THE IMPACT OF STOCHASTIC INTEREST AND MORTALITY RATES ON RUIN PROBABILITY AND ANNUITIZATION DECISIONS FACED BY RETIREES

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Abstract

This dissertation focuses on two issues in retirement planning. The first issue, annuitization problem, provides insight on how interest rates may affect annuitization decisions for retirees under an all-or-nothing framework. The second issue, ruin probability, studies the probability for a retired individual who might run out of money, under a fixed consumption strategy before the end of his/her life under stochastic hazard rates. These two financial problems have been very important in personal finance for both retirees and financial advisors throughout the world, especially in the developed countries as the baby boom generation nears retirement. They are the direct results of both longevity risk and demise of Defined Benefit (DB) pension plans.

The existing literature of the annuitization problem, such as Richard (1975), concludes that it is always optimal to annuitize with no bequest motives under a constant interest rate. To see the effect of stochastic interest rates on the annuitization decisions under a constrained consumption strategy without bequest motives,

we present two life cycle models. They investigate the optimal annuitization strategy for a retired individual whose objective is to maximize his/her lifetime utility under a variety of institutional restrictions, in an all-or-nothing framework. The individual is required to annuitize all his/her wealth in a lump sum at some time at retirement. The first life cycle model we have presented assumes full consumption after annuity purchasing. A free boundary exists in this case upon the assumption of constant spread between the expected return of the risky asset and the riskless interest rate. The second life cycle model applies the optimal consumption strategy after annuitization, and numerical analysis shows that it is always optimal to annuitize no matter what the current interest rate is. This conclusion is based on the assumption of constant risk premium, no loads and no bequest motives.

Historical data show that mortality rates for human beings behave stochastically. Motivated by this, we study the ruin probability for a retired individual who withdraws \$1 per annum with various initial wealth for log-normal mortality with constant drift and volatility, which is a special form of the most widely accepted Lee-Carter model. This problem is converted to a Partial Differential Equation (PDE) and solved numerically by the Alternative Direction Implicit (ADI) method. For any given initial wealth, ruin probability can be obtained for various initial hazard rates. The correlation between the wealth process and the mortality process slightly affects the ruin probability at time zero.

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1 Introduction

1.1 Introduction and Motivation

This dissertation focuses on two issues in retirement planning. The first issue, annuitization problem, provides insight on how interest rates may affect annuitization decisions for retirees under an all-or-nothing framework. The second issue, ruin probability, studies the probability for a retired individual who might run out of money under a fixed consumption strategy before the end of their life under stochastic hazard rates. These two financial problems have been very important in personal finance for both retirees and financial advisors throughout the world, especially in the developed countries as the baby boom generation nears retirement. They are the direct results of both longevity risk and demise of Defined Benefit (DB) pension plans.

Longevity risk is the risk that an individual will outlive his/her retirement savings due to a longer life span. For example, if one's retirement consists of personal savings only, the possibility exists that the money will run out before one dies under a fixed consumption strategy. According to Berkeley human mortality database (http://www.mortality.org/hmd/), life expectancies at birth for Canadians increased by more than 20 years from 1929 to 2009 due to enhancements of diet, life style and medical care. By the time of late adulthood, one's chances of survival to a very old age are quite good. For example, although the life expectancy for those born in Canada in 1989 is 77.12 years, those who live to age 65 will have an average of almost 18 additional years left to live, making their life expectancy almost 83 years. The risk to retirees who do not adequately consider these life expectancy gains is potentially very expensive since they are in great danger of becoming financially ruined at retirement, especially in the current situation where the world economy is not doing well after the subprime crisis.

Traditional DB pension plans are becoming less popular worldwide. One of the factors that have contributed to this is the risks associated with long life expectancy. These risks finally translate to higher than expected pay-out-ratios for many pension funds and insurance companies, so more and more institutions are closing DB pension plans to avoid this risk. In 1998, 62.7% of individuals who participated in a retirement plan had a Defined Contribution (DC) plan as their primary plan, compared to 49.8% in 1993 (Copeland 2002). Therefore, more and more people assume all investment and longevity risk, which makes it a great challenge to manage their wealth after retirement.

Many retirees face a dilemma as to whether to choose annuitization from insurers who guarantee a lifelong payment stream, or self-annuitization offering a higher consumption rate by investing more assets in the equity market but with a risk that retirees may outlive the wealth from the self-managed assets. For instance, in Canada, RRSPs must be collapsed by December 31st of the year individuals turn age 71. One option is to cash out all their RRSPs, but it is obviously not the best alternative if the amount of accumulated income in the RRSP is significant because the tax payment is huge. The second option is to purchase a fixed term annuity or life annuity to provide for a steady stream of income over their life or their spouse depending on the plan. The third option is to establish a Registered Retirement Income Fund (RRIF) for which retirees will self-manage the funds while required to make an annual minimum withdrawal based on age. Therefore, most retirees need to make decisions between annuitization and self-annuitization, which is affected by many factors, such as, longevity, risk aversion and existing pensions.

It is well known that the advantage of self-annuitization is high liquidity, on the other hand, it has the risk of outliving one's wealth in case the investment return is below expectation. The risk in this case can be measured by the probability of running out of money before one dies, with current living standard maintained. This interesting problem, known as ruin probability, is our second research project. Some literature on this exists, such as Khorasanee (1996), Milevsky and Robinson

(2000), Albrecht and Maurer (2002), Young (2003) and Huang, Milevsky and Wang (2004). Through analyzing historical mortality data, we find that the mortality rate is better described by the stochastic model, rather than the Gompertz model. Motivated by this finding, we study the individual ruin probability under stochastic hazard rates in which the mortality rate is a state variable. Therefore, for any given initial mortality rate at time 0, which could be either greater than, equal to, or less than the GM mortality rate, we will compare the ruin probabilities under stochastic and GM mortality rates to look into the effect of stochastic mortality on the lifetime ruin.

Annuitization guarantees a certain living standard with its lifelong payment stream, but its obvious disadvantage is the illiquidity, which may not lead to a substantial bequest to survivors and the estate upon the death of the annuitant. In addition to bequest motives, the other factors that may affect the annuitization decision is the personal mortality rate and interest rates. In the real world, the risk-free interest rate is changing over time, which in turn affects the optimal annuitization time. In this dissertation, we study how stochastic interest rates, which are assumed to follow the Cox-Ingersoll-Ross (CIR) process, affect the optimal annuitization timing problem.

1.2 Contributions and Outline of the Dissertation

The contribution of the first project is to study the optimal annuitization time under the all-or-nothing framework when the individual consumes all of his/her annuity payment or consumes optimally, an extension of Milevsky and Young (2007). The contribution of our second project is to study the ruin probability under stochastic hazard rates under a fixed retirement consumption strategy, which is an extension of Huang, Milevsky and Wang (2004).

This dissertation is organized as follows. Chapter 2 studies the annuitization problem for a retired individual whose objective is to maximize his/her lifetime utility under stochastic interest rates by assuming that she will consume all the annuity income after annuitization in an all-or-nothing framework. When the subjective mortality rate is equal to the current interest rate, the results are consistent with previous works done by other researchers. We first study the optimal annuitization time with the exponential mortality rate for constant and stochastic interest rates. Then we move on to the same problem with the GM mortality rate, which is a free boundary problem, quite similar to the American option pricing problem.

In Chapter 3, which is an extension of Chapter 2, by realizing the fact that retirees may not consume all the annuity income, we study the optimal consumption rate by assuming that consumption is part of the annuity income and the remains are used to purchase new annuities. Then we apply the dynamical programming strategy to find the free boundary. Exponential mortality and GM mortality are investigated respectively and numerical results are given in each subsection.

Chapter 4 implements numerical PDE solution techniques to calculate the probability of lifetime ruin, which is the probability that a fixed retirement consumption strategy will lead to financial insolvency under stochastic investment returns and deterministic mortality rates. The ruin probability satisfies a backward Kolmogorove equation and can be solved by finite difference method. Secondly, we obtain the PDE that the ruin probability must solve under stochastic hazard rates. This PDE is two dimensional with cross derivatives. We have checked the consistence of the two PDEs under special conditions and carried out a convergence analysis to prove that our numerics are good, and then provide the numerical results in the end. We find that the ruin probability under stochastic hazard rate is always greater than the ruin probability under Gompertz mortality. The correlation between wealth and hazard rate affects the lifetime ruin for stochastic hazard rate.

Finally, Chapter 5 concludes this dissertation and identifies future research.

- **Remarks on Simulations and Software:** All simulations in this dissertation are performed on MATLAB version 6.5.1 with Lenovo's ThinkPad T43 using MATLAB programming.

2 Optimal Annuitization Timing with Constrained Consumption

2.1 Introduction

The existing literature of the annuitization problem, such as Yaari (1965) and Richard (1975), concludes that it is always optimal to annuitize with no bequest motives under a constant interest rate. In this chapter, we study the effect of stochastic interest rates on the annuitization decisions under a constrained consumption strategy without bequest motives. The various models proposed to describe the behavior of interest rates in literature are equilibrium models and no-arbitrage models. In this chapter, we use the CIR model, a one-factor no-arbitrage model of the short rate, since it has the advantage of avoiding the possibility of negative interest rates, as well as mean reversion and robustness, which can be used in conjunction with any set of initial zero rates to study the optimal annuitization problem after retirement. When the short-term interest rate falls below the long-term average, the short-term interest rate tends to increase towards the long run rate in the future. When the short-rate interest rate is above the long-term average, the short-term interest rate tends to fall towards the long run rate. Another advantage of the CIR model is that the present value of a bond's price can be computed through a neat exponential expression, which can be used to calculate the actuarial annuity factor, the present value of a life annuity that pays $1 per year continuously to an individual at the time of purchase. Although the interest rate models are mature in pricing options, futures and other derivatives, little work has been done on how this might affect the retired individuals with regards to their annuity purchasing decisions at retirement.

In an attempt to help fill this vacuum, we seek to present two life cycle models which investigate the optimal annuitization strategy for a retired individual whose objective is to maximize his/her lifetime utility under a variety of institutional restrictions without bequest motives in an all-or-nothing framework, where the individual is required to annuitize all his/her wealth in a lump sum at some time τ at retirement. We further explore the effect of stochastic interest rates on individual annuity purchasing. Motivated by previous works where researchers often assume full consumption of the annuity payment after annuitization, such as Milevsky and Young (2007) and Yaari (1965), we use the same assumption throughout the whole chapter, i.e., the individual will consume all his/her annuity income after annuitization, which echoes the fact for some retirees in reality. In financial markets, this individual is allowed to invest in a risky asset with constant volatility and a riskless asset whose interest rates obey the CIR process, and the expected equity returns are modeled to be a constant spread above the riskless interest rate, which is reasonable in the sense that the equity return should be always greater than the riskless interest rate, and so far we have not found any research studying this relationship. This assumption means that when the riskless interest rate goes up, the equity return also goes up and vice versa, and Merton's constant is fixed if the risky asset volatility and the individual's risk-aversion coefficient are constants.

In the two primary life cycle models addressed in this chapter, the consumer's preference is represented by the constant relative risk aversion (CRRA) utility function, whose homogeneity allows the value function to take a similar power form. We take advantage of this property from a technical point of view.

For the first model, we present a constant force of mortality to address the optimal annuitization problem with constant interest rates and stochastic interest rates. Given initial wealth w at time zero (retirement), we are looking to see if it is most favorable to annuitize, as well as the optimal annuitization time, if it is necessary upon optimal investment and consumption strategies. In general, the value function (the present discounted utility function from retirement to time of decease) associated with this optimal control problem is a function of time t, wealth w and interest rate r. When the interest rate is static, this value function is independent of time t for any given initial interest rate r (the interest rate will not change over time). Therefore, the value function depends only on the interest rate r, not time t, with the mortality rate λ as a parameter. One step further, for any given r, it will be favorable either to annuitize or never to annuitize.

The second model is actually an extension of the all-or-nothing framework of Milevsky and Young (2007) under GM mortality, in which we modify the constant interest rate by stochastic interest rates. The force of mortality is assumed to be invariable after the maximum age (120) of a human being. This is a plausible assumption because human beings rarely live past the maximum age, and their mortality rate is very high, which means that the effect of mortality after the maximum age is trivial to the value function. In this scenario, the value function can be proved to satisfy a second-order linear Hamilton-Jacob-Bellman (HJB) equation with cross derivatives after applying Ito's lemma and Bellman's principle of optimality. Our problem becomes a free boundary one which is quite similar to the American put option problem since at each time t, we need to determine not only the value function, but also, for each value of r, whether or not the individual need to annuitize. We then transfer this free boundary problem to an equivalent linear complementarity (LCP) problem which has the advantage that the free boundary does not interfere with the solution process, and it can be recovered from the solution after the latter has been found. Then the projected successive over-relaxation (SOR) method is applied to solve the LCP problem since it has the advantage of immediate replacement of the newest values of the unknown variable.

Some literature exists on the annutization decisions at retirement. In the seminal paper of Yaari (1965), he argues that an individual should always annuitize all his/her wealth in the absence of bequest motives, but in reality, the annuitization rates are very low, the so called 'annuity puzzle'. There have been a lot of papers which study 'annuity puzzle' problem such as Brown and Poterba (2000), and Brown and Warshawsky (2001), which documented that the low voluntarily annuitization rate is due to the high loads and fees embedded in annuity prices. Friedman and Warshawsky (1990) and Vidal-Melia and Lejarraga-Garcia (2006) concentrated specially on how bequest motives affect the demand for annuities, both showed that strong enough bequest motives can eliminate purchases of annuities with high enough loads. For more literature review about this topic, we refer the interested reader to the paper by Milevsky and Young (2007). In this chapter we focus on the optimal asset allocation associated with the optimal annuitization timing under two different types of interest rates.

The remainder of this chapter is organized as follows. General notation and basic assumptions coming from the research community are elaborated in Section 2.2. Then we document the annuitization problem under exponential mortality rate for constant and stochastic interest rates respectively in Section 2.3. Next we investigate the annuitization problem under GM mortality and stochastic interest rates in Section 2.4, which is a free boundary problem similar to the American option problem. We convert it to an equivalent LCP problem, and solve it by the projected SOR method, and then recover the free boundary from the solution of the value function. Finally, Section 2.5 concludes this chapter and gives directions for future research.

2.2 General Notation and Basic Assumptions

This section provides a primer on the notation and terminologies used later in the annuitization problem. It aims at providing a consistent nomenclature.

The survival probability for an individual aged x, alive at time t, who survives to a future time s ($s \geq t$), is given by

$$\left({}_{s-t}p_{x+t} \right) = e^{-\int_t^s \lambda_{x+v} dv}, \tag{2.1}$$

where λ_{x+v} stands for the instantaneous force of mortality at age $x + v$. In the case of exponential mortality, i.e., $\lambda_{x+v} = \lambda$, this survival probability simplifies to $e^{-\lambda(s-t)}$. In this chapter, we will study the annuitization problem under constant and variable mortality rate respectively, i.e., the force of mortality is constant and Gompertz.

We further assume that the individual can choose to invest his/her wealth W_v in a financial market composed by a risky asset (a portfolio of stocks with return dS_v) and a riskless asset (with return $R_v dv$), and consumes at a rate c_v, at time v. This riskless asset, X_v, evolves according to the following process

$$\begin{cases} dX_v &= R_v X_v dv, \\ X_t &= x_t, \end{cases} \tag{2.2}$$

where x_t is the amount of riskless asset at time t. Notation R_v is the instantaneous risk-free rate of interest at time v, which obeys the following CIR process (see Chapter 17, Hull (2005))

$$\begin{cases} dR_v &= \theta(\mu_r - R_v)dv + \sigma_r \sqrt{R_v} dB_v^r, \\ R_t &= r, \end{cases} \tag{2.3}$$

where B_v^r represents a standard Brownian motion, the superscript r means the instantaneous riskless asset and the subscript v means time, and θ, μ_r, σ_r are the parameters. θ is the speed of adjustment, μ_r is the long run interest rate and σ_r is the volatility. This dynamic interest rate model was introduced by Cox, Ingersoll and Ross (1985) and has been applied widely in financial economics. For given positive initial interest rate r, R_v will never touch zero if $2\theta\mu_r \geq \sigma_r^2$ holds, otherwise, it will occasionally touch zero. For detailed parameter estimates, we refer the interested reader to Chan, Karolyi, Longstaff and Sanders (1992). Another advantage of the CIR process is that it is mean reverting. When the short-term interest rate falls below the long-term average μ_r, the short-term interest rate tends to increase towards μ_r in the future. When the short-term interest rate is above the long-term average, the short-term interest rate tends to fall towards the long-term average in the future.

As in Black and Scholes (1973) and Merton (1971), the risky asset S_v evolves according to a geometric Brownian motion (GBM)

$$\begin{cases} dS_v &= \mu_s S_v dv + \sigma_s S_v dB_v^s, \\ S_t &= s, \end{cases} \tag{2.4}$$

where B_v^s represents a standard Brownian motion, the subscript v means time and the superscript s means stocks (risky asset). Parameter σ_s is the diffusion term of the risky asset, its typical values fall in the range of $(5\%, 50\%)$. μ_s is the drift term, which is modeled to be stochastic, i.e., $\mu_s(v) = R_v + \delta_1$. This implies that $\mu_s(v)$ is modeled to be a constant spread above the riskless interest rate, which is reasonable in the sense that the equity return should be always greater than the riskless interest rate. This assumption means that when the interest rate goes up, the expected return of the risky asset goes up as well and vice versa. In this chapter, δ_1 is taken to be a constant 0.03. The correlation between dB_v^r and dB_v^s is denoted by ρ_{rs} (a constant), which is independent of time v and ranges from -1 to $+1$. A correlation of $+1$ means a perfect positive correlation, indicating the two variables moving in the same direction together. A zero correlation means that there is no relationship between the two variables. Since changes in wealth are equal to the return from the riskless and risky assets minus the consumption, we obtain the wealth dynamics as

$$\begin{cases} dW_v &= [R_v W_v - c_v + (\mu_s - R_v)\pi_v]dv + \sigma_s \pi_v dB_v^s, \\ W_t &= w, \end{cases} \tag{2.5}$$

where π_v is the amount invested in the risky asset. Note that this variable can be negative, meaning that the individual has shorted the risky asset and invested in the riskless asset.

We also assume that the individual can annuitize all his/her wealth at a time $\tau \geq t$ (if applicable) and obtain an actually fair amount, determined by the objective actuarial annuity factor

$$\bar{a}_{x+\tau}(\tau, R_\tau) = E\left[\int_\tau^\infty e^{-\int_\tau^s R_v dv}({}_{s-\tau}p_{x+\tau})ds | R_\tau = r\right] \tag{2.6}$$

$$= \int_\tau^\infty E_\tau\left[e^{-\int_\tau^s R_v dv} | R_\tau = r\right]({}_{s-\tau}p_{x+\tau})ds \tag{2.7}$$

$$= \int_\tau^\infty P_B(\tau, s, R_\tau)({}_{s-\tau}p_{x+\tau})ds, \tag{2.8}$$

where $P_B(\tau, s, R_\tau)$ describes the price of the zero-coupon bond at time τ with time to maturity s. We have also assumed independence between the bond price and the survival probability so that the expectation can be taken inside the integral directly to the discounted interest rate. We finally assume that the individual will consume the annuity income after annuitizing his/her wealth as Milevsky and

Young (2007) in their all-or-nothing framework. Note that $\bar{a}_{x+\tau}(\tau, R_\tau)$ is a random variable depending on time τ and the corresponding interest rate at that time.

The concave utility function of consumption we are interested in exhibits constant relative risk aversion (CRRA). In specification, it follows

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma}, & \gamma \neq 1, \\ \ln(c), & \gamma = 1, \end{cases} \tag{2.9}$$

in which γ represents the relative risk aversion coefficient and $\frac{1}{\gamma}$ measures the elasticity of substitution between consumption at two points in time. In this dissertation, we only consider the cases when γ is greater than or equal to 1, because low levels of γ imply high leverage ratios which is not allowed at retirement. In fact, for $\gamma \neq 1$, we will use $\frac{c^{1-\gamma}}{1-\gamma}$ for simplicity as it does not affect the optimal solution.

Now that we have finished introducing all the notation and terminologies we are going to use in this chapter, so we are ready to move on to our model calibration part for exponential mortality rate next.

2.3 Model Calibration 1: Exponential Mortality

In this section, we study the annuitization problem for a retired individual whose objective is to maximize his/her lifetime utility under exponential mortality and a variety of institutional restrictions without bequest motives in an all-or-nothing framework. This individual only has initial wealth in the form of a lump sum cash amount (such as an RRSP account in Canada), does not come pre-annuitized with a pre-existing pension or social security and has no remaining lifetime income. In general, the value function associated with this optimal control problem is a function of time t, wealth w and interest rate r, but it is independent of time t under exponential mortality for any given initial interest rate r. Therefore the value function should depend only on r, not time t, with the mortality rate λ as a parameter, which is proved later in this section. This implies that, for any given interest rate r, it will be favorable either to annuitize or never to annuitize at retirement. Based on this observation, we will look at the value functions with and without annuity purchasing at time t, V^a and V^n, for two different interest rates models: constant and stochastic. Then we compare the two value functions to draw the conclusion as to whether it is optimal to annuitize or not for any given interest rate.

Firstly, we investigate the case when the interest rate is constant. In this scenario, the analytical solutions for V^a are obtained under the assumption of full consumption of the annuity income after annuitization. For V^n, the HJB equation that it must satisfy is derived using dynamic programming techniques. Then its analytical solution is acquired in a similar power form as the CRRA utility function. Finally, we compare the two analytical value functions we have obtained and conclude that when the interest rate r is equal to the subjective discount factor ρ, it is optimal to annuitize when the force of mortality is greater than Merton's constant, which is consistent with the results obtained by Milevsky and Young (2007). When the interest rate r is not equal to the subjective discount rate, it is optimal to annuitize when the interest rate is small and it is optimal not to annuitize when the interest rate is large.

Secondly, we describe the optimal control problem with stochastic interest rate. In this case, the analytical solutions for V^a are obtained through the zero-coupon bond price which is derived from the CIR process. For V^n, it satisfies a second-order nonlinear HJB equation which can be derived by applying dynamic programming techniques. At last, we compare these two value functions and find that the results are consistent with what we have obtained under a constant interest rate.

The rest of the section goes into detail about the annuitization problem corresponding to constant and stochastic interest rates respectively. Section 2.3.1 works on the constant interest rate case, and analytic solutions for V^a and V^n are derived. Through comparison of these two functions, we draw our conclusion as to when it is optimal to annuitize. Then the annuitization problem under a more complicated stochastic interest rate situation is considered in Section 2.3.2. Here, one important observation is that the two value functions, V^a and V^n, are both time-independent if a power term $e^{-(\rho+\lambda)t}$ is excluded. Then a comparison is performed between them to find the annuitization boundary, which is an increasing function of the interest rate r and consistent with the constant interest rate case.

2.3.1 Constant Interest Rate

In this section, we study the annuitization problem starting from the most simple short interest rate case in which the return of the riskless asset is fixed all the time. It is known that in this case, the associated value function is independent of time t, and it is a function of wealth w and interest rate r. Therefore we only need to address the optimal control problem at time 0 (age x) to obtain the optimal annuitization strategies without loss of generosity. The homogeneity property of the CRRA utility function allows the value function to take a similar power form, i.e., the wealth w can be factored out. So the value function becomes invariant to the scale of wealth, i.e., the level of wealth does not matter in this specification of utility. Therefore, the only thing that matters is the interest rate. Therefore, for any given interest rate r, it will be favorable either to annuitize or never to annuitize, i.e., if it is optimal not to annuitize at time zero, then it will never be in the future. Hence, it is sufficient to study the two value functions, with annuitization (V^a) and without annuitization (V^n), at time zero, and then compare them to see whether it is optimal to annuitize when the risks faced by the individual includes the longevity risk and the return risk. Next we will illustrate this in much detail.

2.3.1.1 The Value Function with Annuitization

When the force of mortality is assumed to be constant λ, the value function is defined as

$$V(w) = \sup_{c_s} E\left[\int_0^\infty e^{-(\rho+\lambda)s} u(c_s)ds | W_0 = w\right], \tag{2.10}$$

in which c_s is the optimal consumption rate at time s, and E denotes the expectation conditional on $W_0 = w$, and u is an increasing concave utility function of consumption introduced back in section 2.2. Notation ρ is the subjective discount rate which is personal and independent of the economic models for the risky asset and the risk-free asset in the financial market. This parameter is subjective by its own nature despite the fact that people prefer to consume more now rather than more later. Next we will study this value function according to whether the individual annuitizes his/her wealth or not at time zero, denoted by V^a and V^n respectively.

If the individual annuitizes at time zero, the value function of the control problem can be written as

$$\begin{aligned} V^a(w) &= \sup_{c_s} E\left[\int_0^\infty e^{-(\rho+\lambda)s} u(c_s)ds\right] && (2.11) \\ &= \int_0^\infty e^{-(\rho+\lambda)s} u\left(\frac{w}{\bar{a}_x(0,r)}\right)ds. && (2.12) \end{aligned}$$

We have assumed that the individual consumes exactly the annuity payout after annuitization, which is equivalent to $\frac{w}{\bar{a}_x(0,r)}$ at time 0 and thereafter. The annuity factor $\bar{a}_x(0,r)$, the present value of a life annuity that pays \$1 per year continuously to the retiree who is age x at the time of purchase, is computed by

$$\bar{a}_x(0,r) = \int_0^\infty e^{-rs} e^{-\lambda s} ds = \frac{1}{r+\lambda}. \tag{2.13}$$

After plugging this expression and the utility function (2.9) into equation (2.12), the closed-form expression for $V^a(w)$ can be obtained as

$$V^a(w) = u(\frac{w}{\bar{a}_x(0,r)})\bar{a}_x(0,\rho), \quad \gamma \geq 1. \tag{2.14}$$

2.3.1.2 The Value Function without Annuitization

In this subsection, we assume that the individual does not annuitize at time zero, and investigate the value function V^n when both riskless and risky assets are available to invest by applying dynamic programming techniques. In details, we will apply Bellman's optimality principle and Ito's lemma to obtain the HJB equation that the value function must solve. This HJB equation is then solved analytically by making a proper transformation stimulated by the special form of the CRRA utility function.

The expected discounted utility of consumption in this case is defined by

$$V^n(w) = \sup_{c_t,\pi_t} E\left[\int_0^\infty e^{-(\rho+\lambda)t} u(c_t)dt | W_0 = w\right], \tag{2.15}$$

with the following budget constraint

$$\begin{cases} dW_v &= [rW_v - c_v + (\mu_s - r)\pi_v]dv + \sigma_s \pi_v dB_v^s, \\ W_t &= w. \end{cases} \tag{2.16}$$

Note that borrowing is allowed in this circumstance. To apply the dynamic programming techniques, we denote $V(t,w)$ the value function starting in state w at time t and controlling the system optimally from then until time ∞ and divide the value function $V(t,w)$ into two sub-integrals. Specifically,

$$\begin{aligned} V(t,w) &= \sup_{c_s,\pi_s} E\left[\int_t^\infty e^{-(\rho+\lambda)s}u(c_s)ds|w_t = w\right] \\ &= \sup_{c_s,\pi_s}\left[\int_t^{t+dt} e^{-(\rho+\lambda)s}u(c_s)ds + V(w_{t+dt}, t+dt)\right] \\ &= \sup_{c_s,\pi_s}\left[\int_t^{t+dt} e^{-(\rho+\lambda)s}u(c_s)ds + V(w_t, t) + dV\right]. \end{aligned} \tag{2.17}$$

It can be easily observed that $V^n(w) = V(0,w)$. Then Bellman's optimality principle and Ito's lemma are applied to obtain the following HJB equation for $V(t,w)$, see Bjork (2004, Chapter 14).

$$V_t + \sup_{c_t,\pi_t}\{e^{-(\rho+\lambda)t}u(c_t) - c_t V_w + (rw + (\mu_s - r)\pi_t)V_w + \frac{1}{2}\sigma_s^2\pi_t^2 V_{ww}\} = 0, \tag{2.18}$$

subject to the terminal condition $V(\infty, w) = 0$. Let $V(t,w) = \frac{w^{1-\gamma}}{1-\gamma}h(t)$ $(\gamma \neq 1)$, then the optimal consumption $\tilde{c}_t$ and investment $\tilde{\pi}_t$ can be obtained from the first order necessary conditions

$$\begin{cases} \tilde{c}_t = e^{-\frac{(\rho+\lambda)t}{\gamma}}\tilde{w}h^{-\frac{1}{\gamma}}, \\ \tilde{\pi}_t = \frac{\mu_s - r}{\sigma_s^2\gamma}\tilde{w}, \end{cases} \tag{2.19}$$

where $\tilde{w}$ is the optimally controlled wealth. Substituting the two admissible controls into the HJB equation (2.18), we obtain the following PDE that $h(t)$ must satisfy

$$h_t + (1-\gamma)\eta h + \gamma e^{-\frac{(\rho+\lambda)t}{\gamma}} h^{1-\frac{1}{\gamma}} = 0, \tag{2.20}$$

subject to the terminal condition $h(\infty) = 0$. The notation η is the sum of the current interest rate r and $(\mu_s - r)^2/\sigma_s^2$ scaled by double γ, i.e., $\eta = r + \frac{(\mu_s - r)^2}{2\gamma\sigma_s^2}$. This Bernoulli ordinary differential equation (ODE) can be solved by making a transformation $h = y^\gamma$. After some mathematical manipulation, the analytic solution for $h(t)$ (when $\rho + \lambda + (\gamma - 1)\eta > 0$) is

$$h(t) = (\frac{e^{-(\rho+\lambda+k-k\gamma)t} - e^{-(\rho+\lambda+k-k\gamma)T}}{\rho+\lambda+k-k\gamma})^\gamma e^{(r\gamma - r)t}. \tag{2.21}$$

So that

$$\begin{aligned} V^n(w) &= V(w(0), 0) && (2.22) \\ &= \frac{w^{1-\gamma}}{1-\gamma}(\frac{1-e^{-(\rho+\lambda+k-k\gamma)T}}{\rho+\lambda+k-k\gamma})^\gamma. && (2.23) \end{aligned}$$

Similarly we can obtain the expression for V^n when the utility function takes the form of the natural logarithm

$$V^n(w) = \frac{\ln(w) + \ln(\rho+\lambda) + \frac{\eta}{\rho+\lambda} - 1}{\rho+\lambda}, \ \gamma = 1. \tag{2.24}$$

2.3.1.3 Optimal Annuitization Strategy

In this subsection, we discuss whether it is optimal to annuitize at retirement by comparing the two value functions V^a and V^n which are time independent to give the annuitization boundaries for γ greater than 1. The reason why we are interested in these γ values is historical, such as Feldstein and Ranguelova (2001) documented that the risk aversion constant is less than 3, while Campbell and Viceira (2002) suggested that risk aversion levels may be higher. The constant spread between the expected equity returns and the risk-free interest rates are set to be 0.03, which is reasonable in the circumstance of our current low interest rates. The equity volatility is taken to be 0.2, which is roughly in line with Ibbotson Associates (2001). From Section 2.3.1.1 and 2.3.1.2, we have obtained the analytic solutions for V^a and V^n, hence we have

$$V^a - V^n \propto \begin{cases} (\rho+\lambda)(r+\lambda)^{\gamma-1} - (\frac{\rho+\lambda+(\gamma-1)\eta}{\gamma})^{\gamma}, & \gamma > 1, \\ \frac{\rho+\lambda-\eta}{\rho+\lambda} - \ln\frac{\rho+\lambda}{r+\lambda}, & \gamma = 1. \end{cases} \tag{2.25}$$

When the interest rate r is equal to the subjective discount rate ρ, the condition for $V^a \geq V^n$ is simplified to

$$\lambda \geq \frac{(\mu_s - r)^2}{2\gamma\sigma_s^2}, \gamma \geq 1. \tag{2.26}$$

This means that it is optimal to annuitize today when the force of mortality is greater than Merton's constant, which is consistent with the results obtained by Milevsky and Young (2007). Since we have assumed constant spread between the expected return of the risky asset and the riskless interest rate, this means that it is optimal to annuitize for any interest rate when the hazard rate is greater than Merton's constant. In contrast, it is always optimal not to annuitize when the hazard rate is less than Merton's constant. Therefore, the size of the force of mortality decides if the individual need to annuitize at retirement.

