the partition $\{A_1, A_2, A_3\}$. Thus the bar-joint framework in Figure 3.5c is not infinitesimally rigid even though it is rigid (as is its projection on the sphere). However, any generic point-line realization of G is flexible.

According to the correspondence established in Theorem 3.3.1 between point-hyperplane and bar-joint frameworks in $\mathbb{E}^n$, a point-hyperplane framework with different sets of parallel hyperplanes in $\mathbb{E}^n$ is infinitesimally equivalent to a bar-joint framework in $\mathbb{E}^n$ with the sets of vertices corresponding to each set of parallel hyperplanes realized *coincident* and conversely. For example, two parallel hyperplanes are projected to two coincident points. This type of bar-joint framework is an interesting non-generic case to be explored. For the case when only two vertices realized coincident in a bar-joint framework in the plane, a characterization of the infinitesimal rigidity and a count matroid is given in [19, Theorem 15]. This theorem states that a graph G with two distinct vertices u, v may be realized as an infinitesimally rigid bar-joint framework in the plane with the vertices u and v coincident if and only if $G - uv$ and G_{uv} are both infinitesimally rigid in $\mathbb{E}^2$ where $G - uv$ is the graph G with the edge $\{u, v\}$ deleted and G_{uv} is the graph obtained from G by contracting the vertices u, v. By theorem 3.3.3, this result immediately characterizes the rigidity of a point-line framework whose underlying graph is G with two vertices u, v as the exactly two line-vertices ℓ_1, ℓ_2 of G realized as two parallel lines in the plane. Such a framework is infinitesimally rigid if and only if the deletion of the edge $\{\ell_1, \ell_2\}$ (representing an angle constraint between ℓ_1, ℓ_2, if it exists) and contraction of the pair ℓ_2, ℓ_1 of vertices both result in two infinitesimally rigid point-line frameworks.

3.6 Some remarks on the inductive construction of point-line frameworks

In general, it can be a bit more involved to conclude whether a given point-line graph is isostatic or not. *Henneberg methods* are one of the powerful tools to detect isostatic bar-joint frameworks in the plane. Considering our geometric insight of point-line frameworks as a special non-generic bar-joint frameworks, we will see that applying Henneberg methods to point-line graphs needs some care. We do not aim to establish an inductive method that captures isostatic point-line frameworks in the plane but there are some observations to share.

Due to the special geometry of point-line configurations as bar-joint configurations with collinear points, Henneberg moves might fail to preserve independence or minimal rigidity for these frameworks.

Consider the point-line graph G in Figure 3.5a and let G_1 be the subgraph $G - \{\ell_2\}$. G_1 is an isostatic point-line graph because it is isostatic as a bar-joint graph. But adding a 2-valent line-vertex ℓ_2, adjacent to two line-vertices ℓ_1 and ℓ_3 results in a dependent point-line graph G. This is always the case: Adding a 2-valent line-vertex adjacent to two line-vertices of an isostatic point-line graph will result in a dependent graph. Except for this case, 2-addition moves preserve minimal rigidity of point-line graphs.

The following proposition is a consequence of [61, Proposition 3.1.] for bar-joint frameworks and Theorem 3.5.1.

Proposition 3.6.1. *Let $G = (V, E)$ be a point-line graph with two distinct vertices i and j and $G' = (V', E')$ be a graph obtained from G by attaching a new 2-valent vertex k and edges ik and jk to G such that at least one of i, j, k is a point-vertex. Then G is generically minimally rigid if and only if G' is generically minimally rigid.*

Just like bar-joint frameworks in the plane, removing any 2-valent vertex v_0 from a point-line graph will preserve minimal rigidity of the graph. To see this, one can realize the graph by a framework then the removal of v_0 will leave us with a set of $2(v - 1) - 3$ independent edges on $v - 1$ vertices, which is an isostatic point-line framework, by definition. From this argument, we can conclude that there is no isostatic point-line graph with a 2-valent line-vertex ℓ_0 adjacent to two line-vertices, say ℓ_1 and ℓ_2. Otherwise the removal of the 2-valent vertex ℓ_0 would give rise to an isostatic framework. That is, the edge $\{\ell_1, \ell_2\}$ is induced by the edge set $E - \{\{\ell_0, \ell_1\}, \{\ell_0, \ell_2\}\}$, which contradicts that fact that the initial graph was isostatic.

Let's now consider the removal of a 3-valent vertex v_0 from an isostatic point-line graph G. Let $(G, \mathbf{p})$ is an isostatic realization of G as a bar-joint framework with exactly all line-vertices realized collinear on a line ℓ^* (point vertices are not on ℓ^*). We might encounter the following two cases:

1. v_0 is adjacent to three vertices, not all are line-vertices. So we may assume that these three vertices are not collinear in the corresponding bar-joint framework with collinear joints. Then, by the proof of Proposition 3.3 in [61], the removal of v_0 and inserting *a* new edge among the adjacent vertices yields an isostatic framework. Therefore the new graph obtained this way from G is an isostatic point-line graph.

2. v_0 is adjacent to three line-vertices ℓ_1, ℓ_2 and ℓ_3. Let $E' = E - \{\{v_0, \ell_1\}, \{v_0, \ell_2\}, \{v_0, \ell_3\}\}$. We distinguish two cases:

   Case 1. There is no tree on the line-vertices ℓ_1, ℓ_2 and ℓ_3 induced by E'. That is, at most one of the edges $\{\ell_1, \ell_2\}$, $\{\ell_1, \ell_3\}$ or $\{\ell_2, \ell_3\}$ may be induced by E'. In this case, the removal of v_0 and the addition of one of the edges $\{\ell_1, \ell_2\}$, $\{\ell_1, \ell_3\}$ or $\{\ell_2, \ell_3\}$ that are not induced by E' preserves independence and implies that the resulting graph is an isostatic point-line graph.

   Case 2. There is a tree on the line-vertices ℓ_1, ℓ_2 and ℓ_3 induced by E'. Then, v_0 has to be a point-vertex. We may assume that the vertices ℓ_1, ℓ_2 or ℓ_3 are of valence at least 3. If the edges $\{\ell_1, \ell_2\}, \{\ell_2, \ell_3\}$ exist in E' then there is a new vertex v adjacent to ℓ_1 or ℓ_3. If v is a point-vertex adjacent to, say ℓ_1 then the edges $\{v, \ell_2\}, \{v, \ell_3\}$ are not both induced by E'. So the removal of v_0 and its 3 incident edges and addition of one of the edges $\{v, \ell_2\}, \{v, \ell_3\}$ will results in a smaller isostatic point-line graph. If v is a line-vertex adjacent to say ℓ_1 then there must be a point-vertex p adjacent to v otherwise v is adjacent to some line vertices $\ell_4, \ell_5, \ldots, \ell_n$ for some $n \in \mathbb{N}$. Then the

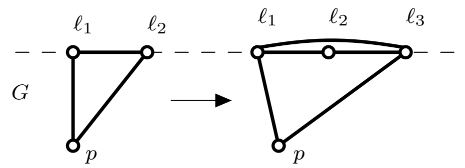


Figure 3.6: An edge-split move that fails to preserve the independence of a point-line graph.

edges $\{v, \ell_4, \}, \ldots, \{v, \ell_n\}$ could be replaced by $\{v, \ell_4, \}, \{\ell_4, \ell_5\}, \ldots, \{\ell_{n-1}, \ell_n\}$ respectively, without affecting the minimal rigidity. But this results in a 2-valent line-vertex adjacent to two line vertices ℓ_1, ℓ_4 in an isostatic graph, which is impossible. Therefore a point p is adjacent to v. Now the edges $\{p, v\}, \{p, \ell_1\}, \{p, \ell_2\}, \{p, \ell_3\}$ are not all induced by E'. Remove v_0 and add one of the edges $\{p, v\}, \{p, \ell_1\}, \{p, \ell_2\}, \{p, \ell_3\}$ that is not induced by E'. If the edges $\{\ell_1, \ell_2\}, \{\ell_2, \ell_3\}$ do not exist in E' but induced by it then we do not know whether $\{\ell_1, \ell_2\}, \{\ell_2, \ell_3\}$ are necessarily induced by some isostatic blocks in the graph or no. This should be clear before one can proceed.

Consider the triangle graph G with 1 point-vertex and 2 line-vertices as a point-line graph shown in Figure 3.6. Splitting the edge $\{p, \ell_2\}$ by a line-vertex ℓ_3 does not produce an isostatic point-line graph. An exploration of the inductive methods for point-line graphs is a possible project for the future work. This example shows that an *edge-split* move might fail to preserve isostatic graphs.

3.7 Alternative presentation of point-hyperplane frameworks in $\mathbb{E}^n$

Before we move on to the next section, we would like to explain a slightly different approach to present point-hyperplane frameworks in $\mathbb{E}^n$ that has also been adopted in [17]. As mentioned before, normalizing normal vectors to hyperplanes simplifies the algebraic expressions of the constraints and some connections to spherical motions at the cost of an extended set of constraints and a larger rigidity matrix.

Suppose $(G, \mathbf{p}, \ell)$ is a point-hyperplane framework in $\mathbb{E}^n$. All the points p_i, $i \in V_P$ are assigned their affine coordinates $p_i = (x_{i,1}, \ldots, x_{i,n}, 1)$ in the affine hyperplane $x_{n+1} = 1$ and hyperplanes $i \in V_L$ are coordinated as $\ell_i = (a_{i,1}, \ldots, a_{i,n}, a_{i,n+1})$ where

$$a_{i,1}^2 + \ldots + a_{i,n}^2 = 1, \tag{3.7.0.1}$$

so that ℓ_i has the equation $a_{i,1}x_1 + \ldots + a_{i,n}x_n + a_{i,n+1} = 0$. The vector $(a_{i,1}, \ldots, a_{i,n}) \in \mathbb{R}^n$ determines the orientation of ℓ_i. So ℓ_i can be written as $\ell_i = (\vec{\ell_i}, a_{i,n+1})$. Now let the variables $(x_{i,1}, \ldots, x_{i,n})_{i \in V_P}$ and $(a_{i,1}, \ldots, a_{i,n}, a_{i,n+1})_{i \in V_L}$ vary over time $t \in [0, 1)$ so that $\mathbf{p}(0) = \mathbf{p}$ and $\ell(0) = \ell$ while (3.7.0.1) is respected for $t \in [0, 1)$. The point-line constraints may be written as:

$$\begin{aligned}
\langle p_i(t) - p_j(t), p_i(t) - p_j(t) \rangle &= \langle p_i - p_j, p_i - p_j \rangle && \text{for every } ij \in E_{PP}, \\
\langle p_i(t), \ell_j(t) \rangle &= \langle p_i, \ell_j \rangle && \text{for every } ij \in E_{PL}, \\
\langle \vec{\ell_i}(t) - \vec{\ell_j}(t), \vec{\ell_i}(t) - \vec{\ell_j}(t) \rangle &= \langle \vec{\ell_i} - \vec{\ell_j}, \vec{\ell_i} - \vec{\ell_j} \rangle && \text{for every } ij \in E_{LL}, \\
\langle \vec{\ell_i}(t), \vec{\ell_i}(t) \rangle &= 1 && \text{for every } i \in V_L, \\
\langle p_i(t), e_{n+1} \rangle &= 1 && \text{for every } i \in V_P.
\end{aligned}$$

Taking the derivative gives the first-order constraints as the following:

$$\langle p_i - p_j, p_i' - p_j' \rangle = 0 \qquad \text{for every } ij \in E_{PP}, \tag{3.7.0.2}$$

$$\langle p_i, \ell_j' \rangle + \langle p_i', \ell_j \rangle = 0 \qquad \text{for every } ij \in E_{PL}, \tag{3.7.0.3}$$

$$\langle \vec{\ell_i} - \vec{\ell_j}, \vec{\ell_i}' - \vec{\ell_j}' \rangle = 0 \qquad \text{for every } ij \in E_{LL}, \tag{3.7.0.4}$$

$$\tag{3.7.0.5}$$

and in addition,

$$\langle p_i', e_{n+1} \rangle = 0 \qquad \text{for every } i \in V_P, \tag{3.7.0.6}$$

$$\langle \vec{\ell_i}, \vec{\ell_i}' \rangle = 0 \qquad \text{for every } i \in V_L. \tag{3.7.0.7}$$

where $\ell'_i = (\vec{\ell'_i}, a'_{i,n+1})$ for every $i \in V_L$. We refer to (3.7.0.2)–(3.7.0.7) as the $(n+1)$-dimensional representation of point-hyperplane frameworks in $\mathbb{E}^n$. Recall that, by our notation, $\ell_i^\circ = (\vec{\ell_i}, 0) \in \mathbb{R}^{n+1}$ is the coordinates of the hyperplane parallel to ℓ_i through the origin, for every $i \in V_L$. Thus the coefficient matrix of (3.7.0.2)–(3.7.0.7) is the following $(e+v) \times (n+1)v$ matrix:

$$R(G,\mathbf{p},\ell) = \begin{array}{c} \begin{array}{ccccccccc} & p_h & & p_i & & \ell_j & & \ell_k & \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & p_h - p_i & \cdots & p_i - p_h & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \ell_j & \cdots & \mathbf{0} & \cdots & p_h & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \ell_j^\circ - \ell_k^\circ & \cdots & \ell_k^\circ - \ell_j^\circ & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & e_{n+1} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \ell_j^\circ & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{pmatrix} \end{array}, \qquad (3.7.0.8)$$

It is obvious, from the matrix $R(G,\mathbf{p},\ell)$ and also (3.7.0.2)–(3.7.0.7) (the last coordinate of $p'_i \in \mathbb{R}^n$ is zero), that the last component $a_{i,n+1}$ of each hyperplane $\ell_i = (a_{i,1}, \ldots, a_{i,n}, a_{i,n+1})$ does not affect the rank of $R(G,\mathbf{p},\ell)$ and we take it to be zero. Therefore

$$\text{rank } R(G,\mathbf{p},\ell) = \text{rank } R(G,\mathbf{p},\ell^\circ),$$

for any point-hyperplane configuration $(\mathbf{p},\ell) \in \mathbb{R}^{(n+1)v}$. Using some elementary row-column operations the matrix $R(G,\mathbf{p},\ell)$ can be turned into (3.1.0.9). By the equivalence of (3.1.0.9) and the spherical matrix, we can equivalently describe the infinitesimal rigidity of a point-hyperplane framework in $\mathbb{E}^n$ using any of these matrices:

$$\text{rank } R_{\mathbb{P}}(G,\tilde{\mathbf{p}}) = \text{rank } R_{\mathbb{S}}(G,\hat{\mathbf{p}}) = \text{rank } R(G,\mathbf{p},\ell),$$

where $\tilde{\mathbf{p}}$ and $\hat{\mathbf{p}}$ are defined in (3.1.0.3)–(3.1.0.6) and (3.3.0.2). Notice that, by replacing ℓ_i by ℓ_i° in the above matrix, we make $R(G, \mathbf{p}, \ell)$ have a uniform shape for the configuration $(\mathbf{p}, \ell^\circ)$ where all hyperplanes are through the origin.

This approach eases the expression of the point-hyperplane constraints and the correspondence between the infinitesimal motions of spherical frameworks and those of point-hyperplane framework (see [17]). We will use this treatment in Chapter 5 when we want to describe *tensegrity constraints* for point-hyperplane framework.

Chapter 4

Sliders and point-line frameworks in the plane

In this chapter, we reflect further on the connection between slider mechanisms and point-hyperplane frameworks in the plane. Through examples, we will see that point-line frameworks are natural candidates for analyzing slider mechanisms in the plane.

As a simplification of a difficult problem, it is natural and useful to study point-hyperplane frameworks with restricted hyperplane-motions simply because the rigidity matrix becomes simpler. This restrictions consist of forcing the hyperplanes to only translate or only rotate in $\mathbb{E}^n$ but not both, simultaneously. In the former case, the normals to the hyperplanes are being fixed (fixed-normal rigidity) and in the latter case, the hyperplanes are to maintain their distance from a reference point in $\mathbb{E}^n$.

In this chapter, we mainly focus on point-line frameworks with restricted motions of the lines in the plane. This study will lead to some interesting connections to other types of structures (such as scene analysis and incidence structures) and moreover, interesting results on sliders. Also, in the light of the connection we made between point-hyperplane and bar-joint frameworks in Chapter 3, these results on point-line frameworks (or point-hyperplane frameworks) will have twin results for bar-joint frameworks with co-hyperplanar joints with restricted motions of the co-hyperplanar joints. Again, by the geometric and algebraic understanding of the motion of hyperplanes established in Chapter 3, the restricted motions of the co-hyperplanar joints are: the motion in the hyperplane containing the joints or the motion perpendicular to it.

We also consider point-line frameworks in which lines are pinned. This is used to model pinned slider frameworks.

The main results in this chapter have appeared in the joint work [17].

4.1 Slider joints in the plane and point-line frameworks

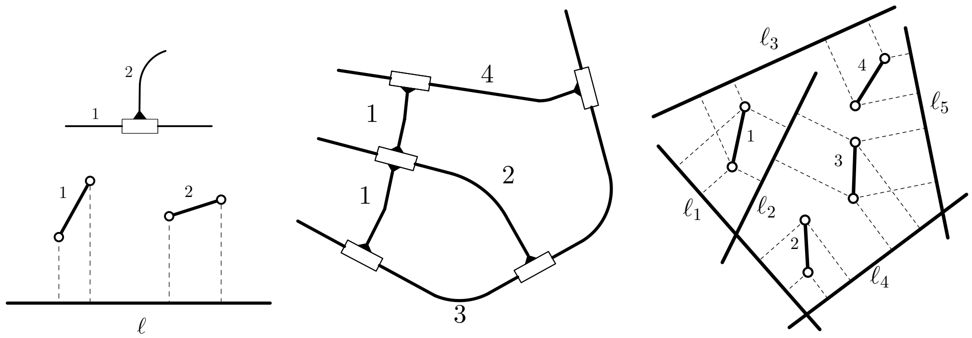
Given two rigid bodies in the plane with translational relative motion, we may replace each by a single bar (as a planar rigid body) that is constrained to slide along a line. This gives rise to a point-line framework that models the relative translational motion of two rigid bodies in the plane (see Figure 4.1a). Note that the type of the relative motion (translation in a specific direction determined by a line ℓ), degrees of freedom of the substructures (bodies) and the degrees of freedom of the entire system are correspondingly preserved in such a point-line model.

Figure 4.1b[1] shows a system of sliders. We will analyze the rigidity of this mechanism and determine its degrees of freedom using a point-line model. This example has also appeared in [17].

The system consists of 4 rigid bodies connected in a more complicated way. But the same rule still holds: each slider restricts the relative motion of the connected bodies to a translation in a specific direction.

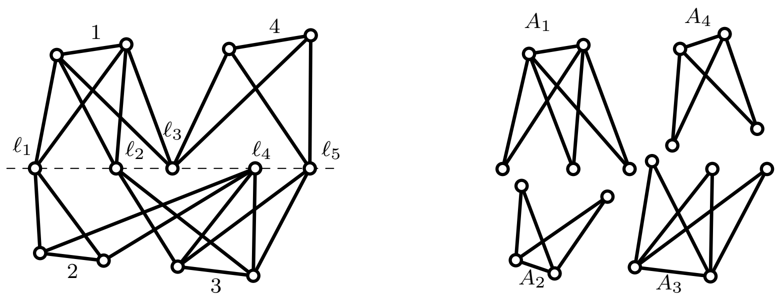
Based on what we explained for Figure 4.1a, we model this slider system as a point-line framework shown in Figure 4.1c. In fact, pairs of bodies with relative

[1] This figure is adopted from [49, p. 278].



(a) (b) (c)

Figure 4.1: The slider system illustrated in (b) is modeled by the point-line framework shown in (c).



(a) The point-line graph of the framework in Figure 4.1c.

(b) Rigid blocks $A_1, \dots, A_4$ of the graph in (a).

Figure 4.2: The dependent point-line graph associated to the slider system in Figure 4.1b.

sliding motions are described as point-line frameworks according to Figure 4.1a.

By Theorem 3.3.4, corresponding to the point-line framework in Figure 4.1c, there is a bar-joint framework with collinear joints shown in Figure 4.2a. It has $v = v_p + v_l = 8 + 5 = 13$ vertices with $e = 24$ edges.

Let $G = (V, E)$ be the point-line graph corresponding to the framework in Figure 4.1c. Let A_1, A_2, A_3 and A_4 be the edge sets incident to the bodies 1, 2, 3 and 4, respectively, which are also the maximal rigid blocks of G (Figure 4.2b). So $\nu_P(A_1) = \nu_P(A_2) = \nu_P(A_3) = \nu_P(A_4) = 2$, $\nu_L(A_1) = \nu_L(A_3) = 3$

and $\nu_L(A_2) = \nu_L(A_4) = 2$. Applying the count in Theorem 3.5.1, part c to the partition $\{A_1, A_2, A_3, A_4\}$ of E, we conclude that

$$r_{PL}(E) \leq (2\times2+3-2)+(2\times2+2-2)+(2\times2+3-2)+(2\times2+2-2)+(5-1) = 22,$$

where $r_{PL}(E)$ is the rank of E in the point-line rigidity matroid. If we delete an edge from A_2 and an edge from A_4 in G the remaining edges are independent because the resulting subgraph is constructed by 2-addition moves only (Proposition 3.6.1). Since this subgraph is a spanning graph as well, we have $r_{PL}(E) = 22$. This means the system has $2v - 3 - r_{PL}(E) = 23 - 22 = 1$ degree of freedom (as finite motion).

In addition, it is easy to check that $A_1 \cup A_2 \cup A_3$, which corresponds to the slider cycle 1-2-3, is also dependent. This corresponds to a cycle of three sliders which has only one degree of freedom. By removal of 3-valent vertices from $A_1 \cup A_2 \cup A_3$ (see Section 3.6), we can reduce it to a collinear triangle which is obviously dependent with one degree of freedom as a point-line graph. In fact, every cycle of sliders in the system corresponds to a dependent subgraph in G as a point-line graph.

4.2 Pinned-slider frameworks in the plane

Pinned frameworks are regarded as an important class of frameworks and have been studied in the context of different types of frameworks such as bar-joint frameworks and body-bar frameworks. Pinned frameworks arise naturally because, in practice, frameworks are attached to rigid bases such as ground, walls etc. and therefore, they are pinned.

In this section, using the known results on the pinned bar-joint frameworks, we will give a combinatorial characterization of the rigidity of point-line frameworks with all lines pinned in the plane.

A point-line framework $(G, \mathbf{p}, \ell)$ is *pinned* by prescribing points $p_i \in \mathbb{R}^2$ or lines $\ell_j \in \mathbb{R}^2$ for all points p_i and lines ℓ_j in a subset V_{pi} of the vertex set V as fixed positions for the point-vertices and line-vertices in V_{pi}. The vertices in V_{pi} are called *pinned* vertices and the others are called *inner* vertices, denoted by V_{in}. Indeed, the velocities at pinned vertices are set to be zero and therefore, the columns corresponding to pinned vertices will be removed from the corresponding rigidity matrices. This modified rigidity matrix is called a *pinned rigidity matrix* of $(G, \mathbf{p}, \ell)$. If we use (3.5.0.1) as the point-line rigidity matrix then we may delete 2 columns under the pinned vertices. If (3.1.0.9) or (5.3.2.1) is used then we delete three columns under each pinned vertex and its associated row to obtain the pinned rigidity matrix of the pinned point-line framework. In the following we assume $|V_{pi}| \geq 2$.

By a *pinned-slider framework* in the plane, we mean a point-line framework $(G, \mathbf{p}, \ell)$ in which all the lines are pinned, $V_L \subseteq V_{pi}$; its associated point-line graph is called a *pinned point-line graph.* When G is understood from the context, we also denote it by $G = (V_{in}, V_{pi}, E)$ indicating the pinned vertices in contrast to the inner vertices regardless of whether they are point-vertices or line-vertices. In addition, there is no edge connecting two pinned vertices in a pinned-line graph G.

Under the assumption $|V_{pi}| \geq 2$, infinitesimal rigid motions are automatically excluded from the kernel of pinned rigidity matrices. Therefore, the rigidity of pinned-slider frameworks is described as the following.
A pinned-slider framework $(G, \mathbf{p}, \ell)$ is called *infinitesimally rigid* if the kernel of its associated pinned rigidity matrix is trivial; it is *isostatic* if it is infinitesimally rigid and removing any edge from G yields an infinitesimally flexible framework or equivalently, its pinned rigidity matrix is invertible.

The well-known result on isostatic pinned bar-joint graphs (see [57, Theorem 4]) gives a combinatorial characteristic of the realizability of these graphs as isostatic pinned bar-joint frameworks up to arbitrary configurations with at least two distinct locations for pinned, and generic configurations for inner vertices. The proof uses Henneberg methods.

We now directly employ this result [57] to prove the following theorem.

Theorem 4.2.1. *A pinned-slider graph $G = (V_{in}, V_{pi}, E)$ can be realized as an isostatic pinned-slider framework in the plane if and only if*

(a) $|E| = 2|V_{in}|$,

(b) $|E'| \leq 2|V_{in}(E')|$ *for all* $E' \subseteq E$ *with* $|V(E') \cap V_{pi}| \geq 2$,

(c) $|E'| \leq 2|V_{in}(E')| - 1$ *for all* $E' \subseteq E$ *with* $|V(E') \cap V_{pi}| = 1$,

(d) $|E'| \leq 2|V_{in}(E')| - 3$ *for all* $E' \subseteq E(V_{in})$.

Proof. Suppose $G = (V_{in}, V_{pi}, E)$ is a pinned-slider graph satisfying conditions (a)-(d) . By Theorem [57, Theorem 4], conditions (a)-(d) hold for G if and only if G has a realization $\mathbf{q}$ as an isostatic pinned bar-joint framework where the pinned vertices realized arbitrarily with at least two distinct positions. In particular, $\mathbf{q}$ may be assumed to realize only the pinned line-vertices collinear. By theorem 3.3.4, the infinitesimal rigidity of $(G, \mathbf{q})$ is equivalent to that of an isostatic point-line framework with all lines pinned. This proves the theorem. $\square$

If we have at least two pinned line-vertices in an isostatic pinned-slider graph G then G can be realized as an isostatic pinned-slider framework with all pinned line-vertices realized coincident to any two non-parallel lines. Also note that the above theorem holds even when there are some points pinned. If G contains a pinned point-vertex and a pinned line-vertex, then all the pinned line-vertices can be realized coincident while the pinned-slider framework remains isostatic.

A version of the above result was proved in [59] by L. Theran and I. Streinu using the known results on the realization of direction networks in the plane (see [65], [68], [70] by W. Whiteley). Another version of the result was proved in [34] by N. Katoh and S. Tanigawa using techniques from matroid theory.

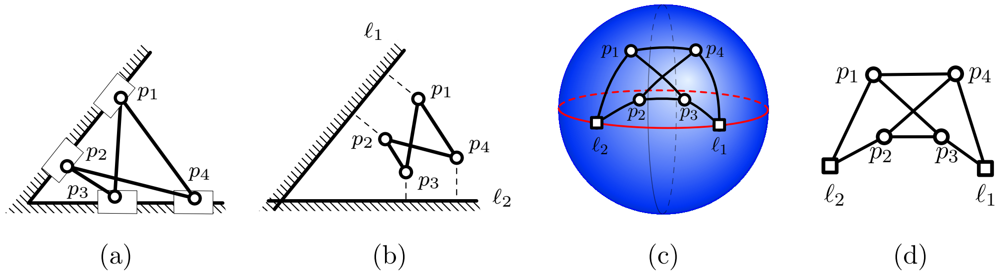


Figure 4.3: A generically infinitesimally rigid pinned-slider framework (a), its corresponding point-line framework (b), its corresponding spherical framework (c) and its associated graph (d).

Following the convention in [17], we draw squares for the pinned vertices in a pinned point-line graph.

Example 4.2.1. Figure 4.3a illustrates an example of a pinned-slider framework in the plane. The four boxes indicate the restricted sliding motions of the joints p_1, p_2, p_3, p_4 along the two pinned lines. The pinned point-line framework of Figure 4.3a with pinned lines is shown in Figure 4.3b, which in turn, is equivalent to a pinned bar-joint framework on the sphere with two pinned points on the equator, by Theorem 3.3.1 (see Figure 4.3c). The first-order rigidity of this spherical pinned framework is equivalent to the first-order rigidity of a pinned bar-joint framework in the plane. The associated pinned point-line graph is shown in Figure 4.3d. The number of pinned vertices, inner vertices and edges are $|V_{pi}| = 2$, $|V_{in}| = 4$ and $e = 8$, respectively. One can easily check that conditions (b)-(d) are also satisfied. So the pinned-slider framework is infinitesimal rigid.

Note that the realization of the inner vertices is not arbitrary. In particular, the realization of the unpinned points on the pinned lines may yield a flexible framework. As an example, in Figure 4.3a, one can check that if we place the points p_1, p_2 on ℓ_1 and the points p_3, p_4 on ℓ_2 then the framework becomes infinitesimally flexible.

4.3 Fixed-normal sliders in the plane

A *fixed-normal slider framework* is a point-line framework in which lines may only move parallel to themselves but not change their normals. Namely, the lines will only have translational motions so their normals will be fixed[2].

Let $(G, \mathbf{p}, \ell)$ be a point-line framework in the plane where $p_i = (x_i, y_i) \in \mathbb{R}^2$ for all points $i \in V_P$ and the line $\ell_i = (a_i, b_i) \in \mathbb{R}^2$ has the equation $x = a_i y + b_i$, for all lines $i \in V_L$, as described in Section 3.5. The point-line framework $(G, \mathbf{p}, \ell)$ is a fixed-normal slider framework when $a_i' = 0$, for every $i \in V_L$. For a fixed-normal slider framework, there is no angle constraints between the lines as all the angles are automatically preserved. So $E_{LL} = \emptyset$ for fixed-normal slider graphs.

A fixed-normal slider framework is *infinitesimally rigid* if translations of the entire plane are the only motions of the framework. A rigidity matrix of a fixed-normal slider framework $(G, \mathbf{p}, \ell)$ can be obtained from (3.5.0.1) by deletion of the first column under each line $\ell_i, i \in V_L$ to obtain the following $e \times (2v_p + v_l)$ matrix:

$$\begin{array}{c} \\ \text{Point-point} \\ \\ \text{Point-line} \end{array} \begin{array}{c} \begin{array}{ccccccccc} & \multicolumn{2}{c}{p_h} & & \multicolumn{2}{c}{p_i} & & \ell_j & \end{array} \\ \begin{pmatrix} \dots & x_h - x_i & y_h - y_i & \dots & x_i - x_h & y_i - y_h & \dots & 0 & \dots \\ & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \\ \dots & 1 & -a_j & \dots & 0 & 0 & \dots & -1 & \dots \end{pmatrix} \end{array}, \tag{4.3.0.1}$$

Alternatively, a rigidity matrix for fixed-normal slider frameworks may be obtained from (3.1.0.9) by deletion of the first two columns and the row corresponding to each point $\tilde{p}_i$ at infinity, which determines a normal to a line ℓ_i, for every $i \in V_L$. Also, the fixed-normal constraint equations can be given in terms of the system

[2]This type of rigidity is named *fixed-slope rigidity* in [32] and *fixed-normal rigidity* of a point-line framework in [17].

of constraints (3.7.0.2)–(3.7.0.7) for a point-line framework $(G, \mathbf{p}, \ell)$ in the plane (see [17]) by setting $\vec{\ell}_j' = 0$, for every line $j \in V_L$ while d_j' may vary for every line $\ell_j = (\vec{\ell_j}, d_j)$, $j \in V_L$, where $\vec{\ell_j} \in \mathbb{R}^2$ determines the orientation of the line ℓ_j and $d_j \in \mathbb{R}$. Therefore, the fixed-normal constraint equations are:

$$\langle p_i - p_j, p_i' - p_j' \rangle = 0 \qquad ij \in E_{PP}, \tag{4.3.0.2}$$

$$\langle p_i', \vec{\ell_j} \rangle + d_j' = 0 \qquad ij \in E_{PL}. \tag{4.3.0.3}$$

Hence a rigidity matrix of a fixed-normal slider framework $(G, \mathbf{p}, \ell)$ can be obtained from (5.3.2.1) for $n = 2$ after the deletion of the first two columns under each line, in addition to the deletion of the row corresponding to each line $i \in V_L$. This yields a $(e + v_p) \times (2v_p + v_l)$ matrix. Equivalently, one may also use the spherical bar-joint rigidity matrix (2.3.1.3) to study these constraints if they remove the first and second columns along with the associated row, corresponding to each point on the equator.

A fixed-slope slider framework is *isostatic* if it is infinitesimally rigid and the removal of any edge results in an infinitesimally flexible framework.

