Complex Powers of a Fourth-Order Operator: Heat Kernels, Green Functions and $L^p - L^{p'}$ Estimates

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Abstract

We first construct the minimal and maximal operators of the Hermite operator.Then we apply a classical reslult by Askey and Wainger, to prove that for $4/3 < p < 4$. This implies that the Hermite operator is essentially self-adjoint, which means that its minimal and maximal operators coincide. Using the asymptotic behaviour of the L^p-norms of the Hermite functions and essentially the same method as in the proof of $4/3 < p < 4$, the same results are true for $1 \leq p \leq \infty$. We also compute the spectrum for the minimal and the maximal operator for $4/3 < p < 4$. Then we construct a fourth-order operator, called the twisted bi-Laplacian, from the Laplacian on the Heisenberg group, namely, the twisted Laplacian. Using spectral analysis, we obtain explicit formulas for the heat kernel and Green function of the twisted bi-Laplacian. We also give results on the spectral theory and number theory associated with it. We then consider all complex powers of the twisted bi-Laplacian and compute their heat kernels and Green functions, and moreover, we obtain $L^p - L^{p'}$ estimates for the solutions of the initial value problem for the heat equation and the Poisson equation governed by complex powers of the twisted bi-Laplacian.

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1 Introduction

The Hermite operator plays an important role in both mathematics and physics. It is known as the quantum harmonic oscillator, or the simple harmonic oscillator [25],[26], [30], and is given by

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^n,$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, It maps the Schwartz space $\mathcal{S}$ into $\mathcal{S}$. In quantum mechanics, an arbitrary Schrödinger operator can be approximated locally by H, and one can compute its eigenvalues and eigenfunctons from it. That H is called the Hermite operator is due to the fact that Hermite functions are the eigenfunctions of H. See, for instance, Section 6.4 in [29]. We begin this dissertation by proving that the minimal and the maximal operators of the Hermite operator agree with each other on $L^p(\mathbb{R}^n)$, $4/3 < p < 4$. We denote the minimal operator on $L^p(\mathbb{R}^n)$ by $H_{0,p}$ and the maximal operator by $H_{1,p}$. The extension of H from $\mathcal{S}$ to $L^2(\mathbb{R}^n)$ has been well understood, and in fact, $H_{0,2} = H_{1,2}$. This means that H

is essentially self-adjoint. The spectrum $\Sigma(H_{0,2})$ of $H_{0,2}$ is given by

$$\Sigma(H_{0,2}) = \{2|\alpha| + 1 : \alpha \in \mathbb{N}_0^n\},$$

where $\mathbb{N}_0^n$ is the set of all multi-indices. However, we are interested in knowing the spectrum of $H_{0,p}$ and $H_{1,p}$ for general p. In this thesis we show that the spectrum of the Hermite operator is equal to that of its minimal (maximal) operator for $4/3 < p < 4$. Moreover, the two operators are the same for $4/3 < p < 4$, or in other words, the Hermite operator is essentially self-adjoint for p between 4/3 and 4.

In Section 2 of Chapter 3, by proving that the Hermite operator H is closable on $L^p(\mathbb{R}^n),\ 1 < p < \infty$, it follows that $H_{0,p}$ is the minimal operator, (i.e., the smallest closed extension) of H on $L^p(\mathbb{R}^n)$. In order to prove our desired result, we first raise the Hermite operator to its Nth power for some positive integer N, which enables us to compute explicitly the spectrum for $1 \leq p < \infty$. Secondly, we compute explicitly the spectrum of the minimal and maximal operator of H on $L^p(\mathbb{R}^n)$, and finally via the functional calculus, we prove that the two operators are the same on $L^p(\mathbb{R}^n),\ 1 \leq p < \infty$. In addition, we give an estimate for the L^p norm of the solution to the initial value problem for the heat equation governed by the minimal (maximal) operator for $1 \leq p < \infty$. The result so far is correct and relys on a classical result by Askey and Wainger in [2]. In the last section, we use the asymptotic behaviour of the L^p norms of the Hermite functions to replace the Askey and Wainger's result to show that the same results hold for $1 \leq p \leq \infty$.

Related results of the heat equation associated with the Hermite operator can be found in [6, 7, 11, 40, 41].

When $n = 2$ an important operator related to H is L, the twisted Laplacian L on $\mathbb{R}^2$,

$$L = -\Delta + \frac{1}{4}(x^2 + y^2) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Thus, the twisted Laplacian L is the Hermite operator

$$H = -\Delta + \frac{1}{4}(x^2 + y^2)$$

perturbed by the partial differential operator $-iN$, where

$$N = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the rotation operator.

N is the rotation operator because in polar coordinates,

$$N = \frac{\partial}{\partial \theta}.$$

The twisted Laplacian, which has been studied extensivelyl, appears in harmonic analysis naturally in the context of Wigner transforms and Weyl transforms [13, 38], which we will recall in Chapter 2. In the paper [10], it is shown that L is essentially self-adjoint, and the spectrum $\Sigma(L_0)$ of the closure L_0 is given by a sequence of eigenvalues, which are odd natural numbers, *i.e.*,

$$\Sigma(L_0) = \{2k + 1 : k = 0, 1, 2, \dots\}.$$

It should be noted, however, that each eigenvalue has infinite multiplicity. And this is a disadvantage in applications. So we introduce the twisted bi-Laplacian of which each eigenvalue has finite multiplicity.

Renormalizing the twisted Laplacian L to the partial differential operator P given by

$$P = \frac{1}{2}(L + 1), \tag{1.2}$$

we see that the eigenvalues of P are the natural numbers $1, 2, \dots$, and each eigenvalue, as in the case of L, has infinite multiplicity.

Now, the conjugate $\overline{L}$ of the twisted Laplacian L is given by

$$\overline{L} = -\Delta + \frac{1}{4}(x^2 + y^2) + i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{1.3}$$

and after renormalization, we get the conjugate Q of P given by

$$Q = \frac{1}{2}(\overline{L} + 1). \tag{1.4}$$

One of the goals of the thesis is to analyze the heat kernels and Green functions of complex powers of the twisted bi-Laplacian M defined by

$$M = QP = PQ = \frac{1}{4}(H - iN + 1)(H + iN + 1), \tag{1.5}$$

where P and Q commute because it can be shown by easy computations that H and N commute, i.e., $HNf = NHf$ for all functions f in $C^\infty(\mathbb{R}^2)$.

It is proved in [15] that M is essentially self-adjoint on $L^2(\mathbb{R}^2)$. The unique self-adjoint extension of M on $L^2(\mathbb{R}^2)$ is again denoted by M. To see how the operator M arises, we need to recall some analysis on $\mathbb{H}^1$, the Heisenberg group.

For the sake of simplifying the notation and making the thesis more clear, we have chosen to work in the one-dimensional Heisenberg group only. However, the results in this thesis are also valid for the n-dimensional Heisenberg group.

If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via the obvious identification

$$\mathbb{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \mathbb{C},$$

and we let

$$\mathbb{H}^1 = \mathbb{C} \times \mathbb{R},$$

then $\mathbb{H}^1$ becomes a noncommutative Lie group when equiped with the multiplication $\cdot$ given by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{4}[z, w]\right), \quad (z, t), (w, s) \in \mathbb{H}^1,$$

where $[z, w]$ is the symplectic form of z and w defined by

$$[z, w] = 2\,\mathrm{Im}(z\overline{w}).$$

We call $\mathbb{H}^1$ the one-dimensional Heisenberg group.

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$. Then a basis for $\mathfrak{h}$ is given by X, Y and T, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t},$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t},$$

and

$$T = \frac{\partial}{\partial t}.$$

It is easy to check that

$$[X, Y] = -T$$

and all other commutators are zero. The sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^1$ is defined by

$$\mathcal{L} = -(X^2 + Y^2).$$

The connection between the sub-Laplacian on the Heisenber group and the Hermite operators can first be attributed to Greiner [14]. Based on this observation, the Laguerre calculus has been constructed [3, 4, 5] and used by [36] to construct the heat kernel, wave kernel and the Green function of the twisted Laplacian.

By a result of Hörmander in [22], $\mathcal{L}$ is hypoelliptic on $\mathbb{H}^1$. A simple computation gives

$$\mathcal{L} = -\Delta - \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In Chapter 4, we will obtain explicit formulas for the heat kernel and Green function of the fourth-order operator $\mathcal{L}_+\mathcal{L}_-$ on

$\mathcal{H}^1$, where $\mathcal{L}_+ = \mathcal{L}$ and $\mathcal{L}_-$ is defined by

$$\mathcal{L}_- = -\Delta - \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial t^2} - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}.$$

In fact,

$$\mathcal{L}_- = -(X_-^2 + Y_-^2),$$

where

$$X_- = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial t},$$

and

$$Y_- = \frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial t}.$$

Since

$$[X_-, Y_-] = T,$$

it follows from Hörmander's result in [22] again that $\mathcal{L}_-$ is also hypoelliptic on $\mathbb{H}^1$. Thus the hypoellipticity of $\mathcal{L}_+\mathcal{L}_-$ on $\mathbb{H}^1$ follows easily from the hypoellipticity on $\mathbb{H}^1$ of $\mathcal{L}_+$ and $\mathcal{L}_-$. The heat kernel K_ρ for $\rho > 0$ and the Green function G of $\mathcal{L}_+\mathcal{L}_-$ are functions on $\mathbb{H}^1$ such that for all suitable functions f on $\mathbb{H}^1$,

$$e^{-\rho(\mathcal{L}_+\mathcal{L}_-)}f = K_\rho *_{\mathbb{H}^1} f \tag{1.6}$$

and

$$(\mathcal{L}_+\mathcal{L}_-)^{-1}f = G *_{\mathbb{H}^1} f, \tag{1.7}$$

where the convolution $f *_{\mathbb{H}^1} g$ of two functions f and g on $\mathbb{H}^1$ is given by

$$(f *_{\mathbb{H}^1} g)(z,t) = \int_{-\infty}^{\infty}\int_{\mathbb{C}} f((z,t)\cdot(w,s)^{-1})g(w,s)\,dw\,ds \tag{1.8}$$

for all (z,t) in $\mathbb{H}^1$.

We will first give a self-contained and detailed construction of the heat kernels and Green functions of twisted bi-Laplacians obtained from the sub-Laplacian on the Heisenberg group. It is worth noting that the heat kernels can be expressed in terms of theta functions [24]. The spectral theory and the number theory of the twisted Laplacian L_1L_{-1} can be found in the works [16, 17].

In Chapter 5, we will show how to transform $\mathcal{L}_+\mathcal{L}_-$ to a family of twisted bi-Laplacians $L_\tau L_{-\tau}$ on $\mathbb{C}$ parametrized by $\tau \in \mathbb{R}\setminus\{0\}$. This has the advantage of reducing the number of independent variables of the sub-Laplacian from three to two and can be seen as a method of descent and the parameter τ can be seen as Planck's constant. The τ-Fourier–Wigner transforms of Hermite functions are developed, which can then be used to construct the heat kernel of L_τ. The Green function of L_τ is constructed in Section 4 of chapter 4. The heat kernel and Green function of $L_\tau L_{-\tau}$ for $\tau \in \mathbb{R}\setminus\{0\}$ are given in, respectively, Section 6 and Section 7. The heat kernel and Green function of the fourth-order operator $\mathcal{L}_+\mathcal{L}_-$ are given in Section 8. A related fourth-order operator on the Heisenberg group has also been studied using Laguerre calculus in [9].

The last part of the thesis describes the spectral properties of M precisely. Let us first recall that the Fourier–Wigner transform $V(f,g)$ of two functions f and g in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ is the function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$ given by

$$V(f,g)(q,p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{iqy} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy$$

for all q and p in $\mathbb{R}$. For $k = 0, 1, 2, \ldots$, the Hermite function e_k of order k is defined on $\mathbb{R}$ by

$$e_k(x) = \frac{1}{(2^k k!\sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R}, \tag{1.9}$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k e^{-x^2}, \quad x \in \mathbb{R}. \tag{1.10}$$

Now, for $j, k = 0, 1, 2, \ldots$, we define the function $e_{j,k}$ on $\mathbb{R}^2$ by

$$e_{j,k}(x,y) = V(e_j, e_k)(x,y), \quad x, y \in \mathbb{R}. \tag{1.11}$$

It can be shown that $\{e_{j,k} : j,k = 0,1,2,\ldots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$. See, for example, Theorem 21.2 in [38].

The following result is Theorem 1.1 in [15].

Theorem 1.0.1. *The eigenvalues and the eigenfunctions of the twisted bi-Laplacian M are, respectively, the natural numbers $1, 2, 3, \ldots,$ and the functions $e_{j,k}$, $j, k = 0, 1, 2, \ldots.$*

More precisely, for $n = 1, 2, 3, \ldots,$ *the eigenfunctions corresponding to the eigenvalue* n *are all the functions* $e_{j,k}$ *where* $j, k = 0, 1, 2, \ldots,$ *such that*

$$(j+1)(k+1) = n.$$

By means of Theorem 1.0.1, we see that the multiplicity of each eigenvalue n of the twisted bi-Laplacian is equal to the number $d(n)$ of divisors of the positive integer n. We give as Corollary 1.2 in [15] an estimate on the counting function $N(\lambda)$ defined as the number of eigenvalues of M less than or equal to λ. In fact, we can see that the following result, which is Corollary 1.2 in [15], is the well-known result of asymptotic behavior of the Dirichlet divisors in the perspective of the counting function of the twisted bi-Laplacian, in which the multiplicity of each eigenvalue is taken into account.

Theorem 1.0.2. *For all* λ *in* $[0, \infty)$,

$$N(\lambda) = \sum_{n \le \lambda} d(n) = \lambda \ln \lambda + (2\gamma - 1)\lambda + E(\lambda), \tag{1.12}$$

where γ *is Euler's constant and*

$$E(\lambda) = O(\sqrt{\lambda})$$

as $\lambda \to \infty$.

