NORMAL SUPERCHARACTER THEORIES

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Abstract

Classification of irreducible characters of some families of groups, for example the family of the groups of unipotent upper-triangular matrices, is a “wild” problem. To have a tame and tractable theory for the groups of unipotent-upper triangular matrices André and Yan introduced the notion of supercharacter theory. Diaconis and Issacs axiomatized the concept of supercharacter theory for any group.

In this thesis, for an arbitrary group G, by using sublattices of the lattice of normal subgroups containing the trivial subgroup and G, we build a family of integral supercharacter theories, called ***normal supercharacter theories*** (abbreviated NSCT). We present a recursive formula for supercharacters in a NSCT. The finest NSCT is constructed from the whole lattice of normal subgroups of G, and is a mechanism to study the behavior of conjugacy classes by the lattice of normal subgroups. We will uncover a relation between the finest NSCT, faithful irreducible characters, and primitive central idempotents. We

argue that NSCT cannot be obtained by previous known supercharacter theory constructions, but it is related to $*$-products of some certain supercharacter theories.

We also construct a NSCT for the family of groups of unipotent upper-triangular matrices. These groups are crucial to the supercharacter theory. The supercharacters of the resulting NSCT are indexed by Dyck paths, which are combinatorial objects that are central to several areas of algebraic combinatorics. Finally, we show that this supercharacter construction is identical to Scott Andrews' construction after gluing the superclasses and the supercharacters by the action of the torus group.

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Introduction

Representation theory is undeniably at the heart of many areas of mathematics and science. For example, it is a fundamental part of Fourier analysis on groups and a powerful method to study problems in abstract algebra. Also, there is a connection between representation theory and particle physics in which the different quantum states of an elementary particle give rise to an irreducible representation of the Poincaré group.

Studying the character theory of a group individually is always possible using known techniques and computer programming. Unfortunately, the classification of all irreducible representations for some families of groups is intractable. These families of groups are known as wild. For one such family of groups, André (1995) and Yan (2001) introduced the notion of a supercharacter theory as a new tractable tool that would allow us to still get useful information about the representation theory of the groups.

Inspired by André's and Yan's work, Diaconis and Isaacs (2008) axiomatized the concept of supercharacter theory for arbitrary groups. A given group can have multiple supercharacter theories. The problem is then to describe, in a useful way, how to obtain such supercharacter theories with enough information about the representations.

Denote by 1 and $\mathbb{1}$ the trivial element and the trivial character of group G, respectively. More precisely, a supercharacter theory is a pair $(\mathcal{X}, \mathcal{K})$, where $\mathcal{K}$ is a partition of elements of a finite group G containing $\{1\}$ and $\mathcal{X}$ is a partition of irreducible characters of G containing $\{\mathbb{1}\}$ such that all characters $\sum_{\psi \in X} \psi(1)\psi$ $(X \in \mathcal{X})$ are constant on every $K \in \mathcal{K}$ and $|\mathcal{K}| = |\mathcal{X}|$. We call a supercharacter theory $(\mathcal{X}, \mathcal{K})$ ***integral*** if $\sum_{\psi \in X} \psi(1)\psi(g) \in \mathbb{Q}$ for every $X \in \mathcal{X}$ and $g \in G$.

Diaconis and Isaacs (2008) also mentioned two supercharacter theory constructions for finite groups G, one comes from the action of a group A on G by automorphisms and another one comes from the action of the cyclotomic field $\mathbb{Q}[\zeta_{|G|}]$. They also generalized André's original construction to define a supercharacter theory for ***algebra groups***, groups of the form $1 + J$ where J is a finite dimensional nilpotent associative algebra over a finite field $\mathbb{F}_q$. Later, Hendrickson (2012) provided some other supercharacter theories for finite groups G. Arias-Castro et al. (2004) used Yan's work in place of the usual irreducible character theory to analyze a random walk on $UT_n(q)$, the group of $n \times n$ unipotent upper-triangular matrices over a finite field $\mathbb{F}_q$. Also Aguiar et al. (2012) obtained a relationship between the supercharacter theories of all unipotent upper-triangular matrices over a finite field $\mathbb{F}_q$ and the combinatorial Hopf algebra of symmetric functions in non-commuting variables.

Let $\mathbf{Norm}(G)$ be the set of all normal subgroups of G. Then $\mathbf{Norm}(G)$ is a lattice in which the least upper bound and the greatest lower bound of two normal subgroups N and M are the product and the intersection of M and N respectively. Let $\mathcal{L}$ be an arbitrary sublattice of $\mathbf{Norm}(G)$ containing the trivial subgroup and G. Let N° be the set of those elements of N that do not belong to any subgroup $H \in \mathcal{L}$ with $H \subset N$.

Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of G. Recall that the kernel of a character χ of G is the set $\ker \chi = \{g \in G : \chi(g) = \chi(1)\}$. Define $\mathcal{X}^{N^{\bullet}}$ to be the set of all irreducible characters ψ such that the smallest normal subgroup in $\mathcal{L}$ and $\ker \psi$ is N. We will show in Theorem 3.2.3 that for an arbitrary sublattice $\mathcal{L} \subseteq \mathbf{Norm}(G)$,

$$\left(\{\mathcal{X}^{N^{\bullet}} \neq \emptyset : N \in \mathcal{L}\}, \{N^{\circ} \neq \emptyset : N \in \mathcal{L}\}\right)$$

is an integral supercharacter theory of G.

In chapter 1, we give a quick review of the basic theorems in character theory. Next, we present the concept of the lattice of normal subgroups and the Möbius inversion formula and the dual of the Möbius inversion formula.

In chapter 2, we first review the definitions for supercharacter theories, then we discuss the main methods for constructing supercharacter theories.

In chapter 3, we present our main supercharacter theory construction for finite groups. Next, we study the relation between normal supercharacter theories and some other supercharacter theory constructions. We show how the normal supercharacter theory can be built by a set of $*$-product supercharacter theories. Moreover, we indicate how we can combine normal supercharacter theories using $*$-products.

In the last chapter, we obtain a normal supercharacter theory for the set of normal pattern subgroups of unipotent upper-triangular matrices. We also construct another supercharacter theory by gluing the superclasses and supercharacters of Andrews' construction by the action of the torus group. Then we show that these two supercharacter theories are identical.

1 Preliminaries

We start to provide the background needed for understanding the concepts in this thesis. In this chapter we introduce some basic terminology and notation. At the end of this chapter, we mention the basics of lattice theory and the Möbius inversion formula.

1.1 Character Theory

We begin by the study of group representations and then we turn to the theory of characters. Let G be a group with identity 1.

1.1.1 Representations and G-modules

A representation can be thought of as a way to model a group with a concrete group of matrices. After giving the precise definition, we look at some examples. Let $\mathbb{C}$ denote the complex numbers. The ***general linear group*** of degree d, $GL_d(\mathbb{C})$ is the set of all $d \times d$ invertible matrices over $\mathbb{C}$.

Definition. *A* ***representation*** *of a group G is a group homomorphism*

$$X: \quad G \quad \rightarrow \quad GL_d(\mathbb{C}).$$

Equivalently, to each $g \in G$ is assigned $X(g) \in GL_d(\mathbb{C})$ such that

1. *$X(1) = I_d$ the identity matrix in GL_d, and*

2. *$X(gh) = X(g)X(h)$ for all $g, h \in G$.*

The parameter d is called the ***degree****, or* ***dimension****, of the representation.*

In the remainder of this document, we only say a matrix representation without mentioning the group G, if it is clear that we are using G.

Example 1.1.1. *All groups have the* ***trivial representation****, which is the one sending every $g \in G$ to the matrix (1). This is clearly a representation because $X(1) = (1)$ and*

$$X(gh) = (1) = (1)(1) = X(g)X(h)$$

for all $g, h \in G$. We often use the notation $\mathbf{1}$ to stand for the trivial representation of G.

Example 1.1.2. *Let $G = C_n$ the cyclic group of order n. Let g be a generator for C_n, i.e.,*

$$C_n = \{1, g, g^2, \cdots, g^{n-1}\}.$$

We aim to find all one-dimensional representations of C_n. To identify a group homomorphism form C_n to $GL_d(\mathbb{C})$, it is enough to give $X(g)$. Assume that $X(g) = (c)$ be a one-dimensional representation. Then $X(1) = X(g^n) = X(g)^n = (c)^n = (c^n) = 1$. Therefore, c must be a nth root of unity, and it is clear that for every root of unity we have a one-dimensional representation.

Because matrices correspond to linear transformations, we can think of representations in these terms. This is the idea of a G-module.

Definition. *Let V be a vector space over $\mathbb{C}$ and let G be a group. Then V is a G-**module** if there is an action*

$$\begin{array}{rrcl} .: & G \times V & \to & V \\ & (g, v) & \mapsto & g.v \end{array}$$

such that

1. $g.(cv + dw) = c(g.v) + d(g.w)$,
2. $(gh).v = g.(h.v)$, *and*
3. $1.v = v$

for all $g, h \in G$; and scalars $c, d \in \mathbb{C}$.

Equivalently, V is a G-module if there is a group homomorphism

$$\rho : G \to GL(V),$$

where $GL(V)$ is the group of all linear transforms of V. Also, in the future, "G-module" will be shortened to "module" when it is clear we are using G.

Example 1.1.3. *Let $V = \mathbb{C}$-Span$\{e_i : i = 1, 2, \ldots, n\}$. Let S_n be the symmetric group of $[n] := \{1, 2, \ldots, n\}$. Then V is a S_n-module with the following action*

$$\begin{array}{ccc} S_n \times V & \rightarrow & V \\ (\sigma, c_1e_1 + \ldots + c_ne_n) & \mapsto & c_1e_{\sigma(1)} + \ldots + c_ne_{\sigma(n)}. \end{array}$$

If the dimension of V as a vector space over $\mathbb{C}$ is d, it is well-known that $GL(V) \cong GL_d(\mathbb{C})$. Therefore, from the definitions of G-modules and matrix representations we can see whenever we have a G-module we can have a matrix representation and vice versa. So the following theorem follows from the definitions of G-modules, matrix representation and the fact that $GL(V) \cong GL_d(\mathbb{C})$.

Theorem 1.1.4. *For every group G we have the following.*

1. *Given a matrix representation X of degree d, then $\mathbb{C}^d := \mathbb{C} \times \ldots \times \mathbb{C}$ is a G-module with the following action*

$$\begin{array}{ccc} G \times \mathbb{C}^d & \rightarrow & \mathbb{C}^d \\ (g, v) & \mapsto & X(g)v. \end{array}$$

2. *Let V be a module with a basis $\{v_1, \ldots, v_d\}$. Then there is a representation X such that $X(g) = (x_{ij})$ where*

$$g.v_j = \sum_{i=1}^{d} x_{ij}v_i.$$

We now describe regular representation which is one of the most important representations of any group.

Definition. *Let G be a finite group. Then*

$$\mathbb{C}[G] = \{c_1 g_1 + \cdots + c_n g_n : c_i \in \mathbb{C} \ \ \textit{for all } i\}$$

is a module with the following actions

$$\begin{array}{ccc} G \times \mathbb{C}[G] & \to & \mathbb{C}[G] \\ (g, c_1 g_1 + \cdots + c_n g_n) & \mapsto & c_1(gg_1) + \cdots + c_n(gg_n). \end{array}$$

*The corresponding representation to the module $\mathbb{C}[G]$, denoted by P_G, is called the **regular representation** of G.*

1.1.2 Irreducible Representations

Irreducible representations are the building blocks of representation theory since every representation can be written as a direct sum of irreducible representations. Here we give the basic concept of irreducible representations.

Definition. *Let V be a G-module. A **submodule** of V is a subspace W that is G-**invariant subspace**, i.e., for every $w \in W$ and $g \in G$, $g.w \in W$. We write $W \leq V$ if W is a submodule of V.*

We say a G-module V is the ***direct sum*** of two submodules U and W, denoted by $V = U \oplus W$, if every element $v \in V$ can be written uniquely as a sum

$$v = u + w, \qquad u \in U, w \in W.$$

Also, if X is a matrix, then X is the ***direct sum*** of matrices A and B, written as $X = A \oplus B$, if X can be written as the block diagonal form

$$X = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right).$$

Every G-module V contains two trivial submodules, itself and the zero submodule. We call a G-module ***irreducible*** if it does not contain any non-trivial submodules. A representation of G is called ***irreducible*** if its corresponding module is irreducible. We now state the Maschke's Theorem which shows that the irreducible modules are the building blocks of the G-modules. For a proof of the Maschke's Theorem refer to (Sagan 2001, Theorem 1.5.3).

Theorem 1.1.5. *(Maschke's Theorem) Let G be a finite group and let V be a non-zero G-module. Then*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

where each W_i is an irreducible submodule of V. Equivalently, if X is a non-zero representation, then

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_k,$$

where each X_i is an irreducible representation.

As every module can be written as a direct sum of irreducible submodules, it is important to see when two irreducible submodules are isomorphic. This yields Schur's Lemma (Theorem 1.1.6) which states that a G-homomorphism between irreducible modules is either an isomorphism or zero.

Definition. *Let V and W be G-modules. Then a G-homomorphism (or simply a homomorphism) is a linear transformation*

$$\theta : \quad V \quad \rightarrow \quad W$$

such that

$$\theta(g.v) = g.\theta(v)$$

for all $g \in G$ and $v \in V$. We also say that θ preserves or respects the action of G.

We translate the above definition to the language of representations. Let X be a representation corresponding to V and Y be a representation corresponding to W. If there is a G-morphism between V and W, that means that there is a matrix T such that $X(g)T = TY(g)$ for every $g \in G$.

Definition. *Let V and W be modules for a group G. A G-isomorphism is a G-homomorphism $\theta : V \rightarrow W$ that is bijective. In this case we say that V and W are G-isomorphic, or G-equivalent, written $V \cong W$. Otherwise we say that V and W are G-inequivalent.*

Also, a translation of the G-isomorphism to representations implies that whenever we have a G-isomorphism between V and W, with corresponding representations X and Y respectively, then there is an invertible matrix T such that $X(g)T = TY(g)$ for every $g \in G$, and we write $X \cong Y$.

Now we state ***Schur's Lemma*** which characterizes G-homomorphisms of irreducible modules. For a proof of Schur's Lemma refer to (Sagan 2001, Theorem 1.6.5).

Theorem 1.1.6. *(Schur's Lemma) Let V and W be two irreducible G-modules. If $\theta : V \to W$ is a G-homomorphism, then either*

1. *θ is a G-isomorphism, or*

2. *θ is the zero map.*

Proposition 1.1.7. *(Sagan 2001, Proposition 1.10.1) Let G be a finite group and suppose*

$$\mathbb{C}[G] = \bigoplus_i m_i V_i,$$

where the V_i form a complete list of pairwise inequivalent irreducible G-modules. Then

1. *$m_i = \mathrm{dim} V_i$,*

2. *$\sum_i (\mathrm{dim} V_i)^2 = |G|$, and*

3. the number of V_i equals the number of conjugacy classes of G.

As the number of V_i is equal to both the cardinality of pairwise inequivalent irreducible G-modules and the number of conjugacy classes of G, it turns out the number of inequivalent irreducible G-modules is finite.

1.1.3 Group Characters

Much of the information about representations can be obtained by the traces of the corresponding matrices. This is the theory of characters and we spend the rest of this chapter on character theory.

Definition. *Let G be a group and X be a representation of G. Then the* **character** *of X is*

$$\chi(g) = \mathrm{tr} X(g),$$

where tr *denotes the trace of a matrix. Also, a character of a G-module V is the character of a matrix representation corresponding to V.*

Example 1.1.8. *Every group has a trivial character which takes every element of G to 1. We denote the trivial character by $\mathbb{1}$.*

Example 1.1.9. *The character ρ_G corresponding to the regular representation P_G is called regular character. Note that $P_G(1) = I_n$ the $n \times n$ identity matrix where n is the cardinality of G. To compute $\rho_G(g)$ for a non-identity element*

$g \in G$, take the standard basis $\{g_1, \ldots, g_n\}$ for $\mathbb{C}[G]$. Then the trace of P_G is the number of $h \in G$ such that $hg = h$, but since g is not identity this number is 0. Thus we have

$$\rho_G(g) = \begin{cases} |G| & \textit{if } g = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Let X be a representation of a group G of degree d with character χ. We state some basic properties of character χ.

- By the definition of a representation, we have $X(1) = I_d$, so the trace of $X(1)$ is d. Therefore, $\chi(1) = d$.

- Let C_g be the conjugacy class of group G containing g. Let $h \in C_g$. Then $h = tgt^{-1}$ for some $t \in G$. We have

$$\chi(h) = \mathrm{tr}X(tgt^{-1}) = \mathrm{tr}X(t)X(g)X(t^{-1}) = \mathrm{tr}X(g) = \chi(g),$$

  where the second equality is because X is a group homomorphism and the third equality is because for all matrices A and B, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Therefore, every character is constant on elements of a conjugacy class.

- If Y is a representation with character ψ and $X \cong Y$, then there is an invertible matrix T such that $T^{-1}X(g)T = Y(g)$ for all $g \in G$. Thus,

$$\mathrm{tr}(Y(g)) = \mathrm{tr}(T^{-1}X(g)T) = \mathrm{tr}(X(g)).$$

Therefore,

$$X \cong Y \Rightarrow \chi(g) = \psi(g)$$

for all $g \in G$.

Moreover, we have the following theorem.

Theorem 1.1.10. *(Sagan 2001, Corollary 1.9.4) Let X and Y be matrix representations of G with characters χ and ψ respectively. Then*

$$X \cong Y \text{ if and only if } \chi(g) = \psi(g)$$

for all $g \in G$.

This theorem shows that the number of inequivalent irreducible G-modules and the number of inequivalent irreducible characters of G are equal and so the number of inequivalent irreducible characters of G is finite.

Also, a character of a representation X is said to ***irreducible*** if X is irreducible. Two characters are ***inequivalent*** if their representations are inequivalent. We denote by $\mathrm{Irr}(G)$ the set of all inequivalent irreducible characters of G.

As it follows from above, whenever we have a conjugacy class C_g and an irreducible character χ, then χ is constant on C_g. This gives us the definition of character table.

Definition. *Let G be a group. The character table of G is an array with rows indexed by inequivalent irreducible characters and columns indexed by conjugacy classes, and the table entry of row indexed by χ and column C_g is $\chi(h)$ for an arbitrary element $h \in C_g$. By convention, the first row corresponds to the the trivial character, and the first column corresponds to the class containing the identity, $C_1 = \{1\}$.*

As an example we present the character table of an abelian group.

Example 1.1.11. *Let $G = C_3 \times C_3 = \langle a \rangle \times \langle b \rangle$. The following is the character table of G. Let $\zeta_3 = e^{2\pi i/3}$.*

	1	b	b^2	a	ab	ab^2	a^2	a^2b	a^2b^2
$\mathbb{1}$	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	$-\zeta_3 - 1$	$-\zeta_3 - 1$	$-\zeta_3 - 1$	ζ_3	ζ_3	ζ_3
χ_3	1	1	1	ζ_3	ζ_3	ζ_3	$-\zeta_3 - 1$	$-\zeta_3 - 1$	$-\zeta_3 - 1$
χ_4	1	$-\zeta_3 - 1$	ζ_3	1	$-\zeta_3 - 1$	ζ_3	1	$-\zeta_3 - 1$	ζ_3
χ_5	1	ζ_3	$-\zeta_3 - 1$	1	ζ_3	$-\zeta_3 - 1$	1	ζ_3	$-\zeta_3 - 1$
χ_6	1	$-\zeta_3 - 1$	ζ_3	$-\zeta_3 - 1$	ζ_3	1	ζ_3	1	$-\zeta_3 - 1$
χ_7	1	ζ_3	$-\zeta_3 - 1$	ζ_3	$-\zeta_3 - 1$	1	$-\zeta_3 - 1$	1	ζ_3
χ_8	1	$-\zeta_3 - 1$	ζ_3	ζ_3	1	$-\zeta_3 - 1$	$-\zeta_3 - 1$	ζ_3	1
χ_9	1	ζ_3	$-\zeta_3 - 1$	$-\zeta_3 - 1$	1	ζ_3	ζ_3	$-\zeta_3 - 1$	1

A character of degree 1 is called ***linear***. When G is finite, the kernel of the character χ is the normal subgroup:

$$\ker \chi := \{g \in G \mid \chi(g) = \chi(1)\}$$

which by the following lemma, is precisely the kernel of the representation X corresponding to χ.

Lemma 1.1.12. *Let X be a representation whose character is χ. Then $\ker\chi = \ker X$.*

Proof. Let g be an arbitrary element of G. If $X(g) = X(1)$, then $\chi(g) = \chi(1)$. So we can see that $\ker\chi \subseteq \ker X$. Now let $\chi(g) = \chi(1)$, we want to show that $X(g) = X(1)$. Note that the restriction of X to $\langle g\rangle$ is a representation of $\langle g\rangle$. As $\langle g\rangle$ is a cyclic group, we see in Example 1.1.2 that the irreducible representations of $\langle g\rangle$ are one dimensional representations and so they are identical with their characters. Decompose $X_{\langle g\rangle}$ as the direct sum of one dimensional irreducible representations, i.e., $X_{\langle g\rangle}(g) = d_1X_1(g) \oplus \ldots \oplus d_kX_k(g)$. Note that

$$X_{\langle g\rangle}(g) = d_1X_1(g) \oplus \ldots \oplus d_kX_k(g) = d_1(\chi_1(g)) \oplus \ldots \oplus d_k((\chi_k(g)),$$

where χ_i is the corresponding irreducible character to X_i. We have $\chi(g) = \chi(1)$, thus $\chi(g) = d_1\chi_1(g) + \ldots + d_k\chi_k(g) = d_1\chi_1(1) + \ldots + d_k\chi_k(1) = d_1 + \ldots + d_k$. Since for some positive integer n, $\chi_i(g)^n = 1$, it follows $|\chi_i(g)| \leq 1$, furthermore, d_i is positive for every i, we can see that $\chi_i(g) = \chi_i(1)$. Since $g \in \ker\chi$,

$$X(g) = X_{\langle g\rangle}(g) = d_1X_1(g) \oplus \ldots \oplus d_kX_k(g) =$$

$$d_1(\chi_1(g)) \oplus \ldots \oplus d_k(\chi_k(g)) = d_1(\chi_1(1)) \oplus \ldots \oplus d_k(\chi_k(1)) =$$

$$d_1X_1(1) \oplus \ldots \oplus d_kX_k(1) = X_{\langle g \rangle}(1) = X(1).$$

$\square$

A character is said to be ***faithful*** if its kernel is trivial. A ***class function*** of G is a function from G to $\mathbb{C}$ which takes a constant value on each conjugacy class. The set of irreducible characters of a given group G forms a basis for

$$\mathbf{cf} := \mathbb{C}\text{-Span}\{f : f \text{ is a class function of } G\}.$$

The space of class functions of a finite group G has a natural inner-product

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)},$$

where $\overline{\beta(g)}$ is the complex conjugate of $\beta(g)$. With respect to this inner product, the irreducible characters form an orthonormal basis for the space of class-functions, i.e, for irreducible characters χ and ψ we have

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \chi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.1.13. *(Sagan 2001, Corollary 1.9.4) Let χ be a character of G. Then*

- *Each character χ can be written as*

$$\chi = m_1\chi_1 + m_2\chi_2 + \ldots + m_k\chi_k,$$

where each m_i is an integer and the χ_i are pairwise inequivalent irreducible characters.