To see when it is optimal to annuitize when the interest rate r is not equal to the subjective discount rate ρ, we solve the equation $V^a = V^n$ to obtain the annuitization boundary that the interest rate must satisfy. The conditions for $V^a \geq V^n$ are summarized in Table 2.1 for $\gamma = 2$ and Table 2.2 for $\gamma = 3$ respectively. Merton's constant ($\frac{(\mu_s - r)^2}{2\gamma\sigma_s^2}$) corresponding to $\gamma = 2$ and $\gamma = 3$, give 0.005625 and 0.00375 for fixed values of σ_s (0.2) and $\mu_s - r$ (0.03), which means that all the force of mortality values in both tables are greater than these two constants. Note that the maximum interest rate we are going to consider is 0.4000, which is never reached in reality in developed countries where people pay attention to retirement planning after retirement. Two different subjective discount factors $(0, 0.02)$ are investigated in the two tables so that we can observe the annuitization boundaries more consistently.

From Table 2.1 and Table 2.2, we can see that the annuitization boundary is an increasing function of the mortality rate when λ is small. Note that we have taken the subjective discount rate to be zero for comparison purposes. The maximum interest rate we are considering is 0.4, which is an artifact because in reality it is too large to be attained in western countries. We observe from the table that when λ is big enough, it is always optimal to annuitize. This means that for individuals who believe that their mortality rate is relatively high, it is optimal to annuitize immediately, and for individuals whose mortality rate is relatively low, it is optimal to annuitize when the interest rate is less than the annuitization boundary and optimal not to annuitize when the interest rate is greater than the annuitization boundary. There are three factors that may have contributed to these numerical results: the interest rate is not equal to the subjective interest rate, full consumption after annuity purchasing, and constant spread between the risky asset return and the riskless asset return. Then we observe that higher levels of the subjective discount rate leads to higher levels of annuitization boundaries, which means that when ρ is higher, individuals more likely choose to annuitize. Intuitively, this is because the discounted utilities at retirement are in fact very different for two subjective discount rates.

Comparing Table 2.1 to Table 2.2, we see that higher levels of risk-aversion coefficient implies higher levels of annuitization boundary. This is a reflection of the fact that risk averse individuals are more likely to annuitize at retirement if applicable. And once the force of mortality is big enough, it is always optimal to annuitize for interest rates less than 0.4.

Table 2.1: Conditions for $V^a \geq V^n$ When $\gamma = 2$

λ	domain of r ($\rho = 0$)	domain of r ($\rho = 0.02$)
0.0200	$r \leq 0.0483$	$r \leq 0.0823$
0.0500	$r \leq 0.1386$	$r \leq 0.1758$
0.0800	$r \leq 0.2287$	$r \leq 0.2668$
0.1000	$r \leq 0.2887$	$r \leq 0.3273$
0.1242	$r \leq 0.3613$	$r \leq 0.4000$
0.1371	$r \leq 0.4000$	$\forall \quad r$
> 0.1371	$\forall \quad r$	$\forall \quad r$

Table 2.2: Conditions for $V^a \geq V^n$ When $\gamma = 3$

λ	domain of r ($\rho = 0$)	domain of r ($\rho = 0.02$)
0.0200	$r \leq 0.0651$	$r \leq 0.1035$
0.0500	$r \leq 0.1767$	$r \leq 0.2187$
0.0800	$r \leq 0.2885$	$r \leq 0.3313$
0.0984	$r \leq 0.3569$	$r \leq 0.4000$
0.1100	$r \leq 0.4000$	$\forall \quad r$
> 0.1100	$\forall \quad r$	$\forall \quad r$

2.3.2 Stochastic Interest Rates

In the previous section, we have studied the optimal annuitization timing problem under constant interest rate and concluded that for $r = \rho$, it is optimal to annuitize when the mortality rate is greater than Merton's constant, and for $r \neq \rho$, it is optimal to annuitize when the interest rate is smaller than the annuitization boundary, and optimal not to annuitize when the interest rate is greater than the annuitization boundary if the mortaliy rate is relatively small. Otherwise, it is always optimal to annuitze at any interest rate. But the assumption of a constant interest rate is not true in the real world since it fluctuates over time. Therefore, we move on to investigate the same annuitization problem under a much more real interest rate in which the return of the riskless asset obeys the CIR process (see equation (2.3)).

In this section, we look at the annuitization value function V^a first, which is proved to be time-independent if the power term $e^{-(\rho+\lambda)t}$ is factored out. Secondly, we study the non-annuitization function V^n via dynamic programming, which is also time-independent if $e^{-(\rho+\lambda)t}$ is factored out. Finally we compare these two value functions to find the free boundaries and present the numerical results in tables for two different risk aversion coefficients $\gamma = 2$ and $\gamma = 3$.

2.3.2.1 The Value Function with Annuitization

If the retiree annuitizes his/her wealth at time t, the associated value function can be written as

$$\begin{aligned} V^a(t,w,r) &= \sup_{c_s} E\left[\int_t^\infty e^{-(\rho+\lambda)s}u(c_s)ds\right] \\ &= \int_t^\infty e^{-(\rho+\lambda)s}u(\frac{w}{\bar{a}_{x+t}(t,r)})ds. \end{aligned} \tag{2.27}$$

Note that we have assumed full annuity payout consumption after annuitization. After some mathematical manipulation, we obtain

$$V^a(t,w,r) = e^{-(\rho+\lambda)t}u(\frac{w}{\bar{a}_{x+t}(t,r)})\bar{a}_{x+t}(t,\rho), \quad \gamma \geq 1. \tag{2.28}$$

The annuity factor $\bar{a}_{x+t}(t,R_t)$ is determined by

$$\bar{a}_{x+t}(t,R_t) = E\left[\int_t^\infty e^{-\int_t^s R_v dv}({}_{s-t}p_{x+t})ds\right] = \int_t^\infty P_B(t,s,R_t)({}_{s-t}p_{x+t})ds. \tag{2.29}$$

The notation $P_B(t,s,R_t)$ is the zero-coupon bond price at time t with maturity s. According to Cox, Ingersoll and Ross (1985), it is computed by

$$\begin{aligned} P_B(t,s,R_t) &= A(t,s)e^{-B(t,s)R_t}, \\ B(t,s) &= \tfrac{2(e^{\xi(s-t)}-1)}{(\xi+\theta)(e^{\xi(s-t)}-1)+2\xi}, \\ A(t,s) &= [\tfrac{2\xi e^{(\theta+\xi)(s-t)/2}}{(\xi+\theta)(e^{\xi(s-t)}-1)+2\xi}]^{\frac{2\theta\mu_r}{\sigma_r^2}}, \\ \xi &= \sqrt{\theta^2+2\sigma_r^2}. \end{aligned} \tag{2.30}$$

If we make a transformation $z = s-t$, then the above annuity factor becomes

$$\bar{a}_{x+t}(t,R_t) = \int_0^\infty e^{-\lambda z}A(t,t+z)e^{-B(t,t+z)R_t}dz. \tag{2.31}$$

Since $A(t, t+z)$ and $B(t, t+z)$ are functions of z only, the annuity factor does not depend on time t. It is the interest rate at time t decides the size of the annuity factor. It is known that the return of the riskless asset progresses as a stochastic process with a set mean and experiences random deviations from its mean that are not known beforehand. Therefore, the specific interest rate at time t can be any positive value which is a state variable in our optimal problem, and the value function V^a is time-independent if the power term $e^{-(\rho+\lambda)t}$ is factored out as the constant interest rate case. In next section, we will show that V^n shares the same property.

2.3.2.2 The Value Function without Annuitization: Dynamic Programming Method

If we assume the retiree never annuitizes at retirement, then the discounted utility of consumption the individual is seeking to maximize is defined by

$$V^n(t, w, r) = \sup_{c_s, \pi_s} E\left[\int_t^\infty e^{-(\rho+\lambda)s} u(c_s) ds\right]. \tag{2.32}$$

By applying Bellman's optimality principle and Ito's lemma, we obtain the following HJB equation (superscript n is omitted in V^n hereafter in this section) that

$V^n(t,w,r)$ must satisfy

$$\begin{aligned} &V_t + \sup_{c_t,\pi_t}\left[e^{-(\rho+\lambda)t}u(c_t) + (rw-c_t)V_w + \theta(\mu_r - r)V_r + \tfrac{1}{2}r\sigma_r^2 V_{rr}\right] \\ &+\sup_{\pi_t}\left[(\mu_s - r)\pi_t V_w + \tfrac{1}{2}\pi_t^2\sigma_s^2 V_{ww} + \rho_{rs}\sigma_r\sigma_s\sqrt{r}\pi_t V_{wr}\right] = 0. \end{aligned} \tag{2.33}$$

If we postulate $V(t,w,r) = \frac{w^{1-\gamma}}{1-\gamma}h(t,r)$ like before, then the optimal consumption and investment strategies can be obtained via its first order derivatives as

$$\bar{c}_t = e^{-\frac{(\rho+\lambda)t}{\gamma}}\tilde{w}h^{-\frac{1}{\gamma}}, \tag{2.34}$$

$$\bar{\pi}_t = \frac{(\mu_s - r)h + \rho_{rs}\sigma_r\sigma_s\sqrt{r}h_r}{\gamma\sigma_s^2 h}\tilde{w}. \tag{2.35}$$

where $\tilde{w}$ is the optimally controlled wealth. Substituting them back into the HJB equation (2.33), we obtain the following nonlinear PDE of h.

$$\begin{aligned} h_t + (1-\gamma)rh + \gamma e^{-\frac{(\rho+\lambda)t}{\gamma}}h^{1-\frac{1}{\gamma}} &+ \theta(\mu_r - r)h_r + \tfrac{1}{2}r\sigma_r^2 h_{rr} \\ &+ \tfrac{(1-\gamma)((\mu_s-r)h+\rho_{rs}\sigma_r\sigma_s\sqrt{r}h_r)^2}{2\sigma_s^2\gamma h} = 0, \end{aligned} \tag{2.36}$$

with terminal condition $h(\infty, r) = 0$. For simplicity, we will consider the case when there is no correlation between the Brownian motions that drive the risky asset and the return of the riskless asset, i.e., $\rho_{rs} = 0$. In this case, equation (2.36) collapses to

$$h_t + (1-\gamma)\eta h + \gamma e^{-\frac{(\rho+\lambda)t}{\gamma}}h^{1-\frac{1}{\gamma}} + \theta(\mu_r - r)h_r + \frac{1}{2}r\sigma_r^2 h_{rr} = 0, \tag{2.37}$$

where $\eta = r + \frac{(\mu_s - r)^2}{2\sigma_s^2\gamma}$. For numerical calculation purposes, the computational domain is truncated to be $[0,T]\times[0,r_{max}]$, in which T is the maximum lifespan of the individual minus his/her current age x, and r_{max} is the maximum interest rate that the riskless asset can attain. The terminal and boundary conditions imposed at time $t = T$, $r = 0$, and $r = r_{max}$ are respectively

$$\begin{cases} t = T: & h(T,r) = 0, \\ r = 0: & h_t + (1-\gamma)\frac{(\mu_s - r)^2}{2\sigma_s^2\gamma}h + \gamma e^{-\frac{(\rho+\lambda)t}{\gamma}}h^{1-\frac{1}{\gamma}} + \theta\mu_r h_r = 0, \\ r = r_{max}: & h_{rr} = 0. \end{cases} \tag{2.38}$$

We make an attempt to explain these conditions intuitively. Firstly, the zero terminal condition at time T is due to the fact that the integration of V is zero when T is fairly large. Secondly, the boundary condition at $r = 0$ is obtained from the PDE (2.36) by setting $r = 0$ on both sides of the equation, which is a natural boundary condition. Thirdly, the Neumann boundary condition at $r = r_{max}$ is provided on the observation that the second-order derivative at this point is close to zero for constant interest rates.

Now we are ready to solve this equation system (2.37) and (2.38) numerically. Unfortunately, the right hand sides of the new discretized equation system are all trapped to zero due to the zero terminal condition, thence zero solutions are obtained, which is not what we are looking for. To seek a non-zero solution, we make the transformation $h(t,r) = y^\gamma(t,r)$ as before, then the non-zero solution $y(t,r)$ must solve the following nonlinear second-order PDE

$$y_t + \frac{(1-\gamma)}{\gamma}\eta y + e^{-\frac{(\rho+\lambda)t}{\gamma}} + \theta(\mu_r - r)y_r + \frac{1}{2}r\sigma_r^2(y_{rr} + \frac{\gamma-1}{y}y_r^2) = 0. \tag{2.39}$$

Since the force of mortality is constant, the value function V can be shown to be independent of time t if the time factor $e^{-(\rho+\lambda)t}$ is excluded, just as the annuitization function V^a. Therefore, we make a new transformation $y(t,r) = e^{\frac{-(\rho+\lambda)t}{\gamma})}\tilde{y}(t,r)$, and substitute it into equation (2.39), we obtain the following PDE that $\tilde{y}(t,r)$ must satisfy

$$\tilde{y}_t+(\frac{1-\gamma}{\gamma}(r+\frac{(\mu_s-r)^2}{2\gamma\sigma_s^2})-\frac{\rho+\lambda}{\gamma})\tilde{y}+(\theta(\mu_r-r)+\frac{1}{2}r\sigma_r^2\frac{\gamma-1}{\tilde{y}}\tilde{y}_r)\tilde{y}_r+\frac{1}{2}r\sigma_r^2\tilde{y}_{rr}+1=0. \tag{2.40}$$

The corresponding terminal and boundary conditions become

$$\begin{aligned} t=T: &\qquad \tilde{y}(T,r)=0, \\ r=0: &\qquad \tilde{y}_t+(\tfrac{1-\gamma}{\gamma}(\tfrac{(\mu_s-r)^2}{2\gamma\sigma_s^2})-\tfrac{\rho+\lambda}{\gamma})\tilde{y}+\theta\mu_r\tilde{y}_r+1=0, \\ r=r_{max}: &\qquad \tilde{y}_{rr}=0. \end{aligned} \tag{2.41}$$

We have solved this equation system of $\tilde{y}$ in two different ways. The first way is to solve it directly with the above terminal and boundary conditions applying the implicit finite difference method. The solution we have obtained in this way is time invariant when time t is away from the zero terminal condition, which means that the effect of the zero terminal condition can actually be eliminated after some time. The second way is to apply the condition $\tilde{y}_t = 0$ first, and then solve the ODE by finite difference method using the iterative method. The two solutions obtained using these two different ways are in perfect agreement after elimination of the effect of the zero terminal condition from the first method. Therefore, $\tilde{y}$ depends only on the interest rates r. After solving this equation system, we are able to obtain all the numerical solutions of $V^n(t, w, r)$ on its grids.

The value function, V, is the maximum of the two value functions V^n and V^a, i.e., $V = \sup(V^n, V^a)$. We run into a problem when we take the maximum of the two if they intersect with each other. This means that at this intersection point, V^n equals V^a, while their derivatives are not equal. Motivated by the classical Stefan velocity for phase-change models (see Donaldson and Wetton (2006)), we move the intersection leftward so that both the value function values and their derivatives are equal on it. Next we will illustrate this procedure in much detail.

It is known that the two value functions with and without annuitization are respectively

$$\begin{cases} V^n(t, w, r) = \frac{w^{1-\gamma}}{1-\gamma} h(t, r) = \frac{w^{1-\gamma}}{1-\gamma} y^\gamma(t, r) = \frac{w^{1-\gamma}}{1-\gamma} e^{-(\rho+\lambda)t} \tilde{y}^\gamma(t, r), \\ V^a(t, w, r) = \frac{w^{1-\gamma}}{1-\gamma} e^{-(\rho+\lambda)t} \frac{1}{\rho+\lambda} \bar{a}_{x+t}(t, r)^{\gamma-1}. \end{cases} \tag{2.42}$$

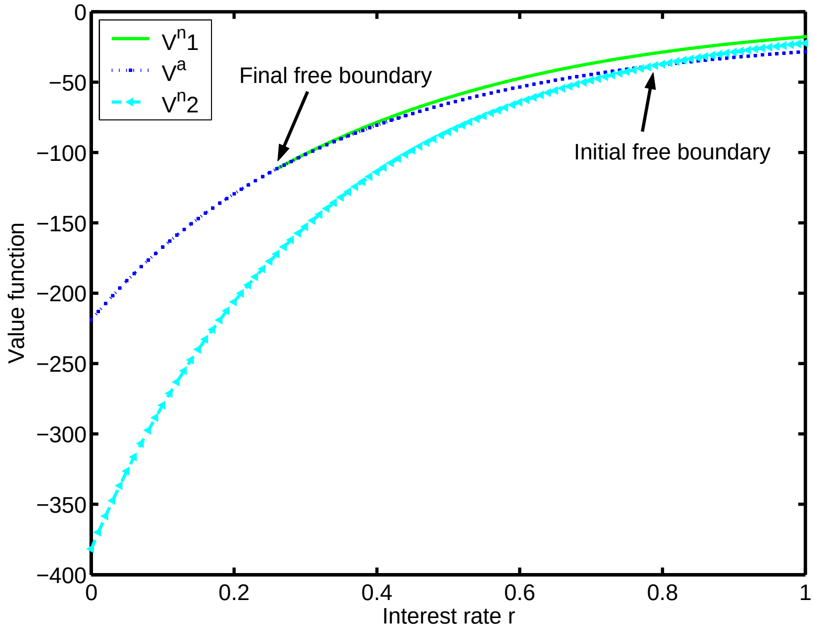
We can see that V^n and V^a are both independent of time t if the exponential term $e^{-(\rho+\lambda)t}$ is excluded because the annuity factor $\bar{a}_{x+t}(t, r)$ and the function $\tilde{y}(t, r)$ are independent of time t, so the annuitization boundary does not change over time. Therefore, for each fixed time t, we only need to compare $\frac{\tilde{y}^\gamma(t,r)}{1-\gamma}$ and $\frac{\bar{a}_x(t,r)^{\gamma-1}}{(\rho+\lambda)(1-\gamma)}$ to find the annuitization boundary. Without loss of generality, we do this comparison at time $t = 0$ ($s = T$).

We first compare V^a and V^n to obtain the initial annuitization boundary $r^\star$ (it indeed exists in this case), which divides the whole interest rate domain into two separated regions: in region $[0, r^\star]$, it is optimal to annuitize, and in region $[r^\star, r_{max}]$, it is optimal not to annuitize. On this annuitization boundary, not only the two value functions, but also their derivatives with respect to r should be equal. So we compute the difference of the derivatives $\frac{\partial V^n}{\partial r}|_{r=r^\star} - \frac{\partial V^a}{\partial r}|_{r=r^\star}$, if this difference is equal to zero, then $r^\star$ is the annuitization boundary that we are looking for, otherwise, we need to move $r^\star$ with an explicit time step δr to the left (denote $r^\star = r^\star - \delta r$), and solve the PDE of $\tilde{y}$ in the new domain $[r^\star, r_{max}]$ by setting the boundary condition at $r^\star$ to be the corresponding value so that the two value functions V^n and V^a would be equal on the annuitization boundary (Note that in domain $[0, r^\star]$, it is optimal to annuitize, so $V = V^a$). Then we compare the two derivatives on the new annuitization boundary $r^\star$. If it is equal to zero, then this new $r^\star$ is what we are looking for, otherwise, repeat the above procedure until we find a new $r^\star$ in which V^n and V^a and their derivatives with respect to r are equal on the annuitization boundary. If we continue this procedure and cannot find a solution, then the annuitization boundary does not exist, which means that it is always optimal not to annuitize for any interest rate.

2.3.2.3 Numerical Results

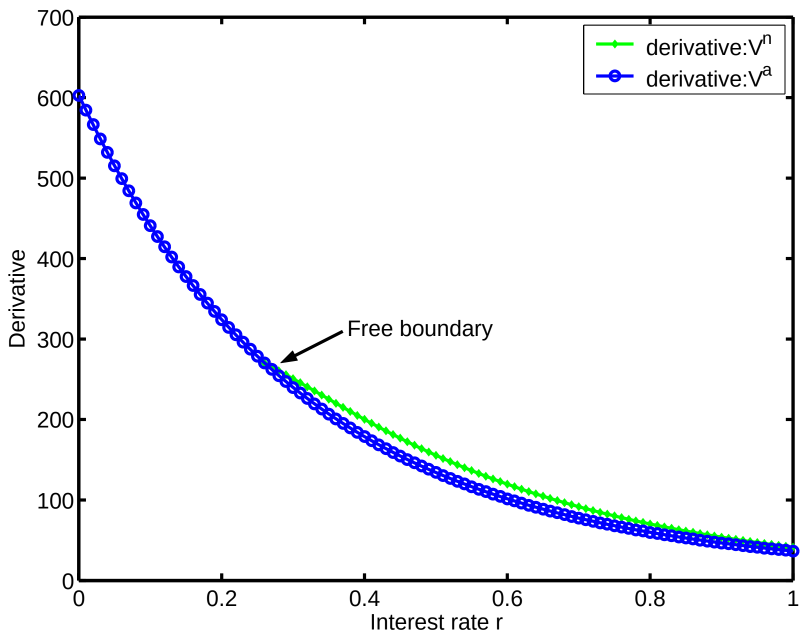
In this section, numerical results are presented for two different levels of risk aversion, $\gamma = 2$ and $\gamma = 3$. As for the financial market parameters, the volatility for the risky asset, σ_s, is assumed to be 0.2, which is roughly in line with numbers provided by Ibbotson Associates (2001), which are widely used by practitioners when simulating long-term investment returns. The drift term μ_s, is assumed to be moving with the interest rate r at any time t, i.e., the expected equity returns are modeled to be 0.03 above risk-free interest rates. The constant mortality rate is assumed to be 0.05, implying that the expected remaining lifetime is 20 years. The maximum life span for the individual is assumed to be $T = 125$ years. The parameters for stochastic interest rates are $\theta = 0.25$, $\mu_r = 0.06$ and $\sigma_r = 0.1$. The volatility σ_r is chosen to satisfy condition $2\theta\mu_r \geq \sigma_r^2$, which guarantees that the interest rate will never touch zero for any given positive initial interest rate. The subjective discount rate ρ is specified to be 0 for comparison purposes, since it is not a real assumption. The correlation ρ_{rs} is taken to be 0 since Munk, Sorensen and Vinther (2004) estimated that the stock index is slightly negatively correlated with the nominal interest rate (-0.06). Parameters described in the algorithm are summarized in Table 2.3. All the parameter values take these typical values unless otherwise specified throughout the entire section.

Figure 2.1: The Annuitization and Non-annuitization Value Functions.



The figure shows both initial and final free boundaries under stochastic interest rates (note that the free boundary here means the annuitization boundary). The parameters used are: constant force of mortality $\lambda = 0.05$, adjustment speed $\theta = 0.25$, long run interest rate $\mu_r = 0.06$, volatility of interest $\sigma_r = 0.1$, volatility of risky asset $\sigma_s = 0.2$ and risk aversion coefficient $\gamma = 2$. V^a denotes the value function with annuitization, V^n2 denotes the value function obtained by solving $\tilde{y}$ in domain $[0, r_{max}]$ in which the boundary condition at $r = 0$ is imposed by equation (2.41), and the initial annuitization boundary is equal to 0.7800. V^n1 is the non-annuitization value function obtained by solving $\tilde{y}$ in domain $[0.2560, r_{max}]$ in which the boundary condition at $r = 0.2560$ is set to be $V^n1 = V^a$.

Figure 2.2: The Derivatives of the Value Functions.



Note that the free boundary here means the annuitization boundary. The parameters used are: constant force of mortality $\lambda = 0.05$, adjustment speed $\theta = 0.25$, long run interest rate $\mu_r = 0.06$, volatility of interest $\sigma_r = 0.1$, volatility of risky asset $\sigma_s = 0.2$ and risk aversion coefficient $\gamma = 2$. The two derivatives intersect at point 0.2560. This is the annuitization boundary we are looking for.

Table 2.3: Typical Parameter Values

Constant mortality λ	0.05
CIR model θ, μ_r, σ_r	0.25, 0.06, 0.1
Risky asset $\mu_s - r, \sigma_s$	0.03, 0.2
Maximal life time T	125
Correlation between B_t^s and B_t^r, ρ_{rs}	0

To give readers some intuition about how the annuitization boundary is obtained, the two value functions and their derivatives are plotted in two separate figures (2.1 and 2.2) for parameters $\theta = 0.25, \mu_r = 0.06, \sigma_r = 0.1$ and $\lambda = 0.05$. In figure 2.1, V^a is the value function with annuitization, V^n2 is the value function obtained by solving $\tilde{y}$ in domain $[0, r_{max}]$ in which the boundary condition at $r = 0$ is imposed by equation (2.41), and V^n1 is the non-annuitization value function obtained by solving $\tilde{y}$ in domain $[0.2560, r_{max}]$ in which the boundary condition at $r = 0.2560$ is set so that $V^n1 = V^a$. The initial annuitization boundary, where V^a intersects V^n2, is at point $r = 0.7800$, we then move it leftward until the final annuitization boundary $r = 0.2560$ is obtained on which both V^a and V^n1 and their derivatives with respect to r are equal. These two derivatives are plotted in Figure 2.2. Note that when r is constant, the annuitization boundary is $r = 0.1386$, so the annuitization boundaries for constant and stochastic interest rates for exponential mortality are not too far away from each other.

Table 2.4: Match of the Annuitization Boundaries

λ	Domain of r (analytical solution)	Domain of r ($\theta = 0$, $\sigma_r = 0$)
0.0200	$r \leq 0.0483$	$r \leq 0.0462$
0.0300	$r \leq 0.0784$	$r \leq 0.0762$
0.0400	$r \leq 0.1086$	$r \leq 0.1063$
0.0500	$r \leq 0.1386$	$r \leq 0.1363$
0.0800	$r \leq 0.2287$	$r \leq 0.2263$
0.1000	$r \leq 0.2887$	$r \leq 0.2838$
0.1371	$r \leq 0.4000$	$r \leq 0.3888$
0.1398	$\forall r$	$r \leq 0.4000$
> 0.1398	$\forall r$	$\forall r$

Notes: The 2^{nd} column denotes the annuitization boundary for constant interest rates, the 3^{rd} column denotes the annuitization boundary for stochastic interest rates when the adjustment speed and volatility are both 0.

To verify our numerical results, we first compare the annuitization boundaries we have obtained by setting the adjustment speed and volatility of the interest rate to be 0 (the stochastic interest rates collapse to constants) and compare them with previous results in section 2.3.1.3, which is summarized in Table 2.4 for $\gamma = 2$ and

$\rho = 0$. It can be easily computed that the maximum absolute difference of the two annuitization boundaries using two different methods is 1.12 percent, so they are in agreement, which gives us confidence that our numerics are good.

Table 2.5: Annuitization Boundaries for Stochastic and Constant Interest

λ	Domain of r ($\theta = 0$, $\sigma_r = 0$)	Domain of r ($\theta = 0.25$, $\sigma_r = 0.1$)
0.0200	$r \leq 0.0462$	$\varnothing$
0.0300	$r \leq 0.0762$	$r \leq 0.1120$
0.0400	$r \leq 0.1063$	$r \leq 0.1920$
0.0500	$r \leq 0.1363$	$r \leq 0.2560$
0.0800	$r \leq 0.2263$	$r \leq 0.3867$
0.1000	$r \leq 0.2838$	$\forall r$
0.1371	$r \leq 0.3888$	$\forall r$
0.1398	$r \leq 0.4000$	$\forall r$
> 0.1398	$\forall r$	$\forall r$

Notes: The 2^{nd} column denotes the annuitization boundary for constant interest rates, and the 3^{rd} column denotes the annuitization boundary for stochastic interest rates.

Then we compare the effect of the stochastic interest rates on the optimal annuitization strategies in Table 2.5. We can see that in general the annuitization boundary for stochastic interest rates lies above the annuitization boundary for constant interest rate (if applicable). For stochastic interest rates, when the mortality rate is 0.02, it is always optimal not to annuitize, and when the mortality rate is greater than 0.10, it is always optimal to annuitize. For a constant interest rate, when the mortality rate is greater than 0.1398, it is always optimal to annuitize. Therefore, both the mortality rate and the interest rate matter when it comes to the decision of annuity purchasing. The intuitive explanation for the rise in the annuitization boundary in this case lies in that the dominant effect of the stochastic interest rate to push up expected interest rates over time.

Now we move on to see the effect of the subjective discount rate on the annuitization boundary. The parameters for the CIR process are $\theta = 0.25, \mu_r = 0.06, \sigma_r = 0.10$ for two different risk-aversion coefficients. The annuitization boundary are summarized in Table 2.6 for $\gamma = 2$ and Table 2.7 for $\gamma = 3$ respectively. In each table, two different subjective discount rates, $\rho = 0$ and $\rho = 0.02$, are considered. It can be seen that the annuitization boundary is an increasing function of λ, and it is optimal to annuitize when the interest rate is smaller than the annuitization boundary (note that the hazard rate is greater than Merton's constant), which is consistent with the constant interest rate case. Note that when $\gamma = 2$ and $\lambda = 0.02$, the annuitization boundary does not exist, which means that it is always optimal not to annuitize no matter what the current interest rate is, which is an extension of the result for constant interest rate. Another important observation is that when the force of mortality is big enough, it is always optimal to annuitize, which is due to a significant survivor credit to be gained in investing in annuities. When the subjective discount rate is larger, the individual tends to annuitize in a larger interest rate domain.

Table 2.6: Annuitization Boundary for $\gamma = 2$

λ	Domain of r ($\rho = 0$)	Domain of r ($\rho = 0.02$)
0.0200	$\varnothing$	$r \leq 0.1250$
0.0300	$r \leq 0.1120$	$r \leq 0.2150$
0.0400	$r \leq 0.1920$	$r \leq 0.2750$
0.0500	$r \leq 0.2560$	$r \leq 0.3250$
0.0800	$r \leq 0.3867$	$\forall r$
0.1000	$\forall r$	$\forall r$

Above all, whether the interest rate is constant or stochastic, it is optimal to annuitize when the interest rate is small and optimal not to annuitize when the interest rate is large when applicable (if the mortality rate beats the risk premium). When the mortality rate is higher, it is always optimal to annuitize due to higher mortality credit. With a participating annuity, premiums paid by those who die earlier than expected contribute to gains of the overall pool and provide a higher

Table 2.7: Annuitization Boundary for $\gamma = 3$

λ	Domain of r ($\rho = 0$)	Domain of r ($\rho = 0.02$)
0.0200	$r \leq 0.1350$	$r \leq 0.2950$
0.0300	$r \leq 0.2550$	$r \leq 0.3650$
0.0400	$r \leq 0.3250$	$\forall r$
0.0500	$r \leq 0.3850$	$\forall r$
0.0800	$\forall r$	$\forall r$

yield or credit to survivors than could be achieved through individual investments outside of the pool. The annuitization boundary that lies between is an increasing function of mortality rate λ. These important observations will shed light on the next section where we study the annuitization problem under Gompertz mortality.

2.3.3 Concluding Remarks

In this section, we have studied the annuitization problem for a retired individual whose objective is to maximize his/her lifetime utility under exponential mortality and a variety of institutional restrictions in the absence of bequest motives. There are two asset classes available to invest in: a risky asset and a riskless asset, in which the return of the riskless asset is constant, or stochastic. The utility function we are interested in exhibits constant relative risk aversion (CRRA), which has been widely used in the insurance economics literature. Since two value functions V^a and V^n are independent of time t if the power term $e^{-(\rho+\lambda)t}$ is excluded, so that the annuitization boundary is deterministic for any constant λ.

First of all, we calibrated the model for a constant interest rate. Analytic solutions for V^a and V^n are obtained using some mathematical techniques. In this case, the mortality rate λ and interest rate r are free parameters. Through comparison of the two value functions, we find that for $r = \rho$, it is optimal to annuitize for any interest rate when the mortality rate is greater than Merton's constant, which is consistent with Milevsky and Young (2007). If the interest rate is not equal to the subjective interest rate, it is optimal to annuitize when the interest rate is small, and optimal not to annuitize when the interest rate is large when applicable, while it is always optimal to annuitize when the force of mortality is higher. This is due to the assumption of constant spread between the expected risky asset return and that of the riskless asset, and the significant survivor credit to be gained in investing in annuities.