More precise results than Theorems 1.0.1 and 1.0.2 can be found in [17]. A complete and classical proof of Theorem 1.0.2 can be based on Theorem 4.12 in [31] and the above-mentioned connection between the Dirichlet divisors and the twisted bi-Laplacian. It is interesting to point out the connection with the divisor problem, which asks for the best number μ such that

$$E(\lambda) = O(\lambda^{\mu})$$

as $\lambda \to \infty$. The conjecture is that $\mu = 1/4$, but it is a result of Hardy [18] that $\mu = 1/4$ does not work. The best result to date due to Huxley [20] is that $\mu = \frac{131}{416}$, which is approximately 0.31490.

Theorem 1.0.2 is used to compute the trace of the heat semigroup of M in Chapter 4, Section 2 and the Dixmier trace of the inverse of M in Section 3. Another theme of Chapter 4 is to compute the zeta function regularizations of the trace and the determinant of the complex power M^{α} of M, where $\alpha \in \mathbb{C}$. To that end, we use the complex-valued function $\zeta_{M^{\alpha}}$ defined *formally* by

$$\zeta_{M^{\alpha}}(s) = \mathrm{tr}((M^{\alpha})^{-s}) = \mathrm{tr}(M^{-\alpha s}), \quad s \in \mathbb{C},$$

in Section 4 of Chapter 4 to compute the zeta function regularizations of the trace and determinant of M^{α}, and give a formula for the zeta function regularization of the determinant of the heat semigroup $e^{-tM^{\alpha}}$.

In the final chapter, using the results obtained in all previous chapters, we are able to obtain the heat kernels and Green functions for all complex powers of the twisted bi-Laplacian M. We prove that these class of operators are Hilbert-Schimidt, and are in the trace class. Lastly, by applying Wong's result on the estimates of the L^p norms of Hermite functions, we give the range of p's for the $L^p - L^{p'}$ estimates to the solutions of partial differential equations governed by M^α, $\alpha \in \mathbb{C}$, are in L^p.

2 Background

2.1 The Heisenberg Group and the Sub-Laplacian

In this section, we introduce the Heisenberg group $\mathbb{H}^1$, a non-commutative group with underlying manifold $\mathbb{R}^3$. For the sake of transparency considering the type of problems we are dealing with in this dissertation, we only look at the one-dimensional Heisenberg group, and the extensions to higher dimensions are easy generalizations.

The Heisenberg group and its connections with quantum mechanics and other branches of mathematics can be found in [12],[23], [29],[35].We begin with the definition of the Heisenberg group. We identify points in $\mathbb{R}^2$ with points in $\mathbb{C}$ through the following law:

$$\mathbb{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \mathbb{C}.$$

Let $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$. Then for all points (z, t), $(w, s) \in \mathbb{H}^1$, we define the group law by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{4}[z, w] \right),$$

where $[z, w]$ is the symplectic form of z, w given by

$$[z, w] = 2\mathrm{Im}(z\overline{w}).$$

Thus, $\mathbb{H}^1$ is a noncommutative Lie group under this group law, with the identity element $(0, 0)$, and the inverse of the element (z, t) is simply $(-z, -t)$. Moreover, the Heisenberg group is unimodular, which means that the left Haar measure and the right Haar measure agree, and equal to the Lebesgue measure on $\mathbb{R}^3$.

A Lie algebra is a real vector space $\mathfrak{g}$ with a binary operation $[\cdot, \cdot]$ which is bilinear and satisfies the Jacobi identity. The Jacobi identity states that

$$[g_1, [g_2, g_3]] + [g_2, [g_1, g_3]] + [g_3, [g_1, g_2]] = 0$$

for all g_1, g_2 and g_3 on $\mathfrak{g}$.

A vector field V on $\mathbb{H}^1$ is said to be left-invariant if

$$VL_{(w,s)} = L_{(w,s)}V$$

for all $(w, s) \in \mathbb{H}^1$, where $L_{(w,s)}$ is the left translation by (w, s) defined by

$$(L_{(w,s)}f)(z, t) = f((w, s) \cdot (z, t)), \quad (z, t) \in \mathbb{H}^1.$$

We now introduce a particular Lie algebra, namely the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$.

Theorem 2.1.1. *[44] Let $\mathfrak{h}^1$ be the set of all left-invariant vector fields on $\mathbb{H}^1$. Then $\mathfrak{h}^1$ is a Lie algebra in which the Lie bracket $[\cdot,\cdot]$ is the commutator given by*

$$[X, Y] = XY - YX$$

for all $X,\ Y \in \mathfrak{h}^1$.

Proof Linearity is obvious. Let $X,\ Y \in \mathfrak{h}^1$, and we need to show firstly that $[X, Y] \in \mathfrak{h}^1$. We write

$$X = a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1 \frac{\partial}{\partial t}$$

and

$$Y = a_2 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + b_3 \frac{\partial}{\partial t},$$

where $a_1,\ b_1,\ c_1,\ a_2,\ b_2,\ c_2$ are C^∞ functions on $\mathbb{H}^1$. Then one can easily check that

$$XY = a_1 a_2 \frac{\partial^2}{\partial x^2} + b_1 b_2 \frac{\partial^2}{\partial y^2} + c_1 c_2 \frac{\partial^2}{\partial t^2} + (a_1 b_2 + a_2 b_1) \frac{\partial^2}{\partial x \partial y} + (b_1 c_2 + b_2 c_1) \frac{\partial^2}{\partial y \partial t}$$
$$+ (a_1 c_2 + a_2 c_1) \frac{\partial^2}{\partial t \partial x} + V_1,$$

where V_1 is a vector field on $\mathbb{H}^1$. By switching subscripts in the second-order terms in XY, we get

$$[X, Y] = XY - YX = V_1 - V_2,$$

where V_2 is another vector field on $\mathbb{H}^1$. To see that $[X, Y]$ is left-invariant, let $(w, s) \in \mathbb{H}^1$, and we use the left-invariance of $X,\ Y$ to check that

$$L_{(w,s)} XY = X L_{(w,s)} Y = XY L_{(w,s)}$$

and

$$L_{(w,s)}YX = YL_{(w,s)}X = YXL_{(w,s)}.$$

Thus, we have

$$[X, Y]L_{(w,s)} = L_{(w,s)}[X, Y],$$

and therefore $[X, Y] \in \mathfrak{h}^1$, as desired. Secondly, we prove Jacobi's identity.

$$[X, [Y, Z]] = [X, YZ - ZY] = XYZ - XZY - YZX + ZYX,$$

$$[Y, [Z, X]] = [Y, ZX - XZ] = YXZ - YZX - ZXY + XZY,$$

$$[Z, [X, Y]] = [Z, XY - YX] = ZXY - ZYX - XYZ + YXZ.$$

Thus,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

and therefore $\mathfrak{h}^1$ is a Lie algebra.

Theorem 2.1.2. *X, Y, T are vector fields on $\mathbb{H}^1$ defined as follows,*

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t},$$

$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t},$$

$$T = \frac{\partial}{\partial t}.$$

Then X, Y, T form a basis for $\mathfrak{h}^1$.

Proof Firstly, we check that $X,\ Y,\ T \in \mathfrak{h}^1$. i.e.,

$$XL_{(w,s)} = L_{(w,s)}X$$

for all $(w, s) \in \mathbb{H}^1$. To see this, we write $w = (u, v),\ z = (x, y)$. Then

$$(L_{(w,s)}f)(z,t) = f((w,s)\cdot(z,t)) = f\left(u+x, v+y, s+t+\frac{1}{2}(vx-uy)\right),$$

where $(z,t) \in \mathfrak{h}^1$. To simplify notation, we denote

$$(...) = \left(u+x, v+y, s+t+\frac{1}{2}(vx-uy)\right).$$

Then, we have

$$\begin{aligned}
&(XL_{(w,s)}f)(z,t)\\
= \quad &\left(\left(\frac{\partial}{\partial x}+\frac{1}{2}y\frac{\partial}{\partial t}\right)(L_{(w,s)}f)\right)(z,t)\\
= \quad &\frac{\partial f}{\partial x}(...)+\frac{1}{2}v\frac{\partial f}{\partial t}(...)+\frac{1}{2}y\frac{\partial f}{\partial t}(...)\\
= \quad &\frac{\partial f}{\partial x}(...)+\frac{1}{2}(v+y)\frac{\partial f}{\partial t}(...).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(L_{(w,s)}Xf)(z,t)\\
= \quad &(Xf)(...)\\
= \quad &\frac{\partial f}{\partial x}(...)+\frac{1}{2}(v+y)\frac{\partial f}{\partial t}(...).
\end{aligned}$$

Thus,

$$XL_{(w,s)} = L_{(w,s)}X.$$

So we have proved that $X \in \mathfrak{h}^1$, and similar arguments show that Y, T are also elements of $\mathfrak{h}^1$.

Moreover, we know that the Lie algebra $\mathfrak{h}^1$ is isomorphic to $T_{(0,0,0)}\mathbb{H}^1$, the tangent space of the Heisenberg group at the origin, and a proof can be found in [44]. Since $T_{(0,0,0)}\mathbb{H}^1$ is a three dimensional vector space, it remains to show that X, Y, T are linearly independent. The see this, we consider, the equation

$$aX + bY + cT = 0,$$

where a, b, c are real numbers. for all f on $\mathbb{H}^1$, we must show that

$$(aX + bY + cT)f = 0 \Leftrightarrow a,\ b,\ c = 0.$$

But this is clear if we pick

$$f(x, y, t) = x; \quad f(x, y, t) = y; \quad f(x, y, t) = t.$$

Therefore, X, Y, T is a basis for $\mathfrak{h}^1$.

Lastly, we explain the choice of vector fields X, Y, T as a basis for $\mathfrak{h}^1$.

Theorem 2.1.3. *Let e_1, e_2, e_3 be the coordinate axes and write them in their parameterized form*

$$e_1(s) = (s, 0, 0), \quad s \in \mathbb{R},$$

$$e_2(s) = (0, s, 0), \quad s \in \mathbb{R},$$

$$e_3(s) = (0, 0, s), \quad s \in \mathbb{R}.$$

Then for all C^∞ functions f on $\mathbb{H}^1$, we have

$$(Xf)(z,t) = \frac{d}{ds}|_{s=0} f((z,t) \cdot e_1(s)),$$

$$(Yf)(z,t) = \frac{d}{ds}|_{s=0} f((z,t) \cdot e_2(s)),$$

$$(Tf)(z,t) = \frac{d}{ds}|_{s=0} f((z,t) \cdot e_3(s))$$

for all $(z,t) \in \mathbb{H}^1$.

Proof Since

$$\begin{aligned} & \frac{d}{ds}|_{s=0} f((z,t) \cdot e_1(s)) \\ = & \frac{d}{ds}|_{s=0} f\left(x+s, y, t+s+\frac{1}{2}sy\right) \\ = & \frac{\partial f}{\partial x}(x,y,t) + \frac{1}{2}y\frac{\partial}{\partial t}(x,y,t), \end{aligned}$$

We get

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t},$$

as asserted.

Lastly, an observation can be made through the theorem below.

Theorem 2.1.4. $[X, Y] = -T$*, and all other commutators among* X, Y, T *vanish.*

By Theorem 2.14, the vector fields X, Y, and their first-order commutator span the Lie algebra $\mathfrak{h}^1$ on the Heisenberg group. In fact, they are the so-called horizontal vector fields on $\mathbb{H}^1$, and T is known as the missing direction.

Now, we develop the sub-Laplacian on $\mathbb{H}^1$, which will later give rise to a family of linear operators known as the twisted Laplacians on $\mathbb{R}^3$. The sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^1$ is defined by

$$\mathcal{L} = -(X^2 + Y^2).$$

More explicitly,

$$\begin{aligned} X^2 &= \left(\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x \partial t} + \frac{1}{4}y^2\frac{\partial^2}{\partial t^2} \end{aligned}$$

and

$$\begin{aligned} Y^2 &= \left(\frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial y^2} - x\frac{\partial^2}{\partial y \partial t} + \frac{1}{4}x^2\frac{\partial^2}{\partial t^2}. \end{aligned}$$

Thus,

$$\mathcal{L} = -\Delta - \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

2.2 The Wigner Transforms and the Weyl Transforms

In this section, we introduce the Weyl transform and its properties. We begin with introducing the first related transform, the Fourier-Wigner transform.

Let p, q be in $\mathbb{R}$. Define the function $\rho(q,p)f$ on $\mathbb{R}$ by

$$(\rho(q,p)f)(x) = e^{iq\cdot p+\frac{1}{2}iqp} f(x+p), \quad x \in \mathbb{R}.$$

The main properties used in the thesis are stated as follows.

Proposition 2.2.1. *$\rho(q,p) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary operator for all q, p in $\mathbb{R}$.*

The proof of the proposition is straight forward, and we omit it. Note that $\rho(q,p)^{-1} = \rho(-q,-p)$, $q, p \in \mathbb{R}^n$. In fact, ρ is a projective representation, which is a unitary representation up to a phase factor, of the phase space $\mathbb{R}^2$ on $L^2(\mathbb{R})$.

Let f and g be Schwătz functions on $\mathbb{R}$. Then we define the Fourier-Wigner transform $V(f,g)$ on $L^2(\mathbb{R}^2)$ by

$$V(f,g)(q,p) = (2\pi)^{-1/2} < \rho(q,p)f, g >, \quad q, p \in \mathbb{R},$$

where $<,>$ is the inner product in $L^2(\mathbb{R}^2)$.

Now, we give a working formula for the Fourier-Wigner transform.

Proposition 2.2.2. *Let f and g be in $\mathcal{S}(\mathbb{R})$. Then,*

$$V(f,g)(q,p) = (2\pi)^{-1/2} \int_{\mathbb{R}^n} e^{iq\cdot y} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy$$

for all q and p in $\mathbb{R}^n$.

