THETA MAPS FOR COMBINATORIAL HOPF ALGEBRAS

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Abstract

This thesis introduces a way to generalize of peak algebra. There are several equivalent definitions for the peak algebra. Stembridge describes it via enriched P-partitions to generalize marked shifted tableaux and Schur's $\mathcal{Q}$ functions. Nyman shows that it is a the sum of permutations with the same peak set. Aguiar, Bergeron and Sottile show that the peak algebra is the odd Hopf sub-algebra of quasi-symmetric functions using their theory of combinatorial Hopf algebras.

In all these cases, there is a very natural and well-behaved Hopf algebra morphism from quasi-symmetric functions or non-commutative symmetric functions to their respective peak algebra, which we call the theta map. This thesis focuses on generalizing the peak algebra by constructing generalized theta maps for an arbitrary combinatorial Hopf algebra.

The motivating example of this thesis is the Malvenuto-Reutenauer Hopf algebra of permutations. Our main result is a combinatorial description of all of the theta maps of this Hopf algebra whose images are generalizations of the peak algebra. We also give a criterion to check whether a map is a theta map, and we find theta maps for Hopf sub-algebras of quasi-symmetric functions. We also show the existence of theta maps for any commutative and cocommutative Hopf algebras. From there, we study the diagonally symmetric functions and diagonally quasi-symmetric functions. Lastly, we describe theta maps for a Hopf algebra $\mathcal{V}$ on permutations.

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Contents

Abstract **ii**

Acknowledgement **iii**

Table of Contents **iv**

1 Introduction **1**

2 Background **15**

2.1 Hopf algebras 15

2.1.1 The Hopf algebra of symmetric functions 19

2.1.2 The Hopf algebra of quasi-symmetric functions 25

2.1.3 The Hopf algebra of non-commutative symmetric functions 32

2.1.4 Malvenuto-Reutenauer Hopf algebra of permutations . 35

2.1.5 Vargas Hopf algebra of permutations 38

2.2 Combinatorial Hopf algebras 40

2.2.1 Odd Hopf sub-algebras 42

2.3 Theta maps 43

3 The odd Hopf subalgebras for combinatorial Hopf algebras 47

3.1 A Strategy for finding $S_{-}(\mathcal{H}, \zeta)$ 52

3.2 The odd Hopf subalgebra of $\mathfrak{S}\mathsf{Sym}$ 53

3.3 The odd Hopf subalgebra of $\mathcal{V}$ 55

4 Theta maps for combinatorial Hopf algebras 58

4.1 Theta map for NSym in the immaculate basis 60

4.2 Theta maps for Hopf subalgebras of QSym 77

4.3 Theta maps for commutative and co-commutative Hopf algebras 79

4.3.1 Theta maps for diagonally symmetric functions 82

4.3.2 Theta maps for diagonally quasi-symmetric functions . 87

4.4 Theta maps for $\mathcal{V}$ 90

4.5 Theta maps for Malvenuto-Reutenauer Hopf algebra 92

4.6 Convolutions of theta maps 102

A Diagonally quasi-symmetric functions 107

A.1 The F basis 110

A.2 The G basis 111

A.3 The Hilbert Basis 119

A.4 Finitely many variables case 123

Bibliography **128**

Chapter 1

Introduction

Let $\Bbbk[[\mathbf{x}]]$ be the set of power series in variables $\mathbf{x} = x_1, x_2, x_3, \dots$ of bounded degrees. The symmetric functions are the set of invariant power series under permutation of variables i.e. the set of elements f such that

$$f(x_1, x_2, x_3, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)} \dots)$$

for all bijections $\sigma : \{1, 2, \dots\} \to \{1, 2, \dots\}$ that fix all but finitely many numbers. Bases elements of the space of symmetric functions are indexed by partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The monomial symmetric function m_λ is the sum of the monomials in the orbit of $x_1^{\lambda_1} \dots x_k^{\lambda_k}$, e.g.

$$m_{21}(\mathbf{x}) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x1 x_3^2 + x_2^2 x_3 + \cdots$$

Symmetric functions inherit the regular multiplication from power series, and they have a co-multiplication given as follows

$$f(\mathbf{x}) \mapsto f(\mathbf{x},\mathbf{y}) = \sum_{f_1,f_2} f_1(\mathbf{x}) f_2(\mathbf{y}) \mapsto \sum_{f_1,f_2} f_1 \otimes f_2.$$

For example,

$$\begin{aligned} m_{21}(\mathbf{x},\mathbf{y}) &= x_1^2 x_1 + x_1 x_2^2 + \cdots + x_1^2 y_1 + \cdots + x_1 y_1^2 + \cdots + y_1 y_2^2 + \cdots \\ &= m_{21}(\mathbf{x}) + m_2(\mathbf{x}) m_1(\mathbf{y}) + m_1(\mathbf{x}) m_2(\mathbf{y}) + m_{21}(\mathbf{y}) \end{aligned}$$

The symmetric functions can be shown to satisfy the conditions of a Hopf algebra. The symmetric functions are core objects in algebraic combinatorics. They are isomorphic as a Hopf algebra to the class functions of the symmetric groups via Frobenius characteristic map [44]. They also represent special Schubert classes in the colomology of flag varieties and Grassmanians [39].

These applications lead to the fundamental basis of symmetric functions, known as the Schur basis $\{s_\lambda\}$. This is a self-dual basis of Sym i.e. we have a bilinear form $\langle -,- \rangle : \mathsf{Sym} \times \mathsf{Sym} \to \Bbbk$, called Hall inner product, such that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ where δ is the Kronecker delta. In the other words, we have an isomorphism between Sym and its dual space $I_{\mathsf{Sym}} : \mathsf{Sym} \to \mathsf{Sym}^*$ that maps $s_\lambda \mapsto s_\lambda^*$. This is an Hopf isomorphism that preserves product and coproduct.

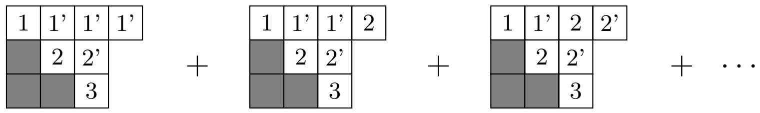
The Schur function s_λ is the generating function for the semi-standard Young tableaux of shape λ i.e. tableaux whose rows are weakly increasing and columns are strictly increasing. For example,

$$s_{21}(\mathbf{x}) = \begin{array}{ccccccccc} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array} & + & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array} & + & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} & + & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} & + & \cdots \\ x_1^2 x_2 & + & x_1 x_2^2 & + & x_1 x_2 x_3 & + & x_1 x_2 x_3 & + & \cdots \end{array}$$

There are combinatorial rules for the product of a Schur with a Schur function (Littlewood-Richardson rule), Schur function with a complete symmetric function (Pieri rule), Schur function with a power sum generator (Murnaghan-Nakayama rule) [25]. The structure constants for these products are weighted sums of tableaux with certain properties. And these can be restated as the product formula for irreducible characters of symmetric groups, intersection formula for certain Schubert cells, etc.

In 1911, Schur defines the $\mathcal{Q}$ functions in the study of projective representations of symmetric groups and alternating groups [47]. The $\mathcal{Q}$ functions show great importance in several different contexts including representation theory of Lie superalgebras, certain cohomology classes, and they play a similar role in projective representations of symmetric groups as the Schur functions do in representation theory of symmetric groups [31, 42]. The $\mathcal{Q}$ functions are also known to play the role of Schur functions in Hecke-Clifford algebra i.e. the $\mathcal{Q}$ functions correspond to simple Hecke-Clifford modules [33].

Combinatorially, the $\mathcal{Q}$ functions are indexed by strict partitions $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$. They are generating functions for marked shifted tableaux [44, 54] i.e. tableaux that are filled with $\{1, 1', 2, 2', \ldots\}$ such that each x (x') appears at most once in each row (column), e.g.

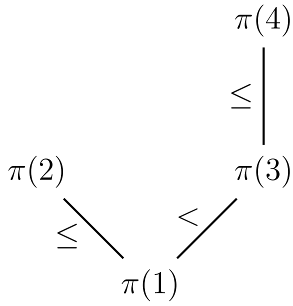
$\mathcal{Q}_{421} =$ 

$$x_1^4 x_2^2 x_3 \quad + \quad x_1^3 x_2^3 x_3 \quad + \quad x_1^2 x_2^4 x_3 \quad + \quad \cdots$$

The Schur's $\mathcal{Q}$ functions are known satisfy analogous Pieri rule and Littlewood-Richardson rule.

Consider the algebra morphism $\Theta_{\mathsf{Sym}} : \mathsf{Sym} \to \mathsf{Sym}$, which maps $s_n \mapsto \mathcal{Q}_n$ and extend the map multiplicatively. This completely defines the map since $\{s_n\}$ is a set of generators of Sym. We call this map the theta map for symmetric functions. It is not only an algebra morphism, but also a Hopf morphism that is self-adjoint with respect to the Hall scalar product. In the language of combinatorial Hopf algebra, it is the unique lifting of the odd character of ζ_{Sym} [5]. We will explain later in detail. This is our starting point of the theory of theta maps.

The $\mathcal{P}$-partitions and quasi-symmetric functions (QSym) are introduced as one of the most important generalizations of semi-standard Young tableaux and symmetric functions [20, 49]. They are generating functions for partially ordered sets whose covering relations are labeled with $<$ or $\leq$ that satisfy certain transitivity properties ($\mathcal{P}$-partitions can be equivalently defined as poset $\mathcal{P}$ with labeled vertices). For example, consider the labeled poset $\mathcal{P}$

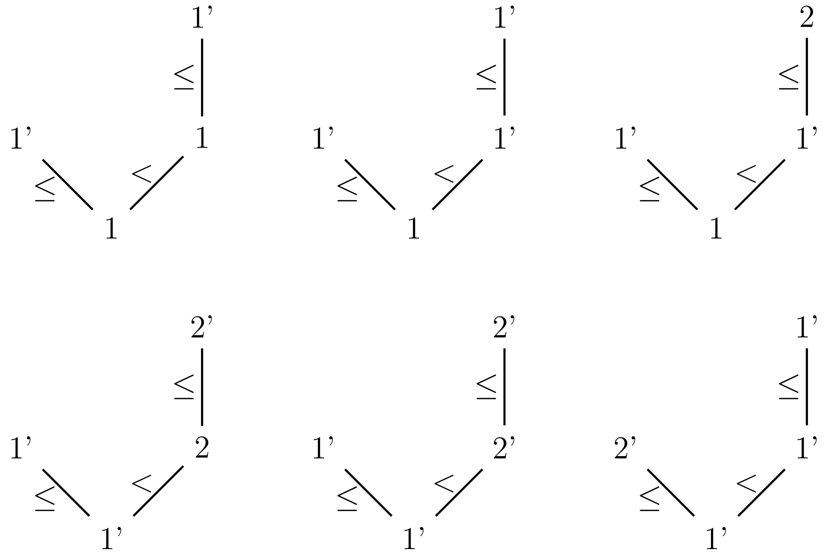


$$L_{\mathcal{P}} = \sum_{\mathcal{P}\text{-partitions } \pi} x^{wt(\pi)} = \sum_{\pi(1)\leq\pi(2),\pi(1)<\pi(3)\leq\pi(4)} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}$$
$$= x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^3 + x_1x_2^2x_3 + x_1x_2^2x_4 + \cdots$$

The linear space spanned by $\mathcal{P}$-partitions is the space of quasi-symmetric functions (QSym). The quasi-symmetric functions can be multiplied as power series, and co-multiplied via introducing a new set of variables $x_1 < x_2 < \cdots < y_1 < y_2 < \cdots$ similar to the way that the coproduct is defined on Sym. Then, QSym also form a Hopf algebra. Note that semi-standard Young tableaux can be views as a partially ordered set whose rows are labeled with $\leq$ and columns are labeled with $<$. We have an embedding ι : Sym $\to$ QSym. In Stanely's thesis, he gives a fundamental lemma that states that $\{L_{\mathcal{P}} : \mathcal{P} \text{ is totally ordered}\}$ forms a basis of QSym. Hence, in the degree n component of QSym, the basis elements are indexed by the places where $<$ occur, i.e. subsets of $\{1, 2, \ldots, n-1\}$.

In the groundbreaking paper by Stembridge [51], he defines enriched $\mathcal{P}$-partitions and the peak algebra that are generalizations of marked shifted tableaux and Schur's $\mathcal{Q}$ functions. An enriched $\mathcal{P}$-partition is a filling of a labeled poset with 1 < 1'< 2 < 2'< $\cdots$ such that no x'< x' or $x \leq x$ occurs.

Then, $K_{\mathcal{P}}$ is the generating function for enriched $\mathcal{P}$-partitions where both i and i' contribute to x_i. For example, consider the same poset $\mathcal{P}$ in the previous example of $\mathcal{P}$-partitions, the following are samples of enriched $\mathcal{P}$ partitions



$$K_{\mathcal{P}} = \sum_{\text{enriched } \mathcal{P}\text{-partitions } \pi} x^{wt(\pi)} = 2x_1^4 + 2x_1^3x_2 + 2x_1^2x_2^2 + \cdots$$

In the paper of Stembridge, he shows that the linear span of $\{K_{\mathcal{P}}\}$, denoted by Π, is spanned by $\{K_{\mathcal{P}} : \mathcal{P}$ is totally ordered$\}$. Moreover, for two totally ordered sets $\mathcal{P}$ and $\mathcal{P}'$, $K_{\mathcal{P}} = K_{\mathcal{P}'}$ if and only if the places of occurrence of intervals of length 2 with labels $a \leq b < c$ in $\mathcal{P}$ and $\mathcal{P}'$ are the same. Therefore, a basis of Π is indexed by intervals of length 2 with labels $a \leq b < c$, which is called *peak*. Hence, Π is called the peak algebra, indexed by the peak sets, and the dimension of the degree n component is the n-th Fibonacci number.

Consider the natural map $\displaystyle\sum_{\mathcal{P}\text{-partitions } \pi} x^{wt(\pi)}$ to $\displaystyle\sum_{\text{enriched } \mathcal{P}\text{-partitions } \pi} x^{wt(\pi)}$ i.e.

$L_{\mathcal{P}} \mapsto K_{\mathcal{P}}$ for any labeled poset $\mathcal{P}$. This map is called the theta map for QSym, $\Theta_{\mathsf{QSym}} : \mathsf{QSym} \to \mathsf{QSym}$, as θ is the symbol Stembridge uses. This map is compatible with Θ_{Sym} via the inclusion map $\iota : \mathsf{Sym} \to \mathsf{QSym}$. Moreover, Θ_{QSym} is a Hopf morphism and Π is a Hopf algebra [16]. Further more, the dual of the image of Θ_{QSym}, is the image of the dual map of Θ_{QSym}. In [17], the authors give the connection between peak algebra and the representation theory of the 0-Hecke-Clifford algebras, analoguous to the isomorphism between QSym and representation theory of 0-Hecke algebra [34]. In the language of combinatorial Hopf algebras, Θ_{QSym} is the unique lifting of the odd character of ζ_{QSym} [5].

The graded dual of QSym is the non-commutative symmetric functions (NSym) [22]. We will explain it in terms of Malvenuto-Reutenauer Hopf algebra of permutations ($\mathfrak{S}\mathsf{Sym}$) [40]. The Hopf algebra $\mathfrak{S}\mathsf{Sym}$ live in the ring $\Bbbk\langle X_1, X_2, \dots \rangle$ of power series of bounded degree in non-commuting variables. As a vector space, $\mathfrak{S}\mathsf{Sym}$ has basis indexed by permutations as follows, for each permutation $\sigma \in \mathfrak{S}_n$, we define

$$F_\sigma = \sum_{std(\omega_1, \omega_2, \dots, \omega_n) = \sigma^{-1}} X_{\omega_1} X_{\omega_2} \cdots X_{\omega_n}$$

where *std* is the standardization of a word to make it a permutation. Let $std(\omega) = \sigma$ for some word ω and permutation σ, then for any $i < j$, $\omega_i \leq \omega_j$ if $\sigma_i < \sigma_j$; $\omega_i > \omega_j$ if $\sigma_i > \sigma_j$.

For example,

$$\begin{aligned} F_{231} &= \sum_{std(\omega_1,\omega_2,\omega_3)=312} X_{\omega_1}X_{\omega_2}X_{\omega_3} \\ &= \sum_{\omega_2\le\omega_3<\omega_1} X_{\omega_1}X_{\omega_2}X_{\omega_3} \\ &= X_2X_1X_1 + X_3X_1X_1 + X_3X_1X_2 + X_3X_2X_2 + \cdots \end{aligned}$$

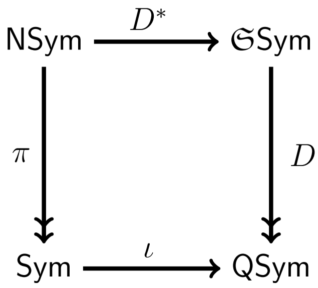
With the same method of introducing a new set of variables, $\mathfrak{S}\mathsf{Sym}$ has a co-multiplication structure and it forms a Hopf algebra. Moreover, by making the variables commutative, we obtain a Hopf morphism $D : \mathfrak{S}\mathsf{Sym} \to \mathsf{QSym}$. It is not hard to see that the image of F_σ only depends on the descent set of σ, $Des(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$. In fact, $D(F_\sigma) = L_{Des(\sigma)}$. Hence, D is frequently referred to as the descent map.

Dually, the graded dual of QSym, i.e. NSym, is a Hopf sub-algebra of $\mathfrak{S}\mathsf{Sym}$. Its degree n component is indexed by subsets of $\{1, 2, \ldots, n-1\}$. For each subset $S \subseteq \{1, 2, \ldots, n-1\}$, the embedding is

$$R_S \mapsto \sum_{Des(\sigma)=S,\sigma\in\mathfrak{S}_n} F^*_\sigma$$

where $\{F^*_\sigma\}$ is the dual basis of $\{F_\sigma\}$ and $\{R_S\}$ is the basis dual to $\{L_S\}$. This embedding is denoted by D^* as it is the dual map of the descent map D.

Moreover, when making the variables commutative, NSym is projected to the symmetric functions Sym, and this projection π is dual to the embedding $\iota : \mathsf{Sym} \to \mathsf{QSym}$. Then, we obtain the following classical commutative square



The dual map of Θ_{QSym} is the theta map for NSym, $\Theta_{\mathsf{NSym}} : \mathsf{NSym} \to \mathsf{NSym}$. In particular, its image is the space spanned by $\left\{ \sum_{\substack{\sigma \in \mathfrak{S}_n \\ peak(\sigma)=S}} F^*_\sigma : S \subseteq \{1, 2, \ldots, n-1\} \right\}$ where the peak set is $peak(\sigma) = \{i : \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$.

The original construction of peak algebra is combinatorial. But after its introduction, it inspired a series of researches and surprising results that connected a number of area together and there have been many attempts at generalizations.

In [15], the authors relate the enriched $\mathcal{P}$-partitions to chains in Eulerian posets and their *cd*-index. They give a combinatorial interpretation of when a function in Π can be expressed positively in enriched $\mathcal{P}$-partitions. Then, they studied the theta map Θ_{QSym} as operators of certain posets and in particular, they show that Θ_{QSym} is diagonalizable on Π.

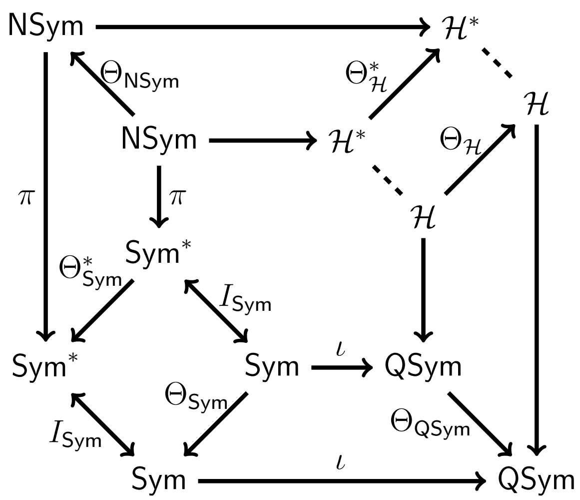
In [41], Nyman proves that the dual of the space of enriched $\mathcal{P}$-partitions is isomorphic to the sub-algebra of the Hopf algebra of permutations spanned by elements of the form $\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{peak}(\sigma)=S}} F^*_\sigma$ where $S \subseteq \{1, 2, \ldots, n-1\}$. In addtion, when we identify 𝔖Sym with the group ring $\Bbbk\mathfrak{S} = \bigoplus_{n \geq 0} \Bbbk\mathfrak{S}_n$, then its subspace NSym is given an internal multiplication for which it is closed under multiplication. The space NSym is isomorphic to Solomon's descent algebra [48]. And Nyman shows that the peak algebra also forms a sub-algebra of Solomon's descent algebra and has many properties analoguous to the descent algebra.

From the relation between descents and peaks, the peak algebra is extended to Coxeter groups of type B in papers [6]. The peak algebra of type B also has a Hopf algebra structure similar to NSym and $\mathfrak{S}\mathsf{Sym}$, and a descent-to-peak theta map is constructed in [29]. And then in [14], the authors further generalize the peak algebra to Mantaci-Reutenauer Hopf algebra of G-colored symmetric groups. However, no similar construction is known for other types of Coxeter groups.

From the combinatorial perspective, in [32], the authors generalize the combinatorics of marked shifted tableaux and enriched $\mathcal{P}$-partitions further to Poirier-Reutenauer Hopf algebra of standard Young tableaux [43]. They introduced the shifted Poirier-Reutenauer Hopf algebra. However, there is no natural ways to give a nice analoguous theta map that is compatible with their construction. The authors give two candidates, but neither is a Hopf morphism.

Many other authors have also explored the Hopf algebra and representation theoretical structure of the peak algebra (see for instance [16, 46]). As listed above, the peak algebra and its generalizations appear in the form of Schur's $\mathcal{Q}$ functions, enriched $\mathcal{P}$-partitions, sum of elements in certain Coxeter groups with given peak set, shifted standard Young tableaux, etc. Each of these constructions, both combinatorial and algebraic, has certain interesting properties. The problem is then as follows, is there a unified way of describing these peak algebras? Can we generalize the peak algebra to an arbitrary Hopf algebra in a meaningful way, and obtain analogous combinatorial and algebraic properties?

In this thesis, we will partially answer this question from the angle of theta maps. More precisely, given a Hopf algebra $\mathcal{H}$ with a Hopf morphism $\Psi : \mathcal{H} \to \mathsf{QSym}$, we define a theta map to be a Hopf morphism $\Theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ that makes the following diagram commute.



Then the image of $\Theta_{\mathcal{H}}$ is a generalized peak algebra. Moreover, if $\mathcal{H}$ is self-dual, we want $\Theta_{\mathcal{H}}$ to be self-adjoint, in which case the two self-adjointness squares in bottom-left and top-right can be omitted and the diagram becomes a cube. Usually, making $\Theta_{\mathcal{H}}$ self-adjoint simplifies the calculations a lot.

The importance of the theta maps comes from the theory of combinatorial Hopf algebras developed in [5]. A *combinatorial Hopf algebra* is a graded connected Hopf $\Bbbk$-algebra $\mathcal{H}$ equipped with a multiplicative and linear map

$\zeta : \mathcal{H} \to \Bbbk$ called *character*. The set of characters forms a group. Two characters can be multiplied via convolution product, and the inverse of a character ζ is $\zeta \circ S_{\mathcal{H}}$ where $S_{\mathcal{H}}$ is the antipode of $\mathcal{H}$. Let $\overline{\zeta}$ be the character such that $\overline{\zeta}(h) = (-1)^{\deg(h)}\zeta(h)$ for homogeneous h. Then $\chi = \overline{\zeta^{-1}}\zeta$ is called the odd character of ζ as $\overline{\chi} = \chi^{-1}$.

Consider the canonical character for Hopf algebras in power series via evaluation $\zeta(f(x_1, x_2, \dots)) = f(1, 0, 0, 0, \dots)$, both in commuting and non-commuting variables. When restricted to Sym, QSym, NSym and $\mathfrak{S}\mathsf{Sym}$, we obtain ζ_{Sym}, ζ_{QSym}, ζ_{NSym} and $\zeta_{\mathfrak{S}\mathsf{Sym}}$ respectively. In [5], the authors show that $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ is the terminal object in the category of combinatorial Hopf algebras i.e. for any pair $(\mathcal{H}, \zeta)$, there exists a unique Hopf morphism $\Phi : \mathcal{H} \to \mathsf{QSym}$ such that $\zeta = \zeta_{\mathsf{QSym}} \circ \Phi$. And similarly, $(\mathsf{Sym}, \zeta_{\mathsf{Sym}})$ is the terminal object in co-commutative combinatorial Hopf algebras. In particular, ζ_{Sym}, ζ_{QSym}, ζ_{NSym}, $\zeta_{\mathfrak{S}\mathsf{Sym}}$ are compatible with the classical square i.e. the maps are combinatorial Hopf morphism.

The theta maps for Sym and QSym are the unique combinatorial Hopf morphisms for $\left(\mathsf{Sym}, \overline{\zeta_{\mathsf{Sym}}^{-1}}\zeta_{\mathsf{Sym}}\right)$, $\left(\mathsf{QSym}, \overline{\zeta_{\mathsf{QSym}}^{-1}}\zeta_{\mathsf{QSym}}\right)$ respectively. Under this construction, the space of Schur's $\mathcal{Q}$ functions and peak algebra are the odd Hopf sub-algebras of Sym and QSym that satisfy the generalized Dehn-Sommerville relations introduced in [10]. By choosing appropriate combinatorial Hopf algebras, relations among flag vectors in Eulerian posets can be obtained from the generalized Dehn-Sommerville relations. For NSym, although it is not a terminal object, its theta map also comes from the odd character i.e. $\overline{\zeta_{\mathsf{NSym}}^{-1}}\zeta_{\mathsf{NSym}} = \zeta_{\mathsf{NSym}} \circ \Theta_{\mathsf{NSym}}$, and the image is the odd Hopf sub-algebra of $(\mathsf{NSym}, \zeta_{\mathsf{NSym}})$.

Our first result is that for any combinatorial Hopf algebra $(\mathcal{H}, \zeta)$, a Hopf morphism $\Theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is a theta map if and only if it is obtained from the odd character i.e. $\overline{\zeta^{-1}}\zeta = \zeta \circ \Theta_{\mathcal{H}}$. Moreover, the image of any theta map must lie inside the odd Hopf sub-algebra of $(\mathcal{H}, \zeta)$. From this result, we compute theta maps for different combinatorial Hopf algebras.

The central candidate of combinatorial Hopf algebras, and our main motivation for considering this problem, is the Malvenuto-Reutenauer Hopf algebra of permutations, also known as the free quasi-symmetric functions. It is a non-commutative, non-cocommutative, self-dual and graded Hopf algebra. Its sub and quotient Hopf algebras contain Sym, QSym, NSym, the peak algebra, the Loday-Ronco Hopf algebra of binary trees, the Poirier-Reutenauer Hopf algebra of standard Young tableaux and many other central objects in algebraic combinatorics.

Our main result is that we are able to give a combinatorial description the theta maps for $\mathfrak{S}\mathsf{Sym}$. In fact, there are infinitely many, and we will give an algorithm to construct all of them. The tool we use is monomial basis introduced in [8]. The monomial basis gives a set of free generators of $\mathfrak{S}\mathsf{Sym}$. We also heavily use the combinatorics of the peak set, global descents and the weak order on permutations.

In addition to this, we also prove related results on other combinatorial Hopf algebras.

In Chapter 2, we provide the background, definitions and notation needed in this thesis.

In Chapter 3, we will study odd Hopf subalgebra of an arbitrary combinatorial Hopf algebra. We give an algorithm to find the odd subalgebra, and we show that the image of a theta map, if it exists, will be contained in the odd subalgebra. In particular, it satisfies the generalized Dehn-Sommerville relations. A main part of this chapter comes from the paper [7] that is a joint work with Farid Aliniaeifard.

In Chapter 4, we first provide the connection between theta maps and characters. Using that, we show that for any Hopf subalgebra of QSym, the theta map, if it exists, must be unique. We also give theta maps for any Hopf algebras that are commutative and cocommutative, and we use the diagonally symmetric functions DSym as example. In particular, we construct a non-trivial Hopf sub-algebra of DSym. After that, we extend it to diagonally quasi-symmetric functions. We also have a structural result on the quasi-symmetric analogue of the diagonal harmonics. We leave it in appendix as it is not directly related with theta maps. Some sections of chapter 4 are also found in my joint paper with Farid Aliniaeifard [7] and some other sections are in [35]. The appendix is from the paper [36].

Then, we study another Hopf algebra of permutations, called $\mathcal{V}$ [53], which is the coradical filtration of $\mathfrak{S}\mathsf{Sym}$. We give the theta maps for $\mathcal{V}$. And we find a particular theta map, whose image is exactly the peak algebra. After that, we will construct and describe the theta maps for the Malvenuto-Reutenauer Hopf algebra of permutations.

Chapter 2

Background

In this section, we provide some the definitions and background used in this thesis to make it self-contained as possible. For more details, a good reference would be [25].

2.1 Hopf algebras

A Hopf algebra $\mathcal{H}$ is a $\mathbb{k}$-vector space with the following $\mathbb{k}$-linear maps.

- product $m : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ that is associative i.e. $m\circ(m\otimes id) = m\circ(id\otimes m)$.
- unit $u : \mathbb{k} \to \mathcal{H}$ such that $u(1)$ is the identity element for multiplication.
- coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ that is coassociative i.e. $(\Delta \otimes id) \circ \Delta = (id\otimes\Delta)\circ\Delta$. It is sometimes convenient to write the coproduct as $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$.
- counit $\epsilon : \mathcal{H} \to \mathbb{k}$ such that $\sum_{(c)} m(c_1 \otimes \epsilon(c_2)) = c$.

These maps satisfy the following compatibility axioms

1. $\Delta \circ m = (m \otimes m) \circ (id \otimes T \otimes id) \circ (\Delta \otimes \Delta)$ where T is the transposition $a \otimes b \mapsto b \otimes a$
2. $m \circ (\epsilon \otimes \epsilon) = \epsilon \circ m$
3. $\Delta \circ u = (u \otimes u) \circ \Delta$
4. $\epsilon \circ u = id$

And there exists a linear map called antipode $S : \mathcal{H} \to \mathcal{H}$ such that $\sum_{(c)} S(c_1)c_2 = u(\epsilon)(c) = \sum_{(c)} c_1 S(c_2)$.

A Hopf algebra $\mathcal{H}$ is graded if

1. the underlying vector space is graded i.e. $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$, $\mathcal{H}_n$ is called the graded component of degree n.
2. given two homogeneous elements $h_1 \in \mathcal{H}_i$ and $h_2 \in \mathcal{H}_j$, then $m(h_1 \otimes h_2) \in \mathcal{H}_{i+j}$,
3. given a homogeneous element $h \in \mathcal{H}_n$, $\Delta(h) \in \bigoplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j$.

A Hopf algebra is graded and connected if it is graded and $\mathcal{H}_0 = \mathbb{k}$.

In this thesis, all Hopf algebras will be graded and connected, and the dimensions in all graded component are finite. In this case, we can drop the antipode axiom because it always exists via the Takeuchi formula [52]

$$S = \sum_{n \geq 0} (-1)^n m^{(n-1)} f^{\otimes n} \Delta^{(n-1)}$$

where $f = id - u\epsilon$, $m^{-1} = u$, $\Delta^{-1} = \epsilon$ and $m^0 = \Delta^0 = id$.

Given two Hopf algebras $\mathcal{H}$ and $\mathcal{H}'$, we say a map $\varphi : \mathcal{H} \to \mathcal{H}'$ is a Hopf morphism if it is a linear map that preserves the product and coproduct i.e.