Secondly, stochastic interest rates are considered, which adds more uncertainty to the interest rates. In this case the annuity factor is much more complex since it involves the price of a bond which matures at certain time. The annuitization value function can be obtained analytically. The non-annuitization value function satisfying an HJB equation can be solved numerically using the upwind scheme and the Crank-Nicolson method. We then use the free boundary refining method to find the annuitization boundary. The annuitization boundaries agree well when the stochastic interest rates collapse to constants. Numerical results show us that when the force of mortality λ is less than Merton's constant, it is always optimal not to annuitize. Otherwise, it is optimal to annuitize when the interest rate is small and optimal not to annuitize when the interest rate is large when applicable. When the mortality rate is higher, which makes the survivor credit significant, it is optimal to annuitize, which agrees with the constant interest rate case. When interest rates are stochastic and current interest rate is high, one should delay annuitizing, earn short term interest, and once interest rates revert to a more realistic level, one will probably be able to buy more annuities than he/she would otherwise. In other words, the annuities one eventually buy will be more expensive, but he/she will be able to buy more of them and actually earn higher income.

The mathematical simplification of the mortality process (exponential) enables us to find a solution with much greater ease. This assumption is memory-less which means that the future mortality rates of the individual are independent of the past mortality rate which is inconsistent with the time varying mortality models and reality. To help overcome this disadvantage, we will look at the same optimal control problem under the GM mortality rate in next section because of its widespread use in the insurance and finance literature (see Milevsky and Young

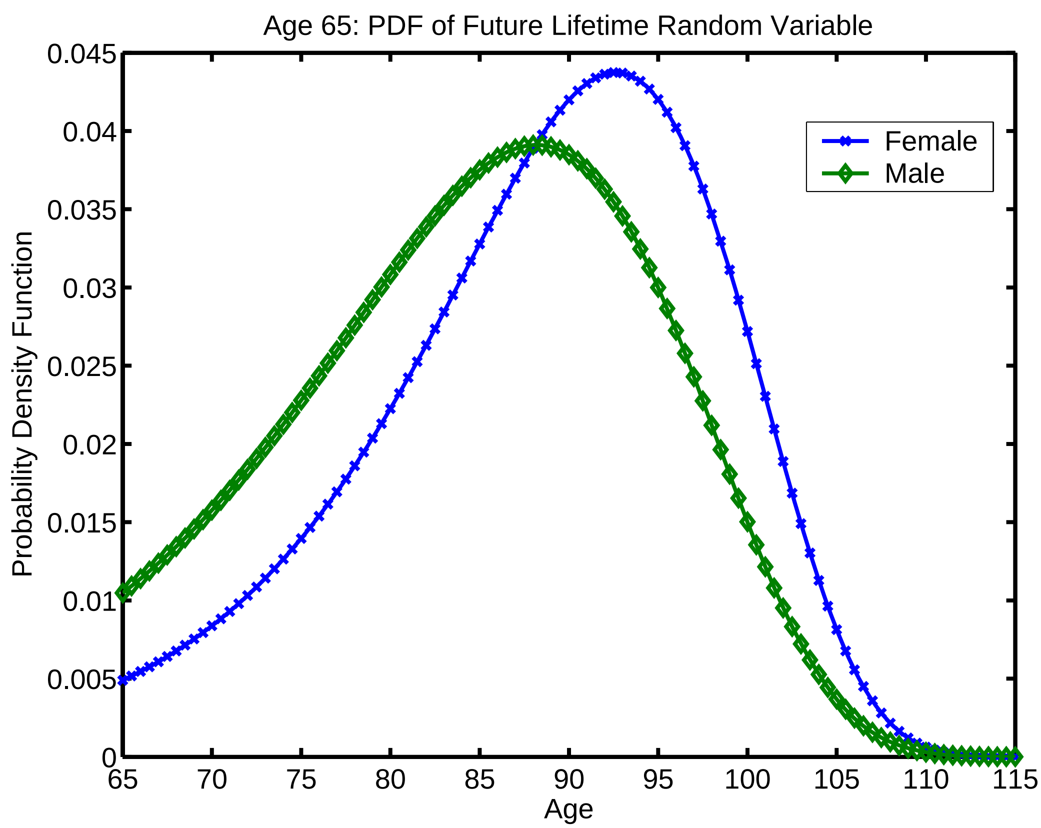
(2007), Horneff, Maurer and Stamos (2008)), which is simple and consistent with the insurer's view on mortality.

2.4 Model Calibration 2: Gompertz Mortality

In the previous section, we have studied the annuitization problem for a retiree who seeks to maximize his/her lifetime utility of consumption after retirement under constant force of mortality and other institutional restrictions in an all-or-nothing framework. In this section, we will investigate the same problem under Gompertz mortality, in the circumstance of stochastic interest rates which follows the CIR process. The Gompertz mortality model ($\lambda_{x+t} = \frac{1}{b}e^{\frac{x+t-m}{b}}, t \in [0, \infty]$) is common in the actuarial literature for annuity pricing (Frees et al., 1996) and in the economics literature for pricing insurance (Johansson, 1996). Milevsky and Young (2007) have fitted the Gompertz distribution to the individual annuity mortality 2000 (basic) table, obtaining estimates of the parameters $(m, b) = (88.18, 10.5)$ for males and $(m, b) = (92.63, 8.78)$ for females. These parameters are the values that we will use for the annuitization problem for males and females. Figure 2.3 plots the probability density function of the future lifetime random variable with above parameters for both males and females.

For mathematical manipulation purposes, our Gompertz mortality λ_{x+t} is modified

Figure 2.3: The Probability Density Function for Males and Females



This is the probability density function of the future lifetime random variable under Gompertz mortality rate. For females, the fitted parameters are $(m, b) = (92.63, 8.78)$ and for males they are $(m, b) = (88.18, 10.5)$.

to be

$$\lambda_{x+t} = \begin{cases} \frac{1}{b} e^{\frac{x+t-m}{b}}, & t \leq T, \\ \\ \lambda_{x+T}, & t \geq T, \end{cases} \tag{2.43}$$

where x denotes the current age of the individual, s is the time the individual is going to survive, m is the mode of the future lifetime, b is the dispersion constant. T is the maximum life time for a human being (in our case, it is taken to be 125). Note that this definition is different from the traditional GM mortality which assumes exponential function all the time. This is an approximation we need to work out the terminal condition without having any practical impact from a technical point of view. This is a plausible assumption because human beings rarely live past the maximum age (the longest unambiguously documented human lifespan is 122 years old), so their mortality rate is very high, which means that the effect of mortality after the maximum age is trivial to the value function. For example, if we take $x = 65, m = 88.18, b = 10.5, T = 125$, then the mortality rate is constant 3.1840 after age 125. One big advantage of this assumption is that we can apply non-zero terminal condition at $t = T$, which can be obtained by applying the same mathematical techniques for exponential mortality as in the previous section.

Instead of including one special point in time (retirement), we will include the whole retirement period (from retirement to death) to see under what conditions should the individual annuitize all his/her wealth. This is a free boundary problem because the value function is specified by a set of constraints which are exactly the properties of a free boundary problem, similar to the American option pricing problem. The main takeaway from the last section is that it is optimal to annuitize when the interest rate is small and not optimal to annuitize when the interest rate is large when the hazard rate is greater than Merton's constant. Therefore, we have come up with the illustration of our free boundary problem: for any given time t, when the interest rate is smaller than the free boundary (optimal annuitization interest rate), it is optimal to annuitize, and when the interest rate is greater than the free boundary, it is optimal not to annuitize. This free boundary problem can be converted to an equivalent linear complementarity problem and solved by the projected successive over-relaxation method.

The rest of this section is organized as follows. In Section 2.4.1, we model and frame the optimal annuitization timing problem when the risk-free rate is driven by CIR process under Gompertz mortality. Then the free boundary problem and its equivalent LCP problem are illustrated in Section 2.4.2 and 2.4.3. Next the projected SOR method is applied to solve the LCP problem in Section 2.4.4, and finally, numerical results are addressed in Section 2.4.5.

2.4.1 The Value Function and HJB Equation

For a retired individual at age x, we look for the optimal asset allocation, consumption, and annuitization strategies to maximize his/her lifetime utility of consumption in an all-or-nothing framework without bequest motives. Mathematically, we wish to find the value function defined as below

$$\begin{aligned} V(t,w,r) = \sup_{\pi_s, c_s, \tau} E \Big[\textstyle\int_t^{\tau} e^{-\rho(s-t)} \ {}_{s-t}p_{x+t} u(c_s)\, ds \\ + \textstyle\int_{\tau}^{\infty} e^{-\rho(s-t)} \ {}_{s-t}p_{x+t} u\left(\frac{W_\tau}{\bar{a}_{x+\tau}(\tau, R_\tau)}\right) ds \Big| W_t = w, R_t = r \Big], \end{aligned} \tag{2.44}$$

in which E denotes the expectation conditional on $W_t = w$ and $R_t = r$, and u is a concave utility function of consumption. Note that the expectation stays outside the integral since $u(c_s)$ and $u(W_\tau/\bar{a}_{x+\tau}(\tau, R_\tau))$ may be correlated with the discount factor. Thus we cannot replace the discount factor by the zero-coupon bond price $P_B(t,s,r)$ inside the integral. τ is the time the individual annuitizes all his/her wealth in a lump sum. The survival probability $({}_{s-t}p_{x+t})$ is defined back in equation (2.1). The actuarial annuity factor is calculated using equation

$$\bar{a}_{x+t}(t, R_t) = \begin{cases} \int_t^T P_B(t,s,r) e^{-\int_t^s \lambda_{x+v} dv} ds + \int_T^\infty P_B(t,s,r) e^{-\lambda_{x+T}(s-t)} ds, & t \le T, \\ \int_t^\infty P_B(t,s,r) e^{-\lambda_{x+T}(s-t)} ds, & t \ge T. \end{cases} \tag{2.45}$$

Note that this annuity factor is different than the traditional one due to the modified GM mortality we have applied.

Next we are going to derive the HJB equation that the value function must satisfy in domain $t \in [0, T]$ by applying Bellman's optimality principle and Ito's lemma.

$$
\begin{aligned}
& V(t, w, \lambda) \\
= \;& \sup_{\pi_s, c_s, \tau} E_{w,r} \left[\int_t^{\tau} e^{-\rho(s-t)} \;{}_{s-t}p_{x+t}^{s} u(c_s)\, ds \; + \int_{\tau}^{\infty} e^{-\rho(s-t)} \;{}_{s-t}p_{x+t}^{s} u\left(\frac{W_\tau}{a_{x+\tau}^{o}}\right) ds \,\right] \\
= \;& \sup_{\pi_s, c_s, \tau} E_{w,r} [\int_t^{t+dt} e^{-\rho(s-t)} \;{}_{s-t}p_{x+t}^{s} u(c_s)\, ds \\
+ \;& e^{-\rho dt} \;{}_{dt}p_{x+t}^{s}\, V(t+dt, w+dW_t, r+dR_t)].
\end{aligned} \tag{2.46}
$$

Since $V(t, w, r)$ has two state variables, wealth w and interest rate r, it is obvious that we can apply Ito's lemma to obtain the stochastic differential equation that V must satisfy

$$
\begin{aligned}
dV(t, w, r) \;&=\; V_t dt + V_w dW_t + V_r dR_t + \frac{1}{2} V_{ww} < dW_t, dW_t > \\
&+\; \frac{1}{2} V_{rr} < dR_t, dR_t > + V_{wr} < dW_t, dR_t > \\
&=\; V_t dt + V_w((R_t W_t + (\mu_s - R_t)\pi_t - c_t)dt + \sigma_s \pi_t dB_t^s) \\
&+\; V_r(\theta(\mu_r - R_t)dt + \sigma_r \sqrt{R_t} dB_t^r) \\
&+\; \frac{1}{2} V_{ww} \sigma_s^2 \pi_t^2 dt + \frac{1}{2} V_{rr} \sigma_r^2 R_t dt + \rho_{rs} \sigma_r \sigma_s \pi_t \sqrt{R_t} V_{wr} dt \\
&=\; V_t dt + \mathfrak{L} V_t dt + \sigma_r \sqrt{R_t} V_r dB_t^r + \sigma_s \pi_t V_w dB_t^s,
\end{aligned} \tag{2.47}
$$

where ρ_{rs} is the correlation between dB_t^s and dB_t^r, and the second-order differential operator $\mathfrak{L}V_t$ is defined as

$$\begin{aligned}\mathfrak{L}V_t = (R_tW_t + (\mu_s - R_t)\pi_t - c_t)V_w &+ \theta(\mu_r - R_t)V_r + \tfrac{1}{2}\sigma_s^2\pi_t^2V_{ww} \\ &+ \tfrac{1}{2}R_t\sigma_r^2V_{rr} + \rho_{rs}\sigma_r\sigma_s\pi_t\sqrt{R_t}V_{wr}.\end{aligned} \tag{2.48}$$

This is equivalent to

$$\begin{aligned}V(t+dt, w+dW_t, r+dR_t) &= V(t,w,r) + V_tdt + \mathfrak{L}V_tdt \\ &+ \sigma_r\sqrt{R_t}V_rdB_t^r + \sigma_s\pi_tV_wdB_t^s.\end{aligned} \tag{2.49}$$

Thus the value function V satisfies the following equation

$$\begin{aligned}V(t,w,r) = \sup_{\pi_s,c_s,\tau} E_{w,r}\Bigg[\int_t^{t+dt} e^{-\rho(s-t)}\ {}_{s-t}p_{x+t}^s u(c_s)\,ds \\ + \ e^{-\rho dt}\ {}_{dt}p_{x+t}^s\left(V + V_tdt + \mathfrak{L}V_tdt + \sigma_r\sqrt{R_t}V_rdB_t^r + \sigma_s\pi_tV_wdB_t^s\right)\Bigg].\end{aligned} \tag{2.50}$$

Moving V to the right-hand side, we arrive at

$$\begin{aligned}\sup_{\pi_s,c_s,\tau} \quad & E_{w,r}\Bigg[\int_t^{t+dt} e^{-\rho(s-t)}\ {}_{s-t}p_{x+t}^s u(c_s)\,ds \\ & + (e^{-\rho dt}\ {}_{dt}p_{x+t}^s - 1)V + e^{-\rho dt}\ {}_{dt}p_{x+t}^s\left(V_tdt + \mathfrak{L}V_tdt\right)\Bigg] = 0.\end{aligned} \tag{2.51}$$

Dividing dt on both sides, letting $dt \longmapsto 0$ and assuming that we can change the order of the limit and expectation, we get the HJB equation for V

$$(\rho + \lambda_{x+t}^s)V = V_t + \sup_{c,\pi} \mathfrak{L}V, \tag{2.52}$$

where the second-order differential operator $\mathfrak{L}V$ is defined by

$$\begin{aligned}\mathfrak{L}V &= u(c) + (rw + (\mu_s - r)\pi - c)V_w + \theta(\mu_r - r)V_r + \tfrac{1}{2}\sigma_s^2\pi^2V_{ww} \\ &+ \tfrac{1}{2}r\sigma_r^2V_{rr} + \rho_{rs}\sigma_r\sigma_s\pi\sqrt{r}V_{wr}.\end{aligned} \tag{2.53}$$

The optimal consumption and asset allocation strategies $\bar{c}$ and $\bar{\pi}$ can be obtained in feedback form

$$\begin{cases} \bar{c} &= V_w^{-\frac{1}{\gamma}}, \\ \bar{\pi} &= -\frac{(\mu_s - r)V_w + \rho_{rs}\sigma_r\sigma_s\sqrt{r}V_{wr}}{\sigma_s^2 V_{ww}}. \end{cases} \tag{2.54}$$

In the next subsection, we will study the free boundary problem in detail.

2.4.2 Free Boundary Problems

In this section, we will solve the HJB equation (2.52) by transferring it into an equivalent free boundary problem. At each time t we need to determine not only $V(t, w, r)$, but also, for each value of r, whether or not the individual needs to annuitize. Typically at each time t there is a particular interest rate r which marks the boundary between two regions: on one side the individual should not annuitize and on the other side the individual should annuitize. The value function $V(t, w, r)$ is specified by a set of constraints:

- The value function must be greater than or equal to the annuitization function, the value of V when the individual annuitizes immediately at time t.
- The HJB equation is replaced by an inequality because the value function is the supreme of all the functions that maximize the individual's utility.
- The value function must be a continuous function of wealth, this can be seen from the definition of the value function.

- The derivatives of the value function are continuous. This is the basic assumption when we are solving the problem.

Therefore, this is a free boundary problem, quite similar to the American put option pricing problem. We denote the free boundary by $r^*(t)$, and refer to it as the annuitization boundary. From Section 2.3, we have known that it is optimal to annuitize when the interest rate is small and it is optimal not to annuitize when the interest rate is large if the optimal annuitization interest rate exists for exponential mortality. Therefore, for a specified time t, it is favorable to annuitize if the interest rate is smaller than the optimal annuitization interest rate, otherwise it is not favorable to annuitize. The mathematical statement of the free boundary problem is given by

$$(\rho + \lambda_{x+t})V - V_t - \mathfrak{L}V > 0, V(t, w, r) = G(t, w, r) \tag{2.55}$$

for $0 < r < r^*(t)$ (optimal to annuitize),

$$(\rho + \lambda_{x+t})V - V_t - \mathfrak{L}V = 0, V(t, w, r) > G(t, w, r) \tag{2.56}$$

for $r^*(t) < r < \infty$ (optimal not to annuitize). Here $r^*(t)$ is the function of free boundary at time t. The notation $G(t, w, r)$ is the value function when it is optimal to annuitize at time t. Since $V(t, w, r)$ is the supreme value of the HJB equation (2.52), it does have a lower bound G, which is the value of V when we annuitize immediately at time t. This bound can be calculated via its definition as below

$$\begin{aligned} G(t,w,r) = & \int_t^\infty e^{-\rho(s-t)} \ {}_{s-t}p_{x+t} u\left(\frac{w}{\bar{a}_{x+t}(t,r)}\right) ds \\ = & u(\frac{w}{\bar{a}_{x+t}(t,r)}) \int_t^\infty e^{-\rho(s-t)} e^{-\int_t^s \lambda_{x+v} dv} ds \\ = & \frac{w^{1-\gamma}}{1-\gamma} g(t,r), \end{aligned} \tag{2.57}$$

in which $g(t,r)$ is a function of t and r, and defined as below

$$g(t,r) = \frac{1}{\bar{a}_{x+t}^{1-\gamma}(t,r)} \begin{cases} \int_t^T e^{-\rho(s-t)} e^{-\int_t^s \lambda_{x+v} dv} ds + \frac{1}{\rho+\lambda_{x+T}}, & t \le T, \\ \frac{1}{\rho+\lambda_{x+T}}, & t \ge T. \end{cases} \tag{2.58}$$

This is due to the assumption that the individual will annuitize all his/her wealth at time τ (if applicable) and consume exactly the annuity payout after annuitization, which is the classical annuity result that has been proved in the absence of bequest motives such as Yaari (1965). Therefore, if the individual annuitizes at time t, he/she will consume the amount $\frac{w}{\bar{a}_{x+t}(t,R_t)}$ thereafter.

If we postulate that the value function can be written in the form $V(t,w,r) = \frac{w^{1-\gamma}}{1-\gamma} h(t,r)$, then the optimal consumption and investment strategies in equation (2.54) can be written as

$$\begin{cases} \bar{c} = h^{-\frac{1}{\gamma}} w, \\ \bar{\pi} = \frac{(\mu_s - r)h + \rho_{rs}\sigma_r\sigma_s\sqrt{r}h_r}{\sigma_s^2 \gamma h} w. \end{cases} \tag{2.59}$$

Plugging them into the HJB equation (2.52), we obtain

$$\begin{aligned} (\rho + \lambda_{x+t})h = h_t + (1-\gamma)rh + \gamma h^{1-\frac{1}{\gamma}} + \theta(\mu_r - r)h_r + \frac{1}{2} r \sigma_r^2 h_{rr} \\ + \frac{1-\gamma}{2\sigma_s^2 \gamma h}((\mu_s - r)h + \rho_{rs}\sigma_r\sigma_s\sqrt{r}h_r)^2. \end{aligned} \tag{2.60}$$

Before seeking a non-zero solution $h(t,r)$, a standard transformation $h(t,r) = y(t,r)^\gamma$ is done first, thus equation (2.60) is converted to

$$\frac{1}{1-\gamma}\left[(\rho+\lambda^s_{x+t})y - \gamma y_t + \mathfrak{L}y\right] = 0. \tag{2.61}$$

The reason the factor $\frac{1}{1-\gamma}$ is kept here will be explained shortly. The second-order differential operator $\mathfrak{L}y$ is given by

$$\begin{aligned}\mathfrak{L}y = \quad & - \quad (1-\gamma)ry - \gamma - \theta(\mu_r - r)\gamma y_r \\ & - \quad \tfrac{1-\gamma}{2\sigma_s^2\gamma}(\mu_s - r)^2 y - \tfrac{1}{2}r\sigma_r^2\gamma(\gamma-1)\tfrac{y_r^2}{y} - \tfrac{1}{2}r\gamma\sigma_r^2 y_{rr} \\ & - \quad \tfrac{1-\gamma}{2\sigma_s^2\gamma}\rho_{rs}^2\sigma_r^2\sigma_s^2 r\gamma^2\tfrac{y_r^2}{y} - \tfrac{1-\gamma}{\sigma_s^2\gamma}(\mu_s - r)\rho_{rs}\sigma_r\sigma_s\sqrt{r}\gamma y_r.\end{aligned} \tag{2.62}$$

Therefore the free boundary problem can be written in terms of the new variable y as below

$$\frac{1}{1-\gamma}\left[(\rho+\lambda^s_{x+t})y - \gamma y_t + \mathfrak{L}y\right] > 0, \quad \frac{1}{1-\gamma}y^\gamma(t,r) = \frac{1}{1-\gamma}g(t,r) \tag{2.63}$$

for $0 < r < r^*(t)$ (optimal to annuitize),

$$\frac{1}{1-\gamma}\left[(\rho+\lambda^s_{x+t})y - \gamma y_t + \mathfrak{L}y\right] = 0, \quad \frac{1}{1-\gamma}y^\gamma(t,r) > \frac{1}{1-\gamma}g(t,r) \tag{2.64}$$

for $r > r^*(t)$ (optimal not to annuitize). The reason why the common factor $\frac{1}{1-\gamma}$ has not been eliminated in the equation is that the statement of the free boundary will have two different forms if we eliminate it (it is positive when $\gamma < 1$ and negative when $\gamma > 1$).

In general, there are two distinct methods for the numerical solutions of free boundary problems. One is to try to track the free boundary as part of the time-stepping process. In our context this is not a particularly attractive method because the free boundary is implicit. We refer the interested reader to Crank (1984) for the numerical solutions of implicit free boundary problems by boundary tracking strategies. The second method is to try to find a transformation that reduces the problem to a fixed boundary problem from which the free boundary can be inferred afterwards. There are many transformations that can do this, but here we only consider the elegant method which involves the linear complementarity formulation. In next section, the free boundary will be converted to an equivalent LCP problem for reasonable risk-aversion coefficients.

2.4.3 The Linear Complementarity Problem (LCP)

It is almost always impossible to find a closed-form solution to any given free boundary problem, so our chief aim is to construct efficient and robust numerical methods for the computation. Since it is difficult to deal with free boundaries, it is worthwhile to reformulate the problem in such a way as to eliminate any explicit dependence on the free boundary. The free boundary does not then interfere with the solution process, and it can be recovered from the solution after the latter has been found.

In mathematical optimization theory, the LCP problem arises frequently in computational mechanics and encompasses the well-known quadratic programming as a special case. It was proposed by Cottle and Dantzig in 1968. We start by considering a simple example of such a reformulation, in the context of the obstacle problem. The reason why we do not consider the American option pricing problem is that it is much more complex than the obstacle problem. We then apply the lessons learnt from the obstacle problem, which has linear complementarity formulations leading to efficient and accurate numerical solution schemes with the desirable property of not requiring explicit tracking of the free boundary, i.e., we are going to convert the free boundary problem into an equivalent LCP problem and then solve it by an iterative numerical method.

2.4.3.1 Linear Complementarity Problem for $\gamma > 1$

In this section, we will illustrate the optimal annuitization problem in the compact linear complementarity form for the risk aversion constant $\gamma > 1$. We can not directly convert the free boundary problem into an LCP problem because the free boundary problem is not written in a standard form. Therefore, we make a transformation $\tilde{y}(t,r) = -y(t,r)$, then the free boundary problem can be restated as

$$(\rho + \lambda_{x+t})\tilde{y} - \gamma\tilde{y}_t + \mathfrak{L}\tilde{y} > 0, \quad \tilde{y}(t,r) = -g^{\frac{1}{\gamma}}(t,r) \tag{2.65}$$

for $0 < r < r^*(s)$,

$$(\rho + \lambda_{x+t})\tilde{y} - \gamma \tilde{y}_t + \mathfrak{L}\tilde{y} = 0, \quad \tilde{y}(t,r) > -g^{\frac{1}{\gamma}}(t,r) \tag{2.66}$$

for $r^*(s) < r < \infty$. The second-order differential operator $\mathfrak{L}\tilde{y}$ is defined by

$$\begin{aligned} \mathfrak{L}\tilde{y} &= -(1-\gamma)r\tilde{y} + \gamma - \theta(\mu_r - r)\gamma\tilde{y}_r - \tfrac{1-\gamma}{2\sigma_s^2\gamma}(\mu_s - r)^2\tilde{y} - \tfrac{1}{2}r\sigma_r^2\gamma(\gamma-1)\tfrac{\tilde{y}_r^2}{\tilde{y}} \\ &-\tfrac{1}{2}r\gamma\sigma_r^2\tilde{y}_{rr} - \tfrac{1-\gamma}{2\sigma_s^2\gamma}\rho_{rs}^2\sigma_r^2\sigma_s^2 r\gamma^2\tfrac{\tilde{y}_r^2}{\tilde{y}} - \tfrac{1-\gamma}{\sigma_s^2\gamma}(\mu_s - r)\rho_{rs}\sigma_r\sigma_s\sqrt{r}\gamma\tilde{y}_r. \end{aligned} \tag{2.67}$$

Note that the only difference between $\mathfrak{L}y$ and $\mathfrak{L}\tilde{y}$ is that the sign in front of γ is opposite. Let $s = T - t$, $\hat{y}(s,r) = \tilde{y}(T-s,r)$, $\hat{g}(s,r) = g(t,r)$, i.e.,

$$\hat{g}(s,r) = \frac{1}{\bar{a}_{x+T-s}^{1-\gamma}(T-s,r)} \begin{cases} \int_{T-s}^{T} e^{-\rho(z-T+s)} e^{-\int_{T-s}^{z}\lambda_{x+v}dv} dz + \frac{1}{\rho+\lambda_{x+T}}, & t \le T, \\ \frac{1}{\rho+\lambda_{x+T}}, & t \ge T. \end{cases} \tag{2.68}$$

Then the above free boundary problem (2.65) and (2.66) can be converted to an equivalent LCP problem

$$\begin{cases} ((\rho + \lambda_{x+T-s})\hat{y} + \gamma\hat{y}_s + \mathfrak{L}\hat{y})(\hat{y} + \hat{g}^{\frac{1}{\gamma}}) = 0, \\ \hat{y} + \hat{g}^{\frac{1}{\gamma}} \ge 0, \\ (\rho + \lambda_{x+T-s})\hat{y} + \gamma\hat{y}_s + \mathfrak{L}\hat{y} \ge 0. \end{cases} \tag{2.69}$$

To solve this LCP problem in domain $s \in [0,T]$, we need to specify its initial and boundary conditions. The boundary conditions imposed on $r = 0$ and $r = r_{max}$ are similar to what we have done before, specifically

$$\begin{cases} r = r_{max} : \hat{y}_{rr} = 0, \\ r = 0 : (\rho + \lambda_{x+T-s})\hat{y}(s,0) - \frac{1-\gamma}{2\sigma_s^2\gamma}\mu_s^2\hat{y}(s,0) + \gamma\hat{y}_s(s,0) + \gamma - \theta\mu_r\gamma\hat{y}_r(s,0) = 0. \end{cases} \tag{2.70}$$

Now we look at the non-zero initial condition at time $s = 0$ by applying the same mathematical techniques for exponential mortality in domain $t \in [T, \infty)$. In this region, the two value functions with and without annuitization, V^a and V^n, are time independent if the power time term $e^{-(\rho+\lambda)t}$ is excluded according to our previous results. So we are able to compare the two value functions to find the initial free boundary and then move it to the place where both value functions and their derivatives are equal, and a time-independent function $\breve{y}(r)$ in which $V^n = \frac{w^{1-\gamma}}{1-\gamma}\breve{y}(r)^\gamma e^{-(\rho+\lambda_{x+T})t}$ as in Section 2.3.2.2. If we denote this final free boundary as r^*, we know that when $r > r^*$, it is optimal not to annuitize, and when $r < r^*$, it is optimal to annuitize. Therefore the initial condition for $\hat{y}(s, r)$ can be derived to be

$$\hat{y}(0, r) = e^{-\frac{(\rho+\lambda_{x+T})T}{\gamma}}\breve{y}(r). \tag{2.71}$$

Note that for large T, λ_{x+T} is usually greater than 1, and the exponential term $e^{-\frac{(\rho+\lambda_{x+T})T}{\gamma}}$ is very close to 0 but not equal to 0. Since $\breve{y}(r)$ is bounded, so $\hat{y}(0, r)$ is close to but not equal to 0 as well.

The advantage of the LCP formulation (2.69) is that it has no explicit mention of the free boundary. If we can solve it, then we can recover the free boundary afterwards. We will solve this LCP problem in the next section by an iterative finite difference method.

2.4.4 The Projected Successive Over-relaxation (SOR) Method

In this section we numerically solve the LCP problem (2.69) by applying the projected SOR method. In numerical linear algebra, the projected SOR method is a variant of the Gauss-Seidel method for solving a linear system of equations, resulting in faster convergence. A similar method can be used for any slowly converging iterative process. It was devised simultaneously by David M. Young, Jr. and by H. Frankel in 1950 for the purpose of automatically solving linear systems on digital computers.