Proof By definition, we write

$$\begin{aligned} V(f,g)(q,p) &= (2\pi)^{-1/2} < \rho(q,p)f, g > \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} e^{iqx+\frac{1}{2}iqtp} f(x+p)\overline{g(x)}dx \end{aligned}$$

for all q, p in $\mathbb{R}$. If we let $x = y - \frac{p}{2}$, in the last equality, we immediately get that the Fourier-Wigner transform is a bilinear mapping.

Proposition 2.2.3. $V : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^2)$ *is a bilinear mapping.*

To prove the propsition, we need a lemma as follows.

Lemma 2.2.4. Let $\phi \in \mathcal{S}(\mathbb{R})$. Then the function Φ on $\mathbb{R}^{2n}$ defined by

$$\Phi(q,p) = \int_{\mathbb{R}} e^{iqy}\phi(y,p)dy, \quad q, p \in \mathbb{R},$$

is also in $\mathcal{S}(\mathbb{R}^2)$.

The proof of Lemma 2.2.4. is straight-forward computation, and we omit it here. Now we give a proof of Proposition 2.2.3.

Proof (Propositi 2.2.3) Note that for all f and g in $\mathbb{R}$, the function ϕ on $\mathbb{R}^2$ defined by

$$\phi(y,p) = f(y)\overline{g(p)}, \quad y, p \in \mathbb{R},$$

is obviously in $\mathbb{R}^2$. Hence the function ψ on $\mathbb{R}^2$ defined by

$$\psi(y,p) = f\left(y + \frac{p}{2}\right)\overline{g\left(y - \frac{p}{2}\right)}, \quad y, p \in \mathbb{R}.$$

Therefore, by Lemma 2.2.4, $V(f,g) \in \mathcal{S}(\mathbb{R}^2)$.

In order to study the Weyl transform, it is necessary to introduce the Wigner transform of two arbitrary L^2 functions on $\mathbb{R}$. The Wigner transform $W(f)$ of a function f on $L^2(\mathbb{R})$ is a tool for studying the nonexisting joint probability distribution of position and momentum in the state f. We will next introduce the Wigner transform as the Fourier transform of the Fourier-Wigner transform, and some of its important properties.

Theorem 2.2.1. *Let f and g be in $\mathcal{S}(\mathbb{R})$. Then*

$$V(f,g)\hat{}(x,\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\xi p} f(x+\frac{p}{2})\overline{g(x-\frac{p}{2})}dp, \quad x,\xi \in \mathbb{R}.$$

Proof For any positive number ϵ, we define the function I_ϵ on $\mathbb{R}^2$ by

$$I_\epsilon(x,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{-\epsilon^2|q|^2}{2}} e^{-ixq-i\xi p} V(f,g)(q,p)dqdp, \quad x,\xi \in \mathbb{R}.$$

Then, using Fubini's theorem and the fact that the Fourier transform of the function ϕ given by

$$\phi(x) = e^{-\frac{|x|^2}{2}}, \quad x \in \mathbb{R},$$

equals to ϕ, we get,

$$\begin{aligned} & I_\epsilon(x,\xi) \\ = \; & (2\pi)^{-1/2} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{\epsilon^2|q|^2}{2}} e^{-ixq-i\xi p} \left\{ \int_{\mathbb{R}} e^{iqy} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy \right\} dp \\ = \; & (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\xi p} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-i(x-y)q} e^{-\frac{\epsilon^2|q|^2}{2}} dq \right) f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dydp \\ = \; & \int_{\mathbb{R}} e^{-i\xi p} \left\{ \int_{\mathbb{R}} \epsilon^{-1} e^{-\frac{|x-y|^2}{2\epsilon^2}} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy \right\} dp. \end{aligned} \tag{2.1}$$

Now, for each p in $\mathbb{R}$, we define the function F_p by

$$F_p(y) = f\left(y + \frac{p}{2}\right)\overline{g\left(y - \frac{p}{2}\right)}, \quad y \in \mathbb{R}.$$

Then, by (2.1) and the definition of F_p we get

$$I_\epsilon(x,\xi) = \int_{\mathbb{R}} e^{-i\xi p}(F_p * \phi_\epsilon)(x)dp, \quad x, \xi \in \mathbb{R},$$

where

$$\phi_\epsilon(x) = \epsilon^{-1}\phi\left(\frac{x}{\epsilon}\right), \quad x \in \mathbb{R}.$$

Note that, for each fixed p in $\mathbb{R}$, we have

$$F_p * \phi_\epsilon \to \left(\int_{\mathbb{R}} \phi(x)dx\right)F_p = (2\pi)^{\frac{1}{2}}F_p$$

uniformly on compact subsets of $\mathbb{R}$ as $\epsilon \to 0$. Now, let N be any positive integer. Then there exists a positive constant C_N such that

$$\begin{aligned}
|(F_p * \phi_\epsilon)(x)| &\leq \|F_p\|_{L^\infty(\mathbb{R})}\|\phi_\epsilon\|_{L^1(\mathbb{R})} \\
&= \|F_p\|_{L^\infty(\mathbb{R})}\|\phi\|_L^1(\mathbb{R}) \\
&\leq (2\pi)^{n/2} sup_{y\in\mathbb{R}} \left\| f\left(y + \frac{p}{2}\right) g\left(y - \frac{p}{2}\right)\right\| \\
&\leq C_N(1 + |p|^2)^{-N}, \quad x, p \in \mathbb{R}^n,
\end{aligned}$$

for all positive numbers ϵ. So, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{\epsilon\to 0} I_\epsilon(x,\xi) &= (2\pi)^{n/2}\int_{\mathbb{R}} e^{-i\xi p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dp \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-ixq-i\xi p} V(f,g)(q,p)\,dq\,dp \\
&= (2\pi)V(f,g)\hat{}(x,\xi), \quad x,\xi\in\mathbb{R},
\end{aligned}$$

and the theorem is proved.

We define the Wigner transform of two functions in $\mathcal{S}(\mathbb{R})$ as the Fourier transform of the Fourier-Wigner transform of two functions, and we give a working formula for the Wigner transform in the following theorem.

Theorem 2.2.2. *For all Schwartz functions f and g on $\mathbb{R}$,*

$$W(f,g)(x,\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dp, \quad x,\xi\in\mathbb{R}.$$

Proof For all $x,\xi\in\mathbb{R}$,

$$\begin{aligned}
& W(f,g)(x,\xi) \\
=\;& (2\pi)^{-3/2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-ixq-i\xi p}\left(\int_{\mathbb{R}^N} e^{iqy} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy\right)dq\,dp \\
=\;& (2\pi)^{-3/2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-iq(x-y)}dq\right)e^{-i\xi p} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy\,dp \\
=\;& (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi p}\left(\int_{\mathbb{R}} \delta(x-y) f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy\right)dp \\
=\;& (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dp,
\end{aligned}$$

where

$$(2\pi)^{-1/2}\int_{\mathbb{R}} e^{iq(y-x)}dq = \delta(y-x) = \delta(x-y)$$

and

$$\int_{\mathbb{R}} \delta(x-y)f(y)dy = f(x).$$

Lemma 2.2.3. *Let f, g be in $L^2(\mathbb{R})$. Then,*

$$W(f,g) = \overline{W(g,f)}.$$

Now we give the Moyal's identity, which will be used in constructing explicit fomulas for the heat kernels and Green functions.

Theorem 2.2.4. *For all functions $f_1,\ f_2$ and $g_1,\ g_2$ in $L^2(\mathbb{R})$. Then*

$$(W(f_1,g_1), W(f_2,g_2)) = (f_1,f_2)\overline{(g_1,g_2)}.$$

Proof Using Plancherel's theorem, we get

$$\begin{aligned}
& (W(f_1,g_1), W(f_2,g_2)) \\
= \;& (V(f_1,g_1)^\wedge, V(f_2,g_2)^\wedge) \\
= \;& (V(f_1,g_1), V(f_2,g_2) \\
= \;& \int_{\mathbb{R}}\int_{\mathbb{R}} f_1\left(x+\frac{p}{2}\right)\overline{g_1\left(x-\frac{p}{2}\right)}f_2\left(x+\frac{p}{2}\right)\overline{g_2\left(x-\frac{p}{2}\right)}dxdp.
\end{aligned}$$

We make a change of variable $u = x + \frac{p}{2}$ and $v = x - \frac{p}{2}$, and get

$$dudv = dxdp$$

Hence,

$$\begin{aligned} & (W(f_1, g_1), W(f_2, g_2)) \\ = & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_1(u)\overline{g_1(v) f_2(u)} g_2(v) du dv \\ = & (f_1, f_2)\overline{(g_1, g_2)}. \end{aligned}$$

Lemma 2.2.5. *The Moyal identity is also true for the Fourier-Wigner transform V.*

Lemma 2.2.6. *$W : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ can be extended uniquely to a bilinear operator*

$$W : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

such that

$$\|W(f, g)\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}$$

for all f and g in $L^2(\mathbb{R})$.

Lemma 2.2.7. *The preceding lemma is also true for the Fourier-Wigner transform.*

Now we introduce the Weyl transform and its connection with the Wigner transform. The role of the Weyl transform in quantization is given at the end of this section. Suppose σ is a function in $L^2(\mathbb{R} \times \mathbb{R})$. Then for all functions f in $L^2(\mathbb{R})$, we define the Weyl transform of f with symbol σ, denoted by $W_\sigma f$ by

$$(W_\sigma f, g) = (2\pi)^{-n/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi, \quad g \in L^2(\mathbb{R}).$$

By the adjoint formula for the Fourier transform, we get

$$\begin{aligned}
& (W_\sigma f, g) \\
= \;& (2\pi)^{-1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(q,p) V(f,g)^{\wedge}(q,p)\,dq\,dp \\
= \;& (2\pi)^{-1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\sigma}(q,p) V(f,g)(q,p)\,dq\,dp \\
= \;& (2\pi)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\sigma}(q,p)(\rho(q,p)f, g)\,dq\,d.
\end{aligned}$$

So,

$$W_\sigma f = (2\pi) \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\sigma}(q,p)\rho(q,p) f\,dq\,dp.$$

In classical mechanics, the phase space used to describe the motion of a particle moving in $\mathbb{R}$ is given by

$$\mathbb{R}^2 = \{(x,\xi) : x, \xi \in \mathbb{R}\},$$

where the variable x and ξ are used to denote the position and momentum of the particle, respectively.

The motivation for studying the Wely Transform comes from the quantization. The in classical mechanics are given by real-valued tempered distributions on $\mathbb{R}^2$. The rules of quantization, with Planck's constant adjusted to 1, describes that a quantum-mechanical model of the motion can be set up using the Hilbert space $L^2(\mathbb{R})$, for the phase space, the multiplication operator on $L^2(\mathbb{R})$ by the funciton x_j for the position variable x_j, and the differential operator D_j for the momentum variable ξ_j. Thus, the quantum-mechanical analogue of the classical mechanical observable $\sigma(x,\xi)$ should be the linear operator $\sigma(x,D)$ by direct substitution, where D is the vector $(D_1, D_2, ..., D_n)$. It can be proved that a good choice for $\sigma(x,D)$ is the weyl Transform W_σ.

Lastly, we introduce the product formula for Weyl transforms.

Lemma 2.2.8. *Let z, w be points in $\mathbb{R}$. Then,*

$$\rho(z)\rho(w) = \rho(z+w)e^{\frac{1}{4}i[z,w]},$$

where $[\cdot,\cdot]$ is the symplectic form of two complex points in $\mathbb{C}$.

Proof We write $z = q + ip$ and $w = v + iu$. Then for all $x \in \mathbb{R}$ and all $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned} & (\rho(z)\rho(w)f)(x) \\ = \quad & e^{iqx+\frac{1}{2}iqp}(\rho(w)f)(x+p) \\ = \quad & e^{iqx+\frac{1}{2}qp}e^{iv(x+p)+\frac{1}{2}ivu}f(x+p+u) \\ = \quad & e^{iqx+\frac{1}{2}qp+ivx+ivp+\frac{1}{2}ivu}f(x+p+u) \end{aligned}$$

On the other hand,

$$\begin{aligned} & (\rho(z+w)e^{\frac{1}{4}i[z,w]}f)(x) \\ = \quad & e^{\frac{1}{2}ipv-\frac{1}{2}iqu+i(q+v)x+\frac{1}{2}i(q+v)(p+u)}f(x+p+u) \\ = \quad & e^{ipv+iqx+ivx+\frac{1}{2}iqp+\frac{1}{2}vu}f(x+p+u) \end{aligned}$$

The proof is complete.

Theorem 2.2.9. *Let σ, τ be in $L^2(\mathbb{R} \times \mathbb{R})$. Then we have*

$$W_\sigma W_\tau = W_\omega,$$

where ω is given by

$$\hat{\omega} = (2\pi)(\hat{\sigma} *_{1/4} \hat{\tau}).$$

3 Spectral Theory of the Hermite Operator on $L^p(\mathbb{R}^n)$

3.1 The Minimal Operator and the Maximal Operator

We first give the definition of a closable operator in general, which can be found in the book [43]. Let A be a linear operator from a Banach space X into a Banach space Y with dense domain $D(A)$.

Definition 3.1.1. The operator $A : X \to Y$ is said to be closable if for any sequence $\{x_k\}$ in $\mathcal{D}(A)$ such that $x_k \to 0$ in X and $Ax_k \to y$ in Y as $k \to \infty$, then we have $y = 0$.

To see that the Hermite operator is closable, we let $\{\phi_k\}$ be a sequence in $\mathcal{S}$ such that $\phi_k \to 0$ in $L^p(\mathbb{R}^n)$ and $H\phi_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Then for all $\psi \in \mathcal{S}$, we have

$$(f, \psi) = \lim_{k\to\infty}(H\phi_k, \psi) = \lim_{k\to\infty}(\phi_k, H\psi) = 0,$$

where

$$(g, h) = \int_{\mathbb{R}^n} g(x)\overline{h(x)}dx, \quad g \in L^p(\mathbb{R}^n),\ h \in L^{p'}(\mathbb{R}^n).$$

This implies that $f = 0$. Therefore H is closable in $L^p(\mathbb{R}^n)$.