The following theorem from the joint work [17] gives a combinatorial characterization of the generic rigidity of fixed-normal slider frameworks. This result was proved in [32] for the special case when there is no point-point distance.

We offer two proofs for this theorem. The first proof is based on the count matroid for point-line frameworks (which is similar to the proof given in [17]). The second proof is a geometric proof that is a consequence of a result of W. Whiteley [65, Theorem 5.2.]. This result states that a Leman graph (a graph satisfying Laman's conditions) may be realized as an infinitesimally rigid bar-joint framework in the plane with *several* sets of collinear edges (or bars) that each forms a tree on the line containing them. As we know, the generic infinitesimal rigidity of a point-line framework is geometrically equivalent to that of a bar-joint framework with *only one* set of collinear edge while these edges do not need to form a tree. But, as we will show, when these edges form a tree, the rigidity of the point-line graph is equivalent to the fixed-normal rigidity of the subgraph without the collinear edges (the tree). This second proof creates a link to *scene analysis*.

We recall that if $(G, \mathbf{p}, \ell)$ is a point-line framework in the plane with the rigidity matrix given in (3.5.0.1) then a translational infinitesimal motion in direction of $t = (t_1, t_2)$ of $(G, \mathbf{p}, \ell)$ has the form $u_i = (t_1, t_2)$ at every points $p_i = (x_i, y_i) \in \mathbb{R}^2$ and $u_i = (0, -t_2 a_i + t_1) \in \mathbb{R}^2$ at every line $\ell_i : x = a_i y + b_i$, $i \in V_L$ while a pure rotation has the form $u_i = (-y_i, x_i)$ at every point p_i, $i \in V_P$ and $u_i = (\alpha, 0)$ at every line ℓ_i, $i \in V_L$ and some $\alpha \in \mathbb{R}$.

Theorem 4.3.1. *Let $G = (V, E), v_p \geq 1, v_l \geq 2$ be a fixed-normal slider graph. G can be realized as an isostatic fixed-normal slider framework if and only if $e = 2v_p + v_l - 2$ and $|F| \leq 2\nu_P(F) + \nu_L(F) - 2$ for all $\phi \neq F \subseteq E$ with $\nu_L(\phi) = -1$.*

Proof. The necessity follows since the rank of the submatrix of the slider fixed-slope rigidity matrix corresponding to $E' \subseteq E$, which is $|E'|$, cannot exceed $2|V_p(E')| + |V_s(E')| - 2$ because of the two-dimensional subspace of translational motions restricted to $V(E')$.

So we prove the sufficiency. Let T be the spanning tree on the line-vertices in G. We show that $G \cup T$ is isostatic as a point-line graph. We first notice that

$$|E(G \cup T)| = |E(G)| + |E(T)| = 2|V_P| + |V_L| - 2 + |V_L| - 1 = 2(|V_P| + |V_L|) - 3.$$

Let F be a non-empty subset of the edge set of $G \cup T$. If $V_L(F) = \phi$ then

$$\begin{aligned} |F| &\leq 2\nu_P(F) - 1 - 2, \\ &= 2\nu_P(F) - 3. \end{aligned}$$

If $V_L(F) \neq \phi$ then we have $|F| = |F \cap E(G)| + |F \cap E(T)|$. If $F \cap E(G)$ is a non-empty subset of $E(G)$, then we have

$$|F \cap E(G)| \leq \rho(F \cap E(G)) \leq \rho(F),$$

where $\rho = \min \sum_{i=1}^{s}(2\nu_P(A_i)+\nu_L(A_i)-2)$ for any partition $\{A_i\}_{i=1}^{s}$ of F. The first inequality follows from the fact that the matroid induced by ρ and $2\nu_P+\nu_L-2$ are identical ([32, Lemma 3.6.]) and the last inequality follows since ρ is a non-decreasing submodular function [32].

Because $|F\cap E(T)| \leq \nu_L(F)-1$, we now have

$$|F| = |F\cap E(G)| + |F\cap E(T)| \leq \rho(F)+\nu_L(F)-1$$

i.e., F is independent in the point-line matroid (which is the matroid induced by the submodular function $\rho+\nu_L-1$). If $F\cap E(G)=\phi$ then $\rho(F)=0$ and $|F\cap E(T)| \leq \nu_L(F)-1$ which means F is independent in the point-line matroid. Therefore $G\cup T$ is an isostatic point-line graph and can be realized as an isostatic point-line framework $(G\cup T, \mathbf{p}, \ell)$. Thus it follows that $(G, \mathbf{p}, \ell)$ is an isostatic fixed-slope slider framework because otherwise, it has a non-trivial (non-translational) infinitesimal motion $m=(m_i)\in\mathbb{R}^{2v_p}\times\mathbb{R}^{v_l}$ with $m_i\in\mathbb{R}^2$ for $i\in V_P$ and $m_i\in\mathbb{R}$ for $i\in V_L$. Then $\tilde{m}=(m_i)_{i=1}^{v}\in\mathbb{R}^{2v_p+2v_\ell}$ with $\tilde{m}_i=m_i, i\in V_P$, $\tilde{m}_j=(0,m_j), j\in V_L$ would be a non-trivial motion of $(G\cup T,\mathbf{p},\ell)$, which is impossible. This completes the proof. $\square$

The second proof comes from a study of *scene analysis* and the interesting derived results from it such as [65, Theorem 5.2.]. This theorem gives necessary and sufficient counting conditions on an incidence graph $G^*=(A,B;I)$ under which it can be realized as bar-joint framework in the plane such that each vertex in A and B is a line and a joint, respectively and the vertices of B incident to a line in A correspond to the joints that are collinear with a tree of bars spanned on them. We use this result below for point-line framework as they are considered bar-joint frameworks with only one set of joints collinear, by Theorem 3.3.4 . The reader is referred to [65] to see more matroids and connections on incidence structures.

An alternative proof of Theorem 4.3.1. The necessity of the statement is similar to the proof above. To prove sufficiency, we interpret the graph G as an incidence graph

$G^* = (A, B; I)$ so that each edge in $E_{PP} \cup E_{PL}$ represents a line-vertex in A with two incidences and $B = V_P \cup V_L$ forms the set of point-vertices in G^*. Moreover, the line-vertices in V_L are all incident to an extra line, denoted by ℓ, that does not correspond to any edges in G. This is the line that all line vertices are incident to it (the line at infinity.) Thus the total number of lines $|A|$ in G^* is $e + 1$, the total number of incidences $|I|$ is $2e + v_l$ and the total number of vertices $|B|$ is $v_p + v_l$. We now show that the incidence graph G^* satisfies the following conditions:

(a) $|I| = |A| + 2|B| - 3$,

(b) $|I'| \leq |A'| + 2|B'| - 3$, for any non-empty subset of incidences $I' \subseteq I$ with at least two points.

It is easy to check (a):

$$|I| = 2e + v_l = e + 1 + (2v_p + v_l - 2) - 1 + v_l = |A| + 2|B| - 3.$$

To see condition (b) suppose I' is a set of $|I'|$ incidences with $|A'|$ lines (edges) and $|B'| \geq 2$ points. Trivially, (b) holds if $|I'| = 2$. If $|I'| > 2$ we use a recursive process to reduce the incidence structure I' to a simple one for which condition (b) can be easily verified. Then we substitute back to recover the original incidence set.

Case 1. Suppose the special line ℓ is not contained in the line set A' of I'. Knowing that no line in A' has more than two incidences, we have either $|I'| = 2|A'| = 2|E'|$ (full incidence) or $|I'| < 2|E'|$ (non-full incidence). In the former case, all edges have their two endpoints in I'. So $|B'| = |V_P(E')| + |V_L(E')|$. Now (b) follows because

$$|I'| = 2|E'| = |E'| + |E'| \leq |E'| + (2\nu_P(E') + \nu_L(E') - 2) \leq |A'| + 2|B'| - 3.$$

That last inequality is using the fact that $\nu_L(E') = -1$ if $V_L(E) = \emptyset$ and $\nu_L(E') = |V_L(E')| \geq 1$ if $V_L(E) \neq \emptyset$.

If $|I'| < 2|E'|$ then there exists a line with only one incidence, say $(e_1, b_1) \in I'$. Now the incidence set $I^{(2)} = I' - \{(e_1, b_1)\}$ has $|E'| - 1$ lines in its line set. If $I^{(2)}$ is full incidence we are done otherwise we continue until we end up with an incidence subset $I^{(n)}$ that is full incidence or there only two points in the point-set of $I^{(n)}$, which in both cases (b) holds for $I^{(n)}$. Note that in each step either a single line or a line and a point incident to it are being removed for each incident. Substituted back the removed incidences, we obtain (b) for the original incidence set I'.

Case 2. The line ℓ in the line set of I' is the only one that can possibly have more than two incidences. Remove the incidences (ℓ, b) in I' that b is only incident to ℓ until we obtain an incidence set $I^{(n)} \subseteq I'$ with points on ℓ that are incident to another line or there is no incidence of ℓ in $I^{(n)}$. Note that in each step we are either removing a single point or a point and the line ℓ for each incidence set. In the latter case, (b) holds for $I^{(n)}$ by Case 1. In the former case, we will have $|B^{(n)}| = |V_P(E^{(n)})| + |V_L(E^{(n)})|$. Thus

$$\begin{aligned}|I^{(n)}| \leq 2|E^{(n)}| + |V_L(E^{(n)})| \leq &(|E^{(n)}| + 1) + (2|V_P(E^{(n)})| + |V_L(E^{(n)})| - 2) + \\ &|V_L(E^{(n)})| - 1 = |A^{(n)}| + 2|B^{(n)}| - 3.\end{aligned}$$

which verifies (b) for $I^{(n)}$. Substituting back the incidences, we obtain (b) for I'. Thus (b) holds for G^*.

Theorem 5.2 in [65] implies that G^* can be a realized as an isostatic bar-joint framework $(G \cup T, \mathbf{q})$ in the plane where T is the spanning tree on the line-vertices incident to the line ℓ and $\mathbf{q}$ is an embedding of the vertices of G. Therefore, by Theorem 3.3.4, there is an isostatic realization of $G \cup T$ as a point-line framework $(G \cup T, \mathbf{p}, \ell)$ with line-vertices as the vertices of T. Similar to the argument in the above proof, $(G, \mathbf{p}, \ell)$ is an isostatic fixed-normal slider framework. □

As an insightful corollary of the above proofs, we have

Corollary 4.3.2. *A fixed-normal slider framework $(G, \mathbf{p}, \ell)$ is isostatic in the plane if and only if $(G \cup T, \mathbf{p}, \ell)$ is an isostatic point-line framework in the plane where T is a spanning tree on the line-vertices V_L.*

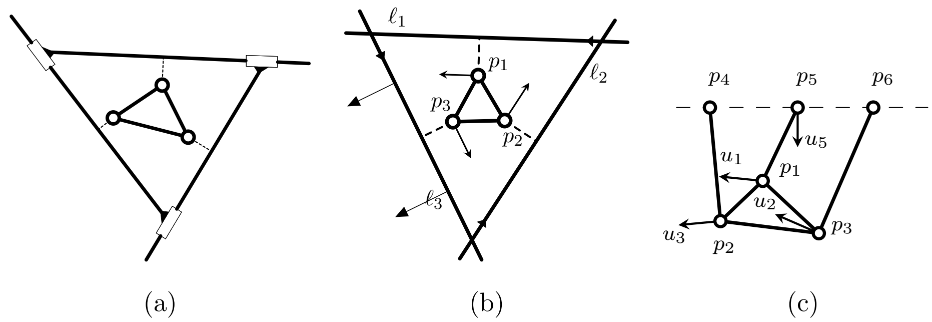


Figure 4.4: Fixed-normal slider frameworks

As the above corollary shows, a basis in the fixed-normal rigidity matroid can be easily turned into a basis element in the point-line rigidity matroid. In this sense, the fixed-normal rigidity is a special case of the general point-line rigidity.

Intuitively, the tree T mentioned above blocks the relative rotational motion of the lines so that if we prevent one line from rotating then none of the remaining lines will rotate. If the framework is now infinitesimally rigid then it is fixed-normal rigid and vice versa.

A projective image of a fixed-normal slider framework (as a fixed-normal point-line framework) as a bar-joint framework with collinear joints in the plane such that the motion at the collinear joints is restricted to be perpendicular to the direction of the line containing them.

We use little black-filled triangles on the lines to indicate that the lines are constrained to have fixed normals, see Figure 4.4b.

Example 4.3.1. Figure 4.4a illustrates a fixed-normal slider framework with 3 revolute joints and 3 prismatic joints. Its point-line framework model is shown in Figure 4.4b with 3 points p_1, p_2, p_3 and 3 fixed-normal lines ℓ_1, ℓ_2, ℓ_3. By Theorem 4.3.1, this fixed-normal slider framework is not infinitesimally rigid since $e = 6 < 2v_p + v_l - 2 = 7$. It has one degree of freedom because the framework is independent by Theorem 4.3.1. This infinitesimal motion leads to a finite motion (in the associated point-line framework) as the configuration is a regular point of the matrix (4.3.0.4). The infinitesimal rigidity of this point-line framework is equivalent to that of a bar-joint framework shown in Figure 4.4c in which the collinear points are restricted to infinitesimally move perpendicular to the line containing them.

One can easily generalize the concept of the fixed-normal rigidity in the plane to higher dimensions for point-hyperplane frameworks. 'Fixed-normal rigidity' for point-hyperplane frameworks in $\mathbb{E}^n$ restricts the motion of the hyperplanes to translations only. This rules out the rotational motions of the entire framework in $\mathbb{E}^n$, which is the $n(n-1)/2$ dimensional space of infinitesimal rotations $so(n)$. A rigidity matrix of the fixed-normal rigidity of a fixed-normal point-hyperplane framework $(G, \mathbf{p}, \ell)$ in $\mathbb{E}^n$ can be obtained from (3.2.0.2) by removing the first $n-1$ columns under each hyperplane, which gives rise to the following matrix:

$$\begin{array}{l} \\ \\ \text{Point-point} \\ \\ \text{Point-hyperplane} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} & p_h & & p_i & & \ell_j & \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & & \vdots & \\ \dots & p_h - p_i & \dots & p_i - p_h & \dots & 0 & \dots \\ & \vdots & & \vdots & & \vdots & \\ \dots & H_j & \dots & 0 & \dots & 1 & \dots \\ & \vdots & & \vdots & & \vdots & \end{pmatrix} \end{array}, \qquad (4.3.0.4)$$

where $H_j = (1, a_{j,1}, \dots, a_{j,n-1})$ represents a normal vector to the hyperplane ℓ_j in $\mathbb{E}^n$. Therefore if $G = (V_P \cup V_L, E)$ is a point-hyperplane graph on v_p point-vertices and v_l hyperplane-vertices then the necessary condition for the edge-set E to be independent in the fixed-normal rigidity matroid in $\mathbb{R}^n$ is

$$|F| \leq n\nu_P(F) + \nu_L(F) - n,$$

for every non-empty subset $F \subseteq E$. The question here is whether this is a sufficient condition for independence or not?

The answer is affirmative for the case when there is no point-point distance constraints, i.e., $E_{PP} = \phi$ in which case G is a bipartite graph with the bipartition

$\{V_P, V_L\}$. In this case, the rigidity matrix (4.3.0.4) is identical to the rigidity matrix of the *parallel n-scene matrix* given by W. Whiteley in [68, p. 210-211] and as a result, its rigidity matroid is the same as the *n-parallel matroid* (see also [65]).

Thus we have the following theorem:

Theorem 4.3.3. *[17] Let $G = (V_p \cup V_L, E)$ be a bipartite graph. Then the following are equivalent:*

(a) *Every realization of G as a n-scene with generic hyperplane normals is trivial*[3].

(b) *Every realization of G as a point-hyperplane framework in $\mathbb{E}^n$ with generic normals is infinitesimally fixed-normal rigid in $\mathbb{E}^n$.*

(c) *G contains a spanning subgraph $G' = (V_P \cup V_L, E')$ such that*

$$|E'| = nv_p + v_l - n,$$
$$|F| \leq n\nu_P(F) + \nu_L(F) - n,$$

for all non-empty $F \subseteq E'$.

The generic rigidity of point-hyperplane frameworks in $\mathbb{E}^n$, $n > 2$ is even more difficult to understand than the generic rigidity of bar-joint frameworks. To see this, consider complete bipartite point-hyperplane graphs. The generic rigidity of a complete bipartite graph $K_{p,q}$ in $\mathbb{E}^n$ is characterized by a theorem of Whiteley in [67]: $K_{p,q}$ is generically rigid in $\mathbb{E}^n$ if and only if $p + q \geq \binom{n+2}{2}$, $p, q > n$. By this theorem, the complete bipartite graph $K_{4,6}$ is generically rigid in $\mathbb{E}^3$ as a bar-joint framework. On the other hand, in [4, Theorem 15], it is shown that a bar-joint framework with the underlying graph $K_{4,6}$ in $\mathbb{E}^3$ has non-trivial infinitesimal motion if and only if the joints all lie on a quadric surface or the 4 points lie in a plane. This result combined with Theorem 3.3.4 implies that a point-plane framework in $\mathbb{E}^3$ with $K_{4,6}$ as its point-plane graph with 4 plane-vertices and 6 point-vertices is

[3] All points in V_P will be realized coincident.

generically infinitesimally flexible because the 4 planes correspond to 4 coplanar points. Again, by a result of Whiteley (see [67, Corollary 1.4]), this point-plane framework has 2 degrees of infinitesimal motions in $\mathbb{E}^3$. This example shows that the difficulty of understanding the generic rigidity of point-hyperplane frameworks in $\mathbb{E}^3$ goes beyond an obstacle such as double banana for bar-joint frameworks as there is not yet any point-point distance edge in $K_{4,6}$.

As expected, by generic normals $H_j = (1, a_{j,1}, \dots, a_{j,n-1}), j \in V_L$, to hyperplanes ℓ_j we mean the generic points $(a_{j,1}, \dots, a_{j,n-1}) \in \mathbb{R}^n, j \in V_L$, in a hyperplane in $\mathbb{E}^n$. For $n = 2$, the normals to the lines are generic points on the line at infinity. We want to show that it is not enough to have distinct normals (distinct points on the line at infinity) in order to maintain the rank of the point-line rigidity matrix. In general, the points on the line at infinity should be 'generic' on the line regardless of the position of the points in $\mathbb{E}^n$.

As the following example shows for the plane, there are some cases for which the genericity of the normals is not necessary for generic rigidity but only the distinction of the normals is enough. However, the genericity of normals is necessary in general (Example 4.3.3). To show this, we use fixed-normal point-line frameworks with naturally bipartite graphs for which the positions of points play *no* role in the rigidity of these frameworks as point coordinates do not appear in the rigidity matrix (4.3.0.4).

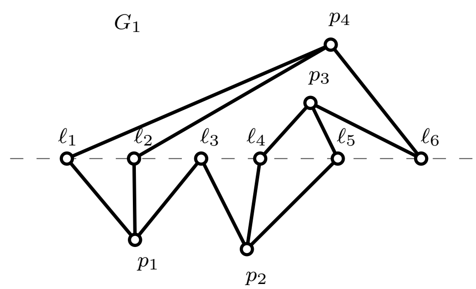
Example 4.3.2. Figure 4.5a illustrates an example of a bipartite graph G_1 that is generically infinitesimally rigid as a fixed-normal slider graph (or fixed-normal point-line graph) by Theorem 4.3.1. But its fixed-normal rigidity in the plane depends only on the distinction of the normals to the lines. Let $R(G_1, \mathbf{p}, \ell)$ be the rigidity matrix of the framework and lines $\ell_1, \dots, \ell_6$ have equations $x = a_i y + b_i$, $1 \leq i \leq 6$. Suppose $M(G, \mathbf{p}, \ell)$ is the principal minor in the rigidity matrix obtained by removing the two columns corresponding to any two lines, say, ℓ_5 and ℓ_6. Then

$$\det M(G, \mathbf{p}, \ell) = \pm(a_1 - a_2)(a_3 - a_6)(a_4 - a_5)(a_5 - a_6),$$

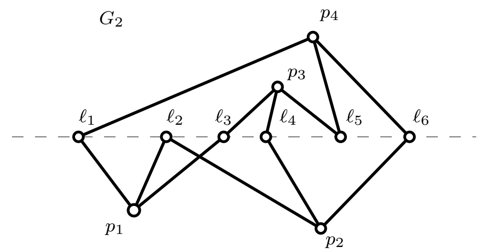
where $\pm$ sign is up to the ordering of the lines as they appear in the rigidity matrix. This indicates the rank of the rigidity matrix will not drop as long as the lines have different normals. So the normals to the lines only need to be distinct. It is not difficult to see why linear terms such as $(a_i - a_j)$ for some $i, j \in V$, appear in the determinant. det $M(G, \mathbf{p}, \ell)$ can be calculated using *Laplace* expansion and some rules of thumb. First, a 2-valent point-vertex can be removed from the graph and as a result, the determinant is factored by $\pm(a_i - a_j)$ if the 2-valent point-vertex is adjacent to lines ℓ_i and ℓ_j. Second, if two point-vertices i, j are adjacent to the same two line-vertices k, l then every edge jm adjacent to j could be replaced by edge im in the graph. In the case of G_1, we first replace the edge $\{p_4, \ell_6\}$ by $\{p_1, \ell_6\}$ then expand the determinant on the columns of p_4. Consequently the factor $a_1 - a_2$ appears. Similarly, replace the edge $\{p_3, \ell_6\}$ by $\{p_2, \ell_6\}$ and remove p_3. This factors det $M(G, \mathbf{p}, \ell)$ by $a_4 - a_5$. Finally we replace edges $\{p_2, \ell_4\}$ and $\{p_2, \ell_5\}$ by $\{p_1, \ell_4\}$ and $\{p_1, \ell_5\}$, respectively and the factor $a_3 - a_6$ will appear, as a result. We are then left with a *cone graph*; point p_1 is connected to all lines and the determinant of this cone graph is $a_5 - a_6$. Using this observation, it can be checked that for fixed-slope point-line frameworks with less than 6 lines, the genericity of the normals is not necessary.

Example 4.3.3. Now consider the graph G_2 shown in Figure 4.5b, which is isostatic as a generic fixed-normal slider framework (or fixed-normal point-line framework) by Theorem 4.3.1. All point-vertices are 3-valent and every two have only one line-vertex in common. If we remove the columns of ℓ_5 and ℓ_6 in the rigidity matrix, the determinant of this minor is

$$\pm(a_5 - a_6)\Big((a_4 - a_5)(a_1 - a_3)(a_2 - a_6) + (a_3 - a_5)(a_1 - a_2)(a_4 - a_6)\Big),$$



(a) G_1 is always independent for distinct line-normals.



(b) G_2 becomes dependent for some distinct line-normals.

Figure 4.5: The genericity of the normals of the lines is necessary for generic rigidity of fixed-normal point-line frameworks.

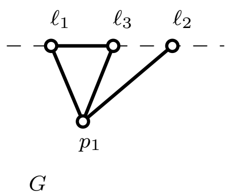
which is not the product of linear factors. In fact, there are some sets of distinct normals for which the rigidity matrix is not full-rank. One can easily check that the rank drops if we choose $a_1 = 1, a_2 = 0, a_3 = 2, a_4 = 4/5, a_5 = 3, a_6 = 1/4$ while it is full-rank for generic normals.

One can adopt Theorem 3.4.2 for fixed-normal point-hyperplane frameworks to conclude that if a point-hyperplane configuration $(\mathbf{p}, \ell)$ is a regular point of the matrix (4.3.0.4) and the rank of this matrix at $(\mathbf{p}, \ell)$ is less than $nv_p + v_l - n$ then the rigidity and the infinitesimal rigidity of the fixed-normal framework $(\mathbf{p}, \ell)$ are equivalent.

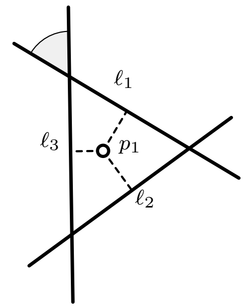
In the next section, we consider the rigidity of a class of point-line frameworks with constrained line motions for which the genericity of line-normals is not necessary.

4.4 Fixed-intercept rigidity

In this section we consider point-line frameworks for which the lines may not translate. To prevent translational motions of the lines we restrict them to maintain their



(a) A generically fixed-intercept rigid graph.



(b) If the center is the point p_1 the framework is not infinitesimally rigid.

Figure 4.6: Fixed-intercept rigidity.

distance from a fixed point in the plane. This fixed point will be the center of the trivial rotational motion of the entire framework, which is a rigid motion. Note that trivial infinitesimal translations are being automatically excluded. So the center of the trivial rotation should be a specific point in the sense that it is not coincident to any other points. With no loss of generality, this center may be assumed to be the origin. In Chapter 3, we realized that the generic rigidity of point-line frameworks does not depend on the position of the lines in the plane. Therefore we may also assume that the lines are concurrent and through the origin. In [17], these frameworks are referred to as *line-concurrent frameworks*. It should be emphasized that the point of concurrency (the center of the allowed rotation) is a special point and no other point should be coincident to this center (see Figure 4.6b).

Let G be a point-line graph and $(\mathbf{p}, \ell)$ be a generic configuration of points and lines in the plane. The framework $(\mathbf{p}, \ell)$ is called *infinitesimally fixed-intercept rigid* if the framework has only the trivial rotational motion. A rigidity matrix capturing the fixed-intercept rigidity of a framework $(G, \mathbf{p}, \ell)$ can be obtained from a point-line rigidity matrix such as (3.5.0.1) by deleting the last column under every line, i.e. the infinitesimal translational component of the motions of the lines is set to zero for line-concurrent frameworks. Hence the rank of the corresponding rigidity matrix does not exceed $2v_p + v_l - 1$.

Fixed-intercept rigidity constraints on a point-line configuration $(\mathbf{p}, \ell)$ in the plane can also be obtained from the constraints (3.7.0.2)–(3.7.0.7) by setting $d_i' = 0$ for every line $\ell_i = (\vec{\ell_i}, d_i), i \in V_L$:

$$\begin{aligned}
\langle p_i - p_j, p_i' - p_j' \rangle = 0 \qquad & ij \in E_{PP}, \\
\langle p_i, \vec{\ell_j}' \rangle + \langle p_i', \vec{\ell_i} \rangle = 0 \qquad & ij \in E_{PL}, \\
\langle \vec{\ell_i}, \vec{\ell_j}' \rangle + \langle \vec{\ell_i}', \vec{\ell_j} \rangle = 0 \qquad & ij \in E_{LL}, \\
\langle \vec{\ell_i}, \vec{\ell_i}' \rangle = 0 \qquad & i \in V_L,
\end{aligned}$$

where $p_i = (x_i, y_i) \in \mathbb{R}^2$ for every $i \in V_P$ and $\vec{\ell_i}$ is a unit vector in $\mathbb{R}^2$ for every line $i \in V_L$.

A combinatorial characterization of the generic fixed-intercept rigidity is given in the following theorem.

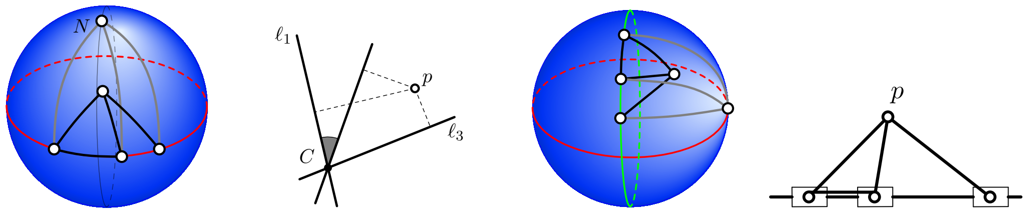
Theorem 4.4.1. *[17] Let G be a point-line graph with $v_l \geq 2$. Then G can be realized as an isostatic fixed-intercept line concurrent framework with arbitrary distinct normals of the lines if and only if $e = 2v_p + v_l - 1$ and*

$$|F| \leq 2\nu_P(F) + \nu_L(F) - 3 + min\{2, \nu_L(F)\},$$

for all non-empty $F \subseteq E$.

Proof. See [17, Theorem 4.3.] for a proof. □

In Chapter 3, we established an intuitive connection (see Figure 3.2) between the motions (translational-rotational components) of lines of a point-line framework in the plane and those of the points on the equator of a bar-joint framework on the sphere $\mathbb{S}^2$. We utilized this conception in the proof of Theorem 4.3.1. This connection will also give us a geometric insight into the rigidity of fixed-intercept frameworks. Suppose $(G, \mathbf{p}, \ell)$ is a point-line framework in the plane and $(G, \mathbf{q})$ is its projection into $\mathbb{S}^2$ with all points $i \in V_L$ on the equator. Blocking the translational



(a) (b) (c) (d)

Figure 4.7: Fixed-intercept point-line frameworks and collinear sliders.

motion of the lines in the plane is equivalent to the blocking of the perpendicular (to the equator of $\mathbb{S}^2$ in $\mathbb{R}^3$) component of the motion of the points on the equator. This is achieved by connecting the north pole to each point on the equator which yields a new bar-joint framework $(G', \mathbf{q})$ on $\mathbb{S}^2$, where $V(G') = V(G) \cup \{N\}$ and $E(G') = E(G) \cup \{e_1, \ldots, e_{v_l}\}$ with $e_i = \{N, \ell_i\}$, $1 \leq i \leq v_l$. The projection of $(G', \mathbf{q})$ to the plane is a point-line framework whose fixed-intercept rigidity is equivalent to the rigidity of $(G', \mathbf{q})$ with the north pole pinned because there is a natural one-one correspondence between the infinitesimal motions of these two frameworks. Applying a rotation that maps only the north pole to the equator results in a bar-joint framework with only one joint on the equator which is pinned. Projecting this framework to the plane gives rise to a point-line framework with one line which is pinned and the collinear points on the pinned line are constrained to remain on the line because they are connected to a point at infinity. See [17, Figure 8] for more examples.

Therefore, the infinitesimal rigidity of a fixed-intercept framework $(G, \mathbf{p}, \ell)$ is equivalent to the infinitesimal rigidity of a bar-joint framework $(G, \mathbf{q})$ where the joints $q_i, i \in V_L$ are collinear and constrained to move on a line.

Example 4.4.1. Consider the graph G in Figure 4.6a. By Theorem 4.4.1, G is infinitesimally fixed-intercept rigid. A point-line framework with the underlying graph G is shown in Figure 4.7b. As was explained, the fixed-intercept infinitesimal rigidity of this framework is equivalent to that of a bar-joint framework on the sphere shown in Figure 4.7a. In turn, the infinitesimal rigidity of this spherical framework is equivalent to the framework shown in Figure 4.7c, under a rigid motion of the sphere. The projection of this last framework to the plane is a bar-joint framework with collinear joints constrained to stay collinear, shown in Figure 4.7d.

Hence Theorem 4.4.1 has an analogue for bar-joint frameworks with one set of collinear joints that are constrained to move along the line containing them. In fact, Theorem 4.4.1 has been extended in [17, Theorem 4.4] to give a combinatorial characterization of fixed-intercept rigidity for the non-generic case when some lines might have the same normals, meaning they could be coincident or parallel.

This result will directly translate to a result about the rigidity of a slider mechanism with a set of sliders on a line so that the slider points could be coincident. We state it here for completeness:

Theorem 4.4.2. *[17, Theorem 4.5.] Let G be a point-line graph with $v_l \geq 2$ and let $x_i \in \mathbb{R}$ for each $i \in V_L$. Then G can be realized as a minimally infinitesimally rigid bar-joint framework in $\mathbb{R}^2$ with V_L as a set of horizontal slider joints such that the coordinate of $i \in V_L$ is $(x_i, 0)$ if and only if*

(a) $e = 2v_p + v_l - 1$,

(b) $x_i \neq x_j$ *for each* $ij \in E_{LL}$, *and*

(c) $|F| \leq 2\nu_P(F) + \nu_L(F) - 1 - \sum_{H \in C(G[F])^P)} max\{0, 2 - |\{x_j : ij \in F \cap E_{PL}, i \in V(H)\}|\})$ *for all non-empty* $F \subseteq E$.

The counting condition in the above theorem simplifies to the counting condition in Theorem 4.4.1 when the lines have distinct normals. See [17] for the proof of these results.

Chapter 5

Tensengrity for point-hyperplane frameworks

In this chapter we introduce and study the type of point-hyperplane frameworks with not only fixed distances on some pair of points but also with the constraints that restrict the motion of some points in some directions. This will lead to the study of the rigidity of a system of constraints with upper-bounded or lower-bounded distance or angle constraints on some pairs of points and hyperplanes. We investigate and characterize the rigidity and infinitesimal rigidity of these types of point-hyperplane frameworks. These results can detect the rigidity or flexibility of a broad class of slider systems.