1. $\varphi \circ m = m \circ (\varphi \otimes \varphi)$
2. $\Delta \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$.

Similarly, a map is an algebra morphism or coalgebra morphism if it is linear and it preserves the product or coproduct respectively.

Given a graded connected Hopf $\Bbbk$-algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$, we can define its graded dual, denoted by $\mathcal{H}^* = \bigoplus_{n \geq 0} \mathcal{H}_n^*$ where $\mathcal{H}_n^*$ is the set of all linear maps from $\mathcal{H}$ to $\Bbbk$ such that all homogeneous elements $h \in \mathcal{H}_i$ are mapped to 0 unless $n = i$.

The graded dual space $\mathcal{H}^*$ is a graded connected $\Bbbk$-vector space and we have a bilinear scalar product

$$\langle -, - \rangle_{\mathcal{H}} : \mathcal{H}^* \times \mathcal{H} \to \Bbbk$$

that takes a pair $(f, h) \in \mathcal{H}^* \times \mathcal{H}$ to $f(h)$.

For each linear map $\phi : \mathcal{H} \to \mathcal{H}$, there is a unique adjoint map $\phi^* : \mathcal{H}^* \to \mathcal{H}^*$ such that for all $a \in \mathcal{H}, b \in \mathcal{H}^*$, we have $\langle b, \phi(a) \rangle_{\mathcal{H}} = \langle \phi^*(a), b \rangle_{\mathcal{H}}$.

The graded dual space has a Hopf structure where the product on $\mathcal{H}^*$ is the adjoint map of the coproduct on $\mathcal{H}$ and the coproduct on $\mathcal{H}^*$ is the adjoint map of the product on $\mathcal{H}$ with respect to the scalar product $\langle -, - \rangle_{\mathcal{H}}$.

It is not hard to check that the product and coproduct on $\mathcal{H}^*$ satisfy the compatiblility axioms of Hopf algebra with its grading.

Let $\{b_1, b_2, \dots\}$ be a homogeneous basis of $\mathcal{H}$, then there is a homogeneous dual basis $\{b_1^*, b_2^* \dots\}$ for $\mathcal{H}^*$ such that $\langle b_i^*, b_j \rangle_{\mathcal{H}} = \delta_{ij}$ where $\delta_{ij} = 1$ when $i = j$, and otherwise $\delta_{ij} = 0$.

A Hopf algebra $\mathcal{H}$ is called self-dual if it is isomorphic to its graded dual space as Hopf algebras.

Given two Hopf algebras A, B and a Hopf morphism $\phi : A \to B$, we have the dual map $\phi^* : B^* \to A^*$ such that $\phi^*(f)(a) = f(\phi(a))$. If $B = A^*$, then the dual map ϕ^* is the adjoint map of ϕ with respect to the scalar product $\langle -, - \rangle_A$.

Given a Hopf algebra $\mathcal{H}$ and two Hopf morphisms $f, g : \mathcal{H} \to \mathcal{H}$, we define their convolution product, denoted by $f*g$, to be the map composition $m \circ (f \otimes g) \circ \Delta$. The convolution product is a linear map, but it fails to be a Hopf morphism in general.

Remark 2.1.1. The convolution product can be defined for any linear maps between a coalgebra C and an algebra A. In fact, it makes $Hom(C, A)$ an algebra. But we will not use it in this thesis.

The rest of this section recalls the central Hopf algebras we will use in this thesis including the symmetric functions (Sym), the quasi-symmetric functions (QSym), the non-commutative symmetric functions (NSym), the free quasi-symmetric functions ($\mathfrak{S}\mathsf{Sym}$) and Vargas Hopf algebra of permutations ($\mathcal{V}$). For more details, we refer to [25].

2.1.1 The Hopf algebra of symmetric functions

In this section, we recall symmetric functions and their basis properties. More details can be found in [45, 38]

A partition λ of n, written $\lambda \vdash n$ is a finite tuple of positive integers $(\lambda_1, \dots, \lambda_\ell)$ such that $\lambda_1 + \cdots + \lambda_\ell = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. We say the length of λ is $\ell(\lambda) = \ell$ and the size of λ is $|\lambda| = n$.

For partitions μ, ν, and positive integer s, we say $\mu \prec_s \nu$, or ν/μ is a horizontal s-strip, if

1. $|\nu| = |\mu| + s$,
2. $\mu_j \leq \nu_j$ for all $1 \leq j \leq \ell(\mu)$,
3. If $\nu_i > \mu_i$ and $\nu_j > \mu_j$ for two different integers i, j, then either $\nu_i \leq \mu_j$ or $\nu_j \leq \mu_i$.

By convention, we let $\mu_j = 0$ for $j > \ell(\mu)$.

The ring of symmetric functions, $\mathsf{Sym} = \bigoplus_{n \geq 0} \mathsf{Sym}_n$, is sub-ring of the power series $\Bbbk[[x_1, x_2, \dots]]$ with bounded degree that are fixed under symmetric group action where Sym_n is the space of homogeneous symmetric functions of degree n. More precisely, $f \in \mathsf{Sym}$ if $f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$ for all $\sigma \in \mathfrak{S}_{(\infty)}$ where $\mathfrak{S}_{(\infty)}$ is the set of all bijections from the set of positive integers to itself that fix all but finitely many numbers.

The symmetric functions clearly form a graded vector space, and the degree n component is clearly has a basis, called the monomial basis m_λ, the sum of all monomials in the orbit of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}}$, for all $\lambda \vdash n$.

Example 2.1.2. *For $n = 3$, we have*

- $m_{(3)}(\mathbf{x}) = x_1^3 + x_2^3 + x_3^3 + \cdots$
- $m_{(2,1)}(\mathbf{x}) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots$
- $m_{(1,1,1)}(\mathbf{x}) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + \cdots$

The symmetric functions form a ring with the regular multiplication of power series. It is, in fact, a free commutative ring with generators $\{h_1, h_2, \dots\}$ where $h_n = \sum_{\lambda \vdash n} m_\lambda$. The $\{h_n\}$ are called the complete homogeneous functions. For a partition λ, we denote $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$.

Proposition 2.1.3. *[45] $\{h_\lambda : \lambda \vdash n\}$ is a basis of Sym_n.*

We define the coproduct on Sym to be $\Delta(h_n) = \sum_{i+j=n} h_i \otimes h_j$ and extended multiplicatively. Then Sym becomes a Hopf algebra.

We will then define the most important basis $\{s_\lambda\}$ using tableaux.

Definition 2.1.4. A tableau is a finite collection of cells, arranged in left-justified rows and filled with positive integers e.g.

$T =$

1	1	2
2		
2	3	

Let T be a tableau. The shape of T, denoted by $sh(T)$, is the integer vector whose i-th entry is the length of row i, reading from top to bottom. The content of T, denoted by $c(T)$, is the integer vector whose j-th entry is the number of times j appears in T. Since all but finitely many of the entries in $sh(T)$ and $c(T)$ are zero, we identify them with a finite sequence of non-negative integers.

The reading word of T, denoted by $read(T)$, is the word of entries read starting in the top row from right to left, then proceeding down the rows. In the example above, $sh(T) = (3,1,2)$, $c(T) = (2,3,1)$, and $read(T) = (2,1,1,2,3,2)$.

For compositions α, β with $\alpha_i \geq \beta_i$ for all i, a *skew tableau* of shape α/β is a tableau of shape α with cells of β removed from the upper left corner e.g.

$$T' = \begin{array}{|c|c|c|c|} \hline \blacksquare & 1 & 1 & 2 \\ \hline \blacksquare & \blacksquare & 2 \\ \cline{1-3} 2 & 3 \\ \cline{1-2} \end{array}$$

has skew shape $(4,3,2)/(1,2)$.

In this case, β is called the *inner shape* of T' and α is called the *outer shape* of T', denoted by $outsh(T')$.

A tableau is called *semi-standard* if its rows are weakly increasing from left to right and its columns are strictly increasing from top to bottom. A tableau is said to be *immaculate* if its rows are weakly increasing from left to right and its first column is strictly increasing from top to bottom. A tableau T is called *Yamanouchi* if in $read(T)$, for every positive integer j and every prefix d, there are at least as many occurrences of j as there are of $j+1$ in d.

All the definitions above for tableaux apply verbatim to skew tableaux.

Definition 2.1.5. The Schur functions $s_\lambda(\mathbf{x}) = \sum_{\substack{\text{semi-standard tableaux} \\ \text{T of shape } \lambda}} \mathbf{x}^{c(T)}$ where

$\mathbf{x}^{c(T)} = x_1^{c(T)_1} x_2^{c(T)_2} \cdots$

Example 2.1.6.

$$s_{21}(\mathbf{x}) = \begin{array}{ccccccccc} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & + & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & + & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & + & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & + & \cdots \\ x_1^2 x_2 & + & x_1 x_2^2 & + & x_1 x_2 x_3 & + & x_1 x_2 x_3 & + & \cdots \end{array}$$

Proposition 2.1.7. *([25], Proposition 2.11) The Schur functions are symmetric i.e.* $s_\lambda \in \mathsf{Sym}$*. Moreover,* $\{s_\lambda : \lambda \vdash n\}$ *forms a basis of* Sym_n*.*

The Schur functions are positive, self-dual basis. Let λ, μ, ν be partitions and $c^\lambda_{\mu\nu}$ be the number of skew semi-standard Yamanouchi tableaux of shape λ/μ and content ν. Then we have the following Pieri rule and Littlewood-Richardson rule that are nicely explained in [25].

Theorem 2.1.8. *([25], Theorem 2.58) For a partition* μ *and a positive integer* n*,*

$$s_\mu h_n = \sum_{\mu \prec_n \lambda} s_\lambda.$$

Theorem 2.1.9. *[37] For partitions* λ*,* μ *and* ν*,*

$$s_\mu s_\nu = \sum_{\lambda \vdash |\mu|+|\nu|} c^\lambda_{\mu\nu} s_\lambda.$$

$$\Delta(s_\lambda) = \sum_{\lambda \vdash |\mu|+|\nu|} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu.$$

These show that Sym is a self-dual Hopf algebra via the isomorphism

$$\begin{aligned} I_{\mathsf{Sym}} : \quad \mathsf{Sym} &\rightarrow \mathsf{Sym}^* \\ s_\lambda &\mapsto s^*_\lambda \end{aligned}$$

and we have the Hall scalar product on Sym, $\langle -,- \rangle : \mathsf{Sym} \times \mathsf{Sym} \to \Bbbk$ such that $\langle s_\lambda, s_\mu \rangle = \langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$.

The importance of positivity and self-duality was studied by Zelevinsky and that results in beautiful descriptions of irreducible representations of many families of finite groups including the symmetric groups and general linear groups.

One can also obtain Schur functions directly using the homogeneous basis via the Jacobi-Trudi formula.

Theorem 2.1.10. *[30] For a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$*,*

$$s_\lambda = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & \cdots & h_{\lambda_k} \end{bmatrix} = \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma h_{\lambda_1+\sigma_1-1,\ldots,\lambda_k+\sigma_k-k}$$

where for convinience we use σ_i *to denote* $\sigma(i)$ *and the same notation is used later.*

Another important class of symmetric functions is the Schur's $\mathcal{Q}$ functions, defined using marked shifted tableaux.

A shifted tableau of shape λ is a tableau of shape λ whose row i is shifted to the right by $i-1$ for all rows i.e. of shape $(\lambda + (0, 1, \ldots, \ell(\lambda) - 1))/(0, 1, \ldots, \ell(\lambda) - 1)$ A marked shifted tableau is a shifted tableau that are filled with $\{1, 1', 2, 2', 3, 3', \cdots\}$ such that

1. both rows and columns are weakly increasing with respect to the ordering

$1 < 1\text{'} < 2 < 2\text{'} < 3 < 3\text{'} < \cdots,$

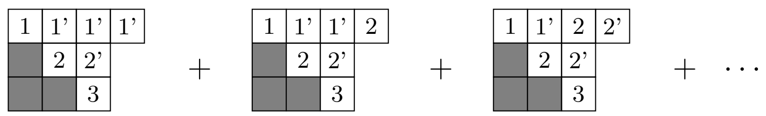
2. there can be at most one x in each row and at most one x' in each column for all numbers x.

The content of a marked shifted tableau is the regular content by reading x' as x for all numbers x.

Definition 2.1.11. The Schur's $\mathcal{Q}$ functions is $\mathcal{Q}_\lambda = \sum_{\substack{\text{marked shifted tableaux} \\ \text{T of shape } \lambda}} \mathbf{x}^{c(T)}$ where $\mathbf{x}^{c(T)} = x_1^{c(T)_1} x_2^{c(T)_2} \cdots$

Example 2.1.12.

$\mathcal{Q}_{421} =$



$$x_1^4 x_2^2 x_3 \quad + \quad x_1^3 x_2^3 x_3 \quad + \quad x_1^2 x_2^4 x_3 \quad + \quad \cdots$$

The Schur's $\mathcal{Q}$ functions form a subspace of Sym, with a basis $\{\mathcal{Q}_\lambda : \lambda_1 > \lambda_2 > \cdots > \lambda_{\ell(\lambda)}\}$ indexed by strict partitions. The set $\{\mathcal{Q}_n\}$ is a set of generators of the space of Schur's $\mathcal{Q}$ functions [38].

The Schur's $\mathcal{Q}$ functions play a central role in the projective representations of symmetric groups, similar to Schur functions in regular representation theory. They also have analogous Pieri rule and Littlewood-Richardson rule.

Consider the algebra morphism $\Theta_{\mathsf{Sym}} : \mathsf{Sym} \to \mathsf{Sym}$ that sends $h_n \to \mathcal{Q}_n$ and extended multiplicatively. This map is in fact a self-adjoint Hopf morphism with respect to the Hall scalar product i.e. Θ_{Sym} preserves the coproduct, and it makes the following diagram commute.

$$\begin{array}{ccc} \mathsf{Sym} & \xrightarrow{I_{\mathsf{Sym}}} & \mathsf{Sym}^* \\ \downarrow{\scriptstyle \Theta_{\mathsf{Sym}}} & & \downarrow{\scriptstyle \Theta^*_{\mathsf{Sym}}} \\ \mathsf{Sym} & \xrightarrow{I_{\mathsf{Sym}}} & \mathsf{Sym}^* \end{array}$$

It is not hard to see that $\mathcal{Q}_n = 2\sum_{k=1}^{n} s_{(k,1,1,\ldots,1)}$ where each term has $n-k$ 1's. Therefore, we may write $\Theta_{\mathsf{Sym}}(h_n) = 2 \sum_{\text{hook } \lambda \text{ of size } n} s_\lambda$.

2.1.2 The Hopf algebra of quasi-symmetric functions

A composition α of n, written $\alpha \models n$, is a finite tuple of positive integers $(\alpha_1, \ldots, \alpha_\ell)$ where $\alpha_1 + \ldots + \alpha_\ell = n$. We say the length of α, $\ell(\alpha) = \ell$ and the size of α, $|\alpha| = n$.

For compositions α, β, and positive integer i, we say $\alpha \subset_i \beta$ if

1. $|\beta| = |\alpha| + i$,
2. $\alpha_j \leq \beta_j$ for all $1 \leq j \leq \ell(\alpha)$,
3. $\ell(\beta) \leq \ell(\alpha) + 1$.

Denote by $\mathcal{Q}_n$ the set of all subsets of $[n-1] := \{1, 2, \ldots, n-1\}$. There is a one-to-one correspondence I between $\mathcal{Q}_n$ and the set of compositions of n, where

$$I(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \ldots + \alpha_{\ell(\alpha)-1}\}.$$

Then we have a refinement order on the set of compositions of n. For two compositions $\alpha, \beta \models n$, $\alpha \leq \beta$ if $I(\alpha) \subseteq I(\beta)$. For a composition α of n, let

$$M_\alpha := \sum_{i_1 < \ldots < i_{\ell(\alpha)}} x_{i_1}^{\alpha_1} \cdots x_{i_{\ell(\alpha)}}^{\alpha_{\ell(\alpha)}}.$$

This is an element of the commutative algebra of formal power series in variables $\{x_i\}_{i \geq 1}$. By convention, $M_{()} = 1$, where $()$ denotes the unique composition of 0 with no parts. The multiplication is inherited from the ring of power series.

The ring of quasisymmetric functions is denoted by QSym and is defined as follows

$$\mathsf{QSym} = \bigoplus_{n \geq 0} \mathsf{QSym}_n,$$

where

$$\mathsf{QSym}_n = \mathbb{k}\text{-}\{M_\alpha : \alpha \models n\}.$$

The ring of quasisymmetric functions is indeed a Hopf algebra with co-multiplication

$$\Delta(M_\alpha) = \sum_{\alpha = \beta \odot \gamma} M_\beta \otimes M_\gamma,$$

where $\beta \odot \gamma$ is the concatenation of compositions β and γ i.e. a composition of length $\ell(\beta) + \ell(\gamma)$ whose i-th entry is β_i if $i \leq \ell(\beta)$ and $\gamma_{i-\ell(\beta)}$ otherwise.

Example 2.1.13. $(2,1,3) \odot (3,1) = (2,1,3,3,1)$ *and*

$$\Delta(M_{(2,1,3)}) = 1 \otimes M_{(2,1,3)} + M_{(2)} \otimes M_{(1,3)} + M_{(2,1)} \otimes M_{(3)} + M_{(2,1,3)} \otimes 1.$$

Clearly, QSym is commutative but not co-comutative.

The Hopf algebra QSym contains Sym has a Hopf sub-algebra as follows.

$$m_\lambda = \sum_{sort(\alpha)=\lambda} M_\alpha$$

where $sort(\alpha)$ is the partition obtained by reordering entries of α in decreasing order. This gives a natural embedding

$$\iota : \mathsf{Sym} \to \mathsf{QSym}\,.$$

Another well-known linear basis of QSym_n is obtained by defining for each $\alpha \models n$,

$$L_\alpha = \sum_{\alpha \leq \beta} M_\beta.$$

The sum is over all β that refines α. The set $\{L_\alpha\}$ is called the fundamental basis that can be more naturally described using the theory of $\mathcal{P}$-partitions.

Let $\mathcal{P}$ be a finite labelled partially ordered set whose underlying vertices are totally ordered. For convenience, we assume the underlying set is some finite set of integers.

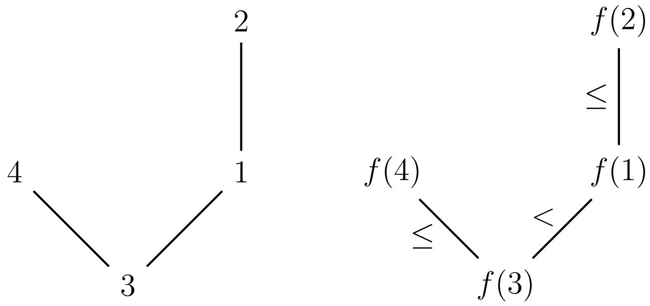
This definition is equivalent to the definition in introduction where edges of $\mathcal{P}$ are labeled by $<$ and $\leq$. Because the inequalities on the edges are determined by the labellings on the vertices.

Definition 2.1.14. A $\mathcal{P}$-partition is a function $f : \mathcal{P} \to \{1, 2, 3, \ldots\}$ such that

1. $i <_P j$ and $i <_{\mathbb{Z}} j$ implies $f(i) \leq f(j)$,

2. $i <_P j$ and $i >_{\mathbb{Z}} j$ implies $f(i) < f(j)$.

Example 2.1.15. *Consider the following poset $\mathcal{P}$, then a $\mathcal{P}$ partition f must satisfy*



The generating functions

$$L_{\mathcal{P}} = \sum_{\mathcal{P}\text{-partitions } f} \mathbf{x}^{wt(f)} = \sum_{\mathcal{P}\text{-partitions } f} \left(\prod_{i \in \mathcal{P}} x_{f(i)} \right)$$

are quasi-symmetric functions.

When $\mathcal{P}$ is totally ordered, we obtain the fundamental basis $L_{I(Des(\mathcal{P}))} = L_{\mathcal{P}}$. Without loss of generality, we can write $\mathcal{P}$ as a permutation σ of $[n]$ i.e. $\sigma(1) <_{\mathcal{P}} \sigma(2) <_{\mathcal{P}} \cdots <_{\mathcal{P}} \sigma(n)$. We say σ has a descent at i if $\sigma(i) > \sigma(i+1)$. Let $Des(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$ denote the descent set of σ, then,

$$L_{\mathcal{P}} = L_{I^{-1}(Des(\sigma))}.$$

Example 2.1.16. *Let $\mathcal{P}$ be the totally ordered set* 35412*. Then its descent set is* $\{2, 3\}$ *and* $L_{\mathcal{P}} = L_{(2,1,2)}$.

The next proposition is the Stanley's main lemma in $\mathcal{P}$-partition theory.

Theorem 2.1.17. *[49] Let $\mathcal{P}$ be a poset and $\mathcal{L}(\mathcal{P})$ be the set of all linear* *extensions of* $\mathcal{P}$*, then*

$$L_{\mathcal{P}} = \sum_{\sigma \in \mathcal{L}(\mathcal{P})} L_{\sigma}.$$

The semi-standard tableaux can be viewed as special labelled posets and its generating Schur functions can be written as linear combinations of the fundamental basis in QSym.

In 1995, Stembridge gave a analogous generalization for Schur's $\mathcal{Q}$ functions using enriched $\mathcal{P}$-partitions.

Definition 2.1.18. Consider the totally order set $1 < 1\text{'} < 2 < 2\text{'} < \cdots$. Given a labelled poset $\mathcal{P}$, an enriched $\mathcal{P}$-partition is a function $f : \mathcal{P} \to \{1, 1\text{'}, 2, 2\text{'}, \dots\}$ such that for all $i <_{\mathcal{P}} j$, we have

1. $f(i) \leq f(j)$.
2. $f(i) = f(j)$ and $f(i)$ is of the form x' for some number x implies $i <_{\mathbb{Z}} j$.
3. $f(i) = f(j)$ and $f(i)$ is not of the form x' for some number x implies $i >_{\mathbb{Z}} j$.

The enriched $\mathcal{P}$-partitions generalize the marked shifted tableaux and its generating functions,

$$K_{\mathcal{P}} = \sum_{\text{enriched } \mathcal{P}\text{-partitions } f} \mathbf{x}^{wt(f)} = \sum_{\text{enriched } \mathcal{P}\text{-partitions } f} \left(\prod_{i \in \mathcal{P}} x_{\overline{f(i)}} \right)$$

where $\overline{f(i)}$ is the underlying integer of $f(i)$, generalize the Schur's $\mathcal{Q}$ functions.

We also have the following fundamental lemma.

Lemma 2.1.19. *[51] Let* $\mathcal{P}$ *be a poset and* $\mathcal{L}(\mathcal{P})$ *be the set of all linear* *extensions of* $\mathcal{P}$*, then*

$$K_{\mathcal{P}} = \sum_{\sigma \in \mathcal{L}(\mathcal{P})} K_\sigma.$$

It is still true that K_σ depends only on the peak set of σ where the peak set is $peak(\sigma) = \{i : \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$.

Therefore, the space of enriched $\mathcal{P}$-partitions is a linear subspace of QSym whose degree n component is the n-th Fibonacci number $(1, 1, 2, 3, 5, 8, \dots)$.

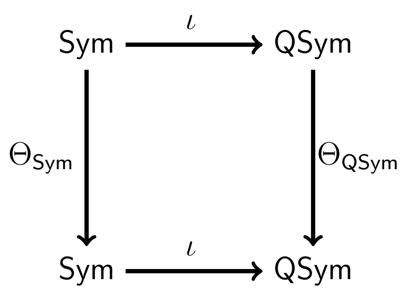
Moreover, the results by Stembridge [51] and Bergeron et al. [16] show that it is also closed under product and coproduct inherited from QSym.

Theorem 2.1.20. *[16] The space of enriched* $\mathcal{P}$*-partitions is a Hopf sub-algebra of* QSym*.*

Therefore, it is also referred to as the peak algebra.

In fact, it is the image of the map $\Theta_{\mathsf{QSym}} : \mathsf{QSym} \to \mathsf{QSym}$ that sends $L_{\mathcal{P}}$ to $K_{\mathcal{P}}$ for all labelled poset $\mathcal{P}$.

The map Θ_{QSym} is a Hopf morphism, and it is also compatible with Θ_{Sym} via the embedding i.e. we have the following commutative diagram.



We will work with the space enriched $\mathcal{P}$-partitions and Θ_{QSym} extensively in this thesis, and try to generalize them into other Hopf algebras.

Another basis of QSym we will use is the shuffle basis $\{S_\alpha\}$, first introduced in [40]. In this case, we assume $\Bbbk$ contains $\mathbb{Q}$ as a sub-field. For two compositions $\alpha, \beta \models n$ and $\alpha \leq \beta$, let $d^\alpha_\beta = \dfrac{1}{n_1!n_2!\dots n_k!}$ where $\beta = (\alpha_1 + \cdots + \alpha_{n_1}, \alpha_{n_1+1} + \cdots + \alpha_{n_1+n_2}, \dots, \alpha_{n-n_k+1} + \cdots + \alpha_{\ell(\alpha)})$.

Definition 2.1.21.

$$S_\alpha = \sum_{\beta \leq \alpha} d^\beta_\alpha M_\beta.$$

Example 2.1.22.

$$S_{(1,1)} = M_{(1,1)} + \frac{1}{2} M_{(2)}.$$

By triangularity, $\{S_\alpha\}$ forms a basis of QSym. And its Hopf structure is isomorphic to the shuffle algebra.

Let $Sh_{n,m}$ be the following subset of permutation group $\mathfrak{S}_{n+m}$

$$\{\sigma \in \mathfrak{S}_{n+m} : \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n); \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m)\}.$$

For two compositions α, β of length n, m respectively, we define $\alpha \sqcup\!\sqcup \beta$ to be the multi-set $\{\omega_{\sigma(1)}, \omega_{\sigma(2)}, \dots, \omega_{\sigma(n+m)} : \sigma \in Sh_{n,m}\}$ where ω is the concatenated composition $\alpha \odot \beta$.

Proposition 2.1.23. *[28]*

$$S_\alpha S_\beta = \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta} S_\gamma,$$

$$\Delta(S_\alpha) = \sum_{\alpha = \beta \odot \gamma} S_\beta \otimes S_\gamma.$$

Example 2.1.24.

$$S_1 \cdot S_1 = M_1 \cdot M_1 = 2M_{(1,1)} + M_2 = 2S_{(1,1)}$$

$$\begin{aligned}
\Delta(S_{(1,1)}) &= \Delta\left(M_{(1,1)} + \frac{1}{2}M_2\right) \\
&= 1 \otimes M_{(1,1)} + M_1 \otimes M_1 + M_{(1,1)} \otimes 1 + \frac{1}{2} \otimes M_2 + M_2 \otimes \frac{1}{2} \\
&= 1 \otimes S_{(1,1)} + S_1 \otimes S_1 + S_{(1,1)} \otimes 1.
\end{aligned}$$

2.1.3 The Hopf algebra of non-commutative symmetric functions

The graded dual of QSym is the non-commutative symmetric functions, NSym. The dual basis of $\{M_\alpha\}$ is denoted by $\{H_\alpha\}$ i.e. $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha\beta}$.

In fact, $\mathsf{NSym} = \langle H_1, H_2, \cdots, \rangle$ is the non-commutative algebra freely generated by infinitely many elements $\{H_n\}_{n\geq 1}$. The algebra NSym is a graded connected Hopf algebra with comultiplication

$$\Delta(H_n) = \sum_{i+j=n} H_i \otimes H_j.$$

For a composition $\alpha = (\alpha_1, \ldots, \alpha_n)$, let $H_\alpha = H_{\alpha_1} \ldots H_{\alpha_n}$, and by convention we let $H_0 = 1$. Then we have

$$\mathsf{NSym} = \bigoplus_{n\geq 0} \mathsf{NSym}_n,$$

where

$$\mathsf{NSym}_n = \Bbbk\text{-Span}\{H_\alpha : \alpha \models n\}.$$

Recall that there is an embedding $\iota : \mathsf{Sym} \to \mathsf{QSym}$. Dually, we have a surjective Hopf morphism.

$$\begin{aligned} \pi : \quad \mathsf{NSym} &\to \mathsf{Sym} \\ H_n &\mapsto h_n \end{aligned}$$

Hence, $\{H_\alpha\}$ is called the homogeneous basis of NSym.

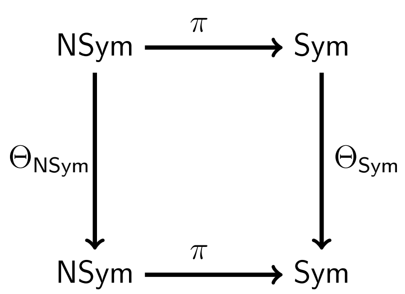
The dual basis of $\{L_\alpha\}$ is called the non-commutative ribbon basis $\{R_\alpha\}$ with $\langle R_\alpha, L_\beta \rangle = \delta_{\alpha\beta}$. Their relate with homogeneous basis in the following way

$$H_\alpha = \sum_{\beta \leq \alpha} R_\beta,$$

$$R_\alpha = \sum_{\beta \leq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} H_\beta.$$

If we dualize the map Θ_{QSym}, we obtain $\Theta_{\mathsf{NSym}} : \mathsf{NSym} \to \mathsf{NSym}$. It maps H_n to $2\sum_{k=1}^{n} R_{(1,1,\ldots,1,k)}$ where each term has $n-k$ 1's.

This map is compatible with Θ_{Sym} i.e. it makes the following diagram commute.



Moreover, the image of Θ_{NSym}, as a Hopf algebra, is isomorphic to the graded dual of the image of Θ_{QSym}.

In [12], the authors introduced a new basis $\{\mathfrak{S}_\alpha\}$ for NSym, called the immaculate basis. This basis is one of the best non-commutative analogues of the Schur functions.

The Schur functions can be defined using the Jacobi-Trudi formula, and $\{\mathfrak{S}_\alpha\}$ admits a very similar expression. For a composition α of length k,

$$\mathfrak{S}_\alpha = \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma H_{\alpha_1+\sigma_1-1,\ldots,\alpha_k+\sigma_k-k}. \tag{2.1}$$

The immaculate basis also satisfies a formula analogous to the Pieri rule

Theorem 2.1.25. *([12] Theorem 3.5) For a composition α and a positive integer s,*

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta.$$

It also has an analogous Littlewood-Richardson rule.

Theorem 2.1.26. *([12] Theorem 7.3) For composition α and partition λ,*

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_{\gamma \models |\alpha|+|\lambda|} C^\gamma_{\alpha\lambda} \mathfrak{S}_\gamma$$

where $C^\gamma_{\alpha,\lambda}$ is the number of skew immaculate Yamanouchi tableaux of shape γ/α and content λ.

Unfortunately, we do not have a nice formula for general structure constants because they could be negative, and no general cancellation-free formula is known for a composition β in the place of a partition λ.

Example 2.1.27.

$$\mathfrak{S}_2\mathfrak{S}_{2,4} = \mathfrak{S}_{3,1,4} + \mathfrak{S}_{2,2,4} + \mathfrak{S}_{3,2,3} - \mathfrak{S}_{5,3} - \mathfrak{S}_{4,3,1}.$$

However, we have a cancellation free formula for the left Pieri rule. We will introduce it in section 4.2.

