# FINITE PSEUDO-DIFFERENTIAL OPERATORS, LOCALIZATION OPERATORS FOR CURVELET AND RIDGELET TRANSFORMS

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# Abstract

Pseudo-differential operators can be built from the Fourier transform. However, besides the difficult problems in proving convergence and $L^2$-boundedness, the problem of finding eigenvalues is notoriously difficult. Finite analogs of pseudo-differential operators are desirable and indeed are constructed in this dissertation.

Energized by the success of the Fourier transform and wavelet transforms, the last two decades saw the rapid developments of new tools in time-frequency analysis, such as ridgelet transforms and curvelet transforms, to deal with higher dimensional signals. Both curvelet transforms and ridgelet transforms give the time/position-frequency representations of signals that involve the interactions of translation, rotation and dilation, and they can be ideally used to represent signals and images with discontinuities lying on a curve such as images with edges. Given the resolution of the identity formulas for these two transforms, localization operators on them are constructed.

The later part of this dissertation is to investigate the $L^2$-boundedness of the localization operators for curvelet transforms and ridgelet transforms, as well as their trace properties.

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# 1 Introduction

Let $f$ be a signal in $L^2(\mathbb{R}^n)$. Then the Fourier transform $\hat{f}$ of $f$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

The Fourier inversion formula gives us back the signal $f$ via

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n.$$

This is the basis for pseudo-differential operators on $\mathbb{R}^n$ or sometimes referred to as time-varying filters. Indeed, let $\sigma$ be a suitable function on $\mathbb{R}^n \times \mathbb{R}^n$. Then the pseudo-differential operator $T_\sigma$ is defined by

$$(T_\sigma f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\, \hat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^n.$$

In the case when $\sigma$ is identically equal to 1, then $T_\sigma$ is the identity in view of the Fourier inversion formula. Pseudo-differential operators have been used in quantizations and time-frequency analysis. Their usefulness notwithstanding, these operators are difficult to work with because of the convergence of the integrals. Moreover, useful information such as eigenvalues is difficult or even impossible to compute. So, it is desirable to obtain finite analogs of pseudo-differential operators. First of all, in applications the numerical implementations of pseudo-differential operators require a finite setting. Secondly, finite pseudo-differential operators are finite-dimensional matrices of which the entries are given by the finite Fourier transforms defined in Section 2.6. Thus, the computations of the eigenvalues can be performed using the fast Fourier transforms and available algorithms. Furthermore, issues like $L^p$-boundedness, which pseudo-differential operators have to deal with all the time, are irrelevant to finite pseudo-differential operators. In this dissertation, we are particularly interested in constructing such operators on $L^2(\mathbb{Z}_N)$, where $\mathbb{Z}_N$ is the discretization of a circle. These operators are discrete analogs of pseudo-differential operators on the unit circle $\mathbb{S}^1$ with center at the origin, which have been studied in [26] and [27]. Pseudo-differential operators on the torus $\prod_{j=1}^{n} \mathbb{S}^1$ are routine extensions of the ones on $\mathbb{S}^1$.

Given a signal, we are often interested in its frequency content locally in time. For instance, in a piece of music we want to know the notes (frequency information) that are played at any given moment. The Fourier transform gives a representation of the frequency content of the signal $f$, but information concerning time-localization of certain frequencies cannot be read off easily.

Time-localization can be achieved by cutting off the signal equally in time and then taking its Fourier transform. This can be achieved by the so-called windowed Fourier transform or the Gabor transform. The details are given in Section 2.1. The drawback here is that a window of fixed width is used for all time $b$. It is more accurate and desirable if we can have an adaptive window that gives a wide window for low frequency and a narrow window for high frequency. That this can be done comes from familiarity with the wavelet transform.

The wavelet transform [7, 37, 38], which decomposes a signal into components depending on translations and scales, is a multiscale integral transform. It can be used in time-frequency analysis in which scale and frequency are reciprocal of each other. Its advantage over the ubiquitous Fourier transform in time-frequency analysis is that it can display the information as to when a certain frequency of a signal takes place by using a window with size depending on the frequency. Fundamental to the theory is the resolution of the identity formula that synthesizes the wavelet transform in order to recover the orignal signal. As in the case of the Gabor transform, there is a window $\varphi_{b,a}$ in the wavelet transform. Unlike the case of the Gabor transform, the window $\varphi_{b,a}$ is adjustable in the sense that it is narrow if the scale $a$ is small and wide if the scale $a$ is big. The wavelet transforms as defined are essentially one-dimensional time-frequency tools since the frequency $\xi$ and scale $a$ can be thought of as being related by $a = 1/\xi$.

In an attempt to deal with two-dimensional signal analysis such as image processing, the wavelet transform has been extended to the two-dimensional wavelet transform even with directions taken into account. A version of this two-dimensional wavelet transform, known as the polar wavelet transform, has been developed in [21] and the resolution of the identity formula is also established therein for this mathematical tool. Details on the analysis and applications of wavelets can be found in [7, 11, 38].

Among other developments related to the wavelet transform, an integral transform in [24], orignally used by Stockwell in atmospheric physics, has been used in time-frequency analysis. It is related to, but different from, the wavelet transform in that the time average of the Stockwell transform over all time from the past into the future of a signal gives the Fourier transform of the signal. This property, known as the absolutely referenced phase information, puts the Stockwell transforms into a class in its own right. The usefulness of the phase of the Stockwell transform is illustrated in [12]. The resolution of the identity formula for the one-dimensional Stockwell transform is given in [9]. In [22] can be found the two-dimensional Stockwell transform and the corresponding resolution of the identity formula. Multi-dimensional Stockwell transforms and their resolution of the identity formulas have recently been obtained in [29, 30].

Motivated by the results of two-dimemsional wavelet and Stockwell transforms, a relatively new two-dimensional multiscale integral transform, which is dubbed the curvelet transform [3], has emerged in time-frequency analysis. Like the wavelet transform and the Stockwell transform, the translations, the dilations and the rotations are built into the genesis of the curvelet transform. The important difference of the curvelet transform from the wavelet and Stockwell transforms lies in the fact that non-isotropic instead of isotropic dilations are used. It can be ideally used to represent images with discontinuities lying on a curve such as images with edges. It is an interesting fact that the resolution of the identity formula is now only valid for high-frequency signals. The full resolution of the identity formula for all signals with finite energy requires an additional term to cope with low-frequency signals as well. This additional term turns out to be a wavelet multiplier first studied systematically in [14].

Closely related to multi-dimensional wavelet transforms are ridgelet transforms first introduced by Emmanuel Candés in his 1998 Ph.D. thesis [1,2]. As a matter of historical fact, ridgelets predate curvelets. Like wavelet transforms and curvelet transforms, there is a resolution of the identity formula for ridgelet transforms.

Once we have a resolution of the identity formula, we are interested in studying localization operators on them as in [7, 9, 13, 21, 22, 25, 37]. The idea of a localization operator is to pick out the areas of interests by inserting a weight function or a symbol in a resolution of the identity formula.

Notwithstanding the importance of applications and computations of the operators, the focus of this dissertation is to develop the general theory of finite pseudo-differential operators, localization operators for curvelet transforms and ridgelet transforms for the widest possible classes of symbols and windows. Two properties of these operators are of the greatest interest - the $L^2$-boundedness and the trace. The importance of $L^2$-boundedness lies in the so-called continuous dependence on the initial data in input-output analysis. To wit, in applications, measurements of the input always entail errors and a good model for a filter must ensure that the errors in the output can be controlled in terms of the errors in the input, and a natural means of the measurements is the $L^2$-norm. Applications of the trace in operator theory and operator algebras abound and in this dissertation we note its usefulness in the context of the Gershgorin circle of a bounded and self-adjoint operator on a Hilbert space.

It is a well-known fact that for a compact and self-adjoint operator there is an eigenvalue lying on the Gershgorin circle. Its distance from the origin is equal to the norm of the operator. But the norm of an operator is a difficult quantity to obtain, so we give at least an estimate-an upper estimate in this dissertation. There are estimates coming from the $L^2$-estimates, but the trace, which is computed for all operators in the dissertation is an alternative upper bound. Which one is better can be checked easily on an individual basis. In the case of the finite pseudo-differential operators, all the eigenvalues can be obtained exactly using existing softwares. Nevertheless, the trace formula is still of great interest because it gives the architecture of the eigenvalues of the operator in terms of its symbols and not just an arbitrary set of numbers.

In Section 2 we present without proofs the background material needed for the results on the finite pseudo-differential operators, localization operators for curvelet transforms and localization operators for ridgelet transforms.

In Section 3 we introduce the finite pseudo-differential operators. The main results are the two representations of finite pseudo-differential operators as matrices in terms of the Fourier basis and the unit impulse basis, and a trace formula in Theorem 3.1.1 to show that the eigenvalues are related to the symbol in a very elegant way.

In Section 4 we give a streamlined proof of the resolution of the identity formulas for curvelet transforms based on the paper [3] by Candès and Donoho. Due to the special constructions of curvelet transforms, the corresponding localization operators are built with two parts - one for high frequencies and one for low frequencies. The localization operator for high frequencies $L_\tau : L^2_{2/a_0}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with symbol $\tau$ is defined by

$$(L_\tau f, g) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a,b,\theta)(f,\gamma_{ab\theta})(\gamma_{ab\theta},g)\frac{da}{a^3}db\,d\theta \tag{1.1}$$

for all $f$ in $L^2_{2/a_0}(\mathbb{R}^2)$ and $g$ in $L^2(\mathbb{R}^2)$. The wavelet multiplier $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with symbol $\sigma$ that makes up for the low-frequencies is given by

$$(T_\sigma f, g) = \int_{\mathbb{R}^2} \sigma(b)(f,\Phi_b)(\Phi_b,g)db \tag{1.2}$$

and

$$(T_\tau f, g) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a,b,\theta)(f, \gamma_{ab\theta})(\gamma_{ab\theta}, g) \frac{da}{a^3} db\, d\theta \tag{1.3}$$

for all $f$ and $g$ in $L^2(\mathbb{R}^2)$. The new results are the $L^2$-boundedness of the localization operators for high frequencies, the wavelet multipliers and the curvelet localization operators for all signals with finite energy. To be explicit, $L^2$-boundedness for high-frequencies is in Theorems 4.2.1 and 4.2.2; $L^2$-boundedness for low frequencies is in Theorems 4.3.2 and 4.3.3; and the one for curvelet localization operators is in Theorems 4.4.1 and 4.4.2. The trace class properties for wavelet multipliers and localization operators for all signals in $L^2(\mathbb{R}^2)$ are also investigated. Furthermore, we give a self-contained treatment of trace class operators from a closed subspace of an infinite-dimensional, separable and complex Hilbert space $X$ into $X$ and then obtain in Theorem 4.6.4 trace class localization operators for high-frequency signals.

In Section 5, we give two kinds of localization operators. We give conditions on the symbols to guarantee that each kind of localization operator for ridgelet transforms is a bounded linear operator on $L^2(\mathbb{R}^n)$. The main results are in Theorem 5.1.1 and Theorem 5.2.1. The trace formula for the second kind of localization operator in the trace class $S_1$ is given in Theorem 5.2.2. The second kind of trace class localization operator is reminiscent of the Landau–Pollak–Slepian operators [15, 31–34] and wavelet multipliers [5, 8, 14, 40, 41].

# 2 Background Materials

## 2.1 One-Dimensional Gabor Transforms

For a signal $f$ in $L^2(\mathbb{R})$, the Gabor transform or the short-time Fourier transform $G_\varphi f$ of $f$ with respect to a window $\varphi$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$(G_\varphi f)(b,\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\overline{\varphi(x-b)}dx, \quad b,\xi \in \mathbb{R}.$$

Let us note that

$$(G_\varphi f)(b,\xi) = (2\pi)^{-1/2}(f, M_\xi T_{-b}\varphi)_{L^2(\mathbb{R})}, \quad b,\xi \in \mathbb{R},$$

where $M_\xi$ and $T_{-b}$ are the modulation operator and the translation operator given by

$$(M_\xi h)(x) = e^{ix\xi}h(x) \tag{2.1}$$

and

$$(T_{-b}h)(x) = h(x-b) \tag{2.2}$$

for all measurable functions $h$ on $\mathbb{R}$ and all $x$ in $\mathbb{R}$. We call the function $M_\xi T_{-b}\varphi$ the Gabor wavelet generated from $\varphi$ by translation $T_{-b}$ and modulation $M_\xi$.

More interesting results on modulation $M_\xi$ and translation $T_{-b}$ on $\mathbb{R}^n$ can be found in Proposition 3.4. [36].

**Proposition 3.4.** *Let $f \in L^1(\mathbb{R}^n)$. Then the functions $T_{-y}f$ and $M_y f$ defined in 2.2 and 2.1 respectively are in $L^1(\mathbb{R}^n)$. Moreover,* *(i)* $\widehat{(T_{-y}f)}(\xi) = (M_{-y}\hat{f})(\xi), \quad \xi \in \mathbb{R}^n,$

*(ii)* $\widehat{(M_{y}f)}(\xi) = (T_{-y}\hat{f})(\xi), \quad \xi \in \mathbb{R}^n.$

The usefulness of the Gabor windows in signal analysis is enhanced by the following resolution of the identity formula, which allows the reconstruction of a signal from its Gabor transform.

**Theorem 2.1.1.** *Suppose that* $\|\varphi\|_2 = 1$*, where* $\|\ \|_2$ *is the norm in* $L^2(\mathbb{R})$*. Then for all* $f$ *and* $g$ *in* $L^2(\mathbb{R})$*,*

$$(f,g)_{L^2(\mathbb{R})} = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(f, M_\xi T_{-x}\varphi)_{L^2(\mathbb{R})}(M_\xi T_{-x}\varphi, g)_{L^2(\mathbb{R})}\,dx\,d\xi.$$

Another way of looking at Theorem 2.1.1 is that for all $f$ in $L^2(\mathbb{R})$,

$$f = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(f, M_\xi T_{-x}\varphi)_{L^2(\mathbb{R})}M_\xi T_{-x}\varphi\,dx\,d\xi,$$

which is also known as a continuous inversion formula for the Gabor transform.

In signal analysis, $(G_\varphi f)(b,\xi)$ gives the time-frequency content of a signal $f$ at time $b$ and frequency $\xi$ by placing the window $\varphi$ at time $b$.