This study is inspired by the analogous constraint system on points in $\mathbb{E}^n$. These frameworks drew attention from engineers and architects in the last century. A physical model of these structures could be composed of some bars, cables and struts connecting specific pairs of points so that bars keep the distance between their endpoints fixed while cables (respectively, struts) allow this distance to reduce (respectively, increase) only. These structures were called *tensegrity frameworks* by Buckminster Fuller.

We first establish tensegrity frameworks in $\mathbb{S}^n$ and then 'project' them into the Euclidean space $\mathbb{E}^n$ in order to understand tensegrity constraints for point-hyperplane frameworks in $\mathbb{E}^n$. To this end, we first develop the *Static Theory* of frameworks in spherical spaces. This will give us insight into the static analysis of point-hyperplane frameworks.

5.1 Statics of bar-joint frameworks in $\mathbb{S}^n$

In this section, the static theory of rigidity is developed as the dual concept to the infinitesimal theory of rigidity in n-dimensional spherical space. We will see that the static rigidity of frameworks in $\mathbb{S}^n$ is equivalent to their infinitesimal rigidity.

In the entire section, by a framework we mean a spherical framework.

We recall that for a spherical configuration $\mathbf{p} = (p_1, \ldots, p_v)$ with $p_i \in \mathbb{S}^n$, $1 \leq i \leq v$, the linear subspace

$$\mathcal{T}(\mathbf{p}) = \{(p'_1, \ldots, p'_v) \in \prod_{i=1}^{v} T_{p_i}\mathbb{S}^n \mid p'_i = Sp_i, S^t = -S, S \in M_{(n+1)\times(n+1)}\},$$

of $\mathbb{R}^{(n+1)v}$ is the linear space of *trivial infinitesimal motions* at $\mathbf{p}$.

If $\mathbf{p}$ generates a vector space of dimension at least n then the dimension of $\mathcal{T}(\mathbf{p})$ is $\binom{n+1}{2}$; the dimension of the space of skew-symmetric $(n+1)\times(n+1)$ matrices i.e., the dimension of the linear space of *infinitesimal rotations* in $\mathbb{R}^{n+1}$. In this chapter, we assume that all configurations $\mathbf{p}$ have this property.

In differential geometry, forces at a point p are elements of the *cotangent space* $T_p^*\mathbb{S}^n$ at p; the space of *1-forms* at p, which is isomorphic to $\mathbb{R}^n$. In addition, forces are pulled back by smooth maps as opposed to velocities that are pushed forward by smooth maps. In projective geometry, a force at a point p in $\mathbb{P}^n$ is a 2-form $f \wedge p$ for some $f \in \mathbb{R}^{n+1}$, see [16] and [33]. For our purposes, we define

Definition 5.1.1. A *force f_i at a point p_i* on $\mathbb{S}^n$ is an element of $\mathbb{R}^{n+1}$ such that $\langle f_i, p_i \rangle = 0$. The exterior 2-form $f_i \wedge p_i$ determines a line called the *line of force f_i* on $\mathbb{S}^n$.

Therefore, by definition, forces determine *decomposable* 2-extensors in the language of projective geometry and conversely, any decomposable 2-extensor in $\mathbb{R}^{n+1}$ determines a force at a point on sphere $\mathbb{S}^n$.

A force at a configuration $\mathbf{p} = (p_1, \ldots, p_v)$ is a v-tuple $\mathbf{F} = (f_1, \ldots, f_v)$ of forces f_i at points p_i on $\mathbb{S}^n$, $i = 1, \ldots, v$. We define the linear space of *equilibrium forces* at configuration $\mathbf{p} = (p_1, \ldots, p_v)$ as

$$\mathcal{E}(\mathbf{p}) = \{\mathbf{F} \in \mathbb{R}^{(n+1)v} | \langle \mathbf{F}, \mathbf{p}' \rangle = 0, \mathbf{p}' \in \mathcal{T}(\mathbf{p})\}.$$

In other words, $\mathcal{E}(\mathbf{p})$ is the orthogonal complement of $\mathcal{I}(\mathbf{p})$ in a nv-dimensional space, so dim $\mathcal{E}(\mathbf{p}) = nv - \binom{n+1}{2}$. The following proposition gives an alternative description of $\mathcal{E}(\mathbf{p})$.

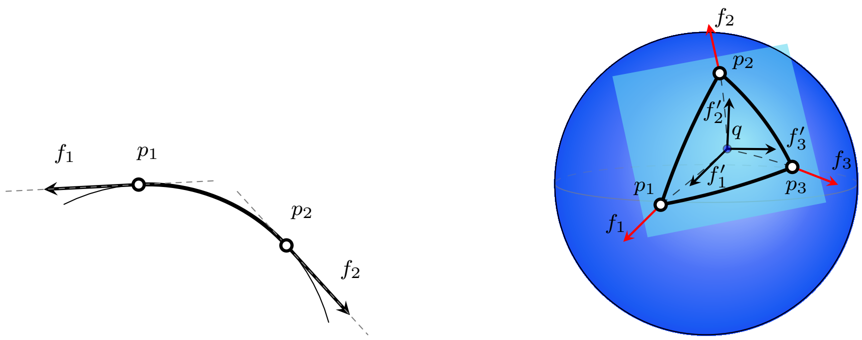
Proposition 5.1.1. *A force* $\mathbf{F} = (f_1, \ldots, f_v)$ *at a spherical configuration* $\mathbf{p} = (p_1, \ldots, p_v)$ *is in equilibrium if and only if*

$$\sum_{i=1}^{v} f_i \wedge p_i = 0. \tag{5.1.0.1}$$

Proof. Let $\mathbf{F} = (f_1, \ldots, f_v)$ and $\mathbf{p}' = (p'_1, \ldots, p'_v)$ be respectively, an equilibrium load and a trivial infinitesimal motion at $\mathbf{p}$. It is useful to regard a skew symmetric matrix $S \in M_{(n+1)\times(n+1)}$ as a extensor $(n+1) - 2$ form so that $Sp_i = S \wedge p_i$ is an extensor n form. So $\langle f_i, S \wedge p_i \rangle = -\langle f_i \wedge p_i, S \rangle$. Hence,

$$\begin{aligned}
\langle \mathbf{F}, \mathbf{p}' \rangle = \sum_{i=1}^{v} \langle f_i, p'_i \rangle = \sum_{i=1}^{v} \langle f_i, S \wedge p_i \rangle = & \\
&= \Big(\sum_{i=1}^{v} f_i \wedge S \wedge p_i \Big) \\
&= \Big(-(\sum_{i=1}^{v} f_i \wedge p_i) \wedge S \Big).
\end{aligned}$$

This implies $\sum_{i=1}^{v} f_i \wedge p_i = 0$ if only if $\langle \mathbf{F}, \mathbf{p}' \rangle = 0$. □



(a) Spherical equilibrium loads on two points p_1 and p_2.

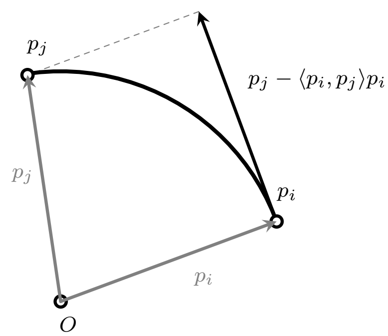
(b) Spherical equilibrium forces on three points p_1, p_2 and p_3.

Figure 5.1: Equilibrium forces on $\mathbb{S}^2$.

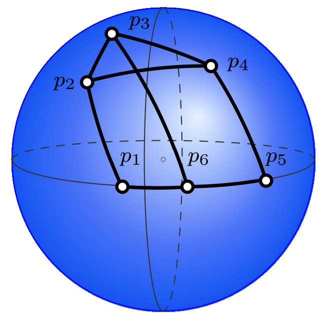
For example, an equilibrium load $\mathbf{F} = (f_1, f_2)$ of a single bar on $\mathbb{S}^2$ is a pair of forces f_1 and f_2 applied to the end points of the bar so that they both are tangent to the great circle containing the bar and moreover, their parallel transports to any point on the line add up to zero. Algebraically, this is expressed by the equality $f_1 \times p_1 = -f_2 \times p_2$ where $\times$ is the cross product of two vectors in $\mathbb{R}^3$. Similarly, a triple of forces (f_1, f_2, f_3) applied to three points p_1, p_2 and p_3 on $\mathbb{S}^2$ is in equilibrium if their lines of forces (great circles) are concurrent and the parallel transport of the forces to the point of concurrency add up to zero. To see this, note first that $f_1 \times p_1 + f_2 \times p_2 + f_3 \times p_3 = 0$ implies the lines of forces of f_1, f_2 and f_3 through p_1, p_2 and p_3 respectively, intersect at a point, say q. The parallel transport of each f_i, $i = 1, 2, 3$ along their lines of force to point q are forces f_1', f_2' and f_3' at q such that:

$$f_i \times p_i = f_i' \times q \qquad i = 1, 2, 3.$$

Since $\sum_{i=1}^{3} f_i \times p_i = 0$ we conclude $f_1' + f_2' + f_3' = 0$. In general, equation (5.1.0.1) assures that equilibrium forces do not generate a net momentum around any axis (through the origin.)



(a) Stress definition



(b) An independent framework without non-trivial stresses.

Figure 5.2: Stresses in $\mathbb{S}^2$.

Given a configuration $\mathbf{p} = (p_1, \ldots, p_v)$ for every $1 \le i < j \le v$, the force

$$\begin{aligned}\mathbf{F}_{ij} &= (0, \ldots, 0, f_i, 0, \ldots, 0, f_j, 0, \ldots, 0)\\ &= (0, \ldots, 0, p_j - \langle p_i, p_j \rangle p_i, 0, \ldots, 0, p_i - \langle p_i, p_j \rangle p_j, 0, \ldots, 0) \in \mathbb{R}^{(n+1)v},\end{aligned}$$

is an example of an equilibrium force at $\mathbf{p}$.

Let $G = (V, E)$ be a simple graph with $|V| = v$, $|E| = e$.

Definition 5.1.2. A *self-stress* of a framework $(G, \mathbf{p})$ is an assignment $\omega : E \to \mathbb{R}$ of scalars $\omega_{ij} = \omega_{ji}$ to each edge $ij \in E$ such that for every $i \in V$,

$$\sum_{ij \in E} \omega_{ij}(p_j - \langle p_i, p_j \rangle p_i) = 0. \qquad (5.1.0.2)$$

Note that $\langle (p_j - \langle p_i, p_j \rangle p_i), p_i \rangle = 0$ for every $i \in V$, as Figure 5.2a shows. Clearly, the set of stresses of a framework $(G, \mathbf{p})$ is a subspace of $\mathbb{R}^E$; we denote it by $\mathcal{S}(E)$.

Every stress of $(G, \mathbf{p})$ corresponds to a linear dependence of the rows of the spherical rigidity matrix $R_{\mathbb{S}}(G, \mathbf{p})$. More specifically, for any nontrivial stress $\omega = (\omega_{ij})_{ij \in E}$,

$$\sum_{ij \in E} \omega_{ij} p_j + \omega_i p_i = 0 \qquad \text{for every } i \in V,$$

where $\omega_i = -\sum_{ij\in E}\omega_{ij}\langle p_j, p_i\rangle$.

Given a framework $(G, \mathbf{p})$, the *stress transformation* of $(G, \mathbf{p})$ is defined to be the linear transformation $T : \mathbb{R}^E \to \mathbb{R}^{(n+1)v}$, $T(\omega) = (T_1(\omega), \ldots, T_v(\omega))$ where $T_i(\omega)$ is given by

$$T_i(\omega) = \sum_{ij\in E}\omega_{ij}(p_j - \langle p_i, p_j\rangle p_i).$$

Note that $\ker(T) = \mathcal{S}(E)$, by definition 5.1.2. Also, the image of the stress transformation T of $(G, \mathbf{p})$ is contained in $\mathcal{E}(\mathbf{p})$ because

$$\sum_{i\in V} T_i(\omega) \wedge p_i = \sum_{i\in V}\left(\sum_{ij\in E}\omega_{ij}(p_j - \langle p_i, p_j\rangle p_i)\right) \wedge p_i = \sum_{i\in V}\sum_{j\in V}\omega_{ij} p_j \wedge p_i = 0,$$

for any $\omega \in \mathbb{R}^E$.

An equilibrium force $\mathbf{F} = (f_1, \ldots, f_v)$ is called *resolvable* by the spherical framework $(G, \mathbf{p})$ if there exists $\omega \in \mathbb{R}^E$ such that $T(\omega) = \mathbf{F}$. Such a stress ω is called a *resolution* of $\mathbf{F}$ by framework $(G, \mathbf{p})$. If every equilibrium force of a framework $(G, \mathbf{p})$ is resolvable then the framework is called *statically rigid*. In this case, the rank of T achieves its maximum possible value $nv - \binom{n+1}{2}$.

The edge set E of a framework $(G, \mathbf{p})$ is called *independent* if and only if the stress transformation T of $(G, \mathbf{p})$ is injective, $\ker T = \mathcal{S}(E) = \{\mathbf{0}\}$; otherwise E is called *dependent*. In this case, the rank of T is less than the number of bars. Finding non-trivial dependency relations, or showing that none exists, is a useful method for deciding the independence of an edge set. The framework shown in Figure 5.2b with points p_1, p_5 and p_6 collinear is independent since it only admits the trivial self-stress. To see this, notice that the force $p_3 - \langle p_3, p_6\rangle p_6$ at point p_6 cannot be a non-trivial linear combination of forces $p_1 - \langle p_1, p_6\rangle p_6$ and $p_5 - \langle p_5, p_6\rangle p_6$ at p_6. Hence $\omega_{36} = \omega_{16} = \omega_{56} = 0$. Consequently, $\omega_{ij} = 0$ for all $ij \in E$. Now if we add the edge $\{1, 5\}$ to the framework then we will have a non-trivial stress on edges $\{1, 5\}$, $\{1, 6\}$

and $\{5,6\}$. In general, if $p_6 = \lambda_1 p_1 + \lambda_2 p_5$ then $\omega_{16} = \lambda_1, \omega_{15} = -\lambda_1\lambda_2, \omega_{56} = \lambda_2$ and, $\omega_{ij} = 0$ for the rest of edges, is a stress of the new framework.

The rigidity matrix $R_{\mathbb{S}}(G, \mathbf{p})$ is a matrix representation of the rigidity transformation $R : \prod_{i=1}^{v} T_{p_i}\mathbb{S}^n \to \mathbb{R}^E$, $R = (R_{ij})_{ij\in E}$ where

$$R_{ij}(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_v) = \langle p_i, u_j\rangle + \langle p_j, u_i\rangle,$$

for every $ij \in E$. It is crucial to note that operators R and T are transpose of each other, $R^* = T$. To see this, one can verify

$$\langle R(u), w\rangle = \langle u, T(w)\rangle \qquad \text{for all } u \in \prod_{i=1}^{v} T_{p_i}\mathbb{S}^n, w \in \mathbb{R}^E.$$

Now if u is an infinitesimal motion of $(G, \mathbf{p})$, $R(u) = 0$ then $\langle u, T(w)\rangle = 0$ for any $w \in \mathbb{R}^E$, meaning the space of infinitesimal motions of framework $(G, \mathbf{p})$ is orthogonal to the space of resolvable forces. We now express the two important results of this section:

Theorem 5.1.2. *For a spherical framework $(G, \mathbf{p})$, static rigidity is equivalent to infinitesimal rigidity.*

Proof. The fact that $R^* = T$ implies

$$\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(R)).$$

So $(G, \mathbf{p})$ is statically rigid if and only if dim im(T)= $nv - \binom{n+1}{2} = \dim \mathrm{im}(R)$. This is true if and only if $(G, \mathbf{p})$ is infinitesimally rigid. $\square$

We conclude this section with Maxwell's theorem for $\mathbb{S}^n$:

Theorem 5.1.3. *Let $(G, \mathbf{p})$ be a spherical framework in $\mathbb{S}^n$. Then*

$$dim(\mathcal{S}) \geq e - nv + \binom{n+1}{2}$$

The equality holds if and only if the framework is statically rigid or equivalently, it is infinitesimally rigid.

Proof. Since $\dim(\prod_1^v T_{p_i}\mathbb{S}^n) = nv$ and $\dim(\mathbb{R}^{|E|}) = e$, we have

$$\dim(\mathcal{S}) + \dim(\mathrm{im}(T)) = e,$$
$$\dim(\ker(R)) + \dim(\mathrm{im}(R)) = nv,$$

and so

$$\dim(\ker(R)) = n|V| - |E| + \dim \mathcal{S} \geq \binom{n+1}{2}.$$

Equality holds if and only if $\dim(\ker(R)) = \binom{n+1}{2}$ and this is true if and only if $(G, \mathbf{p})$ is infinitesimally rigid or equivalently, statically rigid. $\square$

This theorem explains the term 'isostatic' used by structural engineers: a framework $(G, \mathbf{p})$ is called *isostatic* if $e = nv - \binom{n+1}{2}$ and there is no non-trivial stress, $\dim(\mathcal{S}) = 0$. In other words, $\ker(T) = \{\mathbf{0}\}$ and $\mathrm{im}(T) = \mathcal{E}(\mathbf{p})$. At this point, adding more edges (or bars) creates stresses but no effect on the first-order rigidity of $(G, \mathbf{p})$.

5.2 Tensegrity in $\mathbb{S}^n$

Definition 5.2.1. A *signed graph* $G^{\pm} = (V, E)$ is a simple graph whose edge set E is partitioned into three subsets $E_-, E_\circ$ and E_+ called *cables, bars* and *struts*, respectively. A signed graph is denoted by $G^{\pm} = (V; E_-, E_\circ, E_+)$.

Definition 5.2.2. A *tensegrity framework* $(G^{\pm}, \mathbf{p})$ on $\mathbb{S}^n$ consists of a signed graph $G^{\pm} = (V; E_-, E_\circ, E_+)$ and an embedding $\mathbf{p} : V \to \mathbb{S}^n$ of vertices so that $p_i \neq -p_j$, for all $i, j \in V$.

An *infinitesimal motion* of $(G^{\pm}, \mathbf{p})$ is an assignment $\mathbf{p}' : V \to \prod_{i=1}^v T_{p_i}\mathbb{S}^n$, $\mathbf{p}'(i) = p'_i \in T_{p_i}\mathbb{S}^n$ of velocities to the points p_i, $i \in V$ such that

$$\langle p_i - p_j, p'_i - p'_j \rangle \leq 0 \qquad ij \in E_-, \tag{5.2.0.1}$$
$$\langle p_i - p_j, p'_i - p'_j \rangle = 0 \qquad ij \in E_\circ, \tag{5.2.0.2}$$
$$\langle p_i - p_j, p'_i - p'_j \rangle \geq 0 \qquad ij \in E_+. \tag{5.2.0.3}$$

These conditions are respectively driven from the following:

$$\begin{aligned}
\frac{d}{dt}\cos^{-1}\langle p_i, p_j\rangle &\leq 0,\\
\frac{d}{dt}\cos^{-1}\langle p_i, p_j\rangle &= 0,\\
\frac{d}{dt}\cos^{-1}\langle p_i, p_j\rangle &\geq 0,\\
\frac{d}{dt}\langle p_i, p_i\rangle &= 0,
\end{aligned}$$

where points p_i, $i \in V$ are smooth functions of time t.

Trivial infinitesimal motions of $(G^{\pm}, \mathbf{p})$ are the infinitesimal motions at $\mathbf{p}$ induced by the group of *orthogonal transformations* $SO(n+1)$ of the entire sphere $\mathbb{S}^n$. A tensegrity framework $(G^{\pm}, \mathbf{p})$ is called *infinitesimally rigid* if every infinitesimal motion of $(G^{\pm}, \mathbf{p})$ is trivial that is, $p_i' = Sp_i$ for all $i \in V$ and a skew symmetric $(n+1) \times (n+1)$ matrix S.

A *self-stress* $\omega = (\omega_{ij})_{ij \in E}$ of the tensegrity framework $(G^{\pm}, \mathbf{p})$ is a self-stress of $(G, \mathbf{p})$ such that $\omega_{ij} \geq 0$ for $ij \in E_-$ and $\omega_{ij} \leq 0$ for $ij \in E_+$. A self-stress ω of $(G^{\pm}, \mathbf{p})$ is called *strict* if $\omega_{ij} > 0$ for all $ij \in E_-$ and $\omega_{ij} < 0$ for all $ij \in E_+$.

A tensegrity framework $(G^{\pm}, \mathbf{p})$ is *statically rigid* if for every equilibrium force $\mathbf{F} = (f_i)_{i \in V}$ of $\mathbf{p}$ there are scalars $\omega_{ij} = \omega_{ji}, ij \in E$ with $\omega_{ij} \geq 0$ for $ij \in E_-$ and $\omega_{ij} \leq 0$ for $ij \in E_+$ such that

$$\sum_{ij \in E} \omega_{ij}(p_j - \langle p_j, p_j\rangle p_i) + f_i = 0, \tag{5.2.0.4}$$

for every $i \in V$. In this case, $\mathbf{F}$ is said to be *resolvable* by $(G^{\pm}, \mathbf{p})$ and ω is called a *stress* that resolves $\mathbf{F}$. As (5.2.0.4) indicates, a self-stress of $(G^{\pm}, \mathbf{p})$ demonstrates how the framework responds to an external force $\mathbf{F}$ by creating *tensions* (on cables E_-) and *compressions* (on struts E_+) along the edges in a particular pattern dictated by the sign pattern on the edge set E in order to neutralize the effect of $\mathbf{F}$. In addition,

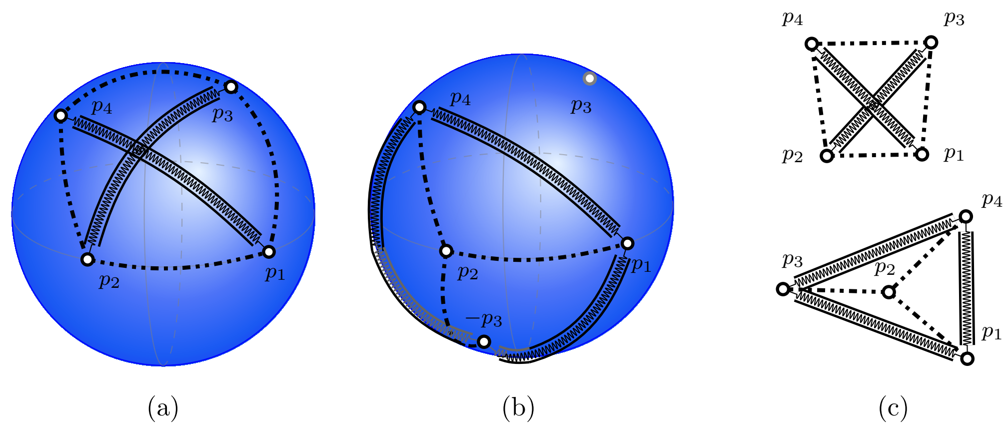


Figure 5.3: A tensegrity framework on $\mathbb{S}^2$.

Proposition 5.2.1. *Suppose $(G^{\pm}, \mathbf{p})$ is a statically rigid tensegrity framework in $\mathbb{S}^n$. The framework $(G'^{\pm}, \mathbf{q})$ with $q_i = -p_i$ for some i and $q_j = p_j$ for all $j \neq i$ whose signed graph $G'^{\pm}$ is obtained by reversing the sign on all edges in $G^{\pm}$ adjacent to the vertex i, is statically rigid if and only if $(G^{\pm}, \mathbf{p})$ is statically rigid.*

Proof. According to (5.2.0.4), changing p_i to $-p_i$ for some i while replacing f_i by $-f_i$ leads to a new set of consistent equations confirming static rigidity of $(G'^{\pm}, \mathbf{q})$ as it resolves all equilibrium forces of $\mathbf{q}$. $\square$

Throughout the chapter, we use dash-dotted lines to represent cable elements. We will use springs with double-lines to represent strut elements and, solid lines to represent bars, as usual.

Example 5.2.1. Suppose K_4 is the complete graph on 4 vertices, We define a signed graph $K_4^{\pm}$ of K_4 by $\{e_{12}, e_{24}, e_{43}, e_{31}\} = E_-$ and $\{e_{14}, e_{23}\} = E_+$. Let $(K_4^{\pm}, \mathbf{p})$ be a tensegrity framework with configuration $\mathbf{p} = (p_1, p_2, p_3, p_4)$ where

$$p_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), p_2 = (\frac{1}{3}, \frac{-2\sqrt{2}}{3}, 0), p_3 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), p_4 = (-\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}).$$

It is easy to check that $(K_4^{\pm}, \mathbf{p})$ admits a strict self-stress

$$\omega = (\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}),$$

where

$$\omega_{12} = \frac{9}{2(1+2\sqrt{2})}, \qquad \omega_{13} = \frac{3}{2}, \qquad \omega_{14} = -\sqrt{2},$$
$$\omega_{24} = \frac{9}{6-\sqrt{2}}, \qquad \omega_{23} = -\frac{27\sqrt{2}}{4(6-\sqrt{2})}, \qquad \omega_{34} = \frac{27\sqrt{2}+3}{32-11\sqrt{2}}.$$

This generates the 1-dimensional stress space of $(K_4^{\pm}, \mathbf{p})$. Figure 5.3a shows a schematic model of the tensegrity framework $(K_4^{\pm}, \mathbf{p})$ on sphere. By a rigid motion of $\mathbb{S}^2$ we can transfer $(K_4^{\pm}, \mathbf{p})$ to $\mathbb{S}^2_+$. The projection of the resulting framework to the plane (Figure 5.3c on top) is statically rigid and infinitesimally rigid. Therefore $(K_4^{\pm}, \mathbf{p})$ is statically and infinitesimally rigid in $\mathbb{S}^2$ (See [54]).

Figure 5.3b shows a tensegrity framework obtained from $(K_4, \mathbf{p})$ after replacing p_3 by its antipodal $-p_3$. This gives rise to a new statically rigid tensegrity framework if we reverse the signs on the edges adjacent to p_3 and the new self-stress is:

$$\omega' = (\omega_{12}, -\omega_{13}, \omega_{14}, -\omega_{23}, \omega_{24}, -\omega_{34}).$$

Figure 5.3c shows the projective image (under the central projection) of the spherical tensegrity frameworks in Figures 5.3a and 5.3b as rigid tensegrity frameworks in the plane. A practical model of the spherical frameworks shown in Figure 5.3 can be made by designing rigid pieces that connect the vertices on the sphere to a central point such that the vertices can freely rotate around that center with cables and struts among them. This process is called *coning*. We employ this idea to transfer statics from $\mathbb{E}^{n+1}$ to $\mathbb{S}^n$ in Lemma 5.2.3 later.

An immediate consequence of (5.2.0.4) is that interchanging the cables and struts of a tensegrity framework does not change the classification of its infinitesimal rigidity. As an example, all the frameworks shown in Figure 5.3, both in plane and on the sphere, lead to new rigid tensegrity framework with cables replaced by struts and vice versa.

We are now going to transfer some key results in the theory of Euclidean tensegrity frameworks to spherical spaces. The following result of Roth and Whiteley [50] connects the infinitesimal rigidity and the static rigidity of tensegrity frameworks.

Theorem 5.2.2. *[50, Theorem 4.3.] A tensegrity framework $(G^{\pm}, \mathbf{p})$ in $\mathbb{E}^n$ is infinitesimally rigid if and only if it is statically rigid.*

We want to conclude the analogous result in $\mathbb{S}^n$. Before that, the following lemma is needed.

Lemma 5.2.3. *Suppose $(\check{G}^{\pm}, \check{\mathbf{p}})$ is the tensegrity framework obtained from $(G^{\pm}, \mathbf{p})$ by adding a new vertex at the origin v_0 plus v bars connecting v_0 to each vertex $v_i \in V$. The following are true:*

(a) *$(G^{\pm}, \mathbf{p})$ is statically rigid on $\mathbb{S}^n$ if and only if $(\check{G}^{\pm}, \check{\mathbf{p}})$ is statically rigid in $\mathbb{E}^{n+1}$.*

(b) *$(G^{\pm}, \mathbf{p})$ is infinitesimally rigid on $\mathbb{S}^n$ if and only if $(\check{G}^{\pm}, \check{\mathbf{p}})$ is infinitesimally rigid in $\mathbb{E}^{n+1}$.*

Proof of a. Suppose $(G^{\pm}, \mathbf{p})$ is statically rigid. We will show any equilibrium force $\mathbf{F} = (f_0, f_1, \dots, f_v) \in \mathbb{R}^{(n+1)(v+1)}$ of $(\check{G}^{\pm}, \check{\mathbf{p}})$ in $\mathbb{E}^{n+1}$ will be resolved by a stress of $(\check{G}^{\pm}, \check{\mathbf{p}})$. Since $\mathbf{F}$ is an equilibrium force of $(\check{G}^{\pm}, \check{\mathbf{p}})$ then $\sum_{i=1}^{v} f_i \wedge p_i = 0$ and therefore,

$$\mathbf{F}' = (f_1 - \langle f_1, p_1 \rangle p_1, \dots, f_v - \langle f_v, p_v \rangle p_v),$$

is an equilibrium force of $(G^{\pm}, \mathbf{p})$ in $\mathbb{S}^n$. So there exists a stress $\omega = (\omega_{ij})_{ij \in E}$ of $(G^{\pm}, \mathbf{p})$ that resolves this force, meaning

$$\sum_{ij \in E} \omega_{ij}(p_j - \langle p_i, p_j \rangle p_i) + f_i - \langle f_i, p_i \rangle p_i = 0,$$

for every $0 \neq i \in V$. A simple manipulation yields

$$\sum_{ij \in E} \omega_{ij}(p_j - p_i) + \left(\sum_{ij \in E} (\omega_{ij} - \langle p_i, p_j \rangle) - \langle f_i, p_i \rangle \right) p_i + f_i = 0, \tag{5.2.0.5}$$

for every $0 \neq i \in V$. At vertex v_0, we have

$$\sum_i \left(\sum_{ij \in E} (\omega_{ij} - \langle p_i, p_j \rangle) - \langle f_i, p_i \rangle \right) p_i - f_0 = 0. \tag{5.2.0.6}$$

This follows from (5.2.0.5) considering $f_0 = -\sum_{i=1}^{v} f_i$. Now (5.2.0.5) and (5.2.0.6) imply $\mathbf{F}$ is resolved by framework $(\check{G}^{\pm}, \check{\mathbf{p}})$ for some stresses. So $(\check{G}^{\pm}, \check{\mathbf{p}})$ is statically rigid.

Conversely, suppose $(\check{G}^{\pm}, \check{\mathbf{p}})$ is statically rigid. If $(f_1, \dots, f_v)$ in $\mathbb{R}^{(n+1)v}$ is an equilibrium force of $(G^{\pm}, \mathbf{p})$ then $\mathbf{F} = (-\sum_i f_i, f_1, \dots, f_v)$ in $\mathbb{R}^{(n+1)(v+1)}$ is an equilibrium force of $(\check{G}^{\pm}, \check{\mathbf{p}})$. So there exists a stress $\omega \in \mathbb{R}^{e+v}$ of $(\check{G}^{\pm}, \check{\mathbf{p}})$ that resolves $\mathbf{F}$. In particular,

$$\sum_j \omega_{ij}(p_j - p_i) + f_i = 0 \qquad 1 \leq i \leq v.$$

Taking the dot product by p_i, we have $\sum_j (\omega_{ij} \langle p_j, p_i \rangle - \omega_{ij}) = 0$, for all $1 \leq i \leq v$ as $\langle p_i, f_i \rangle = 0$, $\langle p_i, p_i \rangle = 1$ for all $1 \leq i \leq v$. This implies

$$\sum_j \omega_{ij}(p_j - \langle p_j, p_i \rangle p_i) + f_i = 0.$$

So $\omega_{ij} \in \mathbb{R}^e$ is a resolving stress. Thus $(G^{\pm}, \mathbf{p})$ is statically rigid.