We divide the (s, r)-plane into a regular finite mesh with step sizes δs and δr, and use a finite-difference approximation for the derivatives with respect to s and r. The truncated domain of my choice is $[0, T] \times [0, r_{max}]$ with $T = 125 - x$ and $r_{max} = 0.4$. The underlying reasons for these numbers are that we believe life expectancy for a human being should not exceed 125 years and the risk-free interest rate is less than 0.4, which is extremely large compared to its normal values. We start with an initial guess for $\hat{y}$ that is certainly above $\hat{g}^{\frac{1}{\gamma}}$, generates a sequence of more accurate approximations to the exact solution. During each iteration the constraint is implemented by resetting $\hat{y}$ to equal $\hat{g}^{\frac{1}{\gamma}}$ if values of $\hat{y}$ is less than $\hat{g}^{\frac{1}{\gamma}}$. For better stability and convergence, the second-order accuracy Crank-Nicolson method is applied. At point $(s_{n+\frac{1}{2}}, r_i)$, the discretization of y (The $\hat{}$ on $\hat{y}$ has been omitted for simplicity hereafter) and its derivatives are,

$$\begin{aligned}
y(s_{n+\frac{1}{2}}, r_i) &= \frac{y_i^{n+1} + y_i^n}{2}, \\
y_s(s_{n+\frac{1}{2}}, r_i) &= \frac{y_i^{n+1} - y_i^n}{\delta s}, \\
y_r(s_{n+\frac{1}{2}}, r_i) &= \frac{y_{i+1}^{n+1} + y_{i+1}^n - y_{i-1}^{n+1} - y_{i-1}^n}{4\delta r}, \\
y_{rr}(s_{n+\frac{1}{2}}, r_i) &= \frac{y_{i+1}^{n+1} + y_{i-1}^{n+1} - 2y_i^{n+1} + y_{i+1}^n + y_{i-1}^n - 2y_i^n}{2\delta r^2},
\end{aligned}$$

where $y_i^n = y(n\delta s, i\delta r)$, is the approximation of $y(s, r)$ at every grid. Hence the partial differential equation $y(s, r)$ must solve is approximated by

$$(\rho + \lambda_{x+T-s_n-0.5ds})\frac{y_i^{n+1} + y_i^n}{2} + \gamma\frac{y_i^{n+1} - y_i^n}{\delta s} + \mathfrak{L}y_i^{n+\frac{1}{2}} = 0. \tag{2.72}$$

Notice how we have discretized the nonlinear term $\frac{y_r^2}{y}$. We discretize one y_r term explicitly with the known values of y at time level n and another y_r term with Crank-Nicolson method using y values at time level n and $n+1$, which has successfully avoided solving a nonlinear equation system. About the denominator y, We discretize it with y_i^n when it is not equal to 0, otherwise, setting the whole nonlinear term $\frac{y_r^2}{y}$ equals 0. This is reasonable because y equals 0 if and only if at time $s = 0$

and the value $\frac{y_r^2}{y}$ is very close to 0 when $y=0$. Specifically, $\mathfrak{L}y_i^{n+\frac{1}{2}}$ takes the form

$$\begin{aligned}
\mathfrak{L}y_i^{n+\frac{1}{2}} = \; & -(1-\gamma)r_i\tfrac{y_i^{n+1}+y_i^n}{2}+\gamma-\tfrac{1-\gamma}{2\sigma_s^2\gamma}(\mu_s-r_i)^2\tfrac{y_i^{n+1}+y_i^n}{2} \\
& -\theta(\mu_r-r_i)\gamma\tfrac{y_{i+1}^{n+1}+y_{i+1}^n-y_{i-1}^{n+1}-y_{i-1}^n}{4\delta r} \\
& -\tfrac{1-\gamma}{\sigma_s^2\gamma}(\mu_s-r_i)\rho_{rs}\sigma_r\sigma_s\sqrt{r}_i\gamma\tfrac{y_{i+1}^{n+1}+y_{i+1}^n-y_{i-1}^{n+1}-y_{i-1}^n}{4\delta r} \\
& -(\tfrac{1}{2}r_i\sigma_r^2\gamma(\gamma-1)+\tfrac{1-\gamma}{2\sigma_s^2\gamma}\rho_{rs}^2\sigma_r^2\sigma_s^2r_i\gamma^2)\tfrac{\frac{y_{i+1}^n-y_{i-1}^n}{2\delta r}}{y_i^n}\tfrac{y_{i+1}^{n+1}+y_{i+1}^n-y_{i-1}^{n+1}-y_{i-1}^n}{4\delta r} \\
& -\tfrac{1}{2}r_i\gamma\sigma_r^2\tfrac{y_{i+1}^{n+1}+y_{i-1}^{n+1}-2y_i^{n+1}+y_{i+1}^n+y_{i-1}^n-2y_i^n}{2\delta r^2}
\end{aligned} \tag{2.73}$$

for $y_i^n \neq 0$, and

$$\begin{aligned}
\mathfrak{L}y_i^{n+\frac{1}{2}} = \; & -(1-\gamma)r_i\tfrac{y_i^{n+1}+y_i^n}{2}+\gamma-\tfrac{1-\gamma}{2\sigma_s^2\gamma}(\mu_s-r_i)^2\tfrac{y_i^{n+1}+y_i^n}{2} \\
& -\theta(\mu_r-r_i)\gamma\tfrac{y_{i+1}^{n+1}+y_{i+1}^n-y_{i-1}^{n+1}-y_{i-1}^n}{4\delta r} \\
& -\tfrac{1-\gamma}{\sigma_s^2\gamma}(\mu_s-r_i)\rho_{rs}\sigma_r\sigma_s\sqrt{r}_i\gamma\tfrac{y_{i+1}^{n+1}+y_{i+1}^n-y_{i-1}^{n+1}-y_{i-1}^n}{4\delta r} \\
& -\tfrac{1}{2}r_i\gamma\sigma_r^2\tfrac{y_{i+1}^{n+1}+y_{i-1}^{n+1}-2y_i^{n+1}+y_{i+1}^n+y_{i-1}^n-2y_i^n}{2\delta r^2},
\end{aligned} \tag{2.74}$$

for $y_i^n=0$. In order to apply the projected SOR method, we then write y_i^{n+1} in terms of all the other terms

$$\begin{aligned}
y_i^{n+1} = \; & (c_1+\tfrac{\gamma}{\delta s}+\tfrac{r_i\gamma\sigma_r^2}{2\delta r^2})^{-1}\{-c_1y_i^n+c_2(y_{i+1}^{n+1}+y_{i+1}^n-y_{i-1}^{n+1}-y_{i-1}^n) \\
+ \; & \tfrac{\gamma}{\delta s}y_i^n+\gamma+\tfrac{r_i\gamma\sigma_r^2}{4\delta r^2}(y_{i+1}^{n+1}+y_{i-1}^{n+1}+y_{i+1}^n+y_{i-1}^n-2y_i^n)\},
\end{aligned} \tag{2.75}$$

where c_1 and c_2 are given by the following expressions

$$\begin{aligned}
c_1 &= \tfrac{1}{2}(\rho+\lambda_{x+T-s_n-0.5ds}-(1-\gamma)r_i-\tfrac{1-\gamma}{2\sigma_s^2\gamma}(\mu_s-r_i)^2), \\
c_2 &= (c_3+\theta(\mu_r-r_i)\gamma+\tfrac{1-\gamma}{\sigma_s^2\gamma}(\mu_s-r_i)\rho_{rs}\sigma_r\sigma_s\sqrt{r}_i\gamma)/4/\delta r,
\end{aligned} \tag{2.76}$$

and where

$$c_3 = \begin{cases} \frac{r_i\sigma_r^2\gamma(\gamma-1)(y_{i+1}^n - y_{i-1}^n)}{4\delta r y_i^n} + \frac{(1-\gamma)\rho_{rs}^2\sigma_r^2\sigma_s^2 r_i\gamma^2(y_{i+1}^n - y_{i-1}^n)}{4\sigma_s^2\gamma\delta r y_i^n}, & \text{if } y_i^n \neq 0, \\ 0, & \text{if } y_i^n = 0. \end{cases} \quad (2.77)$$

The corresponding initial and boundary conditions for $y(s,r)$ imply that

$$\begin{cases} y_i^1 = e^{-\frac{\rho+\lambda_{x+T}}{\gamma}T}\breve{y}(r_i), i = 1, 2, \cdots, I+1, \\ y_{I+1}^n = 2y_I^n - y_{I-1}^n, \\ (\frac{\rho+\lambda_{x+T-s_n-0.5ds}}{2} + \frac{1-\gamma}{4\sigma_s^2\gamma}\mu_s^2 + \frac{\gamma}{\delta s})y_1^{n+1} = (\frac{1-\gamma}{4\sigma_s^2\gamma}\mu_s^2 - \frac{\rho+\lambda_{x+T-s_n-0.5ds}^s}{2})y_1^n \\ \qquad +\gamma + \frac{\gamma}{\delta s}y_1^n - \theta\mu_r\gamma\frac{y_2^n - y_1^n}{\delta r}. \end{cases} \quad (2.78)$$

Here $n = 1$ corresponds to time $t = 0$, in which the initial condition is posed. $i = 1$ and $i = I+1$ correspond to the interest rate $r = 0$ and $r = r_{max}$, which are the boundaries of the truncated computational domain for calculating y. We write $g_i^n = g(n\delta s, i\delta r)$ ($\hat{}$ on $\hat{g}$ is omitted for simplicity) for the discretized annuity function, we will return to its discretization shortly. Hence the projected SOR algorithm is to iterate (on k) the equations

$$\begin{aligned} z_i^{n+1,k+1} &= (c_1 + \frac{\gamma}{\delta s} + \frac{r_i\gamma\sigma_r^2}{2\delta r^2})^{-1}(-c_1 y_i^n + c_2(y_{i+1}^{n+1,k} + y_{i+1}^n - y_{i-1}^{n+1} - y_{i-1}^n) \\ &+ \frac{\gamma}{\delta s}y_i^n + \gamma + \frac{r_i\gamma\sigma_r^2}{4\delta r^2}(y_{i+1}^{n+1,k} + y_{i-1}^{n+1,k+1} + y_{i+1}^n + y_{i-1}^n - 2y_i^n)), \\ y_i^{n+1,k+1} &= \sup(y_i^{n+1,k} + \omega(z_i^{n+1,k+1} - y_i^{n+1,k}), g_i^{n+1}), \end{aligned} \quad (2.79)$$

The parameter ω ($1 < \omega < 2$) is the over-relaxation parameter, which guarantees the convergence of the algorithm. We repeat the above procedure until the error such as

$$||y^{n+1,k+1} - y^{n+1,k}||^2 = \sum_i (y_i^{n+1,k+1} - y_i^{n+1,k})^2 \tag{2.80}$$

is small enough for us to consider any further iterations as unnecessary. Notice that the constraint is enforced at the same time as the iterate $y_i^{n+1,k+1}$ is calculated, the effect of the constraint is immediately felt in the calculation of $y_{i+1}^{n+1,k+1}$, $y_{i+2}^{n+1,k+1}$, etc. The projected SOR method is an iterative method which starts with an initial guess for the solution and successively improves it until it converges to the true solution. One advantage of the projected SOR method is that during the process of searching for the true solution, it can apply the constraints directly without affecting other same time level values, which is impossible if direct methods are applied. Another advantage is that it is easier to program. A disadvantage of the projected SOR method is that it is somewhat slower than direct methods since it usually takes many iterations to complete the searching procedure.

Now we look at the discretization of the function $\hat{g}(s,r)$. From equation (2.68), we know that $\hat{g}(s,r)$ is a product of the actuarial annuity factor to the power $\gamma - 1$ and a piecewise function. The annuity factor and the integral in the piecewise function, can be computed by Simpson's rule as before. We then arrive at all the numerical results of $\hat{g}(s,r)$ at any time s, and ready to solve the LCP problem with known lower bound.

After solving the LCP problem (2.69) to obtain all the values of $\hat{y}(s,r)$, we are ready to recover the free boundary. We will look at the values of $\hat{y}(s,r) + \hat{g}(s,r)^{\frac{1}{\gamma}}$ for each fixed time s. The free boundary lies where this function switches from zero to nonzero. The set of these interest rates form the free boundary. Note that we need to transform back to the (t,r)-plane after the free boundary is obtained.

2.4.5 Numerical Results

In this section, numerical results are presented for two different levels of risk aversion parameters $\gamma = 2$ and $\gamma = 3$. The other financial market parameters used are $\theta = 0, \mu_r = 0.06, \sigma_r = 0.1, \mu_s - r = 0.03, \sigma_s = 0.2, T_{max} = 125, \rho = 0.02, \rho_{rs} = 0$. All the parameter values take these typical values unless otherwise specified throughout the entire section. The Gompertz parameters (before age 125) are taken to be $(m, b) = (88.18, 10.5)$ for males, and $(m, b) = (92.63, 8.78)$ for females as Milevsky and Young (2007), which are fitted to the individual annuity mortality 2000 table with projection scale G. Under this typical GM model the exact instantaneous force of mortality at various ages are listed in Table 2.8. We can see that the force of mortality for males is greater than that of females at first, and when time exceeds 115 and beyond, it becomes less than that of females.

Note that we have treated $\mu_s - r$ as one variable which leads to Merton's constant ($\frac{(\mu_s - r)^2}{2\gamma\sigma_s^2} = 0.0056$) fixed when risk-aversion coefficient and risky asset volatility are both constants. Table 2.9 shows us the annuitization interest rate domain for both

Table 2.8: Force of Mortality Table for Males and Females

Age	65	75	85	95	105	115	≥ 125
λ_{males}	0.0105	0.0271	0.0704	0.1823	0.4726	1.2250	3.1749
$\lambda_{females}$	0.0049	0.0153	0.0478	0.1492	0.4660	1.4555	4.5463

exponential mortality and Gompertz mortality for $\gamma = 2$ and $\rho = 0$. It can be easily observed that for stochastic interest rates, this annuitization domain for GM mortality is much higher than that for exponential mortality when applicable, and it is always optimal to annuitize when the force of mortality is big enough. The intuitive explanation for this rise in the annuitization boundary lies in the fact that Gompertz mortality has a higher force of mortality later at various ages, which adds the survivor credit later on, so that the individual will be better off if he/she annuitizes in a larger interest rate domain.

In the rest of this section we will demonstrate our numerical results in various plots and do sensitivity analysis for the CIR parameters θ, σ_r and μ_r.

2.4.5.1 Annuitization Boundaries for Different Risk Aversion Coefficient γ

Figure 2.4 displays the free boundaries for $\gamma = 2$ and $\gamma = 3$ respectively for a male ($m = 88.15, b = 10.5$) in which the maximum interest rate 0.4 is an artifact.

Table 2.9: Annuitization Boundaries for Exponential and Gompertz Mortalities

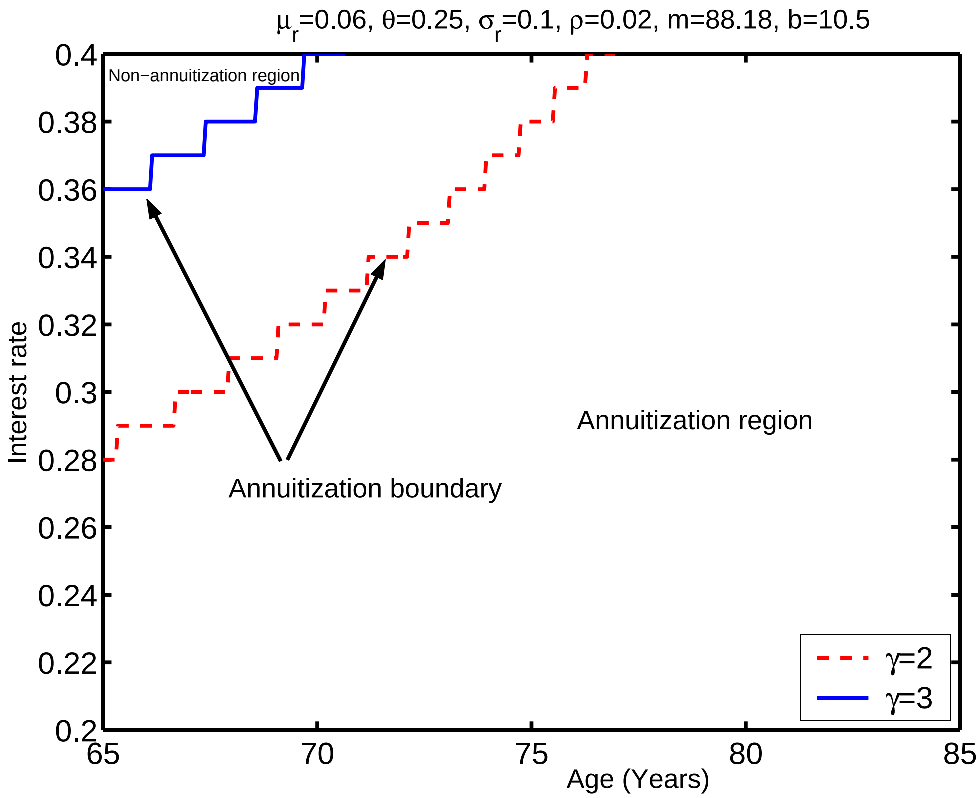
Mortality rate λ	Exponential mortality	Gompertz mortality
0.02	$\varnothing$	$r \leq 0.1575$
0.03	$r \leq 0.1120$	$r \leq 0.2400$
0.04	$r \leq 0.1920$	$r \leq 0.3000$
0.05	$r \leq 0.2560$	$r \leq 0.3500$
0.08	$r \leq 0.3867$	$\forall r$
$[0.10, \infty)$	$\forall r$	$\forall r$

Notes: The 2^{nd} column denotes the annuitization boundary for exponential mortality, the 3^{rd} column denotes the annuitization boundary for GM mortality.

We can observe that the annuitization boundary starts to emerge from age 65 and increases over time, and after some age, whre the mortality rate is much more higher, it becomes always optimal to annuitize for all the interest rates we are considering. These numerical results are consistent with our previous results for constant mortality rates.

On one hand, as time goes by, the individual with higher levels of risk aversion has higher annuitization boundary, which means that the individual tends to annuitize in a larger interest rate domain (from 0 to the annuitization boundary),

Figure 2.4: Free Boundaries for $\gamma = 2$ and $\gamma = 3$



which is consistent with our intuition: if it is optimal to annuitize for $\gamma = 2$, then it must be optimal to annuitize for $\gamma = 3$, but the opposite is not necessarily true. On the other hand, the riskless interest rate in reality seldom reaches 25 percent, so these numerical results merely mean that it is always optimal to annuitize if the risk premium is constant and there are no loads and no bequest motives.

Note that we have drawn the graph as staircases, which has been verified by both the LCP and the free boundary refining method, even for small time steps and space steps. There are two things that have contributed to this result. One is the assumption of constant spread between the risky asset and the riskless asset, which makes $\mu_s - r$ constant while r is a state variable. The other one is that when time changes, the mortality rate won't change big enough to move the free boundary during some period of time.

2.4.5.2 Annuitization Boundaries for Males and Females

Figure 2.5: Free Boundaries for Males and Females

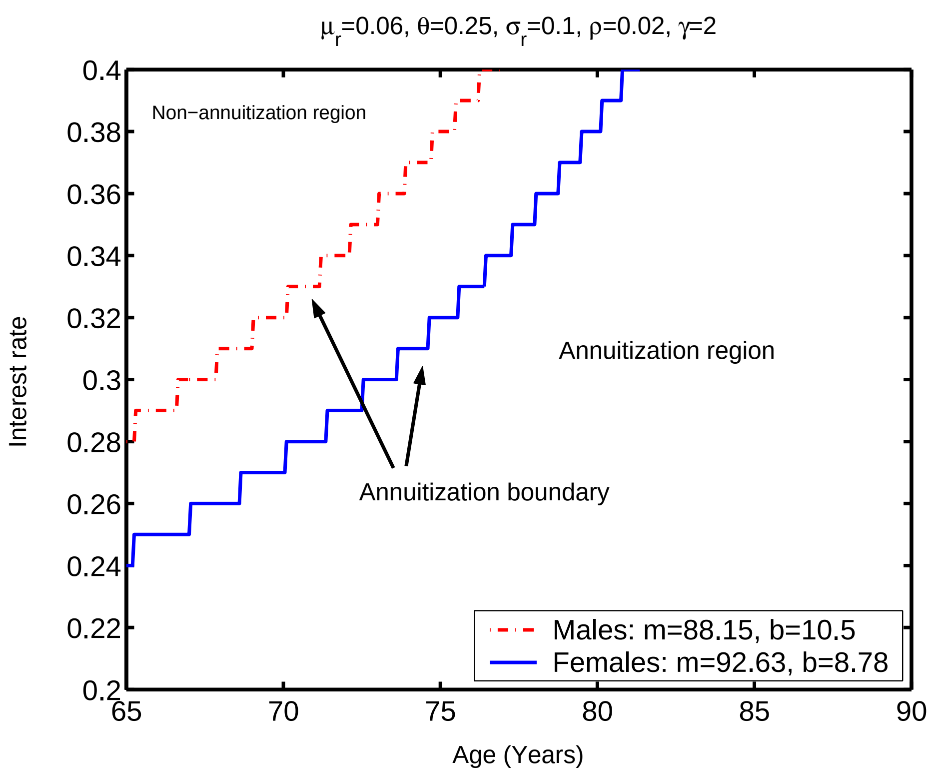


Figure 2.5 provides two annuitization boundaries for males ($m = 88.15, b = 10.5$)

and females ($m = 92.63, b = 8.78$). It can be observed from this figure that the annuitization boundary for males is always above the annuitization boundary for females because the mortality rate of males is higher at each given age. This is equivalent to saying that the annuitization domain for males is always greater than or equal to the annuitization domain for females, i.e., if it is optimal to annuitize for females, then it must be optimal to annuitize for males.

2.4.5.3 Sensitivity Analysis

Figure 2.6: Sensitivity Analysis of Parameter θ

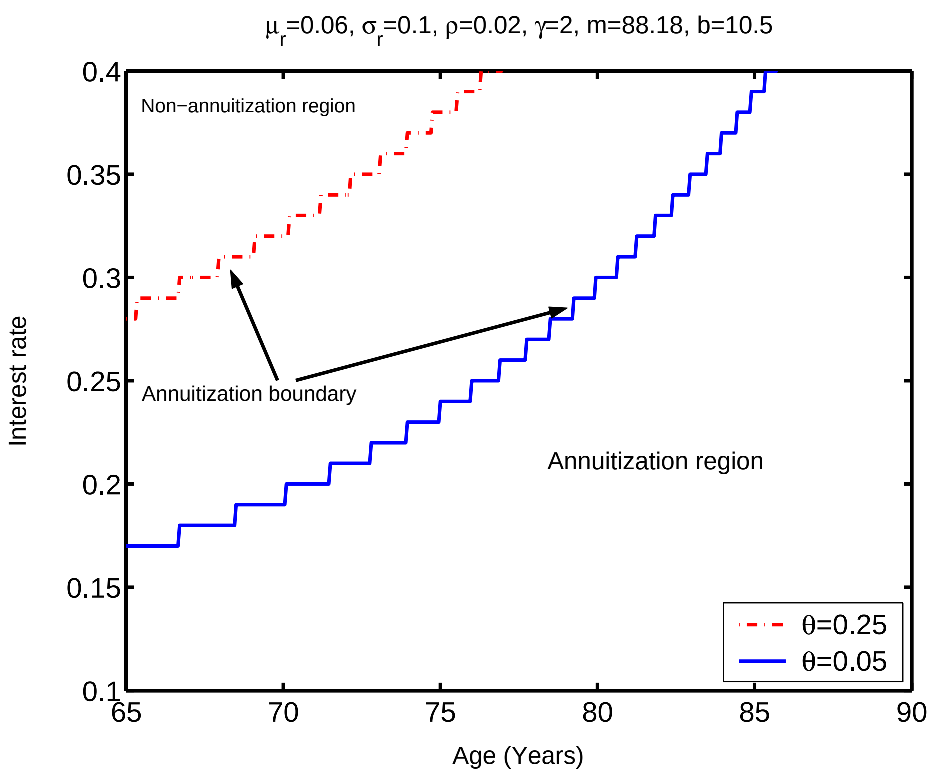
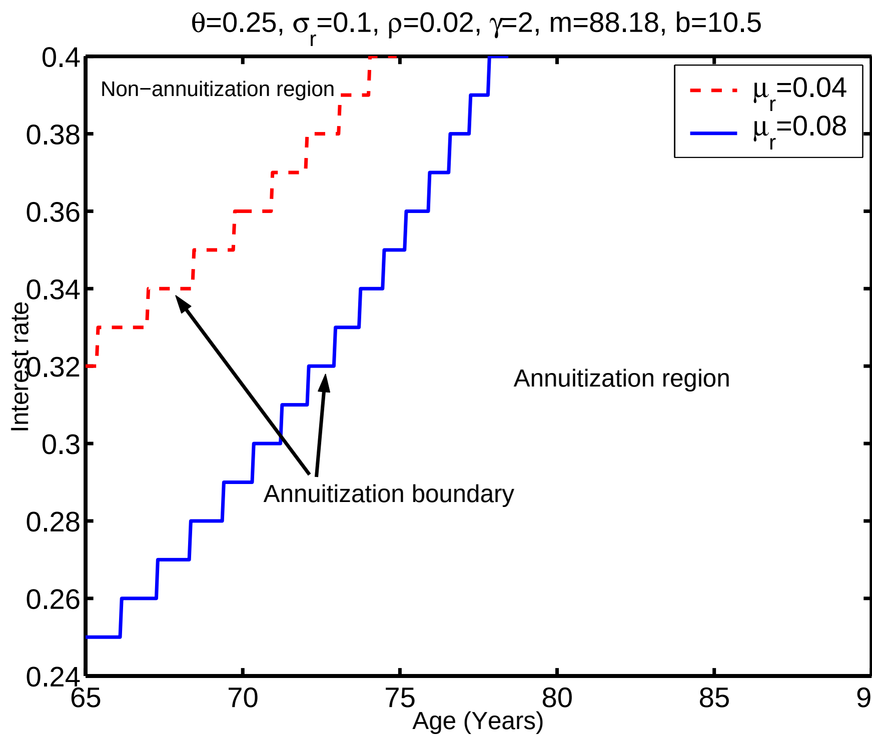
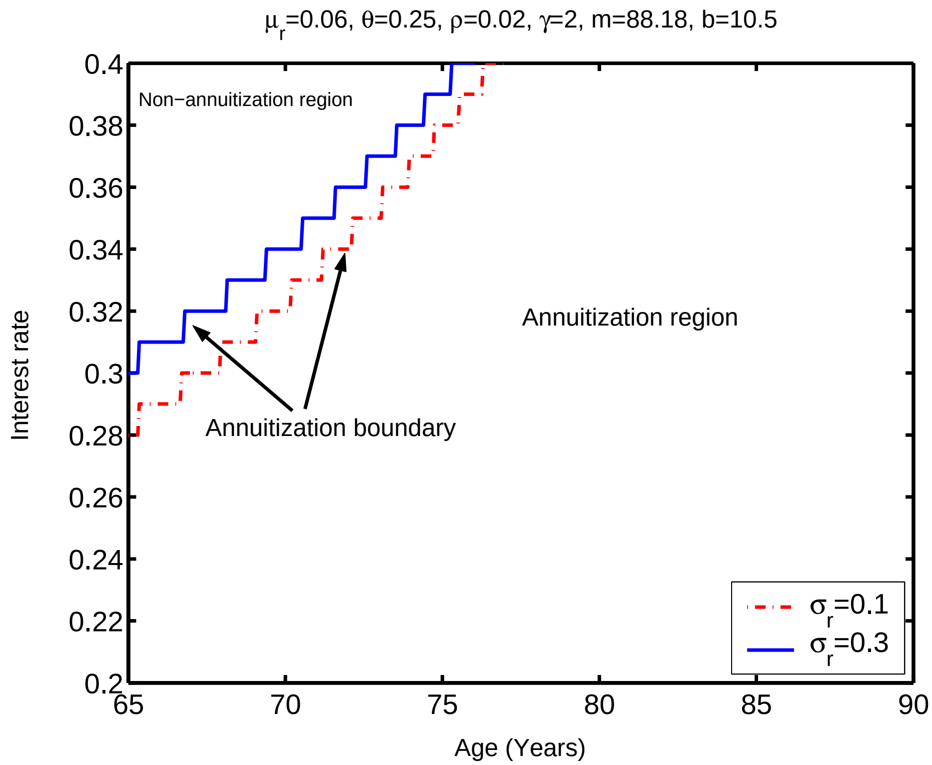


Figure 2.7: Sensitivity Analysis of Parameter μ_r



To further understand the behavior of the annuitization boundary, sensitivity analysis is performed to the three CIR parameters θ, μ_r and σ_r. From Figure 2.6, 2.7 and 2.8, we can see that higher adjustment speed, lower mean and higher volatility have higher annuitization boundaries. Firstly, when the adjustment speed is higher, which means that the interest rate will return to its long run mean sooner (with the current spot interest rate very high), individuals will be better off to annuitize immediately considering the high annuitization boundary. Secondly, when the long term rate drops, this moves interest rates down more quickly, which

Figure 2.8: Sensitivity Analysis of Parameter σ_r



cuts into the benefit of delaying. so there are fewer interest rates at which we delay, if we move the long-term down. Thirdly, the effect of the volatility σ_r is trivial because intuitively its value does not affect the value function significantly. Since the annuitization boundaries are very high compared to real interest rates, we can draw the conclusion that it is optimal to annuitize in reality.

2.5 Concluding Remarks

In this chapter, we have studied the optimal annuitization timing problem for a retired individual whose objective is to maximize his/her lifetime utility of consumption under a variety of institutional restrictions in the absence of bequest motives for exponential and Gompertz mortalities. There are two asset classes available to invest in the financial market, one is the risky asset and the other is the riskless asset.

First of all, we have calibrated two models for exponential mortality: constant and stochastic interest rates. When the interest rate is constant, analytic solutions for V^a and V^n can be obtained using mathematical techniques. If the interest rate equals the subjective discount rate, it is optimal to annuitize when the mortality rate is greater than Merton's constant, which is consistent with Milevsky and Young (2007). If the interest rate is not equal to the subjective discount rate, it is optimal to annuitize when the interest rate is small, and it is optimal not to annuitize when the interest rate is large. When the interest rate is stochastic, the annuity factor is much more complex since it involves the bond price which matures at a future time. It is shown that both V^a and V^n are independent of time t if the power term $e^{-(\rho+\lambda)t}$ is excluded. Numerical results show us that when the force of mortality λ is less than Merton's constant, it is always optimal not to annuitize. Otherwise, it is optimal to annuitize when the interest rate is small, and it is optimal not to annuitize when the interest rate is large. Another important observation is that the annuitization boundary is an increasing function of mortality rate λ. These important results for exponential mortality have shed light on the optimal annuitization timing problem under Gompertz mortality.

Secondly, we have modeled the optimal annuitization problem when the risk-free rate is driven by CIR process under Gompertz mortality. This is a free boundary problem, which is similar to the American put option problem. Its equivalent LCP problem is formulated and the projected SOR method is applied to solve it numerically. Due to the fact that the Gompertz mortality rate increases exponentially with time, the annuitization boundary is an increasing function of time, which echoes the results for exponential mortality when mortality rate is adjusted accordingly. One more finding is that the free boundaries are higher for Gompertz mortality than that of exponential mortality.

No matter the mortality rate is exponential or Gompertz, there is always an annuitization boundary for stochastic interest rate. This means that it is optimal not to annuitize even if r is high. One should delay annuitizing, earn short term interest, and once interest rates revert to a more realistic level, one will probably be able to buy more annuities than he/she would otherwise. In other words, the annuities he/she eventually buy will be more expensive, but he/she will be able to buy more of them and actually earn higher income.

Although we have used the more realistic Gompertz model, there are still some ways that we can improve it. One natural follow up would be to relax the assumption of the mortality rate to be stochastic. This view has been widely accepted since the advent of the stochastic mortality model proposed by Lee and Carter (1992). This complicates our model by introducing one more state variable in the HJB equation, and we leave it for further research in the future. Another natural extension of our model would be to incorporate various stocks, bonds and variable annuities, which would allow the individual to buy annuities in lump sums or continuously, instead of the all-or-nothing framework.

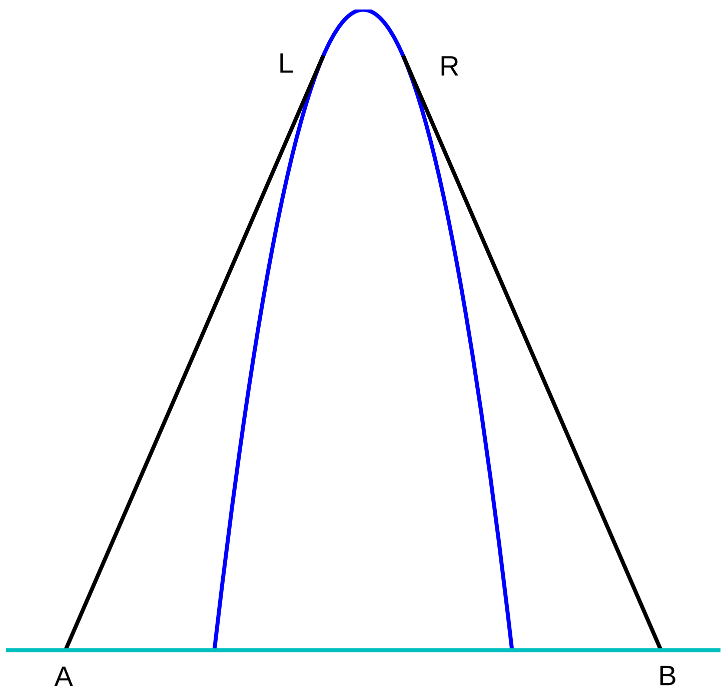
2.6 Appendix

2.6.1 The Obstacle Problem

In this section, the obstacle problem is introduced and illustrated. The free boundary problem and LCP problem corresponding to the obstacle problem are mostly adapted from Wilmott, Howison and Dewynne (1995).