## 2.2 One-Dimensional Continuous Wavelet Transfroms

Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty,$$

where $\hat{\varphi}$ is the Fourier transform of $\varphi$. Such a function $\varphi$ is said to satisfy the admissibility condition and is sometimes called the mother affine wavelet. The adjective *affine* comes from the connection with the affine group that is the underpinning of the wavelet transforms. See Chapter 18 of [38] in this connection.

Let $\varphi \in L^2(\mathbb{R})$ be a mother affine wavelet. Then for all $b$ in $\mathbb{R}$ and $a$ in $\mathbb{R}\setminus\{0\}$, we define the affine wavelet $\varphi_{b,a}$ by

$$\varphi_{b,a}(x) = \frac{1}{\sqrt{|a|}} \varphi\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}.$$

We note that $\varphi_{b,a}$ is generated from the function $\varphi$ by translation and dilation. To put things in perspective, let $b \in \mathbb{R}$ and let $a \in \mathbb{R}\setminus\{0\}$. Then we see that the wavelet $\varphi_{b,a}$ can be expressed as

$$\varphi_{b,a} = T_{-b} D_{1/a} \varphi,$$

where $D_{1/a}$ is the dilation operator defined by

$$(D_{1/a}h)(x) = \frac{1}{\sqrt{|a|}} h\left(\frac{x}{a}\right)$$

for all measurable functions $h$ on $\mathbb{R}$ and all $x$ in $\mathbb{R}$.

Let $\varphi$ be a mother affine wavelet. Then the wavelet transform $\Omega_\varphi f$ of a function $f$ in $L^2(\mathbb{R})$ is defined to be the function on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ by

$$(\Omega_\varphi f)(b,a) = (f, \varphi_{b,a})_{L^2(\mathbb{R})}$$

for all $b$ in $\mathbb{R}$ and $a$ in $\mathbb{R} \setminus \{0\}$. The analysis of the wavelet transform is based on the following resolution of the identity formula, which is also a continuous inversion formula.

**Theorem 2.2.1.** *Let $\varphi$ be a mother affine wavelet. Then for all functions $f$ and $g$ in $L^2(\mathbb{R})$,*

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} (\varphi_{b,a}, g)_{L^2(\mathbb{R})} \frac{db\,da}{a^2},$$

*where*

$$c_\varphi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi.$$

**Remark 2.2.2.** *It can also be proved that a necessary condition for the continuous inversion formula to hold is that $\varphi$ has to be a mother affine wavelet. Indeed, suppose that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} (\varphi_{b,a}, g)_{L^2(\mathbb{R})} \frac{db\,da}{a^2}$ exists for all $f$ and $g$ in $L^2(\mathbb{R})$. Then, letting $f = g = \varphi$, we get*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\varphi, \varphi_{b,a})_{L^2(\mathbb{R})}|^2 \frac{db\,da}{a^2} < \infty,$$

*which can be shown to be the same as*

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

The resolution of the identity formula leads to the reconstruction formula, which says that

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} \varphi_{b,a} \frac{db\,da}{a^2}$$

for all $f$ in $L^2(\mathbb{R})$. In other words, we have a continuous inversion formula for the signal $f$ from a knowledge of its time-scale spectrum.

## 2.3 Curvelet Transforms

We begin with the frequency plane in which the polar coordinates of a point $\xi$ is denoted by $(r,\omega)$, $r>0, -\pi\leq\omega<\pi$. Let $W:(0,\infty)\to(0,\infty)$ be a function such that

$$\mathrm{supp}(W)\subset\left(\frac{1}{2},2\right) \tag{2.3}$$

and

$$\int_0^\infty W(r)^2\frac{dr}{r}=1. \tag{2.4}$$

Let $V:(-\infty,\infty)\to(0,\infty)$ be a function such that

$$\mathrm{supp}(V)\subseteq[-1,1] \tag{2.5}$$

and

$$(2\pi)^2\int_{-1}^{1}V(\omega)^2d\omega=1. \tag{2.6}$$

We call $W$ the radial window and $V$ the angular window in the frequency space $\mathbb{R}^2$.

For a fixed scale $a\in(0,a_0)$, where $a_0<\pi^2$, we define the function $\gamma_{a00}:\mathbb{R}^2\to\mathbb{C}$ by

$$\widehat{\gamma_{a00}}(r,\omega)=W(ar)V\left(\frac{\omega}{\sqrt{a}}\right)a^{3/4},\quad (r,\omega)\in\mathbb{R}^2.$$

Geometrically, the support of $\widehat{\gamma_{a00}}$ is made up of polar wedges defined by the supports of $W$ and $V$.

Let $a\in(0,a_0)$. Then for $b\in\mathbb{R}^2$ and $\theta\in[-\pi,\pi]$, the curvelet $\gamma_{ab\theta}$ generated by translation $b$ and rotation $R_\theta$ is defined by

$$\gamma_{ab\theta}(x)=\gamma_{a00}(R_\theta(x-b)),\quad x\in\mathbb{R}^2.$$

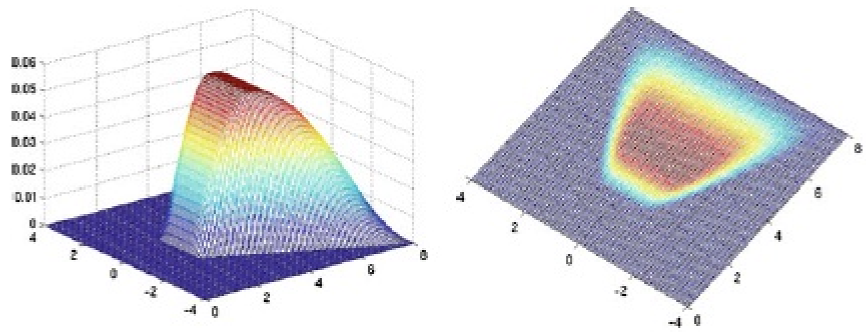

Figure 2.1: Curvelet and its support in the frequency plane

Figure 2.1 shows an example of a curvelet and its compact support.

Now for every function $f$ in $L^2(\mathbb{R}^2)$, we can define the curvelet transform $\Gamma f$ of $f$ to be the function on $(0,a_0) \times \mathbb{R}^2 \times [-\pi,\pi]$ by

$$(\Gamma f)(a,b,\theta) = (f, \gamma_{ab\theta}), \quad a \in (0,a_0), b \in \mathbb{R}^2, \theta \in [-\pi,\pi).$$

The inner product and the norm in $L^2(\mathbb{R}^2)$ are denoted by $(\ ,\ )$ and $\|\ \|$ respectively.

The following result is known as the resolution of the identity formula for high-frequency images.

**Theorem 2.3.1.** *Let $f \in L^2(\mathbb{R}^2)$ be such that*

$$\hat{f}(\xi) = 0, \quad |\xi| < 2/a_0.$$

*Then*

$$(f,g) = \int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_0}(f,\gamma_{ab\theta})(\gamma_{ab\theta},g)\,\frac{da}{a^3}\,db\,d\theta$$

*for all g in* $L^2(\mathbb{R}^2)$.

In order to have a complete resolution of the identity formula, we need to take care of the low frequencies as well. To this end, we let $\Psi$ be the function on $\mathbb{R}^2$ defined by

$$\Psi(\xi)^2 = (2\pi)^{-2}\int_0^{a_0|\xi|} W(a)^2\frac{da}{a}, \quad \xi \in \mathbb{R}^2. \tag{2.7}$$

Let $\Phi$ be a father wavelet, i.e., the nonnegative function on $\mathbb{R}^2$ such that

$$(2\pi)^2(\hat{\Phi}^2+\Psi^2) = 1. \tag{2.8}$$

If, for all $b \in \mathbb{R}^2$, we let $\Phi_b$ be the wavelet on $\mathbb{R}^2$ defined by

$$\Phi_b(x) = \Phi(x-b), \quad x \in \mathbb{R}^2,$$

then we have the following full resolution of the identity formula for curvelet transforms.

**Theorem 2.3.2.** *For all* $f \in L^2(\mathbb{R}^2)$,

$$f = \int_{\mathbb{R}^2}(f,\Phi_b)\Phi_b\,db + \int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_0}(f,\gamma_{ab\theta})\gamma_{ab\theta}\,\frac{da}{a^3}\,db\,d\theta.$$

So, the inversion formula is made up of curvelets at fine scales (high frequencies) and isotropic wavelets at coarse scales (low frequencies).

## 2.4 Radon Transforms

The Radon transform, named after an Austrian mathematician Johann Radon, was first developed in 1917 as an integral transform consisting of the integral of a function over straight lines. He also introduced the inverse transform and the Radon transform was later generalized to higher-dimensional Euclidean spaces.

The Radon transform is widely applicable to tomography. If a function $f$ represents an unknown density, the Radon transform represents the scattering data obtained as the output of a tomographic scan. The inverse of the Radon transform can be used to reconstruct the original density from the scattering data. Thus it forms the mathematical underpinning for tomographic reconstruction. The two-dimensional Radon transform is known as the X-ray transform and the three-dimensional Radon transform is used in computered tomographic scanning.

Let us look at the Radon Transform in $\mathbb{R}^n$ [35], which is defined by

$$(R_u f)(t) = \int_{\mathbb{R}^{n-1}} f(tu + sv)ds, \tag{2.9}$$

where $v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$, $u$ is a unit vector in $\mathbb{S}^{n-1}$, $t, s_1, \ldots, s_{n-1} \in \mathbb{R}$ and $\{v_1, \ldots, v_{n-1}, u\}$ forms an orthonomal basis for $\mathbb{R}^n$.

Then $\forall x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(x)dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} f(tu+sv)dsdt. \tag{2.10}$$

**Theorem 2.4.1.** *If $f \in \mathscr{S}(\mathbb{R}^n)$, then $R_u f \in \mathscr{S}(\mathbb{R})$ for each fixed u. Moreover,*

$$\widehat{R_u f}(\xi) = (2\pi)^{\frac{n-1}{2}} \hat{f}(u\xi). \tag{2.11}$$

## 2.5 Ridgelet Transforms

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a smooth univariant function with sufficient decay such that the admissibility condition

$$K_\Psi = \int_{-\infty}^{\infty} \frac{|\hat{\Psi}(\xi)|^2}{|\xi|^n} d\xi < \infty$$

holds. Such a function $\Psi$ is appropriately called a mother ridgelet.

For a point $x \in \mathbb{R}^n$, the ridgelet $\Psi_\gamma : \mathbb{R}^n \to \mathbb{R}$ generated by scale $a$, projection $u$ and translation $b$ is defined by

$$\Psi_\gamma(x) = a^{-1/2}\Psi\left(\frac{u \cdot x - b}{a}\right).$$

The phase space $\Gamma$ that is relevant to the ridgelet transforms is given by

$$\Gamma = \{\gamma = (a,u,b); a,b \in \mathbb{R}, a > 0, u \in \mathbb{S}^{n-1}\},$$

where $\mathbb{S}^{n-1}$ is the unit sphere centered at the origin in $\mathbb{R}^n$. Points in $\Gamma$ are sometimes denoted by $\gamma$ and the measure $d\gamma$ on $\Gamma$ is given by

$$d\gamma = \frac{da}{a^{n+1}} du\, db,$$

where $du$ is the surface measure of $\mathbb{S}^{n-1}$.

For $y \in \mathbb{R}$, define

$$\Psi_a(y) = a^{-1/2}\Psi(\frac{y}{a})$$

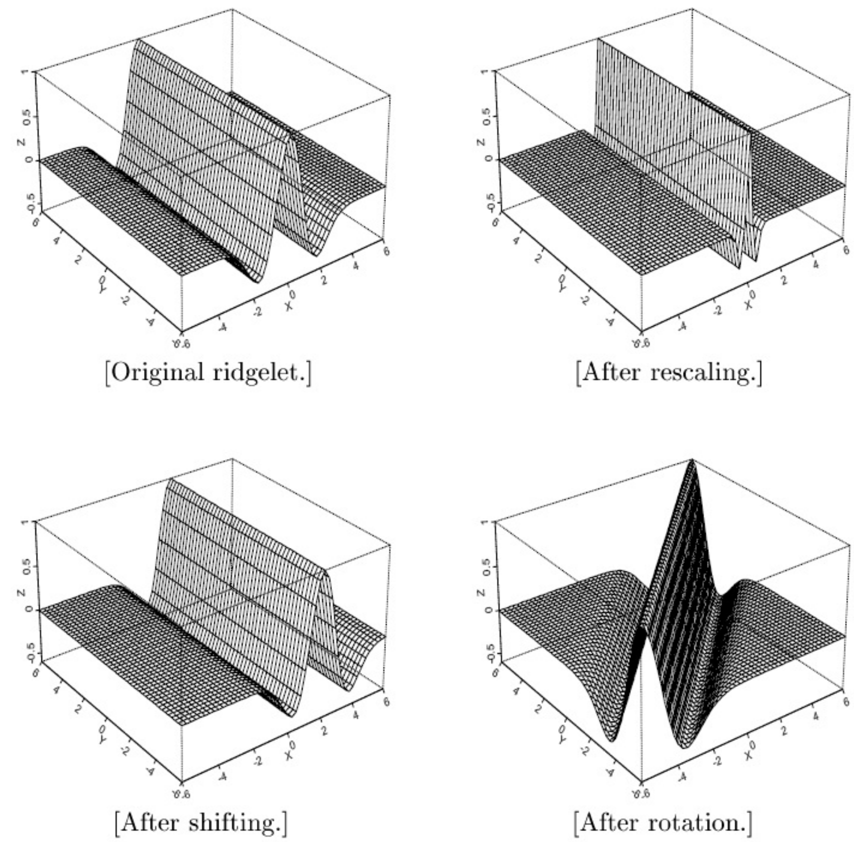


Figure 2.2: Construction of Ridgelets [1]

and

$$\tilde{\Psi}(y) = \Psi(-y).$$

The following is the resolution of identity formula for ridgelet transforms [1].