Proof of b. The statement follows from the fact that if $(p'_1, \dots, p'_v)$ is an infinitesimal motion of $(G^{\pm}, \mathbf{p})$ then $(0, p'_1, \dots, p'_v)$ is an infinitesimal motion of $(\check{G}^{\pm}, \check{\mathbf{p}})$. Conversely, if $(p'_0, p'_1, \dots, p'_v)$ is a motion of $(\check{G}^{\pm}, \check{\mathbf{p}})$ then $(p'_1 - p'_0, \dots, p'_v - p'_0)$ is a motion of $(G^{\pm}, \mathbf{p})$. This establishes a isomorphism between the space of motions of the two framework and so, the equivalence of their infinitesimal rigidity. □

Corollary 5.2.4. *A tensegrity framework $(G^{\pm}, \mathbf{p})$ in $\mathbb{S}^n$ is infinitesimally rigid if and only if it is statically rigid.*

Proof. It follows from Theorem 5.2.2 and Lemma 5.2.3. □

The proof of Lemma 5.2.3 shows us that stresses on non-radial edges in the tensegrity framework $(\check{G}^{\pm}, \check{\mathbf{p}})$ in $\mathbb{E}^d$ are the same as those on their corresponding edges of the spherical tensegrity framework $(G^{\pm}, \mathbf{p})$. In particular, strict stresses of $(\check{G}^{\pm}, \check{\mathbf{p}})$ are transferred to strict stresses of $(G^{\pm}, \mathbf{p})$ and conversely. Also, as indicated in the proof of Lemma 5.2.3, stresses on radial edges in $(\check{G}^{\pm}, \check{\mathbf{p}})$ are dependent on the stresses on non-radial edges. Therefore the dimension of the stress space of the underlying bar-joint framework $(\check{G}, \check{\mathbf{p}})$ in $\mathbb{E}^{n+1}$ is equal to the dimension of the stress space of the bar-joint framework $(G, \mathbf{p})$ in $\mathbb{S}^n$. As a result, the framework $(\check{G}, \check{\mathbf{p}})$ is statically rigid if and only if $(G, \mathbf{p})$ is statically rigid.

Thus, the following important theorem is a corollary of Theorem 2.2.5:

Theorem 5.2.5. *For a tensegrity framework $(G^{\pm}, \mathbf{p})$ in $\mathbb{S}^n$ the following are equivalent:*

(a) $(G^{\pm}, \mathbf{p})$ *is statically rigid.*

(b) $(G^{\pm}, \mathbf{p})$ *has a strict self-stress and the bar-joint framework $(G, \mathbf{p})$ is statically rigid.*

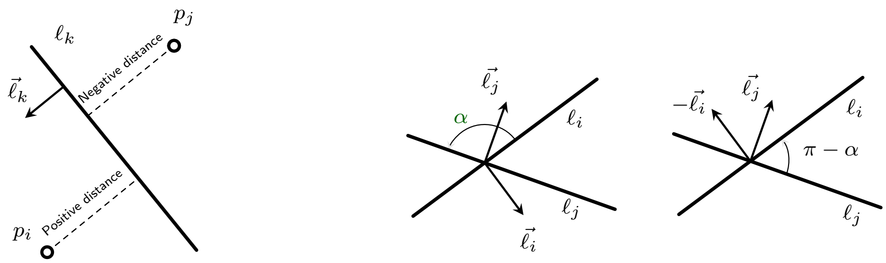
Proof. (a) is equivalent to the static rigidity of the coning framework $(\check{G}, \check{\mathbf{p}})$ in $\mathbb{E}^{n+1}$, by Lemma 5.2.3. This, in turn, is equivalent to the existence of a strict self-stress of $(\check{G}^{\pm}, \check{\mathbf{p}})$ and infinitesimal rigidity of $(\check{G}, \check{\mathbf{p}})$, by Theorem 2.2.5. Thus the equivalence of (a) and (b) follows from the proof of Lemma 5.2.3. $\square$

5.3 Point-hyperplane tensegrity frameworks in $\mathbb{E}^n$

Let $\mathbf{p} = (p_i)_{i=1}^{v_p} \in \mathbb{R}^{nv_p}$ be a configuration of points $p_i = (x_{i,1}, \ldots, x_{i,n}) \in \mathbb{R}^n$ and we have a collection of hyperplanes with equations $x_1 + a_{i,1}x_2 + \cdots + a_{i,n-1}x_n + a_{i,n} = 0$ for each hyperplane ℓ_i. In general, the sign of the point-hyperplane distance formula

$$\frac{x_{i,1} + a_{j,1}x_{i,2} + \cdots + x_{i,n}a_{j,n-1} + a_{j,n}}{\sqrt{1 + a_{j,1}^2 + \cdots + a_{j,n-1}^2}},$$

between point p_i and hyperplane ℓ_j depends on the *orientation* $\vec{\ell_j}$ of ℓ_j and the side on which the point p_i is located. If $\vec{\ell_i}$ is towards the point p_i the sign of the distance



(a) The sign of the point-line distance depends on the orientation the line and the relative position of the point with respect to the line.

(b) Changing the orientation of ℓ_i results in measurement of the complement angle.

Figure 5.4: Orientation of lines and distance-angle measurements.

is positive otherwise it is negative (Figure 5.4a). The angle between two lines ℓ_i and ℓ_j are measured by the angle between their normal vectors as $\alpha = \cos^{-1}\langle \vec{\ell_i}, \vec{\ell_j} \rangle$. If we change the orientation of one line then the complement angle $\pi - \alpha$ will be measured (Figure 5.4b). Increasing the angle α is equivalent to decreasing its complement $\pi - \alpha$. Throughout this chapter, we assume the hyperplanes are oriented.

Recall that $(G, \mathbf{p}, \ell^\circ)$ is a point-hyperplane framework obtained from $(G, \mathbf{p}, \ell)$ by translating all the hyperplanes ℓ_i, $i \in V_L$ to the origin while the points remain in their positions. Therefore the orientation of hyperplanes in $(G, \mathbf{p}, \ell^\circ)$ and $(G, \mathbf{p}, \ell)$ are the same, $\vec{\ell_i^\circ} = \vec{\ell_i}$ but $a_{i,n} = 0$ for all hyperplanes in $(G, \mathbf{p}, \ell^\circ)$. In fact, $\ell_i^\circ = (\vec{\ell_i^\circ}, 0) = (\vec{\ell_i}, 0)$. As we saw before, the frameworks $(G, \mathbf{p}, \ell)$ and $(G, \mathbf{p}, \ell^\circ)$ are equivalent in terms of infinitesimal rigidity. In addition, their infinitesimal rigidity can be examined using the matrix (3.1.0.9) as well as the rigidity matrix $(G, \mathbf{p}, \ell)$:

$$R(G,\mathbf{p},\ell) = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccccc} & p_h & & p_i & & \ell_j & & \ell_k & \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & p_h - p_i & \cdots & p_i - p_h & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \ell_j & \cdots & \mathbf{0} & \cdots & p_h & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \ell_j^\circ - \ell_k^\circ & \cdots & \ell_k^\circ - \ell_j^\circ & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & e_{n+1} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \ell_j^\circ & \cdots & \mathbf{0} & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{pmatrix} \end{array},$$

where $e_{n+1} = (0, \cdots, 1)$ is a unit vector in $\mathbb{R}^{n+1}$ where $p_i = (x_{i,1}, \ldots, x_{i,n}, 1)$ for points and $\ell_i = (\vec{\ell_i}, a_{i,n})$ with $\|\vec{\ell_i}\|^2 = 1$ for hyperplanes.

Similar to spherical frameworks, we can see (proof of Proposition 5.3.1 below) that any linear dependence relation of the rows of the rigidity matrix $R(G,\mathbf{p},\ell)$ can be uniquely determined by the coefficients of the *principal* rows r_{ij}, $ij \in E$ (corresponding to the edges of the graph) in that linear relation.

So we define:

Definition 5.3.1. A *self-stress* of a point-hyperplane framework $(G,\mathbf{p},\ell)$ is an assignment of scalars $\omega_{ij} = \omega_{ji}$ to every edge $ij \in E$ such that for every vertex $i \in V$, if $i \in V_p$,

$$\sum_{j \in V_P, ij \in E} \omega_{ij}(p_i - p_j) + \sum_{j \in V_L, ij \in E} \omega_{ij}\ell_j + \omega_i e_{n+1} = 0,$$

for some scalar ω_i and, if $i \in V_L$,

$$\sum_{j \in V_P} \omega_{ij} p_j + \sum_{j \in V_L} \omega_{ij}(\ell_i^\circ - \ell_j^\circ) + \omega_i \ell_i^\circ = 0,$$

for some scalar ω_i.

Self-stresses are the coefficients of the principal row vectors in a linear dependence relations of the rows of the matrix $(G, \mathbf{p}, \ell)$. They form a linear space that we denote by $\mathcal{S}(G, \mathbf{p}, \ell)$. It is vital to note that any self-stress of $(G, \mathbf{p}, \ell)$ is a self-stress of $(G, \mathbf{p}, \ell^{\circ})$ and vice versa. Therefore $\mathcal{S}(G, \mathbf{p}, \ell) = \mathcal{S}(G, \mathbf{p}, \ell^{\circ})$. To analyze the stresses of a framework $(G, \mathbf{p}, \ell)$, we work with the framework $(G, \mathbf{p}, \ell^{\circ})$ instead as its rigidity matrix has a closer appearance a spherical rigidity matrix.

Next, we see that self-stresses of point-hyperplane frameworks can be obtained from the corresponding spherical frameworks under the central projection as described in Chapter 3.

5.3.1 Transfer of stresses from spherical frameworks to point-hyperplane frameworks

Suppose $(G, \mathbf{p})$, is a framework in $\mathbb{S}^n_+ \cup V_{\text{eq}}$ and V_{eq} contains a non-empty set of joints in V that are located on the equator of $\mathbb{S}^n$. We use $i \in V_{\text{eq}}$ for $p_i \in V_{\text{eq}}$ and $i \notin V_{\text{eq}}$ for $p_i \notin V_{\text{eq}}$.

Let $p_i = (x_{i,1}, \ldots, x_{i,n+1})$, $x_{i,n+1} > 0$, for $i \in V \backslash V_{\text{eq}}$ and $p_i = (x_{i,1}, \ldots, x_{i,n}, 0)$, $i \in V_{\text{eq}}$ be the points on the equator $\mathbb{S}^n$. Projecting $(G, \mathbf{p})$ to the affine hyperplane $x_{n+1} = 1$, we have an oriented point-hyperplane framework $(G, \tilde{\mathbf{p}}, \ell^{\circ})$ with all lines passing through the origin so that $\tilde{p}_i = (x_{i,1}/x_{i,n+1}, \ldots, x_{i,n}/x_{i,n+1}, 1)$ are the points in the hyperplane $x_{n+1} = 1$ if $i \in V \backslash V_{\text{eq}}$ and the oriented hyperplane ℓ_i° has the equation $x_{i,1}x_1 + \cdots + x_{i,n}x_n = 0$ for $i \in V_{\text{eq}}$ corresponding to the point $p_i = (x_{i,1}, \ldots, x_{i,n}, 0)$ on the equator.

We have the following proposition:

Proposition 5.3.1. *Let $(G, \mathbf{p})$ be a framework on $\mathbb{S}^n$ with some vertices on the equator and $(G, \tilde{\mathbf{p}}, \ell^{\circ})$ is its projected point-hyperplane framework in $\mathbb{E}^n$. If $\omega =$*

$(\omega_{ij})_{ij\in E}$ *is a self-stress of* $(G, \mathbf{p})$ *in* $\mathbb{S}^n$ *then* $\tilde{\omega} = (\tilde{\omega}_{ij})_{ij\in E}$ *is a self-stress of* $(G, \tilde{\mathbf{p}}, \ell_\circ)$ *in* $\mathbb{E}^n$ *where*

$$\tilde{\omega}_{ij} = \begin{cases} x_{i,n+1}x_{j,n+1}\omega_{ij} & ij \in E_{PP}, \\ -x_{i,n+1}\omega_{ij} & ij \in E_{PL}, \\ \omega_{ij} & ij \in E_{LL}. \end{cases} \tag{5.3.1.1}$$

Conversely, if $\tilde{\omega}_{ij}$ *is a self-stress of* $(G, \tilde{\mathbf{p}}, \ell_\circ)$ *then*

$$\omega_{ij} = \begin{cases} \|\tilde{p}_i\|\|\tilde{p}_j\|\tilde{\omega}_{ij} & ij \in E_{PP}, \\ -\|\tilde{p}_i\|\tilde{\omega}_{ij} & ij \in E_{PL}, \\ \tilde{\omega}_{ij} & ij \in E_{LL}. \end{cases}$$

is a self-stress of the spherical framework $(G, \mathbf{p})$.

Proof. Assume $\omega = (\omega_{ij})_{ij\in E}$ is a self-stress of $(G, \mathbf{p})$ on $\mathbb{S}^n$. For every $i \in V\backslash V_{\text{eq}}$,

$$\sum_{j\notin V_{\text{eq}}} \omega_{ij}p_j + \sum_{k\in V_{\text{eq}}} \omega_{ik}p_k + \omega_i p_i = 0,$$

where $\omega_i = -\sum_{j\in V} \omega_{ij}\langle p_j, p_i\rangle$ with the convention $\omega_{ij} = 0$ if $ij \notin E$.
This yields

$$\sum_{j\notin V_{\text{eq}}} x_{i,n+1}x_{j,n+1}\omega_{ij}\tilde{p}_j + \sum_{k\in V_{\text{eq}}} x_{i,n+1}\omega_{ik}p_k + \omega_i x_{i,n+1}^2\tilde{p}_i = 0, \tag{5.3.1.2}$$

or

$$\begin{aligned} \sum_{j\notin V_{\text{eq}}} x_{i,n+1}x_{j,n+1}\omega_{ij}(\tilde{p}_j - \tilde{p}_i) + \sum_{k\in V_{\text{eq}}} x_{i,n+1}\omega_{ik}p_k \\ + (\omega_i x_{i,n+1}^2 + \sum_{j\notin V_{\text{eq}}} x_{i,n+1}x_{j,n+1}\omega_{ij})e_{n+1} = 0. \end{aligned} \tag{5.3.1.3}$$

For every $i \in V_{\text{eq}}$ on $\mathbb{S}^n$, we have

$$\sum_{j\notin V_{\text{eq}}} \omega_{ij}p_j + \sum_{j\in V_{\text{eq}}} \omega_{ij}p_j + \omega_i p_i = 0,$$

or

$$\sum_{j\notin V_{\text{eq}}} x_{j,n+1}\omega_{ij}\tilde{p}_j + \sum_{j\in V_{\text{eq}}} \omega_{ij}p_j + \omega_i p_i = 0, \tag{5.3.1.4}$$

which can be written as

$$\sum_{j\notin V_{\text{eq}}} -x_{j,n+1}\omega_{ij}\tilde{p}_j + \sum_{j\in V_{\text{eq}}} \omega_{ij}(p_i - p_j) + (\omega_i - \sum_{ij}\omega_{ij})p_i = 0. \tag{5.3.1.5}$$

Now (5.3.1.3) and (5.3.1.5) prove one direction. Because the stress spaces of $(G, \mathbf{p})$ and $(G, \tilde{\mathbf{p}}, \ell_\circ)$ are isomorphic (of the same finite dimension) and 5.3.1.1 is a linear injection from the stress space of $(G, \mathbf{p})$ in $\mathbb{S}^2$ to the stress space of $(G, \tilde{\mathbf{p}}, \ell_\circ)$ in $\mathbb{E}^n$, then it must be a bijection. So (5.3.1.3) and (5.3.1.5) also trace the way back from point-hyperplane stresses to obtain spherical stresses. □

It is worth mentioning that in order to obtain the corresponding linear relations in the rows of (3.2.0.2), we can continue further from equation (5.3.1.2) to substitute for $\omega_i x_{i,n+1}^2$ using

$$\sum_{j\notin V_{\text{eq}}} \omega_{ij}x_{i,n+1}x_{j,n+1} + \omega_i x_{i,n+1}^2 = 0,$$

to get

$$\sum_{j\notin V_{\text{eq}}} x_{i,n+1}x_{j,n+1}\omega_{ij}(\hat{p}_j - \hat{p}_i) + \sum_{k\in V_{\text{eq}}} x_{i,n+1}x_{k,1}\omega_{ik}p_k = 0, \tag{5.3.1.6}$$

where $\hat{p}_i = (x_{i,1}/x_{i,n+1}, \ldots, x_{i,n}/x_{i,n+1})$ for $i \notin V_{\text{eq}}$ and $\hat{p}_i = (1, x_{i,2}/x_{i,1}, \ldots, x_{i,n}/x_{i,1})$ for $i \in V_{\text{eq}}$. For $i \in V_L$, let $\tilde{p}_i = (1, x_{i,2}/x_{i,1}, \ldots, x_{i,n}/x_{i,1}, 0)$ for all $i \in V_{eq}$. Then we solve the equation (5.3.1.4) for ω_i:

$$x_{i,1}\omega_i = -\sum_{j\notin V_{eq}} x_{j,n+1}\omega_{ij}\langle \tilde{p}_j, \frac{\tilde{p}_i}{\|\tilde{p}_i\|^2}\rangle - \sum_{j\in V_{eq}} x_{j,1}\omega_{ij}\langle \tilde{p}_j, \frac{\tilde{p}_i}{\|\tilde{p}_i\|^2}\rangle.$$

and we next substitute this back into (5.3.1.5) and rule out the first component of the vector equation (5.3.1.5):

$$\sum_{j\notin V_{eq}} x_{j,n+1}\omega_{ij}(\tilde{p}_j - \frac{\tilde{p}_j.\tilde{p}_i}{\|\tilde{p}_i\|^2}\tilde{p}_i) + \sum_{j\in V_{eq}} x_{j,1}\omega_{ij}(\tilde{p}_j - \frac{\tilde{p}_j.\tilde{p}_i}{\|\tilde{p}_i\|^2}\tilde{p}_i) = 0. \tag{5.3.1.7}$$

Equations (5.3.1.6) and (5.3.1.7) imply that the corresponding linear dependence relation the matrix (3.2.0.2) is

$$\hat{\omega}_{ij} = \begin{cases} x_{i,n+1}x_{j,n+1}\omega_{ij} & ij \in E_{PP}, \\ -x_{i,n+1}x_{j,1}\omega_{ij} & i \in V_P, j \in V_L, ij \in E_{PL}, \\ x_{i,1}x_{j,1}\omega_{ij} & ij \in E_{LL}. \end{cases} \tag{5.3.1.8}$$

Example 5.3.1. The spherical bar-joint framework in Example (5.2.1) is projected to the oriented point-line framework $(G, \mathbf{p}, \ell)$ in the plane with configuration

$$p_1 = (-\frac{1}{2}, 1), \quad p_2 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}),$$

of points and

$$\ell_1 : x - 2\sqrt{2}y = 0, \quad \ell_2 : x + y = 0.$$

of lines oriented by the normal vectors $\vec{\ell_1} = (1, -2\sqrt{2})/3$ and $\vec{\ell_2} = (1, 1)/\sqrt{2}$. By Proposition 5.3.1, the 1-dimensional stress space $\mathcal{E}$ is generated by

$$(\frac{18 + \sqrt{2}}{32 - 11\sqrt{2}}, \frac{9\sqrt{2}}{2(6 - \sqrt{2})}, 1, \frac{3\sqrt{2} + 1}{\sqrt{2} + 4}, -\frac{9\sqrt{2}}{2(6 - \sqrt{2})}, -1),$$

for the edges $\{p_1, p_2\}, \{p_1, \ell_1\}, \{p_2, \ell_2\}, \{\ell_1, \ell_2\}, \{p_2, \ell_1\}, \{p_1, \ell_2\}$, respectively. Note that the sum of stresses on the edges incident to each line is zero. This can be seen from the matrix $R(G, \mathbf{p}, \ell)$ in (5.3.2.1).

5.3.2 Point-hyperplane tensegrity frameworks in $\mathbb{E}^n$: infinitesimal rigidity

Suppose $(G, \mathbf{p}, \ell)$ be a point-hyperplane framework in $\mathbb{E}^n$ so that hyperplanes have the coordinates

$$\ell_i = (\vec{\ell_i}, d_i) \in \mathbb{R}^n \times \mathbb{R},$$

where the unit- length vector $\vec{\ell_i}$ determines the orientation of the hyperplane ℓ_i in $\mathbb{E}^n$. Note that the point $b_i = -d_i\vec{\ell_i} \in \mathbb{E}^n$ is on ℓ_i for every $i \in V_L$.

If a point-hyperplane configuration $(\mathbf{p}, \ell)$ in $\mathbb{E}^n$ undergoes a rigid motion $(R, s), R \in SO(n), s \in \mathbb{R}^n$, we obtain a new point-hyperplane configuration $(\mathbf{q}, \jmath)$ where

$$q_i = Rp_i + s, \qquad \jmath_i = (R\vec{\ell_i}, -\langle -d_i R\vec{\ell_i} + s, R\vec{\ell_i}\rangle), \qquad i \in V. \tag{5.3.2.1}$$

Here the points p_i are understood in their Euclidean coordinates. Now, for a one-parameter family of rotations $R(t) \in SO(n)$ and translations $s(t) = (s_1(t), \ldots, s_n(t)) \in \mathbb{R}^n$ where $R(0) = I_{n\times n}$ is the $n \times n$ identity matrix and $s(0) = \mathbf{0} \in \mathbb{R}^n$, $(\mathbf{q}, \jmath)$ changes smoothly as a function of t under a rigid motion. Taking the derivative of (5.3.2.1) with respect to t at $t = 0$, we obtain the infinitesimal rigid motions

$$p_i' = Sp_i + s, \tag{5.3.2.2}$$

for points p_i , $i \in V_P$ where $\frac{d}{dt}R(t)|_{t=0} = S$ is a $n \times n$ skew-symmetric matrix and $\frac{d}{dt}s(t)|_{t=0} = s$. Similarly, for a hyperplane ℓ_i, the derivative of (5.3.2.1) at $t = 0$ is

$$\begin{aligned}\frac{d}{dt}(R\vec{\ell_i}, -\langle -d_i R\vec{\ell_i} + s, R\vec{\ell_i}\rangle) &= \frac{d}{dt}(R\vec{\ell_i}, d_i - \langle s, R\vec{\ell_i}\rangle) \\ &= (S\vec{\ell_i}, -\langle s, I\vec{\ell_i}\rangle - \langle \mathbf{0}, S\vec{\ell_i}\rangle) \\ &= (S\vec{\ell_i}, -\langle s, \vec{\ell_i}\rangle).\end{aligned} \tag{5.3.2.3}$$

If we view points p_i's in the affine hyperplane $x_{n+1} = 1$, i.e., $p_i = (x_{i,1}, \ldots, x_{i,n}, 1)$ we then, based on the equations (5.3.2.2) and (5.3.2.3), have the following uniform expression of the infinitesimal rigid motions of a point-hyperplane framework in $\mathbb{E}^n$ in the matrix format:

$$p_i' = \left(\begin{array}{ccc:c} & & & s_1 \\ & S & & \vdots \\ & & & s_n \\ \hdashline 0 & \ldots & 0 & 0 \end{array}\right) p_i, \tag{5.3.2.4}$$

at the point p_i and

$$\ell_i' = \left(\begin{array}{ccc:c} & & & 0 \\ & S & & \vdots \\ & & & 0 \\ \hdashline -s_1 & \dots & -s_n & 0 \end{array}\right) \ell_i. \tag{5.3.2.5}$$

for the hyperplane ℓ_i. It looks like that these two matrices are two pieces of one skew-symmetric $(n+1)\times(n+1)$ matrix, which is a typical infinitesimal rigid motion of $\mathbb{S}^n$. This suggests how the infinitesimal motion of points and hyperplanes can be uniformly described if the hyperplanes are through the origin and the last coordinate of p_i' is changed to $-\langle s, p_i\rangle$ instead of zero. We will see below that this will work perfectly to allow us to transfer between $\mathbb{E}^n$ and $\mathbb{S}^n$.

Definition 5.3.2. A *signed point-hyperplane graph* is a point-hyperplane graph $G = (V_P \cup V_L; E_{PP}, E_{LL}, E_{PL})$ whose edge set $E = E_{PP} \cup E_{LL} \cup E_{PL}$ is partitioned into three edge subsets $E_-, E_\circ$ and E_+ called *signed edges.* We denote the point-hyperplane signed graph by $G^\pm = (V_P \cup V_L; E_-, E_\circ, E_+)$.

A *point-hyperplane tensegrity framework* $(G^\pm, \mathbf{p}, \ell)$ in $\mathbb{E}^n$ consists of a signed point-hyperplane graph $G^\pm = (V_P \cup V_L; E_-, E_\circ, E_+)$ and a point-hyperplane configuration $(\mathbf{p}, \ell)$ in $\mathbb{E}^n$. An *infinitesimal motion* of a framework $(G^\pm, \mathbf{p}, \ell)$ is an assignment of point-hyperplane velocities $(\mathbf{p}', \ell') \in \mathbb{R}^{(n+1)v}$ such that

$$\langle p_i - p_j, p_i' - p_j'\rangle = 0 \qquad ij \in E_\circ \cap E_{PP}, \tag{5.3.2.6}$$

$$\langle p_i - p_j, p_i' - p_j'\rangle \leq 0 \qquad ij \in E_- \cap E_{PP}, \tag{5.3.2.7}$$

$$\langle p_i - p_j, p_i' - p_j'\rangle \geq 0 \qquad ij \in E_+ \cap E_{PP}, \tag{5.3.2.8}$$

$$\langle p_i', \ell_j\rangle + \langle p_i, \ell_j'\rangle = 0 \qquad ij \in E_\circ \cap E_{PL}, \tag{5.3.2.9}$$

$$\langle p_i', \ell_j\rangle + \langle p_i, \ell_j'\rangle \leq 0 \qquad ij \in E_- \cap E_{PL}, \tag{5.3.2.10}$$

$$\langle p_i', \ell_j\rangle + \langle p_i, \ell_j'\rangle \geq 0 \qquad ij \in E_+ \cap E_{PL}, \tag{5.3.2.11}$$

$$\langle \vec{\ell}_i - \vec{\ell}_j, \vec{\ell}'_i - \vec{\ell}'_j \rangle = 0 \qquad ij \in E_\circ \cap E_{LL}, \tag{5.3.2.12}$$

$$\langle \vec{\ell}_i - \vec{\ell}_j, \vec{\ell}'_i - \vec{\ell}'_j \rangle \leq 0 \qquad ij \in E_- \cap E_{LL}, \tag{5.3.2.13}$$

$$\langle \vec{\ell}_i - \vec{\ell}_j, \vec{\ell}'_i - \vec{\ell}'_j \rangle \geq 0 \qquad ij \in E_+ \cap E_{LL}, \tag{5.3.2.14}$$

where

$$\langle p'_i, e_{n+1} \rangle = 0 \qquad i \in V_P, \tag{5.3.2.15}$$

with $e_{n+1} = (0, \ldots, 0, 1)$ and $\ell'_i \in \mathbb{R}^{n+1}$, $\vec{\ell}'_i = \pi \circ \ell'_i$ with $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates. Conditions (5.3.2.12)–(5.3.2.14) use the fact that

$$\langle \vec{\ell}_i, \vec{\ell}'_i \rangle = 0, \qquad i \in V_L, \tag{5.3.2.16}$$

These conditions are the first-order constraints corresponding to the following time-dependent constraints:

$$\|p_i(t) - p_j(t)\| = \|p_i - p_j\| \qquad ij \in E_\circ \cap E_{PP}, \tag{5.3.2.17}$$

$$\|p_i(t) - p_j(t)\| \leq \|p_i - p_j\| \qquad ij \in E_- \cap E_{PP}, \tag{5.3.2.18}$$

$$\|p_i(t) - p_j(t)\| \geq \|p_i - p_j\| \qquad ij \in E_+ \cap E_{PP}, \tag{5.3.2.19}$$

$$\langle p_i(t), \ell_j(t) \rangle = \langle p_i, \ell_j \rangle \qquad ij \in E_\circ \cap E_{PL}, \tag{5.3.2.20}$$

$$\langle p_i(t), \ell_j(t) \rangle \leq \langle p_i, \ell_j \rangle \qquad ij \in E_- \cap E_{PL}, \tag{5.3.2.21}$$

$$\langle p_i(t), \ell_j(t) \rangle \geq \langle p_i, \ell_j \rangle \qquad ij \in E_+ \cap E_{PL}, \tag{5.3.2.22}$$

$$\cos^{-1}\langle \vec{\ell}_i(t), \vec{\ell}_j(t) \rangle = \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j \rangle \qquad ij \in E_\circ \cap E_{LL}, \tag{5.3.2.23}$$

$$\cos^{-1}\langle \vec{\ell}_i(t), \vec{\ell}_j(t) \rangle \leq \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j \rangle \qquad ij \in E_- \cap E_{LL}, \tag{5.3.2.24}$$

$$\cos^{-1}\langle \vec{\ell}_i(t), \vec{\ell}_i(t) \rangle \geq \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j \rangle \qquad ij \in E_+ \cap E_{LL}, \tag{5.3.2.25}$$

with the following additional constraints:

$$\langle p_i(t), e_{n+1} \rangle = 1 \qquad i \in V_P, \tag{5.3.2.26}$$

$$\langle \vec{\ell}_i(t), \vec{\ell}_i(t) \rangle = 1 \qquad i \in V_L, \tag{5.3.2.27}$$

where points p_i's and hyperplanes ℓ_i's move smoothly in the affine hyperplane $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$ over a period of time $t \in [0, 1]$ with $p_i(0) = p_i, \ell_i(0) = \ell_i$.

As the above constraints imply, each element of $E_\circ \cap E_{PP}$, $E_\circ \cap E_{PL}$ and $E_\circ \cap E_{LL}$ preserves the distance between a pair of points, a pair of point and hyperplane and the angle between a pair of hyperplanes, respectively. Each element of $E_- \cap E_{PP}$ and $E_- \cap E_{LL}$ (respectively, $E_+ \cap E_{PP}$ and $E_+ \cap E_{LL}$) places an upper (respectively, lower) bound on the distances between pairs of points or angles between a pair of hyperplanes. However, the elements of $ij \in E_+ \cap E_{PL}$ and $ij \in E_- \cap E_{PL}$ do not necessarily preserve the distance between pairs of points and hyperplanes but, they restrict a point p_i to move in the half-space with border determined by $\vec{\ell_i}$ and $-\vec{\ell_i}$ in $\mathbb{E}^n$, respectively. To see this, one could view $\langle p_i(t), \ell_j(t) \rangle$ as the signed point-hyperplanes distance so that when $\langle p_i(t), \ell_j(t) \rangle$ is a decreasing function of t, point p_i will relatively move towards the hyperplane ℓ_j if $\langle p_i(t), \ell_j(t) \rangle > 0$ for small t and p_i moves away from ℓ_j if $\langle p_i(t), \ell_j(t) \rangle < 0$; in either case, point p_i does not move in the direction of $\vec{\ell_i}$. Similarly, when $\langle p_i(t), \ell_j(t) \rangle$ is an increasing function of t, point p_i will relatively move toward $\vec{\ell_i}$.

Given a point-hyperplane tensegrity framework $(G^\pm, \mathbf{p}, \ell)$ in $\mathbb{E}^n$:

1. A *trivial infinitesimal motion* of $(G^\pm, \mathbf{p}, \ell)$ is a trivial infinitesimal motion of $(G, \mathbf{p}, \ell)$, which is given by (5.3.2.2) and (5.3.2.3).
2. $(G^\pm, \mathbf{p}, \ell)$ is called *infinitesimally rigid* if trivial infinitesimal motions are the only motions of $(G^\pm, \mathbf{p}, \ell)$. Otherwise, it is called *infinitesimally flexible.*
3. A *self-stress* of $(G^\pm, \mathbf{p}, \ell)$ is a self-stress of $(G, \mathbf{p}, \ell)$ such that $\omega_{ij} \geq 0$ for $ij \in E_-$ and $\omega_{ij} \leq 0$ for $ij \in E_+$.

For a point-hyperplane tensegrity framework $(G^\pm, \mathbf{p}, \ell)$ there is an associated tensegrity framework in $\mathbb{S}^n$ whose point configuration is the projection of $(\mathbf{p}, \ell)$ into $\mathbb{S}^n$ as described in Chapter 3 and its signed graph $G'^\pm$ is obtained by reversing the signs on all point-hyperplane edges of $G^{\pm}$. More clearly, define ϕ by the central projection to $\mathbb{S}^n$ for points:

$$\phi(p_i) = \frac{p_i}{\|p_i\|}, \tag{5.3.2.28}$$

for points $p_i = (x_{i,1}, \ldots, x_{i,n}, 1) \in \mathbb{A}^n$, for all $i \in V_P$. For hyperplanes ℓ_i, $i \in V_L$, it is defined as:

$$\phi(\ell_i) = \xi_i = (\vec{\ell_i}, 0). \tag{5.3.2.29}$$

In fact ξ_i is the pole of the hyperplane ℓ_i° through the origin in $\mathbb{E}^n$, which is a point on the equator of $\mathbb{S}^n$.

The infinitesimal motions of points and hyperplanes transfer by ψ, defined by:

$$\psi_{p_i}(p_i') = \frac{1}{\|p_i\|}(p_i' + (0, \ldots, 0, -\langle p_i, p_i' \rangle)) \in \mathrm{T}_{\phi(p_i)}\mathbb{S}^n, \tag{5.3.2.30}$$

at point p_i where $p_i' \in T_{p_i}\mathbb{A}^n$, and

$$\psi_{\ell_i}(\ell_i') = \ell_i' \in T_{\xi_j}\mathbb{S}^n, \tag{5.3.2.31}$$

at every hyperplane ℓ_i, $i \in V_L$. For this, we take $\ell_i' = \xi_i'$ as the velocity at the point ξ on the equator.