- *If* $\chi = m_1\chi_1 + m_2\chi_2 + \ldots + m_k\chi_k$*, then* $\langle \chi, \chi_i \rangle = m_i$*.*

- *The character* χ *is irreducible if and only if* $\langle \chi, \chi \rangle = 1$*.*

Definition. *If we write a character*

$$\chi = m_1\chi_1 + m_2\chi_2 + \ldots + m_k\chi_k,$$

where each m_i *is a positive integer and the* χ_i *are pairwise inequivalent irreducible characters, then each* χ_i *is called a* ***constituent*** *of* χ*.*

When N is a normal subgroup of a group G, then G/N forms a group. In the following chapter we frequently lift the characters of the group G/N to the group G and vice versa. We now give a description of the irreducible characters of G/N and show how they are related to the characters of G.

Example 1.1.14. *Let* N *be a normal subgroup of* G*. If* ψ *is an irreducible character of* G *such that* $N \subseteq \ker\psi$*, then define* $\overline{\psi} : G/N \to \mathbb{C}$ *by* $\overline{\psi}(gN) = \psi(g)$*. Note that* $\overline{\psi}$ *is an irreducible character of* G/N *and we say* $\overline{\psi}$ *has been* ***lifted to*** *the irreducible character* ψ *of* G*. We have the following bijection between irreducible characters* χ *of* G *containing* N *in their kernels and irreducible characters of* G/N*,*

$$\begin{array}{ccc} \{\psi \in Irr(G) : N \subseteq \ker\psi\} & \to & Irr(G/N) \\ \psi & \mapsto & \overline{\psi}. \end{array}$$

In order to state a lemma of Brauer, we need the following definition.

Definition. *Given finite groups A and G, we say that A* ***acts via automorphisms*** *on G if A acts on G as a set, and in addition $a.(gh) = (a.g)(a.h)$ for all $g, h \in G$ and $a \in A$. An action via automorphisms of A on G determines and is determined by a homomorphism $\phi : A \to Aut(G)$. Since $a.1 = (a.1)(a.1)$, $a.1 = 1$ and $1 = a.1 = a.(gg^{-1}) = (a.g)(a.g^{-1})$, $a.g^{-1} = (a.g)^{-1}$. Notice that*

$$a.C_h = \{a.(ghg^{-1}) : g \in G\} = \{(a.g)(a.h)(a.g^{-1}) : g \in G\} = C_{a.h},$$

and we conclude that A permutes the conjugacy classes of G. Also, when A acts via automorphisms on G, it also acts on irreducible characters as follows

$$\begin{aligned} A \times Irr(G) &\to Irr(G) \\ (a, \chi) &\mapsto \chi_{a^{-1}}, \end{aligned}$$

where $\chi_{a^{-1}}(g) = \chi(a^{-1}.g)$ for all $g \in G$. Note that $\chi_{a^{-1}}$ is irreducible since

$$\langle \chi_{a^{-1}}, \chi_{a^{-1}} \rangle = \sum_{g \in G} \chi^2_{a^{-1}}(g) = \sum_{g \in G} \chi^2(a^{-1}.g) = \sum_{a.h \in G} \chi^2(h) = \langle \chi, \chi \rangle = 1.$$

Therefore the action of A on $Irr(G)$ permutes the irreducible characters.

The following theorem is known as ***Brauer's Lemma***.

Theorem 1.1.15. *(Isaacs 1994, Theorem 6.32) Let A act on $Irr(G)$ and on the set of conjugacy classes of G. Let $a.g$ be an element of $a.C_g$. If $a.\chi(a.g) = \chi(g)$ for all $\chi \in Irr(G)$, $a \in A$, and $g \in G$, then for each $a \in A$, the number of fixed irreducible characters of G is equal to the number of fixed classes.*

As a corollary of this theorem we have the following.

Corollary 1.1.16. *(Isaacs 1994, Corollary 6.33) Under the hypotheses of Theorem 1.1.15 the numbers of orbits in the actions of A on the irreducible characters and conjugacy classes of G are equal.*

1.2 Lattice Theory

Our supercharacter theory construction for a finite group G is actually coming from some sublattices of lattice of normal subgroups. In this section we give the definition of a lattice and we discuss the Möbius inversion formula and the dual of the Möbius inversion formula. In the end we present the details of lattice of normal subgroups.

A ***partial order*** is a binary relation $\leq$ over a set P which satisfies for all a, b, and c in P:

- $a \leq a$ (reflexivity),
- if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry),
- if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

A set with a partial order is called a ***partially ordered set***. We abbreviate "partially ordered set" as ***poset***.

If $(P, \leq)$ is a poset, and $S \subseteq P$ is an arbitrary subset, then an element $u \in P$ is said to be an ***upper bound*** of S if $s \leq u$ for each $s \in S$; an element $v \in P$ is said to be a ***lower bound*** of S if $u \leq s$ for each $s \in S$.

Definition. *A **lattice** is a partial ordered set in which every two elements have a unique least upper bound (or **join**) and a unique greatest lower bound (or **meet**). A **sublattice** of a lattice L is a nonempty subset of L that is a lattice with the same meet and join operations as L.*

1.2.1 The Möbius Inversion Formula and Its Dual

We now present the Möbius inversion formula and its dual. This concept is essential in finding the supercharacters of our supercharacter theory construction.

If $(P, \leq)$ is a poset and $\mathbb{C}^{P\times P}$ is the set of all functions $\alpha : P \times P \to \mathbb{C}$, the associated incidence algebra is

$$A(P) = \{\alpha \in \mathbb{C}^{P\times P} : \alpha(s, u) = 0 \text{ unless } s \leq u\}.$$

The ***Möbius function*** $\mu \in A(P)$ is defined recursively by the following rules:

$$\mu(s, s) = 1,$$

and

$$\mu(s, u) = -\sum_{s\leq t<u} \mu(t, u), \quad \text{for all } s < u \text{ in } P.$$

It is immediate from this definition that

$$\sum_{s \leq t \leq u} \mu(t, u) = \begin{cases} 1 & \text{if } s = u, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.2.1. *(Stanley 1986, Proposition 3.7.1 (Möbius inversion formula)) Let P be a finite poset. Let $f, g : P \to \mathbb{K}$, where $\mathbb{K}$ is a field. Then*

$$g(t) = \sum_{s \leq t} f(s), \quad \textit{for all } t \in P,$$

if and only if

$$f(t) = \sum_{s \leq t} g(s)\mu(s, t), \quad \textit{for all } t \in P.$$

Lemma 1.2.2. *(Stanley 1986, Proposition 3.7.2 (Möbius inversion formula, dual form)) Let P be a finite poset. Let $f, g : P \to \mathbb{K}$, where $\mathbb{K}$ is a field. Then*

$$g(t) = \sum_{s \geq t} f(s), \quad \textit{for all } t \in P,$$

if and only if

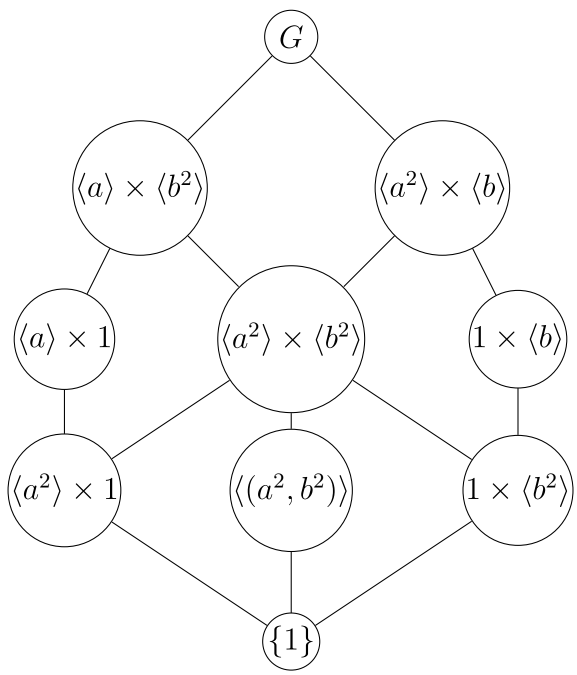
$$f(t) = \sum_{s \geq t} g(s)\mu(t, s), \quad \textit{for all } t \in P.$$

1.2.2 The Lattice of Normal Subgroups of a Group

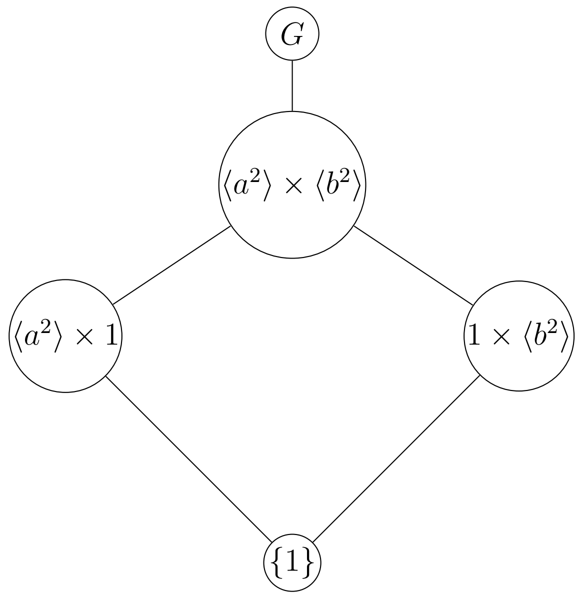
The lattice of normal subgroups of a group G is the lattice whose elements are the normal subgroups of G, with the partial order relation being set inclusion. The least upper bound of two normal subgroups M and N is the product of M

and N, that is $MN = \{mn : m \in M, n \in N\}$ and the greatest lower bound of two normal subgroups M and N is the intersection of M and N.

Example 1.2.3. *The following is the lattice of normal subgroups of $G = C_4 \times C_4 = \langle a \rangle \times \langle b \rangle$.*



Here is a sublattice of the lattice of normal subgroups of G containing identity *subgroup and* G.



2 Supercharacter Theory

2.1 Supercharacter Theory

In this section, we first review the definitions for supercharacter theories, then we discuss the main methods for obtaining supercharacter theories.

We reproduce the definition of supercharacter theory in Diaconis and Isaacs (2008). Throughout this manuscript for every subset $X \subseteq \mathrm{Irr}(G)$ let σ_X be the character $\sum_{\psi \in X} \psi(1)\psi$.

A ***supercharacter theory*** of a finite group G is a pair $(\mathcal{X}, \mathcal{K})$ together with a choice of a set of characters $\{\chi_X : X \in \mathcal{X}\}$, where the elements of X are constituents of χ_X,

- $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$, and
- $\mathcal{K}$ is a partition of G,

such that:

1. The set $\{1\}$ is a member of $\mathcal{K}$.

2. $|\mathcal{X}| = |\mathcal{K}|$.

3. The characters χ_X are constant on the parts of $\mathcal{K}$.

We refer to the characters χ_X as ***supercharacters*** and to the members of $\mathcal{K}$ as ***superclasses***. It is well-known that every supercharacter χ_X is a constant multiple of σ_X (See (Diaconis and Isaacs 2008, Section 2)). Denote by $\mathrm{Sup}(G)$ the set of all supercharacter theories of G.

Every finite group G has two trivial supercharacter theories: the usual irreducible character theory and the supercharacter theory $(\{\{\mathbb{1}\}, \mathrm{Irr}(G) \setminus \{\mathbb{1}\}\}, \{\{1\}, G \setminus \{1\}\})$, where $\mathbb{1}$ is the trivial character of G. A non-trivial example is presented as follows.

Example 2.1.1. *The following table is the character table of S_4.*

	e	$C_{(12)}$	$C_{(12)(34)}$	$C_{(123)}$	$C_{(1234)}$
$\mathbb{1}$	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_2	3	1	-1	0	-1
χ_3	3	-1	-1	0	1
χ_4	2	0	2	-1	0

Let

$$\mathcal{X} = \{\{\mathbb{1}\}, \{\chi_1\}, \{\chi_2, \chi_3, \chi_4\}\}$$

and

$$\mathcal{K} = \{\{e\}, C_{(12)} \cup C_{(1234)}, C_{(123)} \cup C_{(12)(34)}\}.$$

Then $(\mathcal{X}, \mathcal{K})$ *is a supercharacter theory with*

$$\{\mathbb{1}, \chi_1, 3\chi_2 + 3\chi_3 + 2\chi_4\}$$

as the set of supercharacters.

In the following subsections we review known supercharacter theory constructions for G. The supercharacter constructions in Sections 2.2, 2.3, and 2.6 are introduced by Diaconis and Isaacs (2008), and for more details about the other constructions see (Hendrickson 2012, Section 4) and Andrews (2015).

2.2 A Group Acting Via Automorphisms on a Given Group

Suppose that A is a group that acts via automorphisms on our given group G. As we showed before, A permutes both the irreducible characters of G and the conjugacy classes of G. By the lemma of Brauer, the numbers of A-orbits on $\mathrm{Irr}(G)$ and on the set of classes of G are equal (see Theorem 1.1.15 and Corollary

1.1.16). The A-orbit of χ is the set

$$\{\chi_{a^{-1}} : a \in A\}.$$

Also, the union of A-orbits of elements in C_g is

$$\bigcup_{b\in A} b.C_g.$$

We claim that $\sum_{a\in A} \chi_{a^{-1}}(1)\chi_{a^{-1}}$ is constant on each element of $\bigcup_{b\in A} b.C_g$. We have that for every $a \in A$, $\chi(a.1) = \chi(1)$, thus

$$\sum_{a\in A} \chi_{a^{-1}}(1)\chi_{a^{-1}}(g) = \sum_{a\in A} \chi(a^{-1}.1)\chi(a^{-1}.g) = \sum_{a\in A} \chi(1)\chi(a^{-1}.g) =$$
$$\chi(1)\sum_{a\in A} \chi(a^{-1}.g).$$

Moreover, for every $b \in A$,

$$\chi(1)\sum_{a\in A} \chi(a^{-1}.g) = \chi(1)\sum_{a\in A} \chi((a^{-1}b^{-1}).g) = \chi(1)\sum_{a\in A} \chi_{(ba)^{-1}}(g).$$

We showed that χ_a for every $a \in A$ is an irreducible character, therefore,

$$\chi(1)\sum_{a\in A} \chi_{(ba)^{-1}}(g) = \chi(1)\sum_{a\in A} \chi_{(ba)^{-1}}(hgh^{-1}) = \chi(1)\sum_{a\in A} \chi_{a^{-1}}(b^{-1}.hgh^{-1}).$$

Thus, the sum of the characters in an orbit $X \in \mathcal{X}$ is constant on each member of $\bigcup_{b\in A} b.C_g$. Therefore, these orbit decompositions yield a supercharacter theory $(\mathcal{X}, \mathcal{K})$ where the members of $\mathcal{X}$ are the A-orbits on the classes of G and members of $\mathcal{K}$ are the unions of the A-orbits on the classes of G. We denote by $\mathrm{AutSup}(G)$ the set of all such supercharacter theories of G.

Example 2.2.1. *Let $G = C_2 \times C_2 = \langle a \rangle \times \langle b \rangle$. Also define the automorphism $\alpha_1 : G \to G$ as the identity map and $\alpha : G \to G$ is the automorphism of G in which $\alpha(a) = b$ and $\alpha(b) = a$. So $C_2 = \langle x \rangle$ acts on G as follows,*

$$\begin{aligned} C_2 \times G &\to G \\ (x^i, g) &\mapsto \begin{cases} \alpha_1(g) & i = 0 \\ \alpha(g) & i = 1. \end{cases} \end{aligned}$$

Then the set of C_2-orbits on G are

$$\{\{1\}, \{a, b\}, \{ab\}\}.$$

	1	ab	a	b
$\mathbb{1}$	1	1	1	1
χ_1	1	1	-1	-1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1

The set of C_2-orbits on $Irr(G)$ are

$$\{\{\mathbb{1}\}, \{\chi_1\}, \{\chi_2, \chi_3\}\}.$$

Now we can see that

$$(\{\{\mathbb{1}\}, \{\chi_1\}, \{\chi_2, \chi_3\}\}, \{\{1\}, \{a, b\}, \{ab\}\})$$

is a supercharacter theory.

When A acts via automorphisms on G, as this action is determined by a group homomorphism form A to $Aut(G)$, the orbits of this action on G are determined by a subgroup H of $Aut(G)$. More precisely, the orbit of an element $g \in G$ is the set

$$\{\alpha(h) : \alpha \in H, h \in C_g\}.$$

2.3 Action of Automorphisms of The Cyclotomic Field $\mathbb{Q}[\zeta_{|G|}]$

Another general way to construct a supercharacter theory for G uses the action of a group A of automorphisms of the cyclotomic field $\mathbb{Q}[\zeta_{|G|}]$, where $\zeta_{|G|}$ is a primitive $|G|$th root of unity. There is an action on the classes of G, defined as follows. Given $\sigma \in A$, there is a unique positive integer $r < |G|$ such that $\sigma(\zeta_{|G|}) = \zeta_{|G|}^r$, and we let σ carry the class C_g to the class C_{g^r}. Also we have the following action of A on $\mathrm{Irr}(G)$,

$$\begin{array}{rcl} A \times \mathrm{Irr}(G) & \to & \mathrm{Irr}(G) \\ (\sigma, \chi) & \mapsto & \chi_{\sigma^{-1}}, \end{array}$$

where $\chi_{\sigma^{-1}}(g) = \chi(\sigma^{-1}.g)$.

Let $g \in G$. Note that for every $h \in G$,

$$\sigma.\chi(\sigma.hgh^{-1}) = \chi(\sigma^{-1}\sigma.hgh^{-1}) = \chi(hgh^{-1}) = \chi(g),$$

Brauer's lemma (Theorem 1.1.15) shows that the numbers of A-orbits on $\mathrm{Irr}(G)$ and on the set of conjugacy classes of G are equal. We take $\mathcal{X}$ to be the set of A-orbits on $\mathrm{Irr}(G)$, and again, $\mathcal{K}$ is the set of unions of the various A-orbits on conjugacy classes. The A-orbit of $\chi \in \mathrm{Irr}(G)$ is the set

$$\{\chi_{\sigma^{-1}} : \sigma \in A\}.$$

Also, the union of A-orbits of elements in C_g is

$$\bigcup_{\tau \in A} \tau.C_g.$$

We claim that $\sum_{\sigma \in A} \chi_{\sigma^{-1}}(1)\chi_{\sigma^{-1}}$ is constant on each element of $\bigcup_{\tau \in A} \tau.C_g$. We have that for every $\sigma \in A$, $\chi(\sigma.1) = \chi(1)$, thus

$$\sum_{\sigma \in A} \chi_{\sigma^{-1}}(1)\chi_{\sigma^{-1}}(g) = \sum_{\sigma \in A} \chi(\sigma^{-1}.1)\chi(\sigma^{-1}.g) = \sum_{\sigma \in A} \chi(1)\chi(\sigma^{-1}.g) =$$
$$\chi(1) \sum_{\sigma \in A} \chi(\sigma^{-1}.g).$$

Moreover, for every $\tau \in A$,

$$\chi(1) \sum_{\sigma \in A} \chi(\sigma^{-1}.g) = \chi(1) \sum_{\sigma \in A} \chi((\sigma^{-1}\tau^{-1}).g) = \chi(1) \sum_{\sigma \in A} \chi_{(\tau\sigma)^{-1}}(g).$$

We have for every $h \in G$,

$$\chi(1) \sum_{\sigma \in A} \chi_{(\tau\sigma)^{-1}}(g) = \chi(1) \sum_{\sigma \in A} \chi_{(\tau\sigma)^{-1}}(hgh^{-1}) = \chi(1) \sum_{\sigma \in A} \chi_{\sigma^{-1}}(\tau^{-1}.hgh^{-1}).$$

Thus, the sum of the characters in an orbit $X \in \mathcal{X}$ is constant on each member of $\mathcal{K}$. Therefore $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory. We denote by $\mathrm{ACSup}(G)$ the set containing the above supercharacter theories.

Example 2.3.1. *Let $G = C_2 \times C_3 = \langle a\rangle \times \langle b\rangle$. The set of automorphisms of the cyclotomic field $\mathbb{Q}[\zeta_{|G|}] = \mathbb{Q}[\zeta_6]$ is*

$$A := \{\sigma_k : k \in \{1, 5\}\},$$

where $\sigma_k(\zeta_6) = \zeta_6^k$. Then the set of A-orbits on G are

$$\{\{1\}, \{a\}, \{b, b^2\}, \{ab, ab^2\}\}.$$

	1	a	b	ab	b^2	ab^2
$\mathbb{1}$	1	1	1	1	1	1
χ_1	1	-1	1	-1	1	-1
χ_2	1	-1	$-\zeta_3 - 1$	$\zeta_3 + 1$	ζ_3	$-\zeta_3$
χ_3	1	-1	ζ_3	$-\zeta_3$	$-\zeta_3 - 1$	$\zeta_3 + 1$
χ_4	1	1	$-\zeta_3 - 1$	$-\zeta_3 - 1$	ζ_3	ζ_3
χ_5	1	1	ζ_3	ζ_3	$-\zeta_3 - 1$	$-\zeta_3 - 1$

The set of A-orbits on $Irr(G)$ are

$$\{\{\mathbb{1}\}, \{\chi_1\}, \{\chi_2, \chi_3\}, \{\chi_4, \chi_5\}\}.$$

We can see that

$$(\{\{\mathbb{1}\}, \{\chi_1\}, \{\chi_2, \chi_3\}, \{\chi_4, \chi_5\}\}, \{\{1\}, \{a\}, \{b, b^2\}, \{ab, ab^2\}\})$$

is a supercharacter theory for G.

2.4 The $*$-Product

In this subsection, we show that if N is a normal subgroup of G, then some supercharacter theories of N can be combined with supercharacter theories of G/N to form supercharacter theories of the full group G.