2.1.4 Malvenuto-Reutenauer Hopf algebra of permutations

Let $\mathfrak{S}_n$ denote the set of permutations on $\{1, 2, \ldots, n\}$. The elements $\sigma \in \mathfrak{S}_n$ are viewed as word $\sigma(1), \ldots, \sigma(n)$. We denote the length of a word w by $|w|$. If σ, τ are two words, $\sigma \odot \tau = \omega$ is the word such that $\omega(i) = \sigma(i)$ for $1 \le i \le |\sigma|$ and $\omega(i) = \tau(i - |\sigma|)$ for $i > |\sigma|$. If σ, τ are permutations, $\sigma \backslash \tau = \omega$ is the permutation such that $\omega(i) = \sigma(i) + |\tau|$ for $1 \le i \le |\sigma|$ and $\omega(i) = \tau(i - |\sigma|)$ for $i > |\sigma|$.

Let ω be a word of $\mathbb{N}$ such that $\omega(i) \ne \omega(j)$ for all $i \ne j$. The set of inversions of ω is $Inv(\omega) = \{(a, b) : a < b, \omega(a) > \omega(b)\}$. The descent set of ω is $Des(\omega) = \{i : \omega(i) > \omega(i+1)\}$. The set of global descents of ω is $GD(\omega) = \{i : \omega(a) > \omega(b) \text{ for all } a \le i < b\}$. The peak set of ω is $peak(\omega) = \{i : \omega(i-1) < \omega(i) > \omega(i+1)\}$.

The standardization of ω, denoted by $std(\omega)$ is the unique permutation in $\mathfrak{S}_{|\omega|}$ such that $Inv(\omega) = Inv(std(\omega))$. We have a weak order, $<$, on $\mathfrak{S}_n$ by $\sigma < \tau$ if $Inv(\sigma) \subset Inv(\tau)$. If σ, τ are permutations, their shifted shuffle, denoted by $\sigma \overrightarrow{\sqcup\!\sqcup} \tau$, is the set of $\omega \in S_{|\sigma|+|\gamma|}$ such that $std(\omega^{-1}(\sigma(1)), \ldots \omega^{-1}(\sigma(|\sigma|))) = \sigma$ and $std(\omega^{-1}(|\sigma| + \tau(1)), \ldots \omega^{-1}(|\sigma| + \tau(|\tau|))) = \tau$.

The Malvenuto-Reutenauer Hopf algebra, denoted by $\mathfrak{S}\mathsf{Sym}$, is the graded Hopf algebra $\mathfrak{S}\mathsf{Sym} = \bigoplus_{n\geq 0} \mathfrak{S}\mathsf{Sym}_n$ that $\mathfrak{S}\mathsf{Sym}_n = \Bbbk\text{-Span}\{F_\sigma : \sigma \in \mathfrak{S}_n\}$.

The product formula is

$$F_\sigma \cdot F_\tau = \sum_{\gamma \in \sigma \overrightarrow{\sqcup\!\sqcup} \tau} F_\gamma.$$

The coproduct formula is

$$\Delta(F_\sigma) = \sum_{\mu \odot \omega = \sigma} F_{std(\mu)} \otimes F_{std(\omega)}.$$

The product and coproduct formula are obtained from the operations on power series in non-commuting variables, defined in the introduction. They satisfy the axioms of Hopf algebra [40]. In the rest of this thesis, we only use these shifted shuffle and deconcatenation formula to do calculation.

A second basis, introduced in [8], is defined as

$$M_\sigma = \sum_{\tau \geq \sigma} \mu(\sigma, \tau) F_\tau$$

where $\mu(\sigma, \tau)$ is the Möbius function on the weak order.

By Möbius inversion formula,

$$F_\sigma = \sum_{\tau \geq \sigma} M_\tau.$$

The M basis has a particularly nice coproduct formula

$$\Delta(M_\sigma) = \sum_{i \in GD(\sigma) \cup \{0,|\sigma|\}} M_{std(\sigma(1),\ldots,\sigma(i))} \otimes M_{\sigma(i+1),\ldots,\sigma(|\sigma|)}.$$

Let $\{F_\sigma^*\}$ and $\{M_\sigma^*\}$ denote the dual bases of F and M bases respectively, we have

$$M_\sigma^* = \sum_{\tau \leq \sigma} F_\tau^*$$

and

$$M_\sigma^* \cdot M_\tau^* = M_{\sigma \backslash \tau}^*.$$

The self-duality of $\mathfrak{S}\mathsf{Sym}$ follows from the isomorphism $I_{\mathfrak{S}\mathsf{Sym}} : \mathfrak{S}\mathsf{Sym} \to \mathfrak{S}\mathsf{Sym}^*$

$$I_{\mathfrak{S}\mathsf{Sym}}(F_\sigma) = F_{\sigma^{-1}}^*.$$

We have a surjective descent map $D : \mathfrak{S}\mathsf{Sym} \to \mathsf{QSym}$

$$D(F_\sigma) = L_{Des(\sigma)}.$$

The dual map is $D^* : \mathsf{NSym} \to \mathfrak{S}\mathsf{Sym}^*$

$$D^*(H_n) = F_{id_n}^*.$$

As a result, $D^*(H_\alpha) = \sum_{I^{-1}(Des(\sigma)) \leq \alpha} F^*_\sigma$ and $D^*(R_\alpha) = \sum_{I^{-1}(Des(\sigma)) = \alpha} F^*_\sigma$.

Example 2.1.28.

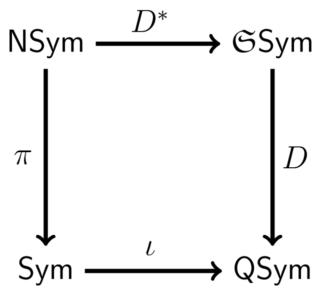
$$D^*(H_{(1,3)}) = F^*_{4123} + F^*_{3124} + F^*_{2134} + F^*_{1234}.$$

$$D^*(R_{(1,3)}) = F^*_{4123} + F^*_{3124} + F^*_{2134}.$$

In this sense, NSym is equivalent to the Solomon descent algebra as vector space.

Consider the elements $\left\{ \sum_{I^{-1}(peak(\sigma)) = \alpha} F^*_\sigma \right\}$. They form a basis for the image space of Θ_{NSym}. Therefore, the image of Θ_{NSym} is called the peak algebra.

Moreover, $\mathfrak{S}$Sym fits nicely into the following commutative square



2.1.5 Vargas Hopf algebra of permutations

In this section, we present another Hopf structure on permutations, the Hopf algebra $\mathcal{V}$ [53]. It also appears as associated graded Hopf algebra to $\mathfrak{S}$Sym [9].

Let

$$\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n,$$

where

$$\mathcal{V}_n = \mathbb{C}\text{-Span}\{v_\sigma : \sigma \in \mathfrak{S}_n\}.$$

By convention when $n = 0$, $\mathfrak{S}_0$ is the set of the permutation of empty set and $\mathcal{V}_0 = \mathbb{C}\text{-Span}\{1\}$.

For each permutation σ, there is a unique way of writing it as $\sigma_1 \backslash \sigma_2 \backslash \cdots \backslash \sigma_n$ such that each σ_i is a permutation with no global descents and σ_i is the standardization of the subword

$$(\sigma(|\sigma_1| + |\sigma_2| + \cdots + |\sigma_{i-1}| + 1), \ldots, \sigma(|\sigma_1| + |\sigma_2| + \cdots + |\sigma_i|)).$$

Example 2.1.29. *The permutation* 6743521 *corresponds to* $12\backslash 213\backslash 1\backslash 1$.

Assume that $\sigma = \sigma_1 \backslash \cdots \backslash \sigma_n$ and $\delta = \delta_1 \backslash \cdots \backslash \delta_m$. Let

$$c_1 \backslash c_2 \backslash \cdots \backslash c_{n+m} = \sigma_1 \backslash \cdots \backslash \sigma_n \backslash \delta_1 \backslash \cdots \backslash \delta_m.$$

We define the shuffle of σ and δ to be the shuffle of blocks,

$$\sigma_1 \backslash \cdots \backslash \sigma_n \sqcup\!\sqcup \delta_1 \backslash \cdots \backslash \delta_m = \{c_{\gamma(1)} \backslash c_{\gamma(2)} \backslash \cdots \backslash c_{\gamma(n+m)} : \gamma \in Sh_{n,m}\}.$$

We define the product and coproduct for $\mathcal{V}$ by

$$v_\sigma v_\delta = \sum_{w\in \sigma_1\backslash\cdots\backslash\sigma_n \sqcup \delta_1\backslash\cdots\backslash\delta_m} v_w$$

and

$$\Delta(v_\sigma) = \sum_{i=0}^{n} v_{\sigma_1\backslash\cdots\backslash\sigma_i} \otimes v_{\sigma_{i+1}\backslash\cdots\backslash\sigma_n}$$

respectively. By this product and coproduct, $\mathcal{V}$ is a Hopf algebra. In face, $\mathcal{V}$ is a shuffle algebra. This follows e.g. from the fact that quasi-symmetric is a Hopf algebra, and it is isomorphism to shuffle algebra [40, 28]. And we have a surjective Hopf morphism.

$$\begin{array}{rccc} \Psi: & \mathcal{V} & \to & \mathsf{QSym} \\ & v_{\sigma_1\backslash\cdots\backslash\sigma_n} & \mapsto & S_{(|\sigma_1|,\ldots,|\sigma_n|)} \end{array}$$

Remark 2.1.30. The Hopf algebra $\mathcal{V}$ is the coradical filtration of $\mathfrak{S}\mathsf{Sym}$. Hence, it is isomorphic to the graded dual of Grossman-Larson Hopf algebra of heap ordered trees [24]. The connection can be found in [9].

2.2 Combinatorial Hopf algebras

Combinatorial Hopf algebras usually refer to Hopf algebras that arise from combinatorial objects.

In this thesis, we will follow the definition in [5]. A combinatorial Hopf algebra is a pair $(\mathcal{H}, \zeta)$ where $\mathcal{H}$ is a graded connected Hopf $\Bbbk$-alegbra and $\zeta : \mathcal{H} \to \Bbbk$ is a multiplicative and linear map. The map ζ is called a character of $\mathcal{H}$.

Let $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$. With abuse of notation, we write $\zeta|_{\mathcal{H}_n}$ to be the function that maps a homogeneous element h to $\zeta(h)$ if $h \in \mathcal{H}_n$, and 0 otherwise. Then, $\zeta|_{\mathcal{H}_n} \in \mathcal{H}_n^*$.

The set of characters of $\mathcal{H}$ forms a group, with convolution product $\zeta * \zeta' = m \circ (\zeta \otimes \zeta') \circ \Delta$. For simplicity, we usually write multiplication of characters as $\zeta\zeta'$. The identity element is the counit ϵ, the inverse is given by $\zeta^{-1} = \zeta \circ S_{\mathcal{H}}$ where $S_{\mathcal{H}}$ is the antipode of $\mathcal{H}$. We define $\bar{\zeta}$ to be the character such that $\bar{\zeta}|_{\mathcal{H}_n} = (-1)^n \zeta|_{\mathcal{H}_n}$. Note that $\bar{\bar{\zeta}} = \zeta$. For more details about the group of characters of a combinatorial Hopf algebra, refer to [5].

The class of combinatorial Hopf algebras forms a category. A *combinatorial Hopf morphism* between two combinatorial Hopf algebras $(\mathcal{H}_1, \zeta_1)$ and $(\mathcal{H}_2, \zeta_2)$ is a Hopf morphism $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\zeta_1 = \zeta_2 \circ \Phi$.

Consider the canonical linear map $\zeta : \Bbbk[[x_1, x_2, \ldots]] \to \Bbbk$ satisfying $\zeta(x_i) = \delta_{i,1}$.

Restricting this character to Sym and QSym gives $\zeta_{\mathsf{Sym}} : \mathsf{Sym} \to \Bbbk$, where $\zeta_{\mathsf{Sym}}(m_\lambda) = \begin{cases} 1 & \text{if } \lambda = (n) \text{ or } (), \\ 0 & \text{otherwise} \end{cases}$ and $\zeta_{\mathsf{QSym}} : \mathsf{QSym} \to \Bbbk$, where $\zeta_{\mathsf{QSym}}(M_\alpha) = \begin{cases} 1 & \text{if } \alpha = (n) \text{ or } (), \\ 0 & \text{otherwise.} \end{cases}$

Theorem 2.2.1. *[5, Theorem 4.1] For any combinatorial coalgebra (Hopf algebra) $(\mathcal{H}, \zeta)$, there exists a unique morphism of combinatorial coalgebras (Hopf algebras)*

$$\Psi : (\mathcal{H}, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}}).$$

Moreover, Ψ is explicitly given as follows. For $h \in \mathcal{H}_n$,

$$\Psi(h) = \sum_{\alpha \models n} \zeta_\alpha M_\alpha$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_k)$, ζ_α is the composite

$$\mathcal{H} \xrightarrow{\Delta^{(k-1)}} \mathcal{H}^{\otimes k} \twoheadrightarrow \mathcal{H}_{\alpha_1} \otimes \ldots \otimes \mathcal{H}_{\alpha_k} \xrightarrow{\zeta^{\otimes k}} \mathbb{k},$$

where the unlabelled map is the canonical projection onto a homogeneous component. Also, if $\mathcal{H}$ is cocommutative, then $\Psi(\mathcal{H}) \subseteq \mathsf{Sym}$.

The theorem above is to say that $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ is the terminal object in the category of combinatorial Hopf algebra.

2.2.1 Odd Hopf sub-algebras

Definition 2.2.2. A character ζ of a graded Hopf algebra $\mathcal{H}$ is said to be even if

$$\overline{\zeta} = \zeta$$

and it is said to be odd if

$$\overline{\zeta} = \zeta^{-1}.$$

Given a character ζ of $\mathcal{H}$, let $S_-(\mathcal{H}, \zeta)$ be the largest graded coalgebra of $\mathcal{H}$ such that

$$\forall h \in S_-(\mathcal{H}, \zeta), \overline{\zeta}(h) = \zeta^{-1}(h),$$

or equivalently

$$\forall h \in S_{-}(\mathcal{H}, \zeta), \overline{\zeta}\zeta(h) = \epsilon(h).$$

Indeed, by [5, Proposition 5.8] $S_{-}(\mathcal{H}, \zeta)$ is a Hopf algebra and is called the odd Hopf subalgebra of $(\mathcal{H}, \zeta)$. Also, $S_{-}(\mathcal{H}, \zeta)$ is the set of all elements $h \in \mathcal{H}$ satisfying one of the following equivalent relations:

$$(id \otimes (\overline{\zeta} - \zeta^{-1}) \otimes id) \circ \Delta^2(h) = 0,$$

$$(id \otimes (\chi - \epsilon) \otimes id) \circ \Delta^2(h) = 0,$$

where $\chi = \overline{\zeta}\zeta$ and ϵ is the counit of $\mathcal{H}$.

These are called the *Generalized Dehn-Sommerville relations* for the combinatorial Hopf algebra $(\mathcal{H}, \zeta)$. When we choose $(\mathcal{H}, \zeta)$ to be $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$, we obtain the relations introduced in [10].

2.3 Theta maps

We have already introduced Θ_{Sym}, Θ_{QSym} and Θ_{NSym}. In this section, we will describe them in terms of combinatorial Hopf algebras.

Consider the combinatorial Hopf algebra $\left(\mathsf{QSym}, \overline{\zeta^{-1}_{\mathsf{QSym}}}\zeta_{\mathsf{QSym}}\right)$. According to the universal property, we have a unique combinatorial Hopf morphism Φ :

$\mathsf{QSym} \to \mathsf{QSym}$ that can be described as follows.

Let $odd(\beta)$ be the composition obtained by adding the entries within each maximal segment of β of the form (even, even, ..., odd). For example, $odd(3,4,2,1,3,2,1) = (3,7,3,3)$. Then by [5, Example 4.9] we have

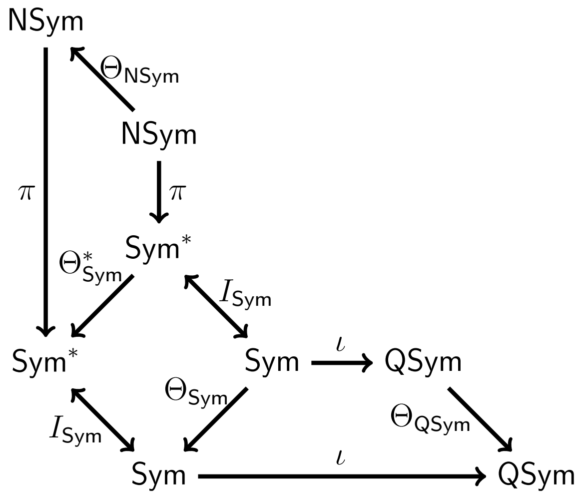
$$\Phi(M_\beta) = \begin{cases} 1 & \text{if } \beta = (), \\ (-1)^{|\beta|+l(\beta)} \sum_{\alpha \leq odd(\beta)} 2^{l(\alpha)} M_\alpha & \text{if the last part of } \beta \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

In fact, this map Φ is exactly the map Θ_{QSym} according to a result by Hsiao [19]. Moreover, its image, the space of enriched P-partitions, is the odd Hopf sub-algebra of $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$.

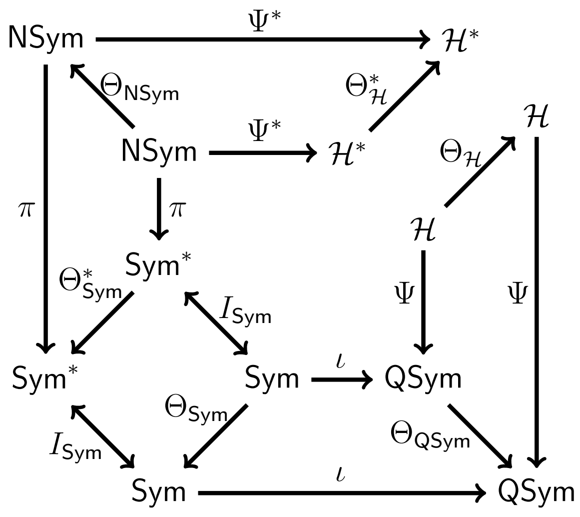
Similarly, for the combinatorial Hopf algebra $\left(\mathsf{Sym}, \overline{\zeta^{-1}_{\mathsf{Sym}}}\zeta_{\mathsf{Sym}}\right)$, we obtain a universal combinatorial Hopf morphism $\Psi : \mathsf{Sym} \to \mathsf{Sym}$ (as Sym is co-commutative). And this map Ψ is exactly the map Θ_{Sym}. Moreover, its image, the space of Schur's $\mathcal{Q}$ functions, is the odd Hopf sub-algebra of $(\mathsf{Sym}, \zeta_{\mathsf{Sym}})$.

Let $\zeta_{\mathsf{NSym}} = \zeta_{\mathsf{Sym}} \circ \pi$. Clearly, this is a character for NSym. Then, the image of Θ_{NSym}, the peak algebra, is the odd Hopf sub-algebra of $(\mathsf{NSym}, \zeta_{\mathsf{NSym}})$.

Up to this point in the thesis, we have the following commutative diagram.



We now can define theta maps for an arbitrary combinatorial Hopf algebra. Let $(\mathcal{H}, \zeta)$ be a combinatorial Hopf algebra, and $\Psi : \mathcal{H} \to \mathsf{QSym}$ be the universal combinatorial Hopf morphism. A map $\Theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is called a theta map if it is a Hopf mophism that makes the following diagram commute.



Moreover, if $\mathcal{H}$ is self-dual, we want $\Theta_{\mathcal{H}}$ to be self-adjoint. In this case, the diagram above can be reduced to a cube.

It is not hard to see that, because of duality, $\Theta_{\mathcal{H}}$ is a theta map if the following square commutes

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Theta_{\mathcal{H}}} & \mathcal{H} \\ \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\Psi} \\ \mathsf{QSym} & \xrightarrow{\Theta_{\mathsf{QSym}}} & \mathsf{QSym} \end{array}$$

The image of $\Theta_{\mathcal{H}}$ will be a lifting of the space of enriched $\mathcal{P}$-partitions. The main objective of this thesis is to study possible ways to lift Θ_{QSym} to $\Theta_{\mathcal{H}}$ and study their images which are potentially generalizations of enriched $\mathcal{P}$-partitions and the peak algebra.

Chapter 3

The odd Hopf subalgebras for combinatorial Hopf algebras

In this section we analyze the image of combinatorial Hopf morphisms, more specifically, we show that the image of a theta map is contained in odd Hopf subalgebra. We will extensively use the map Θ_{QSym}, and we will denote its image, the space of enriched P-partitions, by Π. A main part of this section comes from the paper [7].

Any graded Hopf algebra H carries a canonical automorphism

$$h \mapsto \overline{h} := (-1)^n h$$

for homogeneous elements $h \in \mathcal{H}_n$. This is an involution: $\overline{\overline{h}} = h$. Therefore, it induces an involution $\phi \mapsto \overline{\phi}$ on the character group of $\mathcal{H}$, with

$$\overline{\phi}(h) = (-1)^n \phi(h) \quad \text{for } h \in \mathcal{H}_n.$$

The image of Theta maps for Sym, QSym and NSym are the algebra of Q-Schur functions, the Peak algebra and the dual of peak algebra, respectively, and they are the corresponding odd Hopf subalgebras. We will show that if Θ is a Theta map for $(\mathcal{H},\zeta)$, then the image of Θ must be in $S_-(\mathcal{H},\zeta)$. Before that we need following theorem which also help us to find a strategy for finding the odd Hopf subalgebra of any combinatorial Hopf algebra.

Theorem 3.0.1. *Let $(\mathcal{H},\zeta)$ and $(\mathcal{H}',\zeta')$ be combinatorial Hopf algebras. Let $\alpha : (\mathcal{H},\zeta) \to (\mathcal{H}',\zeta')$ be a combinatorial Hopf morphism.*

1. *Let I be a Hopf ideal of $\mathcal{H}$ such that $I \subseteq \ker(\zeta)$. Then $(\mathcal{H}/I,\zeta_I)$ is a combinatorial Hopf algebra where $\zeta_I(h+I) = \zeta(h)$ for every $h \in \mathcal{H}$.*

2. *Let I be a Hopf ideal of $\mathcal{H}$ such that $I \subseteq \ker(\zeta)$. Then $S_-(\mathcal{H}/I,\zeta_I) = S_-(\mathcal{H},\zeta)/I$.*

3. *If α is surjective, then $\alpha(S_-(\mathcal{H},\zeta)) = S_-(\mathcal{H}',\zeta')$.*

4. *$S_-(\mathcal{H},\zeta) = \alpha^{-1}(S_-(\mathcal{H}',\zeta'))$, where $\alpha^{-1}(S_-(\mathcal{H}',\zeta')) = \{h \in \mathcal{H} : \alpha(h) \in S_-(\mathcal{H}',\zeta')\}$.*

Proof. (1) We first show that ζ_I is well defined. For all $g \in I \subseteq \ker(\zeta)$, we have $\zeta_I(I) = \zeta_I(g+I) = \zeta(g) = 0$.

Since I is a Hopf ideal, $\mathcal{H}/I$ is a Hopf algebra. Also, ζ_I is a character of $\mathcal{H}/I$ because

$$\zeta_I((h+I)(g+I)) = \zeta_I(hg+I) = \zeta(hg) = \zeta(h)\zeta(g) = \zeta_I(h+I)\zeta_I(g+I).$$

(2) Let

$$\begin{array}{rccc} p: & (\mathcal{H},\zeta) & \to & (\mathcal{H}/I,\zeta_I) \\ & h & \mapsto & h+I. \end{array}$$

This is a Hopf morphism and by [5, Proposition 5.6(a)], $p(S_-(\mathcal{H},\zeta)) \subseteq S_-(\mathcal{H}/I,\zeta_I)$ i.e.,

$$(S_-(\mathcal{H},\zeta)+I)/I \subseteq S_-(\mathcal{H}/I,\zeta_I).$$

Since $S_-(\mathcal{H}/I,\zeta_I)$ is a Hopf subalgebra of $\mathcal{H}/I$ ([5], Proposition 5.8), there is a graded coalgebra A of $\mathcal{H}$ such that

$$S_-(\mathcal{H}/I,\zeta_I) = A/I.$$

The Hopf algebra $S_-(\mathcal{H}/I,\zeta_I)$ is the largest graded Hopf subalgebra of $\mathcal{H}$ contained in $\ker(\chi_I - \epsilon_I)$ where ϵ_I is the counit of $\mathcal{H}/I$ and $\chi_I = \overline{\zeta_I}\zeta_I$ is the Euler character of $\mathcal{H}/I$. Let a be a homogeneous element in A. Note that $S_-(\mathcal{H}/I,\zeta_I) = A/I$, thus

$$\chi(a) - \epsilon(a) = \chi_I(a+I) - \epsilon_I(a+I) = 0.$$

Therefore,

$$A \subseteq S_-(\mathcal{H},\zeta).$$

Since $(S_-(\mathcal{H},\zeta)+I)/I \subseteq S_-(\mathcal{H}/I,\zeta_I) = A/I$, we conclude that $S_-(\mathcal{H},\zeta) \subseteq A$, and so

$$S_-(\mathcal{H}/I,\zeta_I) = A/I = S_-(\mathcal{H},\zeta)/I.$$

(3) Since α is a combinatorial Hopf morphism, $\zeta^{'} \circ \alpha = \zeta$. Thus, we have

$\ker(\alpha) \subseteq \ker(\zeta)$, and so $(\mathcal{H}/\ker(\alpha), \zeta_{\ker(\alpha)})$ is a combinatorial Hopf algebra. Therefore,

$$\begin{array}{rccc} \overline{\alpha}: & (\mathcal{H}/\ker(\alpha), \zeta_{\ker(\alpha)}) & \to & (\mathcal{H}^{'}, \zeta^{'}) \\ & h + ker(\alpha) & \mapsto & \alpha(h). \end{array}$$

is well defined and is a Hopf algebra isomorphism. Furthermore, by (2),

$$\alpha(S_{-}(\mathcal{H}, \zeta)) = \overline{\alpha}(S_{-}(\mathcal{H}/ker(\alpha), \zeta_{ker(\alpha)})) = S_{-}(\mathcal{H}^{'}, \zeta^{'}).$$

(4) We first show that $\alpha(S_{-}(\mathcal{H}, \zeta)) = S_{-}(\mathcal{H}', \zeta') \cap \alpha(\mathcal{H})$. By [5, Proposition 5.6], we have

$$\alpha(S_{-}(\mathcal{H}, \zeta)) \subseteq S_{-}(\mathcal{H}', \zeta')$$

and therefore

$$\alpha(S_{-}(\mathcal{H}, \zeta)) \subseteq S_{-}(\mathcal{H}', \zeta') \cap \alpha(\mathcal{H}).$$

For the other direction, observe that $S_{-}(\mathcal{H}', \zeta') \cap \alpha(\mathcal{H})$ is a graded sub-coalgebra of $\alpha(\mathcal{H})$. Moreover, for all $m \in S_{-}(\mathcal{H}', \zeta') \cap \alpha(\mathcal{H})$, we have $m \in S_{-}(\mathcal{H}', \zeta')$ and by definition, $\bar{\zeta}'(m) = \zeta'^{-1}(m)$. Therefore, $S_{-}(\mathcal{H}', \zeta') \cap \alpha(\mathcal{H}) \subseteq S_{-}(\alpha(\mathcal{H}), \zeta'|_{\alpha(\mathcal{H})}) = \alpha(S_{-}(\mathcal{H}, \zeta))$ as the last equality follows from (3).

If $\alpha(h) \in S_{-}(\mathcal{H}', \zeta')$ for some $h \in \mathcal{H}$, we have

$$\alpha(h) \in S_{-}(\mathcal{H}', \zeta') \cap \alpha(\mathcal{H}) = \alpha(S_{-}(\mathcal{H}, \zeta))$$

i.e. there exists some $k \in S_{-}(\mathcal{H}, \zeta)$ such that $\alpha(h) = \alpha(k)$, or equivalently $h - k \in \ker(\alpha)$. But we know that $\ker(\alpha) \subseteq \ker(\zeta) \subseteq S_{-}(\mathcal{H}, \zeta)$ because $\ker(\zeta)$ is a graded sub-coalgebra of $\mathcal{H}$ and for all $g \in \ker(\zeta)$, $\bar{\zeta}(g) = \zeta^{-1}(g) = 0$.

Hence, $h \in S_-(\mathcal{H}, \zeta)$ i.e. $\alpha^{-1}(S_-(\mathcal{H}', \zeta')) \subseteq S_-(\mathcal{H}, \zeta)$. The other direction, $S_-(\mathcal{H}, \zeta) \subseteq \alpha^{-1}(S_-(\mathcal{H}', \zeta'))$, follows from [5, Proposition 5.6(a)].

□

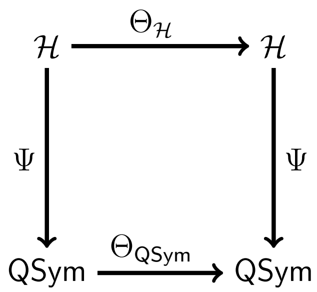
We now show that the image of a theta map for $(\mathcal{H}, \zeta)$ is in the odd Hopf subalgebra of $(\mathcal{H}, \zeta)$.

Corollary 3.0.2. *Let Θ be a theta map for $(\mathcal{H}, \zeta)$. Then the image of Θ is in $S_-(\mathcal{H}, \zeta)$.*

Proof. A Theta map for a combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ is a Hopf algebra map

$$\Theta : \mathcal{H} \to \mathcal{H}$$

that makes the following diagram commutative.



Therefore, we have $\Psi \circ \Theta(\mathcal{H}) \subseteq \Pi$. Since Π is the odd Hopf subalgebra of QSym, by Theorem 3.0.1 (4) we have $\Theta(\mathcal{H}) \subseteq S_-(\mathcal{H}, \zeta)$.

□

3.1 A Strategy for finding $S_-(\mathcal{H}, \zeta)$

In [5, Sections 6,7,8], the authors give dimensions of the odd Hopf subalgebra of Sym, QSym, NSym, $\mathfrak{S}\mathsf{Sym}$, and LR. In the following, we present a strategy to find an explicit basis for the odd combinatorial Hopf subalgebra of an arbitrary combinatorial Hopf algebra.

1. By [5, Theorem 4.1], there is a combinatorial Hopf morphism $\Psi : (\mathcal{H}, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$.

2. By [5, Proposition 5.8],

$$S_-(Img(\Psi), \zeta_{\mathsf{QSym}}|_{\ker \Psi}) = Img(\Psi) \cap \Pi.$$

3. We have $(\mathcal{H}/\ker \Psi, \zeta_{\ker \Psi}) \cong Img(\Psi)$. So by Theorem 3.0.1 (3),

$$S_-(\mathcal{H}, \zeta)/\ker \Psi \cong S_-(Img(\Psi), (\zeta_{\mathsf{QSym}})_{\ker \Psi}) = Img(\Psi) \cap \Pi.$$

4. Let $\{v_i : i \in I\}$ be a basis for $Img(\Psi) \cap \Pi$. Pick a set $\{s_i : i \in I\}$ such that $\overline{\Psi}(s_i + \ker \Psi) = v_i$.

5. Pick a basis $\{k_j : j \in J\}$ for $\ker \Psi$.

6. The set $\{s_i, k_j : i \in I, j \in J\}$ is a basis for $S_-(\mathcal{H}, \zeta)$.

In the next sections, we apply this strategy to find $S_-\mathcal{H}, \zeta$ for $\mathfrak{S}\mathsf{Sym}$ and $\mathcal{V}$.

3.2 The odd Hopf subalgebra of $\mathfrak{S}\mathsf{Sym}$

We now use the above steps to find the odd Hopf subalgebra of $(\mathfrak{S}\mathsf{Sym}, \zeta_{\mathfrak{S}\mathsf{Sym}})$.