It is verified in [17] that $(\mathbf{p}', \ell')$ is an infinitesimal motion of $(G, \mathbf{p}, \ell)$ in $\mathbb{E}^n$ if and only if $(\psi(\mathbf{p}'), \psi(\ell))$ is an infinitesimal motion of $(G, (\phi(\mathbf{p}), \phi(\ell)))$ in $\mathbb{S}^n$. This can also be seen in calculations below in the proof of Theorem 5.3.2. The associated tensegrity framework $(G'^{\pm}, (\phi(\mathbf{p}), \phi(\ell)))$ in $\mathbb{S}^n$ is key to understanding $(G^{\pm}, \mathbf{p}, \ell)$ in $\mathbb{E}^n$.

In the following theorem, we show that conditions (5.3.2.6)–(5.3.2.25) for $(G^{\pm}, \mathbf{p}, \ell)$ are closely related to those of the tensegrity framework $(G'^{\pm}, (\phi(\mathbf{p}), \phi(\ell)))$ in $\mathbb{S}^n$. This is one of our main results.

Theorem 5.3.2. *For a point-hyperplane tensegrity framework $(G^{\pm}, \mathbf{p}, \ell)$ in $\mathbb{E}^n$, the following are equivalent:*

(a) *The tensegrity framework* $(G^{\pm}, \mathbf{p}, \ell)$ *is infinitesimally rigid in* $\mathbb{E}^n$*.*

(b) *The tensegrity framework* $(G'^{\pm}, (\phi(\mathbf{p}), \phi(\ell)))$ *is infinitesimally rigid in* $\mathbb{S}^n$*.*

(c) *The tensegrity framework* $(G'^{\pm}, (\phi(\mathbf{p}), \phi(\ell)))$ *is statically rigid in* $\mathbb{S}^n$*.*

(d) $(G'^{\pm}, (\phi(\mathbf{p}), \phi(\ell)))$ *has a strict self-stress and* $(G, (\phi(\mathbf{p}), \phi(\ell)))$ *is infinitesimally rigid in* $\mathbb{S}^n$*.*

(e) $(G^{\pm}, \mathbf{p}, \ell)$ *has a strict self-stress and* $(G, \mathbf{p}, \ell)$ *is infinitesimally rigid in* $\mathbb{E}^n$*.*

Proof. By (5.3.2.28) and (5.3.2.30), for every $ij \in E_{PP}$ we have

$$\langle p_i - p_j, p'_i - p'_j \rangle = \|p_i\| \|p_j\| \langle \phi(p_i) - \phi(p_j), \psi_{p_i}(p'_i) - \psi_{p_j}(p'_j) \rangle.$$

The right-hand side is a constraint of $ij \in E_{PP}$ in $\mathbb{S}^n$. For every $ij \in E_{PL}$,

$$\begin{aligned} \langle p'_i, \ell_j \rangle + \langle p_i, \ell'_j \rangle =& \|p_i\| \big(\langle \psi_{p_i}(p'_i), \xi_j \rangle + \langle \phi(p_i), \xi'_j \rangle \big) \\ =& - \|p_i\| \langle \phi(p_i) - \xi_j, \psi(p_i) - \xi'_j \rangle, \end{aligned}$$

as $\langle \phi(p_i), \psi(p_i) \rangle = 0$ and $\langle \xi_j, \xi'_j \rangle = 0$ in $\mathbb{S}^n$. The right-hand side is a constraint $ij \in E_{PP}$ with $i \in V_P, j \in V_{\text{eq}}$. Conditions (5.3.2.12)–(5.3.2.14) are equivalent to the following conditions on the points $\phi(\ell_i) = \xi_i$, $i \in V_L$ on the equator of $\mathbb{S}^n$:

$$\langle \xi_i - \xi_j, \xi'_i - \xi'_j \rangle = \langle \vec{\ell}_i - \vec{\ell}_j, \vec{\ell}'_i - \vec{\ell}'_j \rangle.$$

Considering the tensegrity constraints (5.2.0.1)–(5.2.0.3) on $\mathbb{S}^n$, the above equations prove the equivalence of the infinitesimal rigidity of $(G'^{\pm}, (\phi(\mathbf{p}), \phi(\ell)))$ and $(G^{\pm}, \mathbf{p}, \ell)$. The equivalence of the statements (b),(c) and (d) follow from Corollary 5.2.4 and Theorem 5.2.5. The equivalence of (d) and (e) follows from Proposition 5.3.1 and Theorem 3.3.1. $\square$

For a point-hyperplane tensegrity framework $(G^{\pm}, \mathbf{p}, \ell)$ (with $E_- \cup E_+ \neq \phi$) to be infinitesimally rigid, it is necessary to have $e > nv - \binom{n+1}{2}$ by the above theorem. This is because if $e \leq nv - \binom{n+1}{2}$ then $(G, \mathbf{p}, \ell)$ is either isostatic (i.e., no strict stress) or infinitesimally flexible.

The following result is a consequence of Proposition 5.2.1 and Theorem 5.3.2.

Corollary 5.3.3. *Given a point-hyperplane tensegrity framework $(G^{\pm}, \mathbf{p}, \ell)$ in $\mathbb{E}^n$. The framework obtained from $(G^{\pm}, \mathbf{p}, \ell)$ by reversing the orientation of a hyperplane and the signs of the incident edges to that hyperplane-vertex is infinitesimally rigid if and only if $(G^{\pm}, \mathbf{p}, \ell)$ is infinitesimally rigid.*

Before we move on to more explore more about point-hyperplane tensegrity frameworks, we would like to consider the rigidity of these frameworks.

5.4 Rigidity of point-hyperplane tensegrity frameworks

This section closely follows [50] in results and methods.

Let $M(\mathbf{p}, \ell)$ be the algebraic set

$$\begin{aligned}\{(\mathbf{q}, \jmath) \in \mathbb{R}^{(n+1)v_p} \times \mathbb{R}^{(n+1)v_l} :& \|q_i - q_j\|^2 = \|p_i - p_j\|^2, \\ & \langle q_i, \jmath_j \rangle = \langle p_i, \ell_j \rangle, \langle \vec{\jmath_i}, \vec{\jmath_j} \rangle = \langle \vec{\ell_i}, \vec{\ell_j} \rangle, \\ & \|\vec{\jmath_i}\|^2 = 1, \langle q_j, e_{n+1} \rangle = 1,\ \forall i, j \in V\}.\end{aligned}$$

The constraints imposed by members of $E_\circ, E_-$ and E_+ define the set $X(\mathbf{p}, \ell)$ of elements $(\mathbf{q}, \jmath)$ such that $\mathbf{q} = (q_1, \ldots, q_{v_p})$ with $q_i \in \mathbb{R}^{n+1}$ and $\jmath = (\jmath_1, \ldots, \jmath_{v_l})$ with $\jmath_i = (\vec{\jmath_i}, b_i) \in \mathbb{R}^n \times \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned}
\|q_i - q_j\| &= \|p_i - p_j\| && ij \in E_\circ \cap E_{PP}, \\
\|q_i - q_j\| &\le \|p_i - p_j\| && ij \in E_- \cap E_{PP}, \\
\|q_i - q_j\| &\ge \|p_i - p_j\| && ij \in E_+ \cap E_{PP}, \\
\langle q_i, \jmath_j \rangle &= \langle p_i, \ell_j \rangle && ij \in E_\circ \cap E_{PL}, \\
\langle q_i, \jmath_j \rangle &\le \langle p_i, \ell_j \rangle && ij \in E_- \cap E_{PL}, \\
\langle q_i, \jmath_j \rangle &\ge \langle p_i, \ell_j \rangle && ij \in E_+ \cap E_{PL},
\end{aligned}$$

$$
\begin{aligned}
\cos^{-1}\langle \vec{\jmath}_i, \vec{\jmath}_j \rangle &= \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j \rangle && ij \in E_\circ \cap E_{LL}, \\
\cos^{-1}\langle \vec{\jmath}_i, \vec{\jmath}_j \rangle &\leq \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j \rangle && ij \in E_- \cap E_{LL}, \\
\cos^{-1}\langle \vec{\jmath}_i, \vec{\jmath}_i \rangle &\geq \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j \rangle && ij \in E_+ \cap E_{LL}, \\
\|\vec{\jmath}_i\| &= 1 && i \in V_L, \\
\langle q_i, e_{n+1} \rangle &= 1 && i \in V_P.
\end{aligned}
$$

Since rigid motions trivially preserve the constraints of $(G^\pm, \mathbf{p}, \ell)$, $M(\mathbf{p}, \ell) \subseteq X(\mathbf{p}, \ell)$.

Definition 5.4.1. A tensegrity framework $(G^\pm, \mathbf{p}, \ell)$ is *rigid* in $\mathbb{E}^n$ if there exists an open neighbourhood U of $(\mathbf{p}, \ell)$ in $\mathbb{R}^{(n+1)v_p} \times \mathbb{R}^{(n+1)v_l}$ such that

$$X(\mathbf{p}, \ell) \cap U = M(\mathbf{p}, \ell) \cap U.$$

$(G^\pm, \mathbf{p}, \ell)$ is called *flexible* if there exists a continuous path $\gamma : [0, 1] \to \mathbb{R}^{(n+1)v}$ with $\gamma(0) = (\mathbf{p}, \ell)$ and $\gamma(t) \in X(\mathbf{p}, \ell) \backslash M(\mathbf{p}, \ell)$ for all $t \in (0, 1]$.

Proposition 5.4.1. *Suppose $(G^\pm, \mathbf{p}, \ell)$ is a tensegrity framework in $\mathbb{R}^n$. The following are equivalent:*

(a) *$(G^\pm, \mathbf{p}, \ell)$ is not rigid.*

(b) *There exists an analytic path $\gamma : [0, 1] \to \mathbb{R}^{nv}$, with $\gamma(0) = (\mathbf{p}, \ell)$ and $\gamma(t) \in X(\mathbf{p}, \ell) \backslash M(\mathbf{p}, \ell)$ for all $t \in (0, 1]$.*

(c) *$(G^\pm, \mathbf{p}, \ell)$ is flexible.*

Proof. We first construct the algebraic set $\mathcal{A}$ corresponding to semi-algebraic set $X(\mathbf{p}, \ell)$: $\mathcal{A}$ is composed of points $(\mathbf{q}, \jmath, y_1, \ldots, y_{|E_-|}, z_1, \ldots, z_{|E_+|}) \in \mathbb{R}^{(n+1)v+|E_-|+|E_+|}$ such that $(\mathbf{q}, \jmath) \in X(\mathbf{p}, \ell)$ and

$$
\begin{aligned}
\|q_i - q_j\| + y_{ij}^2 &= \|p_i - p_j\|, \\
\langle q_i, \jmath_j \rangle + y_{ij}^2 &= \langle p_i, \ell_j \rangle, \\
\langle \vec{\jmath}_i, \vec{\jmath}_j \rangle - y_{ij}^2 &= \langle \vec{\ell}_i, \vec{\ell}_j \rangle,
\end{aligned}
$$

for all $ij \in E_-$ and

$$\|q_i - q_j\|^2 - z_{ij}^2 = \|p_i - p_j\|^2,$$
$$\langle q_i, \jmath_j\rangle - z_{ij}^2 = \langle p_i, \ell_j\rangle,$$
$$\langle \vec{\jmath}_i, \vec{\jmath}_j\rangle + z_{ij}^2 = \langle \vec{\ell}_i, \vec{\ell}_j\rangle,$$

for all $ij \in E_+$. The last equality follows since cosine inverse is a decreasing function. Thus $(\mathbf{q}, \jmath) \in X(\mathbf{p}, \ell)$ if and only of $(\mathbf{q}, \jmath, y_1, \ldots, y_{|E_-|}, z_1, \ldots, z_{|E_+|}) \in \mathcal{A}$. Now if $(G^\pm, \mathbf{p}, \ell)$ is not rigid then every open neighbourhood of $(\mathbf{p}, \ell)$ contains points in $X(\mathbf{p}, \ell)\backslash M(\mathbf{p}, \ell)$. This means every neighbourhood of $(\mathbf{p}, \ell, 0, \ldots, 0)$ in $\mathcal{A}$ contains points that are not in $M(\mathbf{p}, \ell)$. Let $M(\mathbf{p}, \ell, \mathbf{0})$ be an embedded copy of $M(\mathbf{p}, \ell)$ in $\mathcal{A}$. By the curve selection lemma of Milnor [42, Lemma 3.1, p. 25], there is an analytic curve

$$(\mathbf{q}, \jmath, y_1, \ldots, y_{|E_-|}, z_1, \ldots, z_{|E_+|}) : [0, 1] \to \mathbb{R}^{nv+|E_-|+|E_+|}$$

starting at $(\mathbf{p}, \ell, 0, \ldots, 0)$ and belongs to $\mathcal{A}\backslash M(\mathbf{p}, \ell, \mathbf{0})$ for $t \in (0, 1]$. Thus $(\mathbf{q}(t), \jmath(t))$ is an analytic path in $X(\mathbf{p}, \ell)\backslash M(\mathbf{p}, \ell)$. This proves $a \to b$.
$b \to c$ is obvious. If c holds then there exists t_0 in $[0, 1)$ that $(\mathbf{q}(t_0), \jmath(t_0))$ is the last point of the curve which is in $M(\mathbf{p}, \ell)$. Then there is a rigid motion T such that $T(\mathbf{q}(t_0), \jmath(t_0)) = (\mathbf{p}, \ell)$ but $T(\mathbf{q}(t), \jmath(t)) \in X(\mathbf{p}, \ell)\backslash M(\mathbf{p}, \ell)$ for $t > t_0$. Thus every neighbourhood of $(\mathbf{p}, \ell)$ intersects $X(\mathbf{p}, \ell)\backslash M(\mathbf{p}, \ell)$ so $(G^\pm, \mathbf{p}, \ell)$ is not rigid. $\square$

Suppose $(G, \mathbf{p}, \ell)$ is point-hyperplane framework with points $p_i = (x_{i,1}, \ldots, x_{i,n})$ in $\mathbb{R}^n$ for all $i \in V_P$ and hyperplane ℓ_i, $i \in V_L$ is parametrized by a n-tuples $(a_{i,1}, \ldots, a_{i,n})$ such that its equation is $x_1 + a_{i,1}x_2 + \cdots + a_{i,n-1}x_n + a_{i,n} = 0$. We recall the rigidity map $f_A : \mathbb{R}^{nv} \to \mathbb{R}^{|A|}$ of the subset $A \subseteq E$ by $f_A(\mathbf{p}, \ell) = (\ldots, f_{ij}(\mathbf{p}, \ell), \ldots)$ where

$$f_{ij}(\mathbf{p}, \ell) = \|p_i - p_j\|^2 \qquad ij \in E_{PP} \cap A,$$
$$f_{ij}(\mathbf{p}, \ell) = \frac{x_{i,1} + a_{j,1}x_{i,2} + \cdots + a_{j,n-1}x_{i,n} + a_{j,n}}{\sqrt{1 + a_{j,1}^2 + \cdots + a_{j,n-1}^2}} \qquad ij \in E_{PL} \cap A,$$

$$f_{ij}(\mathbf{p},\ell) = \cos^{-1}\frac{1 + a_{i,1}a_{j,1} + \cdots + a_{i,n-1}a_{j,n-1}}{\sqrt{1 + a_{i,1}^2 + \cdots + a_{i,n-1}^2}\sqrt{1 + a_{j,1}^2 + \cdots + a_{j,n-1}^2)}} \quad ij \in E_{LL} \cap A.$$

A point-hyperplane configuration $(\mathbf{p},\ell)$ is a *regular point* of the tensegrity $(G^{\pm},\mathbf{p},\ell)$ if

$$\text{rank } df_E(\mathbf{p},\ell) = \max\,\{\text{rank } df_E(\mathbf{q},\jmath)|(\mathbf{q},\jmath) \in \mathbb{R}^{nv}\}.$$

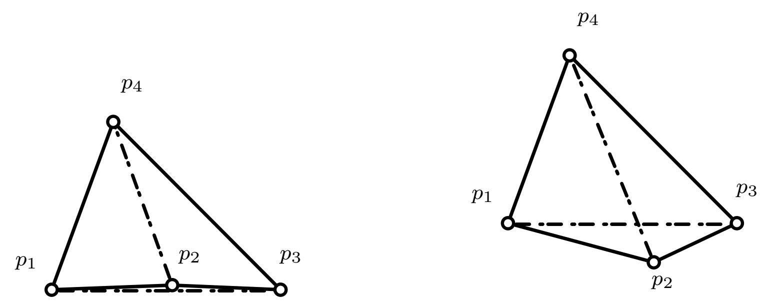
It is important to observe that the set of point-hyperplane configurations $(\mathbf{p},\ell) \in \mathbb{R}^{nv_p}\times\mathbb{R}^{nv_l}$ for which a tensegrity framework $(G^{\pm},\mathbf{p},\ell)$ is infinitesimally rigid is open. This is based on two facts. On one hand, the sign of strict stresses of a framework $(G^{\pm},\mathbf{p},\ell)$ is preserved in an small open set around $(\mathbf{p},\ell)$. The argument is similar to [50, Theorem 5.4.]. On the other hand, the infinitesimal rigidity of $(G,\mathbf{p},\ell)$ is unchanged in an small open set around $(\mathbf{p},\ell)$. So by Theorem 5.3.2, the infinitesimal rigidity of $(G,\mathbf{p},\ell)$ is maintained in an open set around $(\mathbf{p},\ell)$.

The following result was first observed by Connelly [12, Remark 4.1] for tensegrity frameworks.

Theorem 5.4.2. *If a framework $(G^{\pm},\mathbf{p},\ell)$ is infinitesimally rigid then it is rigid.*

Proof. If $(G^{\pm},\mathbf{p},\ell)$ is flexible in $\mathbb{E}^n$ then, by Proposition 5.4.1, there is an analytic path $\gamma : [0,1] \to \mathbb{R}^{nv}$ such that $\gamma(0) = (\mathbf{p},\ell)$ and $\gamma(t) \in X(\mathbf{p},\ell) - M(\mathbf{p},\ell)$ for all $t \in (0,1]$. Thus there exist vertices $k,m \in V$ for which $f_{km}(\gamma(t))$ is not constant on $[0,1]$ for small positive t while $\gamma(t) \in X(\mathbf{p},\ell)$ for all t. Since $f_{km}(\gamma(t))$ is a non-constant real analytic function its derivative is not zero for small enough $t > 0$. Real analyticity of γ implies that $f_{ij}(\gamma(t))$ for every $ij \in E_-$ (resp. $ij \in E_+$) is either constant or its the derivative of $f_{ij}(\gamma(t))$ is negative (resp. positive) for small $t > 0$. Therefore $\gamma'(0)$ is a non-trivial infinitesimal motion of $(G^{\pm},\mathbf{p},\ell)$ and as a result, $(G^{\pm},\mathbf{p},\ell)$ is infinitesimally flexible. $\square$

It turns out that regularity of a configuration $(\mathbf{p},\ell)$ of a tensegrity framework $(G^{\pm},\mathbf{p},\ell)$ is not enough to insure that the set of infinitesimally flexible realization of a framework is a open set. Because tensegrity frameworks have over-braced



(a) A flexible bar-joint tensegrity framework in the plane.

(b) A rigid bar-joint tensegrity framework in the plane.

Figure 5.5: Rigid and flexible tensegrity frameworks with the same underlying signed graph.

underlying framework their sub-frameworks might not be ‘regular’ while the rank of the entire matrix is maximized. Figure 5.5 (borrowed from [50]) shows a simple example in which both realizations are regular by definition but the one in Figure 5.5a (p_1, p_2 and p_3 are collinear) is flexible while any open neighbourhood of its configuration contains rigid frameworks, Figure 5.5b.

Thus we need a more restrictive notion: A configuration $(\mathbf{p}, \ell)$ is said to be *fully regular* for a framework $(G^{\pm}, \mathbf{p}, \ell)$ if

$$\text{rank } df_A(\mathbf{p}, \ell) = \max\, \{\text{rank } df_A(\mathbf{q}, \jmath) | (\mathbf{q}, \jmath) \in \mathbb{R}^{nv}\},$$

for every nonempty $A \subseteq E$.

The following theorem states the rigidity and infinitesimal rigidity are the same for fully regular point-hyperplane tensegrity framework

Theorem 5.4.3. *If $(\mathbf{p}, \ell)$ is fully regular for the framework $(G^{\pm}, \mathbf{p}, \ell)$ in $\mathbb{E}^n$ then $(G^{\pm}, \mathbf{p}, \ell)$ is infinitesimal rigid if and only if it is rigid.*

Proof. One direction is given by Theorem 5.4.2. Conversely, suppose $(G^{\pm}, \mathbf{p}, \ell)$ has a non-trivial infinitesimal motion in $\mathbb{E}^n$. If the framework $(G, \mathbf{p}, \ell)$ is infinitesimally flexible it is flexible since $(\mathbf{p}, \ell)$ is a regular point of $(G^{\pm}, \mathbf{p}, \ell)$. Then $(G^{\pm}, \mathbf{p}, \ell)$ is flexible so we are done. Therefore we assume $(G, \mathbf{p}, \ell)$ is infinitesimally rigid. We will show that $(G^{\pm}, \mathbf{p}, \ell)$ is flexible. Let $\mathcal{I}(\mathbf{p}, \ell)$ be the space of infinitesimal motions of $(G^{\pm}, \mathbf{p}, \ell)$ and

$$A = \{ij \in E \mid F_{ij} \in \mathcal{I}(\mathbf{p}, \ell)^{\perp}\},$$

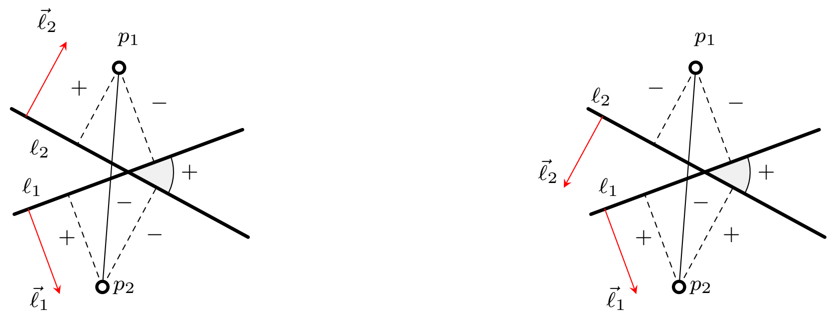
where F_{ij} is the rows ij in the Euclidean matrix (3.2.0.2). We know that $E_{PP} \subseteq A$. Since $(G, \mathbf{p}, \ell)$ is rigid and $(G^{\pm}, \mathbf{p}, \ell)$ is infinitesimally flexible, $A \neq E$. So there exists a non-trivial infinitesimal motion $\mu \in \mathcal{I}(\mathbf{p}, \ell) - \mathcal{T}(\mathbf{p}, \ell)$ such that for some vertices i and j, $\langle \mu, F_{ij} \rangle \neq 0$. Thus we can choose a $(\mathbf{p}', \ell') \in \mathcal{I}(\mathbf{p}, \ell)$ such that $\langle F_{ij}, (\mathbf{p}', \ell') \rangle \neq 0$ for all $ij \in E - A$. If $A = \phi$ then choose the path $\gamma(t) = (\gamma_i(t))_{i \in V}$ to be $\gamma_i(t) = p_i + tp'_i$ for all points $i \in V_P$ and $\gamma_i(t) = (a_{i,1} + tv_{i,1}, \ldots, a_{i,n} + tv_{i,1})$ for the hyperplane parameterized as $(a_{i,1}, \ldots, a_{i,n})$ with velocity $\ell'_i = (v_{i,1}, \ldots, v_{i,n})$. Obviously $\gamma(0) = (\mathbf{p}, \ell)$ and $\gamma'(0) = (\mathbf{p}', \ell')$. Thus $(G^{\pm}, \mathbf{p}, \ell)$ is flexible. If $A \neq \phi$ then $f_A^{-1}(f_A(\mathbf{p}, \ell))$ is a manifold near $(\mathbf{p}, \ell)$ since $(\mathbf{p}, \ell)$ is fully regular. The tangent space to this manifold at $(\mathbf{p}, \ell)$ is ker $df_A(\mathbf{p}, \ell)$ and the infinitesimal motion $(\mathbf{p}', \ell')$ belongs to ker $df_A(\mathbf{p}, \ell)$. So there exists a smooth path $\gamma(t)$ in $f_A^{-1}(f_A(\mathbf{p}, \ell))$ with $\gamma(0) = (\mathbf{p}, \ell)$ and $\gamma'(0) = (\mathbf{p}', \ell')$. The path $\gamma(t)$ automatically satisfies the conditions for $E - A$ as $\gamma'(0) = (\mathbf{p}', \ell')$. This proves that $\gamma(t)$ is a finite flex for $(G^{\pm}, \mathbf{p}, \ell)$ and therefore it is flexible. □

Corollary 5.4.4. *If $(G, \mathbf{p}, \ell)$ is an isostatic framework in $\mathbb{E}^n$ then the tensegrity framework $(G^{\pm}, \mathbf{p}, \ell)$ obtained from $(G, \mathbf{p}, \ell)$ by replacing any edge by a signed edge is flexible.*

Proof. Since $(G, \mathbf{p}, \ell)$ is isostatic then $(\mathbf{p}, \ell)$ is fully regular and it does not admit any non-trivial stress. Therefore $(G^{\pm}, \mathbf{p}, \ell)$ is not infinitesimally rigid by Theorem 5.3.2 and then, by Theorem 5.4.3 it is flexible. □

5.5 Point-line tensegrity frameworks and sliders

Now let's consider an example of a point-line tensegrity framework in the plane.



(a) A rigid point-line tensegrity framework. Minus signed edges are cables and positive edges are struts.

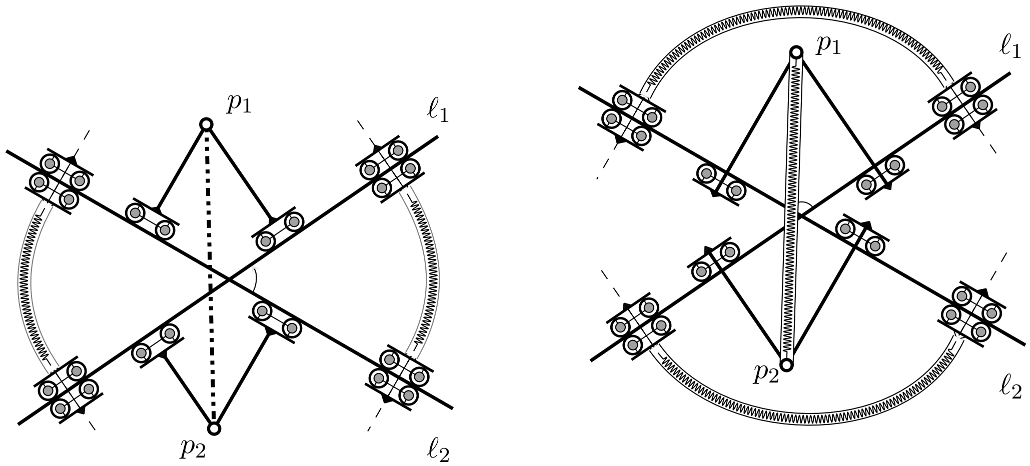
(b) A rigid point-line tensegrity framework obtained from (a) by reversing the orientation of ℓ_2.

Figure 5.6: Point-line tensegrity frameworks.

Example 5.5.1. Figure 5.6a shows a point-line tensegrity framework in the plane. The cable elements E_- and strut elements E_+ are illustrated by minus signs $-$ and plus signs $+$, respectively. Indeed, $E_- = \{\{p_1, p_2\}, \{p_1, \ell_1\}, \{p_2, \ell_2\}, \{\ell_1, \ell_2\}\}$ and $E_+ = \{p_2, \ell_1\}, \{p_1, \ell_2\}\}$. By Example 5.3.1, there exists a strict stress for the framework in Figure 5.6a. The underlying point-line framework is infinitesimally rigid because K_4 is an infinitesimally rigid graph. Thus by Theorem 5.3.2, the point-line tensegrity framework in Figure 5.6a is infinitesimally rigid and by Theorem 5.4.2, it is rigid.

The framework in Figure 5.6b is obtained from (a) by reversing the orientation of ℓ_2 and the reversing the sign on the edges incident to ℓ_2 except for the angle that we did not change while we could. This is because increasing an angle α in equivalent to decreasing the angle $\pi - \alpha$ and vice versa. Therefore the framework in Figure 5.6b is rigid as well, by Corollary 5.3.3.

We have demonstrated a mechanical model of the point-line tensegrity framework in Figure 5.6a using sliders and rollers shown in Figure 5.7a. The rollers attached to the point p_1 restrict the relative motion of p_1 to the lines ℓ_1 and ℓ_2 to be against $\vec{\ell}_1$ and in the direction $\vec{\ell}_2$, respectively. Similarly, the rollers attached to p_2 restrict the



(a) The rigid slider-roller model of the tensegrity framework in Figure 5.6a.

(b) The rigid slider-roller model of the tensegrity framework in Figure 5.6a by reversing all stress signs.

Figure 5.7: A slider-roller model of point-line tensegrity frameworks.

relative motion of p_2 to the lines ℓ_1 and ℓ_2 to be respectively in the direction $\vec{\ell_1}$ and against $\vec{\ell_2}$. The two pairs of double rollers constrain the illustrated angle between two lines to increase only (or equivalently, the angle between $\vec{\ell_1}$ and $\vec{\ell_2}$ to decrease only). This model is infinitesimally rigid (and rigid) by Example 5.5.1. Note that the arrangement of the rollers and sliders does not depend on the orientation of the lines by Corollary 5.3.3. Another infinitesimally rigid model derived from (a) by reversing the sign of all stresses of the framework in Figure 5.6a, is demonstrated in Figure 5.7b. Note that the rollers are drawn on the opposite side compared with Figure 5.7a.

In fact, these tensegrity frameworks are considered various 'projective images' of a quadrilateral tensegrity framework on four points. An arbitrary projective image of a point-hyperplane framework in $\mathbb{E}^n$ is a point-hyperplane framework in $\mathbb{E}^n$ as explained in Section 3.3. It was observed in [50] that a projective image of an infinitesimally rigid bar-joint tensegrity framework is infinitesimally rigid provided that the signed graph is adjusted appropriately. Namely, the bars cut by the hyperplane at infinity under the projective transformation must change signs. This can also be seen from Proposition 5.3.1 using $\mathbb{S}^n$. In fact, the possible sign change occurs as a result of transferring the points from $\mathbb{S}^n_-$ to $\mathbb{S}^n_+$. The finite points mapped to infinity become new oriented hyperplanes in the Euclidean space under the projective transformation.

In the next section we will see that the sign of stresses in a point-hyperplane tensegrity framework may have an interpretation as tension or compression.

5.6 Point-hyperplane tensegrity frameworks in $\mathbb{E}^n$ with point-hyperplane distances

As we saw in the previous section, the constraints (5.3.2.21) and (5.3.2.22) on a point-hyperplane tensegrity framework do not place upper bounds or lower bounds on the point-hyperplane distances. But we are not far from there. In order to accomplish this, we simply need to adjust those constraints by considering their absolute values.