Let G and H be groups and let G act on H by automorphisms. We say that $(\mathcal{X}, \mathcal{K}) \in \mathrm{Sup}(H)$ is G-invariant if the action of G fixes each part $K \in \mathcal{K}$ setwise. We denote by $\mathrm{Sup}_G(H)$ the set of all G-invariant supercharacter theories of H.

For each subset $L \subseteq G/N$ let $\widetilde{L} := \bigcup_{Ng \in L} Ng$. Extend this notation to a set $\mathcal{L}$ of subsets of G/N by $\widetilde{\mathcal{L}} := \{\widetilde{L} : L \in \mathcal{L}\}$, and let $\mathcal{L}^\circ$ denote $\mathcal{L} \setminus \{\{N\}\}$.

Recall that if N is a normal subgroup of G and $\psi \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\psi)$ denotes the set of irreducible characters χ of G such that $\langle \chi_N, \psi \rangle > 0$, where χ_N is the restriction of χ to N. If $Z \subseteq \mathrm{Irr}(N)$ is a union of G-orbits, then define the subset Z^G of $\mathrm{Irr}(G)$ to be $\bigcup_{\psi \in Z} \mathrm{Irr}(G|\psi)$. Extend this notation to a set $\mathcal{Z}$ of subsets of $\mathrm{Irr}(N)$ by letting $\mathcal{Z}^G := \{Z^G : Z \in \mathcal{Z}\}$, and let $\mathcal{Z}^\circ := \mathcal{Z} \setminus \{\{\mathbb{1}_N\}\}$.

Define

$$\begin{array}{rccc} *_N : & \mathrm{Sup}_G(N) \times \mathrm{Sup}(G/N) & \to & \mathrm{Sup}(G) \\ & ((\mathcal{X}, \mathcal{K}), (\mathcal{Y}, \mathcal{L})) & \mapsto & \left((\mathcal{X}^\circ)^G \cup \mathcal{Y}, \mathcal{K} \cup \widetilde{\mathcal{L}^\circ} \right), \end{array}$$

then by (Hendrickson 2012, Theorem 5.3) the map $*_N$ is well-defined. Denoted by $\mathrm{Sup}^*_N(G)$ the image of $*_N$ and let $\mathrm{Sup}^*(G) = \bigcup_{N \in \mathbf{Norm}(G)} \mathrm{Sup}^*_N(G)$.

Example 2.4.1. *The following is the character table of alternating group A_4.*

	1	$C_{(12)(34)}$	$C_{(123)}$	$C_{(124)}$
$\mathbb{1}$	1	1	1	1
ψ_1	1	1	$-\zeta_3 - 1$	ζ_3
ψ_2	1	1	ζ_3	$-\zeta_3 - 1$
ψ_3	3	-1	0	0

Let

$$\mathcal{K} = \{\{1\}, \{(12)(34)\}, \{(123), (124)\}\}$$

and

$$\mathcal{X} = \{\{\mathbb{1}\}, \{\psi_1, \psi_2\}, \{\psi_3\}\}.$$

Then $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory in $Sup_{S_4}(A_4)$. We have

$$(\mathcal{Y}, \mathcal{L}) = (\{\{\mathbb{1}\}, \{\chi\}\}, \{\{A_4\}, (12)A_4\})$$

is a supercharacter theory for S_4/A_4, where $\chi(A_4) = 1$ and $\chi((12)A_4) = -1$. Then

$$\widetilde{\mathcal{L}^{\circ}} = \{C_{(12)} \cup C_{(1234)}\}$$

and

$$\mathcal{K} = \{\{1\}, C_{(12)(34)}, C_{(123)}\}.$$

Also, as we can see from the character table of S_4 in Example 2.1.1,

$$\mathcal{Y} = \{\{\mathbb{1}\}, \{\chi_1\}\}$$

and

$$\{\psi_1, \psi_2\}^{S_4} = \{\chi_4\},$$

$$\{\psi_3\}^{S_4} = \{\chi_2, \chi_3\},$$

and so

$$(\mathcal{X}^\circ)^{S_4} = \{\{\chi_2, \chi_3\}, \{\chi_4\}\}.$$

Therefore,

$$(\mathcal{X}, \mathcal{K}) *_{A_4} (\mathcal{Y}, \mathcal{L}) = \left((\mathcal{X}^\circ)^G \cup \mathcal{Y}, \mathcal{K} \cup \widetilde{\mathcal{L}^\circ}\right) =$$

$$\left(\{\{\mathbb{1}\}, \{\chi_1\}, \{\chi_2, \chi_3\}, \{\chi_4\}\}, \left\{\{1\}, C_{(12)} \cup C_{(1234)}, C_{(12)(34)}, C_{(123)}\right\}\right)$$

is a supercharacter theory for S_4.

2.5 The Δ-Product

Let G be a group and let N and M be normal subgroups of G such that $N \subseteq M$. Similar to the $*$-product, we can obtain another supercharacter theory from an operation called the Δ-product, from the supercharacter theory C of M and the supercharacter theory D of G/N, provided that C and D satisfy certain conditions.

Let $\mathsf{m}_G(N)$ be the finest supercharacter theory in $\mathrm{Sup}_G(N)$ and $\mathsf{m}(G/N)$ be the finest supercharacter theory in $\mathrm{Sup}(G/N)$. We need a little notation. If $Z \subseteq \mathrm{Irr}(G)$ is a union of sets of the form $\mathrm{Irr}(G|\psi)$ for various $\psi \in \mathrm{Irr}(N)$, let $f(Z)$

denote the set of all irreducible constituents of $\sum_{\chi\in Z}\chi_N$, so that $(f(Z))^G = Z$. Moreover, let $\phi : G \to G/N$ be the canonical homomorphism.

Definition. *Let G be a group, let $\mathsf{C} = (\mathcal{X}, \mathcal{K}) \in \mathrm{Sup}(G)$, and let N be a union of superclasses of C. Suppose $(\mathcal{Y}, \mathcal{L}) = \mathsf{m}_G(N) *_N \mathsf{m}(G/N)$. Writing $(\mathcal{X}\vee\mathcal{Y}, \mathcal{K}\vee\mathcal{L}) = (\mathcal{Z}, \mathcal{M})$ and defining ϕ and f as above, let*

$$\mathsf{C}_N = \Big(\{f(Z) : Z \in \mathcal{Z},\ Z \not\subseteq \mathit{Irr}(G/N)\} \cup \{\{1_N\}\}, \{M \in \mathcal{M} : M \subseteq N\}\Big)$$

and

$$\mathsf{C}^{G/N} = \Big(\{Z \in \mathcal{Z} : Z \subseteq \mathit{Irr}(G/N)\}, \{\phi(M) : M \in \mathcal{M},\ M \not\subseteq N\} \cup \{\{N\}\}\Big).$$

Theorem 2.5.1. *(Hendrickson 2012, Theorem 9.1) Let G be a group with normal subgroups $N \leq M \leq G$. Suppose $\mathsf{C} \in \mathrm{Sup}(M)$ and $\mathsf{D} \in \mathrm{Sup}(G/N)$ such that*

1. *N is a union of the superclasses of C.*

2. *M/N is a union of the superclasses of D.*

3. *The "overlap" of the two theories on M/N is the same, i.e., $\mathsf{C}^{M/N} = D_{M/N}$.*

Then there exists a unique supercharacter theory $E \in \mathrm{Sup}(G)$ such that $E_M = \mathsf{C}$ and $E^{G/N} = D$ and every superclass outside M is a union of N-cosets. Using our earlier notation, if $\mathsf{C} = (\mathcal{X}, \mathcal{K})$ and $\mathsf{D} = (\mathcal{Y}, \mathcal{L})$, then

$$E = \Big(\mathcal{Y} \cup \{X^G : X \in \mathcal{X}, X \not\subseteq \mathit{Irr}(M/N)\}, \mathcal{K} \cup \{\widetilde{L} : L \in \mathcal{L}, L \not\subseteq M/N\}\Big).$$

We call this supercharacter theory the Δ-product of C and D. Denote by $\mathrm{Sup}^{\Delta}(G)$ the set of all supercharacter theory construction by the Δ-product.

Example 2.5.2. *Let $G = C_2 \times C_2 \times C_2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$. Let $N = \langle a \rangle$ and $M = \langle a \rangle \times \langle b \rangle$. The following are the character tables of G, M, and G/N.*

	1	c	b	bc	a	ac	ab	abc
$\mathbb{1}$	1	1	1	1	1	1	1	1
χ_1	1	-1	-1	1	-1	1	1	-1
χ_2	1	-1	-1	1	1	-1	-1	1
χ_3	1	-1	1	-1	-1	1	-1	1
χ_4	1	-1	1	-1	1	-1	1	-1
χ_5	1	1	-1	-1	-1	-1	1	1
χ_6	1	1	-1	-1	1	1	-1	-1
χ_7	1	1	1	1	-1	-1	-1	-1

	1	b	a	ab
$\mathbb{1}$	1	1	1	1
ψ_1	1	-1	-1	1
ψ_2	1	-1	1	-1
ψ_3	1	1	-1	-1

	1	cN	bN	bcN
$\mathbb{1}$	1	1	1	1
$\overline{\chi_1}$	1	-1	-1	1
$\overline{\chi_3}$	1	-1	1	-1
$\overline{\chi_5}$	1	1	-1	-1

Note that

$$\mathsf{C} = (\mathcal{X}, \mathcal{K}) = (\{\{\mathbb{1}\}, \{\psi_1, \psi_2\}, \{\psi_3\}\}, \{\{1\}, \{a\}, \{a, ab\}\})$$

and

$$\mathsf{D} = (\mathcal{Y}, \mathcal{L}) = (\{\{\mathbb{1}\}, \{\overline{\chi_1}, \overline{\chi_3}\}, \{\overline{\chi_5}\}\}, \{\{1\}, \{bN\}, \{cN, bcN\}\})$$

are supercharacter theories for M and G/N respectively. Moreover, N is a union of the superclasses of C, M/N is a union of the superclasses of D, and

$$\mathsf{C}^{M/N} = \left(\left\{\{\mathbb{1}\}, \{\overline{\psi_2}\}\right\}, \{\{N\}, \{bN\}\}\right) = D_{M/N}.$$

Therefore by Theorem 2.5.1, we have

$$E = \left(\mathcal{Y} \cup \{X^G : X \in \mathcal{X}, X \not\subseteq Irr(M/N)\}, \mathcal{K} \cup \{\widetilde{L} : L \in \mathcal{L}, L \not\subseteq M/N\}\right) =$$

$$(\{\{\mathbb{1}\}, \{\chi_2, \chi_6\}, \{\chi_4\}, \{\chi_3, \chi_5, \chi_7\}\}, \{\{1\}, \{a\}, \{b, ab\}, \{c, ac, bc, abc\}\})$$

is a supercharacter theory.

2.6 Supercharacter Theory for Algebra Groups

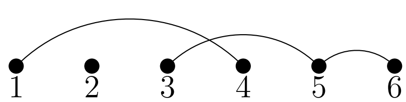
In addition to the generalization of supercharacter theory to arbitrary groups, Diaconis and Isaacs obtained a supercharacter theory for a large class of p-groups, i.e., algebra groups; many authors have studied this supercharacter theory, for instance, see André (1995), André and Neto (2006), Diaconis and Isaacs (2008), Marberg and Thiem (2009), Thiem (2010), Thiem and Venkateswaran (2009), Yan (2001). We now describe this supercharacter theory. We first need the definition of arc diagram of a set partition of $[n] = \{1, 2, \ldots, n\}$.

2.6.1 Set Partition

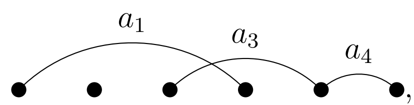
We represent every set partition of $[n]$ by an ***arc diagram***, i.e, if $\{\lambda_i : i \in I\}$ is a set partition of $[n]$, we give it the following arc diagram representation λ,

$$\{i \frown j : \text{ if } i < j \text{ and } i, j \in \lambda_k \text{ for some } \lambda_k\}.$$

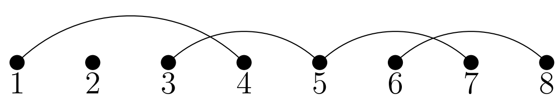
For example the arc diagram representation of $\lambda = \{\{1, 4\}, \{2\}, \{3, 5, 6\}\}$ is

$\lambda =$ 

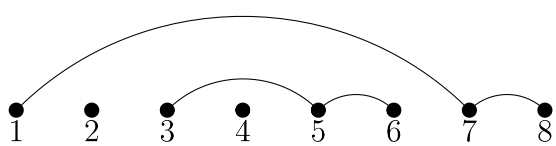
Also, we usually omit the labels from the nodes of arc diagrams. A $\mathbb{F}_q$-***set partition*** of $[n]$ is an arc diagram representation of a set partition in which every arc is labeled by a non-zero element of $\mathbb{F}_q$. For example an $\mathbb{F}_q$-set partition of λ is



Where a_1, a_2, and a_3 are some non-zero elements of $\mathbb{F}_q$. If ν is a set partition of $[n]$, we say that ν is ***nonnesting*** if there is no $k < i < j < l$ such that $k \frown l, i \frown j \in \nu$. The following partition λ of [8] is nonnesting,

$\lambda =$ 

while η is not nonnesting,

$\eta =$ 

2.6.2 A Supercharacter Theory for The Group of Unipotent Upper-Triangular Matrices

Let J be a finite dimensional nilpotent associative algebra over a field $\mathbb{F}$ (without unity), and let $G = \{1 + x : x \in J\}$. It is easy to see that G is a group with the following multiplication

$$(1 + x)(1 + y) = 1 + x + y + xy.$$

The group G, constructed in this way is the ***algebra group*** based on J.

By using a two-sided action of G on J, we define a supercharacter theory $(\mathcal{X}, \mathcal{K})$ for G; the superclasses containing $1 + a$, for $a \in J$, is the set

$$\mathcal{K}_a = \{1 + xay : x, y \in G, a \in J\}.$$

Thus,

$$\mathcal{K} = \{\mathcal{K}_a : a \in J\}.$$

Let the dual of J denoted by J^* be the space of $\mathbb{F}$-linear functionals $\lambda : J \to \mathbb{F}$, i.e., $J^* = \mathrm{Hom}(J, \mathbb{F})$. The algebra group G has a the following two-sided action on J^*,

$$(x\lambda y)(a) = \lambda(x^{-1}ay^{-1}), \qquad \text{for } x, y \in G, a \in J,$$

and the number of two-sided orbits on J and J^* are equal (Diaconis and Isaacs 2008, Lemma 4.1).

Now we construct the supercharacters. Fix a non-trivial homomorphism

$$\theta : \mathbb{F}^+ \to \mathbb{C}^\times.$$

In other words, this map is any of the $|\mathbb{F}| - 1$ non-trivial linear characters of the additive group $\mathbb{F}^+$ of $\mathbb{F}$. Choose an arbitrary $\mathbb{F}$-linear functional λ in a two-sided orbit on J^*, then define the function $\chi_\lambda(x) : G \to \mathbb{C}$ by

$$\chi_\lambda(x) = \frac{|\lambda G|}{|G(x-1)G|} \sum_{a \in G(x-1)G} \theta(\lambda(a)).$$

This function is actually a character of G and is constant on superclasses; also if λ and μ be in the same two-sided orbit on J^*, then $\chi_\lambda = \chi_\mu$. Moreover, if $\mathcal{X}_\lambda$ is the set of irreducible constituent of χ_λ, then $\{\mathcal{X}_\lambda : \lambda \in J^*\}$ is a partition of $\mathrm{Irr}(G)$ (Diaconis and Isaacs 2008, Theorem 5.5 and Theorem 5.8). Therefore,

$$\mathcal{X} = \{\mathcal{X}_\lambda : \lambda \in J^*\}.$$

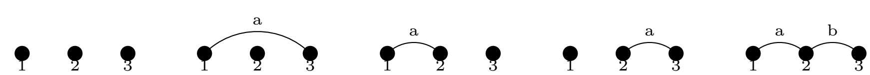
When we consider J as the set of strictly upper-triangular $n \times n$ matrices $\mathfrak{n}_n$, G turns to be $UT_n(q)$, the group of unipotent upper-triangular matrices over the finite field $\mathbb{F}_q$.

We can parametrize the superclasses of $UT_n(q)$ as follows,

$$\left\{ \begin{array}{c} \text{Superclasses} \\ \text{of } UT_n(q) \end{array} \right\}$$
$$\Updownarrow$$
$$\left\{ \begin{array}{c} \text{The unipotent upper-triangular matrices} \\ \text{with at most one nonzero entry in each} \\ \text{row and column strictly above the diagonal} \end{array} \right\}$$
$$\Updownarrow$$
$$\left\{ \mathbb{F}_q\text{-Set Partitions of } n \right\}.$$

Therefore, we index supercharacters and superclasses by $\mathbb{F}_q$-set partitions. The following table lists the correspondences for $n = 3$.

Superclasses	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$



We usually use the notation K_λ and χ_λ for the superclasses and supercharacters of this supercharacter theory respectively.

2.7 Nonnesting Supercharacter Theory for Unipotent Upper-Triangular Matrices

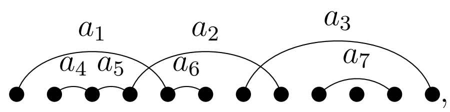
Andrews (2015) constructed a supercharacter theory for $UT_n(\mathbb{F}_q)$. This supercharacter theory is coarser than the one defined by Diaconis and Isaacs (2008). We now describe this supercharacter theory in this section. For proofs of the results in this section refer to Andrews (2015).

Let η be an $\mathbb{F}_q$-set partition, and define

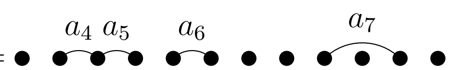
$$\mathrm{sml}(\eta) = \{i \overset{a}{\frown} j \in \eta \mid \text{ there are no } k \overset{b}{\frown} l \in \eta \text{ with } i < k < l < j\} \text{ and}$$

$$\mathrm{big}(\eta) = \{i \overset{a}{\frown} j \in \eta \mid \text{ there are no } k \overset{b}{\frown} l \in \eta \text{ with } k < i < j < l\}.$$

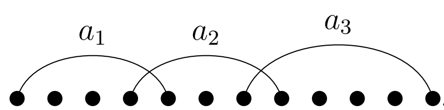
For example when

$\eta =$ 

then we have

$\mathrm{sml}(\eta) =$ 

and

$\mathrm{big}(\eta) =$ 

Note that both $\mathrm{sml}(\eta)$ and $\mathrm{big}(\eta)$ are nonnesting $\mathbb{F}_q$-set partitions. These methods of producing a nonnesting $\mathbb{F}_q$-set partition from an arbitrary $\mathbb{F}_q$-set partition both define equivalence relations on the set of $\mathbb{F}_q$-set partitions, and in both cases the equivalence classes are indexed by the nonnesting $\mathbb{F}_q$-set partitions. For a nonnesting $\mathbb{F}_q$-set partition η, let

$$K_{[\eta]} = \bigcup_{\mathrm{sml}(\nu)=\eta} K_\nu \text{ and}$$

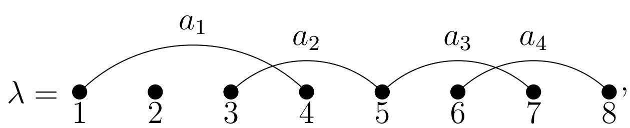
$$\chi_{[\eta]} = \sum_{\mathrm{big}(\nu)=\eta} \chi_\nu,$$

where K_ν and χ_ν are the superclass and supercharacter corresponding to the $\mathbb{F}_q$-set partition ν, respectively.

There is an alternative description of the characters $\chi_{[\eta]}$. We first need to define a subgroup

$$U_\eta = \left\{ g \in UT_n(\mathbb{F}_q) \;\middle|\; \begin{array}{l} g_{ij} = 0 \text{ if there exists } k \overset{a}{\frown} l \in \eta \text{ such that} \\ (i,j) \neq (k,l) \text{ and } k \leq i < j \leq l \end{array} \right\}.$$

For example if



then we have

$$U_\eta = \left\{ \begin{pmatrix} 1 & 0 & 0 & \star & \bullet & \bullet & \bullet & \bullet \\ 0 & 1 & 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 1 & 0 & \star & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 1 & 0 & \star & \bullet \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

where the star entries correspond to the arcs of η.

As a corollary, we can calculate the dimensions of the characters $\chi_{[\eta]}$.

Corollary 2.7.1. *(Andrews 2015, Corollary 5.2) Let η be a nonnesting $\mathbb{F}_q$-set* *partition. Then* $\chi_{[\eta]}(1) = |U : U_\eta|$*, and we have that if*

$$a = \left| \left\{ (i,j) \;\middle|\; \begin{array}{l} i < j \text{ and there exists } k \overset{a}{\frown} l \in \eta \text{ with} \\ \\ (i,j) \neq (k,l) \text{ and } k \leq i < j \leq l \end{array} \right\} \right|,$$

then

$$|U : U_\eta| = q^a.$$

We now calculate the values of the characters $\chi_{[\eta]}$ on the elements of the sets $K_{[\nu]}$.

Proposition 2.7.2. *(Andrews 2015, Proposition 5.3) Let η and ν be nonnesting $\mathbb{F}_q$-set partitions and let $g \in K_{[\nu]}$. Then*

$$\chi_{[\eta]}(g) = \begin{cases} \chi_{[\eta]}(1) \prod\limits_{\substack{i \overset{a}{\frown} j \in \eta \\ i \overset{b}{\frown} j \in \nu}} \theta(ab) & \begin{array}{l} \text{if there are no } i \overset{a}{\frown} j \in \eta \text{ and } k \overset{b}{\frown} l \in \nu \\ \\ \text{with } (i,j) \neq (k,l) \text{and } i \leq k < l \leq j, \end{array} \\ \\ 0 & \text{otherwise.} \end{cases}$$

We can now show that we have constructed a supercharacter theory of $UT_n(\mathbb{F}_q)$.

Theorem 2.7.3. *(Andrews 2015, Theorem 5.4) The sets*

$$\{\chi_{[\eta]} \mid \eta \text{ is a nonnesting } \mathbb{F}_q\text{-set partition}\}$$

and

$$\{K_{[\nu]} \mid \nu \text{ is a nonnesting } \mathbb{F}_q\text{-set partition}\}$$

are the supercharacters and superclasses for a supercharacter theory of $UT_n(\mathbb{F}_q)$.

3 Normal Supercharacter Theories

3.1 Introduction

As we mentioned earlier, in order to find a tractable theory to substitute for the wild character theory of the group of $n \times n$ unipotent upper-triangular matrices over a finite field $\mathbb{F}_q$, André and Yan introduced the notion of supercharacter theory. In this thesis, we construct a supercharacter theory for a finite group from an arbitrary set S of normal subgroups of G. We call such a supercharacter theory the normal supercharacter theory generated by S.