For permutations $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$, recall that we write $\rho = \sigma \backslash \tau$ when

$$\rho = \sigma(1) + q, \sigma(2) + q, \cdots, \sigma(p) + q, \tau(1), \tau(2), \cdots, \tau(q).$$

Every permutation σ can be decomposed to permutation with no global descent, i.e.,

$$\rho = \sigma_1 \backslash \cdots \backslash \sigma_k,$$

where each σ_i has no global descent. Let $k(\sigma) := k$.

Let D be the surjective descent map from $\mathfrak{S}\mathsf{Sym}$ to QSym. Assume that $\sigma \in \mathfrak{S}_n$, by [8, Proposition 1.4] we have that

$$D(M_\sigma) = \begin{cases} M_{(\alpha_1,\dots,\alpha_k)} & \text{if } \sigma = id_{\alpha_1} \backslash id_{\alpha_2} \backslash \cdots \backslash id_{\alpha_k}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\ker D = \bigoplus_{n \geq 2} K\mathfrak{S}_n$$

where

$$K\mathfrak{S}_n = \mathbb{C}\text{-Span}\{M_\sigma : \sigma \neq id_{\alpha_1} \backslash id_{\alpha_2} \backslash \cdots \backslash id_{\alpha_k}, \text{ for all } (\alpha_1, \dots, \alpha_k) \models n\}.$$

A composition $\alpha = (\alpha_1, \dots, \alpha_l)$ of n is said to be odd if each part α_i of α is odd. Given an odd composition β, set

$$\eta_\beta = \sum_{\alpha \le \beta} 2^{\ell(\alpha)} M_\alpha,$$

where $\ell(\alpha)$ is the length of α and the sum is over all compositions $\alpha \le \beta$ in the weak order. In [5, Proposition 6.5] it was shown that $\{\eta_\beta\}_{\beta \text{ odd}}$ is a basis for Π. A permutation σ is said to be odd if $\sigma = id_{n_1} \backslash \ldots \backslash id_{n_k}$ and each n_i is odd. Given an odd permutation σ, set

$$\eta_\sigma = \sum_{\tau \le \sigma} 2^{k(\tau)} M_\tau.$$

Note that for an odd permutation $\sigma = id_{n_1} \backslash \ldots \backslash id_{n_k}$, we have

$$D(\eta_\sigma) = D\left(\sum_{\tau \le \sigma} 2^{k(\tau)} M_\tau\right) = \sum_{\tau \le \sigma} 2^{k(\tau)} D(M_\tau)$$

We see that $D(M_\tau) = 0$ if $\tau \neq id_{\alpha_1} \backslash id_{\alpha_2} \backslash \cdots \backslash id_{\alpha_k}$, for all $(\alpha_1, \ldots, \alpha_k) \models n$. Also, if $\tau = id_{\alpha_1} \backslash id_{\alpha_2} \backslash \cdots \backslash id_{\alpha_k}$, for some $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$, then $k(\tau) = l(\alpha)$. Therefore,

$$D(\eta_\sigma) = \sum_{\tau \le \sigma} 2^{k(\tau)} D(M_\tau) = \sum_{\alpha \le \beta} 2^{l(\alpha)} M_\alpha = \eta_\beta,$$

where $\beta = (n_1, \ldots, n_k)$. To see the equality in the middle, since the sum is over all τ whose inversion set is a subset of that of σ, the only τ such that $D(M_\tau) = 0$ are those of the form $\tau = id_{\alpha_1 + \cdots + \alpha_{i_1}} \backslash id_{\alpha_{i_1+1} + \cdots + \alpha_{i_2}} \backslash \cdots \backslash id_{\alpha_{i_t+1} + \cdots + \alpha_k}$.

We conclude that

$$\{M_\sigma : \sigma \neq id_{\alpha_1} \backslash id_{\alpha_2} \backslash \cdots \backslash id_{\alpha_k}, \text{ for all } (\alpha_1, \ldots, \alpha_k) \models n\} \sqcup \{\eta_\sigma\}_{\sigma \text{ odd}}$$

is a basis for $S_-(\mathfrak{S}\mathsf{Sym}, \zeta_{\mathfrak{S}\mathsf{Sym}})$, and also

$$S_-(\mathfrak{S}\mathsf{Sym}, \zeta_{\mathfrak{S}\mathsf{Sym}}) = \ker D \oplus \mathbb{C}\text{-Span}\{\eta_\sigma\}_{\sigma \text{ odd}}.$$

3.3 The odd Hopf subalgebra of $\mathcal{V}$

Let $\sigma_1 \backslash \cdots \backslash \sigma_n$ be the unique decomposition of a permutation such that each σ_i has no global descent. We define the following function,

$$\begin{array}{rccc} \zeta_{\mathcal{V}}: & \mathcal{V} & \to & \mathbb{C} \\ & v_{\sigma_1 \backslash \cdots \backslash \sigma_n} & \mapsto & 1/n!. \end{array}$$

Lemma 3.3.1. *The function $\zeta_{\mathcal{V}}$ is a character for $\mathcal{V}$.*

Proof. To show the character is multiplicative, we compute

$$\zeta_{\mathcal{V}}(\nu_\sigma \nu_\delta) = \sum_{w \in \sigma_1 \backslash \cdots \backslash \sigma_n \sqcup \delta_1 \backslash \cdots \backslash \delta_m} \zeta_{\mathcal{V}}(v_w) = \sum_{w \in \sigma_1 \backslash \cdots \backslash \sigma_n \sqcup \delta_1 \backslash \cdots \backslash \delta_m} \frac{1}{(m+n)!} =$$

$$\frac{1}{(m+n)!}\binom{m+n}{n} = \frac{1}{n!}\frac{1}{m!} = \zeta_{\mathcal{V}}(\nu_\rho)\zeta_{\mathcal{V}}(\nu_\delta).$$

□

For each composition $\alpha = (\alpha_1, \ldots, \alpha_n) \leq \beta$, let $d^\alpha_\beta = \frac{1}{n_1! \ldots n_k!}$ where $(\alpha_1 + \ldots + \alpha_{n_1}, \alpha_{n_1+1} + \ldots + \alpha_{n_1+n_2}, \ldots, \alpha_{n-n_k+1} + \ldots + \alpha_n)$. By Theorem 2.2.1, the character $\zeta_{\mathcal{V}}$ yields a Hopf algebra morphism Ψ from $\mathcal{V}$ to QSym as follows

$$\begin{array}{rccc} \Psi: & \mathcal{V} & \to & \mathsf{QSym} \\ & v_{\rho_1 \backslash \cdots \backslash \rho_n} & \mapsto & S_{(|\rho_1|, \cdots, |\rho_n|)}, \end{array}$$

where for every composition $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$S_\alpha = \sum_{\beta \leq \alpha} d^\alpha_\beta M_\beta.$$

Since $\{S_\alpha : \alpha \models n\}$ is a basis for QSym_n, we can write

$$M_\alpha = \sum_{\beta \models n} c^\alpha_\beta S_\beta$$

for some coefficients c^α_β in $\mathbb{C}$.

If $\sigma \in \mathfrak{S}_n$, let $|\sigma| := n$. Define a new basis $\{M_\sigma : \sigma \in \sqcup_{n\geq 0}\mathfrak{S}_n\}$ for $\mathcal{V}$, where

$$M_{\sigma_1\backslash\cdots\backslash\sigma_k} := \begin{cases} \sum_{\beta\models n} c_\beta^{(|\sigma_1|,\ldots,|\sigma_k|)} \nu_{id_{\beta_1}\backslash\cdots\backslash id_{\beta_{l(\beta)}}} & \text{if } \sigma = id_{|\sigma_1|}\backslash\cdots\backslash id_{|\sigma_k|} \\ \nu_{\sigma_1\backslash\cdots\backslash\sigma_k} - \nu_{id_{|\sigma_1|}\backslash\cdots\backslash id_{|\sigma_k|}} & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\Psi(M_{\sigma_1\backslash\cdots\backslash\sigma_k}) = \begin{cases} M_{(|\sigma_1|,\ldots,|\sigma_k|)} & \text{if } \sigma = id_{|\sigma_1|}\backslash\cdots\backslash id_{|\sigma_k|} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\ker\Psi = \bigoplus_{n\geq 2} K\mathcal{V}_n$$

where

$$K\mathcal{V}_n = \mathbb{C}\text{-Span}\{M_\sigma : \sigma \neq id_{\alpha_1}\backslash id_{\alpha_2}\backslash\cdots\backslash id_{\alpha_k}, \text{ for every } (\alpha_1,\ldots,\alpha_k)\models n\}.$$

Note that if we restrict Ψ to $\mathbb{C}$-Span$\{M_{id_{n_1}\backslash\cdots\backslash id_{n_k}} : n_i \in \mathbb{N}, k \in \mathbb{N}_0\}$ we have a

Hopf algebra isomorphism, thus we have

$$\mathbb{C}\text{-Span}\{M_{id_{n_1}\backslash\cdots\backslash id_{n_k}} : n_i \in \mathbb{N}, k \in \mathbb{N}_0\} \cong \mathsf{QSym}\,.$$

Also,

$$\mathcal{V} = \ker\Psi \oplus \mathbb{C}\text{-Span}\{M_{id_{n_1}\backslash\cdots\backslash id_{n_k}} : n_i \in \mathbb{N}, k \in \mathbb{N}_0\}.$$

Given an odd permutation $\sigma = id_{n_1}\backslash \ldots \backslash id_{n_k}$, set

$$\eta^{\mathcal{V}}_\sigma := \sum_{\alpha \le (n_1,\ldots,n_k)} 2^{l(\alpha)} M_{id_{\alpha_1}\backslash\ldots\backslash id_{\alpha_{l(\alpha)}}}.$$

Note that for an odd permutation $\sigma = id_{n_1}\backslash \ldots \backslash id_{n_k}$, we have

$$\Psi(\eta^{\mathcal{V}}_\sigma) = \Psi(\sum_{\alpha \le (n_1,\ldots,n_k)} 2^{l(\alpha)} M_{id_{\alpha_1}\backslash\ldots\backslash id_{\alpha_{l(\alpha)}}}) = \sum_{\alpha \le (n_1,\ldots,n_k)} 2^{l(\alpha)} \Psi(M_{id_{\alpha_1}\backslash\ldots\backslash id_{\alpha_{l(\alpha)}}}) =$$

$$\sum_{\alpha \le (n_1,\ldots,n_k)} 2^{l(\alpha)} M_{\alpha_1,\ldots,\alpha_{l(\alpha)}} = \eta_{(n_1,\ldots,n_k)}.$$

We conclude that

$$\{M_\sigma : \sigma \neq id_{\alpha_1}\backslash id_{\alpha_2}\backslash \cdots \backslash id_{\alpha_k},\ \text{for every } (\alpha_1,\ldots,\alpha_k) \models n\} \sqcup \{\eta_\sigma\}_{\sigma \text{ odd}}$$

is a basis for $S_-(\mathcal{V},\zeta_{\mathcal{V}})$, and also

$$S_-(\mathcal{V},\zeta_{\mathcal{V}}) = \ker\Psi \oplus \mathbb{C}\text{-Span}\{\eta^{\mathcal{V}}_\sigma\}_{\sigma \text{ odd}},$$

where

$$\Pi \cong \mathbb{C}\text{-Span}\{\eta^{\mathcal{V}}_\sigma\}_{\sigma \text{ odd}}.$$

Chapter 4

Theta maps for combinatorial Hopf algebras

In this section, the necessary and sufficient conditions for existence of a Theta map will be presented. Later on, we will find Theta maps for several families of Hopf algebras and also $\mathcal{V}$ and $\mathfrak{S}\mathsf{Sym}$. A part of this chapter comes from the paper [7].

The following theorem gives the necessary and sufficient conditions for a Hopf map to be a Theta map.

Theorem 4.0.2. *Let $(\mathcal{H}, \zeta_{\mathcal{H}})$ be a combinatorial Hopf algebra and let $\Psi : (\mathcal{H}, \zeta_{\mathcal{H}}) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ be the canonical Hopf morphism. There exists a combinatorial Hopf morphism $\Theta_{\mathcal{H}} : (\mathcal{H}, \overline{\zeta_{\mathsf{QSym}}^{-1}}\zeta_{\mathsf{QSym}} \circ \Psi) \to (\mathcal{H}, \zeta_{\mathcal{H}})$ if and only if the following diagram commutes,*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Theta_{\mathcal{H}}} & \mathcal{H} \\ \Big\downarrow \Psi & & \Big\downarrow \Psi \\ \mathsf{QSym} & \xrightarrow{\Theta_{\mathsf{QSym}}} & \mathsf{QSym} \end{array}$$

i.e., $\mathcal{H}$ has a theta map.

Proof. We have

$$\overline{\zeta^{-1}_{\mathsf{QSym}}}\zeta_{\mathsf{QSym}} \circ \Psi = \zeta_{\mathcal{H}} \circ \Theta_{\mathcal{H}} \qquad (\Theta_{\mathcal{H}} \text{ is a combinatorial Hopf morphism})$$

$$\Updownarrow$$

$$\overline{\zeta^{-1}_{\mathsf{QSym}}}\zeta_{\mathsf{QSym}} \circ \Psi = \zeta_{\mathsf{QSym}} \circ \Psi \circ \Theta_{\mathcal{H}} \qquad \left(\begin{array}{c} \Psi \text{ is a combinatorial Hopf morphism} \\ \text{and so } \zeta_{\mathcal{H}} = \zeta_{\mathsf{QSym}} \circ \Psi \end{array} \right)$$

$$\Updownarrow$$

$$\zeta_{\mathsf{QSym}} \circ \Theta_{\mathsf{QSym}} \circ \Psi = \zeta_{\mathsf{QSym}} \circ \Psi \circ \Theta_{\mathcal{H}} \qquad (\text{because } \overline{\zeta^{-1}_{\mathsf{QSym}}}\zeta_{\mathsf{QSym}} = \zeta_{\mathsf{QSym}} \circ \Theta_{\mathsf{QSym}})$$

$$\Updownarrow$$

$$\Theta_{\mathsf{QSym}} \circ \Psi = \Psi \circ \Theta_{\mathcal{H}}.$$

The last equation is because there is a unique combinatorial morphism from $(\mathcal{H}, \zeta_{\mathcal{H}})$ to $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ for any character $\zeta_{\mathcal{H}}$. We have $\Theta_{\mathsf{QSym}} \circ \Psi$ and $\Psi \circ \Theta_{\mathcal{H}}$ are combinatorial Hopf morphisms from $(\mathcal{H}, \zeta_{\mathsf{QSym}} \circ \Theta_{\mathsf{QSym}} \circ \Psi)$ to $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$, thus they must be equal.

$\square$

4.1 Theta map for NSym in the immaculate basis

In this section, we study the map Θ_{NSym} in terms of the immaculate basis. We begin with proving some nice properties. A main part of this section comes from the paper [35].

In [18], the authors gave a formula for left Pieri Rule using the dual Hopf algebra, quasi-symmetric functions (QSym). We will give another proof of that using a combinatorial approach.

Unlike the commutative case, for immaculate functions, the left Pieri rule is much different from the right Pieri rule. As shown in Example 2.1.27, the structure constants could be negative.

We start with the following property.

Let $C^{\gamma}_{\alpha,\beta}$ be the multiplicative structure constants for immaculate basis i.e.

$$\mathfrak{S}_\alpha \mathfrak{S}_\beta = \sum_{\gamma} C^{\gamma}_{\alpha,\beta} \mathfrak{S}_\gamma.$$

Theorem 4.1.1. *For compositions α, β, γ, v with $\ell(v) \leq \ell(\alpha)$, we have $C^{\gamma}_{\alpha,\beta} = C^{\gamma+v}_{\alpha+v,\beta}$.*

Proof. Let $\beta = (\beta_1, \ldots, \beta_m)$. Using the definition (2.1) of $\mathfrak{S}_\beta$, we have

$$\mathfrak{S}_\alpha \mathfrak{S}_\beta = \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma \mathfrak{S}_\alpha H_{\beta_1+\sigma_1-1, \beta_2+\sigma_2-2, \ldots, \beta_m+\sigma_m-m}.$$

An iterative use of the right Pieri rule (Theorem 2.1.25) gives

$$\mathfrak{S}_\alpha H_\tau = \sum_{\substack{sh(T)=\gamma/\alpha \\ c(T)=\tau}} \mathfrak{S}_\gamma$$

where τ is an integer vector and the sum is over all skew immaculate tableaux T.

Combining the two equations above yields

$$\mathfrak{S}_\alpha \mathfrak{S}_\beta = \sum_{\sigma \in \mathfrak{S}_m} \sum_{\substack{sh(T)=\gamma/\alpha \\ c(T)=\beta+\sigma-Id}} (-1)^\sigma \mathfrak{S}_\gamma.$$

Let $\mathfrak{T}_\alpha^\beta$ be the set of skew immaculate tableaux of inner shape α for which $c(T) - \beta + Id$ is a permutation in S_m (in one-line notation), where $Id = (1, 2, \ldots., m)$. In this case, entries in T must be in $\{1, 2, \ldots, m\}$ and $c(T)$ means the content vector of length m by padding 0's. Let $\sigma(T) = c(T) - \beta + Id$, we have:

$$\mathfrak{S}_\alpha \mathfrak{S}_\beta = \sum_{T \in \mathfrak{T}_\alpha^\beta} (-1)^{\sigma(T)} \mathfrak{S}_{outsh(T)} \tag{4.1}$$

Example 4.1.2. *Let* $\alpha = (1)$, $\beta = (1, 3, 1)$, *then*

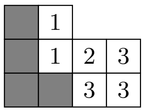
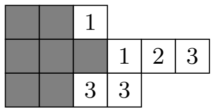
$$T = \begin{array}{|c|c|c|} \hline \blacksquare & 2 & \\ \hline 1 & 2 & 2 \\ \hline 2 & & \\ \hline \end{array} \in \mathfrak{T}_\alpha^\beta$$

and $\sigma(T) = (1, 4, 0) - (1, 3, 1) + (1, 2, 3) = (1, 3, 2)$.

For each $T \in \mathfrak{T}_\alpha^\beta$ with $sh(T) = \gamma/\alpha$, we move each of the first $\ell(v)$ rows of T to the right by a certain number of steps, namely, the i-th row by v_i steps, where $v = (v_1, v_2, \ldots, v_{\ell(v)})$. By this construction, we obtain a tableau $T' \in \mathfrak{T}^{\beta}_{\alpha+v}$ with $sh(T') = (\gamma + v)/\alpha$, and vice versa. Since $\ell(v) \leq \ell(\alpha)$, the first column is preserved under this map. Moreover, $c(T) = c(T')$ and hence $\sigma(T) = \sigma(T')$. Then, the result follows. $\square$

In the case that $\gamma - \alpha$ has negative entries, $C^{\gamma}_{\alpha\beta}$ must be zero by equation (4.1). Therefore, in order to understand $C^{\gamma}_{\alpha\beta}$, it suffices to understand those when α is the n-tuple $(1, 1, \ldots, 1)$ for $n \in \mathbb{N}$.

Example 4.1.3. *Let* $\alpha = (1, 1, 2)$, $\beta = (2, 1, 3)$, $v = (1, 2)$, *then*


$\in \mathfrak{T}^{\beta}_{\alpha} \Leftrightarrow$

$\in \mathfrak{T}^{\beta}_{\alpha+v}$.

Theorem 4.1.1 tells that formulating the left Pieri rule is equivalent to understanding $H_1 \mathfrak{S}_\beta = \mathfrak{S}_1 \mathfrak{S}_\beta$. Equation (4.1) gives a combinatorial interpretation of the coefficients, but with a sign. In [12], the authors proved Theorem 2.1.26 by using a sign-reversing involution. Inspired by that, we now modify that involution and obtain a cancellation-free formula for the coefficients.

Fix a composition β with $\ell(\beta) = n$. For any composition α, we define $\mathfrak{T}^{\beta}_{\alpha}$ to be the set of all skew immaculate tableaux T with inner shape α, entries in $\{1, 2, \ldots, n\}$ and $c(T) - \beta + Id$ is a permutation in S_n (written in one-line notation), where $Id = (1, 2, \ldots, n)$. We define an involution from $\mathfrak{T}^{\beta}_{\alpha}$ to itself.

Definition 4.1.4. For each tableau $T \in \mathfrak{T}^{\beta}_{\alpha}$, we construct a tableau $y(T)$ as follows. For every cell of content r in the i-th row of T, we put a cell of content

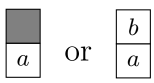
i in the $\sigma(T)(r)$-th row of $y(T)$. We sort the entries of the rows of $y(T)$ in non-decreasing order. In general, $y(T)$ is to be a straight-shape tableau, and might have empty rows.

We define a function Y that maps $T \in \mathfrak{T}_\alpha^\beta$ to the pair $(y(T), \sigma(T))$. Note that Y is injective i.e. fixing α, we can recover T from $(y(T), \sigma(T))$. We define Y^{-1} to be the reversed construction from a pair (T', σ) to T where T' is a tableau with at most n rows and σ is a permutation in S_n. More precisely, $T = Y^{-1}((T', \sigma))$ is constructed as follows: for every cell of content r in the i-th row of T', we put a cell of content $\sigma^{-1}(i)$ in the r-th row of T.

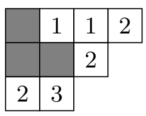
Here, Y^{-1} is not the inverse map of Y because $Y^{-1}(T, \sigma)$ may not be immaculate i.e. the domain of Y is not equal to the image of Y^{-1}.

In this case, $Y^{-1} \circ Y$ is the identity map while $Y \circ Y^{-1}$ is not because Y^{-1} has a much larger domain.

Definition 4.1.5. We say a cell x not in the first row with value a is *nefarious* if the cell above x is either empty or it contains b with $b \geq a$ i.e.



Example 4.1.6. *Let* $\alpha = (1, 2)$ *and* $\beta = (2, 2, 2)$. *Let*

$T =$ 

Note that $\sigma(T) = c(T) - \beta + Id = (2, 3, 1) - (2, 2, 2) + (1, 2, 3) = (1, 3, 2)$, *hence,*

$$y(T) = \begin{array}{|c|c|c}\hline 1 & 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{|c}{} & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

and the nefarious cells in $y(T)$ *are the three cells in the third row.*

The next definition defines a key involution that is a modified version of the Lindstrom-Gessel-Viennot swap, which is usually illustrated on lattice paths, but can be applied to tableaux equally. Details about Lindstrom-Gessel-Viennot swap can be found in chapter 4.5 of [45].

Definition 4.1.7. For each $(y(T), \sigma(T)) \in Y(\mathfrak{T}_\alpha^\beta)$ that contains a nefarious cell x, we define a tableau $\theta_x(y(T))$ and a pair $\Theta_x(y(T), \sigma(T))$ as follows:

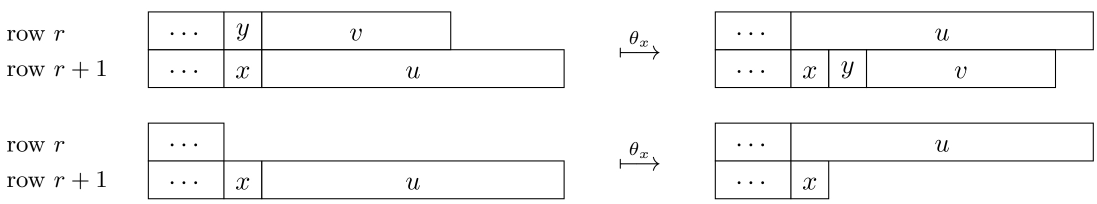
Let the cell x appear in the $(r+1)$-th row of $y(T)$.

1. If the cell y above x is not empty, then define $\theta_x(y(T))$ to be the tableau obtained from $y(T)$ by moving:

   (a) all the cells strictly to the right of x into row r;

   (b) all the cells weakly to the right of y into row $r+1$.

2. Otherwise, move all the cells strictly to the right of x into row r.



Let $t_r = (1, 2, \ldots, r-1, r+1, r, r+2, r+3, \ldots, n)$ be the transposition of r and $r+1$. Then, Θ_x maps the pair $(y(T), \sigma(T))$ to $(\theta_x(y(T)), t_r \circ \sigma(T))$.

Example 4.1.8. *Let x be the second cell in row 2. Then, θ_x maps*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|cc}\hline 1 & 3 & \multicolumn{2}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

or

$$\begin{array}{|c|cc}\hline 1 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 2 \\ \hline \end{array} \; .$$

Definition 4.1.9. For $1 \leq r \leq \ell(\alpha)$ and a cell x in row r of $y(T)$, we say that x is *the most nefarious cell in row* r if it is the left-most nefarious cell in row r such that $Y^{-1} \circ \Theta_x \circ Y(T)$ has the same first column of T. In particular, $Y^{-1} \circ \Theta_x \circ Y(T)$ is immaculate.

Then, for each $1 \leq r \leq n$ we can define a map $\Phi_r : \mathfrak{T}_\alpha^\beta \rightarrow \mathfrak{T}_\alpha^\beta$ by either $\Phi_r(T) = Y^{-1} \circ \Theta_x \circ Y(T)$ where x is the most nefarious cell in row r of $y(T)$, or T is fixed by Φ_r if there is no most nefarious cell in row r of $y(T)$.

For every r, Φ_r has the following properties.

Lemma 4.1.10. *For each $T \in \mathfrak{T}_\alpha^\beta$,*

1. *If there exists a most nefarious cell x in row r of $y(T)$, then it must be the left-most nefarious cell in row r.*

2. *Φ_r is an involution i.e. $\Phi_r^2 = id$.*

3. *T and $\Phi_r(T)$ have the same shape.*

4. *If T is not fixed by Φ_r, then $\sigma(T)$ and $\sigma(\Phi_r(T))$ have opposite sign.*

Proof. For simplicity, we denote $\sigma(T)$ by σ.

(1) Suppose x is the most nefarious cell in row r of $y(T)$, as shown in Figure 1, with entry $c(x)$. Let $c(z)$ denote the entry in cell z in $y(T)$. To obtain $\Phi_r(T)$ from T, it suffices to do the following. For every cell z in row r of $y(T)$ which lies weakly to the left of x, we replace a $\sigma^{-1}(r)$ in row $c(z)$ of T by a $\sigma^{-1}(r-1)$. For every cell z in row $r-1$ of $y(T)$ which lies strictly to the left of x, we replace a $\sigma^{-1}(r-1)$ in row $c(z)$ of T by a $\sigma^{-1}(r)$.

Since $Y^{-1} \circ \Theta_x \circ Y$ fixes the first column of T, and if a cell a_i is nefarious, then we claim that $Y^{-1} \circ \Theta_{a_i} \circ Y$ also fixes the first column of T, because we interchange less cells. If $c(b_k) > c(x)$, $Y^{-1} \circ \Theta_x \circ Y$ changes some $\sigma^{-1}(r-1)$ to $\sigma^{-1}(r)$ in row $c(b_k)$ of T while keeping the first column unchanged. That means the first entry of row $c(b_k)$ of T is no bigger than $\sigma^{-1}(r)$ and $\sigma^{-1}(r-1)$. Therefore, if we interchange less cells as in $Y^{-1} \circ \Theta_{a_i} \circ Y$, the first column remains untouched as the rows in T are weakly increasing and first column is strictly increasing. The argument for the case $c(x) \geq c(b_k)$ is almost identical, just switching $\sigma^{-1}(r)$ and $\sigma^{-1}(r-1)$. Therefore, if x is the most nefarious cell, it must be the left-most nefarious cell.

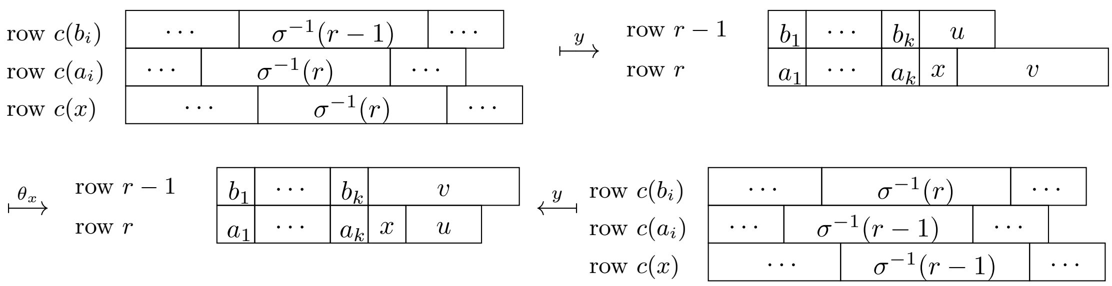


Figure 4.1: $\sigma^{-1}(r), \sigma^{-1}(r-1)$ are entries while a_i, b_i and x stand for cells

(2) If there is no most nefarious cell in row r, then $\Phi_r(T) = T$ and $\Phi_r^2(T) = T$. Otherwise, since Φ_r preserves the first column of T, $Y^{-1}(\theta_x \circ y(T), t_r \circ \sigma(T)) \in \mathfrak{T}_\alpha^\beta$ and hence, $Y^{-1} \circ Y = id$. By part (1), the most nefarious cell remains

unchanged under θ_x. Therefore, $\Phi_r^2(T) = T$ as $\Theta_x^2(T) = T$, and Φ_r is an involution.

(3) $sh(T) = c(y(T))$ and Θ_x preserves the content of $y(T)$.

(4) By the definition of Θ_x, if $\Phi_r(T) \neq T$, then $\sigma(\Phi_r(T)) = t_{r-1} \circ \sigma(T)$. $\square$

Before we continue, consider the case where $\alpha = 0$. Since $\mathfrak{S}_0 = 1$, we know that $\mathfrak{S}_0\mathfrak{S}_\beta = \mathfrak{S}_\beta$. On the other hand, we can also express $\mathfrak{S}_0\mathfrak{S}_\beta$ using (4.1). Hence, there must exist a sign-reversing involution on $\mathfrak{T}_0^\beta$ that cancels everything except the tableau corresponding to $\mathfrak{S}_\beta$, namely the unique immaculate tableau of shape β and content β. For simplicity, we call this involution Φ_0.

Now, we characterize the tableaux that are fixed by all Φ_r. For simplicity, for $\alpha = (1)$, a composition β and $T \in \mathfrak{T}_\alpha^\beta$, we define $\delta(T)$ as $(s+1, \delta_1, \ldots, \delta_n)$ where $n = \ell(\beta)$, s is the number of non-empty cells in the first row of T, and δ_i is the length of row that starts with i, not including the first row as it starts with empty cell. Here, $\delta(T)$ is an integer vector, it may not be a composition as some δ_i could be 0. Then, $sh(T) = \gamma/\alpha = comp(\delta(T))/\alpha$ where $comp(\delta(T))$ is the composition obtained from $\delta(T)$ by removing the zeroes.

Lemma 4.1.11. *Fix $\alpha = (1)$. Let $T \in \mathfrak{T}_\alpha^\beta$ with outer shape γ be fixed by all Φ_r and $\delta(T)$ be defined as above, then*

1. *All entries in the first row of T must be the same, say k, and $\sigma(k) = 1$. In particular, all 1's in $y(T)$ appear in its first row.*

2. *If $\beta_1 > s$, then all entries in the first row of T are 1, and $\delta_1 \leq \beta_1$.*

3. If $\beta_1 < s$*, then* $\sigma(1) = 2$ *and* $\delta_1 > \beta_1$*.*

4. If $\beta_1 = s$*, then*

(a) $\sigma(1) = 1$ *and* $\delta_i = \beta_i$ *for all* $i > 1$*, or*

(b) $\sigma(1) = 2$ *and* $\delta_1 > \beta_1$*.*

5. If $\delta_1 > \beta_1$*, and if* k *is an entry in the second row of* T *and* $k \neq 1$*, then* k *appears in the first row of* T*.*

In particular, the positions all 1*'s in* T *are determined.*

Proof. (1) Let $k_1, \ldots, k_m$ be the m distinct entries in the first row of T. Let $r = \max\{\sigma(k_i)\}$ and suppose $r > 1$. Then, the first cell in row r of $y(T)$ is 1, which must be the most nefarious cell: It is clearly the left-most nefarious cell, and (if we denote it by x) the map $Y^{-1} \circ \Theta_x \circ Y$ fixes the first column of T (since it only changes a single entry in the first row of T, but the first row of T does not intersect the first column).