Now we introduce a proposition in book [43].

Proposition 3.1.2. *A has a closed extension if and only if A is closable.*

In view of the above proposition, having proved the fact that the Hermite operator is closable, we can define the minimal operator $H_{0,p}$ of H on $L^p(\mathbb{R}^n)$ to be the smallest closed extension, or the closure of H, on $L^p(\mathbb{R}^n)$. We end this section with the definition of the maximal operator of the Hermite operator on $L^p(\mathbb{R}^n)$.

Definition 3.1.3. Let u and f be functions in $L^p(\mathbb{R}^n)$. We say that $u \in \mathcal{D}(H_{1,p})$ and $H_{1,p}u = f$ if and only if $(Hu, \phi) = (f, H\phi)$, $\phi \in \mathcal{S}$, and that $\mathcal{D}(H_{1,p}) = \{u : H_{1,p}u \in L^p(\mathbb{R}^n)\}$. Then, $H_{1,p}u = Hu$, $u \in \mathcal{D}(H_{1,p})$.

3.2 The Spectrum of $H^N_{0,p}$ and $H^N_{1,p}$, $4/3 < p < 4$

In this section, we give the spectrum of the operator $H^N_{1,p}$ for some $N \in \mathbb{N}$.

Proposition 3.2.1. *For* $4/3 < p < 4$ *and N large enough, the spectrum* $\Sigma(H_1^{1,p})$ *of the operator* $H^N_{1,p}$ *is given by*

$$\Sigma(H^N_{1,p}) = \{(2|\alpha| + 1)^N : \alpha \in \mathbb{N}^n_0\}.$$

Proof For each $N \in \mathbb{N}$, let $S_N = \{(2|\alpha|+1)^N : \alpha \in \mathbb{N}^n_0\}$. We now show that the resolvent set of the operator $H^N_{1,p}$ is $\mathbb{C} - S_N$. In other words, the spectrum $\Sigma(H^N_{1,p})$ is S_N. For all complex numbers $\lambda \notin S_N$, we claim that the operator $H^N_{1,p} - \lambda$ is bijective. Indeed, for injectivity, let I be the identity operator on $L^p(\mathbb{R}^n)$ and suppose that $(H^N_{1,p} - \lambda I)u = 0$ for some $u \in L^p(\mathbb{R}^n)$. Then

$$((H^N_{1,p} - \lambda I)u)(\phi) = 0, \quad \phi \in \mathcal{S}.$$

Treating u as a distribution, we have

$$((H^N_{1,p} - \lambda I)u)(\phi) = u((H^N - \lambda I)\phi) = 0, \quad \phi \in \mathcal{S}.$$

On the other hand, let $\psi \in \mathcal{S}$. Then we show that there exists $\phi \in \mathcal{S}$ such that $(H^N - \lambda I)\phi = \psi$. Indeed, we define ϕ by

$$\phi = \sum_{\alpha} \frac{1}{(2|\alpha| + 1)^N - \lambda} (\psi, e_\alpha) e_\alpha,$$

where e_α, $\alpha \in \mathbb{N}_0^n$ are the eigenvectors corresponding to the eighenvalue $2|\alpha| + 1$. Then it is clear that $\phi \in \mathcal{S}$, and since

$$(H^N - \lambda I)\phi = \psi,$$

we have

$$u(\psi) = 0, \quad \psi \in \mathcal{S},$$

and $u = 0$, as desired. To see that $H^N_{1,p} - \lambda I$ is surjective, let $f \in L^p(\mathbb{R}^n)$. Then we need to prove that there exists $u \in L^p(\mathbb{R}^n)$ such that $(H^N_{1,p} - \lambda I)u = f$. By Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
\|u\|_p &= \|(H_{1,p}^N - \lambda I)^{-1} f\|_p \leq \sum_{\alpha} \frac{1}{|(2|\alpha| + 1)^N - \lambda|} |(f, e_\alpha)| \, \|e_\alpha\|_p \\
&\leq \sum_{\alpha} \frac{1}{|(2|\alpha| + 1)^N - \lambda|} \|f\|_p \|e_\alpha\|_{p'} \|e_\alpha\|_p,
\end{aligned} \tag{3.1}$$

where p' is the conjugate index of p. And for N large enough (depending on p), by a result from the paper [2], we have

$$\|e_\alpha\|_p \|e_\alpha\|_{p'} = O(1), \quad 4/3 < p < 4, \tag{3.2}$$

or equivalently, this product is bounded for all α, which makes the right hand side of (3.1) finite, given the range for p.

Remark 3.2.1. The range of p for which (3.2) holds is sharp according to the estimate in the paper [2]

3.3 $H_{0,p}^N = H_{1,p}^N$, $4/3 < p < 4$

Before proving the main result of this section, we need two lemmas.

Lemma 3.3.1. *For each $N \in \mathbb{N}$, $H_{0,p}^N \subseteq H_{1,p}^N$.*

Proof We prove the lemma by induction. For $N = 1$, it is clearly true. Suppose it holds for N. Let $u \in \mathcal{D}(H_{0,p}^{N+1})$. Then $H_{0,p} u \in \mathcal{D}(H_{0,p}^N)$. Since $H_{0,p} u = H_{1,p} u$ and $\mathcal{D}(H_{0,p}^N) \subseteq \mathcal{D}(H_{1,p}^N)$, we have $H_{1,p}u \in \mathcal{D}(H_{1,p}^N)$, and it follows that $u \in \mathcal{D}(H_{1,p}^{N+1})$. Furthermore, we have

$$H_{1,p}^{N+1}u = H_{1,p}^N H_{1,p}u = H_{0,p}^N H_{0,p}u = H_{0,p}^{N+1}u.$$

Thus, $H_{0,p}^N \subseteq H_{1,p}^N$.

Lemma 3.3.2. $(H_{0,p}^N)^{-1} = (H_{1,p}^N)^{-1}$.

Proof By definition, $H^N = H_{1,p}^N$ in distribution sense, and $0 \notin \Sigma(H^N)$, so $(H_{1,p}^N)^{-1}$ exists. By the spectral mapping theorem, the spectrum of the operator $(H_{1,p}^N)^{-1}$ is given by

$$\Sigma((H_{1,p}^N)^{-1}) = \left\{ \frac{1}{(2|\alpha| + 1)^N} : \alpha \in \mathbb{N}_0^n \right\}^c.$$

On the other hand, $(H_{0,p}^N)^{-1}$ clearly exists because 0 is not in the set of eigenvalues of $H_{0,p}^N$. Moreover, $(H_{0,p}^N)^{-1}$ and $(H_{1,p}^N)^{-1}$ are bounded linear operators on $L^p(\mathbb{R}^n)$. Now, let $v \in L^p(\mathbb{R}^n)$. Suppose $(H_{0,p}^N)^{-1}v = f$ and $(H_{1,p}^N)^{-1}v = g$ for some f and $g \in L^p(\mathbb{R}^n)$. Then we have $H_{0,p}^N f = v$ and $H_{1,p}^N g = v$. But we have shown in the previous lemma that $H_{0,p}^N \subseteq H_{1,p}^N$, so, in particular,

$$H_{0,p}^N f = H_{1,p}^N f = v.$$

Also, since $H_{1,p}^N$ is injective, we conclude that $f = g$. And it follows that $(H_{0,p}^N)^{-1} = (H_{1,p}^N)^{-1}$.

Now, we prove our main theorem of this section.

Theorem 3.3.3. $H_{0,p}^N = H_{1,p}^N$, $4/3 < p < 4$.

Proof Let $u \in \mathcal{D}(H^N_{1,p})$ and $H^N_{1,p}u = f$ for some $f \in L^p(\mathbb{R}^n)$. Then $(H^N_{1,p})^{-1}f = u$. But by Lemma 4.2, $(H^N_{0,p})^{-1}f = (H^N_{1,p})^{-1}f = u$. Thus, $u \in \mathcal{D}(H^N_{0,p})$ and $H^N_{0,p}u = f$. It follows that $H^N_{0,p} = H^N_{1,p}$.

3.4 The Spectrum of $H_{1,p}$, $4/3 < p < 4$

The goal in this section is to show that $\Sigma(H_{0,p}) = \Sigma(H_{1,p})$, $4/3 < p < 4$. To this end, we use the following result of Taylor [32], [33].

Theorem 3.4.1. *Let A be a closed linear operator and f be a holomorphic function on a neighbourhood of $\Sigma(A)$. Then the spectrum $\Sigma(f(A))$ of the operator $f(A)$ is given by*

$$\Sigma(f(A)) = \{f(\lambda) : \lambda \in \Sigma(A)\}.$$

In view of the theorem, we define a function f on $\mathbb{C} - (-\infty, 0]$ by

$$f(\lambda) = \lambda^{1/N}, \quad N \in \mathbb{N}, \quad \lambda \in \mathbb{C} - (-\infty, 0],$$

where the principal branch is taken. Secondly, we let the operator A be given by

$$A = H^N_{0,p} = H^N_{1,p}.$$

Then we have

$$\Sigma(A) = \left\{(2|\alpha| + 1)^N : \alpha \in \mathbb{N}^n_0\right\}^c.$$

Lastly, since A^N is closed, and f is holomorphic on a neighbourhood of the spectrum of A, we apply Theorem 5.1 to the function f and operator A, and get

$$\Sigma(f(A)) = \{f(\lambda) : \lambda \in \Sigma(A)\}.$$

Therefore

$$\begin{aligned} \Sigma(H_{1,p}) &= \Sigma(H_{0,p}) \\ &= \left\{\lambda^{1/N} : \lambda = (2|\alpha| + 1)^N, \quad \alpha \in \mathbb{N}_0^n\right\}^c \\ &= \{2|\alpha| + 1 : \alpha \in \mathbb{N}_0^n\}^c . \end{aligned}$$

Having computed explicitly the spectrum for $H_{0,p}$ and $H_{1,p}$, we can apply Taylor's theorem to the two operators, and by functional calculus, we see easily that

$$H_{0,p} = A^{1/N} = H_{1,p}, \quad 4/3 < p < 4.$$

So, this means that the Hermite operator is essentially self-adjoint on $L^p(\mathbb{R}^n)$, for p between 4/3 and 4.

3.5 An Initial Value Problem

In this last section, we give the L^p-estimate of the solution to the initial value problem for the heat eqation governed by $H_{0,p}$, which is equal to that of $H_{1,p}$, for p between $4/3$ and 4, i.e.,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = (-H_{0,p}u)(x,t), & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}^n,\ f \in L^p(\mathbb{R}^n),\ 4/3 < p < 4. \end{cases}$$

We have

$$u(\cdot,\ t) = e^{-H_{0,p}t} f, \quad t > 0,$$

and therefore

$$u(\cdot,\ t) = \sum_{\alpha} e^{-(2|\alpha|+1)t}(f,\ e_\alpha)e_\alpha, \quad t > 0,$$

where the convergence is in the space $\mathcal{S}'$ of tempered distributions [44]. So, for $t > 0$,

$$\|u(\cdot,\ t)\|_p \leq \sum_{\alpha} e^{-(2|\alpha|+1)t}\|f\|_p\|e_\alpha\|_p\|e_\alpha\|_{p'}.$$

Let K_p be defined by

$$K_p = \sup_{\alpha \in N_0^n} \|e_\alpha\|_p\|e_\alpha\|_{p'}.$$

Since

$$\|e_\alpha\|_p\|e_\alpha\|_{p'} = O(1) \leq K, \quad 4/3 < p < 4,$$

where K is some positive constant, and

$$\sum_{\alpha} e^{-(2|\alpha|+1)t} = e^{-t}\left(\sum_{j=0}^{\infty} e^{-2jt}\right)^n = \frac{e^{(n-1)t}}{2^n \sinh^n t},$$

we see that

$$u(\cdot,\ t) \in L^p(\mathbb{R}^n), \quad 4/3 < p < 4,$$

and that

$$\|u(\cdot,\, t)\|_p \leq \frac{K_p e^{(n-1)t}}{2^n \sinh^n t} \|f\|_p, \quad t > 0,\ 4/3 < p < 4.$$

3.6 An Improvement

The starting point is the following set of asymptotics in [34] given by

$$\|e_k\|_p \sim k^{\frac{1}{2p}-\frac{1}{4}}, \quad 1 \leq p \leq 4,$$

$$\|e_k\|_p \sim k^{-\frac{1}{8}} \ln k, \quad p = 4,$$

$$\|e_k\|_p \sim k^{-\frac{1}{6p}-\frac{1}{12}}, \quad 4 < p \leq \infty.$$

Then we have the following proposition.

Proposition 3.6.1. *For* $1 \leq p \leq \infty$, *there exists a positive number* ϵ_p *such that*

$$\|e_k\|_p \sim k^{\epsilon_p}, \quad p \to \infty.$$

Then series (3.1) is convergent by the above proposition and the whole proof as for $4/3 < p < 4$ goes through, and we have among others the following theorem.