An elastic string is held fixed at two ends, A and B, and passes over a smooth object which protrudes between the two ends (Figure 2.9). We do not know a priori the region of contact between the string and the obstacle, only that either

Figure 2.9: The Classical Obstacle Problem



The classical obstacle problem: the string is held fixed at A and B and must pass smoothly over the obstacle in between.

the string is in contact with the obstacle, in which case its position is known; or it must satisfy an equation of motion, which, in this case, says that it must be straight. This simply says that the string must lie above or on the obstacle, combined with the equation of motion, the curvature of the string must be negative or zero. In summary,

- the string must be above or on the obstacle;
- the string must have negative or zero curvature;
- the string must be continuous;
- the string slope must be continuous.

Under these constraints, the solution to the obstacle problem can be shown to be unique. The string and its slope are continuous, but in general the curvature of the string, and hence its second derivative, has discontinuities.

To derive the LCP illustration for the obstacle problem, we take the ends of string to be at $z = \pm 1$ and write $d(z)$ for the string displacement and $h_o(z)$ for the height of the obstacle, both for $-1 \leq z \leq 1$. We assume that $h_o(\pm 1) < 0$, and that $h_o(z) > 0$ at some points between -1 and $+1$, so that there definitely is a contact region. We also assume, at least initially, that $\frac{\partial^2 h_o}{\partial x^2} < 0$, thereby guaranteeing that there is only one contact region. The free boundary is then the set of points, marked as L ($z = z_L$) and R ($z = z_R$) in Figure 2.9, where the string first meets the obstacle. These are priori unknowns, and have to be determined as part of the solution.

In the contact region, $d = h_o$, where the string is not in contact with the obstacle it is straight, so $d'' = 0$. Normally, one would need just two boundary conditions to determine the straight portions of the string uniquely, and the values of d at the two ends of each straight portion would certainly do. However, because L and R are unknown, we need two more boundary conditions than usual in order to determine these points, and here a physical argument based on a force balance shows that at points such as L and R, d' must be continuous as well as d. Now we can write this particular example as the problem of finding $d(z)$ and the points L, R such that

$$\begin{aligned}
&d(-1) = 0, && \\
&d'' = 0, && -1 < z < z_L, \\
&d(z_L) = h_o(z_L), && d'(z_L) = f'(z_L), \\
&d(z) = h_o(z), && z_L < z < z_R, \\
&d(z_R) = h_o(z_R), && d'(z_R) = f'(z_R), \\
&d'' = 0, && z_R < z < 1, \\
&d(1) = 0.
\end{aligned} \tag{2.81}$$

Given any particular $h_o(z)$ it is straightforward in principle to show that $d(z)$, L and R are uniquely determined by this problem, and to find them. The procedure is tedious, and for all but specially simple h_o, L and R must be determined numerically as solutions of an algebraic or transcendental equation.

An alternative approach to the problem is to note that the string either lies above the obstacle, $d > h_o$, in which case it is straight, $d'' = 0$, or is in contact with the obstacle, $d = h_o$, in which case $d'' = h_o'' < 0$. This means that we can write the problem as what is call a linear complementarity problem

$$d'' \cdot (d - h_o) = 0, \quad -d'' \geq 0, \quad (d - h_o) \geq 0, \tag{2.82}$$

subject to the boundary conditions

$$d(-1) = d(1) = 0, \quad d, d' \text{are continuous.} \tag{2.83}$$

This statement of the problem has a tremendous advantage over the free boundary version (2.81) because there is no explicit mention of the free boundary points L and R. They are still present, but only implicity via the constraint $d \geq h_o$. If we can devise an algorithm to solve the constrained problem, we just have to look at the resulting values of $d - h_o$: the free boundaries are where this function switches from being zero to nonzero.

It is beyond the scope of this dissertation to prove that the LCP formulation is equivalent to the free boundary formulation, nor do we show that there is a unique solution to the former. The proofs use techniques of functional analysis, in particular the theory of variational inequalities, but the basic idea is simply minimization of the appropriate energy functional over the convex space of all suitably smooth functions $v(z)$ that satisfy the constraint $v \geq h_o$.

2.6.2 Algorithm

We give the algorithm for the LCP problem with $0 < \gamma < 1$ in this section. After the transformation $s = T - t$, how do we obtain $y^{n+1} = (y^{n+1,1}, y^{n+1,2}, \cdots, y^{n+1,I+1})$ from $y^n = (y^{n,1}, y^{n,2}, \cdots, y^{n,I+1})$? To answer this question, we need to fulfill the following five steps.

- step 1: Calculate the annuitization function at time level $n+1$, g^{n+1}.

- step 2: Using the upwind scheme, we obtain the boundary value for $r = 0$ at time level $n+1$ by:

$$\begin{aligned} y_1^{n+1} &= \left(\frac{\rho + \lambda^s_{x+T-s_n-0.5ds}}{2} - \frac{1-\gamma}{4\sigma_s^2\gamma}\mu_s^2 + \frac{\gamma}{\delta s}\right)^{-1} \\ & \quad \left[\left(\frac{1-\gamma}{4\sigma_s^2\gamma}\mu_s^2 - \frac{\rho + \lambda^s_{x+T-s}}{2}\right)y_1^n + \gamma + \frac{\gamma}{\delta s}y_1^n + ab\gamma\frac{y_2^n - y_1^n}{\delta r}\right]. \end{aligned}$$

- step 3: Given y^n, start with the initial guess $y_i^{n+1,1} = \sup(y_i^n, g_i^{n+1})$;

- step 4: In increasing i-indicial order, we calculate $\hat{y}^{n+1,2}$, its components are,

$$\begin{aligned} \hat{y}_i^{n+1,2} &= \left(c_1 + \frac{\gamma}{\delta s} + \frac{r_i\gamma\sigma_r^2}{2\delta r^2}\right)^{-1}\Big(-c_1 \times y_i^n + \frac{\gamma}{\delta s}y_i^n + \gamma \\ &+ c_2 \times (y_{i+1}^{n+1,1} + y_{i+1}^n - y_{i-1}^{n+1,2} - y_{i-1}^n) \\ &+ \frac{r_i\gamma\sigma_r^2}{4\delta r^2} \times (y_{i+1}^{n+1,1} + y_{i-1}^{n+1,2} + y_{i+1}^n + y_{i-1}^n - 2y_i^n)\Big). \end{aligned}$$

and let

$$y^{n+1,2} = \sup(y^{n+1,1} + \omega(\hat{y}^{n+1,2} - y^{n+1,1}), g^{n+1}) \tag{2.84}$$

where the coefficients c_1, c_2 are

$$\begin{aligned}
c_1 &= 0.5(\rho + \lambda^s_{x+T-s_n-0.5ds} - (1-\gamma)r_i - \frac{1-\gamma}{2\sigma_s^2\gamma}(\mu_s - r_i)^2); \\
c_2 &= (c_3 + \theta(\mu_r - r_i)\gamma + \frac{1-\gamma}{\sigma_s^2\gamma}(\mu_s - r_i)\rho_{rs}\sigma_r\sigma_s\sqrt{r}_i\gamma)/4/dr; \\
c_3 &= \frac{r_i\sigma_r^2\gamma(\gamma-1)(y^n_{i+1} - y^n_{i-1})}{4\delta r y^n_i} + \frac{(1-\gamma)\rho_{rs}^2\sigma_r^2\sigma_s^2 r_i\gamma^2(y^n_{i+1} - y^n_{i-1})}{4\sigma_s^2\gamma\delta r y^n_i}, if \quad y^n_i \neq 0; \\
c_3 &= 0, \quad if \quad y^n_i = 0.
\end{aligned}$$

- step 5: Test whether the error $||y^{n+1,2} - y^{n+1,1}||$ is small enough. If yes, put $y^{n+1,1} = y^{n+1,2}$, this is the solution we are seeking. Otherwise let $y^{n+1,1} = y^{n+1,2}$ and return to step 4.

2.6.3 Free Boundary Refining Method to Find the Free Boundary for GM Mortality with Constrained Consumption after Annuitization

In this section, we will illustrate an alternative way (we refer it as free boundary refining method) to find the free boundary for GM mortality, i.e., for any fixed time t, we need to find a critical interest rate, under which it is optimal to annuitize and above which it is optimal not to annuitize. To achieve the specified goal, we need to find the value function $V(t, w, r)$ in which it is optimal not to annuitize and the annuitization function V^a in which it it optimal to annuitize. Then we compare them to find the original free boundary and check if their derivatives are equal. We need to move this free boundary to the left and set the value functions equal on the new point and then check the derivatives again until we obtain a point where both value functions and their derivatives are equal.

If we assume the retiree does not annuitize at time t, the value function $V(t, w, r)$ is defined by

$$V(t, w, r) = \sup_{\pi_s, c_s, \tau} E\left[\int_t^\tau e^{-\rho(s-t)} \ {}_{s-t}p_{x+t}^s u(c_s)\, ds\right]. \tag{2.85}$$

The HJB equation that $V(t, w, r)$ must solve is

$$(\rho + \lambda_{x+t})V = V_t + \sup_{c,\pi} \mathfrak{L}V, \tag{2.86}$$

where the second-order differential operator $\mathfrak{L}V$ is defined by

$$\begin{aligned} \mathfrak{L}V &= u(c) + (rw + (\mu_s - r)\pi - c)V_w + \theta(\mu_r - r)V_r + \tfrac{1}{2}\sigma_s^2\pi^2 V_{ww} \\ &+ \tfrac{1}{2}r\sigma_r^2 V_{rr} + \rho_{rs}\sigma_r\sigma_s\pi\sqrt{r}V_{wr}. \end{aligned} \tag{2.87}$$

Let $V(t, w, r) = \frac{w^{1-\gamma}}{1-\gamma}h(t, r)$, $h(t, r) = y(t, r)^\gamma$, $\bar{y}(s, r) = y(t, r)$, where $s = T - t$. So $\bar{y}(s, r)$ satisfies the following equation

$$(\rho + \lambda_{x+T-s}^s)\bar{y} + \gamma\bar{y}_s + \mathfrak{L}\bar{y} = 0, \tag{2.88}$$

Where $\mathfrak{L}\bar{y}$ is defined as

$$\begin{aligned}\mathfrak{L}\bar{y} = \; & -(1-\gamma)r\bar{y} - \gamma - \theta(\mu_r - r)\gamma\bar{y}_r - \tfrac{1-\gamma}{2\sigma_s^2\gamma}(\mu_s - r)^2\bar{y} - \tfrac{1}{2}r\sigma_r^2\gamma(\gamma-1)\tfrac{\bar{y}_r^2}{\bar{y}} \\ & -\tfrac{1}{2}r\gamma\sigma_r^2\bar{y}_{rr} - \tfrac{1-\gamma}{2\sigma_s^2\gamma}\rho_{rs}^2\sigma_r^2\sigma_s^2 r\gamma^2\tfrac{\bar{y}_r^2}{\bar{y}} - \tfrac{1-\gamma}{\sigma_s^2\gamma}(\mu_s - r)\rho_{rs}\sigma_r\sigma_s\sqrt{r}\gamma\bar{y}_r.\end{aligned} \tag{2.89}$$

The boundary conditions for the above PDE are

$$\begin{aligned}& r = r_{max} : \bar{y}_{rr} = 0; \\ & r = 0 : (\rho + \lambda^s_{x+T-s})\bar{y}(s,0) - \tfrac{1-\gamma}{2\sigma_s^2\gamma}\mu_s^2\bar{y}(s,0) + \gamma\bar{y}_s(s,0) - \gamma - \theta\mu_r\gamma\bar{y}_r(s,0) = 0.\end{aligned} \tag{2.90}$$

These two boundary conditions have been used many times before and readers can refer to Section 2.3 for its detailed explanation.

Now let's look at the initial condition at $s = 0$ $(t = T)$. It is very complicated so we must be cautious. Note the assumption that the hazard rate is constant in domain $[T, \infty)$, therefore the value functions have nothing to do with time t if the time term $e^{-(\rho+\lambda)t}$ is excluded. We will be able to find the critical interest rate r^* and a time-independent function $\bar{y}(r)$ using the same technique as we have applied in Section 2.3. Please note that when $r \geq r^*$, it is optimal not to annuitize, and when $r \leq r^*$, it is optimal to annuitize. So that the initial condition for $\bar{y}(s,r)$ is derived to be

$$\bar{y}(0,r) = e^{-\frac{\rho+\lambda}{\gamma}T}\bar{y}(r). \tag{2.91}$$

To solve the second-order nonlinear PDE of $\bar{y}$, the following quotients are applied for a second-order accuracy.

$$
\begin{aligned}
\bar{y}(s_{n+\frac{1}{2}}, r_i) &= \frac{\bar{y}_i^{n+1} + \bar{y}_i^n}{2}; \\
\bar{y}_s(s_{n+\frac{1}{2}}, r_i) &= \frac{\bar{y}_i^{n+1} - \bar{y}_i^n}{\delta s}; \\
\bar{y}_r(s_{n+\frac{1}{2}}, r_i) &= \frac{\bar{y}_{i+1}^{n+1} + \bar{y}_{i+1}^n - \bar{y}_{i-1}^{n+1} - \bar{y}_{i-1}^n}{4\delta r}; \\
\bar{y}_{rr}(s_{n+\frac{1}{2}}, r_i) &= \frac{\bar{y}_{i+1}^{n+1} + \bar{y}_{i-1}^{n+1} - 2\bar{y}_i^{n+1} + \bar{y}_{i+1}^n + \bar{y}_{i-1}^n - 2\bar{y}_i^n}{2\delta r^2}.
\end{aligned}
$$

Substituting them into equation (2.88), we obtain

$$
\begin{aligned}
\bar{y}_i^{n+1} &= (c_1 + \tfrac{\gamma}{\delta s} + \tfrac{r_i \gamma \sigma_r^2}{2\delta r^2})^{-1}\{-c_1 * \bar{y}_i^n + c_2 * (\bar{y}_{i+1}^{n+1} + \bar{y}_{i+1}^n - \bar{y}_{i-1}^{n+1} - \bar{y}_{i-1}^n) \\
&+ \tfrac{\gamma}{\delta s}\bar{y}_i^n + \gamma + \tfrac{r_i \gamma \sigma_r^2}{4\delta r^2} * (\bar{y}_{i+1}^{n+1} + \bar{y}_{i-1}^{n+1} + \bar{y}_{i+1}^n + \bar{y}_{i-1}^n - 2\bar{y}_i^n)\},
\end{aligned}
\tag{2.92}
$$

in which c_1 and c_2 are given by

$$
\begin{aligned}
c_1 &= 0.5(\rho + \lambda_{x+T-s}^s - (1-\gamma) r_i - \tfrac{1-\gamma}{2\sigma_s^2 \gamma}(\mu_s - r_i)^2); \\
c_2 &= (c_3 + \theta(\mu_r - r_i)\gamma + \tfrac{1-\gamma}{\sigma_s^2 \gamma}(\mu_s - r_i)\rho_{rs}\sigma_r\sigma_s\sqrt{r}_i\gamma)/4/\delta r,
\end{aligned}
\tag{2.93}
$$

and where

$$
c_3 = \begin{cases} \frac{r_i \sigma_r^2 \gamma(\gamma-1)(\bar{y}_{i+1}^n - \bar{y}_{i-1}^n)}{4\delta r \bar{y}_i^n} + \frac{(1-\gamma)\rho_{rs}^2 \sigma_r^2 \sigma_s^2 r_i \gamma^2 (\bar{y}_{i+1}^n - \bar{y}_{i-1}^n)}{4\delta r \sigma_s^2 \gamma \bar{y}_i^n}, & \text{if} \quad \bar{y}_i^n \neq 0; \\ c_3 = 0, & \text{if} \quad \bar{y}_i^n = 0. \end{cases}
\tag{2.94}
$$

This is reasonable because $\bar{y}$ equals 0 if and only if at time $s = 0$ and the value $\frac{\bar{y}_r}{\bar{y}}$

is very close to 0 at $\bar{y}=0$.

$$\begin{aligned}
s=0 \quad &: \quad \bar{y}_i^1 = 0,\ i=1,2,\cdots,I+1; \\
r=r_{max} \quad &: \quad \bar{y}_{I+1}^n + \bar{y}_{I-1}^n - 2\bar{y}_I^n = 0 \Longrightarrow \bar{y}_{I+1}^n = 2\bar{y}_I^n - \bar{y}_{I-1}^n; \qquad (2.95) \\
r=0 \quad &: \quad \left(\frac{\rho+\lambda_{x+T-s_n-0.5ds}^s}{2} - \frac{1-\gamma}{4\sigma_s^2\gamma}\mu_s^2 + \frac{\gamma}{\delta s}\right)\bar{y}_1^{n+1} \\
&= \quad \left(\frac{1-\gamma}{4\sigma_s^2\gamma}\mu_s^2 - \frac{\rho+\lambda_{x+T-s_n-0.5ds}^s}{2}\right)\bar{y}_1^n + \gamma + \frac{\gamma}{\delta s}\bar{y}_1^n + \theta\mu_r\gamma\frac{\bar{y}_2^n - \bar{y}_1^n}{\delta r}.
\end{aligned}$$

The annuitization function $V^a = \frac{w^{1-\gamma}}{1-\gamma} g(t,r)$, where $g(t,r)$ is defined as

$$g(t,r) = \frac{1}{(\bar{a}_{x+t}^o)^{1-\gamma}} \int_t^\infty e^{-\rho(s-t)} e^{-\int_t^s \lambda_{x+t}^s dv} ds. \qquad (2.96)$$

We make a transformation $t = T-s$, and define $\bar{g}(s,r) = g(T-s,r)$ to compare it with $\bar{y}$.

2.6.4 Strategies to Find the Free Boundary

The hazard rate is assumed to follow a modified GM mortality.

$$\lambda_{x+t} = \begin{cases} \frac{1}{b} e^{\frac{x+t-m}{b}}, & \text{if} \quad t < t_{max} \\ \lambda_{x+t_{max}}, & \text{if} \quad t \geq t_{max}. \end{cases} \qquad (2.97)$$

This is a reasonable assumption because when $t \geq t_{max}$, the mortality rate is very large, which means that the probability of surviving to that age is negligible. Therefore, the constant force of mortality after t_{max} is resonable.

To compute the free boundary, the first step is to calculate the terminal condition at $t = t_{max}$ (the initial condition at $s=0$). Due to the mortality assumption, we can calculate the value of $\bar{y}$ by the same method as before. Note that $\bar{y}(s,r) = e^{-\frac{\rho+\lambda}{\gamma}(T-s)}\tilde{y}(r)$.

Suppose we have known the free boundary at time level $s(n)$. How can we obtain the free boundary at time level $s(n+1)$? First, we calculate $y(n+1,:)$ by the projected SOR method using $y(n,:)$, and obtain $V_n(n+1,:)$. Second, we compare $V_n(n+1,:)$ and $V^a(n+1,:)$ to see if there exists an r^*, in which we have $V_n(n+1,r^*) = V^a(n+1,r^*)$. If yes, we then compare their derivatives w.r.t. time t. If their derivatives are equal, then r^* is the free boundary we are looking for. Otherwise, move r^* leftward, and repeat the above procedure by replacing the boundary condition at the new point r^* to be V^a.

3 Optimal Annuitization Timing and Optimal Consumption

3.1 Introduction

In this chapter, we investigate the annuitization problem for a retired individual whose objective is to maximize his/her lifetime utility after retirement with the optimal consumption strategy, instead of what we have done in the previous chapter, where we assumed that the consumption rate is equal to the annuity payout. We also assume that this individual only has initial wealth in the form of a lump sum cash amount, and does not come pre-annuitized with a pre-existing pension or social security and has no remaining lifetime income. To calculate the optimal consumption rate, we assume that this rate is a fraction (α_t) of the annuity income A_t, and the remainder $(1-\alpha_t)A_t$ is used to purchase more annuities at each time t without management fees. Two different mortality models, exponential and GM mortality, are calibrated to study the optimal control problem in a similar way as what we have done in the previous chapter.

The rest of this chapter is organized as follows. Section 3.2 studies the optimal control problem under exponential mortality for both constant and stochastic interest rates. In section 3.3 GM mortality under stochastic interest rate is investigated to see its effect on the value function. Finally, conclusions are addressed in section 3.4.

3.2 Model Calibration 1: Exponential Mortality

In this section, the force of mortality is assumed to be constant λ, which allows us to find the analytic solutions of the value functions V^a and V^n with much greater ease. Comparison of the two value functions shows us that it is always optimal to annuitize no matter what the interest rates are, which differs from the numerical results we have obtained in the previous chapter. The reason lies in the fact that the optimal consumption strategy has been executed, which leads to the value function V^a to be much higher than the previous one with full annuity income consumption. Next we document this optimal control problem for two different types of interest rates (constant and stochastic) to obtain the optimal annuitization strategy for the retired individual.

3.2.1 Constant Interest Rates

In this subsection, we study the annuitization problem for an idealized interest rate case, i.e., constant, which means that the return of the riskless asset is invariant over time. It is known that when the force of mortality is constant, the associated value function is independent of time with full consumption after annuitization, but this is not true under the optimal consumption strategy any more because this strategy depends on time. Since the purpose of this section is to gain useful insight into the optimal annuitization strategy, it is enough to investigate our problem at time zero (age x) for simplicity. Next we will study the two value functions, with and without annuitization (V^a and V^n), and compare them to obtain the optimal annuitization strategy at time 0.

- **The value function with annuitization under the optimal consumption strategy**

  The purpose of this subsection is to find out the optimal consumption strategy at retirement applying the calculus of variations (CV) method if the individual chooses to annuitize at time zero, and then obtain the closed-form solution of the associated value function (see appendix 3.5.1 for the consistency verification using dynamic programming techniques).

  First we look at the discounted lifetime utility of consumption the retiree is seeking to maximize, which is defined as

$$V(w) = \sup_{c_t} \int_0^\infty e^{-(\rho+\lambda)t} u(c_t) dt, \tag{3.1}$$

in which ρ is the subjective discount factor, λ is the constant force of mortality and c_t is the consumption rate. Notice that the mortality rate is high when individuals are getting older, all people will die after some time T. Therefore we will consider a finite domain $[0, T]$ since the integral of the value function from T to ∞ is zero. We first look at this value function with annuitization (denoted as V^a) at age x under the optimal consumption strategy. We assume that the consumption rate is a fraction ($0 \leq \alpha_t \leq 1$) of the annuity income A_t, i.e.,

$$c_t = \alpha_t A_t, \tag{3.2}$$

where α_t is time varying. Note A_t is the only annuity income after annuitization because there is no pre-existing pension or social security. The remainder $(1 - \alpha_t)A_t$ is used to purchase more annuities, so A_t satisfies the following first-order linear ordinary differential equation (ODE)

$$\frac{dA_t}{dt} = \frac{(1 - \alpha_t)A_t}{\bar{a}_{x+t}}, \tag{3.3}$$

in which $\bar{a}_{x+t}$ is the actuarial annuity factor at time t, i.e., age $x + t$. This annuity factor is a constant $(\frac{1}{\lambda+r})$ when both interest rates and the force of mortality are constant. To apply the CV method to obtain the optimal consumption strategy, we rewrite the above ODE of A_t as

$$\alpha_t A_t = A_t - \frac{\dot{A}_t}{\lambda + r}. \tag{3.4}$$

Here the dot denotes the derivative with respect to time t. Substitute into the discounted lifetime utility function (3.1), then V becomes a function of A_t, and it takes the following form

$$V^a(A_t) = \int_0^T e^{-(\rho+\lambda)t} u(A_t - \frac{\dot{A}_t}{\lambda + r}) dt. \tag{3.5}$$

Let $\phi(t, A_t, \dot{A}_t) = e^{-(\rho+\lambda)t} u(A_t - \frac{\dot{A}_t}{\lambda+r})$, we see that ϕ is a functional of function A_t. Next we seek to find a particular path A_t from time zero to T so that the integral reaches its maximum value. First we add a perturbation δA_t to A_t and expand V^a using Taylor expansion

$$\begin{aligned} V^a(A_t + \delta A_t) &= \int_0^T e^{-(\rho+\lambda)t} u(A_t + \delta A_t - \frac{\dot{A}_t + \delta \dot{A}_t}{\lambda + r}) dt \\ &= \int_0^T (\phi(t, A_t, \dot{A}_t) + \frac{\partial \phi}{\partial A_t} \delta A_t + \frac{\partial \phi}{\partial \dot{A}_t} \delta \dot{A}_t + O(\delta A_t)) dt, \end{aligned} \tag{3.6}$$

in which notation $O(\delta A_t)$ means higher order with respect to δA_t, i.e., it goes to zero faster than δA_t as δA_t approaches zero. Therefore, we have

$$V^a(A_t + \delta A_t) - V^a(A_t) = \int_0^T (\frac{\partial \phi}{\partial A_t} \delta A_t + \frac{\partial \phi}{\partial \dot{A}_t} \delta \dot{A}_t + O(\delta A_t)) dt. \tag{3.7}$$

Applying integration by parts, we obtain

$$V^a(A_t + \delta A_t) - V^a(A_t) = \int_0^T (\frac{\partial \phi}{\partial A_t}\delta A_t - \frac{d}{dt}\frac{\partial \phi}{\partial \dot{A}_t}\delta A_t + O(\delta A_t))dt + \frac{\partial \phi}{\partial \dot{A}_t}\delta A_t|_0^T. \tag{3.8}$$

Since A_0 is given, $\delta A_0 = 0$. So we have

$$V^a(A_t + \delta A_t) - V^a(A_t) = \int_0^T (\frac{\partial \phi}{\partial A_t}\delta A_t - \frac{d}{dt}\frac{\partial \phi}{\partial \dot{A}_t}\delta A_t + O(\delta A_t))dt + \frac{\partial \phi}{\partial \dot{A}_t}|_{t=T}\delta A_T. \tag{3.9}$$

The assumption of no bequest motives leads to zero wealth at the horizon, so the fraction of consumption is 100% at time T, meaning that there is no annuity income left to purchase more annuities. Therefore the boundary condition becomes $\frac{dA_t}{dt}|_{t=T} = 0$. So we have

$$\frac{\partial \phi}{\partial \dot{A}_t}|_{t=T} = (-\frac{e^{-(\rho+\lambda)t}}{\lambda + r}(A_t - \frac{\dot{A}_t}{\lambda + r}))^{-\gamma}|_{t=T}. \tag{3.10}$$

This term approaches zero since ρ and λ are both positive numbers, and A_T is bounded. Therefore the necessary condition for the integral to reach its maximum is given by the Euler-Lagrange equation

$$\frac{d}{dt}(A_t - \frac{\dot{A}_t}{\lambda + r}) = \frac{r - \rho}{\gamma}(A_t - \frac{\dot{A}_t}{\lambda + r}). \tag{3.11}$$

After some mathematical manipulation, A_t must satisfy the following second-order linear homogenous differential equation over the values for which $A_t \neq 0$.

$$\ddot{A}_t - (\frac{r - \rho}{\gamma} + \lambda + r)\dot{A}_t + \frac{r - \rho}{\gamma}(\lambda + r)A_t = 0 \tag{3.12}$$

in domain $[0, T]$. The coefficients in this equation are time independent and the method of undetermined coefficients can be used to find the general solution. Note that the two roots of the characteristic equation

$$z^2 - (\frac{r-\rho}{\gamma} + \lambda + r)z + \frac{r-\rho}{\gamma}(\lambda + r) = 0 \tag{3.13}$$

are $\frac{r-\rho}{\gamma}$ and $\lambda + r$, so the general solution to ODE (3.12) is

$$A_t = k_1 e^{\frac{r-\rho}{\gamma}t} + k_2 e^{(\lambda+r)t}. \tag{3.14}$$

To obtain the analytic solution for A_t, we impose the terminal boundary condition at a large enough time T to be $\frac{\partial A_t}{\partial t}|_{t=T} = 0$. This is reasonable because people die at a finite age and the integral for the value function after T is neglectable. To solve the two free constants k_1 and k_2, we apply the initial condition A_0 (known). In mathematics, we have

$$\begin{cases} k_1 + k_2 = A_0, \\ k_1 \frac{r-\rho}{\gamma} e^{(\frac{r-\rho}{\gamma})T} + k_2(\lambda + r)e^{(\lambda+r)T} = 0. \end{cases} \tag{3.15}$$

After some algebraic manipulations, we have

$$\begin{cases} k_1 = \frac{A_0\gamma(\lambda+r)e^{(\lambda+r)T}}{\gamma(\lambda+r)e^{(\lambda+r)T} - (r-\rho)e^{\frac{r-\rho}{\gamma}T}}, \\ k_2 = \frac{A_0(\rho-r)e^{\frac{r-\rho}{\gamma}T}}{\gamma(\lambda+r)e^{(\lambda+r)T} - (r-\rho)e^{\frac{r-\rho}{\gamma}T}}. \end{cases} \tag{3.16}$$

Then the fraction of consumption rate α_t is deterministic and given by

$$\alpha_t = \frac{\gamma(\lambda + r) + \rho - r}{\gamma(\lambda + r) + (\rho - r)e^{(\lambda + r - \frac{r-\rho}{\gamma})(t-T)}}. \tag{3.17}$$

It can be easily observed that α_t is a monotonic function of time t. After applying the constrain $0 \le \alpha_t \le 1$, we obtain the optimal consumption strategy $\alpha_t^\star$ for the retiree for $\gamma \ge 1$

$$\alpha_t^\star = \begin{cases} 1, & r \in (0, \rho], \\ \frac{\gamma(\lambda+r)+\rho-r}{\gamma(\lambda+r)+(\rho-r)e^{(\lambda+r-\frac{r-\rho}{\gamma})(t-T)}}, & r \in [\rho, \infty). \end{cases} \tag{3.18}$$

We see that when the interest rate is less than the subjective discount factor, it is optimal to consume all the annuity income. When the interest rate is greater than the subjective discount factor, it is optimal to consume part of the annuity income depending on time t. This consumption ratio is an increasing function of time t, i.e., it gradually increases to 100 percent upon the decease of the individual. This optimal strategy is consistent with the case in which the consumption ratio α_t is constant, which is left in the appendix.

Intuitively, it is possible that α_t will hit 1 when $t = t^\star < T$ and then stay over the interval $[t^\star, T]$. Below we prove that this scenario will never happen in practice. To this end, we take $t^\star$ as a parameter, compute the corresponding value function $V^a(t^\star)$ for $t^\star \in [0, T]$ and then find that the critical value of V^a always occurs at time $t = T$ through first-order condition.