Let $\mathbf{Norm}(G)$ be the set of all normal subgroups of G. The product of two normal subgroups is a normal subgroup. Therefore, $\mathbf{Norm}(G)$ is a semigroup. Let $S \subseteq \mathbf{Norm}(G)$. We define $A(S)$ to be the smallest subsemigroup of $\mathbf{Norm}(G)$ such that

1. $\{1\}, G \in A(S)$.

2. $S \subseteq A(S)$.

3. $A(S)$ is closed under intersection.

Note that $A(S)$ always exists since $\mathbf{Norm}(G)$ satisfies 1, 2, and 3, and also every element $N \in A(S)$ is a normal subgroup of G. We define for an element $N \in A(S)$,

$$N^{\circ}_{A(S)} := N \setminus \bigcup_{H \in A(S), H \subset N} H,$$

In other words, $N^{\circ}_{A(S)}$ is the set of elements g for which the smallest normal subgroup in $A(S)$ containing g is N. For simplicity of notation, we write N° instead of $N^{\circ}_{A(S)}$ when it is clear that N is in $A(S)$.

A subgroup of G is normal if and only if it is the union of a set of conjugacy classes of G. We have an equivalent characterization of normality in terms of the kernels of irreducible characters. Recall that the kernel of a character χ of G is the set $\ker \chi = \{g \in G : \chi(g) = \chi(1)\}$.

A subgroup of G is normal if and only if it is the intersection of the kernels of some finite set of irreducible characters (James and Liebeck 1993, Proposition 17.5); thus the normal subgroups of G are subgroups that we can construct from the character table of G.

For each $N \in A(S)$, let

$$\mathcal{X}^N := \{\psi \in \mathrm{Irr}(G) : N \subseteq \ker \psi\}$$

and

$$\chi^N := \sum_{\psi \in \mathcal{X}^N} \psi(1)\psi.$$

Indeed, $\mathcal{X}^N$ is the set of irreducible characters of G lifting from irreducible characters of G/N. Therefore,

$$\chi^N(g) = \rho_{G/N}(gN),$$

where $\rho_{G/N}$ is the regular character of G/N. Define for every $N \in A(S)$,

$$\mathcal{X}^{N^\bullet}_{A(S)} := \mathcal{X}^N \setminus \bigcup_{\substack{K \in A(S): \\ N \subset K}} \mathcal{X}^K,$$

then

$$N = \bigcap_{\psi \in \mathcal{X}^N} \ker \psi,$$

and

$$\chi^{N^\bullet}_{A(S)} := \sum_{\psi \in \mathcal{X}^{N^\bullet}_{A(S)}} \psi(1)\psi.$$

By convention $\chi^{N^\bullet}_{A(S)} = 0$, whenever $\mathcal{X}^{N^\bullet}_{A(S)} = \emptyset$. For simplicity of notation, we write $\mathcal{X}^{N^\bullet}$ and $\chi^{N^\bullet}$ instead of $\mathcal{X}^{N^\bullet}_{A(S)}$ and $\chi^{N^\bullet}_{A(S)}$ respectively when it is clear that N is in $A(S)$.

In Theorem 3.2.3, we will show that for an arbitrary subset $S \subseteq \mathbf{Norm}(G)$,

$$\Big(\{\mathcal{X}^{N^\bullet} \neq \emptyset : N \in A(S)\}, \{N^\circ \neq \emptyset : N \in A(S)\}\Big)$$

is a supercharacter theory of G. We call this supercharacter theory the ***normal supercharacter theory*** generated by S.

Note that when we have a larger set of normal subgroups, the normal supercharacter theory we obtain will be finer. In particular the finest normal supercharacter theory is obtained when we consider the set of all normal subgroups of G, and is related to a partition of G given by certain values on the primitive central idempotents.

The lattice of normal subgroups has been well studied (for example see Baer (1938b), Birkhoff (2012), Jońsson (1954)); we show that every sublattice of the lattice of normal subgroups of G containing the trivial subgroup and G yields a normal supercharacter theory and vice versa.

A character χ of a group G is said to be ***integral*** if $\chi(g) \in \mathbb{Q}$ for every element $g \in G$; a supercharacter theory is said to be ***integral*** if its supercharacters are integral.

In Section 3.2, we show that the normal supercharacter theory generated by an arbitrary subset $S \subseteq \mathbf{Norm}(G)$ is integral. Furthermore, we provide a recursive formula for the supercharacters of any normal supercharacter theory. In Section 3.3, we study the finest normal supercharacter theory and we show that the finest normal supercharacter theory is obtained by considering certain values of the primitive central idempotents. Also, It is shown that $\mathcal{X}^{N^\bullet}$ is equal to the set of all faithful irreducible characters of G/N. In the end of this section, we provide an algorithm to construct the finest normal supercharacter theory from the character table. In section 3.4, we argue that, in general, the normal supercharacter theory cannot be obtained by the constructions described in Sections 2.2, 2.3, and 2.4. In the end, we study the relation between normal supercharacter theory and Andrews' construction.

3.2 Normal Supercharacter Theories

In this section we show that our constructions are indeed supercharacter theories of G. Furthermore, we show that these supercharacter theories are integral.

Proposition 3.2.1. *Let G be a group.*

1. *For any arbitrary subset S of $\mathbf{Norm}(G)$, $A(S)$ is a sublattice of the lattice of normal subgroups containing identity subgroup and G.*

2. *Every sublattice of the lattice of normal subgroups that contains the identity subgroup and G is of the form $A(S)$.*

Proof. (1) Note that $A(S)$ is a poset with inclusion as the binary relation. Let $N, H \in A(S)$. Then the least upper bound of N and H exists and that is $NH \in A(S)$; also the the greater lower bound of N and H exists and that is $N \cap H \in A(S)$. Therefore, $A(S)$ is a sublattice of the lattice of normal subgroups of G containing the trivial subgroup and G.

(2) Let L be a a sublattice of the lattice of normal subgroups of G containing the trivial subgroup and G. Suppose S is the set of all elements of L. Then $A(S) = L$. $\square$

Lemma 3.2.2. *Let G be a group. Then the following holds for an arbitrary subset $S \subseteq \mathbf{Norm}(G)$.*

1. $\left\{\mathcal{X}^{N^\bullet} \neq \emptyset : N \in A(S)\right\}$ *is a partition of Irr(G).*

2. $\{N^\circ \neq \emptyset : N \in A(S)\}$ *is a partition of G.*

Proof. Recall that for every $N \in A(S)$,

$$N^\circ = N \setminus \bigcup_{H \in A(S), H \subset N} H.$$

Therefore, $g \in N^\circ$ if and only if N is the smallest normal subgroup in $A(S)$ containing g. This implies that $N^\circ \cap K^\circ = \emptyset$ for every pair of non-equal normal subgroups $N, K \in A(S)$. Also, since $G \in A(S)$, for every $g \in G$ there exists a normal subgroup $N \in A(S)$ such that $g \in N^\circ$. This completes the proof of (2).

Recall that for every $N \in A(S)$, $\mathcal{X}^N := \{\psi \in \operatorname{Irr}(G) : N \subseteq \ker\psi\}$ and

$$\mathcal{X}^{N^\bullet} := \mathcal{X}^N \setminus \bigcup_{\substack{K \in A(S): \\ N \subset K}} \mathcal{X}^K.$$

Therefore, $\mathcal{X}^{N^\bullet}$ is the set of irreducible characters ψ of G such that if for a normal subgroup $H \in A(S)$, $H \subseteq \ker\psi$, then $H \subseteq N$. This implies that $\mathcal{X}^{N^\bullet} \cap \mathcal{X}^{K^\bullet} = \emptyset$ for every pair of non-equal normal subgroups $N, K \in A(S)$. Also, since $\mathcal{X}^{\{1\}} = \operatorname{Irr}(G)$, for every $\psi \in \operatorname{Irr}(G)$ there exists a normal subgroup $N \in A(S)$ such that $\psi \in \mathcal{X}^{N^\bullet}$. This completes the proof of (1).

□

Theorem 3.2.3. *Let G be a group. Then for an arbitrary subset $S \subseteq \mathbf{Norm}(G)$,*

$$\Big(\{\mathcal{X}^{N^\bullet} \neq \emptyset : N \in A(S)\}, \{N^\circ \neq \emptyset : N \in A(S)\}\Big)$$

is an integral supercharacter theory of G. Moreover,

$$\chi^{N^\bullet}(g) = \sum_{\substack{M \in A(S): \\ g \in M, N \subseteq M}} \mu(N, M)\frac{|G|}{|M|}.$$

Proof. Let $\mathcal{X} = \{\mathcal{X}^{N^\bullet} \neq \emptyset : N \in A(S)\}$ and $\mathcal{K} = \{N^\circ \neq \emptyset : N \in A(S)\}$. By Lemma 3.2.2, $\mathcal{X}$ is a partition of $\operatorname{Irr}(G)$ and $\mathcal{K}$ is a partition of G. Recall that

$$\chi^{N^\bullet} := \sum_{\psi \in \mathcal{X}^{N^\bullet}_{A(S)}} \psi(1)\psi.$$

We aim to show that $(\mathcal{X}, \mathcal{K})$ is an integral supercharacter theory of G with the set of supercharacters $\{\chi^{N^\bullet} \neq 0 : N \in A(S)\}$. Therefore, we only need to show the following holds.

1. The set $\{1\}$ is a member of $\mathcal{K}$.

2. $|\mathcal{X}| = |\mathcal{K}|$.

3. The characters $\chi^{N^\bullet}$ are integral and constant on the parts of $\mathcal{K}$.

Since $\{1\} = \{1\}^\circ$, $\{1\}$ is a member of $\mathcal{K}$. Therefore, (1) holds.

Define for every $N \in A(S)$,

$$f^{N^\circ}(g) := \begin{cases} 1 & \text{if } g \in N^\circ, \\ 0 & \text{Otherwise.} \end{cases}$$

Thus for $K \in A(S)$ we have,

$$\sum_{\substack{N \in A(S): \\ N \subseteq K}} f^{N^\circ}(g) = \begin{cases} 1 & \text{if } g \in K, \\ 0 & \text{Otherwise.} \end{cases}$$

Recall that

$$\chi^K(g) = \rho_{G/K}(gK) = \begin{cases} \frac{|G|}{|K|} & \text{if } g \in K, \\ 0 & \text{Otherwise.} \end{cases}$$

Therefore,

$$\chi^K(g) = \frac{|G|}{|K|} \sum_{\substack{N \in A(S): \\ N \subseteq K}} f^{N^\circ}(g).$$

It follows from Möbius inversion formula (Lemma 1.2.1) that

$$f^{K^\circ}(g) = \sum_{\substack{N \in A(S): \\ N \subseteq K}} \frac{|N|}{|G|} \mu(N, K) \chi^N(g).$$

Since

$$\chi^N = \sum_{\substack{M \in A(S): \\ N \subseteq M}} \chi^{M^\bullet},$$

we conclude that

$$f^{K^\circ} = \sum_{\substack{N \in A(S): \\ N \subseteq K}} \left(\frac{|N|}{|G|} \sum_{\substack{M \in A(S): \\ N \subseteq M}} \mu(N, K) \chi^{M^\bullet} \right).$$

Therefore,

$$\mathbb{C}\text{-Span}\{\chi^{N^\bullet} : N \in A(S)\} = \mathbb{C}\text{-Span}\{f^{N^\circ} : N \in A(S)\}.$$

Note that

$$|\mathcal{X}| = \dim \left(\mathbb{C}\text{-Span}\{\chi^{N^\bullet} : N \in A(S)\} \right) = \dim \left(\mathbb{C}\text{-Span}\{f^{N^\circ} : N \in A(S)\} \right) = |\mathcal{K}|.$$

We can see that (2) holds.

Since

$$\chi^N = \sum_{\substack{M \in A(S): \\ N \subseteq M}} \chi^{M^\bullet},$$

by the dual of Möbius inversion formula (Lemma bbc1.2.2) we have

$$\chi^{N^\bullet} = \sum_{\substack{M \in A(S): \\ N \subseteq M}} \mu(N, M) \chi^M.$$

Note that

$$\chi^M(g) = \rho_{G/M}(gM) = \begin{cases} \frac{|G|}{|M|} & \text{if } g \in M, \\ \\ 0 & \text{Otherwise.} \end{cases}$$

It follows that

$$\chi^{N^\bullet}(g) = \sum_{\substack{M \in A(S): \\ g \in M, N \subseteq M}} \mu(N, M) \frac{|G|}{|M|}.$$

Let $h, h_1 \in H^\circ \in \mathcal{K}$. Then for every normal subgroup M of G, $h \in M$ if and only if $H^\circ \subseteq M$ if and only if $h_1 \in M$. So $\chi^{N^\bullet}(h) = \chi^{N^\bullet}(h_1)$. Thus, $\chi^{N^\bullet}$ is integral and constant on each part of $\mathcal{K}$. Therefore, (3) holds. $\square$

In the following theorem we give a recursive formula for supercharacter $\chi^{N^\bullet}$ of the normal supercharacter theory generated by $S \subseteq \mathbf{Norm}(G)$.

Theorem 3.2.4. *Let G be a group and let S be an arbitrary subset of $\mathbf{Norm}(G)$. Then for any normal subgroup $N \in A(S)$*

$$\chi^{N^\bullet}(g) = \begin{cases} \displaystyle\sum_{\psi \in \mathcal{X}^{N^\bullet}} \psi(1)^2 & g \in N, \\ \\ \displaystyle -\sum_{\substack{K \in A(S): \\ N \subset K}} \chi^{K^\bullet}(g) & g \notin N. \end{cases}$$

Proof. Let $N \in A(S)$. Consider the following two cases.

Case 1. $g \in N$. We have

$$\chi^{N^\bullet}(g) = \sum_{\psi \in \mathcal{X}^{N^\bullet}} \psi(1)\psi(g).$$

For every $\psi \in \mathcal{X}^{N^\bullet}$, we see that $N \subseteq \ker \psi$. Thus, $\psi(g) = \psi(1)$. Therefore,

$$\chi^{N^\bullet}(g) = \sum_{\psi \in \mathcal{X}^{N^\bullet}} \psi(1)^2.$$

Case 2. $g \notin N$. Note that

$$\operatorname{Irr}(G/N) = \{\overline{\psi} : \psi \in \mathcal{X}^N\} = \bigcup_{\substack{K \in A(S): \\ N \subseteq K}} \{\overline{\psi} : \psi \in \mathcal{X}^{K^\bullet}\}.$$

Therefore,

$$\rho_{G/N} = \sum_{\substack{K \in A(S): \\ N \subseteq K}} \sum_{\psi \in \mathcal{X}^{K^\bullet}} \overline{\psi}(N)\overline{\psi}.$$

Thus,

$$0 = \rho_{G/N}(gN) = \sum_{\substack{K \in A(S): \\ N \subseteq K}} \sum_{\psi \in \mathcal{X}^{K^\bullet}} \overline{\psi}(N)\overline{\psi}(gN) =$$

$$\sum_{\substack{\psi \in \mathcal{X}^{K^\bullet}: \\ N \subset K \in A(S)}} \overline{\psi}(1)\overline{\psi}(gN) + \sum_{\psi \in \mathcal{X}^{N^\bullet}} \overline{\psi}(1)\overline{\psi}(gN) =$$

$$\sum_{\substack{\psi \in \mathcal{X}^{K^\bullet}: \\ N \subset K \in A(S)}} \psi(1)\psi(g) + \sum_{\psi \in \mathcal{X}^{N^\bullet}} \psi(1)\psi(g) =$$

$$\sum_{\substack{\psi \in \mathcal{X}^{K^\bullet}: \\ N \subset K \in A(S)}} \psi(1)\psi(g) + \chi^{N^\bullet}(g).$$

It follows that

$$\chi^{N^\bullet}(g) = - \sum_{\substack{\psi \in \mathcal{X}^{K^\bullet}: \\ N \subset K \in A(S)}} \psi(1)\psi(g).$$

Also,

$$\sum_{\substack{\psi \in \mathcal{X}^{K^\bullet}: \\ N \subset K \in A(S)}} \psi(1)\psi(g) = \sum_{N \subset K \in A(S)} \chi^{K^\bullet}(g).$$

Thus,

$$\chi^{N^\bullet}(g) = -\sum_{N \subset K \in A(S)} \chi^{K^\bullet}(g).$$

We can conclude from Cases 1 and 2 that

$$= \begin{cases} \displaystyle\sum_{\psi \in \mathcal{X}^{N^\bullet}} \chi(1)^2 & g \in N, \\ \\ \displaystyle -\sum_{N \subset K \in A(S)} \chi^{K^\bullet}(g) & g \notin N. \end{cases}$$

$\square$

Example 3.2.5. *Let $G = C_4 \times C_4 = \langle a \rangle \times \langle b \rangle$. The following is the character table of G.*

	1	b	b^2	b^3	a	ab	ab^2	ab^3	a^2	a^2b	a^2b^2	a^2b^3	a^3	a^3b	a^3b^2	a^3b^3
$\mathbf{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
χ_2	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
χ_4	1	-1	1	-1	$-\zeta_4$	ζ_4	$-\zeta_4$	ζ_4	-1	1	-1	1	ζ_4	$-\zeta_4$	ζ_4	$-\zeta_4$
χ_5	1	-1	1	-1	ζ_4	$-\zeta_4$	ζ_4	$-\zeta_4$	-1	1	-1	1	$-\zeta_4$	ζ_4	$-\zeta_4$	ζ_4
χ_6	1	1	1	1	$-\zeta_4$	$-\zeta_4$	$-\zeta_4$	$-\zeta_4$	-1	-1	-1	-1	ζ_4	ζ_4	ζ_4	ζ_4
χ_7	1	1	1	1	ζ_4	ζ_4	ζ_4	ζ_4	-1	-1	-1	-1	$-\zeta_4$	$-\zeta_4$	$-\zeta_4$	$-\zeta_4$
χ_8	1	$-\zeta_4$	-1	ζ_4	-1	ζ_4	1	$-\zeta_4$	1	$-\zeta_4$	-1	ζ_4	-1	ζ_4	1	$-\zeta_4$
χ_9	1	ζ_4	-1	$-\zeta_4$	-1	$-\zeta_4$	1	ζ_4	1	ζ_4	-1	$-\zeta_4$	-1	$-\zeta_4$	1	ζ_4
χ_{10}	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	ζ_4	1	$-\zeta_4$	-1	$\zeta_4 4$	1	$-\zeta_4$	-1	ζ_4
χ_{11}	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$	1	ζ_4	-1	$-\zeta_4$
χ_{12}	1	$-\zeta_4$	-1	ζ_4	$-\zeta_4$	-1	ζ_4	1	-1	ζ_4	1	$-\zeta_4$	ζ_4	1	$-\zeta_4$	-1
χ_{13}	1	ζ_4	-1	$-\zeta_4$	ζ_4	-1	$-\zeta_4$	1	-1	$-\zeta_4$	1	ζ_4	$-\zeta_4$	1	ζ_4	-1
χ_{14}	1	$-\zeta_4$	-1	ζ_4	ζ_4	1	$-\zeta_4$	-1	-1	ζ_4	1	$-\zeta_4$	$-\zeta_4$	-1	ζ_4	1
χ_{15}	1	ζ_4	-1	$-\zeta_4$	$-\zeta_4$	1	ζ_4	-1	-1	$-\zeta_4$	1	ζ_4	ζ_4	-1	$-\zeta_4$	1

The following is the supercharacter table of the normal supercharacter theory generated by

$$S = \{\langle a^2\rangle \times \langle b^2\rangle, \langle a^2\rangle \times 1, 1 \times \langle b^2\rangle\}.$$

For the lattice of $A(S)$ see Example 1.2.3. Let $\chi_{i_1,\ldots,i_k} := \chi_{i_1} + \ldots + \chi_{i_k}$.

	$\{1\}$	$\{a^2\}$	$\{b^2\}$	$\{a^2b^2\}$	$G \setminus \langle a^2\rangle \times \langle b^2\rangle$
$\mathbb{1}$	1	1	1	1	1
$\chi_{1,2,3}$	3	3	3	3	-1
$\chi_{4,5,6,7}$	4	-4	4	-4	0
$\chi_{8,9,10,11}$	4	4	-4	-4	0
$\chi_{12,13,14,15}$	4	-4	-4	4	0

3.3 Primitive Central Idempotents, Faithful Irreducible Characters, and The Finest Normal Supercharacter Theory

In this section we investigate the connection between primitive central idempotents, faithful irreducible characters, and the finest normal supercharacter theory.

3.3.1 Primitive Central Idempotents and The Finest Normal Supercharacter Theory

Let C_g be the conjugacy class of group G containing g. For every subset $K \subseteq G$, let $\widehat{K} = \sum_{k \in K} k$. By (Lam 1991, Proposition 8.15 (1)) every character $\chi \in \mathrm{Irr}(G)$ has a corresponding primitive central idempotent in the group algebra $\mathbb{C}G$

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g.$$

These idempotents are orthogonal, i.e. $e_\chi e_\psi = 0$ when $\chi, \psi \in \mathrm{Irr}(G)$ and $\chi \neq \psi$. Also by (Lam 1991, Proposition 8.15 (2))

$$\widehat{C_g} = \sum_i \frac{|C_g|}{\chi_i(1)} \chi_i(g) e_{\chi_i}.$$

Therefore,

$$|C_g|1 - \widehat{C_g} = |C_g|1 - \sum_i \frac{|C_g|}{\chi_i(1)} \chi_i(g) e_{\chi_i} = |C_g| \left(1 - \sum_i \frac{1}{\chi_i(1)} \chi_i(g) e_{\chi_i}\right) =$$

$$|C_g| \left(\sum_i e_{\chi_i} - \sum_i \frac{1}{\chi_i(1)} \chi_i(g) e_{\chi_i}\right) = |C_g| \left(\sum_i \left(1 - \frac{1}{\chi_i(1)} \chi_i(g)\right) e_{\chi_i}\right).$$

Thus,

$$1 - \frac{1}{|C_g|}\widehat{C_g} = \sum_i \left(1 - \frac{\chi_i(g)}{\chi_i(1)}\right) e_{\chi_i} = \sum_i \left(1 - \frac{\chi_i(g)}{\chi_i(1)}\right) e_{\chi_i}.$$

Using the last equation i.e., $1-\frac{1}{|C_g|}\widehat{C_g}=\sum_i\left(1-\frac{\chi_i(g)}{\chi_i(1)}\right)e_{\chi_i}$, we will produce some subsets of primitive irreducible characters of G. Define for an element $g\in G$,

$$E_g := \left\{e_{\chi_i} : 1-\frac{\chi_i(g)}{\chi_i(1)} \neq 0\right\} \text{ and } K_g := \bigcup_{E_g=E_h} C_h.$$

In other words, K_g is corresponding to all elements $|C_g|1-\widehat{C_g}$ with the same annihilator in $\mathbb{C}G$.