Applying Φ_r gives an involution that cancels it, because in $\Phi_r(T)$, there is a $\sigma^{-1}(r-1)$ in row 1, and that cell again corresponds to a most nefarious cell and r is still the new $\max\{t_{r-1} \circ \sigma(k_i)\}$ for $\Phi_r(T)$. Therefore, Φ_r is indeed a involution, $T \neq \Phi_r(T)$, they have opposite sign and get canceled by Φ_r. This contradict to our choice of T.

(2) If $\beta_1 > s$ and the entries in the first row of T is $k \neq 1$, then all 1's must be in the second row of T, because T is immaculate. We claim that $\sigma(1) = 2$. If not, $\sigma(1) > 2$, and the first 2 in row $\sigma(1)$ of $y(T)$ is the most nefarious cell. The involution $\Phi_{\sigma(1)}$ fixes the first column because there are at least two 1's in the second row of T, but only one of them is changed to $\sigma^{-1}(\sigma(1) - 1)$.

That means in $y(T)$, there are at least $\beta_1+1 \geq s+2$ many 2's in row 2. Since there are only s 1's appearing in $y(T)$, the $s+1$ cell, counting from the left, in row 2 must be the nefarious cell. Applying Φ_2 gives an involution that maps it to the following situation, that is, we have $(t_1 \circ \sigma)(1) = 1$, all s entries in the first row are changed 1, and $s+1$ 1's in the second row are changed to $\sigma^{-1}(1)$. Therefore, the first column of T remain unchanged.

In this case, we have $\sigma(1) = 1$, all entries in the first row are 1 and the remaining 1's are in the second row. Now, we can consider the 1's as empty cells, and we are in a similar situation where $\alpha' = \beta_1 - s$, $\beta' = (\beta_2, \ldots, \beta_n)$ and $s' = \delta_1 - \beta_1 + s$. By part (1), if it is fixed by all Φ_r, all entries in the second row must be the same k and $\sigma(k) = 2$. That means there are s' 2's in the second row of $y(T)$. But there are s 1's in the first row of $y(T)$. If $\delta_1 > \beta_1$, then $s' > s$ and the $s+1$ cell in the second row of $y(T)$ becomes the most nefarious cell. The first column of T is fixed under Φ_2 because the smallest entry in the second row is always 1. Therefore, the only tableaux that are fixed by Φ_2 are those as defined in the statement.

(3) If $\beta_1 < s$, then it is not possible to fill the first row with 1 while keeping $\sigma(1) = 1$ since $c(T) = \sigma(T) + \beta - Id$. Therefore, all the 1's in T must appear in the second row. Using the same argument as in the proof of part 2, we must have $\sigma(1) = 2$ Moreover, since there are $\beta_1 + 1$ 1's in the second row of T, we must have $\delta_1 > \beta_1$.

(4) If $\beta_1 = s$, then there are two cases. If $\sigma(1) = 1$, then all 1's appear in the first row of T and we are in the case that $\alpha' = 0$ and $\beta' = (\beta_2, \ldots, \beta_n)$. Applying Φ_0 gives the desired result.

If $\sigma(1) > 1$, then we are in the same case as (3). Hence, $\sigma(1) = 2$ and $\delta_1 > \beta_1$.

(5) If $\delta_1 > \beta_1$, by (2),(3) and (4), we have $\sigma(1) = 2$ and all 1's in T appear in its second row. Therefore, if there is some k in the second row of T that $k \neq 1$ and k does not appear in the first row of T, we must have $\sigma(k) > 2$. In this case, let k to be the one with maximal $\sigma(k)$ among all entries j in the second row. It corresponds to a 2 in row $\sigma(k)$ of $y(T)$. This must be the most nefarious cell because by (1), all 1's in $y(T)$ are in the first row of $y(T)$. $\Phi_{\sigma(k)}$ also fixes the first column of T because the first entry in the second row of T remains 1.

□

We now use Lemma 4.1.11 iteratively. If a tableau T is in situation (2) of the Lemma, we can consider all the 1's in T as empty cells. To be precise, we consider the tableau $T' \in \mathfrak{T}_{(1)}^{(\beta_2,\ldots,\beta_n)}$ obtained in the following way. Start with T, remove all but one 1's in the second row of T, then remove its first row, and finally relabel the entries $(1, 2, \ldots, n)$ to $(\text{empty cell}, 1, 2, \ldots, n-1)$. Clearly, T can be obtained from $(sh(T), T')$. We can now apply Lemma 4.1.11 to determine the 1's in T', which leads to the 2's in T.

Suppose a tableau T is in situation (3). By claim (5), there must be exactly two types of entries in the first two rows, let them be 1 and k. We consider the tableau $T' \in \mathfrak{T}_{(1)}^{(\beta_2,\ldots,\beta_n)}$ obtained in the following way. Start with T, move all the k's in its first row (the same as all non-empty cells in the first row) to the second row, remove all but two 1's in the second row, change one of the 1's into k, then remove its first row, and finally relabel the entries $(1, 2, \ldots, n)$ to $(\text{empty cell}, 1, 2, \ldots, n-1)$. Again T can be obtained from $(sh(T), T')$ and we can apply Lemma 4.1.11 to determine the 1's in T', which leads to the 2's in T. The involutions on T' can be transferred to involutions on T because the first column will remain unchanged.

Corollary 4.1.12. *If $\alpha = (1)$, let $T \in \mathfrak{T}_\alpha^\beta$ with outer shape γ be fixed by all Φ_r, and δ be defined as above, then,*

1. *if $\beta_i < s + \sum_{j=1}^{i-1}(\delta_j - \beta_j)$, then $\beta_i < \delta_i$, $\sigma(T)(i) = i+1$ and all i's in T are in row $i+1$.*
2. *if $\beta_i > s + \sum_{j=1}^{i-1}(\delta_j - \beta_j)$, then $\beta_i \geq \delta_i \geq \sum_{j=1}^{i}\beta_j - \sum_{j=1}^{i-1}\delta_j - s$, $\sigma(T)(i) = \max\left\{k+1 \mid k < i, \beta_i > s + \sum_{j=1}^{i-1}(\delta_j - \beta_j) \text{ or } k = 0\right\}$ and all cells above row $i+1$ in T are filled with $\{1, 2, \ldots, i\}$.*
3. *if $\beta_i = s + \sum_{j=1}^{i-1}(\delta_j - \beta_j)$, then*
   (a) $\beta_i < \delta_i$, $\sigma(T)(i) = i+1$ and all i's in T are in row $i+1$. (same as case 1), or
   (b) $\delta_i = 0$, $\sigma(T)(i)$ is the same as in case 2 and for all $i < j \leq n$, $\beta_j = \delta_j$, $\sigma(j) = j$ and all j's are in row $j+1$ of T.

Let Z_β^γ be the set of all integer vectors $\delta(T) = (s+1, \delta_1, \ldots, \delta_n)$ that satisfy these three conditions and $\mathrm{comp}(\delta) = \gamma$, then we have $C_{\alpha,\beta}^\gamma = \sum_{\delta \in Z_\beta^\gamma} \mathrm{sgn}(\beta - \delta)$, where $\mathrm{sgn}(\beta - \delta) = (-1)^k$ and k is the number of negative terms in $\beta - \delta$.

Proof. (1) In the iterative use of Lemma 4.1.11 as mentioned above, in both cases, after we fill in entry i, we remove β_i cells. That means if $\beta_i < s +$

$\sum_{j=1}^{i-1}(\delta_j - \beta_j)$, then after filling in $\{1, 2, \ldots, i-1\}$, we are in the situation of claim (3) of Lemma 4.1.11. Hence, $\beta_i < \delta_i$, $\sigma(T) = i+1$ and all i's appear in row $i+1$.

(2) Similarly, if $\beta_i > s + \sum_{j=1}^{i-1}(\delta_j - \beta_j)$, that means after filling $\{1, 2, \ldots, i-1\}$, we are in the situation of claim (2) of Lemma 4.1.11. Hence, $\beta_i > \delta_i$. And $\delta_i \geq \sum_{j=1}^{i} \beta_j - \sum_{j=1}^{i-1} \delta_j - s$ because we need enough space to put $\{1, 2, \ldots, i\}$ into the first $i+1$ rows of T. To find $\sigma(i)$, we need to trace back and find the last time where $\sigma(k) \neq k+1$ among all $k < i$, or 1 if k does not exist. Therefore, $\sigma(T)(i) = \max\left\{k+1 \mid k < i, \beta_i > s + \sum_{j=1}^{i-1}(\delta_j - \beta_j) \text{ or } k = 0\right\}$.

(3) If $\beta_i = s + \sum_{j=1}^{i-1}(\delta_j - \beta_j)$, that means after filling $\{1, 2, \ldots, i-1\}$, we are in the situation of claim (4) of Lemma 4.1.11, and the result follows.

To sum up, each time we have a $\beta_i < \delta_i$, that corresponds to a $\sigma(i) = i+1$. Therefore, the sign of $\sigma(T)$ is $\operatorname{sgn}(\beta - \delta)$. □

Clearly, if $\ell(\gamma) < \ell(\beta)$ or $\ell(\gamma) > \ell(\beta)+1$, then $Z_\beta^\gamma = \emptyset$. If $\ell(\gamma) = \ell(\beta)+1$, then either $Z_\beta^\gamma = \emptyset$ or $Z_\beta^\gamma = \{\gamma\}$. If $\ell(\gamma) = \ell(\beta)$, it could happen that $\delta \neq \delta'$ but comp(δ) =comp(δ').

Suppose $\delta = (\delta_1, \ldots, \delta_n) \in Z_\beta^\gamma$ and $\ell(\delta) = \ell(\beta)$. Let $1 \leq k \leq n$ be the smallest integer such that $\beta_j = \delta_j$ for all $j > k$. Let $k \leq r \leq n$ be the largest integer such that $\beta_j < \beta_{j+1}$ for all $k \leq j < r$.

Since $\delta \in Z_\beta^\gamma$, the composition $(\delta_1, \ldots, \delta_k, 0, \delta_{k+1}, \ldots, \delta_n)$ will satisfy the conditions in Corollary 4.1.12. If $\beta_k < \beta_{k+1} = \delta_{k+1}$, by condition (3), we can always interchange $\sigma(k)$ to $k+1$ and obtain $(\delta_1, \ldots, \delta_{k+1}, 0, \delta_{k+2}, \ldots, \delta_n)$ which also satisfies the conditions in Corollary 4.1.12. However, for $j > r$, we cannot have the composition because condition $(3.a)$ fails at row r. Since the compositions proceed in alternating signs, they cancel each other in pairs. Therefore, if $r-k$ is odd, then everything cancels and if $r-k$ is even, the first composition is left.

Finally, we have the following criterion to determine the structure constants, and Lemma 4.1.11 and Corollary 4.1.12 give an algorithm to construct the corresponding tableau. At each step of filling numbers, if Corollary 4.1.12 fails, then Lemma 4.1.11 gives a corresponding Φ_r that cancels it, or an implicit involution using $\mathfrak{S}_0$. Therefore, combining with the argument above, we can assign each tableau in the above cases their corresponding involutions, and the remaining tableaux in $\mathfrak{T}_\alpha^\beta$ are left fixed. This indeed gives an involution because all the cases are disjoint and the involution sends the tableaux into tableaux in the same case, clear from the construction in Lemma 4.1.11.

For each outer shape γ, there can be at most one tableau fixed by this involution, which shows the left-Pieri rule is multiplicity free.

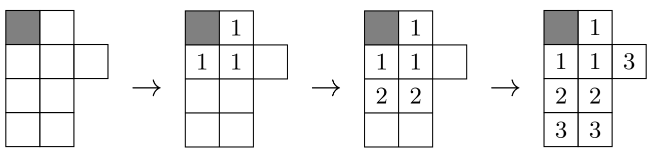
Theorem 4.1.13. *For $\alpha = (1)$ and a partition β of length n,*

$$\mathfrak{S}_1 \mathfrak{S}_\beta = \sum_\gamma C_\beta^\gamma \mathfrak{S}_\gamma \textit{ and}$$

$$C_\beta^\gamma = \begin{cases} \operatorname{sgn}(\beta_1 - \gamma_2, \ldots, \beta_n - \gamma_{n+1}) & \textit{if } \ell(\gamma) = \ell(\beta) + 1 \textit{ and } \gamma \in Z_\beta^\gamma \\ \\ \operatorname{sgn}(\beta_1 - \gamma_2, \ldots, \beta_{k-1} - \gamma_k) & \textit{if } \ell(\gamma) = \ell(\beta), r - k \textit{ is even and} \\ & (\gamma_1, \ldots, \gamma_k, 0, \gamma_{k+1}, \ldots, \gamma_n) \in Z_\beta^\gamma \\ \\ 0 & \textit{otherwise} \end{cases}$$

where k and r are as defined above.

Example 4.1.14. *Let $\alpha = (1)$, $\beta = (3, 1, 4)$ and $\gamma = (2, 3, 2, 2)$. As shown below, since $\gamma_1 - 1 < \beta_1$ and $\gamma_2 \leq \beta_1$, we have $\sigma(1) = 1$ and all 1's appear in rows 1 and 2 of T. Then, since $\gamma_2 - 2 = \beta_2$ and $\gamma_3 > \beta_2$, we have $\sigma(2) = 3$ and all 2's appear in row 3 of T. Finally, $\sigma(3) = 2$. Therefore, $\sigma = (1, 3, 2)$ and $C_{\alpha,\beta}^\gamma = -1 = \operatorname{sgn}(1, -1, 2)$.*



This result is equivalent to the one in [18], but here we give an explicit combinatorial interpretation and an algorithm for constructing the tableaux corresponding to the structure constants.

We now give the theta map in terms of the immaculate basis.

Proposition 4.1.15.

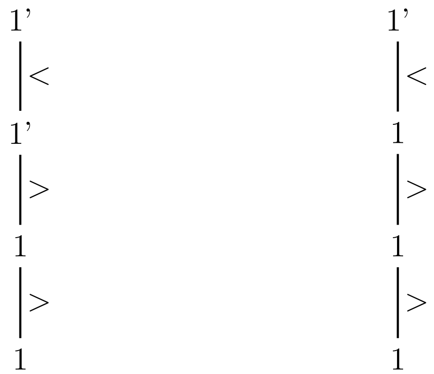
$$\Theta_{\mathsf{NSym}}(\mathfrak{S}_n) = 2\sum_{\alpha\models n} \mathfrak{S}_\alpha.$$

Proof. We first claim that $\Theta_{\mathsf{NSym}}(H_n) = 2\sum_{k=1}^{n} R_{(1,1,\ldots,1,k)}$ with $n-k$ 1's. This follows from the adjointness

$$\langle \Theta_{\mathsf{NSym}}(H_n), L_\alpha\rangle = \langle H_n, \Theta_{\mathsf{QSym}}(L_\alpha)\rangle = \begin{cases} 2 & \text{if } \alpha = (1,1,\ldots,1,k) \\ 0 & \text{otherwise} \end{cases}$$

The last equality comes from the definition of enriched P partitions. If α has a peak, then it is not possible to label the poset with only x and x' for any x. Whereas if α has no peak, then we can label the poset with only x and x', and we can use either x or x' at the bottom.

For example, consider the following totally ordered set P and possible enriched P partitions



Then, we use a result from [12]. An immaculate tableau is standard if its content is $(1,1,\ldots,1)$. A standard immaculate tableau T has a descent at i if the row that contains $i+1$ is strictly lower than the row that contains i.

We write $Des(T)$ to be the set of descents of T. Let $L_{\alpha,\beta}$ be the number of standard immaculate tableaux with shape α and descent set corresponding to the composition β. Then,

$$R_\beta = \sum_\alpha L_{\alpha,\beta} \mathfrak{S}_\alpha.$$

For example,

$$R_{23} = \begin{array}{ccccc} \begin{array}{|c|c|c} \hline 1 & 2 & \multicolumn{1}{|c}{} \\ \hline 3 & 4 & \multicolumn{1}{|c|}{5} \\ \hline \end{array} & + & \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} \end{array} & + & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & \multicolumn{3}{|c}{} \\ \cline{1-1} \end{array} \\ \mathfrak{S}_{23} & + & \mathfrak{S}_{32} & + & \mathfrak{S}_{41} \end{array}$$

In our case, when $\beta = (1, 1, \ldots, 1, k)$, the only possible tableaux must be of height $\ell(\beta)$, first column is filled with $1, 2, \ldots, \ell(\beta)$, and the rest numbers are filled increasingly from bottom to top. And there is exactly one tableau of each shape.

Therefore, let $(1, 1, \ldots, 1, k)$ be a hook of size n, we have

$$R_{(1,1,\ldots,1,k)} = \sum_{\alpha \models n, \ell(\alpha)=\ell(\beta)} \mathfrak{S}_\alpha$$

Summing up all possible hooks, we get the proposition

$$\Theta_{\mathsf{NSym}}(H_n) = 2 \sum_{k=1}^{n} R_{(1,1,\ldots,1,k)} = 2 \sum_{\alpha \models n} \mathfrak{S}_\alpha.$$

$\square$

4.2 Theta maps for Hopf subalgebras of QSym

We show that the theta map of any combinatorial Hopf sub-algebra of $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$, if it exists, is unique.

Theorem 4.2.1. *Let $(\mathcal{H}, \zeta)$ be a combinatorial Hopf algebra, and assume there is a one-to-one combinatorial Hopf morphism β from $(\mathcal{H}, \zeta)$ to $(\mathsf{QSym}, \zeta_{\mathsf{QSym}})$. Then $(\mathcal{H}, \zeta)$ has a Theta map if and only if $Img(\beta)$ is Θ_{QSym}-invariant (i.e., $\Theta_{\mathsf{QSym}}(Img(\beta)) \subseteq Img(\beta)$). Moreover, the Theta map for $(\mathcal{H}, \zeta)$ is $\beta^{-1} \circ \Theta_{\mathsf{QSym}} \circ \beta$, and it is unique.*

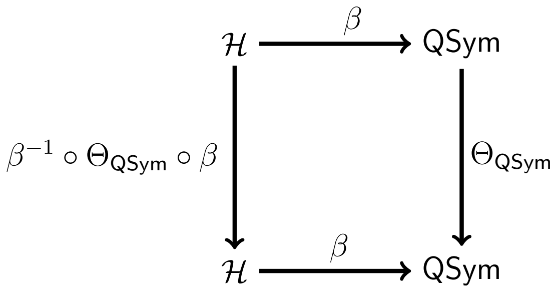
Proof. Let

$$\begin{array}{rccc} \beta : & (\mathcal{H}, \zeta) & \to & (Img(\beta), \zeta_{\mathsf{QSym}}|_{Img(\beta)}) \\ & h & \mapsto & \beta(h). \end{array}$$

Then β is an isomorphism. The map Θ_{QSym} is a combinatorial Hopf morphism, thus the following diagram commutes.

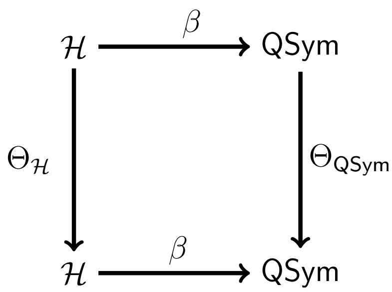
$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\beta} & Img(\beta) & \xrightarrow{i} & \mathsf{QSym} \\ \Big\downarrow{\scriptstyle \beta^{-1} \circ \Theta_{\mathsf{QSym}} \circ \beta} & & \Big\downarrow{\scriptstyle \Theta_{\mathsf{QSym}}} & & \Big\downarrow{\scriptstyle \Theta_{\mathsf{QSym}}} \\ \mathcal{H} & \xrightarrow{\beta} & Img(\beta) & \xrightarrow{i} & \mathsf{QSym} \end{array}$$

therefore, the following diagram commutes,



and so $\beta^{-1}\Theta_{\mathcal{Q}}\beta$ is the theta map for $(\mathcal{H}, \zeta)$.

Conversely, assume that $(\mathcal{H}, \zeta)$ has a Theta map $\Theta_{\mathcal{H}}$. Then the following diagram commutes.



i.e., $\Theta_{\mathsf{QSym}}(\beta(\mathcal{H})) = \beta(\Theta_{\mathcal{H}}(\mathcal{H}))$. Therefore, $Img(\beta)$ is Θ_{QSym}-invariant.

Furthermore, if $(\mathcal{H}, \zeta)$ has a Theta map, then $\Theta_{\mathsf{QSym}} \circ \beta = \beta \circ \Theta_{\mathcal{H}}$, and so $\beta^{-1} \circ \Theta_{\mathsf{QSym}} \circ \beta = \Theta_{\mathcal{H}}$. This shows that there is a unique Theta map for $(\mathcal{H}, \zeta)$.

□

4.3 Theta maps for commutative and co-commutative Hopf algebras

In the case of symmetric function this theta map originate from plethysm and is defined as follows

$$\begin{aligned} \Theta_{\mathsf{Sym}}: \quad \mathsf{Sym} &\to \quad \mathsf{Sym} \\ p_n &\mapsto \begin{cases} 2p_n & n \text{ is odd,} \\ 0 & n \text{ is even.} \end{cases} \end{aligned}$$

We can see that

$$\Theta_{\mathsf{Sym}}(p_n) = m \circ (S \circ R_{-1} \otimes id) \circ \Delta(p_n)$$

where $R_{-1}(f) = (-1)^{\deg(f)} f$.

We will show that for every commutative and co-commutative combinatorial Hopf algebra $m \circ (S \circ R_{-1} \otimes id) \circ \Delta$ is a theta map.

Lemma 4.3.1. *If $\mathcal{H}$ is commutative and co-commutative, then $\Phi = m \circ (S \circ R_{-1} \otimes id) \circ \Delta$ is a Hopf morphism.*

Proof. We first show that Φ is an algebra morphism. Let a, b be two homogeneous elements in $\mathcal{H}$, then

$$\begin{aligned} m \circ \Phi \otimes \Phi(a \otimes b) &= m \left(\left(\sum_{(a)} (-1)^{deg(a_1)} S(a_1) a_2 \right) \otimes \left(\sum_{(b)} (-1)^{deg(b_1)} S(b_1) b_2 \right) \right) \\ &= \sum_{(a),(b)} (-1)^{deg(a_1 + b_1)} S(a_1) a_2 S(b_1) b_2. \end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi \circ m(a,b) &= m \circ (S \circ R_{-1} \otimes id) \circ \Delta \circ m(a \otimes b)\\ &= m \circ (S \circ R_{-1} \otimes id) \circ (m \otimes m) \circ (id \otimes T \otimes id) \circ (\Delta \otimes \Delta)(a \otimes b)\\ &= m \circ (S \circ R_{-1} \otimes id) \left(\sum_{(a),(b)} a_1 b_1 \otimes a_2 b_2 \right)\\ &= \sum_{(a),(b)} (-1)^{deg(a_1+b_1)} S(a_1 b_1) a_2 b_2.\end{aligned}$$

The two sides are the same when $\mathcal{H}$ is commutative. Hence, Φ is an algebra morphism.

The proof of coalgebra morphism is similar so it is omitted. □

Theorem 4.3.2. *Let $(\mathcal{H}, \zeta)$ be a commutative and co-commutative combinatorial Hopf algebra (or $m \circ (S \circ R_{-1} \otimes id) \circ \Delta$ be a Hopf morphism), then there is a theta map for $(\mathcal{H}, \zeta)$ as follows,*

$$\Theta_{(\mathcal{H},\zeta)} = m \circ (S \circ R_{-1} \otimes id) \circ \Delta.$$

Proof. Since $(\mathcal{H}, \zeta)$ is a combinatorial Hopf algebra by [5, Theorem 4.1] we have a combinatorial Hopf algebra morphism Ψ from $(\mathcal{H}, \zeta)$ to $(\mathsf{Sym}, \zeta_{\mathsf{QSym}})$.

Note that

$$m \circ (S \circ R_{-1} \otimes id) \circ \Delta \circ \Psi =$$

$$m \circ (S \circ R_{-1} \otimes id) \circ (\Psi \otimes \Psi) \circ \Delta =$$

$$m \circ (S \circ R_{-1} \circ \Psi \otimes id \circ \Psi) \circ \Delta =$$

$$m \circ (\Psi \otimes \Psi) \circ (S \circ R_{-1} \otimes id) \circ \Delta =$$

$$\Psi \circ m \circ (S \circ R_{-1} \otimes id) \circ \Delta,$$

i.e., the following diagram commutes.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Psi} & \mathcal{S}ym \\ {\scriptstyle m \circ (S \circ R_{-1} \otimes id) \circ \Delta} \downarrow & & \downarrow {\scriptstyle m \circ (S \circ R_{-1} \otimes id) \circ \Delta} \\ \mathcal{H} & \xrightarrow{\Psi} & \mathcal{S}ym \end{array}$$

□

Corollary 4.3.3. *Let (H, ζ) be a commutative and co-commutative combinatorial Hopf algebra. Then there is a theta map as follows*

$$\Theta_{(\mathcal{H},\zeta)}(h) = \sum_{(h)} (-1)^{deg(h_1)} S(h_1) h_2,$$

where $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$.

Remark 4.3.4. From definition, the theta maps are dependent on the character of combinatorial Hopf algebras. In the case of commutative and co-commutative Hopf algebra, the map Φ is always a theta map regardless of the choice of character.

Remark 4.3.5. Even though in the case of non-commutative symmetric functions , $m \circ (S \circ R_{-1} \otimes id) \circ \Delta(H_n) = \Theta_{\mathsf{NSym}}(H_n) = \sum_{k=0}^{n} 2R_{(1^k,n-k)}$, $m \circ (S \circ R_{-1} \otimes id) \circ \Delta$ is not the Theta map for NSym since otherwise the dual must be the Theta map for QSym, but the following example shows that $m \circ (S \circ R_{-1} \otimes id) \circ \Delta$ is not the Theta map for QSym.

Example 4.3.6. *The map* $m \circ (S \circ R_{-1} \otimes id) \circ \Delta$ *is not the theta map for* QSym.

The theta map for QSym *takes* M_{32} *to* 0, *i.e.,* $\Theta_{\mathsf{QSym}}(M_{32}) = 0$ *(see 2.3).*

But,

$$m \circ (S \circ R_{-1} \otimes id) \circ \Delta(M_{32}) =$$

$$m \circ (S \circ R_{-1} \otimes id)(M_{32} \otimes 1 + M_3 \otimes M_2 + 1 \otimes M_{32}) =$$

$$S \circ R_{-1}(M_{32}) + S \circ R_{-1}(M_3)M_2 + M_{32} =$$

$$-M_{23} - M_5 + M_3M_2 + M_{32} =$$

$$-M_{23} - M_5 + M_{32} + M_{23} + M_5 + M_{32} = 2M_{32} \neq 0.$$

4.3.1 Theta maps for diagonally symmetric functions

In this section, we provide an example of a commutative and co-commutative Hopf algebra.

A bipartition λ of length k is a $2 \times k$ matrix $\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1k} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2k} \end{pmatrix}$ such that the columns are ordered in lexicographic order and no column is $\binom{0}{0}$. The size of λ, denoted by $|\lambda|$, is the sum of all its entries. If the size of λ is n, we write $\lambda \vdash n$. There also exists a generalized bipartition with length 0 and size 0, called the zero bipartition, denoted by $\binom{0}{0}$.

Let λ, μ be two bipartitions, we say their disjoint union, $\lambda \uplus \mu$ is the bipartition obtained by taking the disjoint union of columnns of λ and μ, and re-order in decreasing order.

Let $\Bbbk[[\mathbf{x},\mathbf{y}]] = \Bbbk[[x_1, x_2, \ldots, y_1, y_2, \ldots]]$ be the set of power series in two sets of variables with bounded degree.

The diagonally symmetric functions, $\mathsf{DSym} = \bigoplus_{n\geq 0} \mathsf{DSym}_n$, is sub-ring of the power series $\Bbbk[[\mathbf{x},\mathbf{y}]]$ that are fixed under diagonal action of the symmetric group. More precisely, $f \in \mathsf{DSym}$ if

$$f(x_1, x_2, \ldots, y_1, y_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, y_{\sigma(1)}, y_{\sigma(2)}, \ldots)$$

for all $\sigma \in \mathfrak{S}_{(\infty)}$ where $\mathfrak{S}_{(\infty)}$ is the set of all bijections from the set of positive integers to itself that fix all but finitely many numbers.

The diagonally symmetric functions clearly form a graded vector space, and the degree n component is spanned by the monomial basis m_λ indexed by bipartitions, the sum of all monomials in the orbit of $\mathbf{X}^\lambda = x_1^{\lambda_{11}} y_1^{\lambda 21} x_2^{\lambda_{12}} y_2^{\lambda_{22}} \cdots x_{\ell(\lambda)}^{\lambda_{1\ell(\lambda)}} y_{\ell(\lambda)}^{\lambda_{2\ell(\lambda)}}$, for all $\lambda \vdash n$.

Example 4.3.7. *For $n = 2$, we have*

- $m_{\binom{2}{0}} = x_1^2 + x_2^2 + x_3^2 + \cdots$
- $m_{\binom{1}{1}} = x_1y_1 + x_2y_2 + x_3y_3 + \cdots$
- $m_{\binom{10}{01}} = x_1y_2 + x_2y_1 + x_1y_3 + \cdots$

The main reason why DSym is interesting is that the quotient space $\Bbbk[[\mathbf{x},\mathbf{y}]]/\langle \mathsf{DSym}^+ \rangle$ over the ideal generated by diagonally symmetric functions with positive degree is known as the diagonal harmonics. The central result is by Garsia and Haiman [23, 26] where it was used to prove the $n!$ conjecture and Macdonald positivity.

The product is the regular multiplication of power series. The coproduct is as follows

$$\Delta(m_\lambda) = \sum_{\mu \uplus \nu = \lambda} m_\mu \otimes m_\nu.$$

It is not hard to see that DSym is a commutative and co-commutative Hopf algebra. We have a projection given by (λ is a bipartition)

$$\begin{array}{rccc} p: & \mathsf{DSym} & \to & \mathsf{Sym} \\ & m_\lambda & \mapsto & m_{sort(\lambda_{11}+\lambda_{21}, \lambda_{12}+\lambda_{22}, \ldots, \lambda_{1\ell(\lambda)}+\lambda_{2\ell(\lambda)})} \end{array}$$

and an embedding (λ is a partition)

$$\begin{array}{rccc} i: & \mathsf{Sym} & \to & \mathsf{DSym} \\ & m_\lambda & \mapsto & m_{\left(\begin{smallmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{\ell(\lambda)} \\ 0 & 0 & \cdots & 0 \end{smallmatrix}\right)} \end{array}$$

Then, the map $\Theta = i \circ \Theta_{\mathsf{Sym}} \circ p$ is a theta map for DSym. The image of Θ is isomorphic to the image of Θ_{Sym}, the space of Schur's $\mathcal{Q}$ functions.

According the theorem 4.3.2, we have a non-trivial theta map

$$\Theta_{\mathsf{DSym}} = m \circ (S \circ R_{-1} \otimes id) \circ \Delta.$$

The image of Θ_{DSym} is a generalization of Schur's $\mathcal{Q}$ functions into two sets of variables. We now describe the image in more details.