First, we write the value function V^a as a function of $t^\star$,

$$V^a(t^\star) = \sup_{\alpha_t} E\left[\int_0^{t^\star} e^{-(\rho+\lambda)t} u(\alpha_t A_t)dt + \int_{t^\star}^{T} e^{-(\rho+\lambda)t} u(A_{t^\star})dt)\right]. \tag{3.19}$$

Note that the consumption ratio α_t is always equal to 1 over the last time period $t \in [t^\star, T]$, which means that the annuity income will not change in this time interval. Hence we have the following expression for A_t

$$\begin{cases} A_t - \frac{\dot{A}_t}{\lambda+r} = \frac{A_0(\gamma(\lambda+r)-r+\rho)e^{(\lambda+r)t^\star}e^{\frac{r-\rho}{\gamma}t}}{\gamma(\lambda+r)e^{(\lambda+r)t^\star}+(\rho-r)e^{\frac{r-\rho}{\gamma}t^\star}}, & t \in [0, t^\star], \\ A_t = \frac{A_0(\gamma(\lambda+r)-r+\rho)e^{\frac{r-\rho}{\gamma}t^\star}e^{(\lambda+r)t^\star}}{\gamma(\lambda+r)e^{(\lambda+r)t^\star}+(\rho-r)e^{\frac{r-\rho}{\gamma}t^\star}}, & t \in [t^\star, T]. \end{cases} \tag{3.20}$$

Substituting them into equation (3.19), we have

$$\begin{aligned} V^a(t^\star) &= -\frac{A_0^{1-\gamma}(\gamma(\lambda+r)-r+\rho)^{1-\gamma}e^{(1-\gamma)(\lambda+r)t^\star}}{(1-\gamma)(\gamma(\lambda+r)e^{(\lambda+r)t^\star}+(\rho-r)e^{\frac{r-\rho}{\gamma}t^\star})^{1-\gamma}}\times \\ &\quad (\frac{e^{-(\lambda+(1-\frac{1}{\gamma})r+\frac{1}{\gamma}\rho)t^\star}-1}{\lambda+(1-\frac{1}{\gamma})r+\frac{1}{\gamma}\rho} + \frac{(e^{-(\rho+\lambda)T}-e^{-(\rho+\lambda)t^\star})e^{(\frac{1}{\gamma}-1)(r-\rho)t^\star}}{\rho+\lambda}). \end{aligned} \tag{3.21}$$

conditional on $\lambda + r + \frac{1}{\gamma}(\rho - r) > 0$. Denote $M(t^\star)$ to be

$$M(t^\star) = \frac{1}{\gamma-1}(\frac{e^{-(\lambda+(1-\frac{1}{\gamma})r+\frac{1}{\gamma}\rho)t^\star}-1}{\lambda+(1-\frac{1}{\gamma})r+\frac{1}{\gamma}\rho} + \frac{(e^{-(\rho+\lambda)T}-e^{-(\rho+\lambda)t^\star})e^{(\frac{1}{\gamma}-1)(r-\rho)t^\star}}{\rho+\lambda}), \tag{3.22}$$

then $V^a(t^\star)$ can be written as

$$V^a(t^\star) = \frac{A_0^{1-\gamma}(\gamma(\lambda+r)-r+\rho)^{1-\gamma}}{(\gamma(\lambda+r)+(\rho-r)e^{(\frac{r-\rho}{\gamma}-\lambda-r)t^\star})^{1-\gamma}}M(t^\star). \tag{3.23}$$

Note that the fraction before $M(t^\star)$ is a monotonically increasing function of $t^\star$ for $\gamma > 1$, it must attain its maximum value at time $t = T$. Below we will verify that the maximum value of $M(t^\star)$ also occurs at time $t = T$. To this end, we write the first derivative of $M(t^\star)$ with respect to $t^\star$ as

$$\begin{aligned} \frac{\partial M(t^\star)}{\partial t^\star} &= \frac{1}{\gamma-1}\frac{(\frac{1}{\gamma}-1)(r-\rho)}{\rho+\lambda}e^{(\frac{1}{\gamma}-1)(r-\rho)t^\star}(e^{-(\rho+\lambda)T}-e^{-(\rho+\lambda)t^\star}) \\ &= \frac{\rho-r}{\gamma(\rho+\lambda)}e^{(\frac{1}{\gamma}-1)(r-\rho)t^\star}(e^{-(\rho+\lambda)T}-e^{-(\rho+\lambda)t^\star}). \end{aligned} \tag{3.24}$$

No matter whether the interest rate r is greater or less than the subjective factor ρ, the term $\frac{\rho-r}{\gamma(\rho+\lambda)}e^{(\frac{1}{\gamma}-1)(r-\rho)t^\star}$ is an increasing function of $t^\star$, therefore the maximum value of $M(t^\star)$ is attained at time $t^\star = T$ by its first-order condition. This is equivalent to saying that $\alpha_t = 1$ before T is not optimal.

Therefore, the closed-form solution for the value function V^a for $\gamma \geq 1$ can be written as

$$V^a = \begin{cases} u(\frac{w}{\bar{a}_x(0,r)})\bar{a}_x(0,\rho), & r \in (0,\rho], \\ -\frac{w^{1-\gamma}(\lambda+r)^{1-\gamma}(\gamma(\lambda+r)-r+\rho)^{1-\gamma}}{(1-\gamma)(\gamma(\lambda+r)+(\rho-r)e^{(\frac{r-\rho}{\gamma}-\lambda-r)T})^{1-\gamma}}(\frac{e^{-(\lambda+(1-\frac{1}{\gamma})r+\frac{1}{\gamma}\rho)T}-1}{\lambda+(1-\frac{1}{\gamma})r+\frac{1}{\gamma}\rho}), & r \in [\rho,\infty). \end{cases} \tag{3.25}$$

This is due to the fact that in domain $(0,\rho]$, the optimal consumption strategy is $\alpha_t \equiv 1$, so the value function V^a is obtained from section 2.3.1.1 in chapter 2.

- **Optimal annuitization strategy**

In this section, we compare the two value functions V^a and V^n to achieve the optimal annuitization strategy at retirement for γ greater than 1. The analytic solution of V^a for $T = \infty$ (assuming $\lambda + r > \frac{r-\rho}{\gamma}$) can be simplified to

$$V^a = \begin{cases} u(\frac{w}{\bar{a}_x(0,r)})\bar{a}_x(0,\rho), & r \in (0,\rho], \\ \frac{w^{1-\gamma}\gamma^\gamma}{(1-\gamma)(\gamma(\lambda+r)+\rho-r)^\gamma}, & r \in [\rho,\infty). \end{cases} \tag{3.26}$$

The analytic solution for the value function without annuitization, V^n, for

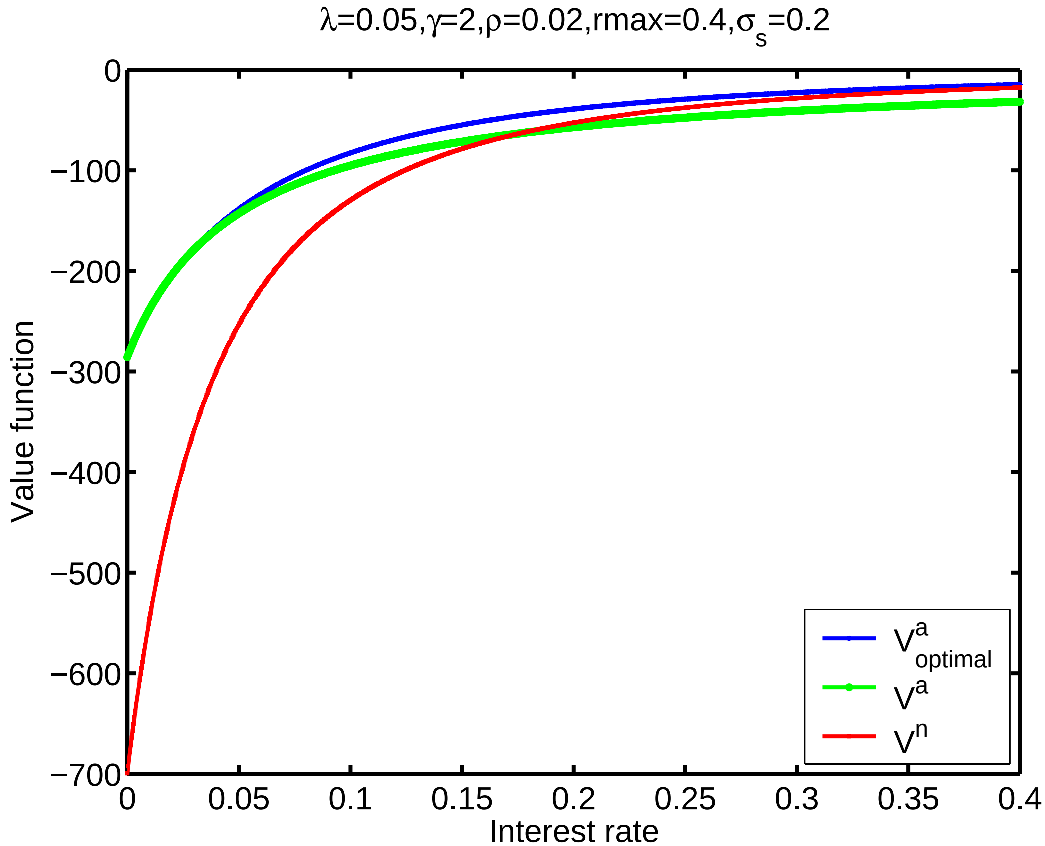
$T = \infty$ is introduced back in section 2.3.1.1 in chapter 2, and takes the following form

$$V^n(w) \;=\; \frac{w^{1-\gamma}}{(1-\gamma)} \frac{1}{(\frac{\rho+\lambda+(\gamma-1)\eta}{\gamma})^{\gamma}},\ \gamma \neq 1.$$

Figure 3.1 and Figure 3.2 display the comparison of V^n and V^a for force of mortality $\lambda = 0.05$ for $\gamma = 2$ and $\gamma = 3$ respectively, in which $V^a_{optimal}$ is the value function with optimal consumption, V^a is the value function with full consumption after annuitization, and V^n is the value function without optimal consumption. We see that all the values of $V^a_{optimal}$ are greater than that of V^a, which is due to the fact that consuming all the annuity income is not always the optimal consumption strategy. It can also be observed that $V^a_{optimal}$ is always greater than V^n, meaning that it is always optimal to annuitize no matter what the current interest rate is. This is different from the numerical results we have obtained in Chapter 2, in which the consumption strategy after annuitization is not optimal. Therefore we recommend the retiree to buy annuities immediately if he/she doesn't have bequest motives and his/her force of mortality is a constant 0.05 based on the assumption of no loading fees.

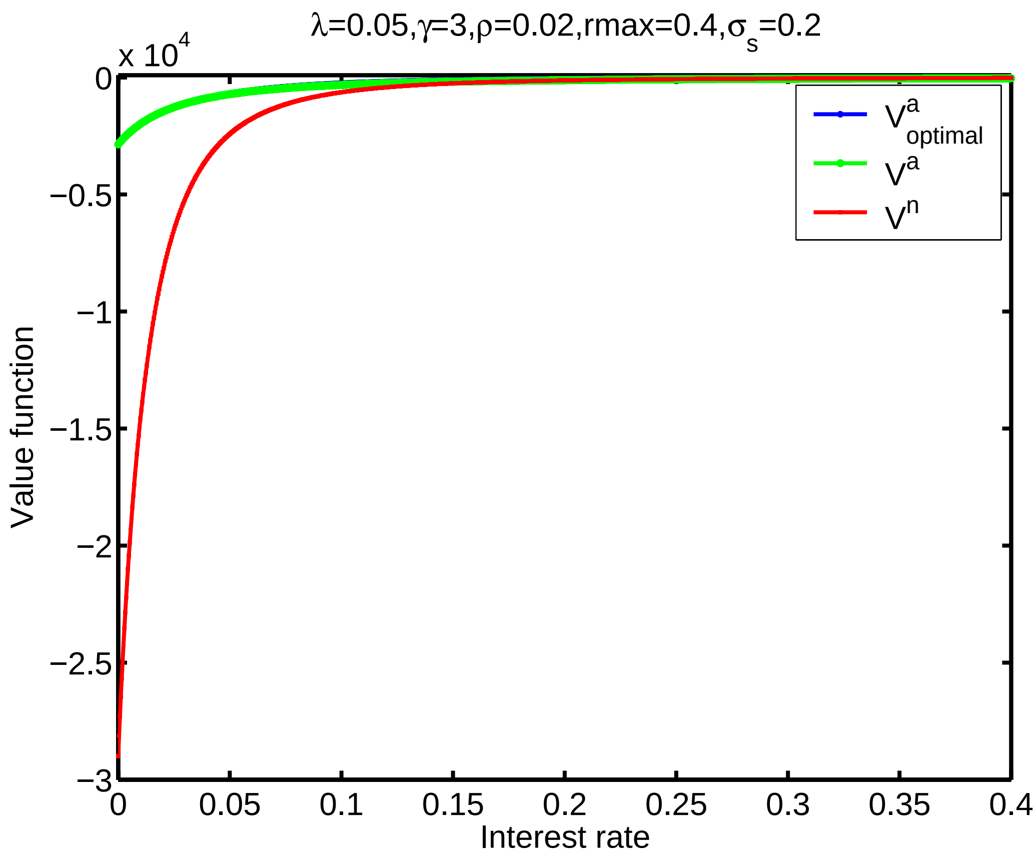
Now we have completed the analysis of the optimal consumption and annuitization strategies for the retired individual whose objective is to maximize

Figure 3.1: Value Function Comparison for $\gamma = 2$



Notes: $V^a_{optimal}$ is the value function with optimal consumption, V^a is the value function with full consumption after annuitization, and V^n is the value function without optimal consumption.

Figure 3.2: Value Function Comparison for $\gamma = 3$



$V^a_{optimal}$ is the value function with optimal consumption, V^a is the value function with full consumption after annuitization, and V^n is the value function without optimal consumption.

his/her lifetime utility from retirement to time of decease in the case of constant interest rates. In the next section, we will investigate the same problem under more realistic interest rate model.

3.2.2 Stochastic Interest Rates

In the previous section, we showed that it is always optimal to annuitize for the retired individual under exponential mortality no matter what the interest rates are (constant) without bequest motives and loading fees. In this section we study the same optimal annuitization problem under stochastic short rate models. To this end, we look at the two value functions V^a and V^n and then compare them to obtain the optimal annuitization strategy at time 0, i.e., age x.

- **The value function with annuitization under optimal consumption**

  In this section, we investigate the optimal fraction of consumption α under exponential mortality rate and stochastic interest rates, which is a function of time t and interest rate r. The discounted utility function (with annuitization) the retiree is seeking to maximize is defined as

$$V^a = \sup_{c_t} E\left[\int_0^\infty e^{-(\rho+\lambda)t} u(c_t)dt\right], \tag{3.27}$$

  where the consumption rate c_t is assumed to be part of the annuity income, i.e., $c_t = \alpha A$, and $\dot{A} = \frac{(1-\alpha)A}{\bar{a}_{x+t}}$ as before. The stochastic interest rate model we applied in this section is still a one-factor CIR interest rate model. We write it here again for convenience's sake.

$$dR_t = \theta(\mu_r - R_t)dt + \sigma_r\sqrt{R_t}dB_t^r. \tag{3.28}$$

Note that the annuity income A depends not only on time, but also on the interest rate due to the fact that r is a state variable when the interest rate is stochastic. The calculus of variations method is not applicable in this scenario due to the stochastic term of the interest rate, so dynamic programming techniques are applied to obtain the HJB equation that V^a must solve. Specifically, we rewrite V^a as a function of time t, annuity income A and interest rate r as below

$$V(t, A, r) = \sup_{c_t} E\left[\int_t^\infty e^{-(\rho+\lambda)s}u(c_s)ds\right]. \tag{3.29}$$

Similarly, we will consider the value function in a limited time domain $[0, T]$ since people will die in a finite time and the utility function is zero after that. After applying Bellman's optimality principle and Ito's lemma, we obtain the following nonlinear HJB equation

$$V_t + \theta(\mu_r - r)V_r + \frac{1}{2}r\sigma_r^2 V_{rr} + \sup_{\alpha}[e^{-(\rho+\lambda)t}u(\alpha A) + V_A\frac{(1-\alpha)A}{\bar{a}_{x+t}}] = 0. \tag{3.30}$$

Note that the consumption strategy α is not only a function of time t, but also a function of interest rate r. Notation $\bar{a}_{x+t}$ is the annuity factor at time

t, i.e., age $x+t$, which is defined by

$$\bar{a}_{x+t}(t,r) = E\left[\int_t^\infty e^{-\int_t^s (R_v+\lambda)dv} ds\right]. \qquad (3.31)$$

If we use a transformation $s = z + t$, then we have

$$\bar{a}_{x+t}(t,r) = E\left[\int_0^\infty e^{-\lambda z} e^{-\int_t^{t+z} R_v dv}\right] = \int_0^\infty e^{-\lambda z} A(t,t+z) e^{-B(t,t+z)R_t} dz. \qquad (3.32)$$

Since $A(t,t+z)$ and $B(t,t+z)$ are independent of time t, the annuity factor $\bar{a}_{x+t}(t,r)$ depends only on interest rate. If the value function takes the power form $V = \frac{A^{1-\gamma}}{1-\gamma} h(t,r)$, then the optimal consumption strategy $\alpha^\star$ is given by the first-order condition

$$\alpha^\star = \left(e^{\rho t+\lambda t} \frac{h(t,r)}{\bar{a}_{x+t}}\right)^{-\frac{1}{\gamma}}. \qquad (3.33)$$

Substituting the expressions of V and $\alpha^\star$ into the HJB equation (3.30), we obtain the following PDE that $h(t,r)$ must solve

$$h_t + \theta(\mu_r - r)h_r + \frac{1}{2} r\sigma_r^2 h_{rr} + \frac{1-\gamma}{\bar{a}_{x+t}} h + \gamma \bar{a}_{x+t}^{\frac{1}{\gamma}-1} e^{-\frac{\rho+\lambda}{\gamma}t} h^{1-\frac{1}{\gamma}} = 0. \qquad (3.34)$$

To solve this PDE numerically, the computational domain is truncated to be $(t,r) \in [0,T] \times [0,r_{max}]$, where T is the maximum life expectancy of the individual minus his/her current age x, and r_{max} is the maximum interest rate that the riskless asset can attain. Terminal and boundary conditions imposed on this PDE are

$$\begin{cases} t = T & : \quad h(T,r) = \bar{a}_{x+T} e^{-\rho T - \lambda T}, \\ r = 0 & : \quad h_t + \theta \mu_r h_r + \frac{1-\gamma}{\bar{a}_{x+t}} h + \gamma \bar{a}_{x+t}^{\frac{1}{\gamma}-1} e^{-\frac{\rho+\lambda}{\gamma} t} h^{1-\frac{1}{\gamma}} = 0, \\ r = r_{max} & : \quad h_{rr} = 0. \end{cases} \tag{3.35}$$

The explanation for the terminal condition is due to that the consumption ratio is 1 at $t = T$. The boundary condition at $r = 0$ is obtained by setting $r = 0$ on both sides of the PDE (3.34). The Neumann boundary condition at $r = r_{max}$ is imposed on the observation that the second-order derivation at this point is close to zero for constant interest rates. To obtain non-zero solutions for $h(t,r)$, we make a transformation $h(t,r) = y(t,r)^{\gamma}$, and reach the following PDE for $y(t,r)$.

$$y_t + \frac{1-\gamma}{\gamma \bar{a}_{x+t}} y + \bar{a}_{x+t}^{\frac{1}{\gamma}-1} e^{-\frac{(\rho+\lambda)t}{\gamma}} + \theta(\mu_r - r) y_r + \frac{1}{2} r \sigma_r^2 (y_{rr} + \frac{\gamma - 1}{y} y_r^2) = 0. \tag{3.36}$$

Let $y(t,r) = e^{-\frac{(\rho+\lambda)t}{\gamma}} \tilde{y}(t,r)$, and substitute it into equation (3.56), we have

$$\tilde{y}_t + (\frac{1-\gamma}{\gamma \bar{a}_{x+t}} - \frac{\rho+\lambda}{\gamma}) \tilde{y} + (\theta(\mu_r - r) + \frac{1}{2} r \sigma_r^2 \frac{\gamma - 1}{\tilde{y}} \tilde{y}_r) \tilde{y}_r + \frac{1}{2} r \sigma_r^2 \tilde{y}_{rr} + \bar{a}_{x+t}^{\frac{1}{\gamma}-1} = 0. \tag{3.37}$$

The corresponding terminal and boundary conditions become

$$\begin{aligned} t = T : & \quad \tilde{y}(T,r) = \bar{a}_{x+T}^{\frac{1}{\gamma}}, \\ r = 0 : & \quad \tilde{y}_t + (\tfrac{1-\gamma}{\gamma \bar{a}_{x+t}} - \tfrac{\rho+\lambda}{\gamma}) \tilde{y} + \theta \mu_r \tilde{y}_r + \bar{a}_{x+t}^{\frac{1}{\gamma}-1} = 0, \\ r = r_{max} : & \quad \tilde{y}_{rr} = 0. \end{aligned} \tag{3.38}$$

We then solve the equation system (3.37) and (3.38) by finite difference method. After solving this equation system, the optimal consumption strategy α can be recovered from the equation (3.33).

Now we are ready to compute the annuitization value function V^a numerically. To this end, we first look at the annuity income A_s at time s, which satisfies the following ODE

$$\frac{dA_s}{ds} = \frac{(1-\alpha_s)A_s}{\bar{a}_{x+s}}. \tag{3.39}$$

Integrating it from time zero to time t, we have

$$\int_0^t \frac{dA_s}{A_s} = \int_0^t \frac{1-\alpha_s}{\bar{a}_{x+s}} ds. \tag{3.40}$$

After some mathematical manipulations, we obtain the following solution

$$A_t = A_0 e^{\int_0^t \frac{1-\alpha_s}{\bar{a}_{x+s}} ds}, \tag{3.41}$$

in which A_0 is the annuity payout at time zero (age x), which is equal to the initial wealth w divided by the actuarial annuity factor at time zero. The value function with annuitization can be computed through formula $V^a = \frac{A_t^{1-\gamma}}{1-\gamma} h(t,r)$ for all t and r.

- **The value function without annuitization**

If the retiree does not annuitize at time t, the value function is defined as

$$V^n(t,w,r) = \sup_{c_s} E\left[\int_t^\infty e^{-(\rho+\lambda)s} u(c_s) ds\right]. \tag{3.42}$$

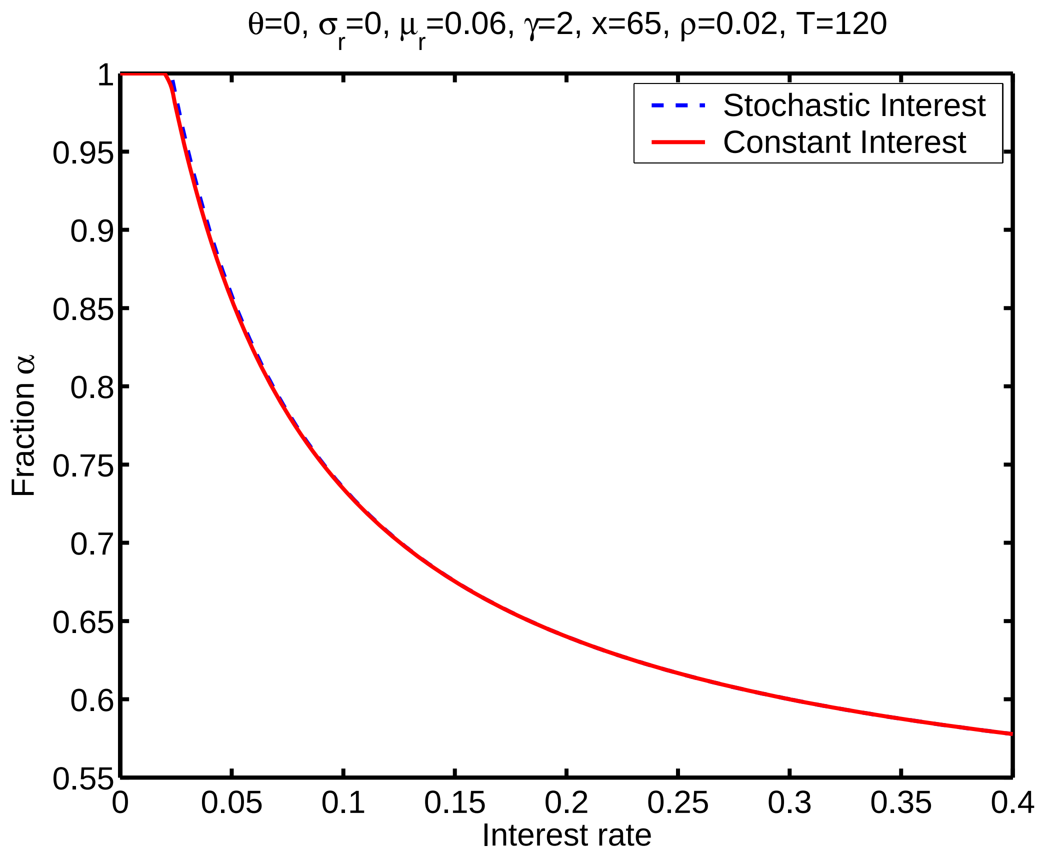
The HJB equation that V^n satisfies and its solution can be obtained by applying the same procedure as in Section 2.3.2.2 in Chapter 2, which is omitted here for simplicity.

Note that V^n is still independent of time t if the exponential power term $e^{-(\rho+\lambda)t}$ is excluded, while V^a does not share the same property with optimal consumption after annuitization. For each fixed time t, we compare V^a and V^n to find the initial free boundary, and if the free boundary does exist, we move it, applying the same method as in Section 2.3.2.2 in Chapter 2 to obtain the final free boundary. This free boundary problem can also be solved by converting the corresponding HJB equation into an equivalent LCP problem, applying the projected SOR method to solve the PDE the value function must solve, and then obtaining the optimal consumption strategy and the optimal annuitization strategy by comparing the value function with its lower bound as before. The numerical results show us that the two methods agree.

- **The optimal consumption strategy** α

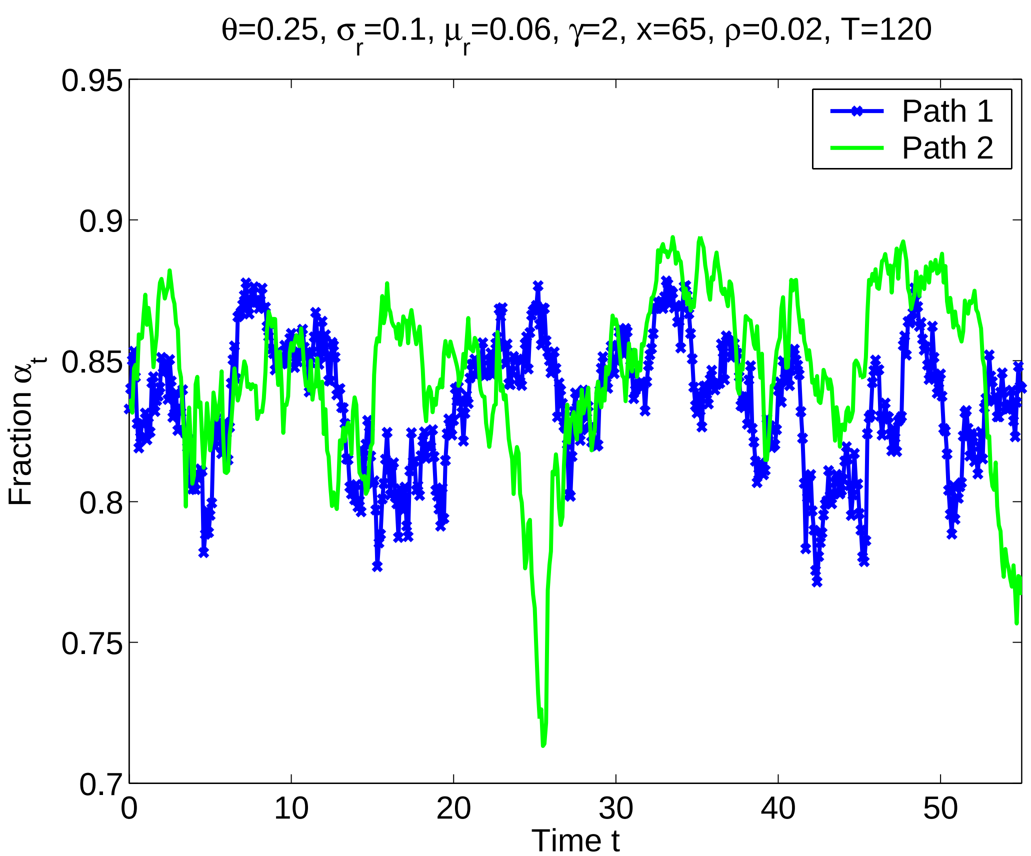
  When the interest rate is stochastic, the analytic optimal consumption ratio α is not available due to the complexity of the PDE that $\tilde{h}$ must solve. From our previous analysis, we have known that both V^a and V^n are independent of time t, therefore, the optimal annuitization strategy is independent of t

Figure 3.3: The Optimal Consumption Strategy Comparison at $t = 0$



When the speed of adjustment θ equals 0, and the volatility σ_r equals 0, stochastic interest rates collapse to constant interest rates. Therefore, the optimal consumption strategies for both stochastic and constant interest rates should agree. This figures compares these two optimal consumption strategies, and the absolute maximum difference is 0.0077.

Figure 3.4: Optimal Consumption Strategies for Stochastic Interest Rates



The figure shows two random paths for the optimal consumption strategy for stochastic interest rates using Monte Carlo simulations for parameters $\gamma = 2$, $\rho = 0.02$, $\theta = 0.25$, $\mu_r = 0.06$, $\sigma_r = 0.1$, with initial interest rate $r(0) = 0.06$.

too. To verify our numerics for stochastic interest rates are correct, Figure 3.3 displays the optimal consumption strategies under stochastic interest rates for $\theta = 0, \sigma_r = 0$ and constant interest rates as a function of r at time $t = 0$. We see that the optimal consumption strategies under two different interest rates models match very well, which gives us confidence that our numerics are good. Figure 3.4 shows two random paths for the optimal consumption strategy for stochastic interest rates using Monte Carlo simulations for parameters $\gamma = 2$, $\rho = 0.02$, $\theta = 0.25$, $\mu_r = 0.06$, $\sigma_r = 0.1$, with initial interest rate $r(0) = 0.06$. Since the interest rate is stochastic, it has many random paths, which leads to different optimal consumption strategies. At any given age, the optimal consumption strategy α_t depends on the spot interest rate realized. It is not an increasing function of time t as for the constant interest rate scenario.

- **The Optimal Annuitization Strategy**

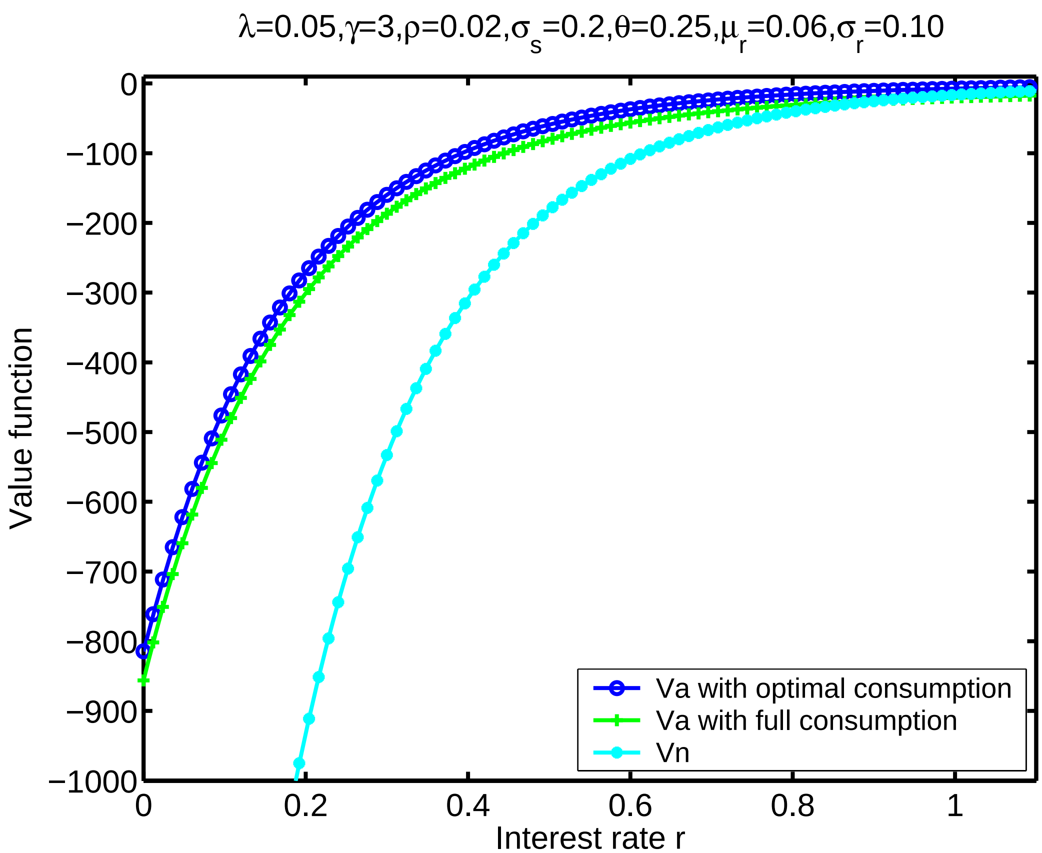
  In this section, we first compare the value functions under stochastic interest rates for $\theta = 0$ and $\sigma_r = 0$ with those of constant interest rates at time 0. It shows that the value functions for them agree very well, which gives us confidence that our numerics are good. We then move on to finish the free boundary seeking procedure for each fixed time t from time zero to T using free boundary refining method. This method and the LCP method both show us that it is always optimal to annuitize no matter what the current interest rate is for $\gamma = 2$ under optimal consumption strategy with continuous annuity purchasing, which is consistent with our previous results for constant interest rates.