Since each irreducible character has a corresponding primitive central idempotent, we will see that some subsets of the primitive central idempotents gives a supercharacter theory. But every partition of primitive central idempotents does not corresponded to a supercharacter theory and it is not clear when a partition of primitive central idempotents is related to a supercharacter theory. We show that the set $\{E_g : g\in G\}$ corresponds to the finest normal supercharacter theory which is the one generated by $S=\mathbf{Norm}(G)$.

Example 3.3.1. *Let $D_8=\langle a,x|a^4=x^2=1, xax=a^3\rangle$. The following is the* *character table of D_8.*

	$\{1\}$	$\{ax, a^3x\}$	$\{x, ax\}$	$\{a, a^3\}$	$\{a^2\}$
$\mathbb{1}$	1	1	1	1	1
χ_1	1	-1	-1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
χ_4	2	0	0	0	-2

We have $E_1 = \{\}$, $E_{ax} = E_{a^3x} = \{\chi_1, \chi_2, \chi_4\}$, $E_x = E_{ax} = \{\chi_1, \chi_3, \chi_4\}$, $E_a = E_{a^3} = \{\chi_2, \chi_3, \chi_4\}$, and $E_{a^2} = \{\chi_4\}$. Also, $K_1 = \{1\}$, $K_{ax} = K_{a^3x} = \{ax, a^3x\}$, $K_x = K_{ax} = \{x, ax\}$, $K_a = K_{a^3} = \{a, a^3\}$, and $K_{a^2} = \{a^2\}$.

Lemma 3.3.2. *Let G be a group and let S be the set of all normal subgroups of G and let $g \in G$. If*

$$N = \bigcap_{\chi \in Irr(G) \setminus \{\chi_i : e_{\chi_i} \in E_g\}} \ker \chi,$$

then $K_g = N^{\circ}_{A(S)}$.

Proof. Let $k \in K_g$. Then $E_k = E_g$, and so $k \in \ker \chi$ for every $\chi \in \mathrm{Irr}(G) \setminus \{\chi_i : e_{\chi_i} \in E_g\}$. Therefore, $k \in N$. Let H be a normal subgroup of G such that $H \subset N$. Then at least there is an irreducible character $\psi \in \mathrm{Irr}(G)$ such that $H \subseteq \ker \psi$, but $N \not\subseteq \ker \psi$. If $k \in H$, then

$$k \in \bigcap_{\chi \in \mathrm{Irr}(G) \setminus \{\chi_i : e_{\chi_i} \in E_g\}} \ker \chi \cap \ker \psi,$$

and so $E_k \neq E_g$. Thus, $k \notin K_g$, yielding a contradiction. Therefore, k is in N, but k is not in any normal subgroup of G such that $H \subset N$, i.e., $k \in N^\circ$. So $K_g \subseteq N^\circ$.

Let $h \in N^\circ$. Then $E_h \subseteq E_g$. If $E_g \neq E_h$, there is an irreducible character $\psi \in \{\chi_i : e_{\chi_i} \in E_g\}$ such that $h \in \ker\psi$. Let $H = N \cap \ker\psi$. Then $h \notin N \setminus (N \cap \ker\psi)$. Therefore, $h \notin N^\circ$, yielding a contradiction. We can conclude that $E_g = E_h$, and so $h \in K_g$. Thus, $N^\circ \subseteq K_g$. □

Theorem 3.3.3. *Let G be a group and let S be the set of all normal subgroups of G.*

1. *For every $g \in G$, there is a normal subgroup N of G such that $K_g = N^\circ$.*

2. *Let N be a normal subgroup of G. If $N^\circ_{A(S)} \neq \emptyset$, then for every $g \in N^\circ_{A(S)}$, $K_g = N^\circ_{A(S)}$.*

Proof. (1) Let

$$N = \bigcap_{\chi \in \mathrm{Irr}(G) \setminus \{\chi_i : e_{\chi_i} \in E_g\}} \ker\chi.$$

Then by Lemma 3.3.2, $K_g = N^\circ$.

(2) Let N be a normal subgroup of G such that $N^\circ \neq \emptyset$. Let $g \in N^\circ$. We show that $K_g = N^\circ$. Assume that $N = \bigcap_{i \in I} \ker\chi_i$. If there is an irreducible character $\chi \in \mathrm{Irr}(G) \setminus \{\chi_i : i \in I\}$ such that $g \in \ker\chi$, then $g \in H = \bigcap_{i \in I} \ker\chi_i \cap \ker\chi$.

Thus, $g \in H \subset N$, and so $g \notin N^{\circ}$, yielding a contradiction. Therefore, $E_g = \{e_\chi : \chi \in \mathrm{Irr}(G) \setminus \{\chi_i : i \in I\}\}$. By Lemma 3.3.2, $K_g = N^{\circ}$. □

As a result of Theorem 3.3.3 we have the following corollary.

Corollary 3.3.4. *Let G be a group. Then the finest normal supercharacter theory of G has*

$$\{K_g : g \in G\}$$

as the set of superclasses.

Let G be a group. Then for every subset X of G we define the ***normal closure*** or ***conjugate closure*** of X by

$$N_G(X) := \langle gxg^{-1} : x \in X, g \in G\rangle = \bigcap_{\substack{N \in \mathbf{Norm}(G): \\ X \subseteq N}} N.$$

When $X = \{g\}$ for some $g \in G$, we denote by g^G the normal closure of X. For every element $g \in G$, let $[g] = \{h \in G : g^G = h^G\}$.

Lemma 3.3.5. *Let G be a group and S be the set of all normal subgroups of G. Let N be a normal subgroup of G. If $N^{\circ}_{A(S)} \neq \emptyset$, then for every $g \in N^{\circ}_{A(S)}$, $[g] = N^{\circ}_{A(S)}$.*

Proof. Let $g \in N^{\circ} \neq \emptyset$. Then $g^G \subseteq N$. Assume that $h \in [g]$ but $h \notin N^{\circ}$. Then there exists a normal subgroup H of G such that $h \in H$, and so $h^G \subseteq H$. Thus $g^G = h^G \subseteq H$, i.e, $g \notin N^{\circ}$, a contradiction. Therefore, $[g] \subseteq N^{\circ}$.

We now show that $N^\circ \subseteq [g]$. Let $h \in N^\circ$. If $h^G \neq g^G$, then h^G is a normal subgroup of G such that $h^G \subset N$. Thus $h \notin N^\circ$, a contradiction. Therefore, $N^\circ \subseteq [g]$ for every $g \in N^\circ$. □

Theorem 3.3.6. *Let G be a group. Then the finest normal supercharacter theory of G has*

$$\{[g] : g \in G\}$$

as the set of superclasses.

Proof. Note that $\{N^\circ \neq \emptyset : N \in \mathbf{Norm}(G)\}$ is a partition of G and also by Lemma 3.3.5 every $N^\circ \neq \emptyset$ is equal to $[g]$ for some $g \in G$. We can conclude that

$$\{[g] : g \in G\} = \{N^\circ \neq \emptyset : N \in \mathbf{Norm}(G)\},$$

which is the set of superclasses of the finest normal supercharacter theory of G. □

3.3.2 Faithful Irreducible Characters and The Finest Normal Supercharacter Theory

A character ψ of a group G is called faithful if $\ker \psi$ is trivial. In this subsection, we investigate the connection between faithful irreducible characters and the finest normal supercharacter theory.

For a normal subgroup N of G, define

$$\mathcal{X}_N := \{\psi \in \mathrm{Irr}(G) : \overline{\psi} \text{ is a faithful irreducible character of } G/N\}$$

and

$$\chi_N = \sum_{\psi \in \mathcal{X}_N} \psi(1)\psi.$$

Proposition 3.3.7. *Let G be a group. Then the set of supercharacters for the finest normal supercharacter theory is*

$$\{\chi_N \neq 0 : N \in \mathbf{Norm}(G)\}.$$

Proof. It is enough to show that

$$\{\mathcal{X}_N \neq \emptyset : N \in \mathbf{Norm}(G)\} = \{\mathcal{X}^{N^\bullet} \neq \emptyset : N \in \mathbf{Norm}(G)\}.$$

For an arbitrary normal subgroup of N, let $\psi \in \mathcal{X}^{N^\bullet}$. By the definition of $\mathcal{X}^{N^\bullet}$, we have $N \subseteq \ker\psi$ but $K \not\subseteq \ker\psi$ for any normal subgroup K of G with $N \subset K$. Since $\ker\psi$ is a normal subgroup, we can conclude that $\ker\psi = N$. Therefore, we have $\overline{\psi}$ is a faithful irreducible character of G/N, and so $\psi \in \mathcal{X}_N$. Now, let $\psi \in \mathcal{X}_N$. Then $N = \ker\psi$, and so $\psi \in \mathcal{X}^{N^\bullet}$. We can conclude that $\mathcal{X}_N = \mathcal{X}^{N^\bullet}$. $\square$

3.3.3 Constructing the Supercharacter Table of the Finest Normal Supercharacter Theory From the Character Table

For a $n \times n$ matrix $T = (t_{ij})$ define

$$T_j = \{(k, j) : t_{kj} = 1\} \text{ and } T^i = \{(i, k) : t_{ik} = 1\}.$$

The following steps constructs the finest normal supercharacter theory of a given group G from its character table.

1. Divide each row i of the character table T of G by $\chi_i(1)$. Denote the new table by A.

2. Rearrange the columns of A such that two columns i and j are consecutive if $A_i = A_j$. Denote the new table by B.

3. Draw some vertical lines in B to distinguish the classification of conjugacy classes with the same A_i. This lines make a partition of G, and this partition is the set of superclasses of the finest normal supercharacter theory of G.

4. Rearrange the rows of B such that two rows i and j are consecutive if $B^i = B^j$. Denote the new table by C.

5. Draw some horizontal lines in C to distinguish the classification of irreducible characters with the same B_i. These lines make a partition of

$\mathrm{Irr}(G)$. This partition is the partition of irreducible characters of the finest normal supercharacter theory of G.

Note that this algorithm works since the first three steps give us the partition $\{K_g : g \in G\}$ and the last three steps give us the partition $\{\mathcal{X}_N : N \in \mathbf{Norm}(G)\}$.

Example 3.3.8. *(1) Let $G = C_3 \times C_3 = \langle a \rangle \times \langle b \rangle$. We construct the supercharacter table for the finest normal supercharacter theory of G.*

	1	b	b^2	a	ab	ab^2	a^2	a^2b	a^2b^2
$\mathbb{1}$	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	$-\zeta_3-1$	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3	ζ_3	ζ_3
χ_3	1	1	1	ζ_3	ζ_3	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	$-\zeta_3-1$
χ_4	1	$-\zeta_3-1$	ζ_3	1	$-\zeta_3-1$	ζ_3	1	$-\zeta_3-1$	ζ_3
χ_5	1	ζ_3	$-\zeta_3-1$	1	ζ_3	$-\zeta_3-1$	1	ζ_3	$-\zeta_3-1$
χ_6	1	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	ζ_3	1	ζ_3	1	$-\zeta_3-1$
χ_7	1	ζ_3	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	1	$-\zeta_3-1$	1	ζ_3
χ_8	1	$-\zeta_3-1$	ζ_3	ζ_3	1	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3	1
χ_9	1	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	1	ζ_3	ζ_3	$-\zeta_3-1$	1

$$\Downarrow$$

	1	b	b^2	a	a^2	ab^2	a^2b	ab	a^2b^2
$\mathbb{1}$	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	ζ_3	ζ_3	$-\zeta_3-1$
χ_3	1	1	1	ζ_3	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3
χ_4	1	$-\zeta_3-1$	ζ_3	1	1	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3
χ_5	1	ζ_3	$-\zeta_3-1$	1	1	$-\zeta_3-1$	ζ_3-1	ζ_3	$-\zeta_3-1$
χ_6	1	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	ζ_3	1	1	ζ_3	$-\zeta_3-1$
χ_7	1	ζ_3	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	1	1	$-\zeta_3-1$	ζ_3
χ_8	1	$-\zeta_3-1$	ζ_3	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3	1	1
χ_9	1	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3	ζ_3	$-\zeta_3-1$	1	1

$$\Downarrow$$

	1	b	b^2	a	a^2	ab^2	a^2b	ab	a^2b^2
$\mathbb{1}$	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	ζ_3	ζ_3	$-\zeta_3-1$
χ_3	1	1	1	ζ_3	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3
χ_4	1	$-\zeta_3-1$	ζ_3	1	1	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3
χ_5	1	ζ_3	$-\zeta_3-1$	1	1	$-\zeta_3-1$	ζ_3-1	ζ_3	$-\zeta_3-1$
χ_6	1	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	ζ_3	1	1	ζ_3	$-\zeta_3-1$
χ_7	1	ζ_3	$-\zeta_3-1$	ζ_3	$-\zeta_3-1$	1	1	$-\zeta_3-1$	ζ_3
χ_8	1	$-\zeta_3-1$	ζ_3	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3	1	1
χ_9	1	ζ_3	$-\zeta_3-1$	$-\zeta_3-1$	ζ_3	ζ_3	$-\zeta_3-1$	1	1

$$\Downarrow$$

The set of supercharacters of the finest normal supercharacter theory is

$$\{\{\mathbb{1}\}, \{\chi_2, \chi_3\}, \{\chi_4, \chi_5\}, \{\chi_6, \chi_7\}, \{\chi_8, \chi_9\}\}$$

The set of superclasses of the finest normal supercharacter theory is

$$\{\{1\}, \{a, a^2\}, \{b, b^2\}, \{ab^2, a^2b\}, \{ab, a^2b^2\}\}.$$

$$\Downarrow$$

Here is the supercharacter table for the finest normal supercharacter theory of G.

	$\{1\}$	$\{b, b^2\}$	$\{a, a^2\}$	$\{ab^2, a^2b\}$	$\{ab, a^2b^2\}$
$\mathbb{1}$	1	1	1	1	1
$\chi_2 + \chi_3$	2	2	-1	-1	-1
$\chi_4 + \chi_5$	2	-1	2	-1	-1
$\chi_6 + \chi_7$	2	-1	-1	2	-1
$\chi_8 + \chi_9$	2	-1	-1	-1	2

(2) We construct the supercharacter table for the finest normal supercharacter theory of S_3.

	e	(12)	(123)
$\mathbb{1}$	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

$$\Downarrow$$

	e	(12)	(123)
$\mathbb{1}$	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

$$\Downarrow$$

	e	(12)	(123)
$\mathbb{1}$	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

This shows that the finest normal supercharacter theory is the same as the character theory of S_3.

(3) We construct the supercharacter table for the finest normal supercharacter theory of S_4.

	e	(12)	$(12)(34)$	(123)	(1234)
$\mathbb{1}$	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_2	3	1	-1	0	-1
χ_3	3	-1	-1	0	1
χ_4	2	0	2	-1	0

$$\Downarrow$$

	e	$(12)(34)$	(123)	(12)	(1234)
$\mathbb{1}$	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	3	-1	0	1	-1
χ_3	3	-1	0	-1	1
χ_4	2	2	-1	0	0

$$\Downarrow$$

	e	$(12)(34)$	(123)	(12)	(1234)
$\mathbb{1}$	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	3	-1	0	1	-1
χ_3	3	-1	0	-1	1
χ_4	2	2	-1	0	0

$$\Downarrow$$

The set of supercharacters of the finest normal supercharacter theory is

$$\{\{\mathbb{1}\}, \{\chi_2\}, \{\chi_2, \chi_3\}, \{\chi_4\}\}$$

The set of superclasses of the finest normal supercharacter theory is

$$\{\{e\}, C_{(12)(34)}, C_{(123)}, C_{(12)} \cup C_{(1234)}\}.$$

$$\Downarrow$$

Here is the supercharacter table for the finest normal supercharacter theory of S_4.

	$\{e\}$	$C_{(12)(34)}$	$C_{(123)}$	$C_{(12)} \cup C_{(1234)}$
$\mathbb{1}$	1	1	1	1
χ_1	1	1	1	-1
$3\chi_2 + 3\chi_3$	18	-6	0	0
$2\chi_4$	4	4	-2	0

3.4 $NSup(G)$ is not a subset of the union of $AutSup(G)$, $ACSup(G)$, and $Sup^*(G)$

In the following example we show that there is a normal supercharacter theory which is not in the union of $AutSup(G)$, $ACSup(G)$, and $Sup^*(G)$. Since two different supercharacter theories have different sets of superclasses, we only work with the set of superclasses. A ***superclass theory*** for a group G is the set of superclasses of a supercharacter theory of G.

Example 3.4.1. *Let $G = C_3 \times C_4 = \langle a \rangle \times \langle b \rangle$. Define*

$$\mathcal{K}_1 := \left\{\{1\}, \{a\}, \{a^2\}, \{b\}, \{b^3\}, \{ab\}, \{ab^3\}, \{a^2b\}, \{a^2b^3\}, \{b^2\}, \{ab^2\}, \{a^2b^2\}\right\},$$

$$\mathcal{K}_2 := \left\{\{1\}, \{a, a^2\}, \{b\}, \{b^3\}, \{ab, a^2b\}, \{ab^3, a^2b^3\}, \{b^2\}, \{ab^2, a^2b^2\}\right\},$$

$$\mathcal{K}_3 := \Big\{\{1\},\{a\},\{a^2\},\{b,b^3\},\{ab,ab^3\},\{a^2b,a^2b^3\},\{b^2\},\{ab^2\},\{a^2b^2\}\Big\},$$

$$\mathcal{K}_4 := \Big\{\{1\},\{a,a^2\},\{b,b^3\},\{ab,a^2b^3\},\{a^2b,ab^3\},\{b^2\},\{ab^2,a^2b^2\}\Big\}.$$

$$\mathcal{K}_5 := \Big\{\{1\},\{a,a^2\},\{b,b^3\},\{ab,ab^3,a^2b,a^2b^3\},\{b^2\},\{ab^2,a^2b^2\}\Big\},$$

$$\mathcal{K}_6 := \Big\{\{1\},\{a\},\{a^2\},\{b,ba,ba^2\},\{b^2,b^2a,b^2a^2\},\{b^3,b^3a,b^3a^2\}\Big\}$$

$$\mathcal{K}_7 := \Big\{\{1\},\{a\},\{a^2\},\{b,ba,ba^2,b^2,b^2a,b^2a^2,b^3,b^3a,b^3a^2\}\Big\},$$

$$\mathcal{K}_8 := \Big\{\{1\},\{a\},\{a^2\},\{b,ba,ba^2,b^3,b^3a,b^3a^2\},\{b^2,b^2a,b^2a^2\}\Big\},$$

$$\mathcal{K}_9 := \Big\{\{1\},\{a,a^2\},\{b,ba,ba^2\},\{b^2,b^2a,b^2a^2\},\{b^3,b^3a,b^3a^2\}\Big\},$$

$$\mathcal{K}_{10} := \Big\{\{1\},\{a,a^2\},\{b,ba,ba^2,b^2,b^2a,b^2a^2,b^3,b^3a,b^3a^2\}\Big\},$$

$$\mathcal{K}_{11} := \Big\{\{1\},\{a,a^2\},\{b,ba,ba^2,b^3,b^3a,b^3a^2\},\{b^2,b^2a,b^2a^2\}\Big\},$$

$$\mathcal{K}_{12} := \Big\{\{1\},\{b^2\},\{b,b^3\},\{a,ab^2,ab,ab^3\},\{a^2,a^2b^2,a^2b,a^2b^3\}\Big\},$$

$$\mathcal{K}_{13} := \Big\{\{1\},\{b^2\},\{a,ab^2,a^2,a^2b^2\},\{b,b^3,ab,ab^3,a^2b,a^2b^3\}\Big\},$$

$$\mathcal{K}_{14} := \Big\{\{1\},\{b^2\},\{b,b^3\},\{a,ab^2,a^2,a^2,a^2b^2,ab,ab^3,a^2b,a^2b^3\}\Big\},$$

$$\mathcal{K}_{15} := \Big\{\{1\},\{b^2\},\{b,b^3\},\{a,ab^2,a^2,a^2b^2\},\{ab,ab^3,a^2b,a^2b^3\}\Big\},$$

$$\mathcal{K}_{16} := \Big\{\{1\},\{b\},\{b^2\},\{b^3\},\{a,ab,ab^2,ab^3\},\{a^2,a^2b,a^2b^2,a^2b^3\}\Big\},$$

$$\mathcal{K}_{17} := \Big\{\{1\},\{b\},\{b^2\},\{b^3\},\{a,ab,ab^2,ab^3,a^2,a^2b,a^2b^2,a^2b^3\}\Big\},$$

$$\mathcal{K}_{18} := \Big\{\{1\},\{b,b^2,b^3\},\{a,ab,ab^2,ab^3\},\{a^2,a^2b,a^2b^2,a^2b^3\}\Big\},$$

$$\mathcal{K}_{19} := \Big\{\{1\},\{b,b^2,b^3\},\{a,ab,ab^2,ab^3,a^2,a^2b,a^2b^2,a^2b^3\}\Big\},$$

$$\mathcal{K}_{20} := \Big\{\{1\}, \{b^2\}, \{a, a^2\}, \{ab^2, a^2b^2\}, \{b, ba, ba^2, b^3, b^3a, b^3a^2\}\Big\},$$

$$\mathcal{K}_{21} := \Big\{\{1\}, \{b, b^2, b^3\}, \{a, a^2\}, \{ab^2, a^2b^2, ba, ba^2, b^3a, b^3a^2\}\Big\},$$

$$\mathcal{K}_{22} := \Big\{\{1\}, \{b^2\}, \{b, b^3\}, \{a, a^2\}, \{ab^2, a^2b^2\}, \{ba, ba^2, b^3a, b^3a^2\}\Big\}.$$

$$\mathcal{K}_{23} := \Big\{\{1\}, \{b^2\}, \{b, b^3, a, a^2, ab^2, a^2b^2, ba, ba^2, b^3a, b^3a^2\}\Big\}.$$

$$\mathcal{K}_{24} := \Big\{\{1\}, \{b^2, a, a^2, ab^2, a^2b^2\}, \{b, b^3, ba, ba^2, b^3a, b^3a^2\}\Big\}.$$

The group of automorphisms of G is the set of the following four automorphisms.

$$\alpha_1 = \begin{cases} a & \mapsto & a \\ b & \mapsto & b \end{cases}, \alpha_2 = \begin{cases} a & \mapsto & a^2 \\ b & \mapsto & b \end{cases}, \alpha_3 = \begin{cases} a & \mapsto & a \\ b & \mapsto & b^3 \end{cases},$$

$$\alpha_4 = \begin{cases} a & \mapsto & a^2 \\ b & \mapsto & b^3. \end{cases}$$

Therefore, $Aut(G)$ has five subgroups

$$\{\alpha_1\}, \{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_3\}, \{\alpha_1, \alpha_4\}, \text{ and } \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$$

Let A act by automorphisms on G, and $\alpha : A \to Aut(G)$ be the corresponding group homomorphism.