Suppose $(\mathbf{p}(t), \ell(t))$, $t \in [0,1]$ with $(\mathbf{p}(0), \ell(0)) = (\mathbf{p}, \ell)$ is a smooth time-dependent point-hyperplane configuration in $\mathbb{E}^n$ and $G^{\pm} = (V_P \cup V_L; E_-, E_\circ, E_+)$ is a signed point-hyperplane graph such that

$$\|p_i(t) - p_j(t)\| = \|p_i - p_j\| \qquad ij \in E_\circ \cap E_{PP}, \tag{5.6.0.1}$$

$$\|p_i(t) - p_j(t)\| \leq \|p_i - p_j\| \qquad ij \in E_- \cap E_{PP}, \tag{5.6.0.2}$$

$$\|p_i(t) - p_j(t)\| \geq \|p_i - p_j\| \qquad ij \in E_+ \cap E_{PP}, \tag{5.6.0.3}$$

$$\langle p_i(t), \ell_j(t)\rangle = \langle p_i, \ell_j\rangle \qquad ij \in E_\circ \cap E_{PL}, \tag{5.6.0.4}$$

$$|\langle p_i(t), \ell_j(t)\rangle| \leq |\langle p_i, \ell_j\rangle| \qquad ij \in E_- \cap E_{PL}, \tag{5.6.0.5}$$

$$|\langle p_i(t), \ell_j(t)\rangle| \geq |\langle p_i, \ell_j\rangle| \qquad ij \in E_+ \cap E_{PL}, \tag{5.6.0.6}$$

$$\cos^{-1}\langle \vec{\ell}_i(t), \vec{\ell}_j(t)\rangle = \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j\rangle \qquad ij \in E_\circ \cap E_{LL}, \tag{5.6.0.7}$$

$$\cos^{-1}\langle \vec{\ell}_i(t), \vec{\ell}_j(t)\rangle \leq \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j\rangle \qquad ij \in E_- \cap E_{LL}, \tag{5.6.0.8}$$

$$\cos^{-1}\langle \vec{\ell}_i(t), \vec{\ell}_i(t)\rangle \geq \cos^{-1}\langle \vec{\ell}_i, \vec{\ell}_j\rangle \qquad ij \in E_+ \cap E_{LL}, \tag{5.6.0.9}$$

for every $t \in [0,1]$. We assume that $\langle p_i, \ell_j\rangle \neq 0$, for $ij \in (E_- \cup E_+) \cap E_{PL}$. Otherwise, we let $ij \in E_\circ$ if $ij \in E_-$; Also $ij \in E_+$ is not considered a constraint as $|\langle p_i(t), \ell_j(t)\rangle| \geq 0$ is trivially true for all t. Now we can take the derivative of the system at $t = 0$ to obtain the corresponding infinitesimal constraints as the following:

$$\langle p_i - p_j, p_i' - p_j'\rangle = 0 \qquad ij \in E_\circ \cap E_{PP}, \tag{5.6.0.10}$$

$$\langle p_i - p_j, p_i' - p_j'\rangle \leq 0 \qquad ij \in E_- \cap E_{PP}, \tag{5.6.0.11}$$

$$\langle p_i - p_j, p_i' - p_j'\rangle \geq 0 \qquad ij \in E_+ \cap E_{PP}, \tag{5.6.0.12}$$

$$\langle p_i', \ell_j\rangle + \langle p_i, \ell_j'\rangle = 0 \qquad ij \in E_\circ \cap E_{PL}, \tag{5.6.0.13}$$

$$\langle p_i, \ell_j\rangle(\langle p_i', \ell_j\rangle + \langle p_i, \ell_j'\rangle) \leq 0 \qquad ij \in E_- \cap E_{PL}, \tag{5.6.0.14}$$

$$\langle p_i, \ell_j\rangle(\langle p_i', \ell_j\rangle + \langle p_i, \ell_j'\rangle) \geq 0 \qquad ij \in E_+ \cap E_{PL}, \tag{5.6.0.15}$$

$$\langle \vec{\ell}_i - \vec{\ell}_j, \vec{\ell}_i' - \vec{\ell}_j'\rangle = 0 \qquad ij \in E_\circ \cap E_{LL}, \tag{5.6.0.16}$$

$$\langle \vec{\ell}_i - \vec{\ell}_j, \vec{\ell}_i' - \vec{\ell}_j'\rangle \leq 0 \qquad ij \in E_- \cap E_{LL}, \tag{5.6.0.17}$$

$$\langle \vec{\ell}_i - \vec{\ell}_j, \vec{\ell}_i' - \vec{\ell}_j'\rangle \geq 0 \qquad ij \in E_+ \cap E_{LL}, \tag{5.6.0.18}$$

where

$$\langle p_i', e_{n+1}\rangle = 0 \qquad i \in V_P, \tag{5.6.0.19}$$

$$\langle \vec{\ell}_i, \vec{\ell}_i'\rangle = 0 \qquad i \in V_L. \tag{5.6.0.20}$$

We denote a point-hyperplane framework with constraints (5.6.0.10) – (5.6.0.18) by $(G^{\pm}, \mathbf{p}, \bar{\ell})$ to distinguish it from the tensgrity framework $(G^{\pm}, \mathbf{p}, \ell)$. Now each element of $E_\circ$ preserves the distance between a pair of points, point-hyperplanes and the angle between a pair of hyperplanes. Also, each element of E_- (respectively,

E_+) places an upper (respectively, lower) bound on the distances between pairs of points or pairs of point-hyperplanes or angles between some pairs of hyperplanes.

A tensegrity framework $(G^{\pm}, \mathbf{p}, \bar{\ell})$ is *infinitesimally rigid* in $\mathbb{E}^n$ if every infinitesimal motion of it is trivial, meaning $(\mathbf{p}', \bar{\ell}')$ is given by (5.3.2.4) and (5.3.2.5). Otherwise, it is called *infinitesimally flexible*.

An assignment of scalars $\omega_{ij} = \omega_{ji}$ to the edges $ij \in E$ is called a *self-stress* of $(G, \mathbf{p}, \bar{\ell})$ if $\omega' = (\omega'_{ij})_{ij \in E}$ is self-stress of $(G, \mathbf{p}, \ell)$ where

$$\begin{aligned} \omega'_{ij} &= \omega_{ij} && ij \in E_\circ \cup E_{PP} \cup E_{LL}, \\ \omega'_{ij} &= \langle p_i, \ell_j \rangle \omega_{ij} && ij \in E_{PL}. \end{aligned} \tag{5.6.0.21}$$

Suppose $(G^{\pm}, \mathbf{p}, \ell)$ is a tensegrity framework and let $(G''^{\pm}, \mathbf{p}, \ell)$ is a framework whose signed graph $G''^{\pm} = (V, E''_- \cup E_\circ \cup E''_+)$ obtained from $G^{\pm}$ as the following:

$$\begin{aligned} &ij \in E''_{PL} \cap E_- && \text{if } ij \in E_{PL} \cap E_+, \langle p_i, \ell_j \rangle < 0, \\ &ij \in E''_{PL} \cap E_+ && \text{if } ij \in E_{PL} \cap E_-, \langle p_i, \ell_j \rangle < 0. \end{aligned} \tag{5.6.0.22}$$

Otherwise, an edge of $G''^{\pm}$ has the same sign as $G^{\pm}$.

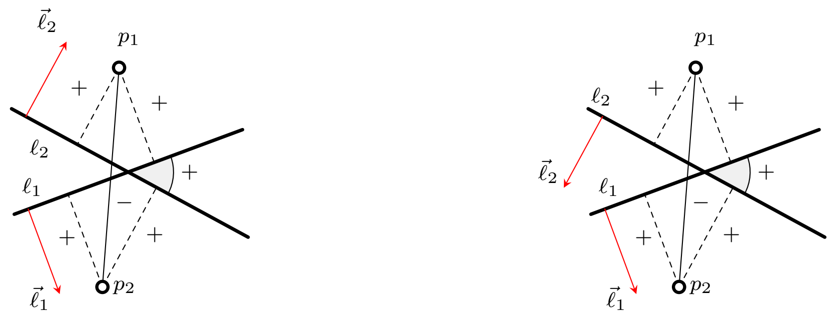
Considering (5.6.0.21), the following Theorem is an immediate consequence of the proof of Theorem 5.3.2.

Theorem 5.6.1. *A point-hyperplane tensegrity framework $(G^{\pm}, \mathbf{p}, \bar{\ell})$ (with absolute value of point-hyperplane distances) is infinitesimally rigid in $\mathbb{E}^n$ if and only if the following hold:*

(a) *The framework $(G''^{\pm}, \mathbf{p}, \ell)$ is infinitesimally rigid in $\mathbb{E}^n$.*

(b) *The framework $(G''^{\pm}, \mathbf{p}, \ell)$ has a strict self-stress and $(G, \mathbf{p}, \ell)$ is infinitesimally rigid.*

The notion of self-stress, as appeared in Definition 5.3.1, does not directly represent *tensions* and *compressions* in the structure of a point-hyperplane framework as one can see in Example 5.5.1. But the geometric adjustment (5.6.0.21) of stresses



(a) A rigid point-line tensegrity framework. Minus signed edges are cables and positive edges are struts.

(b) A rigid point-line tensegrity framework obtained from (a) by reversing the orientation of ℓ_2.

Figure 5.8: Point-line tensegrity frameworks with point-line distances.

provides us with the right pattern of stress signs that can be interpreted as tensions and compressions in a point-hyperplane framework with point-hyperplane distances. Figure 5.8 shows the rigid point-line tensegrity frameworks with point-line distances corresponding to the tensegrity frameworks of Figure 5.6. Note that the sign of stresses (or cables and struts) have changed according to (5.6.0.21). More importantly, the sign of stresses (or equivalently, cables and struts) do not change if we change the orientation of the hyperplanes. This is not accidental. A change of orientation would change the stresses on point-hyperplane edges of the tensegrity framework but that change will be reversed by (5.6.0.21). So the rigidity of the constraints (5.6.0.10) – (5.6.0.18) does not depend on the orientation of the hyperplanes. On the other hand, if we move a hyperplane parallel to itself so that it crosses an adjacent point the sign of stress (tension or compression) will change by (5.6.0.21).

Also note that the slider-roller model of both types of constraints are the same (compare Figure 5.8 and Figure 5.7).

Based on Theorem 5.6.1, the infinitesimal rigidity of a tensegrity framework $(G^{\pm}, \mathbf{p}, \ell)$ is generically equivalent to the rigidity of the associated framework $(G''^{\pm}, \mathbf{p}, \bar{\ell})$

(that respects distance constraints) if $\langle p_i, \ell_j \rangle \neq 0$ for $ij \in E_{PP}$. Assume $\langle p_i, \ell_j \rangle = 0$ for a cable elements $ij \in E_-$. In general, if $(G^{\pm}, \mathbf{p}, \ell)$ is infinitesimally rigid then $(G''^{\pm}, \mathbf{p}, \bar{\ell})$ has to be infinitesimally rigid. If $(G''^{\pm}, \mathbf{p}, \bar{\ell})$ is infinitesimally rigid and $\omega_{ij} = 0$ then $(G^{\pm}, \mathbf{p}, \ell)$ is not infinitesimally rigid by Theorem 5.3.2.

We conclude this chapter with an geometric interpretation of equilibrium forces to a point-line configuration. In fact, we project equilibrium forces from spherical bar-joint configuration with some joints on the equator to the corresponding point-line configuration in the plane, under the central projection.

Suppose $\mathbf{p} = (p_i)_{i=1}^v$ is a configuration of v points on the sphere $\mathbb{S}^2$ with some points on the equator V_{eq}. Let $\mathbb{F} = (f_1, \ldots, f_v)$ be an equilibrium force at $\mathbf{p}$. By the equilibrium condition (5.1.0.1) for spherical configurations, we have:

$$\sum_{i \notin V_{\text{eq}}} (f_{i,1}, f_{i,2}, f_{i,3}) \times (x_i, y_i, z_i) + \sum_{i \in V_{\text{eq}}} (f_{i,1}, f_{i,2}, f_{i,3}) \times (x_i, y_i, 0) = 0,$$

or

$$\sum_{i \notin V_{\text{eq}}} (z_i f_{i,2} - y_i f_{i,3}, -z_i f_{i,1} + x_i f_{i,3}, y_i f_{i,1} - x_i f_{i,2}) + \sum_{i \in V_{\text{eq}}} (-f_{i,3} y_i, f_{i,3} x_i, f_{i,1} y_i - f_{i,2} x_i) = 0.$$

Note that, for all $i \notin V_{\text{eq}}$,

$$(z_i f_{i,2} - y_i f_{i,3}, -z_i f_{i,1} + x_i f_{i,3}, y_i f_{i,1} - x_i f_{i,2}) = (\hat{f}_{i,1}, \hat{f}_{i,2}, 0) \times (x_i/z_i, y_i/z_i, 1).$$

where $(\hat{f}_{i,1}, \hat{f}_{i,2}, 0) = (z_i f_{i,1} - x_i f_{i,3}, z_i f_{i,2} - y_i f_{i,3}, 0)$ is the *vector of the force* $f_i \times p_i$ in the affine plane $z = 1$ at the point $(x_i/z_i, y_i/z_i, 1)$. For every $i \in V_{\text{eq}}$ for which $f_{i,3} \neq 0$, we have

$$(-f_{i,3} y_i, f_{i,3} x_i, f_{i,1} y_i - f_{i,2} x_i) = (-f_{i,3} x_i, -f_{i,3} y_i, 0) \times (f_{i,1}/f_{i,3}, f_{i,2}/f_{i,3}, 1),$$

which is a force at a 'finite' point $(f_{i,1}/f_{i,3}, f_{i,2}/f_{i,3}, 1)$ on the line ℓ_i. Note here that the translation of line i, and as a result, the point $(f_{i,1}/f_{i,3}, f_{i,2}/f_{i,3}, 1)$, anywhere in the direction of $(x_i, y_i, 0)$ (perpendicular tho the line) does not change this equality.

However, in the case $f_{i,3} = 0$, $i \in V_L$, we have

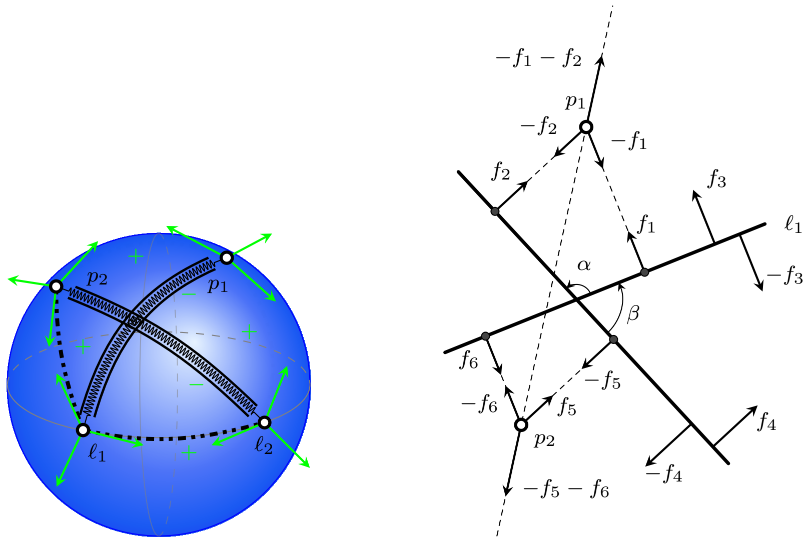
$$(0, 0, f_{i,1}y_i - f_{i,2}x_i) = (-\frac{a}{2}x_i, -\frac{a}{2}y_i, 0) \times (f_{i,1}/a, f_{i,2}/a, 1)$$
$$+ (\frac{a}{2}x_i, \frac{a}{2}y_i, 0) \times (-f_{i,1}/a, -f_{i,2}/a, 1),$$

for any non-zero real number a. This is a *couple* of forces to the line ℓ_i that creates no net linear momentum to the line ℓ_i but a pure angular momentum. In the language of projective geometry, this is a 'force at infinity' which appears as a couple in the Euclidean plane. In general, a couple $\pm\lambda(x_i, y_i, 0)$ of forces should be placed $\|f_i\|/\lambda$ away from each other to have the same effect as the force f_i at infinity. Again, notice that translating this points in any direction does not change the effect of the couple.

Substituting all this back in the sum above, we summarize the equilibrium conditions on point-line configurations as the following:

- All forces to points and lines must add up to zero. Couples do not contribute to this sum.
- The net angular momentum of the whole configuration must be zero. This amounts to a geometric incidence condition on the lines of forces to the system.

Consider the tensegrity framework $(K^{\pm}, \mathbf{p})$ in Example 5.2.1 on sphere with the force diagram shown in Figure 5.9a. Stress pattern on the sphere clearly illustrates compression along the edge with negative stress and tension along the edge with positive stress. Considering (5.6.0.21) and Proposition 5.3.1, we obtain the force diagram of the projected point-line framework in plane as shown in Figure 5.9b. Positive stresses are shown to represent compressions and negative stresses to represent tensions. The couples $\{f_3, -f_3\}$ and $\{f_4, -f_4\}$ compressing the angle α or equivalently impose tension on the complement angle. Considering the fact that a system of forces in the plane is invariant under the translation, we can translate a line so that it passes an adjacent point without changing the force system. Although, this will reverse the tension or compression on the corresponding edge as



(a) (b)

Figure 5.9: Force diagram of a point-line tensegrity framework in the plane.

we knew according to (5.6.0.21).

The geometry of point-hyperplane frameworks in $\mathbb{S}^n$ is neater compared to point-hyperplane frameworks in $\mathbb{E}^n$. The point-hyperplane distance formula in $\mathbb{S}^n$ is $d_{\mathbb{S}}(p, \ell) = |\pi/2 - \cos^{-1}\langle p, \xi\rangle|$, where ξ is a pole of the hyperplane ℓ. This tells us that the generic first-order rigidity of point-hyperplane frameworks in $\mathbb{S}^n$ is equivalent to the generic rigidity of bar-joint frameworks in $\mathbb{E}^n$. In particular, the generic rigidity of a point-line framework in $\mathbb{S}^2$ is characterized by Laman's Theorem. Also the infinitesimal rigidity of their tensegrity constraints in $\mathbb{S}^n$ will be determined by the infinitesimal rigidity of the tensegrity framework obtained by substituting hyperplanes with their poles and the rest will proceed as above.

Inductive constructions of bar-joint tensegrity frameworks has been considered in [14]. This provides the background for an exploration of the inductive techniques for point-hyperplane tensegrity frameworks.

Chapter 6

Body-cad constraints in the Euclidean plane

Geometric constraints in CAD software have motivated a lot of questions and research in Rigidity. In this chapter we study a set of geometric constraints on pairs of points, pairs of lines or a point and a line, each attached to rigid bodies in the plane. These constraints are pairwise coincidences, angle (between two lines) and distance constraints between points or lines attached to rigid bodies, which gives a more diverse set of constraints than the classic body-bar frameworks. These systems of geometric constraints on rigid bodies are called *body-cad* constraints. We will develop the rigidity matrix of these constraints and will give a combinatorial characterization of their minimal rigidity. This will lead to a combinatorial characterization of the rigidity of body-bar frameworks with collinear bars and coincidence constraints. We will combinatorially characterize the rigidity of a special class of rigid bodies with point-point coincidence constraints in 3-space. It turns out that point-point coincidence constraints pose a geometric difficulty in understanding the rigidity of structures. These types of constraints are avoided in the context of body-bar framework.

6.1 Preliminaries

In this section we quickly review the notions and operators in the plane mentioned in Section 2.7.1. We need them in Section 6.3 to derive the algebraic expression of body-cad constraints.

The join $p \vee q$ of the two points p and q in $\mathbb{P}^2$ is the ordered 3-tuple of all 3 minors of the 3×2 matrix M whose columns are the points p and q in their homogeneous coordinates. The standard order of the minors is $|M_{23}|, -|M_{13}|, |M_{12}|$ where $|M_{ij}|, 1 \leq i < j \leq 3$, is the 2×2 minor determined by the rows i and j. So $p \vee q = (|M_{23}|, -|M_{13}|, |M_{12}|)$ as a 3-vector.

Rigid motions in the plane are rotations and translations. We represent an infinitesimal rotation c (the center of the motion) in the plane as a 3-vector $c = (\alpha c_1, \alpha c_2, \alpha)$ where $(c_1, c_2, 1)$ is the affine coordinates of the center (c_1, c_2) in $\mathbb{E}^2$ and $\alpha \neq 0$ is the angular velocity of the infinitesimal rotation. For any point $p = (p_1, p_2, 1)$ in its affine coordinates in the plane, $c \vee p$ is a 3-vector encoding the velocity vector $p' = (p'_1, p'_2, 0)$ at the point p induced by the rotation c as its first two coordinates. More clearly,

$$c \vee p = \begin{vmatrix} \alpha c_1 & p_1 \\ \alpha c_2 & p_2 \\ \alpha & 1 \end{vmatrix} = (p'_1, p'_2, -\langle p', p \rangle) \qquad \text{for every point } p \in \mathbb{P}^2. \tag{6.1.0.1}$$

In particular, an infinitesimal translation in the direction (t_1, t_2) at any point $p = (p_1, p_2, 1)$ in the plane is induced by the infinitesimal rotation $c = (-t_2, t_1, 0)$ whose center is at infinity in $\mathbb{P}^2$, in the direction perpendicular to the vector (t_1, t_2):

$$(-t_2, t_1, 0) \vee (p_1, p_2, 1) = (t_1, t_2, -t_1 p_1 - t_2 p_2), \qquad \text{for every point } p \in \mathbb{P}^2.$$

For three vectors v_1, v_2, v_3 in $\mathbb{R}^3$, $v_1 \vee v_2 \vee v_2$ is a real scalar, which is the determinant $[v_1, v_2, v_3]$ of a 3×3 matrix with the columns v_1, v_2, v_3. In particular,

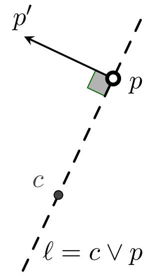


Figure 6.1: $c \vee p$ describes the infinitesimal motion at p, and the line through p and the center of the motion c.

if $q = (q_1, q_2, 0)$ is a direction in the plane then

$$c \vee p \vee q = \begin{vmatrix} \alpha c_1 & p_1 & q_1 \\ \alpha c_2 & p_2 & q_2 \\ \alpha & 1 & 0 \end{vmatrix} = \langle c \vee p, q \rangle = \langle p', q \rangle, \qquad (6.1.0.2)$$

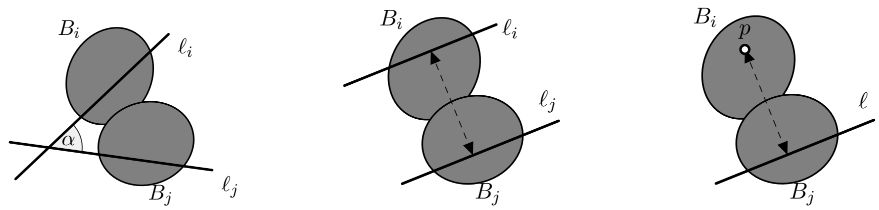
where the determinant is expanded with respect to the third column. We frequently use (6.1.0.2) to express body-cad constraints.

On the other hand, $p \vee q$ represents the *Plücker coordinates* of the line through the points p and q in the plane. The line ℓ joining the two points p and q in the plane is denoted by $\ell = p \vee q$ (see Figure 6.1). The first two coordinates of $\ell = p \vee q$ will represent a vector $\vec{\ell}$ perpendicular to ℓ and the last coordinate will be a scalar multiple $(\pm\|\vec{\ell}\|)$ of the distance of the line from the origin.

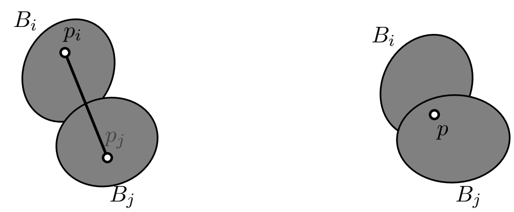
6.2 Body-cad constraints in the plane

A rigid body in the plane can be imagined as a collection of points that are connected to each other in a rigid way. We are not concerned about the shape and the size of rigid bodies but we require that each body, as a collection of points, spans a one dimensional affine space in the plane.

Suppose B_1 and B_2 are two rigid bodies in the plane with some points and lines



(a) Angle constraints. (b) Line-line distance. (c) Point-line distance.



(d) Point-point distance. (e) Point-point coincidence.

Figure 6.2: Body-cad constraints in the plane.

attached to them in a rigid way. We define the following set of constraints on the points and lines on B_1 and B_2:

1. *Angular* constraints.

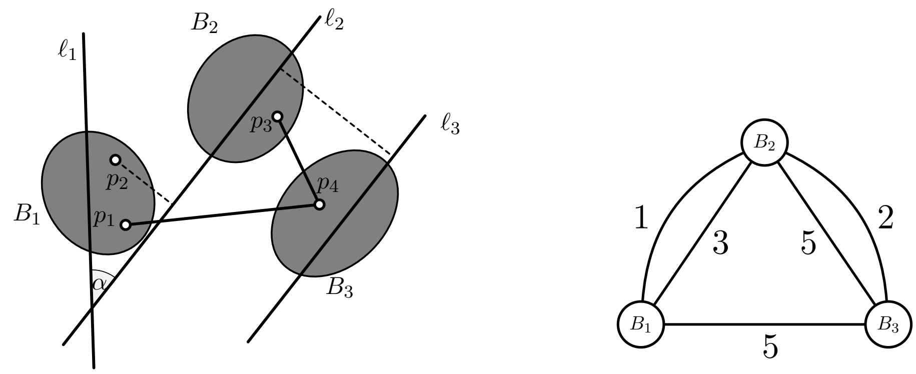
   Assume an arbitrary pair of lines ℓ_i and ℓ_j are attached to the bodies B_i and B_j respectively, in the plane. An angular constraint on the lines ℓ_i and ℓ_j restricts the motions of the bodies B_i and B_j so that the angle between ℓ_i and ℓ_j remains fixed during the motion of the bodies (see Figure 6.2a).

2. *Line-line distance* constraints.

   Given two arbitrary parallel (or coincident) lines ℓ_i and ℓ_i attached to the bodies B_i and B_j respectively, a line-line distance constraint restricts the motions of B_i and B_j so that the distance between the lines ℓ_i and ℓ_j is preserved (see Figure 6.2b).

3. *Point-line distance* constraints.

   For an arbitrary pair of a point p and a line ℓ attached to the bodies B_i and B_j



(a) A body-cad framework in the plane.

(b) The cad graph of the framework in (a).

Figure 6.3: A body-cad framework in (a) and its associated cad graph in (b).

respectively, a point-line distance constraint will restrict the motion of bodies B_i and B_j so that the point-line distance between p and ℓ is maintained (see Figure 6.2c).

4. *Point-point distance* constraints.

   For two arbitrary distinct points p_i and p_j on the bodies B_i and B_j respectively, a point-point distance constraint will restrict the motion of B_i and B_j so that the distance between the points p_i and p_j is preserved (see Figure 6.2d). This single type of constraint is the subject of body-bar frameworks that has been understood in all dimensions (see [60], [64]).

5. *Point-point coincidence* constraints.

   For a common point p on both bodies B_i and B_j, a point-point coincidence constraint restricts the motion of the bodies so that the points remain coincident (see Figure 6.2e).

The above constraints are called *body-cad constraints* in the plane. A set of bodies in the plane interconnected by pairwise body-cad constraints forms a body-cad structure in the plane.

A *cad graph* is a multigraph $G = (V, E)$ with no loops together with an edge colouring function $c : E \to C$, where $C = \{c_1, c_2, c_3, c_4, c_5\}$ is a set of 5 colours corresponding to each of the five body-cad constraints mentioned above. The colouring function naturally partitions the edge set E into 5 partitions $E_1, \ldots, E_5$.

A *body-cad framework* $(G, c, L_1, \ldots, L_5)$ is a cad graph (G, c) along with a family of functions $L_1, \ldots, L_5$, where $L_i(e)$, $e \in E_i$, assigns coordinates of points or lines to two bodies constrained by the geometric constraint corresponding to the edge $e \in E_i$. For example if e is an edge corresponding to a point-line distance constraint in E_3 (number 3 above) on bodies B_i and B_j, then $L_3(e)$ is an element of $\mathbb{R}^2 \times (\mathbb{R}^2 \times \mathbb{R})$ that assigns a point $p_i \in \mathbb{R}^2$ to the body B_i and a line $\ell = (\vec{\ell}, d) \in \mathbb{R}^2 \times \mathbb{R}$, to the body B_j. Figure 6.3 shows a body-cad framework with its associated cad graph.

Now we are not concerned with the realization of these constraints in the plane. Therefore we will need to work with a 'finer' multigraph that represents these constraints. These will be called 'primitive cad graphs'. The next section will show why this is needed.

6.3 Rigidity matrix of body-cad constraints in the plane

Suppose $B_1, B_2, \ldots, B_n$ are rigid bodies in the plane. Let $\mathbf{c} = (c_1, \ldots, c_n) \in (\mathbb{R}^3)^n$ is an assignment of a center $c_i \in \mathbb{R}^3$ of infinitesimal rigid motion in $\mathbb{E}^2$ to each body B_i, $1 \le i \le n$. Then c is an infinitesimal motion of a body-cad framework if c_i and c_j infinitesimally respect the body-cad constraints on the pair of bodies B_i and B_j, $1 \le i, j \le n$. An infinitesimal motion $\mathbf{c} = (c_1, \ldots, c_n)$ is called *trivial* if $c_i = c_j$, for all $1 \le i, j \le n$. A body-cad framework is *infinitesimally rigid* if every infinitesimal motion is trivial; otherwise, it is *infinitesimally flexible*.

A rigid body in the plane has three degrees of freedom, one of which is rotational and two are translational. An angular constraint restricts one degree of freedom on 2 bodies while a line-line distance constraint will reduce two degrees of freedom. A *primitive* constraint on a pair of bodies is the one that may reduce at most one degree of freedom. Therefore a line-line constraint consists of 2 primitive constraints one of which is angular and the other is non-angular (see the next section).

The *rigidity matrix* R for a body-cad framework has 3 columns for each body and there is one row corresponding to each primitive constraint. The rigidity matrix is the coefficient matrix of the linear system of infinitesimal cad constraints (see below). Just like the rigidity matrix of body-bar frameworks, a cad rigidity matrix captures the infinitesimal rigidity of a cad system. The centers of infinitesimal motions of bodies form the kernel of the rigidity matrix, as explained in Section 2.7.2.

Now we start to find the algebraic expression of body-cad constraints and the corresponding rigidity matrix.

Let's start with an *angular constraint* between a pair of lines. Suppose lines ℓ_i and ℓ_j, each rigidly affixed to bodies B_i and B_j respectively, undergo the infinitesimal motions with the centers c_i and c_j respectively. This angular constraint is infinitesimally maintained if and only if the relative infinitesimal motion $c_i - c_j$ is a translation:

$$c_i - c_j \in H^1_\infty, \tag{6.3.0.1}$$

where H^1_∞ is the line at infinity in $\mathbb{P}^2$. Equivalently, the last component of $c_i - c_j$ as a 3-vector must be zero.

Therefore an angular constraint corresponds to the following single row in the rigidity matrix of the associated body-cad system.

$\cdots$	Body B_i			$\cdots$	Body B_j			$\ldots$
$\cdots 0 \cdots$	0	0	1	$\cdots 0 \cdots$	0	0	-1	$\cdots 0 \cdots$

From algebraic and combinatorial (and later geometric, as we understand more) points of view, the pattern of generic zeros appearing in the rows corresponding to an angular constraint is very important. To emphasize this distinction, they are coloured red in contrast to the entries that are generically non-zero which are coloured grey, similar to [25]. We call the rows corresponding to angular constraints in the rigidity matrix *red rows* and others *black rows*. The red rows do not depend on the coordinates of the lines and they have a fixed format. Moreover, we do not need to treat angular constraints for parallel and non-parallel lines separately because the algebraic constraints are identical and geometrically equivalent in the plane. But this is not the case for angle constraints in 3-space because non-parallel angular constraints reduce one degree of freedom while parallel angular constraints reduce two degrees of freedom in the space (see [25]).

The remaining coincidence and distance constraints (*non-angular*[1] constraints) are reduced to some combination of angular and distance constraints as we will see below. There are 3 different distance constraints:

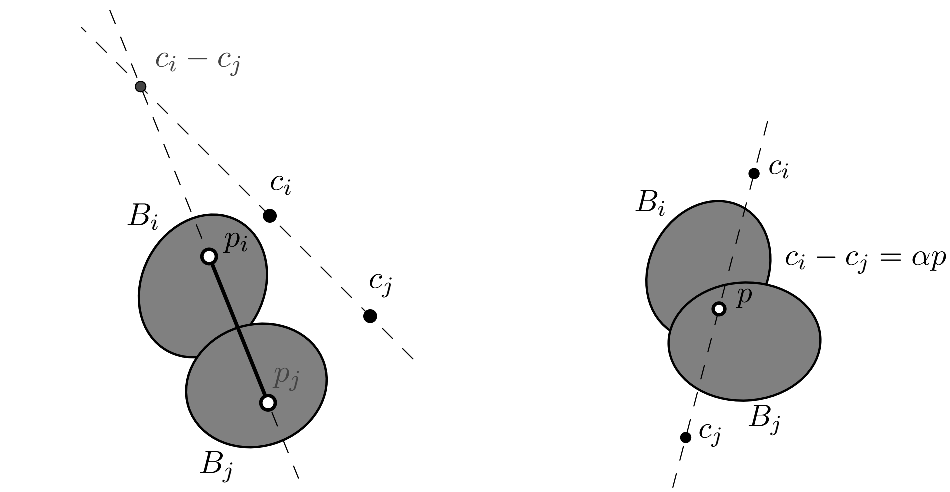
- Point-point distance,
- Point-line distance,
- Line-line distance.