Theorem 3.6.1. *For* $1 \leq p \leq \infty$, $H_{0,p} = H_{1,p}$ *and*

$$\sum(H_{0,p} = \sum(H_{1,p}) = \{2|\alpha| + 1 : \alpha \in \mathbb{N}_0^n\}.$$

4 The Heat Kernel and Green Function of a Fourth-Order Operator on the Heisenberg Group

4.1 The Twisted Bi-Laplacians

Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ be linear partial differential operators on $\mathbb{R}^2$ given by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Let $\tau \in \mathbb{R} \setminus \{0\}$. Then we define the partial differential operators Z_τ and $\overline{Z}_\tau$ by

$$Z_\tau = \frac{\partial}{\partial z} + \frac{1}{2}\tau\bar{z}, \quad \bar{z} = x - iy,$$

and

$$\overline{Z}_\tau = \frac{\partial}{\partial \bar{z}} - \frac{1}{2}\tau z, \quad z = x + iy.$$

Let L_τ be the linear partial differential operator on $\mathbb{R}^2$ defined by

$$L_\tau = -\frac{1}{2}(Z_\tau \overline{Z}_\tau + \overline{Z}_\tau Z_\tau).$$

Then L_τ is an elliptic partial differential operator on $\mathbb{R}^2$ given by

$$L_\tau = -\Delta + \frac{1}{4}(x^2 + y^2)\tau^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau.$$

Thus, L_τ is the ordinary Hermite operator $-\Delta + \frac{1}{4}(x^2 + y^2)\tau^2$ perturbed by the partial differential operator $-iN\tau$, where

$$N = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the rotation operator. As such, we call L_τ the twisted Laplacian. If $\tau = 1$, then we recover the twisted Laplacian studied in detail in [43].

To see the connection of the twisted Laplacian L_τ with the sub-Laplacian, we define for all every function f in $L^1(\mathbb{H}^1)$, the function f^τ on $\mathbb{C}$ by

$$f^\tau(z) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{it\tau} f(z,t)\, dt, \quad z \in \mathbb{C},$$

provided that the integral exists. $f^\tau(z)$ is in fact the inverse Fourier transform of $f(z,t)$ with respect to t evaluated at τ. It is to be noted that the Fourier transform $\hat{F}$ of a function F in $L^1(\mathbb{R}^n)$ is defined by

$$\hat{F}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} F(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

Let $\tau \in \mathbb{R} \setminus \{0\}$. Then for all suitable functions f on $\mathbb{H}^1$, we get

$$(\mathcal{L}f)^\tau = L_\tau f^\tau \tag{4.1}$$

and

$$(\mathcal{L}_+\mathcal{L}_- f)^\tau = L_\tau L_{-\tau} f^\tau. \tag{4.2}$$

4.2 Fourier–Wigner Transforms of Hermite Functions

Let f and g be functions in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$. Then for τ in $\mathbb{R} \setminus \{0\}$, the τ-Fourier–Wigner transform $V_\tau(f, g)$ of f and g is defined by

$$V_\tau(f, g)(q, p) = (2\pi)^{-1/2}|\tau|^{1/2} \int_{-\infty}^{\infty} e^{i\tau q y} f\left(y + \frac{p}{2}\right)\overline{g\left(y - \frac{p}{2}\right)}\,dy$$

for all q and p in $\mathbb{R}$.

For $\tau \in \mathbb{R} \setminus \{0\}$ and for $k = 0, 1, 2, \ldots$, we define e_k^τ to be the function on $\mathbb{R}$ by

$$e_k^\tau(x) = |\tau|^{1/4} e_k(\sqrt{|\tau|}x), \quad x \in \mathbb{R}.$$

For $j, k = 0, 1, 2, \ldots$, we define $e_{j,k}^\tau$ on $\mathbb{R}^2$ by

$$e_{j,k}^\tau = V_\tau(e_j^\tau, e_k^\tau).$$

The connection of $\{e_{j,k}^\tau : j, k = 0, 1, 2, \ldots\}$ with $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$ studied in [43] is given by the following formula.

Theorem 4.2.1. *For $\tau \in \mathbb{R} \setminus \{0\}$ and for $j, k = 0, 1, 2, \ldots,$*

$$e^{\tau}_{j,k}(q, p) = |\tau|^{1/2} e_{j,k}\left(\frac{\tau}{\sqrt{|\tau|}} q,\ \sqrt{|\tau|} p\right), \quad q, p \in \mathbb{R}.$$

Proof For $\tau \in \mathbb{R} \setminus \{0\}$ and for $j, k = 0, 1, 2, \ldots,$

$$\begin{aligned}
& e^{\tau}_{j,k}(q, p) \\
= \;& V_\tau(e^{\tau}_j, e^{\tau}_k)(q, p) \\
= \;& (2\pi)^{-1/2} |\tau|^{1/2} \int_{-\infty}^{\infty} e^{i\tau q y} e^{\tau}_j\left(y + \frac{p}{2}\right) \overline{e^{\tau}_k\left(y - \frac{p}{2}\right)}\, dy \\
= \;& (2\pi)^{-1/2} |\tau| \int_{-\infty}^{\infty} e^{i\tau q y} e_j\left(\sqrt{|\tau|}\left(y + \frac{p}{2}\right)\right) \overline{e_k\left(\sqrt{|\tau|}\left(y - \frac{p}{2}\right)\right)}\, dy \\
= \;& (2\pi)^{-1/2} |\tau|^{1/2} \int_{-\infty}^{\infty} e^{i\tau q y/\sqrt{|\tau|}} e_j\left(y + \sqrt{|\tau|}\frac{p}{2}\right) \overline{e_k\left(y - \sqrt{|\tau|}\frac{p}{2}\right)}\, dy \\
= \;& |\tau|^{1/2} e_{j,k}\left(\frac{\tau}{\sqrt{|\tau|}} q,\ \sqrt{|\tau|} p\right)
\end{aligned}$$

for all q and p in $\mathbb{R}$.

Theorem 4.2.2. $\{e^{\tau}_{j,k} : j, k = 0, 1, 2, \ldots\}$ *forms an orthonormal basis for $L^2(\mathbb{R}^2)$.*

Theorem 4.2.2 follows from Theorem 4.2.1 and Theorem 21.2 in [38] to the effect that $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

Theorem 4.2.3. *For $j, k = 0, 1, 2, \ldots,$*

$$L_\tau e^{\tau}_{j,k} = (2k + 1)|\tau| e^{\tau}_{j,k}.$$

Theorem 4.2.3 can be proved using Theorem 4.2.1 and Theorem 22.2 in [38] telling us that for $j, k = 0, 1, 2, \ldots, e_{j,k}$ is an eigenfunction of L_1 corresponding to the eigenvalue $2k + 1$.

We need the notion of a twisted convolution. Let $\lambda \in \mathbb{R}$, and let f and g be measurable functions on $\mathbb{C}$. Then we define the twisted convolution $f *_\lambda g$ of f and g to be the function on $\mathbb{C}$ by

$$(f *_\lambda g)(z) = \int_{\mathbb{C}} f(z - w)g(w)e^{i\lambda[z,w]}dw, \quad z \in \mathbb{C},$$

provided that the integral exists.

The following formula is the main tool for the construction of the heat kernel of L_τ.

Theorem 4.2.4. *For $\tau \in \mathbb{R} \setminus \{0\}$ and for nonnegative integers α, β, μ and ν,*

$$e^\tau_{\alpha,\beta} *_{\tau/4} e^\tau_{\mu,\nu} = (2\pi)^{1/2}|\tau|^{-1/2}\delta_{\beta,\mu}e^\tau_{\alpha,\nu},$$

where $\delta_{\beta,\mu}$ is the Kronecker delta.

When $\tau = 1$, the formula is the same as that in Theorem 4.1 in [43]. Theorem 4.2.4 can be proved using the formula for $\tau = 1$ and Theorem 4.2.1.

4.3 The Heat Kernel of L_τ.

Using Theorem 4.2.3 and the spectral theorem, we get for all functions f in $L^2(\mathbb{R}^2)$,

$$e^{-\rho L_\tau} f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{-(2k+1)|\tau|\rho}(f, e^\tau_{j,k})e^\tau_{j,k}, \quad \rho > 0.$$

where (,) is the inner product in $L^2(\mathbb{R}^2)$. So, for $\rho > 0$,

$$e^{-\rho L_\tau} f = \sum_{k=0}^{\infty} e^{-(2k+1)|\tau|\rho} \sum_{j=0}^{\infty} (f, e^\tau_{j,k}) e^\tau_{j,k}$$

and our first task is to compute $\sum_{j=0}^{\infty}(f, e^\tau_{j,k})e^\tau_{j,k}$. To this end, we note that for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} f *_{\tau/4} e^\tau_{k,k} &= \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (f, e^\tau_{j,l}) e^\tau_{j,l} *_{\tau/4} e^\tau_{k,k} \\ &= \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (f, e^\tau_{j,l}) (2\pi)^{1/2} |\tau|^{-1/2} \delta_{l,k} e^\tau_{j,k} \\ &= (2\pi)^{1/2} |\tau|^{-1/2} \sum_{j=0}^{\infty} (f, e^\tau_{j,k}) e^\tau_{j,k}. \end{aligned}$$

Hence, for $k = 0, 1, 2, \ldots,$

$$\sum_{j=0}^{\infty} (f, e^\tau_{j,k}) e^\tau_{j,k} = (2\pi)^{-1/2} |\tau|^{1/2} (f *_{\tau/4} e^\tau_{k,k}).$$

Therefore

$$e^{-\rho L_\tau} f = (2\pi)^{-1/2} |\tau|^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1)|\tau|\rho} e^\tau_{k,k} *_{-\tau/4} f, \quad \rho > 0. \tag{4.1}$$

Now, using Theorem 4.2.1 and Mehler's formula, we get for all $z = (q, p)$ in $\mathbb{C}$ and for $\rho > 0$,

$$
\begin{aligned}
& (2\pi)^{-1/2}|\tau|^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)|\tau|\rho} e^{\tau}_{k,k}(q,p) \\
= \;& (2\pi)^{-1/2}|\tau| e^{-|\tau|\rho}\sum_{k=0}^{\infty} e^{-2k|\tau|\rho} e_{k,k}\left(\frac{\tau}{\sqrt{|\tau|}}q,\ \sqrt{|\tau|}p\right) \\
= \;& (2\pi)^{-1}|\tau| e^{-|\tau|\rho}\frac{1}{1-e^{-2|\tau|\rho}} e^{-|\tau|\,|z|^2\frac{1}{4}\frac{1+e^{-2|\tau|\rho}}{1-e^{-2|\tau|\rho}}} \\
= \;& \frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau|\,|z|^2\coth(\tau\rho)}.
\end{aligned}
$$

So, the heat kernel κ^{τ}_{ρ}, $\rho > 0$, of L_τ is given by

$$
\kappa^{\tau}_{\rho}(z,w) = \frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau|\,|z-w|^2\coth(\tau\rho)} e^{-i\frac{\tau}{4}[z,w]}, \quad z,w \in \mathbb{C}.
$$

Hence by (4.1), we have the following result.

Theorem 4.3.1. *For $\rho > 0$ and $\tau \in \mathbb{R}\setminus\{0\}$,*

$$
K^{\tau}_{\rho} = (2\pi)^{-1/2} k^{\tau}_{\rho},
$$

where

$$
k^{\tau}_{\rho}(z) = \frac{1}{4\pi}\frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau|\,|z|^2\coth(\tau\rho)}, \quad z \in \mathbb{C}.
$$

4.4 The Green Function of L_τ

We can obtain a formula for the Green function G_τ of the twisted Laplacian L_τ by integrating the heat kernel of L_τ from 0 to ∞ with respect to time ρ. Indeed, for all z and w in $\mathbb{C}$, we get

$$
\begin{aligned}
G_\tau(z,w) &= \frac{1}{4\pi}\left(\int_0^\infty \frac{\tau}{\sinh(\tau\rho)} e^{-\frac{1}{4}|\tau|\,|z-w|^2\coth(\tau\rho)} d\rho\right) e^{-i\frac{\tau}{4}[z,w]} \\
&= \frac{1}{4\pi}\left(\int_0^\infty \frac{1}{(v^2-1)^{1/2}} e^{-\frac{1}{4}|\tau|\,|z-w|^2 v} dv\right) e^{-i\frac{\tau}{4}[z,w]} \\
&= \frac{1}{4\pi} K_0\left(\frac{1}{4}|\tau|\,|z-w|^2\right) e^{-i\frac{\tau}{4}[z,w]},
\end{aligned}
$$

where K_0 is the modified Bessel function of order 0 given by

$$K_0(x) = \int_0^\infty e^{-x\cosh\delta} d\delta, \quad x > 0.$$

4.5 Heat Kernels of Twisted Bi-Laplacians

Let $\tau \in \mathbb{R} \setminus \{0\}$. We are now interested in computing explicitly the heat kernel W_ρ^τ of the twisted bi-Laplacian $L_\tau L_{-\tau}$.