If the image of α is $\{\alpha_1\}$, $\{\alpha_1,\alpha_2\}$, $\{\alpha_1,\alpha_3\}$, $\{\alpha_1,\alpha_4\}$, or $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ then the set of superclasses theories coming from these actions are $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$, or $\mathcal{K}_5$, respectively. Therefore, the set of superclass theories in $AutSup(G)$ is

$$\{\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4,\mathcal{K}_5\}.$$

To have the list of all superclass theories coming from $*$-product, we need to choose a subgroup.

If we choose $\langle a\rangle\times 1$. Then we have the following set of superclass theories,

$$\{\mathcal{K}_6,\mathcal{K}_7,\mathcal{K}_8,\mathcal{K}_9,\mathcal{K}_{10},\mathcal{K}_{11}\}.$$

If we choose $1\times\langle b^2\rangle$. Then we have the following set of superclass theories,

$$\{\mathcal{K}_{12},\mathcal{K}_{13},\mathcal{K}_{14},\mathcal{K}_{15},\mathcal{K}_{23}\}.$$

If we choose $1\times\langle b\rangle$. Then we have the following set of superclass theories,

$$\{\mathcal{K}_{12},\mathcal{K}_{14},\mathcal{K}_{16},\mathcal{K}_{17},\mathcal{K}_{18},\mathcal{K}_{19}\}.$$

If we choose $\langle a\rangle\times\langle b^2\rangle$. Then we have the following set of superclass theories,

$$\{\mathcal{K}_8,\mathcal{K}_{11},\mathcal{K}_{13},\mathcal{K}_{20},\mathcal{K}_{24}\}.$$

Therefore, the set of superclass theories in $Sup^*(G)$ is

$$\{\mathcal{K}_i : 6\le i\le 20, 23\le i\le 24\}.$$

We now want to classify all supercharacter theories coming from the action of a group of automorphism of $\mathbb{Q}[\zeta_{12}]$*, where* ζ_{12} *is a 12th primitive root of unity. We have*

$$Aut(Q[\zeta_{12}]) = \{\beta_1, \beta_2, \beta_3, \beta_3\}$$

where

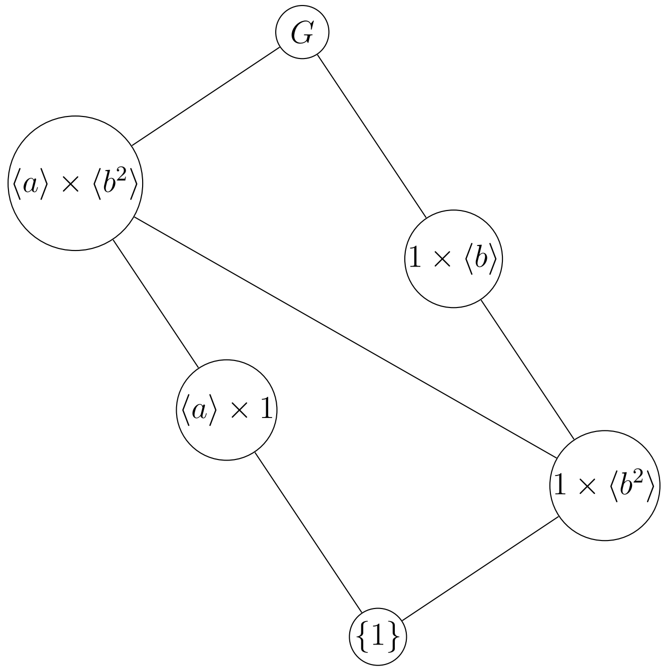
$$\beta_1(\zeta_{12}) = \zeta_{12}, \beta_1(\zeta_{12}) = \zeta_{12}^5, \beta_1(\zeta_{12}) = \zeta_{12}^7, \text{ and } \beta_1(\zeta_{12}) = \zeta_{12}^{11}.$$

If we have $A_1 = \{\beta_1\}$*,* $A_2 = \{\beta_1, \beta_2\}$*,* $A_3 = \{\beta_1, \beta_3\}$*,* $A_4 = \{\beta_1, \beta_4\}$ *and* $A_5 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$*, then the set of superclass theories coming from these actions are* $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ *and* $\mathcal{K}_5$*, respectively. Therefore, the set of superclass theories in* $ACSup(G)$ *is*

$$\{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5\}.$$

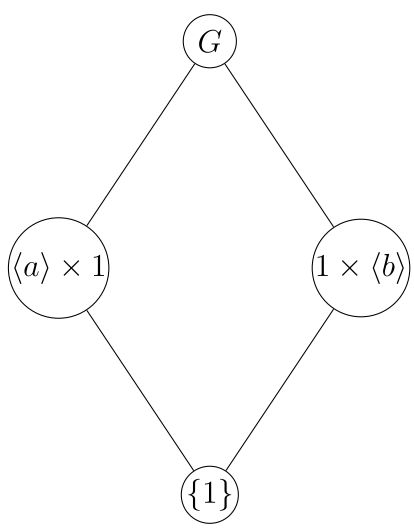
Now we show that $\mathcal{K}_{21}$ *is a superclass theory in* $NSup(G)$*, and so* $NSup(G)$ *is not a subset of the union of* $AutSup(G)$*,* $ACSup(G)$*, and* $Sup^*(G)$*.*

Here is the lattice of normal subgroups of $\langle a\rangle \times \langle b\rangle$*.*



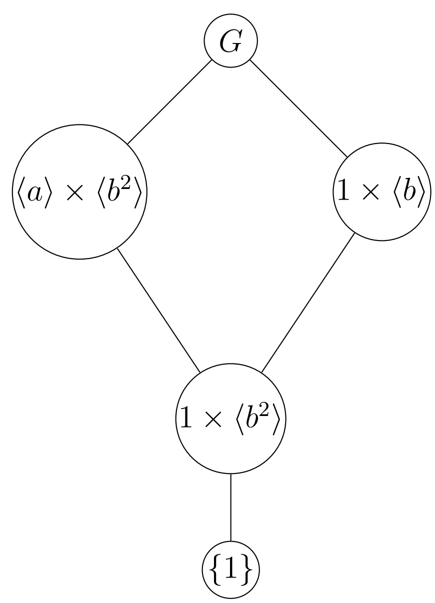
If we choose the whole lattice, then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{22}$.

If we choose the following sublattice



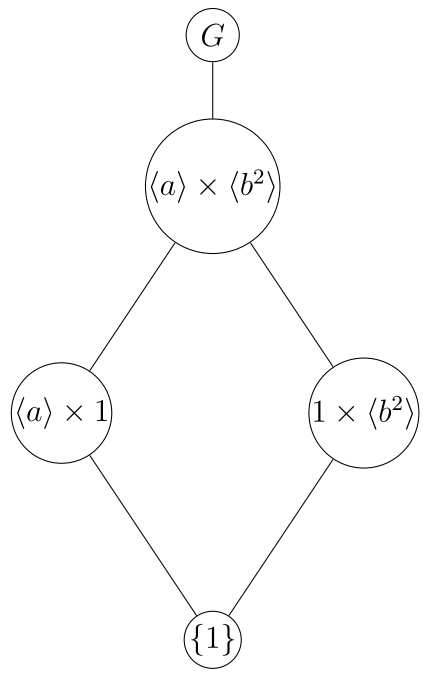
then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{21}$.

If we choose the following sublattice



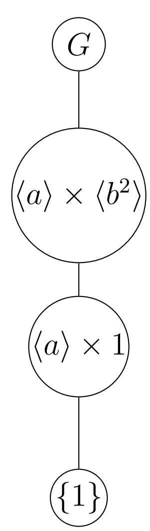
then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{15}$.

If we choose the following sublattice



then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{20}$.

If we choose the following sublattice



then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{11}$.

If we choose the following sublattice

$$\begin{tikzcd}
G \arrow[d, no head] \\
\langle a \rangle \times \langle b^2 \rangle \arrow[d, no head] \\
1 \times \langle b^2 \rangle \arrow[d, no head] \\
\{1\}
\end{tikzcd}$$

then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{13}$.

If we choose the following sublattice

$$\begin{tikzcd} G \arrow[d, dash] \\ 1 \times \langle b \rangle \arrow[d, dash] \\ 1 \times \langle b^2 \rangle \arrow[d, dash] \\ \{1\} \end{tikzcd}$$

then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{15}$.

If we choose the following sublattice

$$\begin{tikzcd} G \arrow[d, dash] \\ 1 \times \langle b^2 \rangle \arrow[d, dash] \\ \{1\} \end{tikzcd}$$

then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{23}$.

If we choose the following sublattice

$$
\begin{tikzcd}
G \arrow[d, no head] \\
1 \times \langle b \rangle \arrow[d, no head] \\
\{1\}
\end{tikzcd}
$$

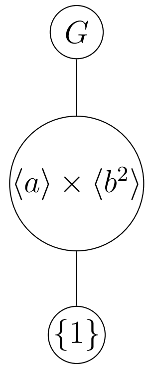
then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{19}$.

If we choose the following sublattice

$$
\begin{tikzcd}
G \arrow[d, no head] \\
\langle a \rangle \times 1 \arrow[d, no head] \\
\{1\}
\end{tikzcd}
$$

then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{10}$.

If we choose the following sublattice



then the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_{24}$.

Therefore, the set of superclass theories in $NSup(G)$ is

$$\{\mathcal{K}_{10}, \mathcal{K}_{11}, \mathcal{K}_{13}, \mathcal{K}_{15}, \mathcal{K}_{19}, \mathcal{K}_{20}, \mathcal{K}_{21}, \mathcal{K}_{22}, \mathcal{K}_{23}, \mathcal{K}_{24}\}.$$

We can see that $\mathcal{K}_{21}$ and $\mathcal{K}_{22}$ is not in the union of $ACSup(G)$, $AutSup(G)$, and $Sup^*(G)$.

We now classify the supercharacter theories for S_4 to show that when we do not have many normal subgroups, we do not have many supercharacter theories.

Example 3.4.2. *Let $G = S_5$. Here is the character table of S_5.*

	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
$1\!\!1$	1	1	1	1	1	1	1
χ_1	1	-1	1	1	-1	-1	1
χ_2	4	-2	0	1	1	0	-1
χ_3	5	-1	1	-1	-1	1	0
χ_4	6	0	-2	0	0	0	1
χ_5	5	1	1	-1	1	-1	0
χ_6	4	2	0	1	-1	0	-1

Define

$$\mathcal{K}_1 := \{\{1\}, \{(12)\}, \{(123)(45)\}, \{(1234)\}, \{(12)(34)\}, \{(123)\}, \{(12345)\}\},$$

$$\mathcal{K}_2 := \{\{1\}, \{(12), (123)(45), (1234)\}, \{(12)(34)\}, \{(123)\}, \{(12345)\}\},$$

$$\mathcal{K}_3 := \{\{1\}, \{(12), (123)(45), (1234)\}, \{(12)(34), (123), (12345)\}\},$$

$$\mathcal{K}_4 := \{\{1\}, \{(12), (123)(45), (1234), (12)(34), (123), (12345)\}\}.$$

Every automorphism of S_n maps every conjugacy class to itself so the only supercharacter theory that comes from actions of automorphisms is $\mathcal{K}_1$.

Given α in a group A of automorphisms of the cyclotomic field $Q[\zeta_{5!}]$, α carries out the class of the permutation σ to itself since the cycle type of σ and σ^r, where r and the order of σ are coprime, are the same. Therefore, the *only supercharacter theory that comes from the action of automorphisms of the cyclotomic field* $\mathbb{Q}[\zeta_{5!}]$ *is* $\mathcal{K}_1$.

We know list the superclass theories in $Sup^*(S_5)$. *The character table of the unique non-trivial normal subgroup of* S_5, *i.e.,* A_5, *is as follows.*

	1	$(12)(34)$	(123)	(12345)	(12354)
$\mathbb{1}$	1	1	1	1	1
χ_1	3	-1	0	$\zeta_5^3+\zeta_5^2+1$	$-\zeta_5^3-\zeta_5^2$
χ_2	3	-1	0	$-\zeta_5^3-\zeta_5^2$	$\zeta_5^3+\zeta_5^2+1$
χ_3	4	0	1	-1	-1
χ_4	5	1	-1	0	0

Let $C_g^{A_5}$ *denote the conjugacy class of* A_5 *containing* g. *The normal subgroup* A_5, *has three superclass theories as follows,*

$$\mathcal{L}_1 = \left\{\{1\}, C^{A_5}_{(12)(34)}, C^{A_5}_{(123)}, C^{A_5}_{(12345)}, C^{A_5}_{(12354)}\right\},$$

$$\mathcal{L}_2 = \left\{\{1\}, C^{A_5}_{(12)(34)}, C^{A_5}_{(123)}, C^{A_5}_{(12345)} \cup C^{A_5}_{(12354)}\right\},$$

and

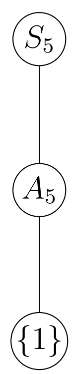
$$\mathcal{L}_3 = \left\{\{1\}, C^{A_5}_{(12)(34)} \cup C^{A_5}_{(123)} \cup C^{A_5}_{(12345)} \cup C^{A_5}_{(12354)}\right\}.$$

Only superclasses in $\mathcal{L}_2$ *and* $\mathcal{L}_3$ *are union of conjugacy classes of* S_5. *Therefore, they can be used to build supercharacter theories for* S_5.

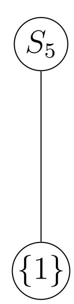
If we choose $\mathcal{L}_2$ for A_5, then we can build $\mathcal{K}_2$ by $*$-product, and if we choose $\mathcal{L}_3$, then we can build $\mathcal{K}_3$ by $*$-product. It follows that the superclass theories coming from $*$-product are

$$\{\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4\}.$$

Now, we will classify all supercharacter theories in $NSup(G)$. Here is the lattice of normal subgroups of S_5.



We have two sublattices containing the trivial subgroup and S_5. The first sublattice is the whole lattice of normal subgroups of S_5, so the set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_3$. The other sublattice is the following,



The set of superclasses of the supercharacter theory coming from this sublattice is $\mathcal{K}_4$.

Therefore, the set of superclass theories in $NSup(S_5)$ is

$$\{\mathcal{K}_3, \mathcal{K}_4\}.$$

3.5 Star Product and Normal Supercharacter Theories

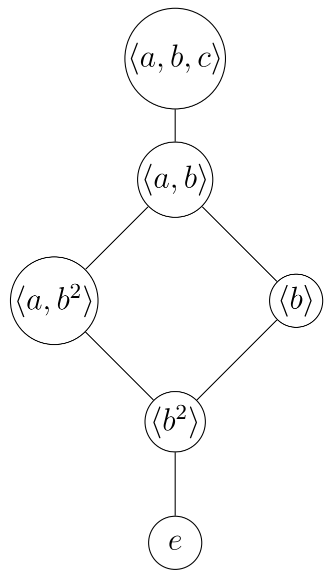
We now investigate the relation between normal supercharacter theories and star product. Moreover we show that normal supercharacter theories are a join of some star products.

3.5.1 Star-Normal Supercharacter Theories

Let S be a set of normal subgroups of a group G. Assume that in the lattice of $A(S)$ we have vertices N and M such that the only vertex right below M is N and the only vertex right above N is M.

For example in the following lattice we can have $N = \langle a, b\rangle$ and $M = \langle a, b, c\rangle$

$(|a| = 4, |b| = 4, |c| = 4)$.



Now let $(\mathcal{Y}, \mathcal{L})$ be a supercharacter theory for M/N such that

$$\overline{Y} = \bigcup_{yN \in Y} yN$$

is a union of conjugacy classes of G (we usually say that $(\mathcal{Y}, \mathcal{L})$ is G-invariant).

For every subset X of G, denote by $\widehat{X}$, the sum of all elements in X.

Let

$$\mathcal{K} = \{H^{\circ}, \overline{Y} : H \in A(S) \setminus \{M\}, Y \in \mathcal{Y} \setminus \{N\}\}$$

and

$$\mathbf{V} = \mathbb{C}\text{-span}\{\widehat{K} : K \in \mathcal{K}\}.$$

It is easy to see that $\mathcal{K}$ is a partition of G, and $\{1\} \in \mathcal{K}$. So if we show that $\mathbf{V}$ is a subalgebra of $Z(\mathbb{C}[G])$, then $\mathcal{K}$ is the set of superclasses of a supercharacter theory (See Proposition 2.4 of Hendrickson (2012)).

Lemma 3.5.1. *The subspace* $\boldsymbol{V}$ *is a subalgebra of* $Z(\mathbb{C}[G])$.

Proof. We only need to show that for arbitrary $H, K \in A(S) \setminus \{M\}$, and $Y, W \in \mathcal{Y} \setminus \{N\}$, we have

(1) $\widehat{H}\widehat{K} \in \mathbf{V}$.

(2) $\widehat{H}\widehat{\widehat{Y}} \in \mathbf{V}$.

(3) $\widehat{\widehat{Y}}\,\widehat{\widehat{W}} \in \mathbf{V}$.

Note that $\widehat{H}\widehat{K} \in \mathbb{C}\text{-span}\{\widehat{N^\circ} : N \in A(S)\}$. Since $\widehat{M^\circ} = \sum_{Y \in \mathcal{Y}\setminus\{N\}} \widehat{\widehat{Y}}$, we have $\mathbb{C}\text{-span}\{\widehat{N^\circ} : N \in A(S)\} \subseteq \mathbf{V}$. Therefore, $\widehat{H}\widehat{K} \in \mathbf{V}$. This completes the proof of (1).

It is clear that

$$\widehat{\widehat{Y}} = \sum_{yN \in Y} y\widehat{N}.$$

Thus,

$$\widehat{H}\widehat{\widehat{Y}} = \widehat{H} \sum_{yN \in Y} y\widehat{N} = \sum_{yN \in Y} y\widehat{N}\widehat{H}.$$

Since the only vertex above N is M in the lattice of $A(S)$, we have two cases:

Case 1. $M \subseteq NH$. Then if $yN \in Y$, $y \in M \subseteq NH$. Therefore, $y\widehat{N}\widehat{H} = n_y\widehat{N}\widehat{H}$ for some positive integer n_y. Thus,

$$\widehat{H}\widehat{\widehat{Y}} = \sum_{yN \in Y} n_y \widehat{N}\widehat{H} \in \mathbb{C}\text{-span}\{\widehat{N^\circ} : N \in A(S)\} \subseteq \mathbf{V}.$$

Case 2. $NH \subset M$. Note that the only vertex above N is M in the lattice of $A(S)$, so we must have $H \subseteq N$. Therefore, $y\widehat{N}\widehat{H} = n_H y\widehat{N}$ for some positive integer n_H. Thus

$$\widehat{H}\widehat{\widehat{Y}} = n_H \sum_{yN \in Y} y\widehat{N} \in \mathbb{C}\text{-span}\{\widehat{\widehat{Y}}\} \subseteq \mathbf{V}.$$

So (2) holds.

We have

$$\widehat{\widehat{Y}}\ \widehat{\widehat{W}} \in \mathbb{C}\text{span-}\{\widehat{\widehat{Y}} : Y \in \mathcal{Y}\}.$$

Also, when $Y = \{N\}$, then

$$\widehat{\widehat{Y}} = \sum_{L \in A(S) : L \subseteq N} L^\circ.$$

We can conclude that

$$\widehat{\widehat{Y}}\ \widehat{\widehat{W}} \in \mathbf{V}.$$

Thus, (3) holds.

$\square$

The following theorem follows from the lemma above.

Theorem 3.5.2. *Let $\mathcal{L}$ be a sublattice of the lattice of normal subgroups of G containing the trivial subgroup and G. Assume that $\mathcal{L}$ has two vertices M and N such that the only vertex right below M is N and the only vertex right above N is M. Now let $(\mathcal{Y}, \mathcal{L})$ be a supercharacter theory for M/N such that*

$$\overline{Y} = \bigcup_{yN \in Y} yN$$

is a union of conjugacy classes of G. Then

$$\mathcal{K} = \{H^{\circ}, \overline{Y} : H \in \mathcal{L} \setminus \{M\}, Y \in \mathcal{Y} \setminus \{N\}\}$$

is a superclass theory for G.

3.5.2 Connection Between Normal Supercharacter Theories and Star Products

If we have a set S, then there is a natural way to make $Part(S)$, the set of all partitions of S, a lattice in which $p \leq q$ whenever blocks of q be a union of blocks of p. Let $Sup(G)$ be the set of all supercharacter theories of G. We define the following ordering on $Sup(G)$. Let $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L}) \in Sup(G)$. Then $(\mathcal{X}, \mathcal{K}) \leq (\mathcal{Y}, \mathcal{L})$ if $\mathcal{K} \subseteq \mathcal{L}$, or equivalently $\mathcal{X} \subseteq \mathcal{Y}$. However, $Sup(G)$ is not a lattice by defining $(\mathcal{X}, \mathcal{K}) \vee (\mathcal{Y}, \mathcal{L}) := (\mathcal{X} \vee \mathcal{Y}, \mathcal{K} \vee \mathcal{L})$

and $(\mathcal{X},\mathcal{K}) \wedge (\mathcal{Y},\mathcal{L}) := (\mathcal{X} \wedge \mathcal{Y}, \mathcal{K} \wedge \mathcal{L})$, because although $(\mathcal{X} \vee \mathcal{Y}, \mathcal{K} \vee \mathcal{L})$ is a supercharacter theory, $(\mathcal{X} \wedge \mathcal{Y}, \mathcal{K} \wedge \mathcal{L})$ is not (see (Hendrickson 2012, Section 3)). However, the meet of arbitrary supercharacter theories of the form $(\mathcal{X}_N, \mathcal{K}_N) = (\{Irr(N) \setminus 1, 1\}, \{\{1\}, N \setminus 1\}) * (\{Irr(G/N) \setminus 1, 1\}, \{\{1\}, G/N \setminus 1\})$, where N is a normal subgroups of G, is a supercharacter theory. More especially we have the following theorem. Also, this theorem has independently been proved by Shawn Burkett, a Ph.D student under the supervision of Professor Nat Thiem from University of Colorado at Boulder. His result builds upon some of the earlier work of the thesis.