We begin with defining two sets of diagonally symmetric functions

Definition 4.3.8. For each pair (a, b), we define the homogeneous and elementary functions $h_{\binom{a}{b}}$, $e_{\binom{a}{b}}$ via the following generating functions

$$H(s,t) = \sum_{a,b} h_{\binom{a}{b}} s^a t^b = \prod_i \frac{1}{1 - x_i s - y_i t},$$

$$E(s,t) = \sum_{a,b} e_{\binom{a}{b}} s^a t^b = \prod_i (1 + x_i s + y_i t).$$

For a bipartition λ,

$$h_\lambda = h_{\binom{\lambda_{11}}{\lambda_{21}}} h_{\binom{\lambda_{12}}{\lambda_{22}}} \dots h_{\binom{\lambda_{1\ell(\lambda)}}{\lambda_{2\ell(\lambda)}}},$$

$$e_\lambda = e_{\binom{\lambda_{11}}{\lambda_{21}}} e_{\binom{\lambda_{12}}{\lambda_{22}}} \dots e_{\binom{\lambda_{1\ell(\lambda)}}{\lambda_{2\ell(\lambda)}}}.$$

These functions are introduced by Gessel [21], and he showed that they are bases for DSym.

Remark 4.3.9. Gessel also defined a third multiplicative bases, the power sum $\{p_\lambda\}$ where $p_{\binom{a}{b}} = m_{\binom{a}{b}}$.

Now we study the antipode of DSym. We start with the following observations.

Lemma 4.3.10. *For each pair (a, b), we have the following coproduct formula*

$$\Delta\left(e_{\binom{a}{b}}\right) = \sum_{i+j=a, k+\ell=b} e_{\binom{i}{k}} \otimes e_{\binom{j}{\ell}},$$

Proof. This follows from the fact that $e_{\binom{a}{b}} = m_{\binom{1\dots10\dots0}{0\dots01\dots1}}$ with a copies of $\binom{1}{0}$ and b copies of $\binom{0}{1}$. □

Proposition 4.3.11. *The antipode S of* DSym *is* $S\left(e_{\binom{a}{b}}\right) = (-1)^{a+b} h_{\binom{a}{b}}$.

Proof. From the generating functions $H(s,t)$ and $E(s,t)$, it is not hard to see that

$$1 = H(s,t)E(-s,-t) = \left(\sum_{a,b} h_{\binom{a}{b}} s^a t^b\right)\left(\sum_{a,b} e_{\binom{a}{b}} (-s)^a (-t)^b\right).$$

Then, for any pair $(a,b) \neq (0,0)$, we have

$$\sum_{i+j=a, k+\ell=b} (-1)^{j+\ell} h_{\binom{i}{k}} e_{\binom{j}{\ell}} = 0$$

On the other hand, from the coproduct formula of e basis, we know that the antipode is determined by the relation

$$\sum_{i+j=a, k+\ell=b} S\left(e_{\binom{i}{k}}\right) e_{\binom{j}{\ell}} = 0$$

for all pair $(a,b) \neq (0,0)$. Comparing these two equations, we will obtain the desired result via induction. $\square$

Therefore, we have the following formula for Θ_{DSym}.

$$\Theta_{\mathsf{DSym}}\left(e_{\binom{a}{b}}\right) = \sum_{i+j=a, k+\ell=b} h_{\binom{i}{k}} e_{\binom{j}{\ell}}.$$

Its generating function is

$$Q(s,t) = \sum_{a,b} \Theta_{\mathsf{DSym}}\left(e_{\binom{a}{b}}\right) s^a t^b = \prod_i \frac{1 + x_i s + y_i t}{1 - x_i s - y_i t}.$$

We denote $\Theta_{\mathsf{DSym}}\left(e_{\binom{a}{b}}\right)$ by $q_{\binom{a}{b}}$. We now give some properties of these $q_{\binom{a}{b}}$

that are analoguous Schur's $\mathcal{Q}$ functions $q_n = \Theta_{\mathsf{Sym}}(e_n)$.

Proposition 4.3.12. *Assuming the base field* $\Bbbk$ *is of characteristic* 0. *The image space* $\Theta_{\mathsf{DSym}}(\mathsf{DSym})$ *is generated by* $\left\{q_{\binom{a}{b}} : a+b \text{ is odd}\right\}$

Proof. Since $Q(s,t) = E(s,t)H(s,t)$, we have that $Q(s,t)Q(-s,-t) = 1$. Therefore, for each $(n,m) \neq (0,0)$, we have

$$\sum_{\substack{a+c=n \\ b+d=m}} (-1)^{a+b} q_{\binom{a}{b}} q_{\binom{c}{d}} = 0.$$

When $n+m$ is even, we have

$$2q_{\binom{n}{m}} = -\sum_{\substack{a+c=n, b+d=m \\ (a,b)\neq(0,0),(c,d)\neq(0,0)}} (-1)^{a+b} q_{\binom{a}{b}} q_{\binom{c}{d}}$$

Therefore, by induction, $q_{\binom{n}{m}} \in \Bbbk\left[q_{\binom{a}{b}} : a \leq n, b \leq m, a+b \text{ is odd}\right]$. ☐

Remark 4.3.13. It presents no difficulty in generalizing this construction to diagonally symmetric functions in more sets of variables.

4.3.2 Theta maps for diagonally quasi-symmetric functions

An element $\tilde{\alpha} = \begin{pmatrix} \tilde{\alpha}_{11} & \tilde{\alpha}_{12} & \cdots \\ \tilde{\alpha}_{21} & \tilde{\alpha}_{22} & \cdots \end{pmatrix} \in \mathbb{N}^{2\mathbb{N}}$ is called a generalized bicomposition if all but finitely many $(\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k})$ are $(0,0)$. Let k be the maximum number such that $(\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k}) \neq (0,0)$. The length of $\tilde{\alpha}$, denoted by $\ell(\tilde{\alpha})$, is k. The size of $\tilde{\alpha}$, denoted by $|\tilde{\alpha}|$, is the sum of all its entries. For simplicity, we usually write $\tilde{\alpha}$ as $\binom{\tilde{\alpha}_{11} \cdots \tilde{\alpha}_{1k}}{\tilde{\alpha}_{21} \cdots \tilde{\alpha}_{2k}}$. There also exists a generalized bicomposition with length 0 and size 0, called the zero bicomposition, denoted by $\binom{0}{0}$.

Every monomial in R can be expressed as $\mathbf{X}^{\tilde{\alpha}} = x_1^{\tilde{\alpha}_{11}} y_1^{\tilde{\alpha}_{21}} \cdots x_k^{\tilde{\alpha}_{1k}} y_k^{\tilde{\alpha}_{2k}}$ for some generalized bicomposition $\tilde{\alpha}$. A generalized bicomposition α is called a bicomposition if $\ell(\alpha) = 0$ or $(\alpha_{1j}, \alpha_{2j}) \neq (0,0)$ for all $1 \leq j \leq \ell(\alpha)$.

In this section, we use Greek letters to denote bicompositions, and Greek letters with tilde to denote generalized bicompositions.

Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be non-zero generalized bicompositions. We write $\tilde{\alpha} = \tilde{\beta}\tilde{\gamma}$ if $\tilde{\alpha}_{ij} = \tilde{\beta}_{ij}$ for all $1 \leq j \leq \ell(\tilde{\alpha}) - \ell(\tilde{\gamma})$, $\tilde{\beta}_{ij} = 0$ for all $j > \ell(\tilde{\alpha}) - \ell(\tilde{\gamma})$ and $\tilde{\alpha}_{i(j+\ell(\tilde{\alpha})-\ell(\tilde{\gamma}))} = \tilde{\gamma}_{ij}$ for all $j \geq 1$. We write $\tilde{\alpha} = \binom{0}{0}\tilde{\beta}$ if $\tilde{\alpha}_{11} = \tilde{\alpha}_{21} = 0$ and $\tilde{\alpha}_{i(j+1)} = \tilde{\beta}_{ij}$ for all $j \geq 2$.

Note that for each generalized bicomposition $\tilde{\alpha}$ that is not a bicomposition, there is a unique way to decompose it into $\tilde{\alpha} = \tilde{\beta}\binom{0}{0}\gamma$ for some generalized bicomposition $\tilde{\beta}$ and bicomposition γ.

The algebra of diagonally quasi-symmetric functions, DQSym, is a subalgebra of $\Bbbk[[\mathbf{x}, \mathbf{y}]]$ spanned by monomials indexed by bicompositions

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_{11}} y_{i_1}^{\alpha_{21}} \cdots x_{i_k}^{\alpha_{1k}} y_{i_k}^{\alpha_{2k}}.$$

As a graded algebra, $\mathsf{DQSym} = \bigoplus_{n \geq 0} \mathsf{DQSym}_n$ where $\mathsf{DQSym}_n = \text{span-}\{M_\alpha : |\alpha| = n\}$ is the degree n component.

The quotient space $\Bbbk[[\mathbf{x}, \mathbf{y}]]/\langle \mathsf{DQSym}^+ \rangle$ over the ideal generated by diagonally quasi-symmetric functions with positive degree is a quasi-symmetric analogue of the diagonal harmonics. It is introduced in [3] where the quotient is called the diagonal Temperley-Lieb harmonics. In [36], we studied the linear structure of this quotient space, and we give a description of its Hilbert basis. This result is not directly related to theta maps, so we leave it in appendix.

Similar to DSym, the space of DQSym admits a Hopf algebra structure where the product is the regular multiplication of power series, and the coproduct is given by deconcatenation

$$\Delta(M_\alpha) = \sum_{\alpha=\beta\gamma} M_\beta \otimes M_\gamma.$$

This gives a commutative but not co-commutative Hopf structure, and we have a projection given by (α is a bicomposition)

$$\begin{aligned} p: \quad \mathsf{DQSym} &\to \mathsf{QSym} \\ M_\alpha &\mapsto M_{(\alpha_{11}+\alpha_{21}, \alpha_{12}+\alpha_{22}, \ldots, \alpha_{1\ell(\alpha)}+\alpha_{2\ell(\alpha)})} \end{aligned}$$

and an embedding (α is a composition)

$$\begin{aligned} i: \quad \mathsf{QSym} &\to \mathsf{DQSym} \\ M_\alpha &\mapsto M_{\left(\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{\ell(\alpha)} \\ 0 & 0 & \cdots & 0 \end{smallmatrix}\right)} \end{aligned}$$

Then, the map $\Theta = i \circ \Theta_{\mathsf{QSym}} \circ p$ is a theta map for DQSym. The image of Θ is isomorphic to the image of Θ_{QSym}, Π.

It is yet unknown whether there exists a theta map for DQSym that is compatible with Θ_{DSym}.

4.4 Theta maps for $\mathcal{V}$

Some combinatorial Hopf algebras can have many theta maps. We show that when a Hopf morphism can be a theta map for $(\mathcal{V}, \zeta_{\mathcal{V}})$.

Recall that we defined the basis $\{M_\sigma : \sigma \in \sqcup_{n\geq 0}\mathfrak{S}_n\}$ for $(\mathcal{V}, \zeta_{\mathcal{V}})$, where

$$\Psi(M_{\sigma_1\backslash\cdots\backslash\sigma_k}) = \begin{cases} M_{(|\sigma_1|,\ldots,|\sigma_k|)} & \text{if } \sigma = id_{|\sigma_1|}\backslash\cdots\backslash id_{|\sigma_k|} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, since $\zeta_{\mathcal{V}} = \zeta_{\mathsf{QSym}} \circ \Psi$, for $\sigma = \sigma_1\backslash\cdots\backslash\sigma_k$ we have that

$$\zeta_{\mathcal{V}}(M_\sigma) = \zeta_{\mathsf{QSym}} \circ \Psi(M_{\sigma_1\backslash\cdots\backslash\sigma_k}) = \begin{cases} \zeta_{\mathsf{QSym}}(M_{(|\sigma_1|,\ldots,|\sigma_k|)}) & \text{if } \sigma = id_{|\sigma_1|}\backslash\cdots\backslash id_{|\sigma_k|}, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\zeta_{\mathsf{QSym}}(M_{(|\sigma_1|,\ldots,|\sigma_k|)}) = \begin{cases} 1 & \text{if } k = 1 \text{ or } \sigma \in \mathfrak{S}_0, \\ 0 & \text{otherwise,} \end{cases}$$

we can conclude that

$$\zeta_{\mathcal{V}}(M_\sigma) = \begin{cases} 1 & \text{if } \sigma = id_n \text{ or } \sigma \in \mathfrak{S}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.4.1. *Let $\Theta_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ be a Hopf morphism and let $\zeta = \overline{\zeta^{-1}_{\mathsf{QSym}}}\zeta_{\mathsf{QSym}} \circ \Psi$. Then the following statements are equivalent.*

1. *$\Psi \circ \Theta_{\mathcal{V}} = \Theta_{\mathsf{QSym}} \circ \Psi$, i.e., $\Theta_{\mathcal{V}}$ is a Theta map for $\mathcal{V}$.*

2. *$\zeta = \zeta_{\mathcal{V}} \circ \Theta_{\mathcal{V}}$.*

3. $\Theta_{\mathcal{V}}^* : \mathcal{V}^* \to \mathcal{V}^*$ *satisfies that* $M_{id_n}^* \mapsto \zeta|_{\mathcal{V}_n}$.

4. $\Psi^* \circ \Theta_{\mathsf{NSym}} = \Theta_{\mathcal{V}}^* \circ \Psi^*$.

Proof. Note that (1) and (4) are equivalent by duality, and by Theorem 4.0.2, (1) and (2) are equivalent.

Now, $\zeta = \zeta_{\mathcal{V}} \circ \Theta_{\mathcal{V}}$ if and only if $\zeta(M_\sigma) = \zeta_{\mathcal{V}} \circ \Theta_{\mathcal{V}}(M_\sigma)$ for all σ. Fix a $\sigma \in \mathfrak{S}_n$, we have that $\zeta(M_\sigma) = \zeta|_{\mathfrak{S}\mathsf{Sym}_n}(M_\sigma)$, thus

$$\zeta_{\mathcal{V}} \circ \Theta_{\mathcal{V}}(M_\sigma) = \langle \Theta_{\mathcal{V}}(M_\sigma), M_{\mathrm{id}_n}^* \rangle = \Theta_{\mathcal{V}}^*(M_{\mathrm{id}_n}^*)(M_\sigma).$$

Therefore, $\zeta = \zeta_{\mathcal{V}} \circ \Theta_{\mathcal{V}}$ if and only if $\Theta_{\mathcal{V}}^*(M_{\mathrm{id}_n}^*)(M_\sigma) = \zeta|_{\mathcal{V}_n}(M_\sigma)$ for all σ if and only if $\Theta_{\mathcal{V}}^*(M_{\mathrm{id}_n}^*) = \zeta|_{\mathcal{V}_n}$. Therefore, (2) and (3) are equivalent. □

Let $\mathsf{QV} := \mathbb{C}\text{-span}\{M_{id_{n_1} \backslash \cdots \backslash id_{n_k}}\}$. Recall that QV is isomorphic to QSym via the isomorphism $I = \Psi|_{\mathsf{QV}}$. And $\mathcal{V} = \mathsf{QV} \oplus \ker(\Psi)$.

For all $b \in \ker(\Psi)$, $\Psi \circ \Theta_{\mathcal{V}}(b) = \Theta_{\mathsf{QSym}} \circ \Psi(b) = 0$ i.e. $\Theta_{\mathcal{V}}(b) \in \ker(\Psi)$.

For all $a \in \mathsf{QV}$, $\Psi \circ \Theta_{\mathcal{V}}(a) = \Theta_{\mathsf{QSym}} \circ \Psi(a) = \Theta_{\mathsf{QSym}} \circ I(a)$. Therefore, $\Theta_{\mathcal{V}}(a) = \Theta_{\mathsf{QSym}} \circ I(a) + b$ for some $b \in \ker(\Psi)$.

Consider the case that $\Theta_{\mathcal{V}}(b) = 0$ for all $b \in \ker(\Psi)$ and $\Theta_{\mathcal{V}}(a) = \Theta_{\mathsf{QSym}} \circ I(a)$ for all $a \in \mathsf{QV}$, then $\Theta_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is a Hopf morphism. Therefore, we have a theta map for $\mathcal{V}$ whose image is isomorphic to the peak algebra Π.

4.5 Theta maps for Malvenuto-Reutenauer Hopf algebra

We now denote $\zeta = \overline{\zeta_{\mathsf{QSym}}^{-1}}\zeta_{\mathsf{QSym}} \circ D$ and $\zeta_{\mathfrak{S}\mathsf{Sym}} = \zeta_{\mathsf{QSym}} \circ D$. In particular, $\zeta|_{\mathfrak{S}\mathsf{Sym}_n} = \sum_{\substack{\sigma \in S_n \\ peak(\sigma)=\emptyset}} 2F_\sigma^*$.

Proposition 4.5.1. *Let $\Theta : \mathfrak{S}\mathsf{Sym} \to \mathfrak{S}\mathsf{Sym}$ be a coalgebra morphism. The following are equivalent.*

1. $D \circ \Theta = \Theta_{\mathsf{QSym}} \circ D$,
2. $\zeta = \zeta_{\mathfrak{S}\mathsf{Sym}} \circ \Theta$,
3. $\Theta^* : \mathfrak{S}\mathsf{Sym}^* \to \mathfrak{S}\mathsf{Sym}^*$ *satisfies that* $M_{id_n}^* \mapsto \zeta|_{\mathfrak{S}\mathsf{Sym}_n}$,
4. $D^* \circ \Theta_{\mathsf{NSym}} = \Theta^* \circ D^*$.

Proof. The same as Proposition 4.4.1. We only need to check that $\zeta_{\mathfrak{S}\mathsf{Sym}} = \zeta_{\mathsf{QSym}} \circ D$ maps M_σ to 1 when $\sigma = id_n$ and 0 otherwise. □

Definition 4.5.2. A map $\Theta : \mathfrak{S}\mathsf{Sym} \to \mathfrak{S}\mathsf{Sym}$ is said to be self-adjoint if $\Theta^* \circ I_{\mathfrak{S}\mathsf{Sym}} = I_{\mathfrak{S}\mathsf{Sym}} \circ \Theta$.

Proposition 4.5.3. *If $\Theta : \mathfrak{S}\mathsf{Sym} \to \mathfrak{S}\mathsf{Sym}$ is a self-adjoint coalgebra morphism, then it is a Hopf morphism.*

Proof. All we need to show is that Θ is an algebra morphism. If Θ is a coalgebra morphism, then Θ^* is an algebra morphism. And the result follows from the composition of algebra morphisms $\Theta = I_{\mathfrak{S}\mathsf{Sym}}^{-1} \circ \Theta^* \circ I_{\mathfrak{S}\mathsf{Sym}}$. □

We will construct graded maps $\Theta^* : \mathfrak{S}\mathsf{Sym}^* \to \mathfrak{S}\mathsf{Sym}^*$ that has three properties:

- P1: $\Theta^*(M^*_{id_n}) = \sum_{\substack{\sigma \in S_n \\ Peak(\sigma)=\emptyset}} 2F^*_\sigma$;
- P2: Θ^* is self-adjoint.
- P3 :Θ^* is an algebra morphism;

Lemma 4.5.4. *Θ^* is self-adjoint if and only if* $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle \Theta^*(F^*_\tau), F_{\sigma^{-1}} \rangle$.

Proof. We know that

$$\left\langle F^*_\sigma, I^{-1}_{\mathfrak{S}\mathsf{Sym}}(F^*_\tau) \right\rangle = \langle F^*_\sigma, F_{\tau^{-1}} \rangle = \delta_{\sigma,\tau^{-1}} = \delta_{\tau,\sigma^{-1}} = \langle F^*_\tau, F_{\sigma^{-1}} \rangle = \left\langle F^*_\tau, I^{-1}_{\mathfrak{S}\mathsf{Sym}}(F^*_\sigma) \right\rangle.$$

Hence, we have

$$\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle F^*_\sigma, \Theta(F_{\tau^{-1}}) \rangle = \left\langle F^*_\sigma, \Theta \circ I^{-1}_{\mathfrak{S}\mathsf{Sym}}(F^*_\tau) \right\rangle$$

and

$$\langle \Theta^*(F^*_\tau), F_{\sigma^{-1}} \rangle = \left\langle \Theta^*(F^*_\tau), I^{-1}_{\mathfrak{S}\mathsf{Sym}}(F^*_\sigma) \right\rangle = \left\langle F^*_\sigma, I^{-1}_{\mathfrak{S}\mathsf{Sym}} \circ \Theta^*(F^*_\tau) \right\rangle$$

□

Therefore, in terms of the structure coefficients, the three properties are as follows:

- C1: $\left\langle \Theta^*(F^*_{id_n}), F_\sigma \right\rangle = \begin{cases} 2 & if\ peak(\sigma) = \emptyset \\ 0 & otherwise \end{cases}$;

- C2: $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle \Theta^*(F^*_\tau), F_{\sigma^{-1}} \rangle$;
- C3: If $\sigma = \mu \backslash \nu$ has global descent, and $M^*_\sigma = M^*_\mu \cdot M^*_\nu$, where μ and ν are non-empty, then $\sum_{\tau \leq \sigma} \langle \Theta^*(F^*_\tau), F_\gamma \rangle = \langle \Theta^*(M^*_\sigma), F_\gamma \rangle = \left\langle \Theta^*(M^*_\mu) \cdot \Theta^*(M^*_\nu), F_\gamma \right\rangle$ is determined by structure coefficients in lower degrees.

Definition 4.5.5. We choose a total order $<_t$ on S_n that satisfies the following two relations:

- if $\sigma < \tau$ in the weak order, then $\sigma <_t \tau$.
- if $GD(\sigma) = \emptyset$ and $GD(\tau) \neq \emptyset$, then $\sigma <_t \tau$.

We need the following lemma to show that $<_t$ is well defined.

Lemma 4.5.6. *For $\sigma, \tau \in S_n$, if $\sigma < \tau$ and $GD(\sigma) \neq \emptyset$, then $GD(\tau) \neq \emptyset$.*

Proof. This follows from the fact that $\{(a, b) : a \leq i, b > i\} \subseteq Inv(\sigma)$ if and only if $i \in GD(\sigma)$, and $\sigma < \tau$ if and only if $Inv(\sigma) \subseteq Inv(\tau)$. □

We construct Θ^* inductively with respect to degree. In degree 1, $\Theta^*(F^*_1) = 2F^*_1$, which satisfies all three conditions. Suppose we have constructed Θ^* for degree $1, \ldots, n-1$, in degree n, we also use induction with respect to $<_t$ as follows:

1. $\left\langle \Theta^*(F^*_{id_n}), F_\sigma \right\rangle = \begin{cases} 2 & if\ peak(\sigma) = \emptyset \\ 0 & otherwise \end{cases}$;
2. $\langle \Theta^*(F^*_\sigma), F_{id_n} \rangle = \left\langle \Theta^*(F^*_{id_n}), F_{\sigma^{-1}} \right\rangle$;
3. if $GD(\sigma) = GD(\tau) = \emptyset$ and $\sigma \leq_t \tau$, then $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle$ is a free variable;

4. if $GD(\sigma) = GD(\tau) = \emptyset$ and $\sigma >_t \tau$, then $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle \Theta^*(F^*_\tau), F_{\sigma^{-1}} \rangle$;

5. if $GD(\sigma) \neq \emptyset$ and $GD(\tau) = \emptyset$, then $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle \Theta^*(M^*_\sigma), F_{\tau^{-1}} \rangle - \sum_{\gamma < \sigma} \langle \Theta^*(F^*_\gamma), F_{\tau^{-1}} \rangle$;

6. if $GD(\sigma) = \emptyset$ and $GD(\tau) \neq \emptyset$, then $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle \Theta^*(F^*_\tau), F_{\sigma^{-1}} \rangle$;

7. if $GD(\sigma) \neq \emptyset$ and $GD(\tau) \neq \emptyset$, then $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle \Theta^*(M^*_\sigma), F_{\tau^{-1}} \rangle - \sum_{\gamma < \sigma} \langle \Theta^*(F^*_\gamma), F_{\tau^{-1}} \rangle$

where σ and τ in steps $(3) - (7)$ are not id_n.

Example 4.5.7. *The following tables are structure coefficients in degree 2 and 3. The entry at row F^*_σ and column F_τ is $\langle \Theta^*(F^*_\sigma), F_\tau \rangle$.*

Degree 2:

	F_{12}	F_{21}
F^*_{12}	2	2
F^*_{21}	2	2

Degree 3:

	F_{123}	F_{132}	F_{213}	F_{312}	F_{231}	F_{321}
F^*_{123}	2	0	2	2	0	2
F^*_{132}	0	a	b	$4-a$	$4-b$	0
F^*_{213}	2	b	c	$2-b$	$2-c$	2
F^*_{231}	2	$4-a$	$2-b$	$a-2$	b	2
F^*_{312}	0	$4-b$	$2-c$	b	$c+2$	0
F^*_{321}	2	0	2	2	0	2

This construction defines Θ^*, in order to show that Θ^* satisfies the three properties, it suffices to check the following:

1. when $GD(\sigma) \neq \emptyset$, $\langle \Theta^*(M^*_\sigma), F_{id_n} \rangle$ satisfies property 3;
2. when $GD(\sigma) \neq \emptyset$ and $GD(\tau) \neq \emptyset$, $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle$ satisfies property 2.

Proof of (1).

By construction, $\langle \Theta^*(M^*_\sigma), F_{id_n} \rangle = \sum_{\tau \leq \sigma} \langle \Theta^*(F^*_\tau), F_{id_n} \rangle = \sum_{\tau \leq \sigma} \left\langle \Theta^*(F^*_{id_n}), F_{\tau^{-1}} \right\rangle = 2 \left| \left\{ \tau \leq \sigma : peak(\tau^{-1}) = \emptyset \right\} \right|$.

When $GD(\sigma) \neq \emptyset$, $M^*_\sigma = M^*_\mu \cdot M^*_\nu$ for some μ, ν non-empty. In this case, property 3 states that

$$\begin{aligned}
\langle \Theta^*(M^*_\sigma), F_{id_n} \rangle &= \left\langle \Theta^*(M^*_\mu) \cdot \Theta^*(M^*_\nu), F_{id_n} \right\rangle \\
&= \left\langle \Theta^*(M^*_\mu) \otimes \Theta^*(M^*_\nu), \Delta(F_{id_n}) \right\rangle \\
&= \left\langle \Theta^*(M^*_\mu) \otimes \Theta^*(M^*_\nu), F_{id_{|\mu|}} \otimes F_{id_{|\nu|}} \right\rangle \\
&= \left\langle \Theta^*(M^*_\mu), F_{id_{|\mu|}} \right\rangle \cdot \left\langle \Theta^*(M^*_\nu), F_{id_{|\nu|}} \right\rangle \\
&= 2 \left| \left\{ \tau \leq \mu : peak(\tau^{-1}) = \emptyset \right\} \right| \cdot 2 \left| \left\{ \tau \leq \nu : peak(\tau^{-1}) = \emptyset \right\} \right|
\end{aligned}$$

where the last equality is by induction on degree. Then, the following lemma completes the proof.

Lemma 4.5.8. *If $M^*_\sigma = M^*_\mu \cdot M^*_\nu$, $\sigma \in S_n$ and μ, ν non-empty, then*

$$\left| \left\{ \tau \leq \sigma : peak(\tau^{-1}) = \emptyset \right\} \right| = 2 \left| \left\{ \tau \leq \mu : peak(\tau^{-1}) = \emptyset \right\} \right| \cdot \left| \left\{ \tau \leq \nu : peak(\tau^{-1}) = \emptyset \right\} \right|.$$

Proof. If $\tau \in S_n$ and $peak(\tau^{-1}) = \emptyset$, then $\tau^{-1}(1) = n$ or $\tau^{-1}(n) = n$. Hence,

$\tau(n) = 1$ or n.

Case 1: $\sigma(n) = 1$. Assume that $\mu \in S_{n-1}$ that $\mu(i) = \sigma(i) - 1$ and $\nu = 1$. Then it suffices to show

$$\left|\{\tau \leq \sigma : peak(\tau^{-1}) = \emptyset\}\right| = 2\left|\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\}\right|.$$

We construct two bijections. One is between $\{\tau \leq \sigma : \tau(n) = 1, peak(\tau^{-1}) = \emptyset\}$ and $\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\}$. The other one is between $\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\}$ and $\{\tau \leq \sigma : \tau(n) = n, peak(\tau^{-1}) = \emptyset\}$.

For each $\tau \in \{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\}$, we assign two elements $\tilde{\tau} = \tau \backslash 1$ and $\tilde{\tau}' = \tau \odot n$.

Since $Inv(\tilde{\tau}') = Inv(\tau)$, $Inv(\tilde{\tau}) = Inv(\tau) \cup \{(a, n) : 1 \leq a < n\}$, $Inv(\tau) \subseteq Inv(\mu)$ and $Inv(\sigma) = Inv(\mu) \cup \{(a, n) : 1 \leq a < n\}$, we have $\tilde{\tau} \leq \sigma$ and $\tilde{\tau}' \leq \sigma$.

Since $\tilde{\tau}(i) = \tau(i) + 1$ for $1 \leq i \leq n-1$ and $\tilde{\tau}(n) = 1$, we have $\tilde{\tau}^{-1}(i) = \tau^{-1}(i-1)$ for $2 \leq i \leq n$. Since $2 \notin peak(\tilde{\tau}^{-1})$ as $\tilde{\tau}^{-1}(1) = n$, it follows that $\tilde{\tau}^{-1}(i-1) < \tilde{\tau}^{-1}(i) > \tilde{\tau}^{-1}(i+1)$ if and only if $\tau^{-1}(i-2) < \tau^{-1}(i-1) > \tau^{-1}(i)$ for $3 \leq i \leq n-1$. Hence, $peak(\tilde{\tau}^{-1}) = \emptyset$.

Since $\tilde{\tau}'(i) = \tau(i)$ for $1 \leq i \leq n-1$ and $\tilde{\tau}'(n) = n$, we have $\tilde{\tau}'^{-1}(i) = \tau^{-1}(i)$ for $1 \leq i \leq n-1$. Since $n-1 \notin peak(\tilde{\tau}'^{-1})$ as $\tilde{\tau}'^{-1}(n) = n$, it follows that $\tilde{\tau}'^{-1}(i-1) < \tilde{\tau}'^{-1}(i) > \tilde{\tau}'^{-1}(i+1)$ if and only if $\tau^{-1}(i-1) < \tau^{-1}(i) > \tau^{-1}(i+1)$ for $2 \leq i \leq n-2$. Hence, $peak(\tilde{\tau}'^{-1}) = \emptyset$.

Conversely, for each $\tau \in \{\tau \leq \sigma : peak(\tau^{-1}) = \emptyset\}$, if $\tau(n) = 1$, we assign

$\tilde{\tau} \in S_{n-1}$ that $\tilde{\tau}(i) = \tau(i) - 1$, if $\tau(n) = n$, we assign $\tilde{\tau} \in S_{n-1}$ that $\tilde{\tau}(i) = \tau(i)$. By a similar argument, $\tilde{\tau} \in \{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\}$. And they are clearly inverse of each other.

If $M^*_\nu = M^*_\omega \cdot M^*_1$ for some non-empty ω, then by induction

$$\left|\left\{\tau \leq \nu : peak(\tau^{-1}) = \emptyset\right\}\right| = 2\left|\left\{\tau \leq \omega : peak(\tau^{-1}) = \emptyset\right\}\right|$$

and

$$\left|\left\{\tau \leq \mu\backslash\omega : peak(\tau^{-1}) = \emptyset\right\}\right| = 2\left|\left\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\right\}\right| \cdot \left|\left\{\tau \leq \omega : peak(\tau^{-1}) = \emptyset\right\}\right|.$$

Therefore,

$$\begin{aligned} & 2\left|\left\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\right\}\right| \cdot \left|\left\{\tau \leq \nu : peak(\tau^{-1}) = \emptyset\right\}\right| \\ =& 4\left|\left\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\right\}\right| \cdot \left|\left\{\tau \leq \omega : peak(\tau^{-1}) = \emptyset\right\}\right| \\ =& 2\left|\left\{\tau \leq \mu\backslash\omega : peak(\tau^{-1}) = \emptyset\right\}\right| \\ =& \left|\left\{\tau \leq \sigma : peak(\tau^{-1}) = \emptyset\right\}\right|. \end{aligned}$$

Case 2: $\sigma(n) > 1$. In this case, $|\nu| \geq 2$. For simplicity, let $\gamma = std(\sigma(1) \odot \cdots \odot \sigma(n-1))$ and $\omega = std(\nu(1) \odot \cdots \odot \nu(|\nu| - 1))$.