The starting point is the spectral analysis of L_τ given by Theorem 4.2.3. By the spectral mapping theorem and the spectral theorem, we get for all suitable functions f on $\mathbb{H}^1$,

$$e^{-\rho L_\tau^2} f = \sum_{k=0}^\infty \sum_{j=0}^\infty e^{-(2k+1)^2\tau^2\rho}(f, e_{j,k}^\tau)e_{j,k}^\tau, \quad \rho > 0.$$

By (4.1), we get for all suitable functions f on $\mathbb{H}^1$,

$$e^{-\rho L_\tau^2} f = (2\pi)^{-1/2}|\tau|^{1/2} \sum_{k=0}^\infty e^{-(2k+1)^2\tau^2\rho} e_{k,k}^\tau *_{-\tau/4} f, \quad \rho > 0.$$

Now, for $\rho > 0$, we introduce the function M_ρ^τ defined by

$$M_\rho^\tau(z) = (2\pi)^{-1/2}|\tau|^{1/2} \sum_{k=0}^\infty e^{-(2k+1)^2\tau^2\rho} e_{k,k}^\tau(z), \quad z \in \mathbb{C},$$

which is some kind of a theta function. Then the heat kernel ω_ρ^τ, $\rho > 0$, of L_τ^2 is given by

$$\omega_\rho^\tau(z, w) = M_\rho^\tau(z - w)e^{-i\frac{\tau}{4}[z,w]}, \quad z, w \in \mathbb{C}. \tag{4.1}$$

Similarly, the heat kernel $\omega_\rho^{-\tau}$, $\rho > 0$, of $L_{-\tau}^2$ is given by

$$\omega_\rho^{-\tau}(z, w) = M_\rho^{-\tau}(z - w)e^{i\frac{\tau}{4}[z,w]}, \quad z, w \in \mathbb{C}. \tag{4.2}$$

We can now derive the heat kernel of the operator $L_\tau L_{-\tau}$ for $\tau \in \mathbb{R} \setminus \{0\}$. Since for $\tau \in \mathbb{R} \setminus \{0\}$,

$$L_\tau - L_{-\tau} = -2iN\tau,$$

it follows that

$$L_\tau^2 + L_{-\tau}^2 - 2L_\tau L_{-\tau} = 4N^2\tau^2.$$

Since L_τ^2, $L_{-\tau}^2$ and N^2 commute, it follows that for $\rho > 0$,

$$e^{-\rho L_\tau L_{-\tau}} = e^{-2\rho N^2\tau^2} e^{-\frac{\rho}{2}L_\tau^2} e^{-\frac{\rho}{2}L_{-\tau}^2}.$$

Lemma 4.5.1. *For all $\rho > 0$,*

$$(e^{-\rho N^2} f)(re^{i\theta}) = (f_r * \Theta_\rho)(\theta), \quad \theta \in [-\pi, \pi],$$

where Θ_ρ is the theta function given by

$$\Theta_\rho(\theta) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{-n^2\rho} e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Proof For all f in $L^2(\mathbb{C})$, we use polar coordinates and write

$$f(z) = f(re^{i\theta}) = f_r(\theta), \quad r > 0,\ \theta \in [-\pi, \pi].$$

For all n in $\mathbb{Z}$, we define the function e_n on $[-\pi, \pi]$ by

$$e_n(\theta) = (2\pi)^{-1/2} e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Then $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2[-\pi, \pi]$ with respect to the inner product $(\,,\,)$ given by

$$(f, g) = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)} d\theta$$

for all f and g in $L^2[-\pi, \pi]$. Using the fact that

$$N = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}$$

and the basics of Fourier series, we get for all $\rho > 0$,

$$\begin{aligned}
(e^{-\rho N^2} f)(z) &= \left(e^{\rho \frac{\partial^2}{\partial \theta^2}} f\right)(z) \\
&= \sum_{n \in \mathbb{Z}} e^{-\rho n^2} (f_r, e_n) e_n(\theta) \\
&= (f_r * \Theta_\rho)(\theta), \quad [-\pi, \pi],
\end{aligned}$$

where

$$(f_r * \Theta_\rho)(\theta) = \int_{-\pi}^{\pi} f_r(\theta - \phi)\Theta_\rho(\phi)\, d\phi.$$

Theorem 4.5.2. *Let $z = re^{i\theta}$ and $z' = se^{i\psi}$, where $r, s \in (0, \infty)$ and $\theta, \psi \in [-\pi, \pi]$. Let W_ρ^τ be the function defined on $\mathbb{C} \times \mathbb{C}$ by*

$$W_\rho^\tau(re^{i\theta}, se^{i\psi}) = \int_{-\pi}^{\pi} \int_{\mathbb{C}} \Theta_{2\rho\tau^2}(\phi - \psi)\omega_{\rho/2}^{\tau}(z, w)\omega_{\rho/2}^{-\tau}(w, se^{i\phi})\, dw\, d\phi.$$

Then for $\rho > 0$,

$$(e^{-\rho(L_\tau L_{-\tau})} f)(z) = \int_{-\pi}^{\pi} \int_0^\infty W_\rho^\tau(z, se^{i\psi}) f(se^{i\psi}) s\, ds\, d\psi$$

for all suitable functions f on $\mathbb{C}$.

Proof Let $z = re^{i\theta} \in \mathbb{C}$. Then

$$(e^{-\rho(L_\tau L_{-\tau})} f)(z) = (e^{-\frac{\rho}{2}L_\tau^2} e^{-\frac{\rho}{2}L_{-\tau}^2} g)(z),$$

where

$$g = e^{2\rho N^2 \tau^2} f.$$

So,

$$\begin{aligned}
(e^{-\rho(L_\tau L_{-\tau})} f)(z) &= (e^{-\frac{\rho}{2}L_\tau^2} e^{-\frac{\rho}{2}L_{-\tau}^2} g)(z) \\
&= \int_{\mathbb{C}} \omega_{\rho/2}^{\tau}(z, w)(e^{-\frac{\rho}{2}L_{-\tau}^2} g)(w)\, dw \\
&= \int_{\mathbb{C}} \omega_{\rho/2}^{\tau}(z, w)\left(\int_{\mathbb{C}} \omega_{\rho/2}^{-\tau}(w, \zeta) g(\zeta)\, d\zeta\right) dw.
\end{aligned}$$

If we let $\zeta = se^{i\phi}$, where $s > 0$ and $\phi \in [-\pi, \pi]$, then by Lemma 4.5.1,

$$g(\zeta) = g(se^{i\phi}) = (f_s * \Theta_{2\rho\tau^2})(\phi).$$

Thus,

$$
\begin{aligned}
&(e^{-\rho(L_\tau L_{-\tau})}f)(z) \\
&= \int_{\mathbb{C}} g(\zeta)\left(\int_{\mathbb{C}} \omega^{\tau}_{\rho/2}(z,w)\omega^{-\tau}_{\rho/2}(w,\zeta)\,dw\right)d\zeta \\
&= \int_{-\pi}^{\pi}\int_0^{\infty}\left(\int_{-\pi}^{\pi}\Theta_{2\rho\tau^2}(\phi-\psi)f_s(\psi)\,d\psi\right)\times \\
&\qquad\qquad \times\left(\int_{\mathbb{C}} \omega^{\tau}_{\rho/2}(z,w)\omega^{-\tau}_{\rho/2}(w,se^{i\phi})dw\right)s\,ds\,d\phi \\
&= \int_{-\pi}^{\pi}\int_0^{\infty} f_s(\psi)W^{\tau}_{\rho}(z,se^{i\psi})s\,ds\,d\psi,
\end{aligned}
$$

where

$$
W^{\tau}_{\rho}(z,se^{i\psi}) = \int_{-\pi}^{\pi}\int_{\mathbb{C}}\Theta_{2\rho\tau^2}(\phi-\psi)\omega^{\tau}_{\rho/2}(z,w)\omega^{-\tau}_{-\rho/2}(w,se^{i\phi})dw\,d\phi.
$$

4.6 Green Functions of Twisted Bi-Laplacians

In contrast with the derivation of the heat kernels, the formulas for the Green functions of twisted bi-Laplacians are much easier to obtain. Indeed, we get for all suitable functions f on $\mathbb{C}$,

$$
\begin{aligned}
((L_{-\tau}L_\tau)^{-1}f)(z) &= (L_\tau^{-1}L_{-\tau}{}^{-1}f)(z) \\
&= \int_C G_\tau(z,\zeta)(L_{-\tau}{}^{-1}f)(\zeta)\,d\zeta \\
&= \int_{\mathbb{C}} G_\tau(z,\zeta)\left(\int_{\mathbb{C}}\overline{G_\tau(\zeta,w)}f(w)\,dw\right)d\zeta \\
&= \int_{\mathbb{C}}\left(\int_{\mathbb{C}} G_\tau(z,\zeta)\overline{G_\tau(\zeta,w)}\,d\zeta\right)f(w)\,dw
\end{aligned}
$$

for all z in $\mathbb{C}$. Since for all z, w and ζ in $\mathbb{C}$,

$$G_\tau(z,\zeta)\overline{G_\tau(\zeta,w)} = \frac{1}{16\pi^2} K_0\left(\frac{1}{4}|\tau|\,|z-\zeta|^2\right) K_0\left(\frac{1}{4}|\tau|\,|\zeta-w|^2\right) e^{-i\frac{\tau}{4}[\zeta,z+w]},$$

it follows that for all suitable functions f on $\mathbb{C}$,

$$\begin{aligned}&((L_{-\tau}L_\tau)^{-1}f)(z)\\ = &\frac{1}{16\pi^2}\int_{\mathbb{C}}\left(\int_{\mathbb{C}} K_0\left(\frac{1}{4}|\tau|\,|z-\zeta|^2\right) K_0\left(\frac{1}{4}|\tau|\,|\zeta-w|^2\right) e^{-i\frac{\tau}{4}[\zeta,z+w]} d\zeta\right)\times\\ &\times f(w)\,dw\end{aligned}$$

for all z in $\mathbb{C}$.

Therefore we have the following theorem.

Theorem 4.6.1. *Let $\tau \in \mathbb{R}\setminus\{0\}$. Then the Green function G_τ^b of the twisted bi-Laplacian $L_\tau L_{-\tau}$ is given by*

$$G_\tau^b(z,w) = \frac{1}{16\pi^2}\int_{\mathbb{C}} K_0\left(\frac{1}{4}|\tau|\,|z-\zeta|^2\right) K_0\left(\frac{1}{4}|\tau|\,|\zeta-w|^2\right) e^{i\frac{\tau}{4}[\zeta,z+w]} d\zeta$$

for all z and w in $\mathbb{C}$.

4.7 The Heat Kernel and Green Function a Fourth-Order Operator on the Heisenberg Group

We begin with the following result that relates the convolution on $\mathbb{H}^1$ to the twisted convolution. A proof can be found in [42].

Theorem 4.7.1. *Let f and g be functions in $L^1(\mathbb{H}^1)$. Then*

$$(f *_{\mathbb{H}^1} g)^\tau = (2\pi)^{1/2} f^\tau *_{\tau/4} g^\tau.$$

By (4.1) and (4.2), we get the following results.

Theorem 4.7.2. *Let $\tau \in \mathbb{R} \setminus \{0\}$. Then for all $\rho > 0$,*

$$e^{-\rho(L_\tau L_{-\tau})} f^\tau = (e^{-\rho(\mathcal{L}_+\mathcal{L}_-)} f)^\tau = (K_\rho *_{\mathbb{H}}^1 f)^\tau = (2\pi)^{1/2} K_\rho^\tau *_{\tau/4} f^\tau.$$

Theorem 4.7.3. *Let $\tau \in \mathbb{R} \setminus \{0\}$. Then*

$$(L_\tau L_{-\tau}) f^\tau = (\mathcal{L}_+\mathcal{L}_- f)^\tau = (G *_{\mathbb{H}}^1 f)^\tau = (2\pi)^{1/2} G^\tau *_{\tau/4} f^\tau.$$

By Theorem 4.5.2 and Theorem 4.7.2, we get for $\rho > 0$ and z in $\mathbb{C}$, respectively,

$$(e^{-\rho(L_\tau L_{-\tau}} f)(z) = \int_{\mathbb{C}} W_\rho^\tau(z, z') f(z')\, dz'$$

for all suitable functions f on $\mathbb{C}$ and

$$e^{-\rho(L_\tau L_{-\tau})} f^\tau = (2\pi)^{1/2} \int_{\mathbb{C}} K_\rho^\tau(z - z') e^{i\frac{\tau}{4}[z,z']} f^\tau(z') dz'$$

for all suitable functions f on $\mathbb{H}^1$. Thus, for all $\rho > 0$, $\tau \in \mathbb{R} \setminus \{0\}$, z and z' in $\mathbb{C}$,

$$W_\rho^\tau(z, z') = (2\pi)^{1/2} K_\rho^\tau(z - z') e^{i\frac{\tau}{4}[z,z']}$$

and hence by (4.1), (4.2) and Theorem 4.5.2,

$$\begin{aligned} & (2\pi)^{1/2} K_\rho^\tau(z) = W_\rho^\tau(z, 0) \\ = \quad & \int_{-\pi}^{\pi} \int_{\mathbb{C}} \Theta_{2\rho\tau^2}(\phi - \psi) M_{\rho/2}^\tau(z - w) e^{-i\frac{\tau}{4}[z,w]} M_{\rho/2}^{-\tau}(w)\, dw\, d\phi \\ = \quad & \left(\int_{-\pi}^{\pi} \Theta_{2\rho\tau^2}(\phi - \psi)\, d\phi \right) (M_{\rho/2}^\tau *_{-\tau/4} M_{\rho/2}^{-\tau})(z). \end{aligned}$$

Since the function $\Theta_{2\rho\tau^2}$ as defined in Lemma 4.5.1 is a 2π-periodic function, it follows that

$$\int_{-\pi}^{\pi} \Theta_{2\rho\tau^2}(\phi - \psi)\, d\phi = \int_{-\pi}^{\pi} \Theta_{2\rho\tau^2}(\theta)\, d\theta = 1.$$

Therefore

$$K_\rho^\tau(z) = (2\pi)^{-1/2}(M_\rho^\tau *_{-\tau/4} M_{\rho/2}^{-\tau})(z), \quad z \in \mathbb{C}.$$

Similarly, by Theorem 4.6.1, and Theorem 4.7.3,

$$G^\tau(z) = \frac{(2\pi)^{-1/2}}{16\pi^2}(K_{0,\tau} *_{-\tau/4} K_{0,\tau})(z), \quad z \in \mathbb{C},$$

where

$$K_{0,\tau}(z) = K_0\left(\frac{1}{4}|\tau|\,|z|^2\right), \quad z \in \mathbb{C}.$$

Thus, we have the following formulas for the heat kernel K_ρ and Green function G of the fourth-order operator $\mathcal{L}_+\mathcal{L}_-$.

Theorem 4.7.4. *Let $\rho > 0$. Then*

$$K_\rho(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\tau}(M_{\rho/2}^\tau *_{-\tau/4} M_{\rho/2}^{-\tau})(z)d\tau$$

for all (z,t) in $\mathbb{H}^1$.

Theorem 4.7.5. *The Green function G of $\mathcal{L}_+\mathcal{L}_-$ is given by*

$$G(z,t) = \frac{1}{32\pi^3}\int_{-\infty}^{\infty} e^{-it\tau}(K_{0,\tau} *_{-\tau/4} K_{0,\tau})(z)\, d\tau$$

for all (z,t) in $\mathcal{H}1$.