Theorem 3.5.3. *Let A be a set of arbitrary normal subgroups. Then*

$$\bigwedge_{N \in A} (\mathcal{X}_N, \mathcal{K}_N)$$

is a supercharacter theory. Moreover, if $(\mathcal{X},\mathcal{K})$ is the normal supercharacter theory generated by S, then

$$(\mathcal{X},\mathcal{K}) = \bigwedge_{N \in A(S)} (\mathcal{X}_N, \mathcal{K}_N).$$

Proof. It is enough to show that

$$\bigwedge_{N \in A(S)} (\mathcal{X}_N, \mathcal{K}_N)$$

has the same superclasses as $(\mathcal{X}, \mathcal{K})$ the normal supercharacter theory generated by S. Precisely, the set of $\bigwedge_{N \in A(S)} (\mathcal{X}_N, \mathcal{K}_N)$ is

$$\bigwedge_{N \in A(S)} \{\{1\}, N \setminus \{1\}, G \setminus N\}.$$

Let $p \in \bigwedge_{N \in A(S)} \{\{1\}, N \setminus \{1\}, G \setminus N\}$. Then there is a normal subgroup $N \in A(S)$ such that $p \cap N^\circ \neq \emptyset$. It is easy to see that $N^\circ = p$. Thus,

$$\bigwedge_{N \in A(S)} (\mathcal{X}_N, \mathcal{K}_N) \subseteq \{N^\circ \neq \emptyset : N \in A(S)\} = \mathcal{K}.$$

Now let $N^\circ \in \mathcal{K}$. Assume that $p \in \bigwedge_{N \in A(S)} \{\{1\}, N \setminus \{1\}, G \setminus N\}$ such that $p \cap N^\circ \neq \emptyset$. Then it is easy to see that $p = N^\circ$. Therefore,

$$\bigwedge_{N \in A(S)} (\mathcal{X}_N, \mathcal{K}_N) = \{N^\circ \neq \emptyset : N \in A(S)\} = \mathcal{K}.$$

$\square$

4 Normal Supercharacter Theories and Non-Nesting Partitions

It's a hard task to classify all the irreducible characters of an algebra group, as we mentioned before the classification of irreducible characters of the family of unipotent upper-triangular matrices is wild. In this chapter we construct a supercharacter theory for unipotent-upper triangular matrices by using the normal pattern subgroups. The supercharacters we obtain for unipotent upper-triangular matrices are indexed by Dyck paths. Andrews (2015) constructed a supercharacter theory for upper-triangular matrices in which superclasses and supercharacters are indexed by $\mathbb{F}_q$-set partitions (for details see section 2.7). The torus group acts on the superclasses and supercharacters of Andrews' construction. We show that the orbits of this action gives a supercharacter theory that is identical with the normal supercharacter theory generated by normal pattern subgroups.

4.1 Background and Definitions

In this section we present some examples of algebra groups.

4.1.1 Some Examples of Algebra Groups

Algebra group is a group of the form $1 + J$ where J is a finite dimensional nilpotent associative algebra over a finite field $\mathbb{F}_q$. Let $\mathbb{I}(J)$ be the set of all two-sided ideals of J.

Example 1. Let

$$UT_n(q) = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1(n-2)} & a_{1(n-1)} & a_{1n} \\ 0 & 1 & a_{23} & \cdots & a_{2(n-2)} & a_{2(n-1)} & a_{2n} \\ 0 & 0 & 1 & \cdots & a_{3(n-2)} & a_{3(n-1)} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} : a_{ij} \in \mathbb{F}_q \right\}.$$

Then $UT_n(q) = 1 + J$ where

$$J = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1(n-2)} & a_{1(n-1)} & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2(n-2)} & a_{2(n-1)} & a_{2n} \\ 0 & 0 & 0 & \cdots & a_{3(n-2)} & a_{3(n-1)} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} : a_{ij} \in \mathbb{F}_q \right\}.$$

4.1.2 Pattern Groups

Pattern groups are some algebra groups which can be defined in terms of partial ordering. Fix a positive integer n and let $\mathcal{P}$ be a partial order on $[n] = \{1, 2, \ldots, n\}$, i.e.,

$$\mathcal{P} \subseteq [\![n]\!] = \{(i, j) : 1 \leq i < j \leq n\}.$$

We can depict Hasse diagram of $\mathcal{P}$, which is the diagram whose vertices are $1, 2, \ldots, n$ and if $(i, j) \in \mathcal{P}$, then a line is drawn upward from i to j. For example let

$$\mathcal{P} = \begin{tikzcd} & 4 \arrow[d, no head] \\ & 3 \arrow[dl, no head] \arrow[d, no head] \\ 1 & 2 \end{tikzcd}$$

where $n = 4$. Then $\mathcal{P} = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

The ***pattern group*** $U_{\mathcal{P}}$ is the subgroup of $UT_n(q)$ given by

$$U_{\mathcal{P}} := \{g \in UT_n(q) : g_{ij} = 0 \text{ if } (i,j) \notin \mathcal{P}\}.$$

Note that UT_n is the pattern group correspond to the poset $\mathcal{P} = [\![n]\!]$.

Another example is coming from the local rings. Let $(R, \mathfrak{m})$ be a Artinian local ring such that $|R/\mathfrak{m}| < \infty$. Then $\mathfrak{m}$ is a finite dimensional nilpotent associative algebra over a finite field, and so $1 + \mathfrak{m}$ is an algebra group. Diaconis and Thiem (2009) used the poset structure to describe a variety of group theoretic structures such as the center, the Frattini subgroups, etc.

4.2 A Normal Supercharacter Theory Corresponding to Dyck Paths

In this section we construct a supercharacter theory in which every superclass and supercharacter is correspond to a Dyck path.

4.2.1 Normal Pattern Subgroups of $UT_n(q)$

Fix a positive integer n and let $[n] = \{1, 2, \ldots, n\}$. Given a labeled poset $\mathcal{P}$ on $[\![n]\!]$, we say that $\mathcal{P}$ is ***normal*** in $[\![n]\!]$ and write $\mathcal{P} \lhd [\![n]\!]$ if

$$(i, l) \notin \mathcal{P} \text{ imlies } (j, k) \notin \mathcal{P}, \text{ for all } \; 1 \leq i \leq j < k \leq l \leq n. \tag{4.2.1}$$

or equivalently, $\mathcal{P} \lhd [\![n]\!]$ if and only if $(j, k) \in \mathcal{P}$ implies $(i, l) \in \mathcal{P}$ for all $1 \leq i \leq j < k \leq l \leq n$. Also, we define ***normal pattern subgroup*** $U_{\mathcal{P}}$ of $UT_n(q)$ by

$$U_{\mathcal{P}} = \{X \in UT_n(q) : X_{ij} = 0 \text{ if } i < j \text{ and } (i, j) \notin \mathcal{P}\}.$$

This lemma shows when normal pattern subgroups are correspond to normal posets.

Lemma 4.2.1. (Marberg 2011, Lemma 4.1) *If $\mathcal{P}$ is a poset on $[\![n]\!]$, then $U_{\mathcal{P}} \lhd UT_n(q)$ if and only if $\mathcal{P} \lhd [\![n]\!]$.*

Let $U_{\mathcal{P}} \subseteq U_{\mathcal{R}}$ be pattern groups with corresponding posets $\mathcal{P}$ and $\mathcal{R}$. Let $\frac{\mathcal{R}}{\mathcal{P}}$ be the set of pairs given by

$$(i, j) \in \frac{\mathcal{R}}{\mathcal{P}}, \text{ if } \; (i, j) \in \mathcal{R} \text{ and } (i, j) \notin \mathcal{P}.$$

Note that $\frac{\mathcal{R}}{\mathcal{P}}$ is not necessarily a poset.

Lemma 4.2.2. (Marberg and Thiem 2009, Lemma 2.1) *Let $U_{\mathcal{P}} \subseteq U_{\mathcal{R}}$ be pattern groups. Then*

$$U_{\mathcal{R}}/U_{\mathcal{P}} = \{u \in U_{\mathcal{R}} : u_{ij} \neq 0 \text{ implies } (i,j) \in \frac{\mathcal{R}}{\mathcal{P}}\}$$

is a set of left coset representatives for $U_{\mathcal{R}}/U_{\mathcal{P}}$ and a set of right coset representatives for $U_{\mathcal{P}} \backslash U_{\mathcal{R}}$.

Now by using the above lemma we show that the product and intersection of normal pattern subgroups of $UT_n(q)$ are normal pattern subgroups of $UT_n(q)$.

Lemma 4.2.3. *The product and intersection of any two normal pattern subgroups of $UT_n(q)$ are normal pattern subgroups of $UT_n(q)$.*

Proof. Let $U_{\mathcal{P}}$ and $U_{\mathcal{R}}$ be normal pattern subgroups with corresponding normal posets $\mathcal{P}$ and $\mathcal{R}$. We show that $\mathcal{R} \cup \mathcal{P}$ is a normal poset and also $U_{\mathcal{R}} U_{\mathcal{P}} = U_{\mathcal{P} \cup \mathcal{R}}$. Assume that $(i,j),(j,k) \in \mathcal{P} \cup \mathcal{R}$, then we have either (without loss of generality) $(i,j) \in \mathcal{P}$ and $(j,k) \in \mathcal{R}$ or (i,j) and (j,k) are in the same poset. In the former case $(i,k) \in \mathcal{P} \cup \mathcal{R}$ since $1 \leq i < j \leq k$ and $\mathcal{P}$ is a normal poset. In the latter case it is clear that $(i,k) \in \mathcal{P} \cup \mathcal{R}$ because both $\mathcal{P}$ and $\mathcal{R}$ are posets. Therefore, $\mathcal{P} \cup \mathcal{R}$ is a poset. To show that $\mathcal{P} \cup \mathcal{R}$ is normal, given an arbitrary pair $(i,j) \in \mathcal{P} \cup \mathcal{R}$, we have (i,j) is in one of the normal posets, thus for every $1 \leq l \leq i < j \leq k \leq n$, we have $(l,k) \in \mathcal{P} \cup \mathcal{R}$. Therefore, $\mathcal{P} \cup \mathcal{R}$ is a normal poset.

Since $U_{\mathcal{P}}, U_{\mathcal{R}} \subseteq U_{\mathcal{P}\cup\mathcal{R}}$, we have $U_{\mathcal{P}}U_{\mathcal{R}} \subseteq U_{\mathcal{P}\cup\mathcal{R}}$. By Lemma 4.2.2,

$$|U_{\mathcal{R}\cup\mathcal{P}}| = |U_{\mathcal{R}\cup\mathcal{P}}/U_{\mathcal{P}}|\,|U_{\mathcal{P}}|$$

and also

$$|U_{\mathcal{R}}U_{\mathcal{P}}| = \frac{|U_{\mathcal{R}}||U_{\mathcal{P}}|}{|U_{\mathcal{R}} \cap U_{\mathcal{P}}|}.$$

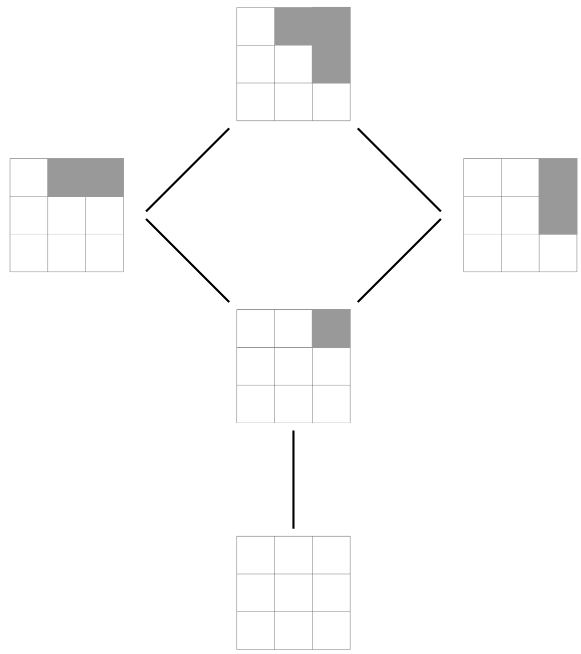
Note that $(\mathcal{R} \cup \mathcal{P})/\mathcal{P}$ is $\{(i,j) \in \mathcal{R} \cup \mathcal{P} : (i,j) \notin \mathcal{P}\}$ which is same as $\{(i,j) \in \mathcal{R} : (i,j) \notin \mathcal{P}\}$. Therefore

$$|U_{\mathcal{R}\cup\mathcal{P}}/U_{\mathcal{P}}| = \frac{|U_{\mathcal{R}}|}{|U_{\mathcal{R}} \cap U_{\mathcal{P}}|}.$$

This implies that $|U_{\mathcal{R}\cup\mathcal{P}}| = |U_{\mathcal{R}}U_{\mathcal{P}}|$ and so $U_{\mathcal{R}\cup\mathcal{P}} = U_{\mathcal{R}}U_{\mathcal{P}}$. Since $\mathcal{P}$ and $\mathcal{R}$ are two arbitrary normal posets, the product of any two normal pattern subgroups of $UT_n(q)$ is a normal pattern subgroup of $UT_n(q)$.

Also, it is clear from the definitions that $U_{\mathcal{P}} \cap U_{\mathcal{R}} = U_{\mathcal{P}\cap\mathcal{R}}$ is a normal pattern subgroup since $\mathcal{P} \cap \mathcal{R}$ is a normal poset on $[\![n]\!]$. $\square$

Example 4.2.4. *Here is the lattice of $A(\{U_{\mathcal{P}} : \mathcal{P} \lhd [\![3]\!]\})$.*



As a result of (Marberg 2011, Corollary 4.1) and (Shapiro 1975, Proposition 2) we have the following corollary.

Corollary 4.2.5. *The number of normal subposets $\mathcal{P} \lhd [\![n]\!]$ is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Corollary 4.2.6. *The normal supercharacter theory generated by*

$$\{U_{\mathcal{P}} : \mathcal{P} \lhd [\![n]\!]\}$$

has C_n superclasses and supercharacter which are

$$\{U_{\mathcal{P}}^{\circ} : \mathcal{P} \lhd [\![n]\!]\} \text{ and } \{\chi^{U_{\mathcal{P}}^{\bullet}} : \mathcal{P} \lhd [\![n]\!]\}$$

respectively.

Proof. Construct the normal supercharacter theory generated by

$$\{U_{\mathcal{P}} : \mathcal{P} \lhd [\![n]\!]\}.$$

Since by Lemma 4.2.3 the product and intersection of two normal pattern subgroups are normal pattern subgroups, we have

$$A(U_{\mathcal{P}} : \mathcal{P} \lhd [\![n]\!]) = \{U_{\mathcal{P}} : \mathcal{P} \lhd [\![n]\!]\}.$$

We show that the normal supercharacter theory generated by

$$\{U_{\mathcal{P}} : \mathcal{P} \lhd [\![n]\!]\}$$

has C_n supercharacters and superclasses. We know that

$$\{U_{\mathcal{P}}^{\circ} : \mathcal{P} \lhd [\![n]\!]\}$$

is the set of superclasses of the normal supercharacter theory generated by

$$\{U_{\mathcal{P}} : \mathcal{P} \lhd [\![n]\!]\}.$$

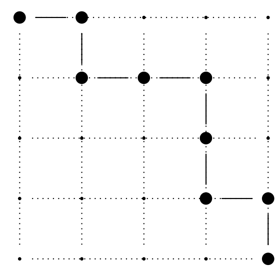
This is true that $U_{\mathcal{P}}^{\circ} \neq \emptyset$ since the matrix $X \in UT_n(q)$ with $X_{ij} \neq 0$ if $(i,j) \in \mathcal{P}$, is in $U_{\mathcal{P}}$, but not in any $U_{\mathcal{Q}}$ where $\mathcal{Q}$ is a normal poset with $U_{\mathcal{Q}} \subset U_{\mathcal{P}}$. By Corollary 4.2.5,

$$|\{U_{\mathcal{P}}^{\circ} : \mathcal{P} \lhd [\![n]\!]\}| = C_n.$$

□

By the ***diagonal*** of a $n \times n$ gird we mean the line from upper left corner to lower right corner of the gird. Every normal pattern subgroup of $UT_n(q)$ is correspond to a path above the diagonal of a $n \times n$ gird, for more details see (Marberg 2011, Section 4). For example,

$$U_{\mathcal{P}} = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & 0 & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\} \quad \text{corresponds to}$$



As a result of Theorems 3.2.3 and 3.2.4, we have the following Proposition.

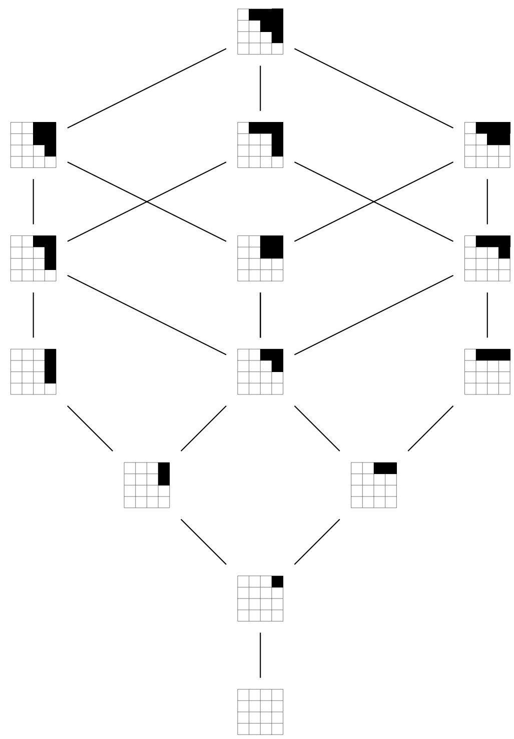
Proposition 4.2.7. *Let* $x \in U_{\mathcal{Q}}^{\circ}$

$$\chi^{U_{\mathcal{P}}^{\bullet}}(x) = \begin{cases} \displaystyle\sum_{\substack{\mathcal{R} \lhd [\![n]\!]: \\ U_{\mathcal{Q}} \le U_{\mathcal{R}}}} \mu(U_{\mathcal{Q}}, U_{\mathcal{R}}) \frac{|UT_n(q)|}{|U_{\mathcal{R}}|} & U_{\mathcal{Q}} \subseteq U_{\mathcal{P}}, \\[2em] \displaystyle -\sum_{\substack{\mathcal{R} \lhd [\![n]\!]: \\ U_{\mathcal{Q}} < U_{\mathcal{R}}}} \left(\sum_{\substack{\mathcal{T} \lhd [\![n]\!]: \\ U_{\mathcal{Q}} \le U_{\mathcal{T}}, U_{\mathcal{R}} \le U_{\mathcal{T}}}} \mu(U_{\mathcal{R}}, U_{\mathcal{T}}) \frac{|UT_n(q)|}{|U_{\mathcal{T}}|} \right) & U_{\mathcal{Q}} \not\subseteq U_{\mathcal{P}}, \end{cases}$$

and

$$\chi^{U_{\mathcal{P}}^{\bullet}}(x) = \sum_{\substack{\mathcal{R} \lhd [\![n]\!]: \\ U_{\mathcal{P}}, U_{\mathcal{Q}} \subseteq U_{\mathcal{R}}}} \mu(U_{\mathcal{P}}, U_{\mathcal{R}}) \frac{|UT_n(q)|}{|U_{\mathcal{R}}|}. \tag{4.2.2}$$

Example 4.2.8. *The Hasse diagram of $A(S)$, where S is the set of all normal pattern subgroups of $UT_4(q)$.*



Example 4.2.9. *The supercharacter table of the normal supercharacter theory generated by all normal pattern subgroups of $UT_4(q)$. Let $t = q - 1$.*

	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	t	t	t	t	t	-1	t	-1	t	t	-1	-1	t	-1
	t	t	t	t	-1	t	t	t	-1	t	-1	t	-1	-1
	t	t	t	t	t	t	t	t	t	-1	t	-1	-1	-1
	t^2	t^2	t^2	t^2	$-t$	$-t$	t^2	$-t$	$-t$	t^2	1	$-t$	$-t$	1
	t^2	t^2	t^2	t^2	t^2	$-t$	t^2	$-t$	t^2	$-t$	$-t$	1	$-t$	1
	t^2	t^2	t^2	t^2	$-t$	t^2	t^2	t^2	$-t$	$-t$	$-t$	$-t$	1	1
	t^3	t^3	t^3	t^3	$-t^2$	$-t^2$	t^3	$-t^2$	$-t^2$	$-t^2$	t	t	t	-1
	q^2t	q^2t	q^2t	$-q^2$	0	q^2t	$-q^2$	$-q^2$	0	0	0	0	0	0
	q^2t	q^2t	$-q^2$	q^2t	q^2t	0	$-q^2$	0	$-q^2$	0	0	0	0	0
	q^2t^2	q^2t^2	$-q^2t$	q^2t^2	$-q^2t$	0	$-q^2t$	0	q^2	0	0	0	0	0
	q^2t^2	q^2t^2	q^2t^2	$-q^2t$	0	$-q^2t$	$-q^2t$	q^2	0	0	0	0	0	0
	q^3t^2	q^3t^2	$-q^3t$	$-q^3t$	0	0	q^3	0	0	0	0	0	0	0
	q^5t	$-q^5$	0	0	0	0	0	0	0	0	0	0	0	0

Another parametrization for superclasses and supercharacters is by non-nesting partitions. Let λ be a non-nesting set partition define two normal pattern subgroups M_λ and N_λ of $UT_n(q)$ as below

$$M_\lambda = \left\{ g \in UT_n(q) : \begin{array}{c} g_{kl} = 0 \text{ unless there exists} \\ i \frown j \in \lambda \text{ with } k \leq i < j \leq l \end{array} \right\}$$

and

$$N_\lambda = \left\{ g \in UT_n(\mathbb{F}_q) \;\middle|\; \begin{array}{l} g_{ij} = 0 \text{ if there exists } k \frown l \in \lambda \text{ such that} \\ (i,j) \neq (k,l) \text{ and } k \leq i < j \leq l \end{array} \right\}.$$

Also for a non-nesting $\mathbb{F}_q$-set partition η we define

$$U_\eta = \left\{ g \in UT_n(\mathbb{F}_q) \;\middle|\; \begin{array}{l} g_{ij} = 0 \text{ if there exists } k \overset{a}{\frown} l \in \eta \text{ such that} \\ (i,j) \neq (k,l) \text{ and } k \leq i < j \leq l \end{array} \right\}.$$

Lemma 4.2.10. *Let λ be a non-nesting set partition. Then M_λ and N_λ are normal pattern subgroups.*

Proof. Let

$$\mathcal{P} := \left\{ (i,j) : i < j, \exists g \in M_\lambda \text{ such that } g_{ij} \neq 0 \right\}.$$

We show that $\mathcal{P}$ is normal in $[\![n]\!]$ and $U_{\mathcal{P}} = M_\lambda$. Let (k,l) and (l,m) be elements of $\mathcal{P}$. Then there exists $i \frown j \in \lambda$ such that $k \leq i < j \leq l$. And since $l < m$, there is an element $g \in M_\lambda$ such that $g_{km} \neq 0$, and so $(k,m) \in \mathcal{P}$ and $\mathcal{P}$ is a poset. Also, it is clear from the definition of M_λ that $\mathcal{P}$ is normal, and $M_\lambda = U_{\mathcal{P}}$. Similarly, we can show that there is a normal poset

$$\mathcal{Q} := \left\{ (i,j) : i < j, \exists g \in N_\lambda \text{ such that } g_{ij} \neq 0 \right\}$$

in $[\![n]\!]$ such that $N_\lambda = U_{\mathcal{Q}}$.