**Theorem 2.5.1.** *Let $f \in L^2(\mathbb{R}^n)$ and $\Psi$ admissible. Then*

$$(f,g) = c_\Psi \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} (f,\Psi_\gamma)(\Psi_\gamma, g) \frac{da}{a^{n+1}} dudb$$

*and*

$$||f||_2^2 = c_\Psi \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} |(f,\Psi_\gamma)|^2 \frac{da}{a^{n+1}} dudb,$$

*where $c_\Psi = (2\pi)^{-\frac{n+1}{2}} K_\Psi^{-1}$, for all $g$ in $L^2(\mathbb{R}^n)$.*

**Proof of Theorem 2.5.1**

Let

$$I = \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} (f,\Psi_\gamma)\Psi_\gamma \frac{da}{a^{n+1}} dudb.$$

We note that by using the Radon transform

$$\begin{aligned}
(f,\Psi_\gamma) &= \int_{\mathbb{R}^n} f(x)\overline{\Psi_a(u\cdot x - b)}dx \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} f(tu+vs)\Psi_a(u\cdot(tu+vs)-b)dsdt \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} f(tu+vs)\Psi_a(t-b)dsdt \\
&= \int_{-\infty}^{\infty} \Psi_a(t-b)\left\{\int_{\mathbb{R}^{n-1}} f(tu+vs)ds\right\}dt \\
&= \int_{-\infty}^{\infty} \tilde{\Psi}_a(b-t)(R_u f)(t)dt \\
&= (\tilde{\Psi}_a * R_u f)(b). \qquad (2.12)
\end{aligned}$$

Therefore, by inverse Fourier transform

$$\begin{aligned}
I &= \int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty}\frac{1}{a}(R_uf*\Psi_{-1/a})(b)\Psi_{1/a}(u\cdot x-b)\frac{da}{a^{n+1}}dudb\\
&= \int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty}\frac{1}{a}(R_uf*\Psi_{-1/a}*\Psi_{1/a})(u\cdot x)\frac{da}{a^{n+1}}du\\
&= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty}e^{i\xi u\cdot x}\frac{1}{a}(R_uf*\Psi_{-1/a}*\Psi_{1/a})^{\wedge}(\xi)\frac{da}{a^{n+1}}dud\xi\\
&= (2\pi)^{-1/2}(2\pi)(2\pi)^{\frac{n-1}{2}}\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty}e^{i\xi u\cdot x}a\hat{f}(\xi u)|\Psi(a\xi)|^2\frac{da}{a^{n+1}}dud\xi.
\end{aligned}$$

Since $K_\Psi=\int_{-\infty}^{\infty}\frac{|\hat{\Psi}(\xi)|^2}{|\xi|^n}d\xi$ and $a>0$, by Fubini's Theorem

$$\begin{aligned}
I &= (2\pi)^{-1/2}(2\pi)((2\pi)^{\frac{n-1}{2}}\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}e^{i\xi u\cdot x}\hat{f}(\xi u)\left\{\int_{-\infty}^{\infty}\frac{|\hat{\Psi}(a\xi)|^2}{|a\xi|^n}|\xi|^n da\right\}dud\xi\\
&= (2\pi)^{-1/2}(2\pi)(2\pi)^{\frac{n-1}{2}}K_\Psi\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}e^{i\xi u\cdot x}\hat{f}(\xi u)|\xi|^{n-1}dud\xi
\end{aligned}$$

Using the change of coordinates, inverse Fourier transform and

$$\hat{f}(-\xi u)=\check{f}(\xi u),$$

we have

$$\begin{aligned}
I &= (2\pi)^{-1/2}(2\pi)(2\pi)^{\frac{n-1}{2}}K_\Psi\frac{1}{2}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}e^{i\xi u\cdot x}\hat{f}(\xi u)|\xi|^{n-1}dud\xi\\
&\quad +(2\pi)^{-1/2}(2\pi)(2\pi)^{\frac{n-1}{2}}K_\Psi\frac{1}{2}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}e^{-i\xi u\cdot x}\hat{f}(-\xi u)|\xi|^{n-1}dud\xi\\
&= (2\pi)^{-1/2}(2\pi)(2\pi)^{\frac{n-1}{2}}K_\Psi\frac{1}{2}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}e^{i\xi u\cdot x}\hat{f}(\xi u)|\xi|^{n-1}dud\xi\\
&\quad +(2\pi)^{-1/2}(2\pi)(2\pi)^{\frac{n-1}{2}}K_\Psi\frac{1}{2}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}e^{-i\xi u\cdot x}\check{f}(\xi u)|\xi|^{n-1}dud\xi\\
&= (2\pi)^{-1/2}(2\pi)(2\pi)^{\frac{n-1}{2}}K_\Psi\int_{\mathbb{R}^n}e^{i\eta\cdot x}\hat{f}(\eta)d\eta\\
&= (2\pi)(2\pi)^{\frac{n-1}{2}}K_\Psi f(x).
\end{aligned}$$

Hence,

$$f(x) = c_\Psi \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} (f, \Psi_\gamma) \Psi_\gamma \frac{da}{a^{n+1}} du db,$$

where $c_\Psi = (2\pi)^{-\frac{n+1}{2}} K_\Psi^{-1}$.

## 2.6 Finite Fourier Transforms

The starting point is the additive group $\mathbb{Z}_N = \{0, 1, ..., N-1\}$, where $N$ is a positive integer greater than or equal to 2 and the group law is addition modulo $N$. It is an abelian group of order $n$ and it is cyclic, which may be viewed as the multiplicative group of $n$-th roots of unity and can be drawn as $n$ equally spaced points on a unit circle. Thus $\mathbb{Z}_N$ is a finite analog of the circle. A function $z : \mathbb{Z}_N \to \mathbb{C}$ is completely specified by $z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}$. We can think of the set of all $n$-tuples with complex entries as functions on $\mathbb{Z}_N$ and we denote it by $L^2(\mathbb{Z}_N)$. The inner product and norm in $L^2(\mathbb{Z}_N)$ are given by

$$(z, w) = \sum_{n=0}^{N-1} z(n)\overline{w(n)}$$

and

$$||z||^2 = (z, z) = \sum_{n=0}^{N-1} |z(n)|^2$$

for all $z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}$ and $w = \begin{pmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{pmatrix}$ in $L^2(\mathbb{Z}_N)$.

An obvious orthonormal basis for $L^2(\mathbb{Z}_N)$ is $\{\varepsilon_0, \varepsilon_1, ..., \varepsilon_{N-1}\}$, where

$$\varepsilon_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad m = 0, 1, ..., N-1,$$

and $\varepsilon_m$ has 1 in the $m^{th}$ position and zeros elsewhere. Another orthonormal basis for

$L^2(\mathbb{Z}_N)$ is $\{e_0, e_1, ..., e_{N-1}\}$, where

$$e_m = \begin{pmatrix} e_m(0) \\ \\ e_m(1) \\ \\ \vdots \\ \\ e_m(N-1) \end{pmatrix}, \quad m = 0, 1, ..., N-1,$$

and

$$e_m(n) = \frac{1}{\sqrt{N}} e^{2\pi imn/N}, \quad n = 0, 1, ..., N-1.$$

**Definition 2.6.1.** *Let $z \in L^2(\mathbb{Z}_N)$. Then we let $\hat{z} \in L^2(\mathbb{Z}_N)$ be defined by*

$$\hat{z} = \begin{pmatrix} \hat{z}(0) \\ \\ \hat{z}(1) \\ \\ \vdots \\ \\ \hat{z}(N-1) \end{pmatrix},$$

*where*

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi imn/N}, \quad m = 0, 1, ..., N-1.$$

*We call $\hat{z}$ the finite Fourier transform of $z$.*

Of particular importance to us is the following inversion formula,

**Theorem 2.6.2.** *Let $z$ and $\hat{z}$ be in $L^2(\mathbb{Z}_N)$. Then*

$$z(n) = \frac{1}{N}\sum_{m=0}^{N-1} \hat{z}(m) e^{2\pi i m n/N}, \quad n = 0, 1, ..., N-1.$$

To simplify the Fourier inversion formula in Theorem 2.6.2, we define

$$F_m = \begin{pmatrix} F_m(0) \\ F_m(1) \\ \vdots \\ F_m(N-1) \end{pmatrix}, \quad m = 0, 1, ..., N-1,$$

in $L^2(\mathbb{Z}_N)$, where

$$F_m(n) = \frac{1}{N} e^{2\pi i m n/N}, \quad n = 0, 1, ..., N-1. \tag{2.13}$$

Obviously, $\{F_0, F_1, ..., F_{N-1}\}$ is orthogonal, but not orthonormal in $L^2(\mathbb{Z}_N)$. Being an orthogonal set of $N$ elements in the $N$-dimensional vector space $L^2(\mathbb{Z}_N)$, $\{F_0, F_1, ..., F_{N-1}\}$ is a basis for $L^2(\mathbb{Z}_N)$ and we call it the Fourier basis for $L^2(\mathbb{Z}_N)$. By Theorem 2.6.2, we get for $k = 0, 1, ..., N-1$,

$$F_k = \sum_{m=0}^{N-1} \widehat{F_k}(m) F_m.$$

Therefore

$$\widehat{F_k}(m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases}$$

and

$$\widehat{F_k}(m) = \varepsilon_m.$$

Using the Fourier basis for $L^2(\mathbb{Z}_N)$ defined in 2.13, the Fourier inversion formula in Theorem 2.6.2 becomes

$$z = \sum_{m=0}^{N-1} \hat{z}(m) F_m. \tag{2.14}$$

## 2.7 Trace and Trace Class

The ideal scenario is that we are able to compute the eigenvalues of an operator explicitly, but in the more often cases the computation is difficult. The trace, the sum of all eigenvalues, will alternatively give us some information about the eigenvalues.

For the sake of self-containedness, we give a brief recall of the basic results on trace class operators and traces in [37,38]. Let $A$ be a compact operator from a complex, separable and infinite-dimensional Hilbert space $X$. Then $(A^*A)^{1/2}$ is a compact and positive operator on $X$, where $A^*$ is the adjoint of $A$. Let $\{\varphi_k : k = 1,2,\dots\}$ be an orthonormal basis of $X$ consisting of eigenvectors of $(A^*A)^{1/2}$, and for $k = 1,2,\dots$, let $s_k$ be the eigenvalue of $(A^*A)^{1/2}$ corresponding to the eigenvector $\varphi_k$. Then we say that $A$ is in the trace class $S_1$ if

$$\sum_{k=1}^{\infty} s_k < \infty.$$

If $A$ is in $S_1$, then for all orthonormal bases $\{\varphi_k : k = 1,2,\dots\}$ of $X$, the series $\sum_{k=1}^{\infty}(A\varphi_k, \varphi_k)$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis $\{\varphi_k : k = 1,2,\dots\}$ for $X$. We define the trace $\mathrm{tr}(A)$ by

$$\mathrm{tr}(A) = \sum_{k=1}^{\infty}(A\varphi_k, \varphi_k),$$

where $\{\varphi_k : k = 1,2,\dots\}$ is any orthonormal basis for $X$. It is a well-known result of Lidskii [20] that the trace $\mathrm{tr}(A)$ of a trace class operator $A$ is the sum of the eigenvalues of $A$, where the multiplicity of each eigenvalue is taken into account.

The following theorem can be found in [38].

**Theorem 2.7.1.** *Let $A : X \to X$ be a positive operator such that*

$$\sum_{k=1}^{\infty} (A\varphi_k, \varphi_k) < \infty$$

*for all orthonormal bases $\{\varphi_k : k = 1, 2, \ldots\}$ of $X$, where $(\,,\,)$ is the inner product in $X$. Then $A \in S_1$.*

## 2.8 The Landau-Pollak-Slepian Operator

Let $\Omega$ and $T$ be positive numbers. A signal $f \in L^2(\mathbb{R})$ is timelimited if $f(x) = 0$ for $|x| > T$. And $f$ is bandlimited if its Fourier transform $\hat{f}(\xi)$ has compact support, i.e.,

$$\hat{f}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx = 0, \quad for \ |\xi| > \Omega. \tag{2.15}$$

In mathematics, bandlimited signals are proven to be extremely smooth. They possesses derivatives of all orders. Indeed, they cannot vanish on any time interval unless they vanish everywhere. Such signals cannot start and stop, but must go on forever. This concludes that theoretically functions cannot be both band- and timelimited.

However, many real-world situations correspond to an effective band- and timelimiting. For instance, a telephone conversation has a finite time duration; at the same time the conversation is transmitted via the wire in a way that frequencies above or below a certain level are lost. Many researchers worked on how to represent a function by simutaneous bandlimiting and timelimiting, until it was solved by the work of Henry Landau, Henry Pollack and David Slepian [15, 31, 34].

Signals in $\mathbb{R}^n$ with a finite time duration transmitted over a bandlimited channel can be modeled as follows: let the linear operators $P_\Omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $Q_T : L^2(\mathbb{R}^n) \to$

$L^2(\mathbb{R}^n)$ be

$$(P_\Omega f)^\wedge(\xi) = \begin{cases} \hat{f}(\xi), & |\xi| \leq \Omega, \\ \\ 0 & |\xi| > \Omega, \end{cases} \tag{2.16}$$

and

$$(Q_T f)(x) = \begin{cases} f(x), & |x| \leq T, \\ \\ 0 & |x| > T, \end{cases} \tag{2.17}$$

for all functions $f$ in $L^2(\mathbb{R}^n)$.These two operators are proven to be self-joint projections [37].

Thus, for all functions $f$ in $L^2(\mathbb{R}^n)$, the function $Q_T P_\Omega f$ can be considered to be a time and band-limited signal. Therefore it is of interest to compare the energy $||Q_T P_\Omega f||^2_{L^2(\mathbb{R}^n)}$ of the time and band-limited signal $Q_T P_\Omega f$ with the engergy $||f||^2_{L^2(\mathbb{R}^n)}$ of the original signal $f$, so as to measure how much energy of the signal is lost after applying the time and band-limited operators on it. Using the fact that $P_\Omega$ and $Q_T$ are self-adjoint and are projections, we get

$$
\begin{aligned}
& sup\left\{\frac{||Q_T P_\Omega f||^2_{L^2(\mathbb{R}^n)}}{||f||^2_{L^2(\mathbb{R}^n)}} : f \in L^2(\mathbb{R}^n), f \neq 0\right\} \\
= & \; sup\left\{\frac{(Q_T P_\Omega f, Q_T P_\Omega f)_{L^2(\mathbb{R}^n)}}{||f||^2_{L^2(\mathbb{R}^n)}} : f \in L^2(\mathbb{R}^n), f \neq 0\right\} \\
= & \; sup\left\{\frac{(P_\Omega Q_T P_\Omega f, f)_{L^2(\mathbb{R}^n)}}{||f||^2_{L^2(\mathbb{R}^n)}} : f \in L^2(\mathbb{R}^n), f \neq 0\right\} \\
= & \; sup\left\{(P_\Omega Q_T P_\Omega f, f)_{L^2(\mathbb{R}^n)} : f \in L^2(\mathbb{R}^n), ||f||^2_{L^2(\mathbb{R}^n)} = 1\right\}. \qquad (2.18)
\end{aligned}
$$

Since $P_\Omega Q_T P_\Omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is self-adjoint,

$$
sup\left\{\frac{||Q_T P_\Omega f||^2_{L^2(\mathbb{R}^n)}}{||f||^2_{L^2(\mathbb{R}^n)}} : f \in L^2(\mathbb{R}^n), f \neq 0\right\} = ||P_\Omega Q_T P_\Omega||_{B(L^2(\mathbb{R}^n))}. \qquad (2.19)
$$

The bounded linear operator $P_\Omega Q_T P_\Omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is called the Landau-Pollak-Slepian operator [15, 31–34], which is in fact a wavelet multiplier by the following theorem in [37].