5 The Dirichlet Divisor Problem, Traces and Determinants for Complex Powers of the Twisted Bi-Laplacian

5.1 The Trace of the Heat Semigroup

Theorem 5.1.1. *For $t > 0$,*

$$\mathrm{tr}(e^{-tM}) = (\gamma - \ln t)t^{-1} + O(t^{\mu}),$$

where $\mu > \frac{1}{4}$.

Proof Since

$$\mathrm{tr}(e^{-tM}) = \int_0^\infty e^{-t\lambda} dN(\lambda),$$

it follows from an integration by parts that for $t > 0$,

$$\mathrm{tr}(e^{-tM}) = e^{-t\lambda}N(\lambda)\Big|_0^\infty + t\int_0^\infty e^{-t\lambda}N(\lambda)d\lambda = t\int_0^\infty e^{-t\lambda}N(\lambda)\,d\lambda. \tag{5.1}$$

So, using the formula for $N(\lambda)$ in Section 1 and (5.1), we get for $t > 0$,

$$\begin{aligned}\operatorname{tr}(e^{-tM}) &= t\int_0^\infty e^{-t\lambda}(\lambda\ln\lambda + (2\gamma-1)\lambda + O(\lambda^\mu))\,d\lambda \\ &= t\int_0^\infty e^{-t\lambda}\lambda\ln\lambda\,d\lambda + (2\gamma-1)t^{-1} + O(t^\mu). \qquad (5.2)\end{aligned}$$

Since

$$\begin{aligned}\int_0^\infty e^{-t\lambda}\lambda\ln\lambda\,d\lambda &= -\frac{d}{dt}\int_0^\infty e^{-t\lambda}\ln\lambda\,d\lambda = \frac{d}{dt}\left[\frac{1}{t}(\gamma+\ln t)\right] \\ &= (1-\gamma-\ln t)t^{-2}, \qquad (5.3)\end{aligned}$$

it follows from (5.2) and (5.3) that for $t > 0$,

$$\operatorname{tr}(e^{-tM}) = (\gamma - \ln t)t^{-1} + O(t^\mu),$$

as required.

We first begin with a version of the Dixmier trace that is tailored for the inverse of the twisted bi-Laplacian M. The book [21] is a comprehensive account of Dixmier traces and related topics. In particular, Chapter 1 of the book [21] contains motivational and background material on Dixmier traces.

Let A be a positive and compact operator on a complex and separable Hilbert space X. Let

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots$$

be the eigenvalues of A arranged in decreasing order with multiplicities counted. For a positive integer k, we say that A is in the k^{th} Dixmier trace class if

$$\left\{\frac{1}{\ln^k N}\sum_{j=1}^{N}\lambda_j(A)\right\}_{N=2}^{\infty} \in l^\infty.$$

If A is in the k^{th} Dixmier trace class such that $\lim_{N\to\infty}\frac{1}{\ln^k N}\sum_{j=1}^{N}\lambda_j(A)$ exists, then the k^{th} Dixmier trace $\mathrm{tr}_k(A)$ of A is given by

$$\mathrm{tr}_k(A) = \lim_{N\to\infty}\frac{1}{\ln^k N}\sum_{j=1}^{N}\lambda_j(A).$$

Using Theorem 1.0.2, we get the following theorem for the Dixmier trace of M^{-1}.

Theorem 5.1.2. *M^{-1} is in the second Dixmier trace class and*

$$\mathrm{tr}_2(M^{-1}) = \frac{1}{2}.$$

Proof Let us compute $\sum_{n\le x}\frac{d(n)}{n}$ for large and positive integers x, say, for $x > 2$. To do this, we use the partial summation formula to the effect that

$$\sum_{n\le x} a_n f(n) = S(x-1)f(x) - \int_1^x S(t)f'(t)\,dt, \tag{5.4}$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence with positive terms, f is a positive and differentiable function on $(0,\infty)$, and S is the function on $[1,\infty)$ given by

$$S(t) = \sum_{n\le t} a_n, \quad t \ge 1. \tag{5.5}$$

Indeed,

$$\begin{aligned}\int_1^x S(t)f'(t)\,dt &= \sum_{n=1}^{x-1}\int_n^{n+1} S(t)f'(t)\,dt\\ &= \sum_{n=1}^{x-1}\int_n^{n+1}\left(\sum_{k=1}^{n} a_k\right)f'(t)\,dt\\ &= \sum_{n=1}^{x-1}\sum_{k=1}^{n} a_k(f(n+1)-f(n)).\end{aligned}$$

Interchanging the order of summation, we get

$$\begin{aligned}\int_1^x S(t)f'(t)\,dt &= \sum_{k=1}^{x-1}\sum_{n=k}^{x-1} a_k(f(n+1)-f(n))\\ &= \sum_{k=1}^{x-1} a_k(f(x)-f(k)).\end{aligned}$$

Therefore

$$S(x-1)f(x) - \int_1^x S(t)f'(t)\,dt = \sum_{n=1}^{x} a_n f(n),$$

which is (5.4). Applying (5.4) and (5.5) with $a_n = d(n)$ and $f(n) = \frac{1}{n}$, and using the asymptotic formula for the function S as given by the Dirichlet divisor problem, we get

$$\begin{aligned}\sum_{n\le x}\frac{d(n)}{n} &= S(x-1)f(x) - \int_1^x S(t)f'(t)\,dt\\ &= \frac{1}{x}((x-1)\ln(x-1) + (2\gamma-1)(x-1) + O(\sqrt{x}))\\ &\quad + \int_1^x\left(\frac{\ln t}{t} + (2\gamma-1)t^{-1} + O(t^{-3/2})\right)dt. \qquad (5.6)\end{aligned}$$

Since

$$(x-1)\ln(x-1) = x\ln x + x + O(\sqrt{x}) \qquad (5.7)$$

as $x \to \infty$, and

$$\int_1^x \frac{\ln t}{t} dt = \frac{1}{2} \ln^2 x. \tag{5.8}$$

it follows from (5.6)–(5.8) that

$$\begin{aligned}
\sum_{n \le x} \frac{d(n)}{n} &= \frac{1}{x}(x \ln x + (2\gamma - 1)x + O(\sqrt{x})) \\
&\quad + \frac{1}{2} \ln^2 x + (2\gamma - 1)\ln x + O(x^{-1/2}) \\
&= \frac{1}{2}\ln^2 x + 2\gamma \ln x + (2\gamma - 1) + O(x^{-1/2})
\end{aligned}$$

as $x \to \infty$. This completes the proof.

5.2 Zeta Function Regularizations

We begin with the following easy observation.

Lemma 5.2.1. *Let $\alpha \in \mathbb{C}$. Then for all s with* $\mathrm{Re}\,(\alpha s) > 1$,

$$\zeta_{M^\alpha}(s) = \zeta^2(\alpha s).$$

Proof Let $s \in \mathbb{C}$ be such that $\mathrm{Re}\,(\alpha s) > 1$. Then using the eigenvalues of$M$, the eigenvalues of $M^{-\alpha s}$ are $n^{-\alpha s}$, $n = 1, 2, \ldots$, and the multiplicity of $n^{-\alpha s}$ is equal to the number $d(n)$ of Dirichlet divisors of n. Therefore

$$\zeta_{M^\alpha}(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^{\alpha s}}.$$

So, a straightforward computation gives

$$\zeta^2(\alpha s) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^{\alpha s}} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{\alpha s}} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha s}} \sum_{\mu\nu=n} 1 = \sum_{n=1}^{\infty} \frac{d(n)}{n^{\alpha s}}.$$

The zeta function regularizations of the trace and the determinant of M^α, denoted by $\mathrm{tr}_R(M^\alpha)$ and $\det_R(M^\alpha)$ respectively, are defined by

$$\mathrm{tr}_R(M^\alpha) = \zeta_{M^\alpha}(-1)$$

and

$$\det_R(M^\alpha) = e^{-\zeta'_{M^\alpha}(0)}.$$

The physical meanings of these quantities can be found in, e.g., the paper [19].

Theorem 5.2.2. *Let* $\alpha \in \mathbb{C} \setminus \{-1\}$. *Then*

$$\mathrm{tr}_R(M^\alpha) = \zeta^2(-\alpha).$$

Proof By Lemma 4.1 and the analytic continuation of the Riemann zeta function to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$, we see that

$$\mathrm{tr}_R(M^\alpha) = \zeta_{M^\alpha}(-1) = \zeta^2(-\alpha).$$

Remark 5.2.3. It is well-known from, say, [37] that

$$\zeta(-1) = -\frac{1}{12}.$$

Hence

$$\mathrm{tr}_R(M) = \frac{1}{144}.$$

Remark 5.2.4. In the case when $\alpha = -1$, the zeta function regularization of the inverse M^{-1} is equal to infinity. The Dixmier trace instead of the trace of the inverse M^{-1} is a finite number.

Theorem 5.2.5. *Let $\alpha \in \mathbb{C}$. Then*

$$\det_R(M^\alpha) = (2\pi)^{-\alpha/2}.$$

Proof As in Theorem 5.2.2,

$$\det_R(M^\alpha) = e^{-\zeta'_{M^\alpha}(0)} = e^{-2\alpha\zeta(0)\zeta'(0)}.$$

It can be found in [37] again that $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$. So,

$$\det_R(M^\alpha) = (2\pi)^{-\alpha/2}.$$

As an application, we can give a formula for the determinants of the heat semigroups of complex powers of the twisted bi-Laplacian.

Theorem 5.2.6. *Let $\alpha \in \mathbb{C} \setminus \{-1\}$. Then for $t > 0$,*

$$\det_R(e^{-tM^\alpha}) = e^{-t\zeta^2(-\alpha)}.$$

Proof By Theorem 1.1, the eigenvalues of $e^{(-tM^\alpha)^{-s}}$ are $e^{tn^\alpha s}$, $n = 1, 2, \ldots$, and the multiplicity of the eigenvalue $e^{tn^\alpha s}$ is $d(n)$. Therefore

$$\zeta_{e^{-tM^\alpha}}(s) = \mathrm{tr}((e^{-tM^\alpha})^{-s}) = \sum_{n=1}^{\infty} d(n) e^{tn^\alpha s}, \quad s \in \mathbb{C}.$$

So,

$$\zeta'_{e^{-tM^\alpha}}(0) = t\sum_{n=1}^{\infty} d(n)n^\alpha = t\zeta^2(-\alpha).$$

Thus,

$$\det_R(e^{-tM^\alpha}) = e^{-\zeta'_{e^{-tM^\alpha}}(0)} = e^{-t\zeta^2(-\alpha)},$$

and this completes the proof.

Remark 5.2.7. By Theorems 5.2.2 and 5.2.6, we see that for $\alpha \in \mathbb{C} \setminus \{-1\}$,

$$\det_R(e^{-tM^\alpha}) = e^{-t\mathrm{tr}_R(M^\alpha)}, \quad t > 0,$$

which is in conformity with the well-known relationship between the determinant and the trace of a square matrix A given by

$$\det(e^A) = e^{\mathrm{tr}(A)}.$$

6 Heat Kernels and Green Functions for Complex Powers of the Twisted Bi-Laplacian and their L^p-Estimates

6.1 The Heat Kernel and Green Function of M^α

Now, we consider the operator M^α, where α is any complex number. We are interested in finding the heat kernel and Green function of M^α, as well as the asymptotic expansions of its counting function.

Firstly, by the spectral mapping theorem, the eigenvalues of M^α are $1^\alpha, 2^\alpha ...$, and the eigenfunctions of each n^α is given by $e_{j,k}$ such that $(j+1)(k+1) = n$.

Then we see immediately that the asymptotic expansion of the counting function can be obtaind for M^α from that of the operator M by a direct change of variable. We define $N(\lambda)$

to be the number of eigenvalues of M^α less or equal to λ, then

$$N(\lambda) = \sum_{n^{Re(\alpha)} \le \lambda} d(n)$$
$$= \sum_{n \le \lambda^{\frac{1}{Re(\alpha)}}} d(n) = (1/\alpha)\lambda^{\frac{1}{Re(\alpha)}} \ln\lambda + (2\gamma - 1)\lambda^{\frac{1}{Re(\alpha)}} + O(\lambda^{\frac{1}{2Re(\alpha)}})$$

Secondly, using the spectral theory, we can obtain the heat kernel for the operator M^α, for $\alpha \in \mathbb{C}$, $Re(\alpha) > 1$. Given $f \in L^2(\mathbb{C})$, we can write

$$e^{-tM^\alpha} f = \sum_{n=1}^{\infty} e^{-tn^\alpha} \sum_{(j+1)(k+1)=n} (f, e_{j,k}) e_{j,k},$$

where we need the product of the indices in the second sum to start from 1, simply because n starts from 1. And it follows that

$$e^{-tM^\alpha} f = \sum_{n=1}^{\infty} e^{-tn^\alpha} \int_{\mathbb{C}} f(w) \sum_{(k+1)(j+1)=n} e_{j,k}(w) e_{j,k}(z) dw, \quad z \in \mathbb{C}.$$

Assuming for now the interchange of the integral and the summation is justified, we can write one more step to get

$$e^{-tM^\alpha} f = \int_{\mathbb{C}} f(z) \left(\sum_{n=1}^{\infty} e^{-tn^\alpha} \sum_{(k+1)(j+1)=n} e_{j,k}(z) e_{j,k}(w) \right) dw, \quad z \in \mathbb{C}.$$

And the last equation gives us the heat kernel of M^α in terms of a series.