Claim 1: if $\tau \leq \sigma$, then $\tau(n) > 1$. If not, $(\sigma^{-1}(1), n) \notin Inv(\sigma)$ but $(\sigma^{-1}(1), n) \in Inv(\tau)$.

Claim 2: $\tau \leq \sigma$ and $\tau(n) = n$ if and only if $\tau \leq \gamma \odot n$. First, $\gamma \odot n \leq \sigma$ as $Inv(\gamma \odot n) = Inv(\sigma) \setminus \{(a, n) : 1 \leq a \leq n-1\}$. Second, if $\tau \leq \gamma \odot n$ and $\tau(n) < n$ then $(\tau^{-1}(n), n) \in Inv(\tau)$ but $(\tau^{-1}(n), n) \notin Inv(\gamma \odot n)$. Therefore, if $\tau \leq \gamma \odot n$, then $\tau \leq \sigma$ and $\tau(n) = n$. Conversely, if $\tau \leq \sigma$ and $\tau(n) = n$, then $Inv(\tau) \subseteq Inv(\sigma) \setminus \{(a,n) : 1 \leq a \leq n-1\} = Inv(\gamma \odot n)$ i.e. $\tau \leq \gamma \odot n$.

Therefore, $\{\tau \leq \sigma : peak(\tau^{-1}) = \emptyset\} = \{\tau \leq \gamma \odot n : peak(\tau^{-1}) = \emptyset\}$ as $peak(\tau) = \emptyset$ only if $\tau(n) = 1$ or n.

We also know that $|\{\tau \leq \gamma \odot n : peak(\tau^{-1}) = \emptyset\}| = |\{\tau \leq \gamma : peak(\tau^{-1}) = \emptyset\}|$ from Case 1. Hence, by induction, we have

$$\begin{aligned}
&2\left|\left\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\right\}\right| \cdot \left|\left\{\tau \leq \nu : peak(\tau^{-1}) = \emptyset\right\}\right| \\
=&2\left|\left\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\right\}\right| \cdot \left|\left\{\tau \leq \omega \odot |\nu| : peak(\tau^{-1}) = \emptyset\right\}\right| \\
=&2\left|\left\{\tau \leq \mu : peak(\tau^{-1}) = \emptyset\right\}\right| \cdot \left|\left\{\tau \leq \omega : peak(\tau^{-1}) = \emptyset\right\}\right| \\
=&\left|\left\{\tau \leq \mu \backslash \omega : peak(\tau^{-1}) = \emptyset\right\}\right| \\
=&\left|\left\{\tau \leq \gamma : peak(\tau^{-1}) = \emptyset\right\}\right| \\
=&\left|\left\{\tau \leq \gamma \odot n : peak(\tau^{-1}) = \emptyset\right\}\right| \\
=&\left|\left\{\tau \leq \sigma : peak(\tau^{-1}) = \emptyset\right\}\right|.
\end{aligned}$$

□

Proof of (2).

Lemma 4.5.9. *The map Θ^* is a coalgebra morphism i.e. $(\Theta^* \otimes \Theta^*) \circ (\Delta_{s,t}(M^*_\sigma)) = \Delta_{s,t} \circ \Theta^*(M^*_\sigma)$ for all $s, t \geq 1$, $s + t = |\sigma|$, where $\Delta_{s,t} : \mathfrak{S}\mathsf{Sym}_{s+t} \to \mathfrak{S}\mathsf{Sym}_s \otimes \mathfrak{S}\mathsf{Sym}_t$ is the co-multiplication followed by projection to degree s and degree t components.*

Proof. The equality clearly holds when $\sigma = 1$. Suppose $|\sigma| \geq 2$ and we use induction on degree.

Case 1: $GD(\sigma) = \emptyset$. The equality clearly holds when $\sigma = id_n$, in which case it is equivalent to that Θ_{NSym} is a coalgebra morphism. When $\sigma \neq id_n$, by induction on $<_t$, it suffices to prove that $(\Theta^* \otimes \Theta^*) \circ (\Delta_{s,t}(F^*_\sigma)) = \Delta_{s,t} \circ \Theta^*(F^*_\sigma)$.

Let $\Delta_{s,t}(F_{\sigma^{-1}}) = F_\delta \otimes F_\epsilon$. For all $\mu \in S_s$, $\nu \in S_t$, we have

$$\begin{aligned}
\langle \Delta_{s,t} \circ \Theta^*(F^*_\sigma), F_\mu \otimes F_\nu \rangle &= \langle \Theta^*(F^*_\sigma), F_\mu \cdot F_\nu \rangle \\
&= \langle \Theta^*(F^*_{\mu^{-1}} \cdot F^*_{\nu^{-1}}), F_{\sigma^{-1}} \rangle \\
&= \langle (\Theta^* \otimes \Theta^*)(F^*_{\mu^{-1}} \otimes F^*_{\nu^{-1}}), \Delta_{s,t}(F_{\sigma^{-1}}) \rangle \\
&= \langle \Theta^*(F^*_{\mu^{-1}}), F_\delta \rangle \cdot \langle \Theta^*(F^*_{\nu^{-1}}), F_\epsilon \rangle \\
&= \langle (\Theta^* \otimes \Theta^*)(F^*_{\delta^{-1}} \otimes F^*_{\epsilon^{-1}}), F_\mu \otimes F_\nu) \rangle \\
&= \langle (\Theta^* \otimes \Theta^*) \circ \Delta_{s,t}(F^*_\sigma), F_\mu \otimes F_\nu) \rangle.
\end{aligned}$$

Case 2: $GD(\sigma) \neq \emptyset$. Let $\sigma = \mu \backslash \nu$, then by induction on degree,

$$\begin{aligned}
\Delta(\Theta^*(M^*_\sigma)) &= \Delta(\Theta^*(M^*_\mu) \cdot \Theta^*(M^*_\nu)) \\
&= \Delta(\Theta^*(M^*_\mu)) \cdot \Delta(\Theta^*(M^*_\nu)) \\
&= \big((\Theta^* \otimes \Theta^*) \circ \Delta(M^*_\mu)\big) \cdot ((\Theta^* \otimes \Theta^*) \circ \Delta(M^*_\nu)) \\
&= (\Theta^* \otimes \Theta^*) \circ \big(\Delta(M^*_\mu) \cdot \Delta(M^*_\nu)\big) \\
&= (\Theta^* \otimes \Theta^*) \circ (\Delta(M^*_\sigma)) \,.
\end{aligned}$$

□

Let $\sigma, \tau \in S_n$ and $GD(\sigma) \neq \emptyset$, $GD(\tau) \neq \emptyset$. Let $\sigma = \mu \backslash \nu$ and $\tau = \delta \backslash \epsilon$ where

$\mu, \nu, \delta, \epsilon$ are non-empty. By construction,

$$\begin{aligned}
\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle &= \langle \Theta^*(M^*_\sigma), F_{\tau^{-1}} \rangle - \sum_{\gamma<\sigma} \langle \Theta^*(F^*_\gamma), F_{\tau^{-1}} \rangle \\
&= \langle \Theta^*(M^*_\sigma), F_{\tau^{-1}} \rangle - \sum_{\gamma<\sigma} \langle \Theta^*(F^*_\tau), F_{\gamma^{-1}} \rangle \\
&= \langle \Theta^*(M^*_\sigma), F_{\tau^{-1}} \rangle - \sum_{\gamma<\sigma} \langle \Theta^*(M^*_\tau), F_{\gamma^{-1}} \rangle + \sum_{\substack{\gamma<\sigma \\ \rho<\tau}} \langle \Theta^*(F^*_\rho), F_{\gamma^{-1}} \rangle.
\end{aligned}$$

Similarly,

$$\langle \Theta^*(F^*_\tau), F_{\sigma^{-1}} \rangle = \langle \Theta^*(M^*_\tau), F_{\sigma^{-1}} \rangle - \sum_{\rho<\tau} \langle \Theta^*(M^*_\sigma), F_{\rho^{-1}} \rangle + \sum_{\substack{\rho<\tau \\ \gamma<\sigma}} \langle \Theta^*(F^*_\gamma), F_{\rho^{-1}} \rangle.$$

Therefore, by induction on $<_t$, $\langle \Theta^*(F^*_\sigma), F_{\tau^{-1}} \rangle = \langle \Theta^*(F^*_\tau), F_{\sigma^{-1}} \rangle$ if and only if

$$\sum_{\gamma\leq\sigma} \langle \Theta^*(M^*_\tau), F_{\gamma^{-1}} \rangle = \sum_{\rho\leq\tau} \langle \Theta^*(M^*_\sigma), F_{\rho^{-1}} \rangle.$$

Let $\Delta_{|\mu|,|\nu|}(M^*_\delta) = \sum_{\alpha,\beta} C^\delta_{\alpha,\beta} M^*_\alpha \otimes M^*_\beta$ and $\Delta_{|\mu|,|\nu|}(M^*_\epsilon) = \sum_{\alpha',\beta'} C^\epsilon_{\alpha',\beta'} M^*_{\alpha'} \otimes M^*_{\beta'}$. Then

$$\Delta_{|\mu|,|\nu|}(M^*_\tau) = \Delta_{|\mu|,|\nu|}(M^*_\delta \cdot M^*_\epsilon) = \sum_{\alpha,\beta,\alpha'\beta'} C^\delta_{\alpha,\beta} C^\epsilon_{\alpha',\beta'} M^*_{\alpha\backslash\alpha'} \otimes M^*_{\beta\backslash\beta'}.$$

By induction on degree, we get

$$
\begin{aligned}
&\sum_{\rho\le\tau}\langle\Theta^*(M^*_\sigma),F_{\rho^{-1}}\rangle=\sum_{\rho\le\tau}\langle\Theta^*(M^*_\sigma),I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(F^*_\rho\right)\rangle=\langle\Theta^*(M^*_\sigma),I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\tau\right)\rangle\\
&=\left\langle\Theta^*(M^*_\mu)\cdot\Theta^*(M^*_\nu),I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\tau\right)\right\rangle\\
&=\left\langle\Theta^*(M^*_\mu)\otimes\Theta^*(M^*_\nu),\Delta_{|\mu|,|\nu|}\left(I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\tau\right)\right)\right\rangle\\
&=\left\langle\Theta^*(M^*_\mu)\otimes\Theta^*(M^*_\nu),\left(I^{-1}_{\mathfrak{S}\mathsf{Sym}}\otimes I^{-1}_{\mathfrak{S}\mathsf{Sym}}\right)\circ\left(\Delta_{|\mu|,|\nu|}\left(M^*_\tau\right)\right)\right\rangle\\
&=\sum_{\alpha,\beta,\alpha',\beta'}C^\delta_{\alpha,\beta}C^\epsilon_{\alpha',\beta'}\left\langle\Theta^*(M^*_\mu),I^{-1}_{\mathfrak{S}\mathsf{Sym}}(M^*_{\alpha\backslash\alpha'})\right\rangle\left\langle\Theta^*(M^*_\nu),I^{-1}_{\mathfrak{S}\mathsf{Sym}}(M^*_{\beta\backslash\beta'})\right\rangle\\
&=\sum_{\alpha,\beta,\alpha',\beta'}C^\delta_{\alpha,\beta}C^\epsilon_{\alpha',\beta'}\left\langle\Theta^*(M^*_{\alpha\backslash\alpha'}),I^{-1}_{\mathfrak{S}\mathsf{Sym}}(M^*_\mu)\right\rangle\left\langle\Theta^*(M^*_{\beta\backslash\beta'}),I^{-1}_{\mathfrak{S}\mathsf{Sym}}(M^*_\nu)\right\rangle\\
&=\left\langle(\Theta^*\otimes\Theta^*)\circ\left(\Delta_{|\mu|,|\nu|}\left(M^*_\tau\right)\right),\left(I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\mu\right)\right)\otimes\left(I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\nu\right)\right)\right\rangle\\
&=\left\langle\Delta_{|\mu|,|\nu|}\left(\Theta^*\left(M^*_\tau\right)\right),\left(I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\mu\right)\right)\otimes\left(I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\nu\right)\right)\right\rangle\\
&=\left\langle\Theta^*\left(M^*_\tau\right),I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(M^*_\sigma\right)\right\rangle=\sum_{\gamma\le\sigma}\langle\Theta^*(M^*_\tau),I^{-1}_{\mathfrak{S}\mathsf{Sym}}\left(F^*_\gamma\right)\rangle=\sum_{\gamma\le\sigma}\langle\Theta^*(M^*_\tau),F_{\gamma^{-1}}\rangle.
\end{aligned}
$$

4.6 Convolutions of theta maps

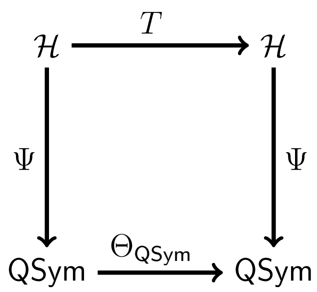
In this section, we study an operation on the set of theta maps.

Let $(\mathcal{H},\zeta_\mathcal{H})$ be a combinatorial Hopf algebra, and Θ_1,Θ_2 be two of its theta maps.

The most natural way of combining two maps is to consider their convolution product $\Theta_1*\Theta_2:\mathcal{H}\to\mathcal{H}$. Unfortunately, $\Theta_1*\Theta_2$ is not a theta map because it is not compatible with Θ_{QSym}.

However, if we take the commutator $[\Theta_1,\Theta_2]=\Theta_1*\Theta_2-\Theta_2*\Theta_1$. This is a non-trivial map in general, when $\mathcal{H}$ is not co-commutative. But when projected down to the level of QSym, it is the commutator of Θ_{QSym} with itself, which is the 0 map. Therefore, if we take any theta map Θ, we obtain an infinite family of maps $T = \Theta + k[\Theta_1, \Theta_2]$ that is compatible with Θ_{QSym}.

Theorem 4.6.1. *Let $(\mathcal{H}, \zeta_{\mathcal{H}})$ be a combinatorial Hopf algebra with Hopf morphism $\Psi : \mathcal{H} \to \mathsf{QSym}$ and $\Theta, \Theta_1, \Theta_2$ be its theta maps. Then $T = \Theta + k[\Theta_1, \Theta_2]$ makes the following square commute.*



Proof. Since Θ_1 and Θ_2 are theta maps, we have

$$\begin{aligned}
\Psi \circ (\Theta_1 * \Theta_2) &= \Psi \circ m \circ (\Theta_1 \otimes \Theta_2) \circ \Delta \\
&= m \circ (\Psi \otimes \Psi) \circ (\Theta_1 \otimes \Theta_2) \circ \Delta \\
&= m \circ (\Psi \circ \Theta_1 \otimes \Psi \circ \Theta_2) \circ \Delta \\
&= m \circ (\Theta_{\mathsf{QSym}} \circ \Psi \otimes \Theta_{\mathsf{QSym}} \circ \Psi) \circ \Delta \\
&= m \circ (\Theta_{\mathsf{QSym}} \otimes \Theta_{\mathsf{QSym}}) \circ (\Psi \otimes \Psi) \circ \Delta \\
&= m \circ (\Theta_{\mathsf{QSym}} \otimes \Theta_{\mathsf{QSym}}) \circ \Delta \circ \Psi \\
&= (\Theta_{\mathsf{QSym}} * \Theta_{\mathsf{QSym}}) \circ \Psi.
\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi \circ [\Theta_1, \Theta_2] &= \Psi \circ (\Theta_1 * \Theta_2) - \Psi \circ (\Theta_2 * \Theta_1) \\ &= (\Theta_{\mathsf{QSym}} * \Theta_{\mathsf{QSym}}) \circ \Psi - (\Theta_{\mathsf{QSym}} * \Theta_{\mathsf{QSym}}) \circ \Psi \\ &= 0\end{aligned}$$

and

$$\Psi \circ T = \Psi \circ (\Theta + k[\Theta_1, \Theta_2]) = \Psi \circ \Theta + k\Psi \circ [\Theta_1, \Theta_2] = \Psi \circ \Theta = \Theta_{\mathsf{QSym}} \circ \Psi.$$

□

Moreover, when $\mathcal{H}$ is self-dual, then T is self-adjoint.

Lemma 4.6.2. *If $\mathcal{H}$ is a self-dual graded Hopf algebra with isomorphism $I : \mathcal{H} \to \mathcal{H}^*$, and we have two self-adjoint map $f, g : \mathcal{H} \to \mathcal{H}$, then $af + bg$ is self-adjoint for any constant a, b.*

Proof. For any $h \in \mathcal{H}, k \in \mathcal{H}^*$, we have

$$\begin{aligned}\langle (af + bg)^*(k), h\rangle &= \langle k, (af + bg)(h)\rangle = a\langle k, f(h)\rangle + b\langle k, g(h)\rangle \\ &= a\langle f^*(k), h\rangle + b\langle g^*(k), h\rangle = \langle (af^* + bg^*)(k), h\rangle\end{aligned}$$

Hence, we have $(af + bg)^* = af^* + bg^*$ and

$$\begin{aligned}I \circ (af + bg) &= a(I \circ f) + b(I \circ g) = a(f^* \circ I) + b(g^* \circ I) \\ &= (af^* + bg^*) \circ I = (af + bg)^* \circ I.\end{aligned}$$

□

Proposition 4.6.3. *When $\mathcal{H}$ is a self-dual combinatorial Hopf algebra, with isomorphism $I : \mathcal{H} \to \mathcal{H}^*$, and $\Theta, \Theta_1, \Theta_2$ are self-adjoint theta maps. Then $T = \Theta + k[\Theta_1, \Theta_2]$ is self-adjoint i.e. $I \circ T = T^* \circ I$.*

Proof. By Lemma 4.6.2, it suffices to show that $\Theta_1 * \Theta_2$ is self-adjoint.

For any $h \in \mathcal{H}, k \in \mathcal{H}^*$, we have

$$\begin{aligned}
\langle (\Theta_1 * \Theta_2)^*(k), h \rangle &= \langle k, (\Theta_1 * \Theta_2)(h) \rangle \\
&= \langle k, m \circ (\Theta_1 \otimes \Theta_2) \circ \Delta(h) \rangle \\
&= \langle \Delta(k), (\Theta_1 \otimes \Theta_2) \circ \Delta(h) \rangle \\
&= \langle (\Theta_1^* \otimes \Theta_2^*) \circ \Delta(k), \Delta(h) \rangle \\
&= \langle m \circ (\Theta_1^* \otimes \Theta_2^*) \circ \Delta(k), h \rangle \\
&= \langle (\Theta_1^* * \Theta_2^*)(k), h \rangle.
\end{aligned}$$

Hence, $(\Theta_1 * \Theta_2)^* = \Theta_1^* * \Theta_2^*$, and we have

$$\begin{aligned}
I \circ (\Theta_1 * \Theta_2) &= I \circ m \circ (\Theta_1 \otimes \Theta_2) \circ \Delta \\
&= m \circ (I \otimes I) \circ (\Theta_1 \otimes \Theta_2) \circ \Delta \\
&= m \circ (\Theta_1^* \otimes \Theta_2^*) \circ (I \otimes I) \circ \Delta \\
&= m \circ (\Theta_1^* \otimes \Theta_2^*) \circ \Delta \circ I \\
&= (\Theta_1^* * \Theta_2^*) \circ I = (\Theta_1 * \Theta_2)^* \circ I.
\end{aligned}$$

□

In general, when $\mathcal{H}$ is not co-commutative, the commutator is non-trivial.

Example 4.6.4. *Consider the Hopf algebra $\mathfrak{S}\mathsf{Sym}$. We described its theta* *maps in the previous section. Consider two theta maps whose multiplicative structure constants on degree* 3 *are*

Θ_1	F_{123}	F_{132}	F_{213}	F_{312}	F_{231}	F_{321}
F^*_{123}	2	0	2	2	0	2
F^*_{132}	0	2	2	2	2	0
F^*_{213}	2	2	0	0	2	2
F^*_{231}	2	2	0	0	2	2
F^*_{312}	0	2	2	2	2	0
F^*_{321}	2	0	2	2	0	2

Θ_2	F_{123}	F_{132}	F_{213}	F_{312}	F_{231}	F_{321}
F^*_{123}	2	0	2	2	0	2
F^*_{132}	0	4	0	0	4	0
F^*_{213}	2	0	2	2	0	2
F^*_{231}	2	0	2	2	0	2
F^*_{312}	0	4	0	0	4	0
F^*_{321}	2	0	2	2	0	2

Then, the commutator $[\Theta_1, \Theta_2]$ *is non-trivial because* $\langle F^*_{1243}, [\Theta_1, \Theta_2](F_{3142})\rangle = 4 \neq 0$.

Remark 4.6.5. These maps in general are not Hopf morphisms. They fail to be algebra morphism and co-algebra morphism because algebra morphisms and co-algebra morphisms are not additive. The best we can do right now is to define them on generators, and extend the map multiplicatively, that give an algebra morphism.

Appendix A

Diagonally quasi-symmetric functions

In this section, we study the diagonally quasi-symmetric functions DQSym. This appendix comes from the paper [36].

In the polynomial ring $\mathbb{Q}[\mathbf{x}_n] = \mathbb{Q}[x_1, \ldots, x_n]$ with n variables, the ring of symmetric polynomials (cf. [38, 44]), Sym_n, is the subspace of invariants under the symmetric group S_n action

$$\sigma \cdot f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

The quotient space $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{Sym}_n^+ \rangle$ over the ideal generated by symmetric polynomials with no constant term is thus called the coinvariant space of symmetric group. Classic results by Artin [1] and Steinberg [50] asserts that this quotient forms an S_n-module that is isomorphic to the left regular representation. Moreover, considering the natural scalar product

$$\langle f, g\rangle = \big(f(\partial x_1, \ldots, \partial x_n)(g(x_1, \ldots, x_n))\big)(0, 0, \ldots, 0),$$

this quotient is equal to the orthogonal complement of Sym_n. In particular, the coinvariant space is killed by Laplacian operator $\Delta = \partial x_1^2 + \cdots + \partial x_n^2$. Hence, it is also known as the harmonic space.

One can show that $\{h_k(x_k, \ldots, x_n) : 1 \leq k \leq n\}$ forms a Gröbner basis of $\langle \mathsf{Sym}_n^+\rangle$ with respect to the usual order $x_1 > \cdots > x_n$, where h_k is the complete homogeneous basis of degree k. As a result, the dimension of $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{Sym}_n^+\rangle$ is $n!$.

One generalization is the diagonal harmonic space. In the context of $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n] = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, the diagonally symmetric functions, DSym_n, is the space of invariants under the diagonal action of S_n

$$\sigma \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}).$$

The diagonal harmonics, $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DSym}_n^+\rangle$, was studied by Garsia and Haiman [23, 26] where it was used to prove the $n!$ conjecture and Macdonald positivity. In particular, its dimension turns out to be $(n+1)^{n-1}$. More interesting results and applications can be found in [11, 13, 27].

The ring of quasi-symmetric functions, QSym, was introduced by Gessel [20] as generating function for Stanley's P-partitions [49]. It soon shows great importance in algebraic combinatorics e.g. [5, 22]. In our context, QSym_n can be defined as the space of invariants in $\mathbb{Q}[\mathbf{x}_n]$ under the S_n-action of Hivert

$$\sigma \cdot \left(x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}\right) = x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$$

where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$ and $\{j_1, \ldots, j_k\} = \{\sigma(i_1), \ldots, \sigma(i_k)\}$.

In a series of papers by Aval, F. Bergeron and N. Bergeron, the authors studied the quotient $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{QSym}_n^+ \rangle$ over the ideal generated by quasi-symmetric polynomials with no constant term, which they called the super-covariant space of S_n. Their main result is that a basis of this quotient corresponds to Dyck paths, and the dimension of the quotient space is the n-th Catalan number C_n [2, 3].

After that, they extended QSym to diagonal setting, called diagonally quasi-symmetric functions, DQSym [4]. They described a Hopf algebra structure on DQSym, and made a conjecture about the linear structure of $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DQSym}_n^+ \rangle$.

In this appendix, we continue the study of the linear structure. We start with the case where there are infinitely many variables i.e. $R = \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ is the ring of formal power series where $\mathbf{x} = x_1, x_2, \ldots$ and $\mathbf{y} = y_1, y_2, \ldots$. The main result is that we give a description of a Hilbert basis for the quotient space R/I where $I = \overline{\mathsf{DQSym}^+}$ is the closure of the ideal generated by DQSym without constant terms. This Hilbert basis gives an upper bound on the degree of $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DQSym}_n^+ \rangle$. We then use it to compute the second column of the Hilbert matrix, which coincides with the conjecture in [4].

A.1 The F basis

We define a partial ordering $\preceq$ on bicompositions: $\alpha \preceq \beta$ and β covers α if there exists a $1 \leq k \leq \ell(\alpha) - 1$ such that either $\alpha_{2k} = 0$ or $\alpha_{1(k+1)} = 0$, and

$$\beta = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1(k-1)} & \alpha_{1k} + \alpha_{1(k+1)} & \alpha_{1(k+2)} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21} & \cdots & \alpha_{2(k-1)} & \alpha_{2k} + \alpha_{2(k+1)} & \alpha_{2(k+2)} & \cdots & \alpha_{2\ell(\alpha)} \end{pmatrix}.$$

By triangularity, $\left\{ F_\alpha = \sum_{\alpha \preceq \beta} M_\beta \right\}$ forms a basis for DQSym. This basis is originally introduced in [4]. For example,

$$F_{\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right)} = M_{\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right)} + M_{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)} + M_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)} + M_{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right)} + M_{\left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right)} + M_{\left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)} + M_{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)} + M_{\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right)}.$$

For convenience, we set $F_{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} = 1$. This basis has the following easy but important properties:

If $\alpha_{11} \geq 1$ and $\alpha_{11} + \alpha_{21} \geq 2$, then

$$F_\alpha = x_1 F_{\left(\begin{smallmatrix} \alpha_{11}-1 & \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)} + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots); \tag{1.1}$$

If $\alpha_{11} = 1$ and $\alpha_{21} = 0$, then

$$F_\alpha = x_1 F_{\left(\begin{smallmatrix} \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots); \tag{1.2}$$

If $\alpha_{11} = 0$ and $\alpha_{21} \geq 2$, then

$$F_\alpha = y_1 F_{\left(\begin{smallmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21}-1 & \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)} + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots); \tag{1.3}$$

If $\alpha_{11} = 0$ and $\alpha_{21} = 1$, then

$$F_\alpha = y_1 F_{\left(\begin{smallmatrix} \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots). \quad (1.4)$$

A.2 The G basis

In this section, we define a basis $\{G_{\tilde{\epsilon}}\}$ indexed by generalized bicompositions for $\mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$.

Base cases: $G_{\binom{0}{0}} = 1$ and $G_{\tilde{\epsilon}} = F_{\tilde{\epsilon}}$ if $\tilde{\epsilon}$ is a bicomposition. Otherwise, let $\tilde{\epsilon} = \tilde{\alpha}\binom{0}{0}\beta$ where β is a non-zero bicomposition. Let $k = \ell(\tilde{\epsilon}) - \ell(\beta) - 1$.

If $\beta_{11} > 0$,

$$G_{\tilde{\epsilon}} = G_{\tilde{\alpha}\beta} - x_{k+1} G_{\tilde{\alpha}\left(\begin{smallmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}. \quad (2.1)$$

If $\beta_{11} = 0$,

$$G_{\tilde{\epsilon}} = G_{\tilde{\alpha}\beta} - y_{k+1} G_{\tilde{\alpha}\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}. \quad (2.2)$$

Inductively, $\{G_{\tilde{\epsilon}}\}$ is defined for all generalized bicomposition $\tilde{\epsilon}$. Clearly $G_{\tilde{\epsilon}}$ is homogeneous in degree $|\tilde{\epsilon}|$. Hence, we have a notion of leading monomial of $G_{\tilde{\epsilon}}$, denoted by $LM(G_{\tilde{\epsilon}})$ with respect to the lexicographic order with $x_1 > y_1 > x_2 > y_2 > \cdots$. To show that $\{G_{\tilde{\epsilon}}\}$ forms a basis, it suffices to prove the leading monomial of $G_{\tilde{\epsilon}}$ is $\mathbf{X}^{\tilde{\epsilon}}$.

Lemma A.2.1. *Let $\tilde{\alpha} = \binom{a}{b}\tilde{\beta}$ be a generalized bicomposition,*

1. *if $a = b = 0$, then $G_{\tilde{\alpha}} = G_{\tilde{\beta}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$,*

2. *if $a > 0$, then $G_{\tilde{\alpha}} = x_1 G_{\binom{a-1}{b}\tilde{\beta}} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)$,*

3. *if* $a = 0$ *and* $b > 0$, *then* $G_{\tilde{\alpha}} = y_1 G_{\binom{0}{b-1}\tilde{\beta}} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)$

for some $P \in \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$.

Proof. We prove by induction on the length of $\tilde{\alpha}$.

1. If $\tilde{\alpha} = \binom{0}{0}$, then $G_{\tilde{\alpha}} = 1$ and we are done.
2. If $\tilde{\beta} = \beta$ is a bicomposition,

   (a) if $a = b = 0$ and β non-zero,

   i. if $\beta_{11} \geq 1$ and $\beta_{11} + \beta_{21} \geq 2$, using (1.1) and (2.1), we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - x_1 G_{\begin{pmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}} \\ &= F_\beta - x_1 F_{\begin{pmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}} \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

   and the lemma follows.

   ii. if $\beta_{11} = 1$ and $\beta_{21} = 0$, using (1.2), (2.1) and induction on $\ell(\tilde{\beta})$, we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - x_1 G_{\begin{pmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ 0 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}} \\ &= G_\beta - x_1 G_{\begin{pmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta - x_1 F_{\begin{pmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{pmatrix}}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

   and the lemma follows.

iii. if $\beta_{11} = 0$ and $\beta_{21} \geq 2$, using (1.3) and (2.2), we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - y_1 G_{\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= F_\beta - y_1 F_{\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

and the lemma follows.

iv. if $\beta_{11} = 0$ and $\beta_{21} = 1$, using (1.4), (2.2) and induction on $\ell(\tilde{\beta})$, we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - y_1 G_{\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ 0 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= G_\beta - y_1 G_{\left(\begin{smallmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta - y_1 F_{\left(\begin{smallmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

and the lemma follows.

(b) if $a \geq 1$ and $a + b \geq 2$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (1.1), we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = x_1 F_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\beta} + F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$$

and the lemma follows, with $P = F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$.

(c) if $a = 1$ and $b = 0$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (1.2) and (2a). we get

$$\begin{aligned} G_{\tilde{\alpha}} = F_{\tilde{\alpha}} &= x_1 F_\beta(x_2, x_3, \dots, y_2, y_3, \dots) + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots) \\ &= x_1 G_{\binom{0}{0}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots) \end{aligned}$$

and the lemma follows with $P = F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$.

(d) if $a = 0$ and $b \geq 2$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (1.3), we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = y_1 F_{\binom{a}{b-1}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$$

and the lemma follows, with $P = F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$.