**Theorem 2.8.1.** *Let $\varphi$ be the function on $\mathbb{R}^n$ defined by*

$$
\varphi(x) = \begin{cases} \frac{1}{\sqrt{\mu(B_\Omega)}}, & |x| \leq \Omega, \\ 0 & |x| > \Omega, \end{cases} \qquad (2.20)
$$

*where $\mu(B_\Omega)$ is the volume of $B_\Omega$, and let $\sigma$ be the characteristic function on $B_T$, i.e.,*

$$
\sigma(\xi) = \begin{cases} 1, & |\xi| \leq T, \\ 0 & |\xi| > T. \end{cases} \qquad (2.21)
$$

*Then the Landau-Pollak-Slepian operator $P_\Omega Q_T P_\Omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitarily equivalent to a scalar multiple of the wavelet multiplier $\varphi T_\sigma \varphi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. In fact,*

$$P_\Omega Q_T P_\Omega = \mu(B_\Omega)\mathscr{F}^{-1}(\varphi T_\sigma \varphi)\mathscr{F}. \tag{2.22}$$

## 2.9 Wavelet Multipliers

Wavelet multipliers are generalized from the Landau-Pollak-Slepian operator, first studied systematically by M. W. Wong and his students. More details see in [14, 37].

Let $\pi : \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$(\pi(\xi)u)(x) = e^{ix\cdot\xi}u(x), \quad x, \xi \in \mathbb{R}^n, \tag{2.23}$$

for all functions $u$ in $L^2(\mathbb{R}^n)$.

The wavelet multiplier $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$((\varphi T_\sigma \bar{\varphi})u, v)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)(u, \pi(\xi)\varphi)_{L^2(\mathbb{R}^n)}(\pi(\xi)\varphi, v)_{L^2(\mathbb{R}^n)} d\xi, \tag{2.24}$$

where $u, v \in \mathscr{S}$, $\sigma$ is a function in $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, and $\varphi$ is a function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $||\varphi||_{L^2(\mathbb{R}^n)} = 1$.

The trace-class property of wavelet multipliers is given by the following theorem.

**Theorem 2.9.1.** *Let $\sigma \in L^1(\mathbb{R}^n)$, and let $\varphi$ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $||\varphi||_{L^2(\mathbb{R}^n)} = 1$. Then the wavelet multiplier $\varphi T_\sigma \bar{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in $S_p$ and*

$$||\varphi T_\sigma \bar{\varphi}||_{S_1} \leq (2\pi)^{-n} ||\sigma||_{L^1(\mathbb{R}^n)}.$$

# 3 Finite Pseudo-Differential Operators

## 3.1 Pseudo-Differential Operators

Now we look at Theorem 2.6.2 more carefully in the perspective of representation theory. Since $\mathbb{Z}_N$ is an abelian group with respect to addition modulo $N$, it follows that the irreducible and unitary representations of $\mathbb{Z}_N$ are one-dimensional. In fact, they are given by the elements in orthonormal basis $\{e_0, e_1, ..., e_{N-1}\}$ for $L^2(\mathbb{Z}_N)$, which can then be identified with $\mathbb{Z}_N$. Thus, the dual group of $\mathbb{Z}_N$ is the group $\mathbb{Z}_N$ itself. We can now give the definition of pseudo-differential operators on the group $\mathbb{Z}_N$.

Let $\sigma$ be a function on the phase space $\mathbb{Z}_N \times \mathbb{Z}_N$. Then $T_\sigma$, the pseudo-differential operator on $\mathbb{Z}_N$ corresponding to the symbol $\sigma$, is defined by

$$(T_\sigma z)(n) = \sum_{m=0}^{N-1} \sigma(n,m)\hat{z}(m)F_m(n),$$

for all $z \in L^2(\mathbb{Z}_N)$, where

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n)e^{-2\pi imn/N}, \quad m = 0, 1, ..., N-1.$$

### 3.1.1 Matrix Representations

We give the matrix of the pseudo-differential opertor $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ with respect to the Fourier basis $\{F_0, F_1, ..., F_{N-1}\}$ for $L^2(\mathbb{Z}_N)$.

For $k = 0,1,...,N-1$, we get

$$\begin{aligned}(T_\sigma F_k)(n) &= \frac{1}{N}\sum_{m=0}^{N-1}\sigma(n,m)\widehat{F_k}(m)e^{2\pi imn/N} \\ &= \frac{1}{N}\sigma(n,k)e^{2\pi ikn/N} \\ &= \sigma(n,k)F_k(n)\end{aligned}$$

for $n = 0,1,...,N-1$. Denoting the Fourier transform of $\sigma$ with respect to the first variable by $\mathscr{F}_1\sigma$, we get by Theorem 2.6.2

$$\begin{aligned}(T_\sigma F_k)(n) &= \sum_{j=0}^{N-1}\mathscr{F}_1\sigma(j,k)F_j(n)F_k(n) \\ &= \frac{1}{N^2}\sum_{j=0}^{N-1}\mathscr{F}_1\sigma(j,k)e^{2\pi i(j+k)n/N}\end{aligned}$$

for $n = 0,1,...,N-1$. Changing the summation index $j$ to $m$ by means of the equation $j+k = m$, and using the periodicity of $\sigma$ with respect to the first variable,

$$\begin{aligned}(T_\sigma F_k)(n) &= \frac{1}{N^2}\sum_{m=k}^{N-1+k}\mathscr{F}_1\sigma(m-k,k)e^{2\pi imn/N} \\ &= \frac{1}{N^2}\sum_{m=0}^{N-1}\mathscr{F}_1\sigma(m-k,k)e^{2\pi imn/N} \\ &= \frac{1}{N}\sum_{m=0}^{N-1}\mathscr{F}_1\sigma(m-k,k)F_m(n)\end{aligned}$$

for $n = 0,1,...,N-1$.

$$T_\sigma F_n = \frac{1}{N}\sum_{m=0}^{N-1}\mathscr{F}_1\sigma(m-n,n)F_m, \quad n = 0,1,...,N-1.$$

So the matrix $(T_\sigma)_F$ of the pseudo-differential operator $T_\sigma$ with respect to the Fourier basis is given by

$$
\begin{aligned}
&(T_\sigma)_F \\
&= \frac{1}{N}\begin{pmatrix}
(\mathscr{F}_1\sigma)(0-0,0) & \dots & (\mathscr{F}_1\sigma)(0-(N-1),0) \\
(\mathscr{F}_1\sigma)(1-0,1) & \dots & (\mathscr{F}_1\sigma)(1-(N-1),1) \\
\vdots & \vdots & \vdots \\
(\mathscr{F}_1\sigma)((N-1)-0,N-1) & \dots & (\mathscr{F}_1\sigma)((N-1)-(N-1)),(N-1))
\end{pmatrix} \\
&= \frac{1}{N}\begin{pmatrix}
(\mathscr{F}_1\sigma)(0,0) & (\mathscr{F}_1\sigma)(N-1,0) & \dots & (\mathscr{F}_1\sigma)(1,0) \\
(\mathscr{F}_1\sigma)(1,1) & (\mathscr{F}_1\sigma)(0,1) & \dots & (\mathscr{F}_1\sigma)(2,1) \\
\vdots & \vdots & \vdots & \vdots \\
(\mathscr{F}_1\sigma)(N-1,N-1) & (\mathscr{F}_1\sigma)(N-2,N-1) & \dots & (\mathscr{F}_1\sigma)(0,N-1)
\end{pmatrix} \\
&= \frac{1}{N}(\mathscr{F}_1\sigma(m-n,n))_{0\le m,n\le N-1}.
\end{aligned}
$$

Similarly, we give the matrix of the pseudo-differential opertor $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ with respect to the unit impulse basis $\{\varepsilon_0, \varepsilon_1, ..., \varepsilon_{N-1}\}$.

For $k = 0, 1, ..., N-1$, we get

$$\begin{aligned}
(T_\sigma \varepsilon_k)(n) &= \sum_{m=0}^{N-1} \sigma(n,m)\widehat{\varepsilon_k}(m)F_m(n) \\
&= \frac{1}{N}\sigma(n,k)\widehat{\varepsilon_k}(m)e^{2\pi imn/N}.
\end{aligned}$$

The entries of the matrix denoted by $[a_{lk}]$ is computed

$$\begin{aligned}
(T_\sigma \varepsilon_k, \varepsilon_l) &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \sigma(n,m)\widehat{\varepsilon_k}(m)F_m(n)\overline{\varepsilon_l} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \sigma(n,m)\widehat{\varepsilon_k}(m)e^{2\pi imn/N}\overline{\varepsilon_l}(n),
\end{aligned}$$

where $l$ is the row index and $k$ is the column index in the matrix.

Since $\varepsilon_k$ has 1 in the $k^{th}$ position and zeros elsewhere,

$$\begin{aligned}
\widehat{\varepsilon_k}(m) &= \sum_{n=0}^{N-1} \varepsilon_k(n)e^{-2\pi imn/N} \\
&= e^{-2\pi ikm/N}.
\end{aligned}$$

Hence, denoting the Fourier transform of $\sigma$ with respect to the second variable by $\mathscr{F}_2\sigma$

$$\begin{aligned}
a_{lk} = (T_\sigma \varepsilon_k, \varepsilon_l) &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \sigma(n,m)e^{-2\pi ikm/N}e^{2\pi imn/N}\overline{\varepsilon_l}(n) \\
&= \frac{1}{N}\sum_{m=0}^{N-1} \sigma(l,m)e^{-2\pi i(k-l)m/N} \\
&= \frac{1}{N}(\mathscr{F}_2\sigma)(l,k-l). \qquad (3.1)
\end{aligned}$$

The matrix $(T_\sigma)_{IU}$ of the pseudo-differential operator $T_\sigma$ with respect to the unit impulse basis is given by

$$(T_\sigma)_{UI}$$

$$= \frac{1}{N}\begin{pmatrix} (\mathscr{F}_2\sigma)(0,0) & (\mathscr{F}_2\sigma)(0,1) & ... & (\mathscr{F}_2\sigma)(0,N-1) \\ (\mathscr{F}_2\sigma)(1,N-1) & (\mathscr{F}_2\sigma)(1,0) & ... & (\mathscr{F}_2\sigma)(1,N-2) \\ \vdots & \vdots & \vdots & \vdots \\ (\mathscr{F}_2\sigma)(N-1,1) & (\mathscr{F}_2\sigma)(N-1,2) & ... & (\mathscr{F}_2\sigma)(N-1,0) \end{pmatrix},$$

where $l,k = 0,1,...,N-1$.

### 3.1.2 Trace of the Pseudo-Differential Operator $T_\sigma$

Using the matrices hitherto computed, we can obtain the explicit eigenvalues using MATLAB or other softwares. But we are still interested in computing the trace of a finite pseudo-differential operator in order to see that the formulas are compatible with the ones for pseudo-differential operators on $\mathbb{R}^n$ under suitable conditions on the symbols. The beauty of the finite analogs is that no restrictions on the symbols are required.

The trace of $T_\sigma$, which is independent from the choice of the bases, can be computed as follows.

**Theorem 3.1.1.** *Let $\sigma$ be a symbol in $L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$. Then the trace* $\mathrm{tr}(T_\sigma)$ *of the linear* *operator $T_\sigma$ associated with the symbol $\sigma$ is given by*

$$\operatorname{tr}(T_\sigma) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m).$$

**Proof.** Let $\{\varphi_0, \varphi_1, ..., \varphi_{N-1}\}$ be any orthonormal basis for $L^2(\mathbb{Z}_N)$. Then

$$\begin{aligned}
\operatorname{tr}(T_\sigma) &= \sum_{j=0}^{N-1}(T_\sigma\varphi_j, \varphi_j) \\
&= \sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m)\hat{z}(m)F_m(n)\bar{\varphi}_j(n).
\end{aligned}$$

Since

$$\hat{z}(m) = N(z, F_m)$$

and

$$F_m(n) = \sum_{j=0}^{N-1}(F_m, \varphi_j)\varphi_j(n),$$

$$\begin{aligned}
\operatorname{tr}(T_\sigma) &= \sum_{j=0}^{N-1}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m)N(\varphi_j, F_m)F_m(n)\bar{\varphi}_j(n) \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m)NF_m(n)\sum_{j=0}^{N-1}(\varphi_j, F_m)\bar{\varphi}_j(n) \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m)NF_m(n)\overline{\sum_{j=0}^{N-1}(F_m, \varphi_j)\varphi_j(n)} \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m)NF_m(n)\bar{F}_m(n) \\
&= N\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m)|F_m(n)|^2 \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\sigma(n,m).
\end{aligned}$$

This completes the proof.

Another way to calculate the trace of $T_\sigma$ is to sum the diagonal entries of the matrix in 3.1, i.e. when $l = k$.

$$\begin{aligned}
\operatorname{tr}(T_\sigma) &= \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sigma(l,m)e^{-2\pi i(k-l)m/N} \\
&= \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sigma(l,m).
\end{aligned}$$

Obviously, the trace is the summation of the chosen symbol.