The next step is to find the Green function, which can be obtained by integrating the heat kernel with respect to t, as we now show,

$$\begin{aligned} G(z,w) &= \int_0^\infty \sum_{n=1}^\infty e^{-tn^\alpha} \sum_{(k+1)(j+1)=n} e_{j,k}(z) e_{j,k}(w) dt \\ &= \sum_{n=1}^\infty \frac{1}{n^\alpha} \sum_{(j+1)(k+1)=n} e_{j,k}(z) e_{j,k}(w) \end{aligned}$$

We are also interested in the trace of the operator $M^{-\alpha}$. To obtain the trace, we use Plancherel's theorem and Moyal's identity for Fourier-Wigner transforms, and get

$$\begin{aligned} tr(M^{-\alpha}) &= \int_{\mathbb{C}} G(z,z) dz \\ &= \sum_{n=1}^\infty \frac{1}{n^\alpha} \sum_{(k+1)(j+1)=n} \int_{\mathbb{C}} e_{j,k}(z) \overline{e_{j,k}(z)} dz \\ &= \sum_{n=1}^\infty \frac{1}{n^\alpha} \sum_{(j+1)(k+1)=n} (e_{j,k}, e_{j,k}) \\ &= \sum_{n=1}^\infty \frac{1}{n^\alpha} \sum_{(j+1)(k+1)=n} (V(e_j, e_k), V(e_j, e_k)) \\ &= \sum_{n=1}^\infty \frac{1}{n^\alpha} \sum_{(j+1)(k+1)=n} (W(e_j, e_k), W(e_j, e_k)) \\ &= \sum_{n=1}^\infty \frac{1}{n^\alpha} \sum_{(j+1)(k+1)=n} (e_j, e_j)(e_k, e_k) \\ &= \sum_{n=1}^\infty \frac{d(n)}{n^\alpha}, \end{aligned}$$

where $W(e_j, e_k)$ is the Wigner transform of the Hermite functions e_j and e_k, and $d(n)$ is the number of divisors of n. From this formula, we see that $M^{-\alpha}$ is a trace class operator for complex α such that $\mathrm{Re}\alpha > 1$.

6.2 $L^p - L^{p'}$-Estimates for M^α

In the following section, for $1 \leq p \leq 2$, we give an estimate of the $L^{p'}$-norm of the solution to the heat equation governed by the operator M^α with $\text{Re}(\alpha) > 0$. i.e.

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = (-M^\alpha u)(x,t), \quad Re(\alpha) > 0, \qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), \qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^n,\ f \in L^p(\mathbb{R}^n),\ 1 < p < 2. \end{cases}$$

We have

$$u(\cdot,\ t) = e^{-M^\alpha t} f, \quad t > 0,$$

and therefore

$$u(\cdot,\ t) = \int_{\mathbb{C}} f(w) \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \sum_{(j+1)(k+1)=n} e_{j,k}(w) e_{j,k}(z) dw \quad t > 0.$$

To begin with, we explore the possible $L^{p'}$-estimate for the solution u in terms of $\|f\|_p$. Let $f \in L^p(\mathbb{C})$. Then by Minkowski's inequality, we have

$$\begin{aligned} \|u(\cdot,t)\|_{p'} &= \left(\int_{\mathbb{C}} \left| \int_{\mathbb{C}} f(w) \sum_{n=1}^{\infty} e^{-tn^\alpha} \sum_{(j+1)(k+1)=n} e_{j,k}(w) e_{j,k}(z) dw \right|^{p'} dz \right)^{\frac{1}{p'}} \\ &\leq \int_{\mathbb{C}} |f(w)| \left(\int_{\mathbb{C}} \left| \sum_{n=1}^{\infty} e^{-tn^\alpha} \sum_{(j+1)(k+1)=n} e_{j,k}(w) e_{j,k}(z) \right|^{p'} dz \right)^{\frac{1}{p'}} dw \\ &\leq \|f\|_p \|g\|_{p'}, \end{aligned}$$

where

$$g(w) = \left(\int_{\mathbb{C}} \left| \sum e^{-tn^\alpha} \sum e_{j,k}(w) e_{j,k}(z) \right|^{p'} dz \right)^{\frac{1}{p'}}.$$

Now, we compute $||g||_{p'}$ and give conditions on p such that it is finite. Using Minkowski's inequality, we get

$$\begin{aligned}
||g||_{p'} &= \left(\int_{\mathbb{C}}\int_{\mathbb{C}}\left|\sum_{n=1}^{\infty} e^{-tn^{\alpha}} \sum_{(j+1)(k+1)=n} e_{j,k}(w)e_{j,k}(z)\right|^{p'} dzdw\right)^{\frac{1}{p'}} \\
&\leq \sum_{n=1}^{\infty} e^{-tn^{Re\alpha}} \sum_{(j+1)(k+1)=n} \left(\int_{\mathbb{C}}\int_{\mathbb{C}} |e_{j,k}(z)e_{j,k}(w)|^{p'} dzdw\right)^{\frac{1}{p'}} \\
&= \sum_{n=1}^{\infty} e^{-tn^{Re\alpha}} \sum_{(j+1)(k+1)=n} ||e_{j,k}||_{p'}^{2}
\end{aligned}$$

We need to explore the conditions on p such that the $||e_{j,k}||_{p'}$'s are uniformly bounded for all j, k. Through Wong's result we can obtain the desired results for $1 \leq p \leq 2$, as we now show. We first prove a theorem which is analogous to the breaking of symmetry obtained by [39]

Theorem 6.2.1. *Let* $\{e_{j,k}\}$, $\quad j, k = 0, 1, 2, ...,$ *be the Fourier-Wigner transform of the Hermite functions* e_j *and* e_k.*Then for* $1 \leq p \leq 2$, *there exists some positive constant* C_p *such that*

$$||e_{j,k}||_{p'} \leq C_p ||e_j||_p ||e_k||_{p'}.$$

Proof. We first rewrite the Fourier-Wigner transform of the Hermite functions e_j and e_k as follows. Recall that

$$e_{j,k}(q, p) = \int_{\mathbb{R}^n} e^{iq\cdot y} e_j(y + \frac{p}{2})\overline{e_k(y - \frac{p}{2})}dy,$$

Let $z = y + \frac{p}{2}$, we can write

$$e_{j,k}(q,p) = e^{-iq\cdot p/2}\mathcal{F}^{-1}(e_j T_{-p} e_k)(q),$$

where $\mathfrak{F}^{-1}$ is the inverse Fourier transform. Now, for $1 \leq p \leq 2$, using the Hausdorff-Young inequality and Minkowski's inequality we get

$$\begin{aligned}
\|e_{j,k}\|_{p'} &= \left(\int_{\mathbb{R}}\left(\left|e^{-iq\cdot p/2}\mathcal{F}^{-1}(e_j T_{-p} e_k)\right|^{p'} dq\right) dp\right)^{\frac{1}{p'}} \\
&\leq C_p\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |e_j(x)\bar{(}T_{-p}\overline{e_k})(x)|^p dx\right)^{\frac{p'}{p}} dp\right)^{\frac{1}{p'}} \\
&= C_p\left(\left(\int_{\mathbb{R}}\left(|e_j(x)(T_{-p}\overline{e_k})(x)|^p dx\right)^{\frac{p'}{p}} dp\right)^{\frac{p}{p'}}\right)^{\frac{1}{p}} \\
&\leq C_p\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |e_j(x)|^p |(T_{-p}\overline{e_k})(x)|^{p'} dp\right)^{\frac{p}{p'}} dx\right)^{\frac{1}{r}} \\
&= C_p\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |e_j(x)|^p |\overline{e_k}(p-x)|^{p'} dp\right)^{\frac{p}{p'}} dx\right)^{\frac{1}{r}} \\
&= C_p\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |e_j(x)|^p |\overline{e_k}(y)|^{p'} dy\right)^{\frac{p}{p'}} dx\right)^{\frac{1}{r}} \\
&= C_p\left(\int_{\mathbb{R}} |e_j(x)|^p \|e_k\|_{p'}^p dx\right)^{\frac{1}{p}} \\
&= C_p \|e_j\|_p \|e_k\|_{p'},
\end{aligned}$$

as desired.

Lemma 6.2.2. *For $j,k = 0,1,2,\ldots$ and $1 \leq p \leq 2$, we have*

$$\|e_{j,k}\|_{p'}^2 \leq C_p^2(\|e_j\|_p\|e_j\|_{p'})(\|e_k\|_p\|e_k\|_{p'}).$$

Proof. Let $z = -y$. We have

$$\begin{aligned} e_{j,k}(q,p) &= \int_{\mathbb{R}} e^{iq\cdot y} e_j(y+\frac{p}{2})\overline{e_k(y-\frac{p}{2})}dy \\ &= \int_{\mathbb{R}} e^{-iqz} e_k(-(z+\frac{p}{2}))\overline{e_j(-(z-\frac{p}{2}))}dz \\ &= V(\widetilde{e_k},\widetilde{e_j})(-q,p). \end{aligned}$$

By the result in Theorem 6.2.1, we immediately get

$$\|e_{j,k}\|_{p'} = \|V(\widetilde{e_k},\widetilde{e_j})\|_{p'} \leq C_p\|\widetilde{e_k}\|_p\|\widetilde{e_j}\|_{p'} = C_p\|e_k\|_p\|e_j\|_{p'}.$$

Thus, by the last inequality, we get

$$\|e_{j,k}\|_{p'}^2 \leq C_p^2(\|e_j\|_p\|e_j\|_{p'})(\|e_k\|_p\|e_k\|_{p'}) = O(n^{2\epsilon_p}),$$

where $(j+1)(k+1) = n$, and ϵ_p is the bound for $\|e_j\|_p\|e_j\|_{p'}$ for all j.

Finally, by Wong's result and the preceding lemma, we immediately get for $p \in [1,2]$,

$$\|e_{j,k}\|_{p'} = O(n^{2\epsilon_p}),$$

where $(j+1)(k+1) = n$.

Now, for $1 \leq p \leq 2$ and $\mathrm{Re}\alpha > 0$, we can give the $L^{p'}$-estimate of the solution u to the heat equation for M^α in terms of the L^p-norm of f. Indeed,

$$\|u(\cdot,t)\|_{p'} \leq \|f\|_p\|g\|_{p'} \leq \|f\|_p \sum_{n=1}^{\infty} e^{-tn^{\mathrm{Re}\alpha}} d(n) C_p \|e_{j,k}\|_{p'},$$

which is clearly convergent because the $L^{p'}$-norms of all the $e_{j,k}$'s are bounded by $n^{2\epsilon_p}$, and that $d(n) = O(n^{\epsilon}), \quad \forall \epsilon > 0$.

Moreover, we can give a separete estimate for the L^{∞}-norm of the solution to the heat equation for M^{α} in terms of the L^1-norm of the function f.

Let $f \in L^1(\mathbb{C})$. Then we can write

$$\|u(\cdot, t)\|_{\infty} \leq \int_{\mathbb{C}} |f(w)| \sum_{n=0}^{\infty} \left|e^{-tn^{\alpha}}\right| \sum_{(j+1)(k+1)=n} |e_{j,k}(z) e_{j,k}(w)| dw.$$

But it is easy to show that the $|e_{j,k}|$ are uniformly bounded by $(2\pi)^{-1/2}$ for all integers j, k. We have

$$\|u(\cdot, t)\|_{\infty} \leq (2\pi)^{-1} \sum_{n=1}^{\infty} e^{-tn^{\mathrm{Re}\alpha}} d(n),$$

which is clearly convergent for all α with $Re\alpha > 0$, as $d(n) = O(n^{\epsilon})$ for all $\epsilon > 0$.

Now, it is possible to give the $L^{p'}$ estimate with $1 \leq p \leq 2$, for the Green function of the operator M^{α} with $\mathrm{Re}\alpha > 0$. The Green function is the kernel of the integral representation of the solution to the Poisson equation

$$M^{\alpha} u = f$$

on $\mathbb{R}^2$ for suitable functions f. Recall that the Green function of M^{α} is given by

$$G(z, w) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \sum_{(j+1)(n+1)=n} e_{j,k}(z) e_{j,k}(w), \quad z, w \in \mathbb{C}.$$

Thus, by Minkowski's inequality, we get

$$\begin{aligned}
\|G(z,w)\|_{p'} &= \int_{\mathbb{C}}\int_{\mathbb{C}}\left(\left|\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}\sum_{(j+1)(n+1)=n} e_{j,k}(z)e_{j,k}(w)\right|^{p'} dwdz\right)^{\frac{1}{p'}} \\
&\leq \sum_{n=1}^{\infty}\frac{1}{n^{\mathrm{Re}\alpha}}\sum_{(j+1)(k+1)=n}\left(\int_{\mathbb{C}}\int_{\mathbb{C}}\left|e_{j,k}(z)e_{j,k}(w)\right|^{p'} dzdw\right)^{\frac{1}{p}} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{\mathrm{Re}\alpha}}\sum_{(j+1)(k+1)=n}\|e_{j,k}\|_p^2
\end{aligned}$$

Using the preceding results that for $1 \leq p \leq 2$,

$$\|e_{j,k}\|_{p'} \leq C_p\|e_j\|_p\|e_k\|_{p'} = O(n^{2\epsilon_p}),$$

for some constant ϵ_p and the fact that $d(n) = O(n^{\epsilon})$, $\forall \epsilon > 0$, we conclude that for $2 \leq p \leq \infty$,

$$\|G(z,w)\|_{p'} \leq K\sum_{n=1}^{\infty}\frac{1}{n^{\mathrm{Re}\alpha}}d(n) < \infty,$$

whenever $\mathrm{Re}\alpha > 1$. Finally, we can give the $L^p - L^{p'}$ estimate of the solution u to the Poisson equation governed by M^{α}.

$$\begin{aligned}
\|u\|_{p'} &= \left(\int_{\mathbb{C}}\left|\int_{\mathbb{C}} G(z,w)f(w)dw\right|^{p'} dz\right)^{\frac{1}{p'}} \\
&\leq \int_{\mathbb{C}}\left(\int_{\mathbb{C}}|G(z,w)|^{p'}|f(w)|^{p'}dz\right)^{\frac{1}{p'}} dw \\
&= \int_{\mathbb{C}}\|G(\cdot, w)\|_{p'}|f(w)|dw \\
&\leq \|f\|_p\|G\|_{p'}.
\end{aligned}$$

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