$\square$

4.3 Identification with Andrews' construction

The Torus group $T_n(q)$ is the group of invertible matrices with only nonzero entries on the diagonal, i.e.,

$$T_n(q) = \{t \in \mathrm{GL}_n(\mathbb{F}_q) : t_{ij} = 0 \text{ if } \ i \neq j\}.$$

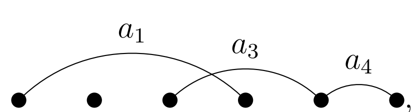
Define an action of $UT_n(q) \times UT_n(q) \times T_n(q)$ on the set

$$\{K_{[\eta]} : \eta \text{ is a non-nesting } \mathbb{F}_q\text{-set partition}\}$$

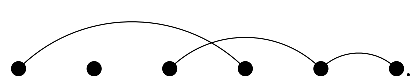
by

$$(a, b, t)K_{[\eta]} = \left\{t(a(u - I_n)b^{-1})t^{-1} : u \in K_{[\eta]}\right\}$$

for $a, b \in UT_n(q)$ and $t \in T_n(q)$. For every $\mathbb{F}_q$-set partition λ, let $\delta(\lambda)$ be the set partition obtained from omitting the labels on arcs in λ. For instance if λ is



then $\delta(\lambda)$ is



Note that $\bigcup_{K \in \mathcal{O}(K_{[\eta]})} K$ is the union of all K_λ such that $\delta(\lambda) = \eta$. For every set partition λ let

$$\mathcal{K}_\lambda := \bigcup_{\substack{K \in \mathcal{O}(K_{[\eta]}): \\ \delta(\eta) = \lambda}} K.$$

We aim to construct a supercharacter theory with the set of superclasses $\{\mathcal{K}_\lambda : \lambda \text{ is a non-nesting set partition}\}$. The supercharacters are also indexed by non-nesting set partitions, and are easiest to describe via their character formula. For every $g \in K_\gamma$ define

$$\Psi_\lambda(g) = \sum_{\substack{\eta \in NNS_n(q): \\ \delta(\eta)=\lambda}} \chi_{[\eta]}.$$

Proposition 4.3.1. *The pair*

$$(\{\mathcal{K}_\lambda : \lambda \in NNS_n\}, \{\Psi_\lambda : \lambda \in NNS_n\})$$

is a supercharacter theory for $UT_n(q)$. Moreover, for every $g \in \mathcal{K}_\gamma$,

$$\Psi_\lambda(g) = \begin{cases} c_\lambda \sum_{a,b \in \mathbb{F}_q} \prod_{\substack{i \frown j \in \lambda \\ i \frown j \in \gamma}} \theta(ab) & \begin{array}{c} \textit{if there are no } i \frown j \in \lambda \textit{ and} \\ k \frown l \in \gamma \textit{ with } (i,j) \neq (k,l) \\ i \leq k < l \leq j \end{array} \\ 0 & \textit{otherwise}, \end{cases}$$

where $c_\lambda = |UT_n(q) : U_\eta|$ for every $\eta \in NNS_n(q)$ with $\delta(\eta) = \lambda$.

Proof. It is clear that $\{\mathcal{K}_\lambda : \lambda \in NNS_n\}$ is a partition of G and $\{\Psi_\lambda : \lambda \in NNS_n\}$ is a set of characters which their constituents are all irreducible characters of $UT_n(q)$ and each pair of these characters have no common irreducible characters in their constituents. So it is enough to show that every Ψ_λ is constant on every $\mathcal{K}_\gamma$ for $\lambda, \gamma \in NNS_n$.

Since all $\mathbb{F}_q$-set partitions η with $\delta(\eta) = \lambda$ have the same shape, By Proposition 2.7.2, for every $g \in \mathcal{K}_\gamma = \bigcup_{\substack{K \in \mathcal{O}(K_{[\eta]}):\\ \delta(\eta)=\gamma}} K$, we have

$$\Psi_\lambda(g) = \begin{cases} \sum_{\substack{\eta,\omega \in NNS_n(q):\\ \delta(\eta)=\lambda \\ \delta(\omega)=\gamma}} \chi_{[\eta]}(1) \prod_{\substack{i \overset{a}{\frown} j \in \eta \\ i \overset{b}{\frown} j \in \omega}} \theta(ab) & \begin{array}{c}\text{if there are no } i \frown j \in \lambda \text{ and} \\ k \frown l \in \gamma \text{ with } (i,j) \neq (k,l) \\ i \leq k < l \leq j \end{array} \\ 0 & \textit{otherwise.} \end{cases}$$

Remember that

$$\chi_{[\eta]}(1) = |UT_n(q) : U_\eta| = q^{\left|\left\{(i,j): i<j \text{ and there exists } k \overset{a}{\frown} l \in \eta \text{ with } (i,j) \neq (k,l) \text{ and } k \leq i < j \leq l\right\}\right|}$$

is independent from labeling. Therefore, for every η_1 and η_2 in $NNS_n(q)$ with $\delta(\eta_1) = \delta(\eta_2) = \lambda$, it follows $\chi_{[\eta_1]}(1) = \chi_{[\eta_2]}(1) = c_\lambda$. Therefore,

$$\Psi_\lambda(g) = \begin{cases} c_\lambda \sum_{\substack{\eta,\omega \in NNS_n(q):\\ \delta(\eta)=\lambda \\ \delta(\omega)=\gamma}} \prod_{\substack{i \overset{a}{\frown} j \in \eta \\ i \overset{b}{\frown} j \in \omega}} \theta(ab) & \begin{array}{c}\text{if there are no } i \frown j \in \lambda \text{ and} \\ k \frown l \in \gamma \text{ with } (i,j) \neq (k,l) \\ i \leq k < l \leq j \end{array} \\ 0 & \textit{otherwise.} \end{cases}$$

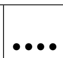
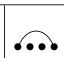
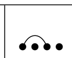
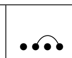
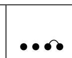
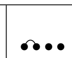
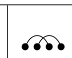
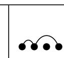
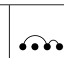
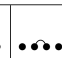
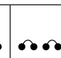
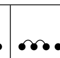
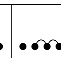
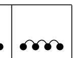
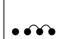
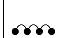
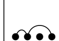
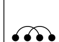
We conclude that

$$\Psi_\lambda(g) = \begin{cases} & \text{if there are no } i \frown j \in \lambda \text{ and} \\ c_\lambda \sum_{a,b\in\mathbb{F}_q} \prod_{\substack{i\frown j\in\lambda \\ i\frown j\in\gamma}} \theta(ab) & k \frown l \in \gamma \text{ with } (i,j) \neq (k,l) \\ & i \le k < l \le j \\ 0 & \textit{otherwise.} \end{cases}$$

The last expression shows that $\Psi_\lambda(g)$ is independent from labeling, and so we can see that $\Psi_\lambda(g)$ is constant for every $g \in \mathcal{K}_\gamma$. □

Example 4.3.2. *The supercharacter table of supercharacter theory*

$$(\{\mathcal{K}_\lambda : \lambda \in NNS_4\}, \{\Psi_\lambda : \lambda \in NNS_4\}).$$

														
	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	t	t	t	t	t	-1	t	-1	t	t	-1	-1	t	-1
	t	t	t	t	-1	t	t	t	-1	t	-1	t	-1	-1
	t	t	t	t	t	t	t	t	t	-1	t	-1	-1	-1
	t^2	t^2	t^2	t^2	$-t$	$-t$	t^2	$-t$	$-t$	t^2	1	$-t$	$-t$	1
	t^2	t^2	t^2	t^2	t^2	$-t$	t^2	$-t$	t^2	$-t$	$-t$	1	$-t$	1
	t^2	t^2	t^2	t^2	$-t$	t^2	t^2	t^2	$-t$	$-t$	$-t$	$-t$	1	1
	t^3	t^3	t^3	t^3	$-t^2$	$-t^2$	t^3	$-t^2$	$-t^2$	$-t^2$	t	t	t	-1
	q^2t	q^2t	q^2t	$-q^2$	0	q^2t	$-q^2$	$-q^2$	0	0	0	0	0	0
	q^2t	q^2t	$-q^2$	q^2t	q^2t	0	$-q^2$	0	$-q^2$	0	0	0	0	0
	q^2t^2	q^2t^2	$-q^2t$	q^2t^2	$-q^2t$	0	$-q^2t$	0	q^2	0	0	0	0	0
	q^2t^2	q^2t^2	q^2t^2	$-q^2t$	0	$-q^2t$	$-q^2t$	q^2	0	0	0	0	0	0
	q^3t^2	q^3t^2	$-q^3t$	$-q^3t$	0	0	q^3	0	0	0	0	0	0	0
	q^5t	$-q^5$	0	0	0	0	0	0	0	0	0	0	0	0

Note that the above table is same as the table in Example 4.2.9 after reordering the rows. This leads us to the following theorem.

Theorem 4.3.3. *The supercharacter theory*

$$(\{\mathcal{K}_\lambda : \lambda \in NNS_n\}, \{\Psi_\lambda : \lambda \in NNS_n\})$$

is the same as the normal supercharacter theory generated by normal pattern *subgroups. More specially*

$$\mathcal{K}_\lambda = M_\lambda^\circ \qquad \textit{and} \qquad \chi^{N_\lambda^\bullet} = \chi_\lambda.$$

Proof. We claim that

$$M_\lambda^\circ = \left\{ g \in UT_n(q) : \begin{array}{c} g_{ij} \neq 0 \text{ for all } i \frown j \in \lambda, \text{ and } g_{kl} = 0 \text{ unless} \\ \text{there exists } i \frown j \in \lambda \text{ with } k \leq i < j \leq l \end{array} \right\}.$$

Let $g \in M_\lambda$. Assume that for some $i \frown j \in \lambda$, $g_{ij} = 0$. Let

$$\lambda_1 = \left\{ k \frown l \in \lambda : k \frown l \neq i \frown j \right\} \cup A$$

where

$$A = \begin{cases} \{i-1 \frown j, i \frown j-1\} & \text{if } j \neq n, i \neq 1 \\ \{i-1 \frown j\} & \text{if } j = n, i \neq 1 \\ \{i \frown j-1\} & \text{if } j \neq, i = 1 \\ \emptyset & \text{if } j = n, i = 1. \end{cases}$$

We can see that $g \in M_{\lambda_1}$ and $M_{\lambda_1} \subset M_\lambda$, so $g \notin M_\lambda^\circ$. This proves the claim.

Note that

$$K_{[\eta]} = \left\{ g \in UT_n(\mathbb{F}_q) \;\middle|\; \begin{array}{c} g_{ij} = a \text{ for all } i \overset{a}{\frown} j \in \eta \text{ and } g_{kl} = 0 \text{ unless} \\ \text{there exists } i \overset{a}{\frown} j \in \eta \text{ with } k \leq i < j \leq l \end{array} \right\}.$$

So it is easy to see that

$$\bigcup_{\substack{\eta \in NNS_n(q): \\ \delta(\eta) = \lambda}} K_{[\eta]} = \left\{ g \in UT_n(q) : \begin{array}{c} g_{ij} \neq 0 \text{ for all } i \frown j \in \lambda, \text{ and } g_{kl} = 0 \text{ unless} \\ \text{there exists } i \frown j \in \lambda \text{ with } k \leq i < j \leq l \end{array} \right\}.$$

Therefore, the supercharacters of normal supercharacter theory generated by normal pattern subgroups are a multiple constant of supercharacters of Andrews' construction glued by two-sided torus action, i.e., $\{\Psi_\lambda : \lambda \in NNS_n(q)\}$. We will show that precisely they are same and

$$\chi^{N_\lambda^\bullet}(g) = \Psi_\lambda(g).$$

First we show that

$$\chi^{N_\lambda^\bullet}(1) = \Psi_\lambda(1).$$

Note that

$$\chi_{[\eta]}(1) = |UT_n(q) : U_\eta|,$$

and so

$$\Psi_\lambda(1) = \sum_{\substack{\eta \in NNS_n(q): \\ \delta(\eta)=\lambda}} \chi_{[\eta]}(1) = (q-1)^{\{\# i \frown j \in \lambda\}} |UT_n(q) : U_\eta|.$$

By Theorem 3.2.3

$$\chi^{N_\lambda^\bullet}(1) = \sum_{N_\lambda \subseteq M} \mu(N_\lambda, M)|UT_n(q) : M| = (q-1)^{\{\# i \frown j \in \lambda\}} |UT_n(q) : N_\lambda| \quad (4.3.1)$$

Since U_η and N_λ has the same cardinality, therefore,

$$\chi^{N_\lambda^\bullet}(1) = \Psi_\lambda(1).$$

This shows that actually the set of two supercharacters $\{\chi^{N_\lambda^\bullet} : \lambda \in NNS_n\}$ and

$\{\Psi_\lambda : \lambda \in NNS_n\}$ are the same. Let $g \in N_\lambda$. Then

$$\chi^{N_\lambda^\bullet}(g) = \chi^{N_\lambda^\bullet}(1) = \Psi_\lambda(1) = \sum_{\substack{\eta \in NNS_n(q): \\ \delta(\eta)=\lambda}} \chi_{[\eta]}(1) = \sum_{\substack{\eta \in NNS_n(q) \\ \delta(\eta)=\lambda}} \sum_{\mathrm{big}(\nu)=\eta} \chi_\nu(1).$$

As a result of (Marberg 2011, Lemma 4.5), we have

$$\sum_{\substack{\eta \in NNS_n(q) \\ \delta(\eta)=\lambda}} \sum_{\mathrm{big}(\nu)=\eta} \chi_\nu(1) = \sum_{\substack{\eta \in NNS_n(q) \\ \delta(\eta)=\lambda}} \sum_{\mathrm{big}(\nu)=\eta} \chi_\nu(g) = \Psi_\lambda(g).$$

Therefore,

$$\chi^{N_\lambda^\bullet}(g) = \chi^{N_\lambda^\bullet}(1) = \Psi_\lambda(1) = \Psi_\lambda(g)$$

for every $g \in N_\lambda$. So $N_\lambda \subseteq \ker \Psi_\lambda$.

From 4.3.1 we have that the supercharacters in $\{\chi^{N_\lambda^\bullet} : \lambda \in NNS_n\}$ and $\{\Psi_\lambda : \lambda \in NNS_n\}$ with same degree are identical, i.e., for a fix positive integer d,

$$\{\chi^{N_\lambda^\bullet} : \chi^{N_\lambda^\bullet}(1) = d, \lambda \in NNS_n\} = \{\Psi_\lambda : \Psi_\lambda(1) = d, \lambda \in NNS_n\}.$$

Moreover the only characters in the both sets that has N_λ in their kernels are $\chi^{N_\lambda^\bullet}$ and Ψ_λ. Therefore,

$$\chi^{N_\lambda^\bullet}(g) = \Psi_\lambda(g)$$

for all $g \in UT_n(q)$.

$\square$

4.3.1 Inflation of Normal Supercharacter Theory Generated by Pattern Subgroups of $UT_n(q)$

Let $\mathbf{CSC}(UT_n(q))$ be the vector space generated by the set of supercharacters of normal supercharacter theory generated by normal pattern subgroups, i.e.,

$$\mathbf{CSC}(UT_n(q)) = \mathbb{C}\text{-Span}\{\chi^{M_\lambda^\bullet} : \lambda \in NNS_n\}.$$

Let

$$\mathbf{CSC} = \bigoplus_{n\geq 0} \mathbf{CSC}(UT_n(q)),$$

where by convention we let

$$\mathbf{CSC}(UT_0(q)) = \mathbb{C}\text{-}\{\chi^{M_{\emptyset_0}^\bullet}\},$$

where $\emptyset_0$ is the empty set partition of the set with 0 elements.

Throughout this section we identify $\chi^{M_\lambda^\bullet}$ by χ^λ. Furthermore, we write $\lambda \leq \omega$ when $M_\lambda \subseteq M_\omega$. Define a product on $\mathbf{CSC}$ by

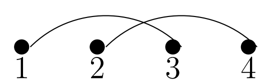
$$\chi \cdot \psi = \mathbf{Inf}_{UT_m(q)\times UT_n(q)}^{UT_{m+n}(q)}(\chi \times \psi) = (\chi \times \psi) \circ \pi,$$

where $\chi \in \mathbf{CSC}(UT_m(q))$, $\psi \in \mathbf{CSC}(UT_n(q))$, and $\mathbf{Inf}$ is the inflation functor coming from the quotient map

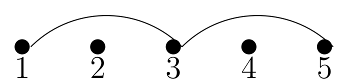
$$\pi \colon UT_{m+n}(q) \longrightarrow \left[\begin{array}{c|c} UT_m(q) & 0 \\ \hline 0 & UT_n(q) \end{array}\right] \cong UT_m(q) \times UT_n(q).$$

Let $\lambda \in NNS_m$ and $\mu \in NNS_n$, denote by $\lambda \frown \mu$ the arc diagram of seating λ and μ side by side and adding m to the nodes of μ and then connect m and $m+1$ by an arc.

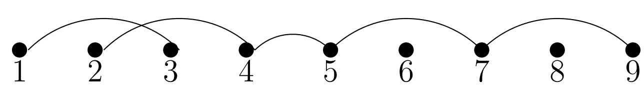
Example 4.3.4. *Let*

$\lambda =$ 

and

$\mu =$ 

Then

$\lambda \frown \mu =$ 

Theorem 4.3.5. *Let $\lambda \in NNS_m$ and $\mu \in NNS_n$. Then*

$$\mathbf{Inf}_{UT_m \times UT_n}^{UT_{m+n}}(\chi^{\lambda} \times \chi^{\mu}) = \chi^{\lambda \frown \mu}.$$

Proof. Let $g \in M_{\nu}^{\circ}$ and $\pi(M_{\nu}^{\circ}) = M_{\nu_1}^{\circ} \otimes M_{\nu_2}^{\circ}$. Then $\pi(g) = g_1 \otimes g_2$ where $g_1 \in M_{\nu_m}^{\circ}$ and $g_2 \in M_{\nu_n}^{\circ}$. Note that $\nu_m = \nu|_{[m]}$ and $\nu_n = \nu|_{[m+n]\setminus[m]}$. By 4.2.2,

$$\chi^{\lambda}(g_1) = \sum_{\substack{\eta_1 \in NNS_m: \\ \nu_n \leq \eta_1, \lambda \leq \eta_1}} \mu(\lambda, \eta_1) \frac{|UT_m|}{|M_{\eta_1}|}$$

and

$$\chi^{\mu}(g_2) = \sum_{\substack{\eta_2 \in NNS_n: \\ \nu_m \leq \eta_2, \mu \leq \eta_2}} \mu(\mu, \eta_2) \frac{|UT_n|}{|M_{\eta_2}|}.$$

Therefore,

$$(\chi^\lambda \times \chi^\mu)(g_1 \otimes g_2) = \chi^\lambda(g_1).\chi^\mu(g_2) =$$

$$\sum_{\substack{\eta_1 \in NNS_m: \\ \nu_m \leq \eta_1, \lambda \leq \eta_1}} \mu(\lambda, \eta_1) \frac{|UT_m|}{|M_{\eta_1}|} \sum_{\substack{\eta_2 \in NNS_n: \\ \nu_n \leq \eta_2, \mu \leq \eta_2}} \mu(\mu, \eta_2) \frac{|UT_n|}{|M_{\eta_2}|} =$$

$$\sum_{\substack{\eta_1 \in NNS_m: \\ \eta_2 \in NNS_n: \\ \nu_m \leq \eta_1, \lambda \leq \eta_1 \\ \nu_n \leq \eta_2, \mu \leq \eta_2}} \mu(\lambda, \eta_1)\mu(\mu, \eta_2) \frac{|UT_m||UT_n|}{|M_{\eta_1}||M_{\eta_2}|}.$$

Note that

$$\frac{|UT_m||UT_n|}{|M_{\eta_1}||M_{\eta_2}|} = \frac{|UT_{m+n}|}{|M_{\eta_1 \frown \eta_2}|},$$

$$\lambda \frown \mu \leq \eta_1 \frown \eta_2 \text{ if and only if } \lambda \leq \eta_1 \text{ and } \mu \leq \eta_2,$$

$$\mu(\lambda, \eta_1)\mu(\mu, \eta_2) = \mu(\lambda \frown \mu, \eta_1 \frown \eta_2)$$

, and

$$\nu_m \leq \eta_1 \text{ and } \nu_n \leq \eta_2 \text{ if and only if } \ \nu \leq \eta_1 \frown \eta_2.$$

Also, $\lambda \frown \nu \leq \eta$ if and only if η can be written as $\eta_1 \frown \eta_2$, where $\eta_1 \in NNS_m$ and $\eta_2 \in NNS_n$. Therefore,

$$(\chi^\lambda \times \chi^\mu)(g_1 \otimes g_2) =$$

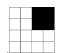
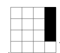
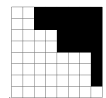
$$\sum_{\substack{\eta \in NNS_{m+n}: \\ \nu \leq \eta, \lambda \frown \mu \leq \eta}} \mu(\lambda \frown \mu, \eta) \frac{|UT_{m+n}|}{|M_\eta|} = \chi^{\lambda \frown \mu}.$$

As our choose of M_ν°, λ, and μ are arbitrary, we conclude that for all $\lambda \in NNS_m$ and $\mu \in NNS_n$,

$$\mathbf{Inf}_{UT_m \times UT_n}^{UT_{m+n}}(\chi^\lambda \times \chi^\mu) = \chi^{\lambda \frown \mu}.$$

$\square$

Example 4.3.6.

$\mathbf{Inf}^{UT_{n+m}}_{UT_n \times UT_m}(\chi$ $\times\ \chi$$) = \chi$.

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