Next we investigate the stochastic interest rates case, in which the adjust speed θ and interest rate volatility σ_r are both positive (0.25, 0.10). Figure 3.5 displays the value functions comparison at time zero, i.e., age x. We see that the annuitization value function V^a with optimal consumption is always above the non-annuitization value function V^n, meaning no annuitization boundaries exist, and the value function V^a with full consumption after annuitization intersects V^n, meaning free boundaries exist in this scenario. It turns out that no annuitization boundaries exist for any time level t_n, and it is always optimal to annuitize for any current interest rate. In other words, if one wants to shift consumption to later years and can rebalance his annuities continuously, he will gain higher income later.

3.2.3 Concluding Remarks

In this section, we have documented the optimal consumption and annuitization strategies for a utility maximizer with exponential mortality rate for constant and stochastic interest rates. The optimal consumption ratio for stochastic interest rates is a little bit greater than that of constant interest rate, while the optimal

Figure 3.5: Value Functions Comparison for $\gamma = 3$ Using the LCP Method



This figure compares the annuitization value functions under optimal and constrained consumption strategies and the non-annuitization value function V^n at time 0. We can observe that no annuitization boundary exists when the consumption strategy is optimized.

annuitization strategy is always the same, i.e., it is always optimal to annuitize no matter what the interest rate is under the assumption of no-bequest, no-loading fees. This is due to the fact that when one sacrifices some of his/her annuity income now, he/she will gain a higher income later, which adds more utilities to the value function.

Although the mathematical simplification of the mortality rate (exponential) makes us to find the solutions with much greater ease, it has the disadvantage of memory-less. To overcome this flaw, we will investigate the same optimal control problem by relaxing the mortality to be GM mortality because it is widely accepted and applied in the insurance and finance literature.

3.3 Model Calibration 2: Gompertz Mortality

In this section, we discuss the optimal consumption and annuitization strategies for a retired individual whose objective is to maximize his/her lifetime consumption utility under the following modified GM mortality rate as in Section 2.4 in Chapter 2.

$$\lambda_{x+t} = \begin{cases} \frac{1}{b} e^{\frac{x+t-m}{b}}, & t \leq T, \\ \lambda_{x+T}, & t \geq T, \end{cases} \tag{3.43}$$

This modified GM mortality enables us to apply the non-zero terminal condition at time $t = T$, which can be computed in domain $[T, \infty]$ by applying the same mathematical techniques for constant force of mortality as in the previous section.

In mathematics, the associated value function the individual is seeking to maximize is defined as

$$V(t,w,r) = \sup_{\pi_s, c_s, \tau} E\left[\int_t^\tau e^{-\rho(s-t)} \ {}_{s-t}p_{x+t} u(c_s)\, ds \right.$$
$$\left. + \int_\tau^\infty e^{-\rho(s-t)} \ {}_{s-t}p_{x+t} u\left(\frac{W_\tau}{\bar{a}_{x+\tau}(\tau, R_\tau)}\right) ds \right| W_t = w, R_t = r\Bigg]. \quad (3.44)$$

This is exactly the same value function as in Section 2.4.1 in Chapter 2. Similarly, this annuitization problem is a free boundary problem and its mathematical statement is given by

$$(\rho + \lambda_{x+t})V - V_t - \mathfrak{L}V > 0, V(t,w,r) = J(t,w,r) \quad (3.45)$$

for $0 < r < r^*(t)$ (optimal not to annuitize),

$$(\rho + \lambda_{x+t})V - V_t - \mathfrak{L}V = 0, V(t,w,r) > J(t,w,r) \quad (3.46)$$

for $r^*(t) < r < \infty$ (optimal to annuitize), in which $\mathfrak{L}V$ is introduced back in equation (2.48). Note that in domain $r \in [0, r^*(t)]$, it is optimal not to annuitize, and in domain $r \in [r^*(t), \infty]$, it is optimal to annuitize, which is different than the free boundary problem stated in Chapter 2. This statement is motivated from the observation that it is always optimal to annuitize for stochastic interest rates under exponential mortality.

If we postulate that $V(t,w,r) = \frac{w^{1-\gamma}}{1-\gamma} h(t,r)$, $h(t,r) = y(t,r)^{\gamma}$, $\tilde{y}(t,r) = -y(t,r)$, then the above free boundary problem is equivalent to

$$(\rho + \lambda_{x+t})\tilde{y} - \gamma \tilde{y}_t + \mathfrak{L}\tilde{y} > 0, \quad \tilde{y}(t,r) = -g^{\frac{1}{\gamma}}(t,r) \tag{3.47}$$

for $0 < r < r^*(s)$,

$$(\rho + \lambda_{x+t})\tilde{y} - \gamma \tilde{y}_t + \mathfrak{L}\tilde{y} = 0, \quad \tilde{y}(t,r) > -g^{\frac{1}{\gamma}}(t,r) \tag{3.48}$$

for $r^*(s) < r < \infty$.

There are two different ways to solve this free boundary problem. The first way is to convert it to an equivalent LCP problem and then solve it by the projected SOR method. The second way is to compare V to the annuitization value function (J below) to obtain the initial free boundary, and then move it to achieve the final free boundary where both value functions and their derivatives are equal. Either way, we need to look at the annuitization value function first. In the next section, we will use dynamic programming techniques to study this annuitization value function.

3.3.1 The Annuitization Value Function Under Stochastic Interest Rates

If the individual annuitizes at time t, the expected utility of discounted lifetime consumption over admissible control α_t that he/she is seeking to maximize is given by the following definition

$$J(t, A_t, R_t) = \sup_{\alpha_s} E\left[\int_t^\infty e^{-\int_0^s(\rho+\lambda_{x+v})dv} u(\alpha_s A_s)ds | A_t = A, R_t = r\right], \tag{3.49}$$

where the consumption rate is assumed to be a fraction ($0 \leq \alpha_t \leq 1$) of the annuity income A_t as before, and the stochastic interest rate R_t follows the CIR process introduced back in equation (2.3). So the annuity factor can be computed through the zero-coupon bond $P_B(t, s, R_t)$ with maturity s

$$\bar{a}_{x+t}(t, R_t) = \int_t^\infty P_B(t, s, R_t)({}_{s-t}p_{x+t})ds. \tag{3.50}$$

We assume that the individual can purchase the annuity at the actuarial fair price $\bar{a}_{x+t}$ per dollar of annuity income at time t and we have $\frac{dA_t}{dt} = \frac{(1-\alpha_t)A_t}{\bar{a}_{x+t}}$. So the HJB equation that $J(t, A, r)$ must satisfy can be derived as

$$J_t + \theta(\mu_r - r)J_r + \frac{1}{2}r\sigma_r^2 J_{rr} + \sup_\alpha [e^{-\rho t - \int_0^t \lambda_{x+v}dv} u(\alpha A) + J_A \frac{(1-\alpha)A}{\bar{a}_{x+t}}] = 0. \tag{3.51}$$

The optimal consumption strategy $\alpha^\star$ is given by the first-order condition

$$\alpha^\star = \frac{\left(\frac{J_A}{\bar{a}_{x+t}} e^{\rho t + \int_0^t \lambda_{x+v}dv}\right)^{-\frac{1}{\gamma}}}{A}. \tag{3.52}$$

Motivated by the CRRA utility function, we postulate that J takes the similar power form as $J = \frac{A^{1-\gamma}}{1-\gamma}h(t, r)$, then the above optimal consumption ratio becomes

$$\alpha^\star = \left(\frac{h(t, r)}{\bar{a}_{x+t}} e^{\rho t + \int_0^t \lambda_{x+v}dv}\right)^{-\frac{1}{\gamma}}, \tag{3.53}$$

and the HJB equation (3.51) collapses to the following partial differential equation for function $h(t,r)$.

$$h_t + \theta(\mu_r - r)h_r + \frac{1}{2}r\sigma_r^2 h_{rr} + \frac{1-\gamma}{\bar{a}_{x+t}}h + \gamma\bar{a}_{x+t}^{\frac{1}{\gamma}-1} e^{-\frac{\rho t+\int_0^t \lambda_{x+v}dv}{\gamma}} h^{1-\frac{1}{\gamma}} = 0. \qquad (3.54)$$

To solve this PDE, we impose the following terminal and boundary conditions

$$\begin{cases} t = T & : \quad h(T,r) = \bar{a}_{x+T}e^{-\rho T-\int_0^T \lambda_{x+v}dv}, \\ r = 0 & : \quad h_t + \theta\mu_r h_r + \frac{1-\gamma}{\bar{a}_{x+t}}h + \gamma\bar{a}_{x+t}^{\frac{1}{\gamma}-1} e^{-\frac{\rho t+\int_0^t \lambda_{x+v}dv}{\gamma}} h^{1-\frac{1}{\gamma}} = 0, \\ r = r_{max} & : \quad h_{rr} = 0, \end{cases} \qquad (3.55)$$

The explanation for conditions $r = 0$ and $r = r_{max}$ are similar to the scenario when λ is constant and r is stochastic. The reason for boundary condition at $t = T$ is due to the fact that the optimal consumption strategy in domain $[T,\infty)$ where the mortality is constant is always 1.

To obtain non-zero solutions for $h(t,r)$, we make a transformation $h(t,r) = y(t,r)^\gamma$, and substitute it into equation (3.54), then we achieve the following PDE for $y(t,r)$.

$$y_t + \frac{1-\gamma}{\gamma\bar{a}_{x+t}}y + \bar{a}_{x+t}^{\frac{1}{\gamma}-1} e^{-\frac{\rho t+\int_0^t \lambda_{x+v}dv}{\gamma}} + \theta(\mu_r - r)y_r + \frac{1}{2}r\sigma_r^2(y_{rr} + \frac{\gamma-1}{y}y_r^2) = 0. \quad (3.56)$$

Let $y(t,r) = e^{-\frac{\rho t+\int_0^t \lambda_{x+v}dv}{\gamma}}\tilde{y}(t,r)$, and substitute it into equation (3.56), we have

$$\tilde{y}_t + (\frac{1-\gamma}{\gamma\bar{a}_{x+t}} - \frac{\rho+\lambda_{x+t}}{\gamma})\tilde{y} + (\theta(\mu_r - r) + \frac{1}{2}r\sigma_r^2\frac{\gamma-1}{\tilde{y}}\tilde{y}_r)\tilde{y}_r + \frac{1}{2}r\sigma_r^2\tilde{y}_{rr} + \bar{a}_{x+t}^{\frac{1}{\gamma}-1} = 0. \quad (3.57)$$

The corresponding terminal and boundary conditions become

$$\begin{cases} t = T: & \tilde{y}(T,r) = \bar{a}_{x+T}^{\frac{1}{\gamma}}, \\ r = 0: & \tilde{y}_t + (\frac{1-\gamma}{\gamma \bar{a}_{x+t}} - \frac{\rho+\lambda_{x+t}}{\gamma})\tilde{y} + \theta\mu_r \tilde{y}_r + \bar{a}_{x+t}^{\frac{1}{\gamma}-1} = 0, \\ r = r_{max}: & \tilde{y}_{rr} = 0. \end{cases} \tag{3.58}$$

We then solve the equation system (3.57) and (3.58) by finite difference method. After solving this equation system, the optimal consumption strategy $\alpha^\star$ can be recovered from equation (3.53).

To compute the value function with annuitization J numerically, we first look at the annuity income A_s at time s, which satisfies the following ODE

$$\frac{dA_s}{ds} = \frac{(1-\alpha_s)A_s}{\bar{a}_{x+s}}. \tag{3.59}$$

Integrating it from time zero to t, we have

$$\int_0^t \frac{dA_s}{A_s} = \int_0^t \frac{1-\alpha_s}{\bar{a}_{x+s}} ds. \tag{3.60}$$

After some mathematical manipulations, we obtain the following expression

$$A_t = A_0 e^{\int_0^t \frac{1-\alpha_s}{\bar{a}_{x+s}} ds}, \tag{3.61}$$

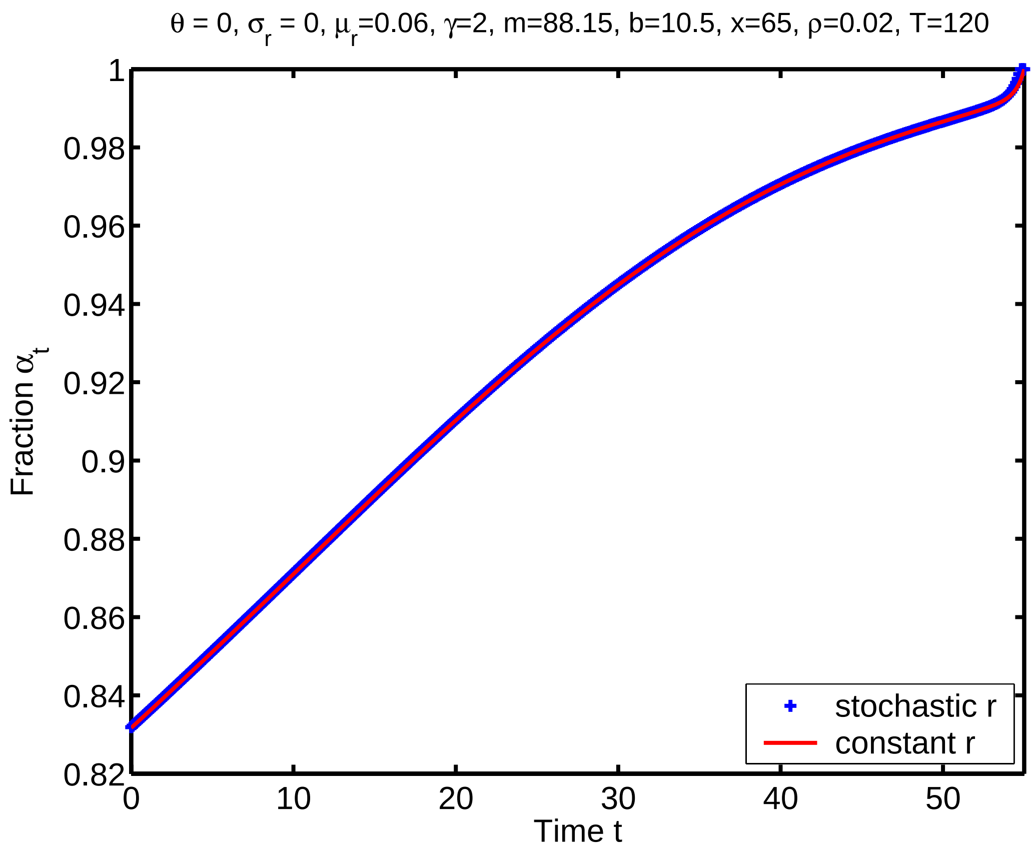
in which A_0 is the annuity payout at time zero (age x), which is equal to the initial wealth w divided by the actuarial annuity factor. Then the annuitization value function J can be calculated via equation $J = \frac{A^{1-\gamma}}{1-\gamma} h(t,r)$ for any time t and interest rate r.

3.3.2 The Optimal Consumption Strategy

In this subsection, we carry out a convergence analysis of our numerical algorithm first. Three different experiments with different time and interest step sizes $([0.1, 0.008], [0.05, 0.004], [0.025, 0.002])$ are performed, and the convergence rate, which is the logarithm of two immediate quotients of the L_2 norms, turns out to be 1. Therefore our algorithm converges to the exact solution as step sizes go to zero. The following optimal consumption comparison further verifies this fact.

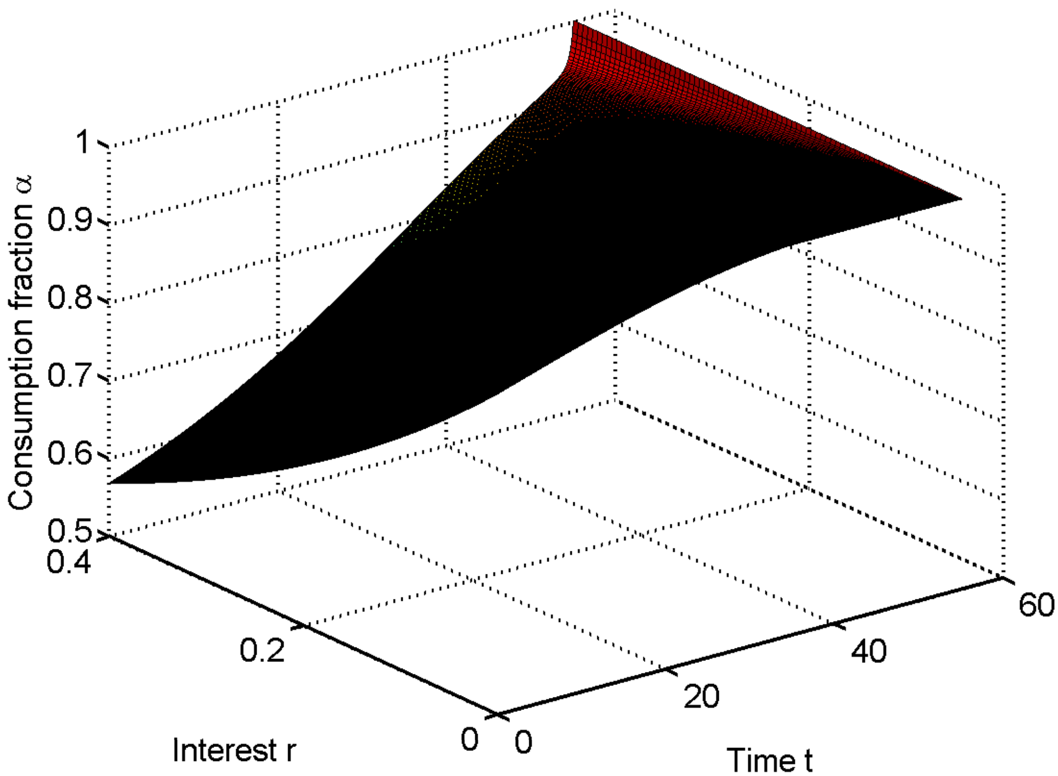
When the interest rate is stochastic, the optimal consumption strategy α_t is not only a function of time t, but also a function of interest rate r. To compare this optimal consumption strategy with the scenario in which the interest rate is constant, we choose a special interest rate $r = \mu_r$, since when $\theta = 0$ and $\sigma_r = 0$, the stochastic interest rate collapses to a constant. Then the optimal consumption ratio α_t is a function of time t, and it should agree with the case in which r is constant. Figure 3.6 displays this comparison for $\gamma = 2$ for both stochastic and constant interest rates, in which the CV method and the dynamic programming techniques are applied to calculate α_t for constant interest (see appendix for its derivation). We can see that α_t agrees very well, meaning that our numerics are good. Now we move on to compute the optimal consumption strategy α for stochastic interest rates. Figure 3.7 plots α as a function of time t and interest rate r for CIR parameters

Figure 3.6: α_t Comparison for Different Interest Rate Models: $\gamma = 2$



When the speed of adjustment θ and volatility σ_r are both 0, stochastic interest rates collapse to constant interest rates. Hence the optimal consumption strategies should agree with each other. This figure verifies this argument for Gompertz mortality.

Figure 3.7: Optimal Consumption Strategy α for $\theta = 0.25, \mu_r = 0.06, \sigma_r = 0.1$



This figure plots the optimal consumption strategy α as a function of time t and interest rate r for Gompertz mortality. When the individual sacrifices some of the annuity income now, in return he/she will be able to consume more later.

$\theta = 0.25, \mu_r = 0.06, \sigma_r = 0.1$, GM parameters $m = 88.15, b = 10.5$, $\gamma = 2$ and $\rho = 0.02$. We see that for any fixed interest rate, α is an increasing function of time t. For any fixed time t, α is a decreasing function of interest rate r, which is intuitively pleasant. The annuity income is also an increasing function of time t, which means that when the individual sacrifices some of the annuity income now, in return he/she will be able to consume more later.

3.3.3 The Optimal Annuitization Strategy

No matter whether we use the LCP method or the free boundary refining method, the first thing we need to handle is the terminal condition at $t = T$. Since the force of mortality is a constant (1.9777) in domain $[T, \infty]$, the free boundaries in this domain is time invariant and we can obtain it by comparing the non-annuitization value function V^n and the annuitization value function V^a as before. When we calculate V^a, which is very time consuming, we store the value function in a matlab file and then we reload it when necessary. It turns out that the optimal consumption ratio is 1 in domain $[T, \infty]$, an intuitively pleasant result, because the mortality rate is a large enough constant so that individuals will have little chance to live past the maximum age.

The parameters used in our experiment are listed below: $\mu_r = 0.06$, $\theta = 0.25$, $\sigma_r = 0.1$, $\sigma_s = 0.2$, $\gamma = 2$, $\rho = 0.02$, $x = 65$, $\rho_{rs} = 0$, $w_0 = 1$, $\delta_1 = 0.03$, the maximum life span of a human being $T_{max} = 125$. Both the LCP method and the free boundary refining method show that it is always optimal to annuitize no matter what the interest rates are, which is consistent with our previous results for exponential mortality. Therefore, if the individual sacrifices some income now and can repurchase annuities at fair prices, he/she will earn higher income later.

3.4 Concluding Remarks

In this chapter, we have studied the optimal annuitization problem for a utility maximizer for exponential and Gompertz mortalities under the optimal consumption strategy.

Firstly, two interest rate models, constant and stochastic, are calibrated under exponential mortality to study the optimal annuitization timing problem. Secondly, stochastic interest rates are imposed under Gompertz mortality to study the optimal consumption and annuitization strategies, which is a free boundary problem, and can be solved using either the LCP method or the free boundary refining method. The results show that it is optimal to annuitize no matter what the interest rate or the mortality rate is. If the individual follows the optimal consumption strategy, he/she will earn higher income if he/she annuitizes immediately upon the assumption of no loading fees and no bequest motives.

All our numerical results show that it is optimal to annuitize even if the interest rate is high, but that one should consume less than what the annuity provides. In other words, one wants to shift consumption to later years. This suggests that if one annuitizes right away, with complete consumption required, then the realized consumption level is higher than optimal. Note that annuities get cheaper when interest rates rise, so in a sense, the annuity is actually too good a deal when the interest rate is high. Optimal behavior is to sacrifice some of that income now, in return for higher income later. In other words, instead of taking that deal, one should delay annuitizing, earn short term interest, and once interest rates revert to a more realistic level, you will probably be able to buy more annuities than you would otherwise. The annuities you eventually buy will be more expensive, but you will be able to buy more of them and actually earn higher income.

3.5 Appendix

3.5.1 Dynamic Programming Techniques for Exponential Mortality and Constant Interest Rates

To apply the dynamic programming techniques to derive the same equation that the annuity income A must satisfy for exponential mortality and constant interest rate, we write the value function V as a function of time t and A as

$$V(t, A) = \int_t^T e^{-(\rho+\lambda)s} u(c_s) ds. \tag{3.62}$$

After applying Bellman's optimality principle and Ito's lemma, we obtain the following HJB equation that V must solve

$$V_t + \sup_{\alpha} V_A(1-\alpha)(\lambda+r)A + e^{-(\rho+\lambda)t}u(\alpha A) = 0. \tag{3.63}$$

Note that α is a function of time t and interest rate r. The optimal consumption strategy can be obtained by its first-order condition, i.e.,

$$\alpha^\star = \frac{(V_A e^{(\rho+\lambda)t}(\lambda+r))^{-\frac{1}{\gamma}}}{A_t}. \tag{3.64}$$

If we postulate $V = \frac{A^{1-\gamma}}{1-\gamma}h(t)$ as before, then the optimal consumption strategy $\alpha^\star$ can be simplified to

$$\alpha^\star = (e^{(\rho+\lambda)t}h(t)(\lambda+r))^{-\frac{1}{\gamma}}. \tag{3.65}$$

Since t and A are both state variables in equation (3.63), which leads to the derivatives of V with respect to t and A are respectively

$$\begin{cases} V_t &= \frac{A^{1-\gamma}}{1-\gamma}h_t, \\ V_A &= A^{-\gamma}h, \end{cases} \tag{3.66}$$

in which h_t denotes the first derivative of $h(t)$ with respect to time t. Substituting from equation (3.65) and (3.66) into equation (3.63), we obtain the following linear homogenous first-order ODE that $h(t)$ must satisfy

$$h_t + (1-\gamma)(\lambda+r)h + \gamma e^{-\frac{\rho+\lambda}{\gamma}t}(\lambda+r)^{1-\frac{1}{\gamma}}h^{1-\frac{1}{\gamma}} = 0. \tag{3.67}$$

From section 3.2.1, we know that A_t must satisfy the following second-order homogenous differential equation (using CV method) over the values for which $A_t \neq 0$.

$$\ddot{A}_t - (\frac{r-\rho}{\gamma} + \lambda + r)\dot{A}_t + \frac{r-\rho}{\gamma}(\lambda + r)A_t = 0. \tag{3.68}$$

To verify that both dynamic programming techniques and CV method lead to the same ODE for $h(t)$, we substitute equation (3.65) into the first and second derivatives of A_t with respect to time t, which yields

$$\begin{cases} \dot{A}_t &= (\lambda + r - (\lambda + r)(e^{(\rho+\lambda)t}h(t)(\lambda + r))^{-\frac{1}{\gamma}})A_t, \\ \ddot{A}_t &= (\lambda + r - (\lambda + r)(e^{(\rho+\lambda)t}h(t)(\lambda + r))^{-\frac{1}{\gamma}})^2 A_t \\ &+ (-(\lambda + r)^{1-\frac{1}{\gamma}}(-\frac{\rho+\lambda}{\gamma}e^{-\frac{\rho+\lambda}{\gamma}t}h^{-\frac{1}{\gamma}} + e^{-\frac{\rho+\lambda}{\gamma}t}(-\frac{1}{\gamma})h^{-\frac{1}{\gamma}-1}h_t))A_t. \end{cases} \tag{3.69}$$

Substituting them into equation (3.68), we obtain the following ODE that $h(t)$ must solve

$$h_t + (1-\gamma)(\lambda + r)h + \gamma e^{-\frac{\rho+\lambda}{\gamma}t}(\lambda + r)^{1-\frac{1}{\gamma}}h^{1-\frac{1}{\gamma}} = 0. \tag{3.70}$$

We see that equation (3.67) and (3.70) are exactly same, which means that we can solve the optimal control problem using either the dynamic programming techniques or the CV method.

3.5.2 Optimal Consumption and Annuitization under the Exponential Mortality Rate and constant interest rates when α_t is Constant

In this section, we study the optimal control problem under exponential mortality rate and constant interest rates when α_t is time invariant. The discounted lifetime utility of consumption the retiree seeks to maximize with annuitization is

$$V^a = \int_0^\infty e^{-(\rho+\lambda)t} u(c_t)dt, \tag{3.71}$$

where the consumption rate is assumed to be a fraction ($0 \le \alpha_t \le 1$) of the annuity income A_t, i.e., $c_t = \alpha_t A_t$. The remainder $(1-\alpha_t)A_t$ is used to purchase more annuities. So the annuity satisfies the following equation

$$\frac{dA_t}{dt} = \frac{(1-\alpha_t)A_t}{\bar{a}_{x+t}}, \tag{3.72}$$

where the annuity factor $\bar{a}_{x+t}$ is constant ($\frac{1}{\lambda+r}$) for constant λ. When α_t is equal to constant α, the above function becomes

$$\frac{dA_t}{A_t} = (1-\alpha)(\lambda+r)dt. \tag{3.73}$$

Integrating this first-order ODE from time zero to t, we obtain the following expression

$$A_t = A_0 e^{(1-\alpha)(\lambda+r)t}, \tag{3.74}$$

in which A_0 is the annuity income at time zero, i.e., age $x+t$. Substituting A_t into equation (3.71), we have

$$V^a(\alpha) = \frac{\alpha^{1-\gamma} A_0^{1-\gamma}}{1-\gamma} \int_0^\infty e^{-(\rho+\lambda-(1-\alpha)(1-\gamma)(\lambda+r))t} dt. \tag{3.75}$$

This improper integration converges if and only if $\rho+\lambda-(1-\alpha)(1-\gamma)(\lambda+r) > 0$ is satisfied, and its limit is

$$V^a(\alpha) = \frac{\alpha^{1-\gamma} A_0^{1-\gamma}}{1-\gamma} \frac{1}{\rho+\lambda-(1-\alpha)(1-\gamma)(\lambda+r)}. \tag{3.76}$$

The derivative of $V^a(\alpha)$ with respect to α can be written as

$$\frac{\partial V^a(\alpha)}{\partial \alpha} = \frac{A_0^{1-\gamma}(\rho + \lambda + (\gamma - 1 - \gamma\alpha)(\lambda + r))}{\alpha^\gamma(\rho + \lambda - (1-\alpha)(1-\gamma)(\lambda + r))^2}. \tag{3.77}$$

Setting it to zero, we obtain the first-order condition as

$$\alpha^\star = \frac{\gamma\lambda + \rho + (\gamma - 1)r}{\gamma(\lambda + r)} = 1 - \frac{r - \rho}{\gamma(\lambda + r)}. \tag{3.78}$$

Note that $\rho + \lambda - (1-\alpha)(1-\gamma)(\lambda + r)$ is always greater than 0 for $\gamma \geq 1$, therefore, the optimal consumption strategy $\alpha^\star$ for $\gamma \geq 1$ is

$$\alpha^\star = \begin{cases} 1, & r \in (0, \rho], \\ 1 - \frac{r-\rho}{\gamma(\lambda+r)}, & r \in [\rho, \infty]. \end{cases} \tag{3.79}$$

Note that $1 - \frac{r-\rho}{\gamma(\lambda+r)} \in [0, 1]$ for all greater than 1 values of the risk aversion coefficients. One step further, we obtain the closed form solution for V^a as

$$V^a = \begin{cases} \frac{1}{(1-\gamma)(\rho+\lambda)(r+\lambda)^{1-\gamma}}, & r \leq \rho, \\ \frac{(\lambda + \frac{\rho}{\gamma} + (1-\frac{1}{\gamma})r)^{-\gamma}}{(1-\gamma)(\lambda+r)^{2-2\gamma}}, & r \geq \rho. \end{cases} \tag{3.80}$$

3.5.3 Optimal Consumption and Annuitization under GM Mortality and Constant Interest Rates

The value function is given by

$$J(c_t) = \sup_{c_t} \int_0^\infty e^{-\int_0^t (\rho + \lambda_{\lambda+s})ds} u(c_t)dt, \tag{3.81}$$

where $c_t = \alpha_t A_t$, and

$$\frac{dA_t}{dt} = \frac{(1-\alpha_t)A_t}{\bar{a}_{x+t}}. \tag{3.82}$$

The Eular-Largarange equation is given by

$$\frac{d}{dt}(A_t - \bar{a}_{x+t}\dot{A}_t) = (\frac{1}{\gamma\bar{a}_{x+t}} - \frac{1}{\gamma}(\rho + \lambda_{x+t}) + \frac{1}{\gamma\bar{a}_{x+t}}\frac{d\bar{a}_{x+t}}{dt})(A_t - \bar{a}_{x+t}\dot{A}_t) \tag{3.83}$$

in domain $(0, T-x)$. Therefore, the annuity income A_t must satisfy the following linear second-order differential equation over the values for which $A_t \neq 0$.