# 4 Localization Operators for Curvelet Transforms

## 4.1 Proof of Resolution of Identity Formulas

For a proof of Theorem 2.3.1, Let $g_{a\theta}$ be the function on $\mathbb{R}^2$ defined by

$$g_{a\theta}(x) = \int_{\mathbb{R}^2} (f, \gamma_{ab\theta})\gamma_{ab\theta}(x)\, db, \quad x \in \mathbb{R}^2.$$

We want to show that

$$f(x) = \int_0^{a_0} \int_{-\pi}^{\pi} g_{a\theta}(x)\, d\theta \frac{da}{a^3}, \quad x \in \mathbb{R}^2.$$

Since

$$\gamma_{ab\theta}(x) = \gamma_{a0\theta}(x-b), \quad x \in \mathbb{R}^2, \tag{4.1}$$

we get

$$\begin{aligned} g_{a\theta}(x) &= \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}^2} f(y)\overline{\gamma_{a0\theta}(y-b)}dy \right\} \gamma_{a0\theta}(x-b)\, db \\ &= \int_{\mathbb{R}^2} \gamma_{a0\theta}(x-b)(\overline{\tilde{\gamma}}_{a0\theta} * f)(b)\, db \\ &= ((\gamma_{a0\theta} * \overline{\tilde{\gamma}}_{a0\theta}) * f)(x) \end{aligned}$$

for all $x$ in $\mathbb{R}^2$. Since

$$(\gamma_{a0\theta} * \overline{\tilde{\gamma}}_{a0\theta})^\wedge(\xi) = 2\pi|\widehat{\gamma_{a0\theta}}(\xi)|^2, \quad \xi \in \mathbb{R}^2,$$

we have

$$\widehat{g_{a\theta}}(\xi) = (2\pi)^2|\widehat{\gamma_{a0\theta}}(\xi)|^2\hat{f}(\xi), \quad \xi \in \mathbb{R}^2.$$

So, for all $\xi$ in $\mathbb{R}^2$,

$$\int_0^{a_0} \int_{-\pi}^{\pi} \widehat{g_{a\theta}}(\xi)\, d\theta\, \frac{da}{a^3} = \hat{f}(\xi)(2\pi)^2 \int_0^{a_0} \int_{-\pi}^{\pi} |\widehat{\gamma_{a0\theta}}(\xi)|^2 d\theta\, \frac{da}{a^3}$$

and we are done if we can prove that

$$(2\pi)^2 \int_0^{a_0} \int_{-\pi}^{\pi} |\widehat{\gamma_{a0\theta}}(\xi)|^2 d\theta \, \frac{da}{a^3} = 1, \quad \xi \in supp(\hat{f}).$$

Since

$$\gamma_{a0\theta}(x) = \gamma_{a00}(R_\theta x), \quad x \in \mathbb{R}^2,$$

we get

$$\widehat{\gamma_{a0\theta}}(\xi) = \widehat{\gamma_{a00}}(R_\theta \xi) = W(ar)V\left(\frac{\omega+\theta}{\sqrt{a}}\right)a^{3/4} \tag{4.2}$$

for all $\xi = (r, \omega)$ in $\mathbb{R}^2$. Therefore for all $\xi$ in $supp(\hat{f})$,

$$(2\pi)^2 \int_0^{a_0} |\widehat{\gamma_{a0\theta}}(\xi)|^2 d\theta \, \frac{da}{a^3} = (2\pi)^2 \int_0^{a_0} \int_{-\pi}^{\pi} W(ar)^2 V\left(\frac{\omega+\theta}{\sqrt{a}}\right)^2 a^{3/2} d\theta \, \frac{da}{a^3}.$$

By the admissibility condition on $V$, we get

$$(2\pi)^2 \int_{-\pi}^{\pi} V\left(\frac{\omega+\theta}{\sqrt{a}}\right)^2 d\theta = a^{1/2}.$$

So we only need to prove that

$$\int_0^{a_0} W(ar)^2 \frac{da}{a} = 1, \quad \xi \in supp(\hat{f}).$$

But for $r > 2/a_0$,

$$\int_0^{a_0} W(ar)^2 \frac{da}{a} = \int_0^{a_0 r} W(a)^2 \frac{da}{a} = \int_{1/2}^{2} W(a)^2 \frac{da}{a} = 1,$$

and the proof of Theorem 2.3.1 is complete.

For $f \in L^2(\mathbb{R}^2)$, we define the function $P_1 f$ on $\mathbb{R}^2$ by

$$(P_1 f)(x) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} (f, \gamma_{ab\theta}) \gamma_{ab\theta}(x) \frac{da}{a^3}\, db\, d\theta, \quad x \in \mathbb{R}^2.$$

Then for all $\xi$ in $\mathbb{R}^2$,

$$\widehat{P_1 f}(\xi) = \hat{f}(\xi) \int_0^{a_0} W(a|\xi|)^2 \frac{da}{a} = (2\pi)^2 \hat{f}(\xi) \Psi(\xi)^2,$$

where $\Psi$ is defined in (2.7). If we define the function $P_0 f$ on $\mathbb{R}^2$ by

$$P_0 f = f - P_1 f,$$

then

$$P_0 f = \Phi * \Phi * f,$$

where $\Phi$ is the function on $\mathbb{R}^2$ defined by (2.8). Then

$$\int_{\mathbb{R}^2} (f, \Phi_b) \Phi_b(x)\, db = (P_0 f)(x), \quad x \in \mathbb{R}^2,$$

and we get

$$f = P_0 f + P_1 f, \quad f \in L^2(\mathbb{R}^2).$$

This completes the proof of Theorem 2.3.2

## 4.2 High-Frequency Signals

For high-frequency signals, we let $L^2_{2/a_0}(\mathbb{R}^2)$ be the closed subspace of $L^2(\mathbb{R}^2)$ given by

$$L^2_{2/a_0}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : \hat{f}(\xi) = 0, |\xi| < 2/a_0\}.$$

Then for suitable measurable functions $\tau$ on the measure space $X$ given by

$$X = (0, a_0) \times \mathbb{R}^2 \times [-\pi, \pi]$$

equipped with the measure $d\mu$ given by

$$d\mu = \frac{da}{a^3} db\, d\theta,$$

we define the localization operator $L_\tau : L^2_{2/a_0}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with symbol $\tau$ by

$$(L_\tau f, g) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a, b, \theta)(f, \gamma_{ab\theta})(\gamma_{ab\theta}, g) \frac{da}{a^3} db\, d\theta \tag{4.1}$$

for all $f$ in $L^2_{2/a_0}(\mathbb{R}^2)$ and $g$ in $L^2(\mathbb{R}^2)$.

**Theorem 4.2.1.** *Let $\tau \in L^\infty(X)$. Then the localization operator*

$$L_\tau : L^2_{2/a_0}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

*with symbol $\tau$ is a bounded linear operator. Moreover,*

$$\|L_\tau\|_{B(L^2_{2/a_0}(\mathbb{R}^2), L^2(\mathbb{R}^2))} \leq \|\tau\|_{L^\infty(X)},$$

*where $\| \, \|_{B(L^2_{2/a_0}(\mathbb{R}^2), L^2(\mathbb{R}^2))}$ is the norm in the Banach space of all bounded linear operators from $L^2_{2/a_0}(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$.*

**Proof** For all $f$ in $L^2_{2/a_0}(\mathbb{R}^2)$ and $g$ in $L^2(\mathbb{R}^2)$, we get by means of the Schwarz inequality and the resolution of the identity formula in Theorem 2.3.1

$$
\begin{aligned}
& |(L_\tau f, g)| \\
= \; & \left| \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a,b,\theta)(f,\gamma_{ab\theta})(\gamma_{ab\theta},g) \frac{da}{a^3} db\, d\theta \right| \\
\leq \; & \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\tau(a,b,\theta)|\,|(f,\gamma_{ab\theta})|\,|(g,\gamma_{ab\theta})| \frac{da}{a^3} db\, d\theta \\
\leq \; & \|\tau\|_{L^\infty(X)} \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |(f,\gamma_{ab\theta})|\,|(g,\gamma_{ab\theta})| \frac{da}{a^3} db\, d\theta \\
\leq \; & \|\tau\|_{L^\infty(X)} \left\{ \int_X |(f,\gamma_{ab\theta})|^2 d\mu \right\}^{1/2} \left\{ \int_X |(g,\gamma_{ab\theta})|^2 d\mu \right\}^{1/2} \\
= \; & \|\tau\|_{L^\infty(X)} \|f\|\, \|g\|.
\end{aligned}
$$

□

Another class of symbols $\tau$ for which $L_\tau : L^2_{2/a_0}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded linear operator is provided by the following theorem.

**Theorem 4.2.2.** *Let $W$ be a radial window such that*

$$
\int_0^\infty W(s)^2 s\, ds < \infty.
$$

*Then for all $\tau \in L^1(X)$, the localization operator*

$$
L_\tau : L^2_{2/a_0}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)
$$

*with symbol $\tau$ is a bounded linear operator. Moreover,*

$$
\|L_\tau\|_{B(L^2_{2/a_0}(\mathbb{R}^2), L^2(\mathbb{R}^2))} \leq (2\pi)^{-2} \int_0^\infty W(s)^2 s\, ds\, \|\tau\|_{L^1(X)}.
$$

In order to prove Theorem 4.2.2, we use the following lemma.

**Lemma 4.2.3.** *Let $W$ be as in Theorem 4.2.2. Then for all $a$ in $(0,a_0)$ and $\theta$ in $[-\pi,\pi]$,*

$$\|\gamma_{a0\theta}\|^2 \leq (2\pi)^{-2}\int_0^\infty W(s)^2 s\,ds.$$

**Proof** Using Plancherel's theorem, polar coordinates and (4.2), we get for all $a$ in $(0,a_0)$ and $\theta$ in $[-\pi,\pi]$,

$$\begin{aligned}
\|\gamma_{a0\theta}\|^2 &= \|\widehat{\gamma_{a0\theta}}\|^2 \\
&= \int_{\mathbb{R}^2} |\widehat{\gamma_{a0\theta}}(\xi)|^2 d\xi \\
&= \int_0^\infty \int_{-\pi}^{\pi} |W(ar)|^2 \left| V\left(\frac{\omega+\theta}{\sqrt{a}}\right)\right|^2 a^{3/2} r\,dr\,d\omega \\
&= \int_0^\infty W(s)^2 a^{3/2} a^{1/2} \left(\int_{(-\pi+\theta)/\sqrt{a}}^{(\pi+\theta)/\sqrt{a}} V(\phi)^2 d\phi\right) \frac{s}{a}\frac{ds}{a} \\
&\leq \int_0^\infty W(s)^2 \left(\int_{-\infty}^{\infty} V(\phi)^2 d\phi\right) s\,ds
\end{aligned}$$

So, by (2.5) and (2.6), we have

$$\begin{aligned}
\|\gamma_{a0\theta}\|^2 &\leq \left(\int_0^\infty W(s)^2 s\,ds\right)\left(\int_{-1}^{1} V(\phi)^2 d\phi\right) \\
&= (2\pi)^{-2}\int_0^\infty W(s)^2 s\,ds
\end{aligned}$$

and this completes the proof. □

**Proof of Theorem 4.2.2** By the Schwarz inequality and Lemma 4.2.3, we have for all $f$

in $L^2_{2/a_0}(\mathbb{R}^2)$ and $g$ in $L^2(\mathbb{R}^2)$,

$$\begin{aligned}
& |(L_\tau f, g)| \\
= & \left| \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a,b,\theta)(f,\gamma_{ab\theta})(\gamma_{ab\theta},g) \frac{da}{a^3} db\, d\theta \right| \\
= & \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\tau(a,b,\theta)| \, |(f,\gamma_{ab\theta})| \, |(\gamma_{ab\theta},g)| \frac{da}{a^3} db\, d\theta \\
\leq & \|f\| \, \|g\| \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\tau(a,b,\theta)| \, \|\gamma_{a0\theta}\|^2 \frac{da}{a^3} db\, d\theta. \qquad (4.2)
\end{aligned}$$

By Plancherel's theorem and (4.1),

$$\begin{aligned}
\|\gamma_{ab\theta}\|^2 &= \|\widehat{\gamma_{ab\theta}}\|^2 = \|(T_b \gamma_{a0\theta})^\wedge\|^2 \\
&= \|M_b \widehat{\gamma_{a0\theta}}\|^2 = \|\widehat{\gamma_{a0\theta}}\|^2 \\
&= \|\gamma_{a0\theta}\|^2 \leq (2\pi)^{-2} \int_0^\infty W(s)^2 s\, ds. \qquad (4.3)
\end{aligned}$$

By (4.2) and (4.3), we get

$$|(L_\tau f, g)| \leq \|f\| \, \|g\| \|\tau\|_{L^1(X)} (2\pi)^{-2} \int_0^\infty W(s)^2 s\, ds.$$

This completes the proof.

**Remark 4.2.4.** *In the proof of Lemma 4.2.3, we use the translation $T_b f$ of a measurable function $f$ on $\mathbb{R}^2$ in the direction $b$ by*

$$(T_b f)(x) = f(x+b), \quad x \in \mathbb{R}^2.$$

*We also use the modulation $M_b f$ of $f$ by $b$ defined by*

$$(M_b f)(x) = e^{ib\cdot x} f(x), \quad x \in \mathbb{R}^2.$$

*It is well-known that for $f$ in $L^1(\mathbb{R}^2)$,*

$$(T_b f)^\wedge(\xi) = (M_b \hat{f})(\xi), \quad \xi \in \mathbb{R}^2,$$

*and*

$$(M_b f)^\wedge \xi) = (T_b \hat{f})(\xi), \quad \xi \in \mathbb{R}^2.$$

*See Proposition 3.4 of [36].*

## 4.3 Wavelet Multipliers

We begin with the following lemma on the father wavelet defined by (2.8).