(e) if $a = 0$ and $b = 1$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (1.4) and (2a). we get

$$\begin{aligned} G_{\tilde{\alpha}} = F_{\tilde{\alpha}} &= y_1 F_\beta(x_2, x_3, \dots, y_2, y_3, \dots) + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots) \\ &= y_1 G_{\binom{0}{0}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots) \end{aligned}$$

and the lemma follows with $P = F_{\tilde{\alpha}}(x_2, x_3, \dots, y_2, y_3, \dots)$.

3. In the general case, let $\tilde{\alpha} = \tilde{\gamma}\binom{0}{0}\beta$ where β is a non-empty bicomposition and $k = \ell(\tilde{\alpha}) - \ell(\beta) - 1$. We prove by induction on k. If $k = 1$, then we are back in case (2a) above. Hence, we assume $k > 1$ and $\tilde{\gamma} = \binom{a}{b}\tilde{\mu}$.

(a) If $a = b = 0$,

i. if $\beta_{11} \geq 1$, by induction and (2.1), we have

$$\begin{aligned} G_{\tilde{\alpha}} =& G_{\binom{0}{0}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{0}\tilde{\mu}\beta} - x_k G_{\binom{0}{0}\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \\ =& G_{\tilde{\mu}\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ & - x_{(k-1)+1} G_{\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ =& G_{\tilde{\mu}\binom{0}{0}\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

and the lemma follows.

ii. if $\beta_{11} = 0$, by induction and (2.2), we have

$$\begin{aligned} G_{\tilde{\alpha}} =& G_{\binom{0}{0}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{0}\tilde{\mu}\beta} - y_k G_{\binom{0}{0}\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \\ =& G_{\tilde{\mu}\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ & - y_{(k-1)+1} G_{\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ =& G_{\tilde{\mu}\binom{0}{0}\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

and the lemma follows.

(b) If $a \geq 1$,

i. if $\beta_{11} \geq 1$, by induction and (2.1), we have

$$\begin{aligned}
G_{\tilde{\alpha}} =& G_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\beta} = G_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\tilde{\mu}\beta} - x_k G_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\
=& x_1 G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\beta} + P_1(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
& - x_k \Big(x_1 G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\
& + P_2(x_2, x_3, \ldots, y_2, y_3, \ldots) \Big) \\
=& x_1 \left(G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\beta} - x_k G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \right) \\
& + P(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
=& x_1 G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\beta} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)
\end{aligned}$$

and the lemma follows with $P = P_1 - x_k P_2$.

ii. if $\beta_{11} = 0$, by induction and (2.2), we have

$$\begin{aligned}
G_{\tilde{\alpha}} =& G_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\beta} = G_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\tilde{\mu}\beta} - y_k G_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\
=& x_1 G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\beta} + P_1(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
& - y_k \Big(x_1 G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\
& + P_2(x_2, x_3, \ldots, y_2, y_3, \ldots) \Big) \\
=& x_1 \left(G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\beta} - y_k G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \right) \\
& + P(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
=& x_1 G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\beta} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)
\end{aligned}$$

and the lemma follows with $P = P_1 - y_k P_2$.

(c) If $a = 0$ and $b \geq 1$,

i. if $\beta_{11} \geq 1$, by induction and (2.1), we have

$$\begin{aligned}
G_{\tilde{\alpha}} =& G_{\binom{0}{b}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{b}\tilde{\mu}\beta} - x_k G_{\binom{0}{b}\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\beta} + P_1(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
& - x_k \Big(y_1 G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \\
& + P_2(x_2, x_3, \ldots, y_2, y_3, \ldots) \Big) \\
=& y_1 \left(G_{\binom{0}{b-1}\tilde{\mu}\beta} - x_k G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \right) \\
& + P(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\binom{0}{0}\beta} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)
\end{aligned}$$

and the lemma follows with $P = P_1 - x_k P_2$.

ii. if $\beta_{11} = 0$, by induction and (2.2), we have

$$\begin{aligned}
G_{\tilde{\alpha}} =& G_{\binom{0}{b}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{b}\tilde{\mu}\beta} - y_k G_{\binom{0}{b}\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\beta} + P_1(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
& - y_k \Big(y_1 G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \\
& + P_2(x_2, x_3, \ldots, y_2, y_3, \ldots) \Big) \\
=& y_1 \left(G_{\binom{0}{b-1}\tilde{\mu}\beta} - y_k G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)} \right) \\
& + P(x_2, x_3, \ldots, y_2, y_3, \ldots) \\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\binom{0}{0}\beta} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)
\end{aligned}$$

and the lemma follows with $P = P_1 - y_k P_2$.

$\square$

Corollary A.2.2. *Let $\tilde{\epsilon}$ be a generalized bicomposition, then the leading monomial of $G_{\tilde{\epsilon}}$ is $\mathbf{X}^{\tilde{\epsilon}}$. Hence, $\{G_{\tilde{\alpha}}\}$ forms a Hilbert basis for R.*

Proof. We prove by induction on $\ell(\tilde{\epsilon})$ and $|\tilde{\epsilon}|$. If $\tilde{\epsilon} = \binom{0}{0}$, by definition $G_{\tilde{\epsilon}} = 1 = X^{\tilde{\epsilon}}$. Otherwise, let $\tilde{\epsilon} = \binom{a}{b}\tilde{\beta}$.

1. If $a = b = 0$ and $\tilde{\beta}$ non-zero, by induction on $\ell(\tilde{\epsilon})$ and Lemma A.2.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM(G_{\tilde{\beta}}(x_2, x_3, \ldots, y_2, y_3, \ldots)) = (x_2, x_3, \ldots, y_2, y_3, \ldots)^{\tilde{\beta}} = \mathbf{X}^{\tilde{\epsilon}}.$$

2. If $a \geq 1$, by induction on $|\tilde{\epsilon}|$ and Lemma A.2.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM\left(x_1 G_{\binom{a-1}{b}\tilde{\beta}}\right) = \mathbf{X}^{\tilde{\epsilon}}.$$

3. If $a = 0$ and $b \geq 1$, by induction on $|\tilde{\epsilon}|$ and Lemma A.2.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM\left(y_1 G_{\binom{0}{b-1}\tilde{\beta}}\right) = \mathbf{X}^{\tilde{\epsilon}}.$$

$\square$

A.3 The Hilbert Basis

The set $\{x^{\tilde{\alpha}}F_\beta\}$ is a spanning set of the ideal I. For each $\tilde{\alpha}$ and β, we write $x^{\tilde{\alpha}}F_\beta$ in terms of the G basis by the following rules.

(1) We reorder the product $x^{\tilde{\alpha}}F_\beta$ as $\cdots\left(x_2^{\tilde{\alpha}_{21}}\left(y_2^{\tilde{\alpha}_{22}}\left(x_1^{\tilde{\alpha}_{11}}\left(y_1^{\tilde{\alpha}_{21}}F_\beta\right)\right)\right)\right)$.

(2) We reduce the above product recursively using (2.1)

$$x_iG_{\tilde{\gamma}} = x_iG_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i} & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} = G_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i}+1 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} \cdots & 0 & \tilde{\gamma}_{1i}+1 & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}; \tag{3.1}$$

or using (2.2) when $\tilde{\gamma}_{1i} = 0$ for some i,

$$y_iG_{\tilde{\gamma}} = y_iG_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i} & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} = G_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i} & \cdots \\ \cdots & \tilde{\gamma}_{2i}+1 & \cdots \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} \cdots & 0 & \tilde{\gamma}_{1i} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i}+1 & \cdots \end{smallmatrix}\right)}. \tag{3.2}$$

(3) When $\tilde{\gamma}_{1i} = a > 0$, we reduce $y_iG_{\tilde{\gamma}}$ as

$$\begin{aligned} y_1G_{\tilde{\gamma}} = y_iG_{\left(\begin{smallmatrix} \cdots & a & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} &= y_i\left(G_{\left(\begin{smallmatrix} \cdots & 0 & a & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} + x_iG_{\left(\begin{smallmatrix} \cdots & a-1 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right) \\ &= y_iG_{\left(\begin{smallmatrix} \cdots & 0 & a & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} + x_i\left(y_iG_{\left(\begin{smallmatrix} \cdots & a-1 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right) = \cdots \\ &= \sum_{k=0}^{a-1} x_i^k\left(y_iG_{\left(\begin{smallmatrix} \cdots & 0 & a-k & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right) + x_i^a\left(y_iG_{\left(\begin{smallmatrix} \cdots & 0 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right). \end{aligned} \tag{3.3}$$

The "$\cdots$" above means $\tilde{\gamma}_{11}\ \cdots\ \tilde{\gamma}_{1(i-1)}$, $\tilde{\gamma}_{1(i+1)}\ \cdots\ \tilde{\gamma}_{1\ell(\tilde{\gamma})}$, $\tilde{\gamma}_{21}\ \cdots\ \tilde{\gamma}_{2(i-1)}$ or $\tilde{\gamma}_{2(i+1)}\ \cdots\ \tilde{\gamma}_{1\ell(\tilde{\gamma})}$ with respect to their positions in the generalized bicomposition.

For example,

$$\begin{aligned}
y_1 F_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = y_1 \left(G_{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} + x_1 G_{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} \right) = y_1 G_{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} + x_1 y_1 G_{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} \\
= G_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)} + x_1 \left(G_{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} \right) \\
= G_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)} + G_{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} + G_{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)}.
\end{aligned}$$

For each of the above rule, we choose a leading basis element $G_{\tilde{\eta}}$ as follows. We define a function ϕ from $(\{x_i\} \times \{G_{\tilde{\gamma}}\}) \cup (\{y_i\} \times \{G_{\tilde{\gamma}}\})$ to $\{G_{\tilde{\gamma}}\}$.

Definition A.3.1. In the case of rules 3.1, 3.2, we choose $\phi(x_i, G_{\tilde{\gamma}}) = G_{\left(\begin{smallmatrix} \cdots & 0 & \tilde{\gamma}_{1i}+1 & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}$ and $\phi(y_i, G_{\tilde{\gamma}}) = G_{\left(\begin{smallmatrix} \cdots & 0 & \tilde{\gamma}_{1i} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i}+1 & \cdots \end{smallmatrix}\right)}$. In the case of rule 3.3, we choose $\phi(y_i, G_{\tilde{\gamma}}) = \phi\left(y_i, G_{\left(\begin{smallmatrix} \cdots & 0 & a & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right) = G_{\left(\begin{smallmatrix} \cdots & 0 & 0 & a & \cdots \\ \cdots & 0 & 1 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}$.

In the other words, at each step of the expansion, we choose the lexicographically smallest $\tilde{\eta}$ such that $G_{\tilde{\eta}}$ appears as a term in the expansion.

Lemma A.3.2. *The process of choosing in Definition A.3.1 is invertible, i.e. ϕ is injective.*

Proof. Since each time we multiply x_i or y_i, the chosen term contains a $\binom{0}{0}$ at position i. Combining this fact with the rule that we have to multiply y_i before x_i, we have the following inverse function.

Let i be the largest number that $(\tilde{\gamma}_{1i}, \tilde{\gamma}_{2i}) = (0,0)$ and $0 < i < \ell(\tilde{\gamma})$.

(1) If $\tilde{\gamma}_{1(i+1)} > 0$, then $\phi^{-1}\left(G_{\left(\begin{smallmatrix} \cdots & 0 & \tilde{\gamma}_{1(i+1)} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2(i+1)} & \cdots \end{smallmatrix}\right)}\right) = x_i G_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1(i+1)}-1 & \cdots \\ \cdots & \tilde{\gamma}_{2(i+1)} & \cdots \end{smallmatrix}\right)}$.

(2) If $\tilde{\gamma}_{1(i+1)} = 0$ and, $\tilde{\gamma}_{1(i+2)} = 0$ or $\tilde{\gamma}_{2(i+1)} > 1$, then

$$\phi^{-1}\left(G_{\left(\begin{smallmatrix}\cdots & 0 & \tilde{\gamma}_{1(i+1)} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2(i+1)} & \cdots\end{smallmatrix}\right)}\right) = y_i G_{\left(\begin{smallmatrix}\cdots & \tilde{\gamma}_{1(i+1)} & \cdots \\ \cdots & \tilde{\gamma}_{2(i+1)-1} & \cdots\end{smallmatrix}\right)}.$$

(3) If $\tilde{\gamma}_{1(i+1)} = 0$, $\tilde{\gamma}_{2(i+1)} = 1$ and $\tilde{\gamma}_{1(i+2)} > 0$, then

$$\phi^{-1}\left(G_{\left(\begin{smallmatrix}\cdots & 0 & 0 & \tilde{\gamma}_{1(i+2)} & \cdots \\ \cdots & 0 & 1 & \tilde{\gamma}_{2(i+2)} & \cdots\end{smallmatrix}\right)}\right) = y_i G_{\left(\begin{smallmatrix}\cdots & \tilde{\gamma}_{1(i+2)} & \cdots \\ \cdots & \tilde{\gamma}_{2(i+2)} & \cdots\end{smallmatrix}\right)}.$$ □

Then, we can construct a map $\Phi : \{X^{\tilde{\alpha}}F_\beta : |\beta| \geq 1\} \to \{G_{\tilde{\gamma}}\}$ that is defined by "composing" ϕ with itself $(|\tilde{\alpha}| - 1)$ times. By the above Lemma, we also have Φ is injective. For simplicity, we define $\phi^{-1}(G_{\tilde{\gamma}})$ (or $\Phi^{-1}(G_{\tilde{\gamma}})$) to be $X^{\tilde{\alpha}}G_{\tilde{\beta}}$ (or $X^{\tilde{\alpha}}F_\beta$) if $\phi(X^{\tilde{\alpha}}G_{\tilde{\beta}}) = G_{\tilde{\gamma}}$ (or $\Phi(X^{\tilde{\alpha}}F_\beta) = G_{\tilde{\gamma}}$ respectively).

Lemma A.3.3. *In the expansion of $X^{\tilde{\alpha}}F_\beta$ in the G basis using the rules 3.1, 3.2 and 3.3, the term $\Phi(X^{\tilde{\alpha}}F_\beta)$ appears only once. In particular, it has coefficients 1 or -1.*

Proof. We begin with the claim that if $\tilde{\mu} \neq \tilde{\nu}$, then $\phi(x_iG_{\tilde{\mu}})$ and $\phi(y_iG_{\tilde{\mu}})$ do not appear in the expansion of $x_iG_{\tilde{\nu}}$ and $y_iG_{\tilde{\nu}}$ respectively.

Let k be the smallest integer such that $(\tilde{\mu}_{k1}, \tilde{\mu}_{k2}) \neq (\tilde{\nu}_{k1}, \tilde{\nu}_{k2})$. In rules (4.1), (4.2) and (4.3), for all $G_{\tilde{\gamma}}$ in the expansion of $x_iG_{\tilde{\mu}}$ or $y_iG_{\tilde{\mu}}$, the first $i-1$ columns of $\tilde{\gamma}$ is the same as that of $\tilde{\mu}$. Hence, the claim follows if $k < i$.

If $k = i$, and if we are multiplying x_i using rules (4.1) or (4.2), then the claim holds because either the i-th or the $i+1$-th columns of $x_iG_{\tilde{\mu}}$ will be different from terms in expansions of $x_iG_{\tilde{\nu}}$. If we are multiplying by y_i, then note that if the $i-th$ column of μ is $(0,0)$, then $\mu_{(i+1)1}$ must be 0 because otherwise, that means we multiplied an x_i or x_j or y_j with $j > i$ before y_i, which violates our rule. And the same condition applies to ν. With this restriction, it is easy to check that the claim holds.

If $k > i$, in both cases, if we choose any term in the expansion that is not $\phi(x_i G_{\tilde{\nu}})$ or $\phi(y_i G_{\tilde{\nu}})$, then the i or $i+1$ column of its index must be different from that of $\phi(x_i G_{\tilde{\mu}})$ or $\phi(y_i G_{\tilde{\mu}})$. If we choose $\phi(x_i G_{\tilde{\nu}})$ or $\phi(y_i G_{\tilde{\nu}})$, we also have $\phi(x_i G_{\tilde{\mu}}) \neq \phi(x_i G_{\tilde{\nu}})$ and $\phi(y_i G_{\tilde{\mu}})\phi(y_i G_{\tilde{\nu}})$ because $\mu \neq \nu$.

Since each term in the expansion of $X^{\tilde{\alpha}} F_\beta$ corresponds to a sequence of choice using rules (4.1), (4.2) or (43), if at some point, we choose a term that is different from the choice in Φ, then a recursive use of the claim asserts that $\Phi(X^{\tilde{\alpha}} F_\beta)$ will not appear again. $\square$

We now define an order ($<_G$) on the set of generalized bicompositions as follows

1. If $\tilde{\alpha}$ and $\tilde{\beta}$ are bicompositions, then $\tilde{\alpha} <_G \tilde{\beta}$ if $\tilde{\alpha} <_{lex} \tilde{\beta}$.
2. If $\tilde{\alpha}$ is a bicomposition and $\tilde{\beta}$ is not, then $\tilde{\alpha} <_G \tilde{\beta}$.
3. If $\tilde{\alpha} = \tilde{\mu}\binom{0}{0}\alpha'$, $\tilde{\beta} = \tilde{\nu}\binom{0}{0}\beta'$ where α' and β' are bicompositions, let $u = \ell(\tilde{\alpha}) - \ell(\alpha') - 1$, $v = \ell(\tilde{\alpha}) - \ell(\beta') - 1$, then $\tilde{\alpha} <_G \tilde{\beta}$ if
   (a) $u < v$, or
   (b) $u = v$, $\alpha'_{11} > 0$ and $\beta'_{11} = 0$, or
   (c) $u = v$, $\alpha'_{11} > 0$, $\beta'_{11} > 0$ (or $\alpha'_{11} = 0$, $\beta'_{11} = 0$) and $\overleftarrow{\phi}(G_{\tilde{\alpha}}) <_G \overleftarrow{\phi}(G_{\tilde{\beta}})$ where we define $\overleftarrow{\phi}(G_{\tilde{\delta}})$ to be $\tilde{\gamma}$ if $\phi(x_i G_{\tilde{\gamma}}) = G_{\tilde{\delta}}$ or $\phi(y_i G_{\tilde{\gamma}}) = G_{\tilde{\delta}}$ for some i.

Lemma A.3.4. *The order $<_G$ is a total order on the set of generalized bicompositions such that if $G_{\tilde{\gamma}} = \Phi(X^{\tilde{\alpha}} F_\beta)$, then for all $G_{\tilde{\delta}}$ that appears in the expansion of $X^{\tilde{\alpha}} F_\beta$, we have $\tilde{\gamma} \geq_G \tilde{\delta}$.*

Proof. Clearly this is a total order. If $\tilde{\alpha} < \tilde{\beta}$ by (1) or (2), then $\tilde{\beta}$ cannot appear in the expansion of $\Phi^{-1}(\tilde{\alpha}) = \tilde{\alpha}$.

If $\tilde{\alpha} < \tilde{\beta}$ by (3a), that means $\phi^{-1}(\tilde{\alpha}) = x_{u+1}G_{\tilde{\gamma}}$ or $y_{u+1}G_{\tilde{\gamma}}$ for some $\tilde{\gamma}$. Hence, $\tilde{\beta}$ cannot appear in the expansion of $\Phi^{-1}(\tilde{\alpha})$ because $\tilde{\beta}_{(v+1)1} = \tilde{\beta}_{(v+1)2} = 0$ cannot be created.

If $\tilde{\alpha} < \tilde{\beta}$ by (3b), that means $\phi^{-1}(\tilde{\alpha}) = x_{u+1}G_{\tilde{\gamma}}$ for some $\tilde{\gamma}$. Hence, $\tilde{\beta}$ cannot appear in the expansion of $\Phi^{-1}(\tilde{\alpha})$ because it is not in that of $x_{u+1}G_{\tilde{\delta}}$ for any $\tilde{\delta}$. □

With this ordering, there is a unique leading $G_{\tilde{\delta}}$ for each expansion of $X^{\tilde{\alpha}}F_{\beta}$.

Theorem A.3.5. *The set* $A = \{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi)\}$ *forms a Hilbert basis for the quotient space* R/I.

Proof. For any polynomial p in R, we write p in terms of the G basis with $<_G$ order. For each term $G_{\tilde{\alpha}} \in Img(\Phi)$, we subtract p by $\Phi^{-1}(G_{\tilde{\alpha}}) \in I$ and $G_{\tilde{\alpha}}$ is cancelled. If we repeat this process (possibly countably many times), we can express p as a series of A. □

A.4 Finitely many variables case

In the case that there are finitely many variables, $R_n = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, the above constructions of $\mathsf{DQSym}(x_1, \ldots, x_n, y_1, \ldots, y_n)$, the F, G bases and the ideal $I_n =< \mathsf{DQSym}^+(x_1, \ldots, x_n, y_1, \ldots, y_n) >$ remain the same by taking

$x_i = y_i = 0$ for $i > n$. In this case, $LM(G_{\tilde{\alpha}}) = X^{\tilde{\alpha}}$ whenever $\ell(\tilde{\alpha}) \leq n$ and hence $\{G_{\tilde{\alpha}} : \ell(\tilde{\alpha}) \leq n\}$ spans R_n.

Let $R_n^{i,j}$ be the span of $\{X^{\tilde{\alpha}} : \ell(\tilde{\alpha}) \leq n, \sum_k \tilde{\alpha}_{1k} = i, \sum_k \tilde{\alpha}_{2k} = j\}$. Since I_n is bihomogeneous in $\mathbf{x}$ and $\mathbf{y}$, $I_n = \bigoplus_{i,j} I_n^{i,j}$ where $I_n^{i,j} = I_n \cap R_n^{i,j}$, and $R_n/I_n = \bigoplus_{i,j} V_n^{i,j}$ where $V_n^{i,j} = R_n/I_n \cap R_n^{i,j}$.

The Hilbert matrix corresponding to R_n/I_n is the matrix $M_n(i,j) = \dim(V_n^{i-1,j-1})$.

The goal of this section is to compute the second column of the Hilbert matrix. The proof is slight generalization of the one in [3].

Lemma A.4.1. *The set* $\{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi), \ell(\tilde{\alpha}) \leq n\}$ *spans the quotient* R_n/I_n.

Proof. Among all $\tilde{\alpha}$ such that $G_{\tilde{\alpha}} \in Img(\Phi)$, $\ell(\tilde{\alpha}) \leq n$ and $G_{\tilde{\alpha}}$ cannot be reduced to 0, let $\tilde{\beta}$ be the smallest one with respect to the $<_G$ order. Then,

$$\begin{aligned} G_{\tilde{\beta}} &= G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}}) + \Phi^{-1}(G_{\tilde{\beta}}) \\ &\equiv G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}}) \mod I_n \end{aligned}$$

But since $G_{\tilde{\beta}}$ is the leading term in $\Phi^{-1}(G_{\tilde{\beta}})$, terms in $G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}})$ are strictly smaller than $G_{\tilde{\beta}}$, and thus they reduce to 0. This contradicts to our assumption on $\tilde{\beta}$. □

Let B_n be the set of generalized bicompositions $\{\tilde{\alpha}\}$ such that $\sum_{i=1}^{k}(\tilde{\alpha}_{1i} + \tilde{\alpha}_{2i}) < k$ for all $1 \leq k \leq n$ and $\ell(\tilde{\alpha}) \leq n$. Clearly from the definition of G basis, if $\tilde{\alpha} \notin B_n$, then $G_{\tilde{\alpha}} \in I_n$. Therefore, the set $\{X^{\tilde{\alpha}} : \tilde{\alpha} \in B_n\}$ spans R_n/I_n, the proof is essentially the same as Lemma A.4.1. In particular, $X^{\tilde{\alpha}} \in I_n$ for all $|\tilde{\alpha}| \geq n$.

Lemma A.4.2. *The set* $\{X^{\tilde{\alpha}} F_\beta : \tilde{\alpha} \in B_n, |\beta| \geq 0\}$ *spans* R_n.

Proof. We already have $X^{\tilde{\epsilon}} \equiv \sum_{\tilde{\alpha} \in B_n} X^{\tilde{\alpha}} \mod I_n$, which means $X^{\tilde{\epsilon}} = \sum_{\tilde{\alpha} \in B_n} X^{\tilde{\alpha}} + \sum_{|\beta| \geq 1} P_\beta F_\beta$ for some polynomial P_β. If we reduce each monomial P_β using the above rule, and write the product of F basis in terms of F basis, the claim will be satisfied in a finite number of steps. $\square$

For a generalized bicomposition $\tilde{\alpha}$ with $\ell(\tilde{\alpha}) \leq n$, we define its reverse $\overline{\alpha}$ to be the generalized bicomposition such that $\overline{\alpha}_{1i} = \tilde{\alpha}_{1(n-i+1)}$ and $\overline{\alpha}_{2i} = \tilde{\alpha}_{2(n-i+1)}$ for all $1 \leq i \leq n$.

We denote the set $\{X^{\tilde{\alpha}} : \overline{\alpha} \in B_n\}$ by A_n. The endomorphism of R_n that sends x_i to x_{n-i+1} and y_i to y_{n-i+1} is clearly an algebra isomorphism that fixes $\mathsf{DQSym}(\mathbf{x}, \mathbf{y})$, in fact, it sends M_α to $M_{\alpha'}$ where α' is the reversed bicomposition of α. Therefore, by Lemma A.4.2, the set $\{X^{\tilde{\alpha}} F_\beta : \tilde{\alpha} \in A_n, |\beta| \geq 0\}$ spans R_n.

Hence, $I_n = \langle F_\gamma : |\gamma| \geq 1 \rangle$ is spanned by $\{X^{\tilde{\alpha}} F_\beta F_\gamma : \tilde{\alpha} \in A_n, |\beta| \geq 0, |\gamma| \geq 1\}$, which means it is spanned by $\{X^{\tilde{\alpha}} F_\beta : \tilde{\alpha} \in A_n, |\beta| \geq 1\}$.

Lemma A.4.3. *For* $X^{\tilde{\alpha}} F_\beta \in R_n^{i,1}$ *with* $\tilde{\alpha} \in A_n$, $|\beta| \geq 1$ *and* $|\tilde{\alpha}| + |\beta| < n$, *let* $G_{\tilde{\gamma}} = \Phi(X^{\tilde{\alpha}} F_\beta)$, *then* $\ell(\tilde{\gamma}) \leq n$.

Proof. First, rules (4.1) and (4.2) increase the length by 1 while (4.3) increase the length by 2. Now, we need to track $\tilde{\gamma}_{\ell(\tilde{\gamma})}$. If $\tilde{\gamma}_{\ell(\tilde{\gamma})}$ comes from $\beta_{\ell(\beta)}$ and gets shifted, since we can use (4.3) at most once, we can make at most $|\tilde{\alpha}|+1$ steps to the right. Therefore, $\ell(\tilde{\gamma}) \leq |\tilde{\alpha}|+1+\ell(\beta) \leq |\tilde{\alpha}|+1+|\beta| \leq n$.

If $\tilde{\gamma}_{\ell(\tilde{\gamma})}$ is 1 which comes from multiplying x_k or y_k to $G_{\tilde{\epsilon}}$ with $k > \ell(\tilde{\epsilon})$, since $\tilde{\alpha} \in A_n$, we have $\sum_{i \geq k}(\tilde{\alpha}_{1i} + \tilde{\alpha}_{2i}) < n-k+1$. In this process, we use rules (4.1) and (4.2) only and each increases the length by 1. Therefore, $\tilde{\gamma}_{\ell(\tilde{\gamma})}$ can be shifted to at most position $k+n-k=n$. □

Corollary A.4.4. *Let M_n be the Hilbert matrix of R_n/I_n, then $M_n(n-1,2) = \frac{1}{n}\binom{2n-2}{n-1}$, $M_n(i,2) = \sum_{1\leq j \leq i, 1 \leq k \leq 2} M_{n-1}(j,k)$ for $1 \leq i \leq n-2$, and $M_n(2,1) = 0$ for $i \geq n$.*

Proof. Lemma A.4.1 shows that $C_i = \{G_{\tilde{\alpha}} \in V_n^{i,1} : G_{\tilde{\alpha}} \notin Img(\Phi)\}$ spans $V_n^{i,1}$. Suppose there is a linear dependence $P = \sum_{G_{\tilde{\alpha}} \in C_i} a_{\tilde{\alpha}} G_{\tilde{\alpha}} \in I_n^{i,1}$. Since $I_n^{i,1}$ is spanned by $D = \{X^{\tilde{\alpha}} F_\beta \in R_n^{i,1} : \tilde{\alpha} \in A_n, |\beta| \geq 1\}$, we have $P = \sum_{X^{\tilde{\alpha}} F_\beta \in D} b_{\tilde{\alpha}\beta} X^{\tilde{\alpha}} F_\beta$. This means the leading term of P when we expand in G basis is some $G_{\tilde{\gamma}}$ such that $\tilde{\gamma} \in Img(\Phi)$ and by Lemma A.4.3 $\ell(\tilde{\gamma}) \leq n$, which is absurd. Therefore, C_i is a linear basis for $V_n^{i,1}$.

Now, $M_n(i,1) = \dim V_n^{i-1,1} = |C_{i-1}|$. Let $G_{\tilde{\gamma}} \in V_n^{i,1}$ and k be the unique number that $\tilde{\gamma}_{k2} = 1$. First, from definition of G, $\tilde{\gamma} \notin B_n$ implies $G_{\tilde{\gamma}} \in I_n$ and $G_{\tilde{\gamma}} \in Img(\Phi)$.

If $i = n-1$, then $|\tilde{\gamma}| = n-1$. If $k < \ell(\tilde{\gamma})$, since $\sum_{j=k+1}^{n} \tilde{\gamma}_{1j} \geq n-k$, we will be using rules (4.3) when applying ϕ^{-1}. This reduces the length by 2 while the size by 1, which means $G_{\tilde{\gamma}} \in Img(\Phi)$. If $k = \ell(\tilde{\gamma})$, we only use rules (4.1) and (4.2) when applying ϕ^{-1}. In this case, $G_{\tilde{\gamma}} \notin Img(\Phi)$ whenever $\tilde{\gamma} \in B_n$.

Therefore, $|C_{n-2}|$ is the Catalan number $\dfrac{1}{n}\dbinom{2n-2}{n-1}$.

If $1 \leq i \leq n-2$, $|\tilde{\gamma}| \leq n-2$. From the definition of ϕ, $G_{\tilde{\gamma}} \notin Img(\Phi)$ if and only if $G_{\begin{pmatrix} \tilde{\gamma}_{11} & \cdots & \tilde{\gamma}_{1(n-1)} \\ \tilde{\gamma}_{21} & \cdots & \tilde{\gamma}_{2(n-1)} \end{pmatrix}} \in V_{n-1}^{j,k} \setminus Img(\Phi)$ for some $1 \leq j \leq i, 1 \leq k \leq 2$. Therefore, $M_n(i,2) = \displaystyle\sum_{1\leq j\leq i, 1\leq k\leq 2} M_{n-1}(j,k)$ for $1 \leq i \leq n-2$. □

By the symmetry $M_n(a,b) = M_n(b,a)$, we obtain the first to rows of the Hilbert matrix, namely $M_n(2, n-1) = \dfrac{1}{n}\dbinom{2n-2}{n-1}$, $M_n(2,i) = \displaystyle\sum_{1\leq j\leq i, 1\leq k\leq 2} M_{n-1}(k,j)$ for $1 \leq i \leq n-2$, and $M_n(2,i) = 0$ for $i \geq n$.

This method can be applied directly to some other terms. To be more specific, for $2i + j \leq n$, the set $\{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi), \ell(\tilde{\alpha}) \leq n\}$ is a linear basis in $V_n^{i,j}$. Therefore, the formula for each column stabilizes when the number of variables is large enough. However, it fails in some other terms and this set is not a linear basis in general.

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