Let's consider a point-point distance constraint. If two points p_i and p_j attached to bodies B_i and B_j undergo infinitesimal motions with the centers c_i and c_j, respectively we have to distinguish two cases here:

1. Two points p_i and p_j are distinct and we want to preserve the non-zero distance between them. The distance between the points p_i and p_j is infinitesimally preserved if the direction of the relative velocity $c_i - c_j$ at p_i is perpendicular to the line $p_i \vee p_j$ (figure 6.4a) which means the center of the rotation $c_i - c_j$ should lie on the line $p_i \vee p_j$:

$$(c_i - c_j) \vee (p_i \vee p_j) = \langle p_i \vee p_j, c_i - c_j \rangle = 0. \tag{6.3.0.2}$$

[1] These constraints are called *blind* in [25].



(a) The centre of the relative rotation $c_i - c_j$ must lie on the line $p_i \vee p_j$.

(b) The center of the relative motion $c_i - c_j$ must be at point p.

Figure 6.4: Point-point constraints in the plane.

As a result, this constraint corresponds to one row of the rigidity matrix:

$\cdots$	Body B_i	$\cdots$	Body B_j	$\cdots$
$\cdots 0 \cdots$	$p_i \vee p_j$	$\cdots 0 \cdots$	$-p_i \vee p_j$	$\cdots 0 \cdots$

2. If $p_i = p_j$, then the distance constraint is a coincidence constraint (Figure 6.4b). A point-point coincidence constraint is infinitesimally maintained if the relative velocity at p induced by the motion $c_i - c_j$ is zero. Equivalently, we should have

$$(c_i - c_j) \vee p = 0, \tag{6.3.0.3}$$

by (6.1.0.1). Note that (6.3.0.3) can be equivalently written as $c_1 - c_2 = \alpha p$, for some scalar $\alpha \in \mathbb{R}$. By (6.1.0.2), this holds if and only if the following two equations are simultaneously satisfied:

$$(c_i - c_j) \vee p \vee (1, 0, 0) = 0,$$
$$(c_i - c_j) \vee p \vee (0, 1, 0) = 0.$$

These equations can be written as follows:

$$\langle c_i - c_j, p \vee (1,0,0) \rangle = 0, \tag{6.3.0.4}$$

$$\langle c_i - c_j, p \vee (0,1,0) \rangle = 0. \tag{6.3.0.5}$$

Thus a point-point coincidence constraint corresponds to 2 rows in the rigidity matrix:

$\cdots$	Body B_i	$\cdots$	Body B_j	$\ldots$
$\cdots 0 \cdots$	$p \vee (1,0,0)$	$\cdots 0 \cdots$	$p \vee (-1,0,0)$	$\cdots 0 \cdots$
$\cdots 0 \cdots$	$p \vee (0,1,0)$	$\cdots 0 \cdots$	$p \vee (0,-1,0)$	$\cdots 0 \cdots$

We now consider a point-line distance constraint. A point p affixed to body B_i is constrained to maintain a fixed distance from a line $\ell = (\vec{\ell}, d)$ affixed to a body B_j. The constraint is preserved if the relative velocity of p lies in the same direction as $\vec{\ell}$ (Figure 6.5a). Therefore the velocity at p must be perpendicular to $\vec{\ell}$, i.e.,

$$(c_i - c_j) \vee p \vee (\vec{\ell}, 0) = 0,$$

or

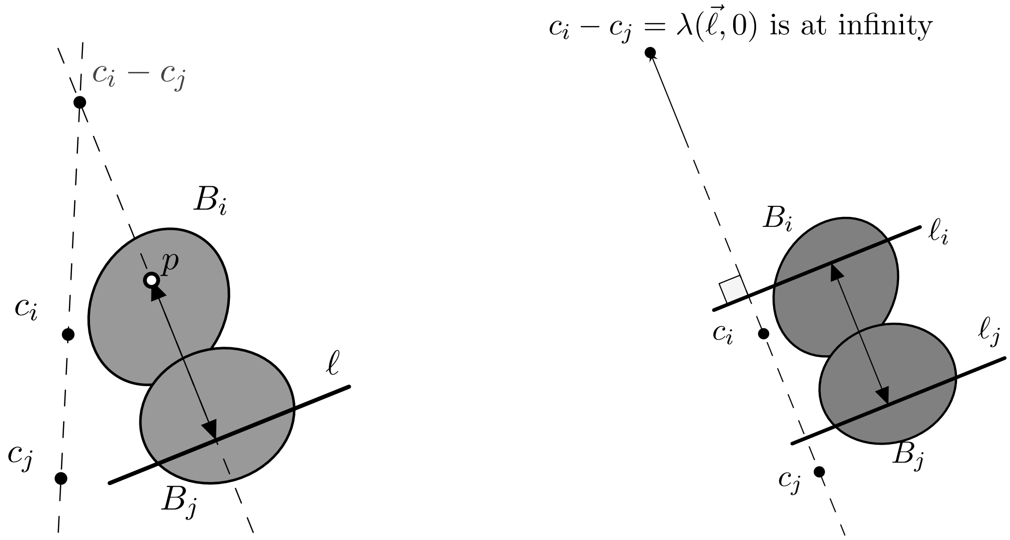
$$\langle c_i - c_j, p \vee (\vec{\ell}, 0) \rangle = 0, \tag{6.3.0.6}$$

using (6.1.0.1) and (6.1.0.2).

Hence a point-line distance constraint corresponds to one row in the rigidity matrix:

$\cdots$	Body B_i	$\cdots$	Body B_j	$\cdots$
$\cdots 0 \cdots$	$p \vee (\vec{\ell}, 0)$	$\cdots 0 \cdots$	$-p \vee (\vec{\ell}, 0)$	$\cdots 0 \cdots$

Note that $(\vec{\ell}, 0)$ is a point at infinity (representing the vector $\vec{\ell}$ in the Euclidean plane). So a point-line distance constraint becomes a point-point distance constraint



(a) The center of the relative motion $c_i - c_j$ is in direction of $\vec{\ell}$ on the line $c_i \vee c_j$.

(b) The center of the relative motion $c_i - c_j$ is in direction of $\vec{\ell}$ on the line $c_i \vee c_j$ at infinity.

Figure 6.5: Point-line and line-line distance constraints.

(compare it with (6.3.0.2)) with one finite point and a point at infinity, which is determined by the orientation of the line.

It is also important to note that the point p can be equivalently realized on the line ℓ. This is equivalent to multiplying $p \vee (\vec{\ell}, 0)$ by a non-zero scalar, which does not alter the generic rigidity of the system. Recall that for point-line frameworks in the plane this may create singularity and violate the generic rigidity of the framework.

Now we consider line-line distance constraints. A pair of parallel lines $\ell_i = (\vec{\ell}, d_i)$ and $\ell_j = (\vec{\ell}, d_j)$ affixed to bodies B_i and B_j respectively, are constrained to have a fixed distance between them. To this end, we place an angular constraint on the lines ℓ_i and ℓ_j to preserve the parallelism i.e., the relative motion $c_i - c_j$ has to be a translation. Then we require the relative velocity at an arbitrary point p on ℓ_i (or

ℓ_j) to be perpendicular to $\vec{\ell}$. This will hold if and only if

$$c_i - c_j = \lambda(\vec{\ell}, 0), \tag{6.3.0.7}$$

for some $0 \neq \lambda \in \mathbb{R}$ (see Figure 6.5b). In fact, (6.3.0.7) can be equivalently expressed by the following two conditions:

$$c_i - c_j = \text{is a translation}, \tag{6.3.0.8}$$

$$\langle c_i - c_j, (\vec{\ell}^{\perp}, 0)\rangle = 0, \tag{6.3.0.9}$$

where $\vec{\ell}^{\perp}$ is $90°$ counterclockwise rotation of $\vec{\ell} \in \mathbb{R}^2$ in the plane. Therefore, the two corresponding rows in the rigidity matrix are:

$\cdots$	Body B_i			$\cdots$	Body B_j			$\cdots$
$\cdots 0 \cdots$	0	0	1	$\cdots 0 \cdots$	0	0	-1	$\cdots 0 \cdots$
$\cdots 0 \cdots$	$\vec{\ell}^{\perp}$		0	$\cdots 0 \cdots$	$-\vec{\ell}^{\perp}$		0	$\cdots 0 \cdots$

By (6.3.0.7), a line-line distance constraint is a point-point coincidence constraint where the coincidence is at infinity. One can see this by comparing (6.3.0.7) with (6.3.0.3).

Note that for line-line constraints, the two lines may be coincident. This will neither change the constraint equations above nor the generic rigidity of the system because the entries of the matrix in the corresponding rows remain the same for this special position of lines. Also, we do not need to pay attention to the zeros in the second row above as they are always paired with a red edge.

Now we have given the algebraic expression of body-cad constraints in the plane. As we noticed the angular constraints are easy to identify in the rigidity matrix because of the pattern of zeros in the corresponding rows. This pattern combinatorially distinguishes them from other constraints.

We have now developed the infinitesimal theory for body-cad rigidity in the plane. The following table summarizes cad constraints in terms of the number of the primitive angular and non-angular constraints for each cad constraints.

Table 6.1: The number of angular and non-angular equations in each body-cad constraint

	Point		**Line**	
	angular	non-angular	angular	non-angular
Point				
distance	0	1	0	1
coincidence	0	2	0	1
Line				
fixed angular			1	0
distance			1	1

6.4 Combinatorics

In this section we give a combinatorial characterization of the generic minimal rigidity of cad graphs in the plane.

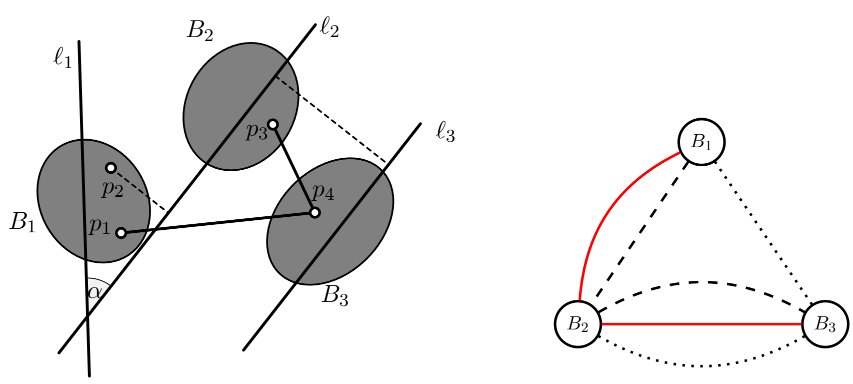
Associated to a cad graph $(G = (V, E), c)$, there is a bicolored (red and black edges) multigraph $H = (V, B \cup R)$ obtained by assigning vertices to bodies and one edge to each primitive cad constraint on bodies i and j such that the primitive angular and non-angular constraints each correspond to two disjoint edge sets R and B, respectively. This bicolored graph $H = (V, E = B \cup R)$ is called the *primitive cad graph*. For example, the primitive cad graph of the framework in Figure 6.6a is illustrated in Figure 6.6b.

The rigidity matrix of a body-cad framework in the plane with a primitive cad graph $H = (V, E = B \cup R)$ is a $|E| \times 3v$ matrix which has a row for every primitive cad constraints, as described in the previous section. By the definition, a body-cad framework in the plane is *infinitesimally minimally rigid* if and only if its rigidity matrix has the rank $3(v-1)$.

A body-cad framework is called *generically rigid* if it is infinitesimally rigid for some families of functions $L_1, \ldots, L_5$ that assign the coordinates of points or lines to the bodies corresponding to each geometric constraint (listed 1 to 5 in Section 6.2).

A body-cad framework is called *generically minimally rigid* if it is generically rigid and the removal of any primitive constraint from the cad system (or an edge from the associated primitive cad graph) would result in an infinitesimally flexible framework. Equivalently, the rank of the cad rigidity matrix is $3(v-1)$.

In the context of point-line frameworks in the plane, we saw that the independence of angle constraints on lines is equivalent to the independence of the corresponding edges in the cycle matroid of the subgraph containing those edges. This followed from the pattern of $0, 1 \ldots 0, -1$ in the line-line edges in the rigidity matrix (3.5.0.1). We had a similar pattern of 0's and 1's for angular constraints in the cad rigidity matrix (red rows). This means that, given a primitive cad graph $H = (V, E = B \cup R)$, a set of red edges $R' \subseteq R$ is independent if and only if R' is a



(a) A body-cad framework in the plane.

(b) A primitive cad graph.

Figure 6.6: A body-cad framework and its primitive cad graph.

forest on the vertex set $V(R')$.

However, if we assume there is any non-angular constraint in a body-cad structure then it is possible that they induce some angular constraint (red edges) in the structure which makes the system dependent. The easiest example occurs when two rigid bodies are linked by two parallel bars, see Figure 6.7a. The two parallel bars linking bodies B_i and B_j have the coordinates of two parallel lines $\ell_1 = (d_1, d_2, d_3)$ and $\ell_2 = (d_1, d_2, d_3')$ in the plane up to a non-zero scalar multiple. Hence they induce a red edge on the bodies B_i and B_j since the red edge is a non-trivial linear combination of the two black rows:

$$
\begin{array}{c}
\begin{array}{ccccccccccc}
 & & & B_i & & & & B_j & & & \\
\end{array} \\
\begin{pmatrix}
 & & & & & \vdots & & & & & \\
0 & \cdots & 0 & (d_1, d_2, d_3) & 0 & \cdots & 0 & -(d_1, d_2, d_3) & 0 & \cdots & 0 \\
0 & \cdots & 0 & (d_1, d_2, d_3') & 0 & \cdots & 0 & -(d_1, d_2, d_3') & 0 & \cdots & 0 \\
0 & \cdots & 0 & (0, 0, 1) & 0 & \cdots & 0 & -(0, 0, 1) & 0 & \cdots & 0 \\
 & & & & & \vdots & & & & &
\end{pmatrix},
\end{array}
$$

So two parallel edges (distance constraints) with an angular constraint together form a dependent set of constraints on two bodies. We have a similar situation in the context of body-bar frameworks: if the lines containing three bars linking two bodies are concurrent or parallel then the system is dependent. Two parallel bars in 3-space and higher dimensions induce a red edge as well. A linear combination of two parallel lines (bars) in 3-space is a red bar, which is a line at infinity in $\mathbb{P}^3$. Note that a red bar in 3-space is characterized by its first three zero coordinates (see [25]).

Figure 6.7b shows another example of black edges producing red edges, which makes the entire system dependent. The black edges (bars) on the pairs of bodies B_1, B_2 and B_1, B_3 induce a bar (the dotted line) on the bodies B_2 and B_3 which happens to be parallel to the single bar that already exists on these bodies because of the mirror symmetry. These constraints induce an angular constraint on B_1 and

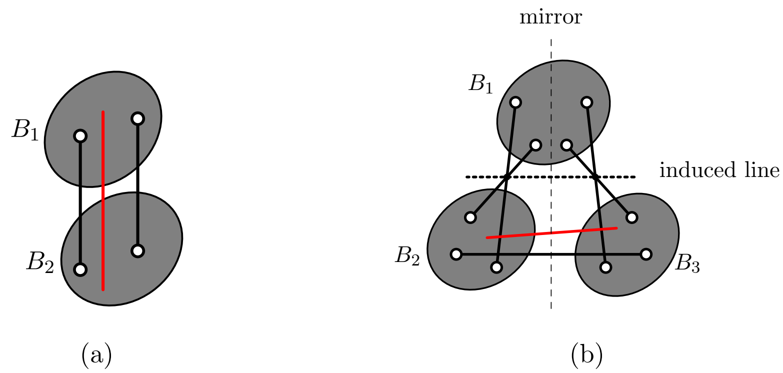


Figure 6.7: Dependent body-cad frameworks in the plane.

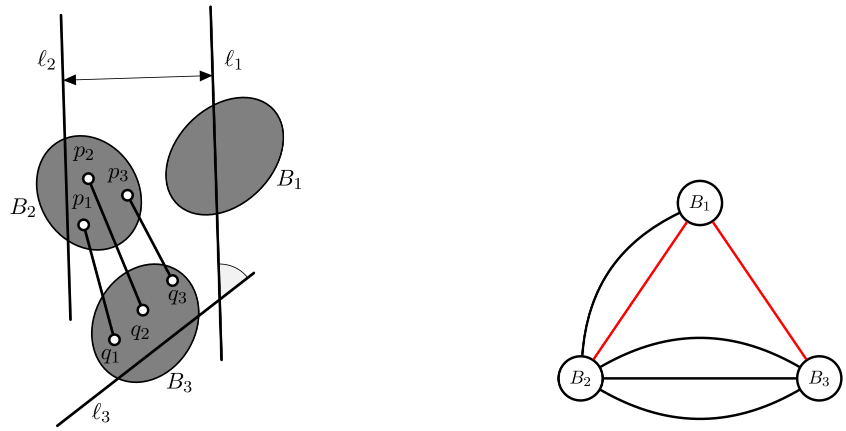
B_3, which results in a dependent system.

We have a necessary condition for a generically independent body-cad system:

Proposition 6.4.1. *A body-cad framework in the plane with the associated multigraph $H = (V, E = B \cup R)$ is generically minimally rigid only if $|E| = 3v - 3$ with $|R| \leq v - 1$ and $|R'| \leq |V(R')| - 1$, for non-empty subsets $E' \subseteq E$ and $R' \subseteq R$.*

Proof. A body-cad framework is minimally infinitesimally rigid if the rank of the rigidity matrix is $3v - 3$. For every non-empty subset $E' \subseteq E$, the corresponding rows in the cad rigidity matrix will be trivially satisfied by the 3-dimensional space of infinitesimal rigid motions so $|E'| \leq 3|V(E')| - 3$. For any subset of red edges $R' \subseteq R$ in E, consider the corresponding rows in the cad rigidity matrix and the last column under each vertex in $V(R')$. This submatrix has a one dimensional space of the rotational rigid motions in its kernel. This implies $|R'| \leq |V(R')| - 1$, for every $\phi \neq R' \subseteq R$. $\square$

The conditions in the above proposition are not enough to detect independence in the cad rigidity matroid. Figure 6.8 provides an example of a body-cad framework satisfying the counting condition in Proposition 6.8 but it is not generically minimally rigid because B_1 may translate independently in the system. This example shows that the edge set in the graph is dependent.



(a) A flexible body-cad framework in the plane.

(b) The non-rigid primitive cad graph of the framework in (a).

Figure 6.8: A flexible body-cad framework satisfying the counting condition in Proposition 6.4.1.

In order to give a combinatorial characterization of the generic rigidity of a body-cad framework we follow the technique of White and Whiteley in [64] used for body-bar frameworks to give a combinatorial criterion for their generic rigidity. Based on this technique used for k-frames, (k, g)-frames for any integer $g, k, g < k$ and a combinatorial characterization of their rigidity is also given in [38] by A. Lee-St.John and J. Sidman.

This technique constructs an abstract model (called frame) of the pattern of the entries in the body-bar rigidity matrix in $\mathbb{R}^n$ and gives the combinatorial characteristic of the generic independence in the rigidity matrix.

A $(3,1)$-*frame* $H(\mathbf{p})$ is a bicolored graph $H = (V, B \cup R)$ with a function $\mathbf{p}$ that assigns a vector in $\mathbb{R}^3$ to every edge such that the first 2 components of these vectors are zero for red edges (see [38]). Associated to each $(3,1)$-frame is a matrix $M(H(\mathbf{p}))$ with e rows and $3v$ columns so that for a row e, $\mathbf{p}(e)$ is in the 3 columns under i and $-\mathbf{p}(e)$ is in the 3 columns under j while the rest of the entries in row

e are zero. A $(3,1)$-frame is a general model of a typical cad rigidity matrix in the plane. A $(3,1)$-frame is a special case of a 3-frame in [64] when the first two components of $\mathbf{p}(e)$ are zero if $e \in R$.

A $(3,1)$-frame $H(\mathbf{p})$ is *minimally rigid* if it becomes flexible after the removal of any edge from H. Every minimally rigid $(3,1)$-frame must have exactly $3v-3$ rows since the trivial motions are obviously in the kernel of the rigidity matrix. For a similar reason we want the number of red edges to be at most $v-1$. The cad rigidity matrix is a $(3v-3)\times 3v$ matrix, which is not square. Again following [64], we append a $3 \times 3v$ matrix $T(3)$ with the identity matrix in the first 3 columns and 0's everywhere else to the bottom of the cad rigidity matrix to form the *basic tie-down* matrix (Figure 6.9). This augmented $3v \times 3v$ matrix is denoted by $M_T(H(\mathbf{p}))$. A $(3,1)$-frame $H(\mathbf{p})$ is called $(3,1)$-*counted* if $|E| = 3(v-1)$.

N. White and W. Whiteley showed that a frame is minimally rigid if and only if its tie-down matrix has a non-zero determinant (see [64, Proposition 2.5.]).

As a consequence of their result, we have,

Proposition 6.4.2. *A* $(3,1)$-*counted* $(3,1)$-*frame* $H(\mathbf{p})$ *is minimally rigid if and only if det* $M_T(H(\mathbf{p})) \neq 0$.

Our proof below follows the same idea as used in [64] and later used in [38] for (k,g)-frames[2] to give a combinatorial description of the independence of the constraints on rigid bodies. But the proof for the planar cad constraints reveals some geometrically interesting results that are true only in the plane. Among them is the fact that point-point coincidence constraints are included in the pattern of $(3,1)$-frames, without violating the combinatorial characterization.

Theorem 6.4.3. *A body-cad framework in the plane with the primitive cad graph* $H = (V, R \cup B)$ *is generically minimally rigid if and only if there is a set of black*

[2]This was motivated by cad constraints in the space (without point-point coincidence constraints) which is a $(6,3)$-frame.

$$
\begin{array}{r|c|c}
 & v_1 & v_2 \quad \cdots \quad v_n \\
\hline
(3n-3)\text{ rows}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. & M_{(3n-3)\times 3} & A_{(3n-3)\times(3n-3)} \\
\hdashline
3\text{ rows}\left\{\begin{array}{c}\\ \end{array}\right. & I_3 & 0_{3\times(3n-3)}
\end{array}
$$

Figure 6.9: The basic tie-down matrix $M_T(H)$.

edges $B' \subseteq B$ such that:

(a) *$R \cup B'$ forms a spanning tree, and*

(b) *$B \backslash B'$ is the edge-disjoint union of 2 spanning trees.*

Proof. ($\rightarrow$) If $H = (V, R \cup B)$ is a minimally rigid cad graph then det $M_T(H(\mathbf{p})) \neq 0$, for some $\mathbf{p}$. Consider a Laplace expansion along the last 3 rows of $M_T(H(\mathbf{p}))$. Since the only non-zero entries in these rows happen in the first 3×3 identity matrix appearing in the tie-down in the first 3 columns, $\det M_T(H(\mathbf{p})) = \det A$, where A is the submatrix of $M_T(H(\mathbf{p}))$ formed by the first $3n-3$ rows and the last $3n-3$ columns (Figure 6.9). So $\det A \neq 0$. Now consider a Laplace expansion of $\det A$ using $(v-1) \times (v-1)$-minors with respect to first columns, second columns and third columns under each vertex i, $i = 2, \ldots, v$. So $\det A$ is a sum of the terms of form $\det A_1 . \det A_2 . \det A_3$ where for $j = 1, 2, 3$, A_j is an $(v-1) \times (v-1)$ submatrix of A with the jth column under each of the $v-1$ remaining vertices (the first vertex is tie-down) and some choices of $v-1$ rows. Since det $A \neq 0$ there is a non-zero term in this expansion, meaning $\det A_1 . \det A_2 . \det A_3 \neq 0$ for some $(v-1) \times (v-1)$ minors A_1, A_2, A_3. This implies that $\det A_1$, $\det A_2$, $\det A_3$ are all non-zero. Each submatrix A_j has one column per vertex and the rows of A_j are just the rows of the incidence matrix of an oriented graph on the edges of H multiplied by non-zero scalars. Since det A_j is non-zero for $j = 1, 2, 3$, then A_j can be regarded as the incidence matrix of a subgraph of H with $v-1$ edges and no cycles on v vertices. Thus A_j describes a spanning tree T_j of H. Therefore A_1, A_2, A_3 provides an edge-disjoint decomposition of E into 3 spanning trees. It is clear that red edges must fall into A_3 because the first two coordinates is always zero. This completes one direction of the proof.

($\leftarrow$) Let H be the generic $(3,1)$-frame, where $H = (V, B \cup R)$ is a $(3,1)$-counted graph satisfying conditions (a) and (b) above. We will show that det $M_T(H)$ is not identically zero, which means there exists a configuration $\mathbf{p}$ for which det $M_T(H(\mathbf{p})) \neq 0$.

The idea is to provide a special $\mathbf{p}$ that decomposes det $M_T(H)(\mathbf{p})$ into blocks A_1, A_2, A_3, each of which corresponds to one of the trees (Figure 6.10) so that the submatrices A_1, A_2 and A_3 (in the manner mentioned above) become the incidence matrices of the given trees. Then it is clear that this particular term detA_1.detA_2.detA_3 in the Laplace expansion of $M_T(H)$ is not zero because A_js are incidence matrices of some edge-disjoint spanning trees. To do this, we show that we can write all the rows corresponding to the edges in tree T_1 so that they look like a non-zero scalar multiple of

$\cdots 0 \cdots$	1	0	0	$\cdots 0 \cdots$	-1	0	0	$\cdots 0 \cdots$

and all the rows corresponding to the edges in tree T_2 so that they look like a non-zero scalar multiple of

$\cdots 0 \cdots$	x	1	0	$\cdots 0 \cdots$	-x	-1	0	$\cdots 0 \cdots$

(x is arbitrary) and all the edges in tree T_3 can be made look like a non-zero scalar multiple of

$\cdots 0 \cdots$	y	z	1	$\cdots 0 \cdots$	$-y$	$-z$	-1	$\cdots 0 \cdots$

where y, z could be zero or nonzero but the third coordinate should be non-zero. If we show all this, then we can reorder and recover the rigidity matrix corresponding to the multigraph $G = T_1 \cup T_2 \cup T_3$ as follows: root every tree at a vertex, say vertex 1. There is a matrix associated to G whose first $n-1$ rows are constraints (all in their new form as above) corresponding to the edges of T_1 and second $n-1$ rows are constraints corresponding to the edges of T_2 and third $n-1$ rows are constraints corresponding to the edges of T_3. We regard vertex 1 as the tie-down vertex in this matrix. So we have recovered a matrix $M_T(G)$ similar to the one in the first part of the proof and a submatrix A of it exactly as explained before. Now reorder the columns of the submatrix A so that its first $n-1$ columns are the first columns under each vertex $2, \ldots, n$ and its second $n-1$ columns are the second columns under each vertex $2, \ldots, n$ and its third $n-1$ columns are the third columns under each vertex $2, \ldots, n$. According to the new form designed for the constraints the reordered matrix A will look like as in Figure 6.10. The determinant of A is $\det A_1 . \det A_2 . \det A_3$ which is nonzero and it is the only non-zero term in the Laplace expansion of A because if you pick the rows in any arrangement other than the one described above we will get a zero row in at least one of the minors located in the place of A_i in Figure 6.10 and then the whole term would be zero.

Now if we look at the equations for all kinds of constraints we will recognize that we can achieve the new form described before for cad constraints.

All edges corresponding to angular constraints can only belong to the "red tree" T_3 because they all are of the form $(0, 0, 1)$.

In point-point coincidence case, the expressions are of the form

$$(p_1, p_2, 1) \vee (1, 0, 0) = (0, 1, -p_2),$$

or

$$(p_1, p_2, 1) \vee (0, 1, 0) = (1, 0, -p_1).$$

The first row can be a member of T_2 or T_3 if we choose the point $p = (0, 0, 1)$

$$
\left.\begin{array}{c:c:c}
A_1 & \mathbf{0}_{(\mathbf{n-1})\times(\mathbf{n-1})} & \mathbf{0}_{(\mathbf{n-1})\times(\mathbf{n-1})} \\
\hdashline
X_{(n-1)\times(n-1)} & A_2 & \mathbf{0}_{(\mathbf{n-1})\times(\mathbf{n-1})} \\
\hdashline
Y_{(n-1)\times(n-1)} & Z_{(n-1)\times(n-1)} & A_3
\end{array}\right.
$$

Figure 6.10: The rearrangement of the rows and columns of the matrix A.

or $p = (0, 1, 1)$, respectively. The second one can be a member of T_1 if we choose $p = (0, 0, 1)$ or T_3 if we choose $p = (1, 0, 1)$.

For point-point distance case, $p \vee q$ could belong to T_1 if we choose $p = (0, 1, 1)$ and $q = (0, 0, 1)$ or it could belong to T_2 if we choose $p = (1, 0, 1)$ and $q = (0, 0, 1)$ or it could belong to T_3 if we choose $p = (1, 0, 1), q = (0, 1, 1)$.

For point-line distance with $\ell = (d_1, d_2, d_3)$, the expression is of the form

$$p \vee (d_1, d_2, 0) = (-d_2, d_1, p_1 d_2 - p_2 d_1),$$

which can be a member of T_1 if we choose $\ell = (0, 1, 0), p = (0, 0, 1)$ or T_2 if we choose $\ell = (1, 0, 0), p = (0, 0, 1)$ or T_3 if we choose $p = (1, 0, 1), \ell = (0, 1, 0)$.

For line-line distance case, one of the rows is a red row so it will belong to T_3. The second row is of the form $(-d_2, d_1, 0)$, which could be in T_1 if $\ell = (1, 0, 0)$ or in T_2 if $\ell = (0, 1, 0)$. This completes the proof. $\square$

Two examples of rigid cad graphs are shown in Figure 6.11. The graphs are decomposed into 3 edge-disjoint spanning trees whose edges are shown by solid lines, dashed lines and dotted lines. Black edges (dashed and dotted) form two edge-disjoint trees and three red edges along with the bolded black edge form the red

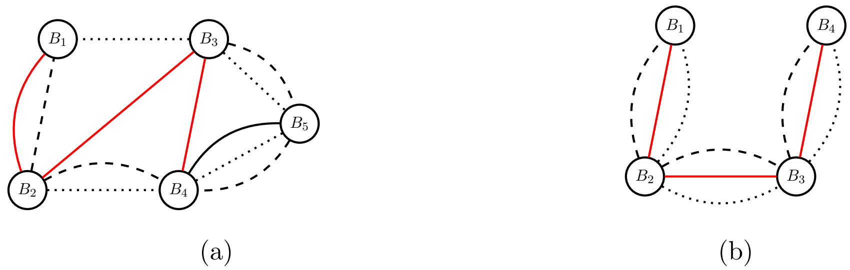


Figure 6.11: Minimally rigid body-cad primitive graphs.

tree.

In summary, the three points $(1, 0, 1), (0, 1, 1), (0, 0, 1)$ together with the two lines $(1, 0, 0), (0, 1, 0)$ and the line at infinity $(0, 0, 1)$ are enough to obtain an independent set of cad constraints.

Even though a point-point coincidence constraint in the plane fits into the combinatorial characterization of the generic rigidity of a cad system in the plane by Theorem 6.4.3, this constraint poses a major problem in 3-space if it is included in the system so that it has to be excluded for the list of the constraints that obey the 3D combinatorial criterion of the rigidity. A famous example of this is the double banana (Figure 6.14a), which is composed of two rigid bodies in space with two point-point coincidence constraints on them. The double banana is flexible because the two coincident points allow a hinge motion along the line connecting the two points. The combinatorial criterion in [38] fails to detect the flexibility of this structure.

The next section shows how the flexibility of a class of 3D structures with point-point coincidence constraints can be determined combinatorially using Theorem 6.4.3.

6.5 Special body-bar frameworks with point-point coincidence constraints in $\mathbb{E}^2$ and $\mathbb{E}^3$

The idea of using the projective invariance of the first-order rigidity of structures is the main theme of this section as well as the entire thesis. As usual, we use spheres to express this idea.

Rigid bodies on the sphere can be imagined as a collection of (infinitely many) points on $\mathbb{S}^2$ that are rigidly attached to each other. Infinitesimal rigid motions on $\mathbb{S}^2$ are the infinitesimal rigid motions in $\mathbb{E}^3$ that fix the origin i.e., the rigid motions of $\mathbb{E}^3$ whose axes are through the origin.

Hence infinitesimal rigid motions on sphere can be presented by a 3-vector $c = (c_1, c_2, c_3)$ in $\mathbb{R}^3$ where the center is the point $c/\|c\|$ on sphere and the induced motion at a point $p \in \mathbb{S}^2 \subseteq \mathbb{R}^3$ is $c \vee p$ as shown in Figure 2.4. So the center of the infinitesimal rigid motions are projected the center of the motion of a rigid body in $\mathbb{S}^2$. For points $p, q \in \mathbb{A}^2$, $p \vee q$ is a scalar multiple of $p/\|p\| \vee q/\|q\|$ for the points $p/\|p\|$ and $q/\|q\|$ on $\mathbb{S}^2$. Therefore, the infinitesimal rigidity of a body-bar framework in the plane is equivalent to that of the projected body-bar framework on the sphere because the rigidity matrix of one framework is obtained from the other by some scalar multiple of the rows.