**Lemma 4.3.1.** *$\Phi$ and $\hat{\Phi}$ are functions in $L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.*

**Proof** By (2.8),

$$\hat{\Phi}(\xi)^2 = (2\pi)^{-2} - \Psi(\xi)^2, \quad \xi \in \mathbb{R}^2.$$

By (2.3), (2.4) and (2.7), we see that $\Psi^2$ is a continuous function on $\mathbb{R}^2$ such that

$$\Psi(\xi)^2 = (2\pi)^{-2}, \quad |\xi| \geq 2/a_0.$$

Thus, $\hat{\Phi}^2$ is a continuous function on $\mathbb{R}^2$ such that

$$\hat{\Phi}^2(\xi) = 0, \quad |\xi| \geq 2/a_0.$$

So, $\hat{\Phi} \in L^2(\mathbb{R}^2)$, and by Plancherel's theorem, $\Phi \in L^2(\mathbb{R}^2)$. To see that $\Phi \in L^\infty(\mathbb{R}^2)$, we note that $\hat{\Phi}^2 \in C_0(\mathbb{R}^2)$ and hence $\hat{\Phi} \in C_0(\mathbb{R}^2)$. Taking the inverse Fourier transform of $\hat{\Phi}$ and invoking the Riemann–Lebesgue lemma, we see that $\Phi \in L^\infty(\mathbb{R}^2)$. $\square$

**Theorem 4.3.2.** *Let $\sigma \in L^\infty(\mathbb{R}^2)$. Then the wavelet multiplier $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by*

$$(T_\sigma f, g) = \int_{\mathbb{R}^2} \sigma(b)(f, \Phi_b)(\Phi_b, g)\, db$$

*for all $f$ and $g$ in $L^2(\mathbb{R}^2)$ is a bounded linear operator. Moreover,*

$$\|T_\sigma\|_* \leq (2\pi)^2 \|\sigma\|_{L^\infty(\mathbb{R}^2)} \|\hat{\Phi}\|^2_{L^\infty(\mathbb{R}^2)},$$

*where $\| \, \|_*$ is the norm in the $C^*$-algebra of all bounded linear operators from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$.*

**Proof** For all $f$ and $g$ in $L^2(\mathbb{R}^2)$, we get by Plancherel's formula

$$\begin{aligned} |(T_\sigma f, g)| &= \left| \int_{\mathbb{R}^2} \sigma(b)(f, \Phi_b)(\Phi_b, g)\, db \right| \\ &\leq \int_{\mathbb{R}^2} |\sigma(b)| \, |(\hat{f}, \widehat{\Phi_b})| \, |(\widehat{\Phi_b}, \hat{g})|\, db. \end{aligned} \tag{4.1}$$

Now,

$$\widehat{\Phi_b}(\xi) = (\pi(-b)\hat{\Phi})(\xi), \quad \xi \in \mathbb{R}^2,$$

where $\pi(-b)\hat{\Phi}$ is the modulation of $\hat{\Phi}$ by $b$ given by

$$(\pi(-b)\hat{\Phi})(\xi) = (M_{-b}\hat{\Phi})(\xi) = e^{-ib\cdot\xi}\hat{\Phi}(\xi), \quad \xi \in \mathbb{R}^2.$$

Let $\Omega$ be the function on $\mathbb{R}^2$ given by

$$\Omega = \frac{\hat{\Phi}}{\|\hat{\Phi}\|}.$$

Then

$$\widehat{\Phi_b}(\xi) = \|\hat{\Phi}\|(\pi(-b)\Omega)(\xi), \quad \xi \in \mathbb{R}^2.$$

We note that

$$(\hat{f}, \widehat{\Phi_b}) = \|\hat{\Phi}\|(\hat{f}, \pi(-b)\Omega) = \|\hat{\Phi}\|(\tilde{f}, T_b\hat{\Omega}),$$

where

$$\tilde{f}(x) = f(-x), \quad x \in \mathbb{R}^2.$$

So,

$$\begin{aligned}
(\hat{f}, \widehat{\Phi_b}) &= \|\hat{\Phi}\| \int_{\mathbb{R}^2} f(-x)\overline{\hat{\Omega}(x+b)}\,dx \\
&= \|\hat{\Phi}\| \int_{\mathbb{R}^2} f(x)\overline{\hat{\Omega}(b-x)}\,dx \\
&= \|\hat{\Phi}\|(f * \overline{\hat{\Omega}})(b), \quad b \in \mathbb{R}^2. \qquad (4.2)
\end{aligned}$$

Similarly,

$$(\widehat{\Phi_b}, \hat{g}) = \|\hat{\Phi}\|\overline{(g * \overline{\hat{\Omega}})(b)}, \quad b \in \mathbb{R}^2. \qquad (4.3)$$

So, by (4.1), (4.2), (4.3) and Plancherel's formula, we get

$$\begin{aligned}
|(T_\sigma f, g)| &\leq \|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\hat{\Phi}\|^2 \int_{\mathbb{R}^2} |(f * \overline{\hat{\Omega}})(b)|\,|(g * \overline{\hat{\Omega}})(b)|\,db \\
&= \|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\hat{\Phi}\|^2 \left\|f * \overline{\hat{\Omega}}\right\| \left\|g * \overline{\hat{\Omega}}\right\| \\
&= (2\pi)^2\|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\hat{\Phi}\|^2 \left\|\hat{\overline{\hat{\Omega}}}\hat{f}\right\| \left\|\hat{\overline{\hat{\Omega}}}\hat{g}\right\| \\
&= (2\pi)^2\|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\hat{\Phi}\|^2 \|\overline{\Omega}\hat{f}\|\,\|\overline{\Omega}\hat{g}\| \\
&= (2\pi)^2\|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\hat{\Phi}\|^2 \|\Omega\|^2_{L^\infty(\mathbb{R}^2)}\|\hat{f}\|\,\|\hat{g}\| \\
&= (2\pi)^2\|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\hat{\Phi}\|^2_{L^\infty(\mathbb{R}^2)}\|f\|\,\|g\|
\end{aligned}$$

for all $f$ and $g$ in $L^2(\mathbb{R}^2)$. This completes the proof. □

We can give another useful class of symbols $\sigma$ for which $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded linear operator.

**Theorem 4.3.3.** *Let $\sigma \in L^1(\mathbb{R}^2)$. Then the wavelet multiplier $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by*

$$(T_\sigma f, g) = \int_{\mathbb{R}^2} \sigma(b)(f, \Phi_b)(\Phi_b, g)\,db$$

*is a bounded linear operator. Moreover,*

$$\|T_\sigma\|_* \leq \|\sigma\|_{L^1(\mathbb{R}^2)}\|\Phi\|^2.$$

**Proof** Using the Schwarz inequality, we get for all $f$ and $g$ in $L^2(\mathbb{R})$,

$$\begin{aligned}
& |(T_\sigma f, g)| \\
= & \left| \int_{\mathbb{R}^2} \sigma(b)(f, \Phi_b)(\Phi_b, g)\, db \right| \\
\leq & \int_{\mathbb{R}^2} |\sigma(b)|\, |(f, \Phi_b)|\, |(\Phi_b, g)|\, db \\
\leq & \left\{ \int_{\mathbb{R}^2} |\sigma(b)|\, \|\Phi_b\|^2 db \right\} \|f\|\, \|g\|. \qquad (4.4)
\end{aligned}$$

But

$$\|\Phi_b\|^2 = \int_{\mathbb{R}^2} |\Phi(x-b)|^2 dx = \int_{\mathbb{R}^2} |\Phi(x)|^2 dx = \|\Phi\|^2 \qquad (4.5)$$

for all $b$ in $\mathbb{R}^2$. So, by (4.4) and (4.5),

$$|(T_\sigma f, g)| \leq \|\sigma\|_{L^1(\mathbb{R}^2)} \|\Phi\|^2 \|f\|\, \|g\|,$$

as required. □

## 4.4 Curvelet Localization Operators

The starting point is the $L^2$-boundedness of curvelet localization operators with symbols in $L^\infty(X)$.

**Theorem 4.4.1.** *Let $\tau \in L^\infty(X)$. Then the curvelet localization operator $T_\tau : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by*

$$(T_\tau f, g) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a, b, \theta)\, (f, \gamma_{ab\theta})(\gamma_{ab\theta}, g) \frac{da}{a^3} db\, d\theta$$

*for all $f$ and $g$ in $L^2(\mathbb{R}^2)$ is a bounded linear operator. Moreover,*

$$\|T_\sigma\|_* \leq \|\tau\|_{L^\infty(X)}(1+\|\hat{\Phi}\|^2_{L^\infty(\mathbb{R}^2)}).$$

**Proof** For all $f$ and $g$ in $L^2(\mathbb{R}^2)$, we get

$$\begin{aligned} |(T_\tau f,g)| &= \left|\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\tau(a,b,\theta)\,(f,\gamma_{ab\theta})\,(\gamma_{ab\theta},g)\frac{da}{a^3}db\,d\theta\right| \\ &\leq \int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\tau(a,b,\theta)|\,|(f,\gamma_{ab\theta}|\,|(\gamma_{ab\theta},g)|\frac{da}{a^3}db\,d\theta \\ &\leq \|\tau\|_{L^\infty(X)}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|(f,\gamma_{ab\theta})|\,|(\gamma_{ab\theta},g)|\frac{da}{a^3}db\,d\theta. \end{aligned} \tag{4.1}$$

Using the Schwarz inequality, we get

$$\begin{aligned} &\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|(f,\gamma_{ab\theta})|\,|(\gamma_{ab\theta},g)|\frac{da}{a^3}db\,d\theta \\ = &\left\{\int_0^{-\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|(f,\gamma_{ab\theta})|^2\frac{da}{a^3}db\,d\theta\right\}^{1/2} \\ &\times\left\{\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|(g,\gamma_{ab\theta})|^2\frac{da}{a^3}db\,d\theta\right\}^{1/2}. \end{aligned} \tag{4.2}$$

Using the full resolution of the identity formula in Theorem 2.3.2 and Plancherel's theorem, we get

$$\begin{aligned} &\int_0^{a_0}\int_{\mathbb{R}^2}\int_{-\pi}^{\pi}|(f,\gamma_{ab\theta})|^2\frac{da}{a^3}db\,d\theta \\ = &\ \|f\|^2-\int_{\mathbb{R}^2}|(f,\Phi_b)|^2db \\ = &\ \|f\|^2-\int_{\mathbb{R}^2}|(\hat{f},\pi(-b)\hat{\Phi})|^2db \\ = &\ \|f\|^2-\int_{\mathbb{R}^2}|(\hat{f},\pi(b)\hat{\Phi})|^2db. \end{aligned} \tag{4.3}$$

Since $\hat{\Phi} \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, it follows from the resolution of the identity formula for wavelet multipliers in Proposition 19.1 in [37] and Plancherel's theorem that

$$\int_{\mathbb{R}^2} |(\hat{f}, \pi(b)\hat{\Phi})|^2 db = \|\hat{\Phi}\hat{f}\|^2 \leq \|\hat{\Phi}\|^2_{L^\infty(X)} \|f\|^2. \tag{4.4}$$

Similarly,

$$\int_{\mathbb{R}^2} |(\hat{g}, \pi(b)\hat{\Phi})|^2_{L^\infty(X)} db \leq \|\hat{\Phi}\|^2_{L^\infty(X)} \|g\|^2. \tag{4.5}$$

Thus, by (4.1)–(4.5), we get

$$|(T_\sigma f, g)| \leq \|\tau\|_{L^\infty(X)} (1 + \|\hat{\Phi}\|^2) \|f\| \, \|g\|$$

for all $f$ and $g$ in $L^2(\mathbb{R}^2)$, and the proof is complete. □

**Theorem 4.4.2.** *Let $W$ be a radial window such that*

$$\int_0^\infty W(s)^2 s \, ds < \infty.$$

*Then for all $\tau \in L^1(X)$, the curvelet localization opertor $T_\tau : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by*

$$(T_\tau f, g) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a, b, \theta) \, (f, \gamma_{ab\theta}) \, (\gamma_{ab\theta}, g) \frac{da}{a^3} db \, d\theta$$

*for all $f$ and $g$ in $L^2(\mathbb{R}^2)$ is a bounded linear operator. Moreover,*

$$\|T_\sigma\|_* \leq (2\pi)^{-2} \|\tau\|_{L^1(X)} \left\{ \int_0^\infty W(s)^2 s \, ds \right\}.$$

**Proof** For all $f$ and $g$ in $L^2(\mathbb{R}^2)$, we get using the Schwarz inequality

$$\begin{aligned}
& |(T_\tau f, g)| \\
= & \left| \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a,b,\theta)\,(f, \gamma_{ab\theta})\,(\gamma_{ab\theta}, g) \frac{da}{a^3} db\, d\theta \right| \\
\leq & \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\tau(a,b,\theta)|\,|(f, \gamma_{ab\theta})|\,|(\gamma_{ab\theta}, g)| \frac{da}{a^3} db\, d\theta \\
= & \left\{ \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\tau(a,b,\theta)|\, \|\gamma_{ab\theta}\|^2 \frac{da}{a^3} db\, d\theta \right\} \|f\|\, \|g\|. \qquad (4.6)
\end{aligned}$$

By (4.1) and Lemma 4.2.3, we have

$$\|\gamma_{ab\theta}\|^2 = \|\gamma_{a0\theta}\|^2 \leq (2\pi)^{-2} \left\{ \int_0^\infty W(s)^2 s\, ds \right\}. \qquad (4.7)$$

So, by (4.6) and (4.7), we get

$$|(T_\sigma f, g)| \leq (2\pi)^{-2} \|\tau\|_{L^1(X)} \left\{ \int_0^\infty W(s)^2 s\, ds \right\} \|f\|\, \|g\|.$$

□

## 4.5 The Trace Class and the Trace

We have the following results.

**Theorem 4.5.1.** *Let $\sigma \in L^1(\mathbb{R}^2)$. Then the wavelet multiplier $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a compact operator.*

**Proof** We first assume that $\sigma$ is nonnegative. Let $\{\varphi_k : k = 1, 2, \ldots\}$ be an orthonormal basis for $L^2(\mathbb{R}^2)$. Then by Fubini's theorem and the Parseval identity,

$$\begin{aligned}\sum_{k=1}^{\infty}(T_\sigma\varphi_k,\varphi_k) &= \sum_{k=1}^{\infty}\int_{\mathbb{R}^2}\sigma(b)|(\varphi_k,\Phi_b)|^2db \\ &= \int_{\mathbb{R}^2}\sigma(b)\sum_{k=1}^{\infty}|(\varphi_k,\Phi_b)|^2db \\ &= \int_{\mathbb{R}^2}\sigma(b)\|\Phi_b\|^2db. \qquad (4.1)\end{aligned}$$

But

$$\|\Phi_b\|^2 = \|\Phi\|^2. \qquad (4.2)$$

So, by (4.1) and (4.2),

$$\sum_{k=1}^{\infty}(T_\sigma\varphi_k,\varphi_k) \le \|\sigma\|_{L^1(\mathbb{R}^2)}\|\Phi\|^2.$$

Hence by Proposition 2.3 in [37], $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is compact. For an arbitrary real-valued symbol $\sigma$ in $L^1(\mathbb{R}^2)$, we can write

$$\sigma = \sigma_+ - \sigma_-,$$

where

$$\sigma_+ = \max(\sigma, 0)$$

and

$$\sigma_- = -\min(\sigma, 0).$$

So, $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a compact operator. Finally, if $\sigma$ is an arbitrary symbol in $L^1(\mathbb{R}^2)$, we can write

$$\sigma = \mathrm{R}e\,\sigma + i\,\mathrm{I}m\,\sigma$$

and the proof is complete. $\square$

Using the same proof of Theorem 4.5.1 and Proposition 2.4 in [37], we can obtain the following result.

**Theorem 4.5.2.** *The wavelet multiplier $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is in the trace class $S_1$.*

We can estimate the trace class norm $\|T_\sigma\|_{S_1}$ of the wavelet multiplier $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ in Theorem 4.5.2.

**Theorem 4.5.3.** *Let $\sigma \in L^1(\mathbb{R}^2)$. Then*

$$\|T_\sigma\|_{S_1} \leq \|\sigma\|_{L^1(\mathbb{R}^2)} \|\Phi\|^2.$$

**Proof** By Theorem 4.5.2, $T_\sigma \in S_1$. Then using the canonical form for compact operators given in Theorem 2.2 in [37], we get

$$T_\sigma f = \sum_{k=1}^{\infty} s_k(T_\sigma)(f, \varphi_k)\psi_k, \quad f \in L^2(\mathbb{R}^2), \tag{4.3}$$

where $s_k(T_\sigma), k = 1,2,\dots,$ are the singular values of $T_\sigma : L^(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $\{\varphi_k : k = 1,2,\dots\}$ is an orthonormal basis for the orthogonal complement of the null space of $T_\sigma$, $\{\psi_k : k = 1,2,\dots\}$ is an orthonormal set in $L^2(\mathbb{R}^2)$ and the convergence of the series is in $L^2(\mathbb{R}^2)$. By (4.3), Fubini's theorem, the Schwarz inequality and Bessel's inequality,

$$\sum_{j=1}^{\infty} (T_\sigma \varphi_j, \psi_j) = \sum_{j=1}^{\infty} s_j(T_\sigma). \tag{4.4}$$

So, using (4.4),

$$\|T_\sigma\|_{S_1} = \sum_{j=1}^{\infty} (T_\sigma \varphi_j, \psi_j). \tag{4.5}$$

Therefore by (4.5),

$$\begin{aligned}
\|T_\sigma\|_{S_1} &= \sum_{k=1}^{\infty} (T_\sigma \varphi_k, \psi_k) \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} \sigma(b)(\varphi_k, \Phi_b)(\Phi_b, \psi_k)\, db \\
&\leq \sum_{k=1}^{\infty} \int_{\mathbb{R}^2} |\sigma(b)|\,|(\varphi_k, \Phi_b)|\,|(\Phi_b, \psi_k)|\, db \\
&= \int_{\mathbb{R}^2} |\sigma(b)| \sum_{k=1}^{\infty} |(\varphi_k, \Phi_b)|\,|(\Phi_b, \psi_k)|\, db \\
&\leq \int_{\mathbb{R}^2} |\sigma(b)| \left\{ \sum_{k=1}^{\infty} |(\varphi_k, \Phi_b)|^2 \sum_{k=1}^{\infty} |(\Phi_b, \psi_k)|^2 \right\}^{1/2} db \\
&\leq \int_{\mathbb{R}^2} |\sigma(b)|\, \|\Phi_b\|^2 db \\
&= \int_{\mathbb{R}^2} |\sigma(b)|\, \|\Phi\|^2 db \\
&= \|\sigma\|_{L^1(\mathbb{R}^2)} \|\Phi\|^2.
\end{aligned}$$

□

The Lidskii's formula [10,16,19] for the trace of the wavelet multiplier $T_\sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is given by the following theorem. The proof is the same as that of Theorem 19.12 in [37].

**Theorem 4.5.4.** *Let* $\sigma \in L^1(\mathbb{R}^2)$. *Then the trace* $\mathrm{tr}(T_\sigma)$ *of the wavelet multiplier* $T_\sigma$ :

$L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *is given by*

$$\mathrm{tr}(T_\sigma) = \|\Phi\|^2 \int_{\mathbb{R}^2} \sigma(b)\, db.$$

Using the same techniques, we can conclude with the following results for the trace class properties of curvelet localization operators.

**Theorem 4.5.5.** *Let $\tau \in L^1(X)$. Then the curvelet localization operator $T_\tau : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by*

$$(T_\tau f, g) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a,b,\theta)(f, \gamma_{ab\theta})(\gamma_{ab\theta}, g) \frac{da}{a^3} db\, d\theta$$

*for all $f$ and $g$ in $L^2(\mathbb{R}^2)$ is in the trace class $S_1$ and*

$$\|T_\tau\|_{S_1} \leq (2\pi)^{-2} \|\tau\|_{L^1(X)} \left\{ \int_0^\infty W(s)^2 s\, ds \right\}.$$

The following Lidskii's formula for the trace is then obvious.

**Theorem 4.5.6.** *Let $\tau \in L^1(X)$. Then the trace $\mathrm{tr}(T_\tau)$ of the curvelet localization operator $T_\tau : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is given by*

$$\mathrm{tr}(T_\tau) = \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \tau(a,b,\theta) \|\gamma_{ab\theta}\|^2 \frac{da}{a^3} db\, d\theta.$$

## 4.6 Trace Class on Closed Subspaces

Let $X$ be an infinite-dimensional, separable and complex Hilbert space in which the inner product and norm are denoted by $(\ ,\ )_X$ and $\|\ \|_X$ respectively. Let $A : X \to X$ be a bounded linear operator. Then its adjoint $A^* : X \to X$ is a bounded linear operator. Let $X_0$ be a closed subspace of $X$. The restriction of $A$ to $X_0$ results in a bounded linear operator $A_0 : X_0 \to X$. The adjoint of $A_0 : X_0 \to X$ is a bounded linear operator $A_0^* : X \to X_0$. So, for all $x$ in $X$, $A^*x$ is a bounded linear functional on $X$ given by an element in $X$ and $A_0^*x$ is a bounded linear functional on $X_0$ given by an element in $X_0$. If $A : X \to X$ is a compact operator, then so is $A_0 : X_0 \to X$ and hence $A_0^*A_0 : X_0 \to X_0$ is a compact and self-adjoint operator. Thus, the absolute value $|A_0|$ defined by

$$|A_0| = \sqrt{A_0^*A}$$

is a compact and self-adjoint operator on $X_0$. So, we can find an orthonormal basis $\{\varphi_k : k = 1,2,\ldots\}$ for $X_0$ consisting of eigenvectors of $|A_0|$. For $k = 1,2,\ldots$, let $s_k(X_0)$ be the eigenvalue of $|A_0|$ corresponding to $\varphi_k$. We say that $A_0 : X_0 \to X$ is in the trace class $S_1(X_0)$ if

$$\sum_{k=1}^{\infty} s_k(X_0) < \infty.$$

It can be proved easily that $A^*A$ is an extension of $A_0^*A_0$.

**Remark 4.6.1.** *Notwithstanding the fact that $A_0 : X_0 \to X$, is in the trace class $S_1$. the trace of the operator $A_0 : X_0 \to X$ is not defined. This is akin to the case that a rectangular matrix that is not a square matrix does not have a trace.*

**Proposition 4.6.2.** *Let $A : X \to X$ be compact. Then $|A|$ is an extension of $|A_0|$.*

**Proof** Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthonormal basis for $X_0$ consisting of eigenvectors of $|A_0|$ and let $s_k(X_0)$ be the eigenvalue of $|A_0|$ corresponding to $\varphi_k$. Since $A^*A$ is an extension of $A_0^*A_0$, we see that for $k = 1,2,\ldots,$ $\varphi_k$ is an eigenvector of $A^*A$ with $s_k(X_0)^2$ as the corresponding eigenvalue. By the spectral mapping theorem, we get

$$|A|\varphi_k = s_k(X_0)\varphi_k = |A_0|\varphi_k.$$

Let $x \in X_0$. Then

$$x = \sum_{k=1}^{\infty} (x, \varphi_k)_X \varphi_k$$

and hence

$$|A|x = \sum_{k=1}^{\infty} (x, \varphi_k)_X |A|\varphi_k = \sum_{k=1}^{\infty} (x, \varphi_k)_X |A_0|\varphi_k = |A_0|x.$$

□

**Corollary 4.6.3.** *If $A : X \to X$ is in the trace class $S_1$, then so is $A_0 : X_0 \to X$.*

**Proof** Since $|A_0|$ is compact and self-adjoint, it follows that we can find an orthonormal basis for $X_0$ consisting of eigenvectors of $|A_0|$. Let $s_k(X_0)$ be the eigenvalue of $|A_0|$ corresponding to $\varphi_k$. Then by Proposition 4.6.2, we see that for $k = 1,2,\ldots,$ $\varphi_k$ is an eigenvector of $|A|$ with $s_k(X_0)$ as the corresponding eigenvalue. Since $A$ is a trace class operator, it follows that

$$\sum_{k=1}^{\infty} s_k(X_0) \le \|A\|_{S_1} < \infty.$$

So, $A_0 : X_0 \to X$ is also a trace class operator as asserted. $\square$

We can now come back to the localization operators for high-frequency signals. The following result is an immediate consequence of Theorem 4.5.5 and Corollary 4.6.3.

**Theorem 4.6.4.** *Let $W$ be a radial window such that*

$$\int_0^\infty W(s)^2 s\, ds < \infty.$$

*Then for all $\tau \in L^1(X)$, the localization operator $L_\tau : L^2_{2/a_0}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with symbol $\tau$ is in the trace class $S_1$.*

# 5 Localization Operators for Ridgelet Transforms

## 5.1 $L^2$-Boundedness of Ridgelet Localization Operators $L_\tau$

We can now give the main result on the $L^2$-boundedness of localization operators for ridgelet transforms.

**Theorem 5.1.1.** *Let $\tau \in L^p(\Gamma)$, $1 \le p \le \infty$. Then the localization operator $L_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with symbol $\tau$ is a bounded linear operator. Moreover,*

$$\|L_\tau\|_* \le ((2\pi)^{n-1} c_\Psi \|\Psi\|^2_{L^2(\mathbb{R})})^{1/p} \|\tau\|_{L^p(\Gamma)},$$

*where $\| \, \|_*$ is the norm in the $C^*$-algebra of all bounded linear operators on $L^2(\mathbb{R}^n)$.*

**Proof** Let $\tau \in L^\infty(\Gamma)$. Then using the Schwarz inequality and the resolution of the identity formula for ridgelet transforms, we have for all $f$ and $g$ in $L^2(\mathbb{R}^n)$

$$\begin{aligned}
& |(L_\tau f, g)_{L^2(\mathbb{R}^n)}| \\
= & \left| c_\Psi \int_\Gamma \tau(\gamma)(f, \Psi_\gamma)_{L^2(\mathbb{R}^n)} (\Psi_\gamma, g)_{L^2(\mathbb{R}^n)} d\gamma \right| \\
\le & \; c_\Psi \int_\Gamma |\tau(\gamma)| \, |(f, \Psi_\gamma)_{L^2(\mathbb{R}^n)} (\Psi_\gamma, g)_{L^2(\mathbb{R}^n)}| d\gamma \\
\le & \; c_\Psi \|\tau\|_{L^\infty(\Gamma)} \int_\Gamma |(f, \Psi_\gamma)_{L^2(\mathbb{R}^n)}| \, |(\Psi_\gamma, g)_{L^2(\mathbb{R}^n)}| d\gamma \\
\le & \; c_\Psi \|\tau\|_{L^\infty(\Gamma)} \left\{ \int_\Gamma |(f, \Psi_\gamma)_{L^2(\mathbb{R}^n)}|^2 d\gamma \right\}^{1/2} \left\{ \int_\Gamma |(\Psi_\gamma, g)_{L^2(\mathbb{R}^n)}|^2 d\gamma \right\}^{1/2} \\
= & \; c_\Psi \|\tau\|_{L^\infty(\Gamma)} \left\{ \frac{\|f\|^2_{L^2(\mathbb{R}^n)}}{c_\Psi} \right\}^{1/2} \left\{ \frac{\|g\|^2_{L^2(\mathbb{R}^n)}}{c_\Psi} \right\}^{1/2} \\
= & \; \|\tau\|_{L^\infty(\Gamma)} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{5.1}$$

Now, let $\tau \in L^1(\Gamma)$. Then for all functions $f$ and $g$ in $L^2(\mathbb{R}^n)$,

$$\begin{aligned}
& |(L_\tau f, g)_{L^2(\mathbb{R}^n)}| \\
= & \left| c_\Psi \int_\Gamma \tau(\gamma)(f, \Psi_\gamma)_{L^2(\mathbb{R}^n)} (\Psi_\gamma, g)_{L^2(\mathbb{R}^n)} d\gamma \right| \\
\leq & \ c_\Psi \int_\Gamma |\tau(\gamma)| \, |(f, \Psi_\gamma)_{L^2(\mathbb{R}^n)}| \, |(\Psi_\gamma, g)_{L^2(\mathbb{R}^n)}| d\gamma. \qquad (5.2)
\end{aligned}$$

For $\gamma = (a, u, b)$, we let $\Psi_a$ be the function on $\mathbb{R}$ defined by

$$\Psi_a(y) = a^{-1/2}\Psi(y/a), \quad y \in \mathbb{R},$$

and we get

$$\begin{aligned}
& (f, \Psi_\gamma)_{L^2(\mathbb{R}^n)} \\
= & \int_{\mathbb{R}^n} f(x) \frac{1}{\sqrt{a}} \Psi_{a^{-1}}((u \cdot x) - b) \, dx \\
= & \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} f\left(tu + \sum_{j=1}^{n-1} v_j s_j\right) \frac{1}{\sqrt{a}} \Psi_{a^{-1}} \left(u \cdot \left(tu + \sum_{j=1}^{n-1} v_j s_j\right) - b\right) dv \, dt \\
= & \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} f\left(tu + \sum_{j=1}^{n-1} v_j s_j\right) \frac{1}{\sqrt{a}} \Psi_{a^{-1}}(t - b) \, dv \, dt \\
= & \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \Psi_{a^{-1}}(t - b) \left\{ \int_{\mathbb{R}^{n-1}} f\left(tu + \sum_{j=1}^{n-1} v_j s_j\right) dv \right\} dt \\
= & \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} \Psi_{-a^{-1}}(b - t)(R_u f)(t) \, dt \\
= & \ \frac{1}{\sqrt{a}} (\Psi_{-a^{-1}} * (R_u f))(b). \qquad (5.3)
\end{aligned}$$