In the case of body-cad structures in the plane where there are some lines involved, we can use our correspondence between points and lines (used in the previous chapters) to project lines to the points on the equator of $\mathbb{S}^2$. All the constraints involving lines in Section 6.3 show that a line could be replaced by a point at infinity.

Suppose $H(\mathbf{p})$ is a body-cad framework in the plane with v bodies where $\mathbf{p}$ is an assignment of points and lines to rigid bodies corresponding to each body-cad constraint in the plane. The projection of a point p to $p/\|p\|$ on the upper hemisphere and a line $\ell = (\vec{\ell}, d)$ to the point $(\vec{\ell}, 0)/\|\vec{\ell}\|$ on the equator gives rise to a body-bar framework on the sphere whose rigidity is equivalent to $H(\mathbf{p})$. We call this body-bar framework the *projection* of the body-cad framework $H(\mathbf{p})$ into the sphere. We denote it by $H(\hat{\mathbf{p}})$.

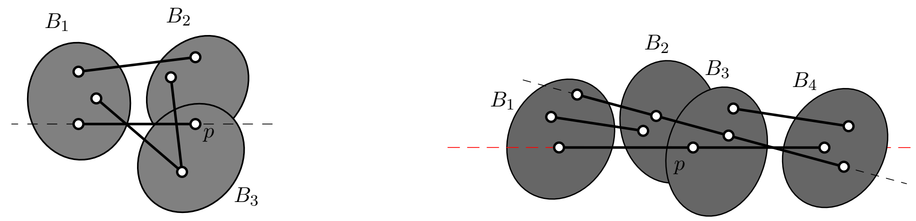
Note that point-point distance or coincidence constraints remain the same point-point distance or coincidence constraints on the projected points on the upper hemisphere under the central projection. A line-line angular constraint on two non-parallel lines $\ell_1 = (\vec{\ell_1}, d_1)$ and $\ell_1 = (\vec{\ell_2}, d_2)$ becomes a point-point distance constraint on the points $(\vec{\ell_1}, 0)/\|\vec{\ell_1}\|$ and $(\vec{\ell_2}, 0)/\|\vec{\ell_2}\|$, on the equator of $\mathbb{S}^2$ (because $(\vec{\ell_1}, 0)/\|\vec{\ell_1}\| \vee (\vec{\ell_2}, 0)/\|\vec{\ell_2}\|$ is a scalar multiple of $(0, 0, 1)$.) A planar point-line distance constraint is a point-point distance constraint on a point in $\mathbb{S}^2_+$ with a point on the equator on sphere after appropriate rescaling (see 6.3.0.6). By comparing (6.3.0.7) and (6.3.0.3), we see that a line-line distance constraint in the plane becomes a point-point coincidence constraint on the equator of the sphere after projection. This shows that $H(\hat{\mathbf{p}})$ is a body-bar framework with some point-point coincidence constraints. Under a rigid motion of the sphere, the configuration $\hat{\mathbf{p}}$ may be replaced by a new configuration $\hat{\mathbf{q}}$ of a body-bar framework in $\mathbb{S}^2_+$ with arbitrary point-point coincidence constraints in addition to a set of collinear point-point distance and point-point coincidence constraints. Projecting this spherical body-bar framework back to the plane, we obtain an equivalently infinitesimally rigid body-bar framework with point-point distance and coincidence constraints with collinear attachments.

This leads us to the following definition:

Definition 6.5.1. Given a body-bar framework in the plane with point-point coincidence constraints and a set of collinear point-point distance or coincidence constraints, its primitive bicolored multigraph $H = (V, R \cup B)$ is a multigraph with no loops whose edge set is partitioned into two subsets R and B such that

(a) there is an edge incident to the vertices i and j for every point-point distance constraint on the bodies B_i and B_j.

(b) there is a parallel edge incident to the vertices B_i and B_j per each point-point



(a) Bodies B_2 and B_3 are coincident at p.

(b) An infinitesimally rigid framework with collinear and coincidence constraint at p.

Figure 6.12: Planar body-bar frameworks with collinear bars.

coincidence constraint on the bodies B_i and B_j.

where R contains all collinear edges (point-point distance constraints), plus one edge (of the parallel edges) per each point-point coincidence constraint on the line containing the collinear edges.

By Theorem 6.4.3, we have

Theorem 6.5.1. *A body-bar framework with coincidence constraints, one set of collinear edges and coincidences in the plane whose primitive multigraph is* $H = (V, R \cup B)$ *is minimally rigid if and only if there is a set of edges* $B' \subseteq B$ *such that:*

(a) $R \cup B'$ *forms a spanning tree, and*

(b) $B \backslash B'$ *is the edge-disjoint union of 2 spanning trees.*

Example 6.5.1. Figure 6.12a shows a body-bar framework with a point-point coincidence constraint at the point p on bodies B_2 and B_3. The rigidity of this framework is equivalent to the rigidity of the body-cad framework shown in Figure 6.8a. Its primitive graph is the same as the graph in Figure 6.6b, which is infinitesimally rigid.

Figure 6.12b shows another infinitesimally rigid body-bar framework whose graph is illustrated in Figure 6.11b, with 2 collinear bars on 3 collinear points on the dashed line.

We conclude this chapter with an application of Theorem 6.5.1.

Given a body-bar framework in $\mathbb{E}^2$, it corresponds to a body-bar framework in $\mathbb{E}^3$ by coning[3] as follows: place the body-bar framework in the affine plane $x_3 = 1$ then connect every rigid body in the plane to the origin of $\mathbb{R}^3$ using three non-coplanar bars. This is a rigid body in the space. This body-bar framework is called the *coning body-bar framework* of a planar body-bar framework. These bodies share a single point in the space and are connected according to the multigraph of the corresponding body-bar framework in the plane.

The coning operation preserves the rigidity of body-bar frameworks. Up to a translation of the entire space, we can assume that the rigid motions for every rigid body is a pure rotation around an axis through the common point. So very motion of the 2D body-bar framework is a motion of the coning body-bar framework in a natural way. These frameworks may have a set of bars or coincidence constraints coplanar. A primitive multigraph for these frameworks is a bicolored multigraph $H = (G, R \cup B)$ whose edge-set is partitioned into two sets R and B where R contains the coplanar edges along with one edge incident to the pair of bodies with a coincidence constraint at a point on the plane and B contains the rest of the edges.

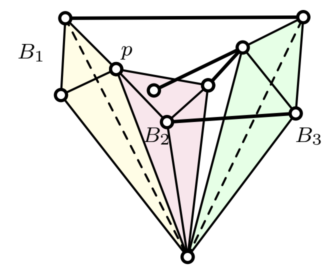
Theorem 6.5.2. *A coning body-bar framework in the space with point-point coincidences, some coplanar set of point-point coincidence constraints and bars with the primitive multigraph $H = (G, R \cup B)$ is minimally rigid in the space if and only if there is a set of edges $B' \subseteq B$ such that:*

(a) *$R \cup B'$ forms a spanning tree, and*

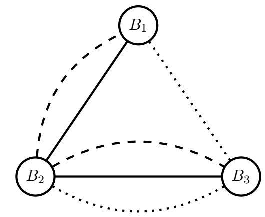
(b) *$B \backslash B'$ is the edge-disjoint union of 2 spanning trees.*

The famous *double banana* is an example of a coning body-bar framework with a single coincidence constraint which has one degree of freedom. Its primitive multi-

[3]The idea of coning has been used in the context of bar-joint frameworks in [13], [66], [54].



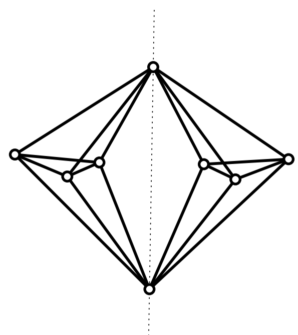
(a) A minimally rigid body-bar framework with coincidence constraints in the space.



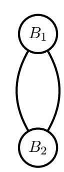
(b) The primitive multigraph of the framework in (a).

Figure 6.13: Rigid frameworks with a common point in $\mathbb{E}^3$.

graph has two vertices with parallel edges on them. This predicts one degree of freedom.



(a) Double banana: a flexible framework in 3-space.



(b) The primitive multigraph of Double Banana.

Figure 6.14: Flexible body-bar frameworks in 3-space.

Figure 6.13a shows three rigid bodies in the space with a common points. The point p is a coincidence constraint of two bodies and the primitive graph is shown in Figure 6.14b.

The results of this section are part of a core idea followed in this thesis. The projective invariance of the first-order rigidity allowed us to switch between different types of constraints and as a result, we could transfer our findings about one set of constraints into another set. Angular constraints provide a pattern of zeros that distinguish them from other constraints. This distinction helped us understand coincidence constraints for special cases in plane and space.

Understanding the combinatorics of point-point coincidences in 3D is more difficult. A plane-plane constraint in 3D has two red edges and one black (see [25]). Two plane-plane distance constraints on two bodies leave one degree of freedom for the bodies just like any two coincidence constraints on two bodies in the space. In the first case it is a translational motion (a rotation at infinity) and in the second case it is a rotational motion. Both types impose 6 primitive constraints on two bodies but in the case of plane-plane distance constraints, the combinatorial criterion (Theorem 2.7.2) is able to detect this flexibility while there is no criterion to detect the flexibility of the two point-point constraints.

We expect that there is connection between these two cases and we anticipate Theorem 2.7.2 can help us understand the point-point coincidence constraints in 3D for a larger class of frameworks (compared to Theorem 6.5.2). This could lie in the fact that a plane-plane distance constraint can be exchanged by a point-point coincidence constraint (and some others correspondingly). However this does not include arbitrary point-point coincidences in the space. To tackle the general case, we need a deeper understanding of the geometry of these constraints in the space.

Chapter 7

Future work

In this chapter we briefly outline some possible directions for the future work that is stimulated by the present work. The thesis provides a valuable simple geometric insight into the rigidity of point-hyperplane frameworks by linking this type of frameworks to bar-joint frameworks, as a special case. Based on this, it becomes natural and easier to apply the methods and techniques available to bar-joint frameworks to explore some fundamental questions about point-hyperplane frameworks.

7.1 Inductive constructions of isostatic point-line graphs

In general, inductive methods are of extreme importance in Rigidity Theory as many proofs and results rely on inductive constructions.

The first fundamental problem is to develop *inductive methods* for isostatic point-line graphs. As it was remarked in Chapter 3, Henneberg moves do not always preserve the minimal rigidity of point-line frameworks if they are simply applied to point-line frameworks as bar-joint frameworks. This is because the special geometry of point-line frameworks as was explained. At the first glance, removing a 3-valent vertex might require us to go deeper into the graph to find an edge to replace the

3-valent vertex. It is not clear to us how to definitely find the appropriate edge for substitution. This makes it harder to develop an inductive method because we need to be able to move backward in an inductive process.

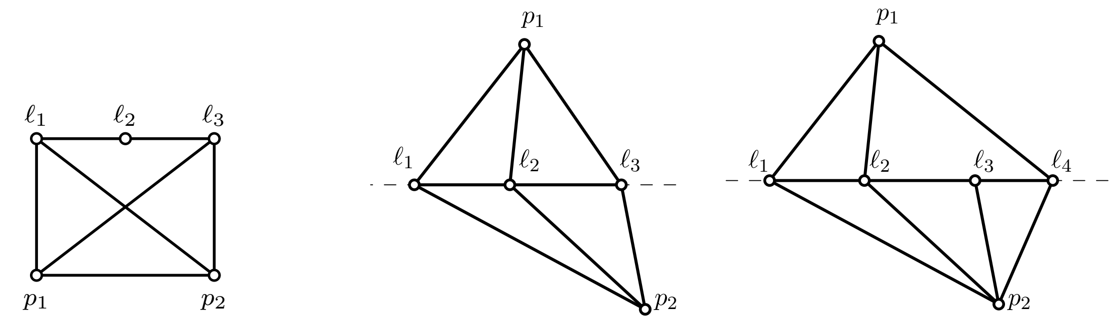
In the proof of the result that certifies the edge-split as an inductive move for bar-joint frameworks, the new 3-valent vertex splitting an edge is placed on the line connecting the two adjacent vertices. In general, there are bad positions [63] of the new vertex that makes an edge-split move fail for bar-joint frameworks. This poses a potential problem for an edge-split move in a point-line framework if we need to split an edge by a line-vertex, as we saw in Figure 3.6 . This is again due to the special geometry of a generic point-line configuration.

Another important step to take is to understand other inductive methods for point-lines such as X-replacement and vertex-split [71].

7.2 Global rigidity of point-line configurations

A fundamental geometric problem in Rigidity Theory is known as *Global Rigidity*, which is concerned with whether a configuration of points with a set of pairwise fixed distances has a unique realization (up to congruence) in $\mathbb{E}^n$. Namely, given a bar-joint framework $(G, \mathbf{p})$ in $\mathbb{E}^n$ if for any framework $(G, \mathbf{q})$ in $\mathbb{E}^n$ where $\|p_i - p_j\| = \|q_i - q_j\|$ for all $ij \in E$, $\mathbf{q}$ is congruent to $\mathbf{p}$ then $(G, \mathbf{p})$ is called globally rigid. It turns out that global rigidity is a generic property of a graph meaning, almost all realizations of a globally rigid graph are globally rigid frameworks. Usually, the term 'generic global rigidity' is used in this case. The generic global rigidity of bar-joint frameworks has found characterizations for generic configurations in $\mathbb{E}^n$. We refer the reader to [28], [31], [9], [22] to see layers of results towards the final characterization of generic global rigidity.

First, inductive proofs play an important role in the derivation of the results in global rigidity. Also, some critical techniques used to tackle global rigidity originated from the study of tensegrity frameworks (see [9] by Connelly).



(a) Not globally rigid point-line framework.

(b) A globally rigid point-line framework.

(c) Not globally rigid point-line framework.

Figure 7.1: Generic global rigidity of point-line frameworks.

It is natural to ask about global rigidity of point-hyperplane frameworks in $\mathbb{E}^n$. Is it a generic property of the graph? If yes, find a characterization of generic global rigidity of point-hyperplane configurations in $\mathbb{E}^n$. This is a major step towards the completion of the understanding of the global geometric rigidity of point-hyperplane graphs. Note that in the whole thesis we have considered generic point-hyperplane frameworks. This condition does not allow parallel hyperplanes and therefore, it rules out hyperplane-hyperplane distance constraints in the generic rigidity and generic global rigidity of point-hyperplane frameworks.

Of course, a class of global rigidity theorems assume genericity of the configurations. It should be studied how much these conditions could be relaxed in general, because generic point-hyperplane configurations are non-generic bar-joint configurations at infinitesimal level. The *stress matrix* technique by Connelly [9] has been extensively used to determine the global rigidity of bar-joint frameworks. The question is: can we applying this technique to the problem of the global rigidity of generic point-hyperplane frameworks? Do we need to extend this technique?

In the key paper [13], R. Connelly and W. Whiteley proved that a bar-joint graph is generically globally rigid in $\mathbb{E}^n$ if and only if it is generically globally rigid in $\mathbb{S}^n$. As the Pogorelov map has played a crucial role in derivation of the results in [13] and an extension of it in this thesis, the natural question is: can we extend the techniques in [13] to extract results on the global rigidity of generic point-hyperplane frameworks? Although the study of the global rigidity needs more sophisticated tools compared to the study of infinitesimal rigidity but *averaging and de-averaging* of bar-joint frameworks is used as a tool in [13] to relate these two concepts. We expect that it is useful to develop the analogous average and de-average point-hyperplane frameworks to get a better understanding of global rigidity of point-hyperplane frameworks.

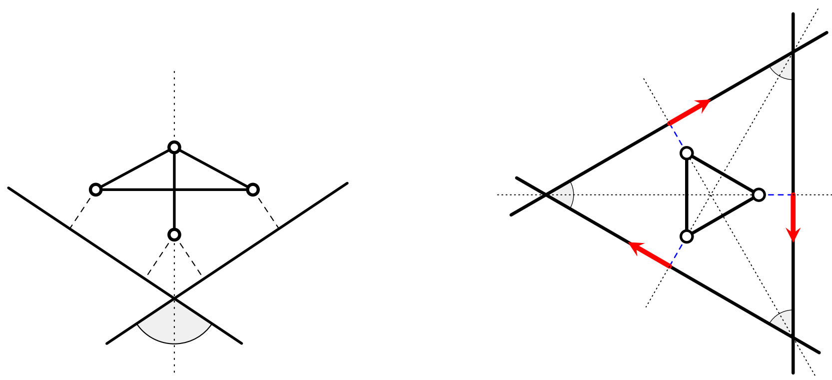
Regarding to the Spherical-Euclidean correspondence between point-hyperplane and bar-joint frameworks, we recall that different but equivalent point-hyperplane frameworks correspond to the same bar-joint framework in $\mathbb{S}^n$. This means the translation of hyperplanes is not directly detected by this correspondence as opposed to the relative rotational motions of hyperplanes that is directly reflected in its projective bar-joint framework in $\mathbb{S}^n$.

Figure 7.1a shows a graph that is globally rigid as a bar-joint graph but as a point-line graph it is not even rigid. In fact, it is flexible as line ℓ_2 may translate independently. Therefore, it can not be globally rigid.

The graph shown in Figure 7.1b is generically globally rigid as a bar-joint graph. But, it is generically infinitesimally rigid (and rigid) as a point-line graph. It is also globally rigid as a point-line graph.

Figure 7.1c shows an infinitesimally rigid point-line graph that is not globally rigid because line ℓ_3 can change its side with respect to the point p_2. This means ℓ_3 may change its orientation with respect to point p_2. On the other hand, the stress on the edge $\{p_2, \ell_3\}$ is zero. The graph is not generically globally rigid as a point-line graph.

As it is pointed out in [13] affine (and therefore, projective) transformations do not preserve global rigidity of bar-joint frameworks, in general. But there are



(a) A singular point-line configuration with mirror symmetry.

(b) A point-line configuration with 3-fold rotational symmetry.

Figure 7.2: Symmetry in point-line configurations.

situations in which projective transformations preserve global rigidity. This might also be helpful to transfer the results on global rigidity of bar-joint frameworks to point-hyperplane frameworks.

7.3 Symmetry and point-hyperplane frameworks

The idea of symmetry is fundamental in geometry and science. In addition, symmetric structures are common in our surrounding. Symmetric bar-joint frameworks have drawn a lot of attention in Geometric Rigidity Theory and been considered for various types of frameworks. Symmetry is usually viewed from two perspectives with respect to the motion of the framework: The configuration tolerate some kind of symmetry (such as mirror symmetry, rotational symmetry) at a moment but the symmetry might break during the motion (*incidental symmetry*). On the other hand, there are symmetric structures that maintain their symmetry during the motions (*forced symmetry*). B. Schulze and W. Whiteley [55] introduced the *orbit matrix* as a tool to study forced symmetric flexes of a bar-joint framework.

The concepts of incidence symmetric and forced-symmetric frameworks can also be considered for point-hyperplane frameworks in $\mathbb{E}^n$. As usual, the presence of symmetry in the incidence structure of a point-hyperplane framework may result in infinitesimal flexibility or even flexibility.

Figure 7.2a shows a point-line framework with mirror symmetry. The incidental symmetry causes singularity and therefore infinitesimal flexibility. But the framework remains rigid.

Figure 7.2b shows a point-line configuration in the plane with 3-fold rotational symmetry. It is infinitesimally flexible and flexible (see Chapter 3). The red vectors show the rotational motion of the lines relative to the triangle of points. The lines may symmetrically rotate around the three vertices of the triangle. This point-line framework corresponds to a bar-joint framework on the sphere with an equilateral triangle in $\mathbb{S}^2_+$ and a collinear triangle on the equator with rotation symmetry about the north pole. The motions shown in Figure 7.2b have correspondence for the spherical framework with rotational symmetry as well (see Chapter 3). Note that both frameworks have one degree of freedom and maintain the symmetry of the motion at any moment. The point-line configuration is regular point of the graph so this infinitesimal motion must lead to a finite motion module a rigid motion of the plane, which has rotational symmetry as well.

The Spherical-Euclidean correspondence of first-order flexes of point-hyperplane and bar-joint frameworks given in this thesis is a great help to understand the symmetric rigidity of point-line frameworks. This correspondence for symmetric bar-joint frameworks has been explored in [54] through conning. It remains to extend these connections to the context of point-hyperplane frameworks in $\mathbb{E}^n$.

Point-line motions are more complex in general. Considering that the symmetries of a point-hyperplane configuration are expected to respect point-vertices and hyperplane-vertices we have more restricted symmetries in the category of generic point-hyperplane frameworks. For example, in the plane, half-turn symmetry would require the existence of parallel lines which is a non-generic point-line configuration as far as we were concerned in this thesis.

Again this raises the significance of the study of *point-hyperplane frameworks with parallel hyperplanes*. These frameworks are common and natural. In addition, their infinitesimal rigidity is equivalent to the infinitesimal rigidity of bar-joint frameworks with coincident joints, as explained before. It sounds very natural to tackle the problem for easier cases such as fixed-normal hyperplanes before facing the general problem. Namely, point-hyperplane frameworks with parallel hyperplanes where the hyperplanes are restricted to maintain their normal, which is an important special case as well.

Bibliography

[1] A.D. Alexandrov. *Convex polyhedra*. GTI, Moscow, 1950. English translation: Springer, Berlin, 2005.

[2] L. Asimow and B. Roth. "The rigidity of graphs". *Trans. Amer. Math. Soc.* 245 (1978), pp. 279–289.

[3] L. Asimow and B. Roth. "The rigidity of graphs, II". *SIAM J. Discrete Math* 68 (1979), pp. 171–190.

[4] E.D. Bolker and B. Roth. "When is a bipartite graph a rigid framework". *Pacific J. Math.* 90 (1980), pp. 27–44.

[5] R. Bricard. "Mémoir sur la théorie de l'octaedre articulé". *J. Math. Pure and Appl.* 5.3 (1897), pp. 113–148.

[6] M.P. Do Carmo. *Differential Geometry of Curves and Surfaces*. Englewood Cliffs, NJ: Prentice-Hall, 1976.

[7] A.L. Cauchy. "Sur les polygones et les polyëdres". *XVIe Cahier* IX (1813), pp. 87–89.

[8] R. Connelly. "A counter example to the rigidity conjecture for polyhedra". *Inst. Haut. Étud. Sci. Publ. Math. Intelligencer* 47 (1978), pp. 333–335.

[9] R. Connelly. "Generic Global Rigidity". *Discrete Comput. Geom.* 33.4 (2005), pp. 549–563.

[10] R. Connelly. *The basic concepts of infinitesimal rigidity.* Draft chapter for The Geometry of Rigid Structures, Cornell University, Dept. of Math., White Hall, Ithaca, NY 14853, USA.: preprint.

[11] R. Connelly. *The basic concepts of static rigidity.* Draft chapter for The Geometry of Rigid Structures, Cornell University, Dept. of Math., White Hall, Ithaca, NY 14853, USA.: preprint.

[12] R. Connelly. "The rigidity of certain cabled frameworks and the second order rigidity of arbitrarily triangulated convex surfaces". *Adv. in Math* 37 (1980), pp. 271–299.

[13] R. Connelly and W. Whiteley. "Global Rigidity: The effect of conning". *Discrete Comput. Geom.* 43 (2010), pp. 717–735.

[14] R. Connelly and W. Whiteley. "Second-order rigidity and pre-stress stability for tensegrity frameworks". *SIAM J. Discrete Math.* 9 (1996), pp. 453–492.

[15] H. Crapo. *On the generic rigidity of structures in the plane.* Rocquencourt: Technical report, RR1278, INRIA, 1990.

[16] H. Crapo and W. Whiteley. "Statics of Frameworks and Motions of Panel Structures, a Projective Geometric Introduction". *Structural Topology* 6 (1982), pp. 43–82.

[17] Y. Eftekhari et al. "Point-hyperplane frameworks, slider joints and rigidity preserving transformations". *Submitted* (2017).

[18] L. Euler. "Opera Postuma I". *Petropoli* (1862), pp. 494–496.

[19] Z. Fekete, T. Jordán, and V.E. Kaszanitzky. "Rigid two-dimensional frameworks with two coincident points". *Graphs and Combinatorics* 31 (2015), pp. 585–599.

[20] A. Frank. "Connections in combinatorial optimization, Oxford Lecture Series in Mathematics and its Applications". *Oxford University Press* (2011).

[21] H. Gluck. "Almost all simply connected closed surfaces are rigid". *Geometric Topology, Lecture Notes in Math., no. 438, Springer-Verlag, Berlin* (1975), pp. 225–239.

[22] S.J. Gortler, A.D. Healy, and D.P. Thurston. "Characterizing Generic Global Rigidity". *Amer. J. Math.* 4.132 (2010), pp. 897–939.

[23] J. Graver, B. Servatius, and H. Servatius. *Combinatorial Rigidity*. Providence RI: Graduate Studies in Mathematics, AMS, 1993.

[24] C. Gunn. *Geometry, Kinematics, and Rigid Body Mechanics in Cayley-Klein Geometries*. PhD Thesis, Universitätsbibliothek, 2011.

[25] K. Haller et al. "Body-and-cad geometric constraint system". *Computational geometry: Theory and applications* 45 (1981), pp. 385–405.

[26] G. Hegedüs, J. Schicho, and H-P. Schröcker. "Factorization of rational curves in the study quadric". *Mechanism and Machine Theory* 69 (2013), pp. 142–152.

[27] G. Hegedüs, J. Schicho, and H-P Schröcker. "The theory of bonds: A new method for the analysis of linkages". *Mechanism and Machine Theory* 70 (2013), pp. 407–424.

[28] B. Hendrickson. "Conditions for unique graph realizations". *SIAM J. Comput.* 21.1 (1992), pp. 65–84.

[29] L. Henneberg. *Die Graphische Statik der Starren Systeme*. Liepzig: B.G. Teubner, 1911.

[30] I. Izmestiev. "Projective background of the infinitesimal rigidity of frameworks". *Geom. Dedicata* 140.1 (2008), pp. 183–203.

[31] B. Jackson and T. Jordán. "Connected rigidity matroids and unique realization graphs". *J. Combin. Theory Ser. B* 94 (2005), pp. 1–29.

[32] B. Jackson and J. Owen. “A characterization of the generic rigidity of 2-dimensional point-line frameworks”. *J. Comb. Theory: Series B* 119 (2016), pp. 96–121.

[33] O. Karpenkov. “The Combinatorial Geometry of Stresses in Frameworks” (2015). preprint, https://arxiv.org/abs/1512.02563.

[34] N. Katoh and S. Tanigawa. “Rooted-tree decompositions with matroid constraints and the infinitesimal rigidity of frameworks with boundaries”. *SIAM J. Discrete Math.* 27.1 (2013), pp. 155–185.

[35] G. Laman. “On graphs and rigidity of plane skeletal structures”. *Journal of Engineering Mathematics* 4.4 (1970), pp. 331–340.

[36] A. Lee and I. Streinu. “Pebble game algorithms and sparse graphs”. *Discrete Math.* 308 (2008), pp. 1425–1437.

[37] J.M. Lee. *Introduction to Smooth Manifolds.* New York: Springer- Verlag, 2003.

[38] A. Lee-St.John and Jessica Sidman. “Combinatorics and the rigidity of CAD systems”. *Computer-Aided Design* 45 (1981), pp. 473–482.

[39] L. Lovász and Y. Yemini. “On generic rigidity in the plane”. *SIAM J. Algebraic Discrete Methods* 3 (1982), pp. 91–98.

[40] J.C. Maxwell. “On reciprocal figures and diagrams of forces”. *Phil. Mag., 4th Series* 26 (1864), pp. 250–261.

[41] J.C. Maxwell. “On reciprocal figures, frames and diagrams of forces”. *Trans. Royal Soc. Edinburgh* 26 (1869-1871), pp. 1–40.

[42] J. Milnor. *Singular points of complex hypersurfaces.* Princeton, N. J.: Ann. of Math. Studies, no. 61, Princeton Univ. Press, 1968.

[43] C.St.J. A. Nash-Williams. “Edge-disjoint spanning trees of finite graphs”. *Journal of the London Mathematical Society* 36 (1961), pp. 445–450.

[44] A. Nixon, J. Owen, and S. Power. "Rigidity of frameworks supported on surfaces". *SIAM J. Discrete Math.* 26.3 (2012), pp. 1733–1757.

[45] A. Nixon and E. Ross. "One brick at a time: a survey of inductive constructions in rigidity theory". *Rigidity and Symmetry, Springer, New York* 170.36 (2014), pp. 303–324.

[46] J. Oxley. *Matroid Theory.* Oxford University Press, 2006.

[47] A. Pogorelov. *Extrinsic geometry of convex surfaces.* AMS, 1973.

[48] A. Recski. "A network approach to the rigidity of skeletal structures. Part 2. Laman's Theorem and topological formulae". *Discrete Appl. Math.* 8 (1988), pp. 63–68.

[49] N. Rosenauer and A.H. Willis. *Kinematics of Mechanisms.* New York: Dover Publications, 1967.

[50] B. Roth and W. Whiteley. "Tensegrity frameworks". *Trans. Amer. Math. Soc.* 256 (1981), pp. 419–446.

[51] P.J. Ryan. *Euclidean and Non-Euclidean Geometry, An analytic approach.* Cambridge University Press, 1986.

[52] F. Saliola and W. Whiteley. "Some notes on the equivalence of first-order rigidity in various geometries" (2007). preprint, https://arxiv.org/abs/0709.3354.

[53] J.B. Sax. *Embeddability of weighted graphs in k-space is strongly NP-hard.* Pittsburgh: Tech. Report CMU-CS-80-102, Computer Science Department, Carnegie-Mellon Univesity, 1979.

[54] B. Schulze and W. Whiteley. "Coning, symmetry and spherical frameworks". *Discrete Comput. Geom.* 48 (2012), pp. 622–657.

[55] B. Schulze and W. Whiteley. "The orbit rigidity matrix of a symmetric framework". *Discrete Comput. Geom.* 46.3 (2010), pp. 561–598.

[56] J.M. Selig. *Geometric Fundamentals of Robotics, Second Edition.* New York: Springer Verlag, 2005.

[57] B. Servatious, O. Shai, and W. Whiteley. "Combinatorial Characterization of the Assur Graphs from Engineering". *European Journal of Combinatorics* 31 (2010), pp. 1091–1104.

[58] I.R. Shafarevich and A.O. Remizov. *Linear Algebra and Geometry.* Berlin Heidelberg: Springer-Verlag, 2013.

[59] I. Streinu and L. Theran. "Slider-pinning rigidity: a Maxwell-Laman-type theorem". *Discrete Comput. Geom.* 44 (2007), pp. 812–837.

[60] T-S. Tay. "Rigidity of multigraphs I: linking rigid bodies in n-space". *J. Comb. Theory* Series B.36 (1984), pp. 95–112.

[61] T-S. Tay and W. Whiteley. "Generating isostatic frameworks". *Structural Topology* 11 (1985), pp. 21–69.

[62] N. White. *Geometric applications of the Grassmann-Cayley algebra, Chapter 59.* Rocquencourt: Handbook of Discrete, Computational Geometry, J. Goodman, and J. O'Rourke (eds), CRC Press LLC, Boca Raton, FL; Second Edition, 2004.

[63] N. White and W. Whiteley. "Algebraic geometry of stresses in frameworks". *SIAM J. Alg. Disc. Meth.* 4 (1983), pp. 53–70.

[64] N. White and W. Whiteley. "The algebraic geometry of motions of bar-and-body frameworks". *SIAM J. Alg. Disc. Meth.* 8.1 (1987), pp. 1–32.

[65] W. Whiteley. "A Matroid on Hypergraphs, with Applications to Scene Analysis and Geometry". *Discrete Comput. Geom.* 4 (1989), pp. 75–95.

[66] W. Whiteley. "Cones, infinity and one-story buildings". *Structural Topology* 8 (1983), pp. 53–70.

[67] W. Whiteley. “Infinitesimal motions of a bipartite framework”. *Pacific J. Math.* 110 (1984), pp. 233–255.

[68] W. Whiteley. “Some matroids from discrete applied geometry”. *Bonin, J. Oxley, J.G. Servatius, B. (eds.) Matroid Theory, Contemporary Mathematics, AMS* 197 (1996), pp. 171–311.

[69] W. Whiteley. *Tensegrity.* Draft chapter for The Geometry of Rigid Structures, Chapter 10: 1987.

[70] W. Whiteley. “The union matroids and the rigidity of frameworks”. *SIAM J. Discrete Math* 1.2 (1988), pp. 237–255.

[71] W. Whiteley. “Vertex splitting in isostatic frameworks”. *Structural Topology* 16 (1991), pp. 23–30.