Since

$$\left\| \frac{1}{\sqrt{a}} \Psi_{a^{-1}} \right\|_{L^2(\mathbb{R})} = \|\Psi\|_{L^2(\mathbb{R})}, \qquad (5.4)$$

it follows from (5.2)–(5.4) and the Schwarz inequality that

$$\begin{aligned}
& |(L_\tau f, g)_{L^2(\mathbb{R}^n)}| \\
\leq\ & c_\Psi \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \int_0^\infty |\tau(a,u,b)| \left| \left( \frac{1}{\sqrt{a}} \Psi_{-a^{-1}} * (R_u f) \right)(b) \right| \\
& \qquad \left| \left( \frac{1}{\sqrt{a}} \Psi_{-a^{-1}} * (R_u g) \right)(b) \right| \frac{da}{a^{n+1}} du\, db \\
=\ & c_\Psi \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \int_0^\infty |\tau(a,u,b)| \, \|R_u f\|_{L^2(\mathbb{R})} \|R_u g\|_{L^2(\mathbb{R})} \|\Psi\|^2_{L^2(\mathbb{R})} \frac{da}{a^{n+1}} du\, db \\
=\ & c_\Psi \|\Psi\|^2_{L^2(\mathbb{R})} \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \int_0^\infty |\tau(a,u,b)| \, \|R_u f\|_{L^2(\mathbb{R})} \|R_u g\|_{L^2(\mathbb{R})} \frac{da}{a^{n+1}} du\, db.
\end{aligned} \tag{5.5}$$

For all Schwartz functions $f$ on $\mathbb{R}^n$, we get by Plancherel's formula

$$\|R_u f\|^2_{L^2(\mathbb{R})} = \|\widehat{R_u f}\|^2_{L^2(\mathbb{R})}. \tag{5.6}$$

Now, we note that the restriction theorem of the Fourier transform holds to the effect that

$$\int_{-\infty}^{\infty} |\hat{f}(\xi u)|^2 d\xi \leq \int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta = \int_{\mathbb{R}^n} |f(x)|^2 dx. \tag{5.7}$$

Indeed,

$$\int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \left| \hat{f}\left( \xi u + \sum_{j=1}^{\infty} v_j s_j \right) \right|^2 d\xi\, dv.$$

Then there exists a vector $v$ in $\mathbb{R}^{n-1}$ such that

$$\int_{-\infty}^{\infty} \left| \hat{f}\left( \xi u + \sum_{j=1}^{n-1} v_j s_j \right) \right|^2 d\xi \leq \int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta.$$

For if this is not true, then for all $v \in \mathbb{R}^{n-1}$,

$$\int_{-\infty}^{\infty} \left| \hat{f}\left( \xi u + \sum_{j=1}^{n-1} v_j s_j \right) \right|^2 d\xi > \int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta &= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \left| \hat{f}\left(\xi u + \sum_{j=1}^{n-1} v_j s_j\right)\right|^2 d\xi\, dv \\ &> \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta\, dv = \infty,\end{aligned}$$

which is a contradiction. So, there exists a $v$ in $\mathbb{R}^{n-1}$ such that

$$\int_{-\infty}^{\infty} |(M_{-\sum_{j=1}^{n-1} v_j s_j} f)^{\wedge}(\xi u)|^2 d\xi \le \int_{\mathbb{R}^n} |\hat{f}(\eta)|^2 d\eta = \int_{\mathbb{R}^n} |f(x)|^2 dx$$

and hence

$$\int_{-\infty}^{\infty} |\hat{f}(\xi u)|^2 d\xi \le \int_{\mathbb{R}^n} |(M_{\sum_{j=1}^{n-1} v_j s_j} f)(x)|^2 dx = \int_{\mathbb{R}^n} |f(x)|^2 dx,$$

where

$$(M_{\pm \sum_{j=1}^{n-1} v_j s_j} f)(x) = e^{\pm ix\cdot \sum_{j=1}^{n-1} v_j s_j} f(x), \quad x \in \mathbb{R}^n.$$

By Theorem 2.4.1 and (5.7),

$$\begin{aligned}\|R_u f\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |\widehat{R_u f}(\xi)|^2 d\xi \\ &= (2\pi)^{n-1} \int_{-\infty}^{\infty} |\hat{f}(\xi u)|^2 d\xi \\ &\le (2\pi)^{n-1} \|\hat{f}\|^2_{L^2(\mathbb{R}^n)} \\ &= (2\pi)^{n-1} \|f\|^2_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{5.8}$$

Thus, by (5.5) and (5.8),

$$|(L_\tau f, g)_{L^2(\mathbb{R})}| \le (2\pi)^{n-1} c_\Psi \|\Psi\|^2_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \|\tau\|_{L^1(\Gamma)}$$

and hence

$$\|L_\tau f\|_{L^2(\mathbb{R}^n)} \leq (2\pi)^{n-1} c_\Psi \|\Psi\|^2_{L^2(\mathbb{R})} \|\tau\|_{L^1(\Gamma)} \|f\|_{L^2(\mathbb{R}^n)}.$$

In order to complete the proof, we need the Riesz–Thorin theorem [39].

**Theorem 5.1.2.** *(The Riesz-Thorin Theorem) Let $(X,\mu)$ be a measure space and $(Y,\nu)$ be $\sigma$-finite measure space. Let $T$ be a linear transformation with domain $\mathscr{D}$ consisting of all $\mu$-simple functions $f$ on $X$ such that*

$$\mu\{s \in X : f(s) \neq 0\} < \infty$$

*and such that the range of $T$ is contained in the set of all $\nu$-measurable functions on $Y$. Suppose that $\alpha_1, \alpha_2, \beta_1$, and $\beta_2$ are real numbers in $[0,1]$ and there exist positive constants $M_1$ and $M_2$ such that*

$$||Tf||_{L^{\frac{1}{\beta_j}}(Y)} \leq M_j ||f||_{L^{\frac{1}{\alpha_j}}(X)}, \quad f \in \mathscr{D}, \quad j = 1,2.$$

*Then, for $0 < \theta < 1$,*

$$\alpha = (1-\theta)\alpha_1 + \theta\alpha_2,$$

*and*

$$\beta = (1-\theta)\beta_1 + \theta\beta_2,$$

*we have*

$$||Tf||_{L^{\frac{1}{\beta}}(Y)} \leq M_1^{1-\theta} M_2^{\theta} ||f||_{L^{\frac{1}{\alpha}}(X)} \quad f \in \mathscr{D}.$$

By using the Riesz–Thorin theorem, we get

$$\begin{aligned}\|L_\tau f\|_{L^2(\mathbb{R}^n)} &\leq ((2\pi)^{n-1}c_\Psi\|\Psi\|^2_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R}^n)})^{1/p}\|f\|^{1-(1/p)}_{L^2(\mathbb{R}^n)}\|\tau\|_{L^p(\Gamma)}\\ &= ((2\pi)^{n-1}c_\Psi\|\Psi\|^2_{L^2(\mathbb{R})})^{1/p}\|\tau\|_{L^p(\Gamma)}\|f\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

and the proof is complete. □

## 5.2 Trace Class Localization Operators $\varphi L_\tau\bar{\varphi}$

In order to investigate the trace properties of the localization operators for ridgelet transforms, we introduce a new localization operator $\varphi L_\tau\bar{\varphi}$. Let $\tau\in L^1(\Gamma)$ and let $\varphi\in\mathscr{S}(\mathbb{R}^n)$ be real, a localization operator based on ridgelet transforms $\varphi L_\tau\varphi$ is defined as

$$\begin{aligned}(\varphi L_\tau\bar{\varphi}f,g) &= (L_\tau\bar{\varphi}f,\bar{\varphi}g)\\ &= c_\Psi\int_\Gamma\tau(a,u,b)(\bar{\varphi}f,\Psi_\gamma)(\Psi_\gamma,\bar{\varphi}g)d\gamma\\ &= c_\Psi\int_\Gamma\tau(a,u,b)(f,\varphi\Psi_\gamma)(\varphi\Psi_\gamma,g)d\gamma.\end{aligned}$$

These non-self-adjoint operators with trace are reminiscent of the Landau–Pollak–Slepian operators [15, 31–34] and wavelet multipliers [6, 8, 14, 40, 41].

**Theorem 5.2.1.** *Let $\tau\in L^p(\Gamma)$, $1\leq p\leq\infty$. Then for all functions $\varphi$ in the Schwartz space $\mathscr{S}$, the localization operator $\varphi L_\tau\overline{\varphi}: L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is a bounded linear operator and*

$$\|\varphi L_\tau\overline{\varphi}\|_*\leq\|\varphi\|^2_{L^\infty(\mathbb{R}^n)}((2\pi)^{n-1}c_\Psi\|\Psi\|^2_{L^2(\mathbb{R})})^{1/p}\|\tau\|_{L^p(\Gamma)}.$$

**Theorem 5.2.2.** *Let $\tau/a \in L^1(\Gamma)$. Then for all functions $\varphi$ in $\mathscr{S}$, the localization operator $\varphi L_\tau \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a trace class operator and*

$$\mathrm{tr}(\varphi L_\tau \overline{\varphi}) = c_\Psi \int_\Gamma \tau(\gamma) \|\varphi \Psi_\gamma\|^2_{L^2(\mathbb{R}^n)} d\gamma.$$

**Proof** We first assume that $\tau$ is a nonnegative real-valued function. Then for all functions $f$ in $L^2(\mathbb{R}^n)$,

$$\begin{aligned}
&(\varphi L_\tau \overline{\varphi} f, f)_{L^2(\mathbb{R}^n)} \\
= \;& (L_\tau \overline{\varphi} f, \overline{\varphi} f)_{L^2(\mathbb{R}^n)} \\
= \;& c_\Psi \int_\Gamma \tau(\gamma) |(\overline{\varphi} f, \Psi_\gamma)_{L^2(\mathbb{R}^n)}|^2 d\gamma \\
= \;& c_\Psi \int_\Gamma \tau(\gamma) |(f, \varphi \Psi_\gamma)_{L^2(\mathbb{R}^n)}|^2 d\gamma.
\end{aligned}$$

Since for $\gamma = (a, u, b)$,

$$\begin{aligned}
&|(f, \varphi \Psi_\gamma)_{L^2(\mathbb{R}^n)}|^2 \\
= \;& \left| \int_{\mathbb{R}^n} f(x) \overline{\varphi}(x) \frac{1}{\sqrt{a}} \Psi\left( \frac{(u \cdot x) - b}{a} \right) dx \right|^2 \\
= \;& \frac{1}{a} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} f\left( tu + \sum_{j=1}^{n-1} v_j s_j \right) \overline{\varphi}\left( tu + \sum_{j=1}^{n-1} v_j s_j \right) \Psi\left( \frac{t-b}{a} \right) dv\, dt \right|^2 \\
\le \;& \frac{1}{a} \|\Psi\|^2_{L^\infty(\mathbb{R})} \left( \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} \left| f\left( tu + \sum_{j=1}^{n-1} v_j s_j \right) \overline{\varphi}\left( tu + \sum_{j=1}^{n-1} v_j s_j \right) \right| dv\, dt \right)^2 \\
= \;& \frac{1}{a} \|\Psi\|^2_{L^\infty(\mathbb{R})} \left( \int_{\mathbb{R}^n} |f(x) \overline{\varphi}(x)| dx \right)^2 \\
\le \;& \frac{1}{a} \|\Psi\|^2_{L^\infty(\mathbb{R})} \|f\|^2_{L^2(\mathbb{R}^n)} \|\varphi\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Therefore

$$(\varphi L_\tau \overline{\varphi} f, f)_{L^2(\mathbb{R}^n)} \geq 0, \quad f \in L^2(\mathbb{R}^n).$$

Thus, $\varphi L_\tau \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a positive operator. Now, let $\{\varphi_k : k = 1, 2, \dots\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then

$$\begin{aligned}
& \sum_{k=1}^{\infty} (\varphi L_\tau \overline{\varphi} \varphi_k, \varphi_k)_{L^2(\mathbb{R}^n)} \\
= \; & \sum_{k=1}^{\infty} (L_\tau \overline{\varphi} \varphi_k, \overline{\varphi} \varphi_k)_{L^2(\mathbb{R}^n)} \\
= \; & c_\Psi \int_\Gamma \tau(\gamma) \sum_{k=1}^{\infty} |(\overline{\varphi} \varphi_k, \Psi_\gamma)_{L^2(\mathbb{R}^n)}|^2 d\gamma \\
= \; & c_\Psi \int_\Gamma \tau(\gamma) \sum_{k=1}^{\infty} |(\varphi_k, \varphi \Psi_\gamma)_{L^2(\mathbb{R}^n)}|^2 d\gamma \\
= \; & c_\Psi \int_\Gamma \tau(\gamma) \|\varphi \Psi_\gamma\|^2_{L^2(\mathbb{R}^n)} d\gamma \\
\leq \; & c_\Psi \left( \int_\Gamma \frac{\tau(\gamma)}{a} d\gamma \right) \|\Psi\|^2_{L^\infty(\mathbb{R})} \|\varphi\|^2_{L^2(\mathbb{R}^n)} < \infty.
\end{aligned}$$

Now let $\tau$ be a complex-valued function in $L^1(\Gamma)$. Let $\tau = \tau_1 + i\tau_2$. Write

$$\tau_1^+ - \tau_1^-$$

and

$$\tau_2^+ - \tau_2^-,$$

where for $j = 1, 2$,

$$\tau_j^+ = \max(\tau_j, 0)$$

and

$$\tau_j^- = -\min(\tau_j, 0),$$

we can conclude that $\varphi L_\tau \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a trace class operator. Finally, let $\{\varphi_k : k = 1, 2, \dots\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then

$$\begin{aligned}\mathrm{tr}(\varphi L_\tau \overline{\varphi}) &= c_\Psi \int_\Gamma \tau(\gamma) \sum_{k=1}^{\infty} |(\varphi_k, \varphi \Psi_\gamma)_{L^2(\mathbb{R}^n)}|^2 d\gamma \\ &= c_\Psi \int_\Gamma \tau(\gamma) \|\varphi \Psi_\gamma\|^2_{L^2(\mathbb{R}^n)} d\gamma,\end{aligned}$$

as asserted. $\square$

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