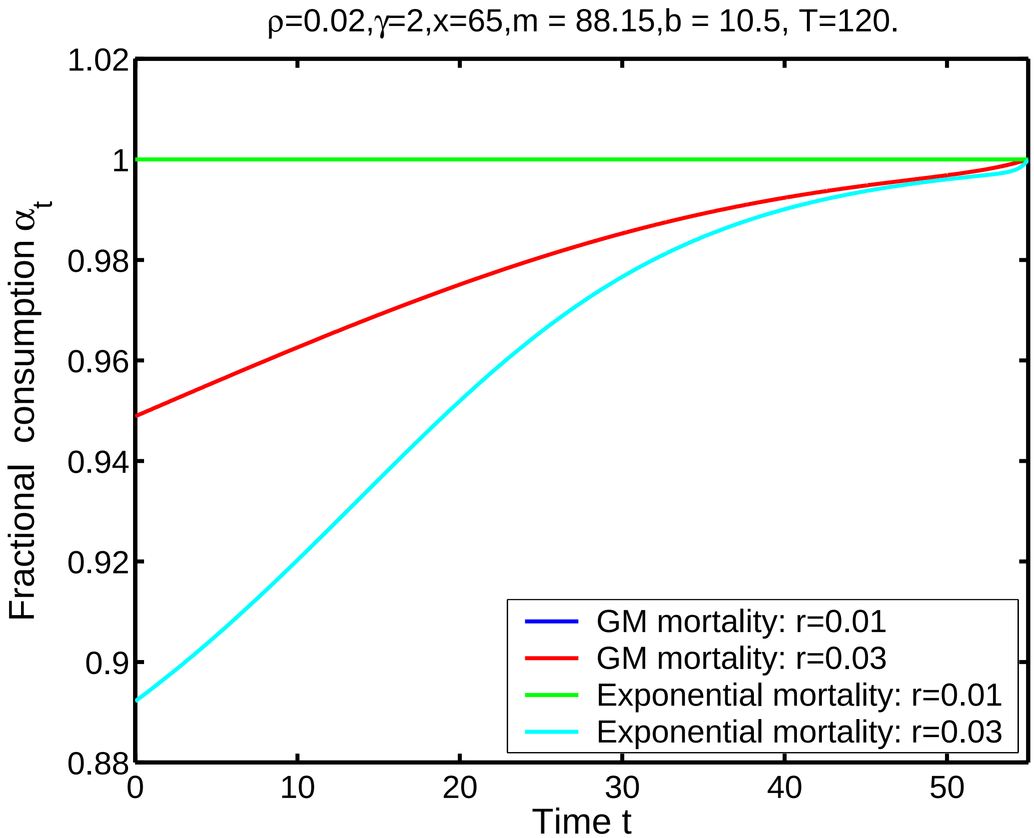
$$\begin{aligned} \bar{a}_{x+t}\ddot{A}_t \quad &+ \quad (\tfrac{d\bar{a}_{x+t}}{dt} - 1 - \tfrac{1}{\gamma} + \tfrac{1}{\gamma}(\rho + \lambda_{x+t})\bar{a}_{x+t} - \tfrac{1}{\gamma}\tfrac{d\bar{a}_{x+t}}{dt})\dot{A}_t \\ &+ \quad (\tfrac{1}{\gamma\bar{a}_{x+t}} - \tfrac{1}{\gamma}(\rho + \lambda_{x+t}) + \tfrac{1}{\gamma\bar{a}_{x+t}}\tfrac{d\bar{a}_{x+t}}{dt})A_t = 0. \end{aligned} \tag{3.84}$$

The initial and terminal conditions for A_t are

$$\begin{cases} A_0 = 1, \\ \frac{dA_t}{dt}|_{t=T-x} = 0. \end{cases} \tag{3.85}$$

This ODE can be solved by the finite difference method with staggered grid. The results are plotted in Figure 3.8 with $\gamma = 2$ for two different interest rates: $r = 0.01$ and $r = 0.03$, one is smaller than ρ (0.02) and the other is greater than ρ. The other parameters are $\gamma = 2$, $x = 65$, $m = 88.15$, $b = 10.5$. Note that α_t for GM mortality hide behind its counterpart for exponential mortality for $r = 0.01$. We can see from these two figures that when $r < \rho$, $\alpha_t \equiv 1$ for both exponential and GM mortality. This is due to the fact that borrowing is not allowed in this circumstance. When α_t is greater than 1, we only allow the individual to consume all the annuity income. Another observation is that α_t for GM mortality is always greater than that for exponential mortality. This may be due to the greater uncertainty of the GM mortality.

Figure 3.8: α_t for Gompertz Mortality and Exponential Mortality for $\gamma = 2$

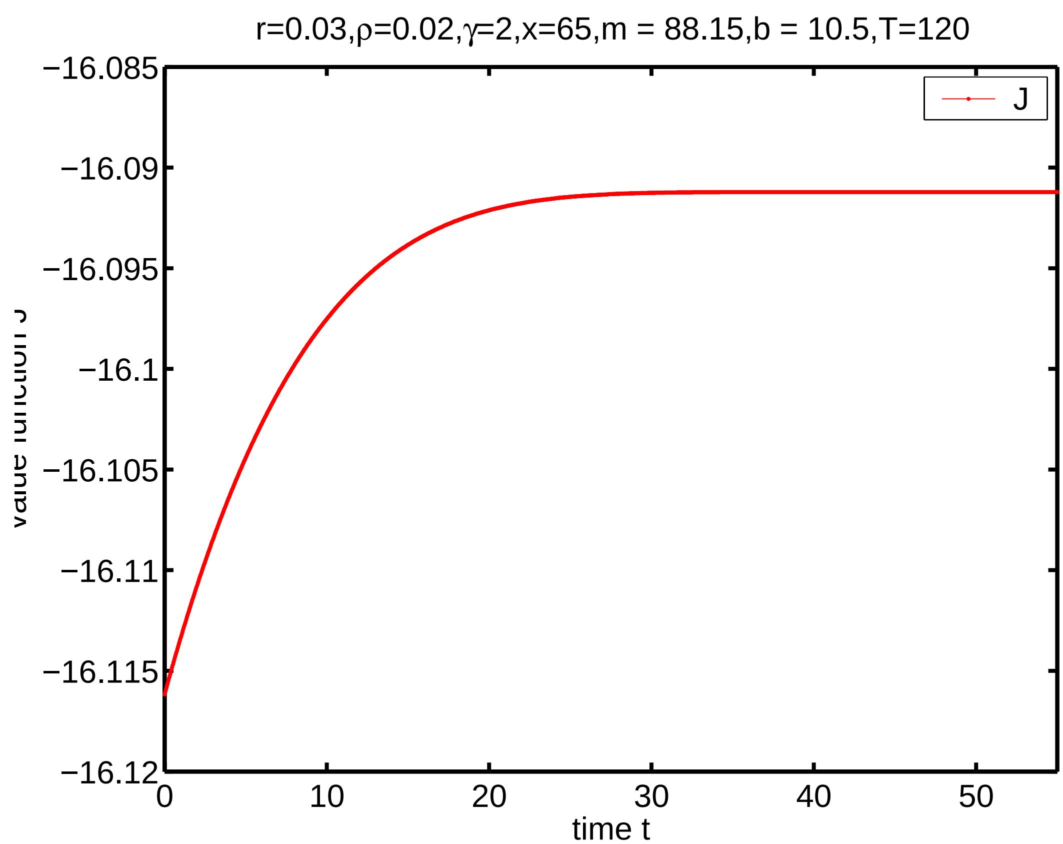


Now we will check if there exists a $t^\star < T$ which makes J attain its maximum at this point and $\alpha_t \equiv 1$ in domain $[t^\star, T]$, which is quite similar to what we have done in Section 3.2. Taking $t^\star$ as a parameter, we have

$$J(t^\star) = \int_0^{t^\star} e^{-\int_0^t (\rho+\lambda_{x+s})ds} u(\alpha_t A_t) dt + \int_{t^\star}^{T} e^{-\int_0^t (\rho+\lambda_{x+s})ds} u(A_{t^\star}) dt). \tag{3.86}$$

Figure 3.9 displays J as a function of $t^\star$ for $\gamma = 2$. The maximum value occurs at time $t^\star = 51.2875$. Actually the difference of $J(51.2875)$ and $J(T - x = 55)$ is $9.9121e-013$, which can be seen as equal. In fact, all the values of J can be treated to be equal after $t^\star = 51.2875$ since the difference is less than $e - 10$. The fraction of consumption at $t^\star = 51.2875$ is 0.9974, and it is increasing to 1 until $t = T - x$ in Figure 3.8. This means that the fraction of consumption is approximately 1 after age $65 + t^\star$.

Figure 3.9: The Value Function J for $\gamma = 2$



4 The Ruin Probability Facing Retirees

4.1 Introduction and Existing Literature

The ruin probability can be traced back to 'The Gambler's Ruin Problem' which emerged more than one hundred years ago. In essence, if two gamblers play a game for stakes, then how likely is it that one gambler will win all the money from the other gambler, before he/she loses all of his/her own money. As time goes by, ruin probability has been studied by insurance companies who want to know the probability of their reserves becoming negative within a certain time period. Lately, as the first baby boom generation reaches the 'standard' retirement age of 65 years in 2011, and the shift from DB pension plans to DC pension plans has occurred in a number of countries, more and more researchers and practitioners are interested in this topic of ruin probability. They studied the probability that individuals will outlive their wealth due to the fact that many of them are not financially prepared for retirement. Therefore, it is very important and meaningful to study the ruin probability which is related to longer than expected life spans.

The average life span has increased more than 30 years during the twentieth century in Canada, and it will continue the upward tendency in the future (see Oeppen and Vaupel (2002)). According to the Berkeley human mortality database, life expectancy at birth for Canadians has gradually increased from 57.94 in 1929 to 81.25 in 2009 (see Table 4.1) for the total population, which accounts for about 24 additional years of life. As well, those who managed to live to age 65 by the year of 2009, will live an average of 20 more years (see Table 4.2). Therefore, it will be a great challenge for those retirees who are without a pre-existing pension or social security, and no remaining income available.

Table 4.1: Life Expectancy at Birth for Canadians

Year	2009	1999	1989	1979	1969	1959	1949	1939	1929
Female	83.39	81.6	80.39	78.59	76.02	73.64	69.85	65.33	59.25
Male	79	76.15	73.87	71.31	69.17	67.98	65.63	62.23	56.72
Total	81.25	78.92	77.12	74.84	72.4	70.61	67.59	63.69	57.94

source: Berkeley human mortality database http://www.mortality.org/hmd/

Although self-annuitization has the advantage of greater liquidity and the opportunity of leaving money for heirs in the event of early decease, its disadvantage is the risk of running out of money before the uncertain date of death. The financial risk associated with self-annuitization is that retirees can outlive their assets in the

Table 4.2: Life Expectancy at 65 Years Old for Canadians

Year	2009	1999	1989	1979	1969	1959	1949	1939	1929
Female	21.68	20.166	19.55	18.80	17.32	15.88	14.89	14.08	13.29
Male	18.61	16.36	15.28	14.54	13.69	13.43	13.26	12.93	12.50
Total	20.26	18.40	17.56	16.74	15.49	14.60	14.02	13.47	12.88

source: Berkeley human mortality database

event of long-run low investment returns and unexpectedly longer life. Therefore, it is very important to find out how much this ruin probability is upon the initial endowment at retirement. Many papers in the field of finance and insurance have studied the 'lifetime ruin probability', the probability the individuals will exhaust their wealth under a fixed consumption strategy, such as Milevsky and Robinson (2000), Huang, Milevsky and Wang (2004), and Moore and Young (2006).

Milevsky and Robinson (2000) studied the approximate distribution of a whole life annuity function. They used Gompertz's law to model mortality and a geometric Brownian motion to model asset price. They fitted the stochastic present value of a continuous whole life annuity with the reciprocal gamma and Type II Johnson distributions and validated these two approximations with numerical results. A numerical case was illustrated to show the impact of asset allocation strategy and gender on the ruin probability. In their example, they showed that a well-diversified portfolio will achieve the lowest ruin probability. Under the same asset allocation strategy, females will have a higher ruin probability than males due to longevity.

Huang, Milevsky and Wang (2004) implemented numerical PDE solution techniques to compute the ruin probability in retirement. They compared their PDE-based values with those quick-and-dirty heuristic approximation methods widely used for ruin problem, such as the reciprocal gamma approximation (RG), the lognormal approximation (LN), and the comonotonic-based lower bound approximation (CLB).

Moore and Young (2006) minimized the ruin probability with varying hazard rates and showed that by updating the hazard rate each year and treating it as a constant, the individual can closely obtain the minimal ruin probability when the true hazard rate is Gompertz. This method results in the ruin probability being close to its minimum.

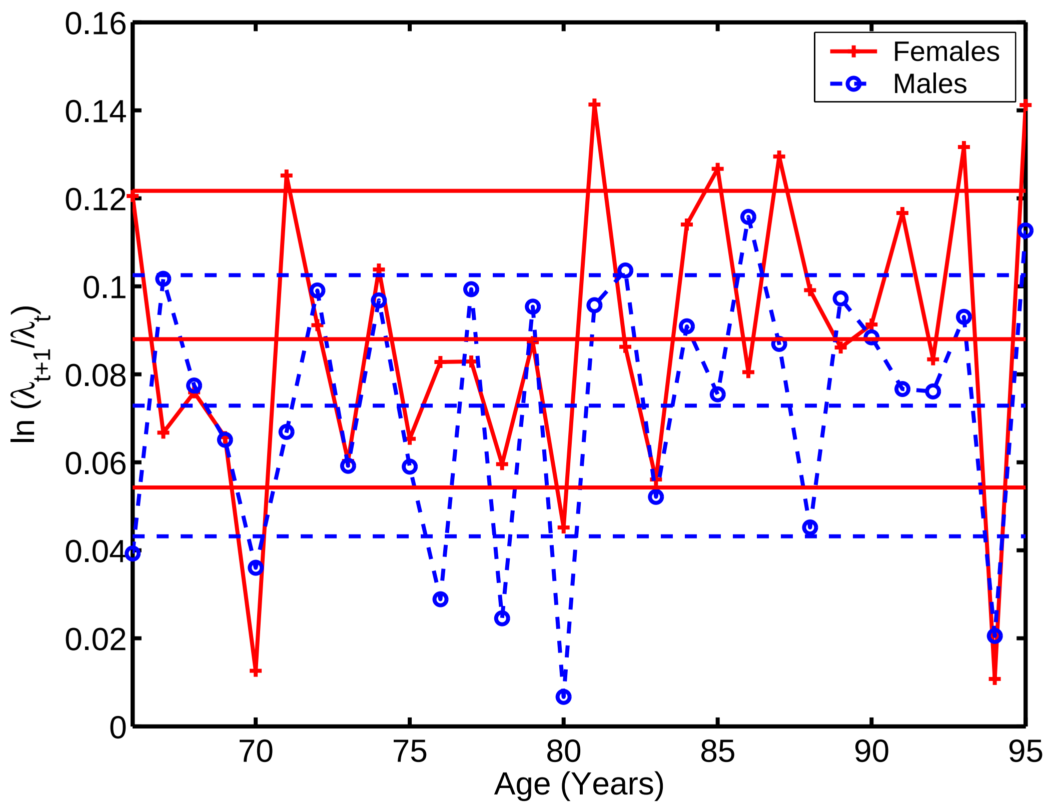
Previous works utilize a constant force of mortality, which is equivalent to assuming that the retiree's future lifetime random variable has exponential distribution, or Gompertz-Makeham (GM) mortality, an exponential function in death rates with age. In reality, the personal mortality rate is much more complex and flexible, and is related to subjective health status, even natural disasters such as earthquakes, epidemics and tornadoes. In this project, we apply a stochastic mortality model. Figure 4.1 plots $\ln(\lambda_{t+1}/\lambda_t)$ as a function of age for the cohort born in

1900. We can observe that $\ln(\lambda_{t+1}/\lambda_t)$ randomly scatters around the average of the mortality rate. As for an individual retiree, his/her mortality rate is much volatile thant the population-wide one, and we assume that personal mortality rates share the same stochastic property as that of the population.

The most widely accepted stochastic mortality model is the Lee-Carter model. For simplicity and ease of handling, we adopt a special form of the Lee-Carter model, i.e., log-normal distribution. We study the effect of this stochastic mortality rate on the ruin probability and compare it with the ruin probability for GM mortality, in which ruin is defined as wealth hitting zero during the lifetime of an individual with various initial wealth, withdrawing \$1 per annum.

The layout of this chapter is as follows. Section 4.2 describes the model of lifetime ruin probability for Gompertz mortality and presents the PDE that governs it, and solves the PDE numerically using the Crank-Nicolson method. Section 4.3 provides the model calibration of lifetime ruin probability under stochastic mortality rate, derives the PDE that the ruin probability must satisfy, solves it numerically using the ADI method, and illustrates the connection between the ruin probabilities for GM mortality and log-normal mortality. The main contributions of this chapter are summarized in Section 4.4, in which we discuss the effect of stochastic mortality on lifetime ruin. Section 4.5 verifies the accuracy of our numerical schemes by performing convergence analysis.

Figure 4.1: $\ln(\lambda_{t+1}/\lambda_t)$ versus Age



The figure shows $\ln(\lambda_{t+1}/\lambda_t)$ versus age for the cohort born in 1900. The middle red solid (blue dashed) line is the average of the mortality rate for females (males), and the other two red solid (blue dashed) lines are this average plus (and minus) the standard deviation. We can observe that $\ln(\lambda_{t+1}/\lambda_t)$ randomly scatters around the average of the mortality rate.

4.2 Lifetime Ruin Probability under Gompertz Mortality

4.2.1 Model Calibration

Existing literature for lifetime ruin probability mainly deals with Gompertz mortality. To have a benchmark for the ruin probability under stochastic mortality which is proven by the historical data, we look at the ruin problem under Gompertz mortality first.

We consider a retiree of age x at time zero. The continuously compounded investment returns are modeled to be normally distributed. This assumption is standard in financial economics, which has been used widely such as Boyle (1976), Black and Scholes (1973). We assume the consumption rate, g, is normalized to 1. The individual's wealth process obeys the following stochastic process

$$dW_t = (\mu_w W_t - g)dt + \sigma_w W_t dB_t^w, \quad W_0 = w, \tag{4.1}$$

where μ_w and σ_w denote the drift and volatility of the investment portfolio, and B_t^w is the Brownian motion driving this process. Note that this investment return may become negative when $\mu_w W_t$ becomes small enough relative to 1, which implies that the process W_t may eventually hit zero, contradicting to the classical geometric Brownian motion (GBM) bounded away from zero in finite time.

We assume the probability that the portfolio holder is still alive at time t is given by

$$ {}_tp_x = e^{-\int_0^t \lambda_{x+s} ds} \tag{4.2} $$

where x denotes the current age of the individual, λ_{x+s} is the hazard function at a future time s, defined by the GM distribution, which is parameterized by three variables (λ_0, m, b):

$$ \lambda_{x+s} = \lambda_0 + \frac{1}{b} e^{\frac{x+s-m}{b}}, \tag{4.3} $$

where s is the time the individual is going to survive, m is the mode of the future lifetime, b is the dispersion constant, and λ_0 is the Makeham term, an age-independent component. In a protected environment where external causes of death are rare (laboratory conditions, low mortality countries, etc.), the age-independent mortality component is often negligible. In this case the formula simplifies to a Gompertz law of mortality (proposed by Benjamin Gompertz in 1825) with exponential increase in death rates with age. The Gompertz-Makeham law of mortality describes the age dynamics of human mortality rather accurately in the age window of about 30-80 years. At more advanced ages, the death rates do not increase as fast as predicted by this mortality law: a phenomenon known as the late-life mortality deceleration, see Olshansky and Carnes (1997). In this section, we use the Gompertz law of mortality since our environment is protected, i.e., $\lambda_0 = 0$.

Now we are ready to define the probability of lifetime ruin:

$$P_G(t, w|\lambda_0, b, m, \mu_w, \sigma_w) = Pr(\inf_{t \le s \le T_x} W_s \le 0 | W_t = w), \tag{4.4}$$

Where T_x is a random variable representing time of death and follows a distribution defined by the Gompertz mortality law. The subscript G on P stands for Gompertz mortality. This is the probability that the net-wealth process W_t hits zero before the retiree dies.

It can be shown that P_G satisfies a Backward Kolmogorov Equation (Subscript G has been dropped on P):

$$\lambda_{x+t} P = P_t + (w\mu_w - g)P_w + \frac{1}{2} w^2 \sigma_w^2 P_{ww}, \tag{4.5}$$

with terminal and boundary conditions:

$$P(t, 0) = 1, P(t, \infty) = 0, P(\infty, w) = 0. \tag{4.6}$$

These conditions are intuitively obvious to even the most casual observer. Firstly, when the individual has zero wealth, i.e., $w = 0$, the probability of lifetime ruin must be 100 percent. Secondly, when the retiree has an infinite amount of money, i.e., $w = \infty$, the chances for him/her to become ruined drop to zero, compared to the standardized consumption rate g and longevity risk. Finally, the probability of lifetime ruin is zero at time ∞ is due to the fact that the hazard rate is so large at advanced ages, so individuals die right away, without having time to get ruined.

By now we have obtained the PDE that the ruin probability must satisfy and prescribed its corresponding boundary and terminal conditions. We are ready to address a numerical method for finding solutions of the second-order differential equations, which is described in Section 4.2.2.

4.2.2 Numerical Schemes

In this section, we illustrate the numerical method that solves PDE (4.5), which is second-order linear with cross derivatives. The Crank-Nicolson method is applied due to second order accuracy for w, and the upwind scheme is chosen for the first order derivative which avoids oscillations of the solution. The truncated computational domain for P_G is $[t, w] \in ([0, 60], [0, 50])$, i.e., $T = 60$ and $w_{max} = 50$. A uniform grid with equal spacing Δt and Δw is used. The PDE can be discretized using Crank-Nicolson method as below

$$\begin{aligned}
&\frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{\sigma_w^2 w_j^2}{2} \frac{P_{j+1}^{n+1} + P_{j-1}^{n+1} - 2P_j^{n+1} + P_{j+1}^n + P_{j-1}^n - 2P_j^n}{2\Delta w^2} \\
&+(\mu_w w_j - g)\frac{P_j^{n+1} - P_{j-1}^{n+1} + P_j^n - P_{j-1}^n}{2\Delta w} = \lambda_{x+t_n} P_i^n, \quad \mu_w w_j - g \leq 0, \quad (4.7) \\
&\frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{\sigma_w^2 w_j^2}{2} \frac{P_{j+1}^{n+1} + P_{j-1}^{n+1} - 2P_j^{n+1} + P_{j+1}^n + P_{j-1}^n - 2P_j^n}{2\Delta w^2} \\
&+(\mu_w w_j - g)\frac{P_{j+1}^{n+1} - P_j^{n+1} + P_{j+1}^n - P_j^n}{2\Delta w} = \lambda_{x+t_n} P_i^n, \quad \mu_w w_j - g \geq 0, \quad (4.8)
\end{aligned}$$

where P_j^n is the grid function that approximates $P(t, w)$ on grid points (t_n, w_j).

The computational boundary conditions at $j = 1$ and $j = J + 1$ and terminal conditions at $n = N+1$ must be provided when solving PDE (4.5) with an implicit numerical method. They can be derived as,

$$P_j^{N+1} = 0, \quad j = 1, 2, \cdots, \quad J+1; \tag{4.9}$$

$$P_1^n = 1, \quad n = 1, 2, \cdots, \quad N+1; \tag{4.10}$$

$$P_{J+1}^n = 0, \quad n = 1, 2, \cdots, \quad N+1. \tag{4.11}$$

$j = 1$ and $j = J+1$ correspond to $w = 0$ and $w = w_{max}$. With these discretized terminal and boundary conditions, the discrete equations (4.7)-(4.8) can be solved by matching from time t_{n+1} to t_n, starting from $n = N+1$. In this uniform grid, we can solve for all the probabilities on all the grid points by iteration.

4.2.3 Numerical Examples

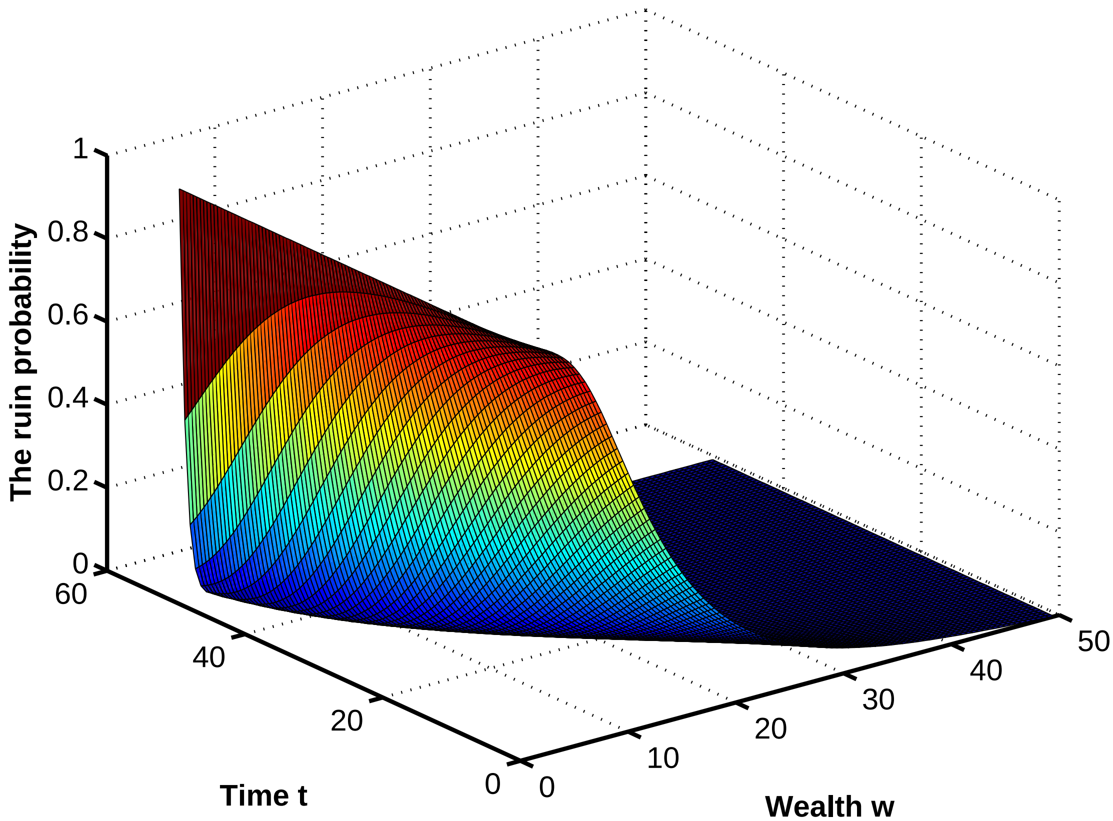
Now we are ready to compute the ruin probability under Gompertz mortality as well as study the effects of some important parameters related to the investment portfolio.

Figure 4.2 displays the lifetime ruin probability P_G in 3D as a function of time t and wealth w using the numerical PDE method. It can be observed that for any fixed time t, P_G is a decreasing function of wealth w. This is intuitively pleasant because when w is larger and all other parameters are kept fixed, the chances to become ruined tend to be smaller. The market parameters for the stochastic process driving wealth are $\mu_w = 0.07$ and $\sigma_w = 0.20$, which are consistent with historical evidence based on the behavior of a broad portfolio of common equities during the last 75 years, reported by Ibbotson Associates (2002). The parameters for the Gompertz mortality are $m = 87.8$ and $b = 9.5$, which are based on a Gompertz approximation to the unisex RP-2000 mortality table compiled by the U.S.-based Society of Actuaries (see Wang (2006)).

We are mainly interested in the ruin probability at time zero since it will give individuals a hint about how much money to save before retirement in order to have a ruin probability that is acceptable to them. Thus we will focus on the ruin at time zero starting from now, although we can obtain all the ruin probabilities for any time and any wealth. In addition to the mortality rate, the main factors that decide the lifetime ruin are the drift and volatility of the investment portfolio.

Figure 4.3 displays the lifetime ruin probability for a 65-year old individual, as a function of their initial wealth (\$10 to \$30) when the volatility takes the values (0.10, 0.15, 0.20, 0.25, 0.30). The other parameters used for this figure are $\mu_w = 0.07$, $m = 87.8$, $b = 9.5$, and $g = 1$. We can see that for any given initial wealth, when the volatility is higher, which means that the wealth process has more chances to hit zero, lifetime ruin is higher. If individuals invest more in stable assets (lower volatility), although the wealth has less chance to grow, it has less chance to hit zero as well, which will lead to lower ruin probability. Therefore, more risk-

Figure 4.2: The Ruin Probability under Gompertz Mortality



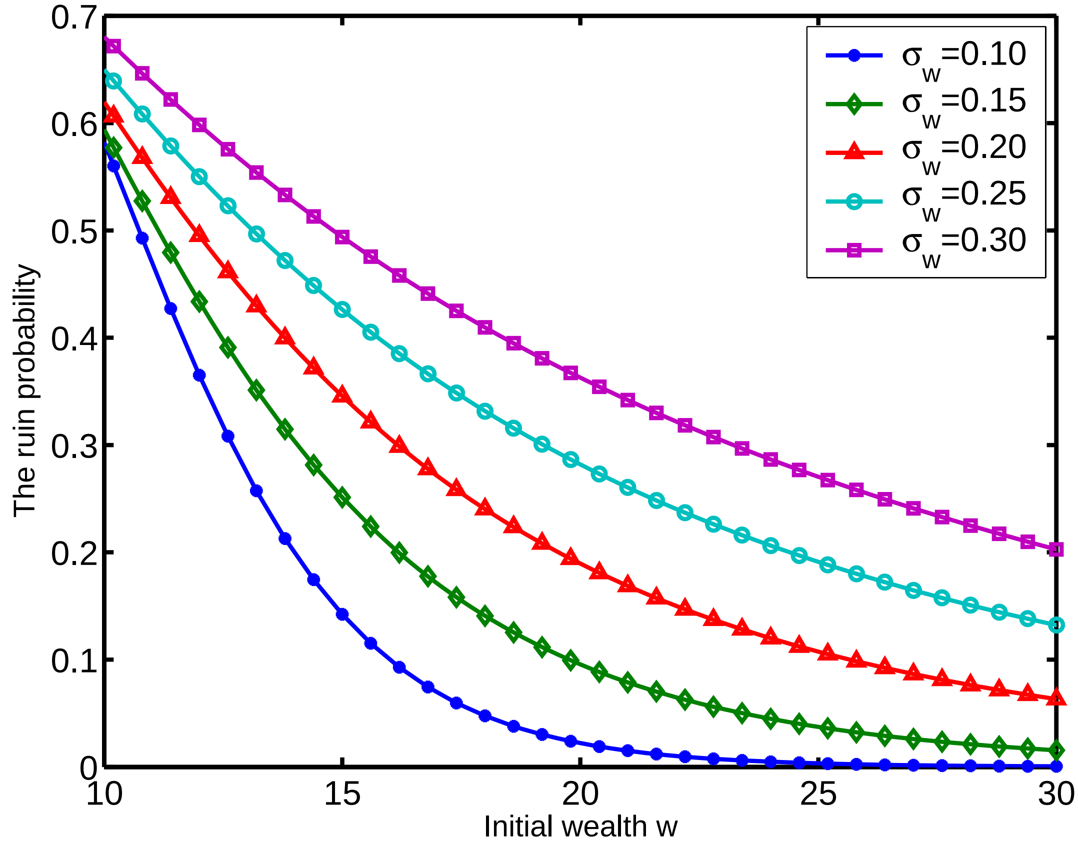
The figure shows the ruin probability under a Gompertz mortality rate that is fitted to the unisex $RP-2000$ mortality table compiled by the U.S.-based Society of Actuaries for a retiree who is 65 years old. The fitted parameters are $(m, b) = (87.8, 9.5)$. The market parameters for the stochastic wealth process are μ_w=0.07 and $\sigma_w = 0.20$, which is consistent with historical evidence. The withdrawal rate is \$1 per annum.

averse individuals should choose to invest in lower volatility assets to avoid higher ruin probabilities. Note that this does not imply that safer asset allocations are necessarily preferable, since in actual portfolios, decreasing volatility also implies decreasing growth rates. Another observation from this figure is that higher initial wealth levels lead to lower ruin probabilities if the expected return and withdrawal rate are the same, which is intuitively pleasant. Therefore, individuals need to save more money before retirement to have a lower ruin probability after retirement.

Figure 4.4 displays the lifetime ruin probability as a function of wealth w with five different drifts (μ_w=0.03, 0.05, 0.07, 0.09, 0.11) for parameters $\sigma_w = 0.25$, $m = 87.8$, $b = 9.5$, and $g = 1$. Note that we have varied the values of μ_w around its expected value 0.07 to see how sensitive the lifetime ruin is to it. It can be observed that the ruin probability is a decreasing function of initial wealth w, which is consistent with our intuition. We can also see that higher levels of expected investment return leads to lower lifetime ruin probability. Therefore, retirees will choose to invest in assets with higher expected returns to obtain lower ruin probability in their remaining lifetime. Therefore, it is a trade-off for retirees with a given initial wealth whether to choose to invest in risky assets or riskless assets since higher expected returns usually come with higher volatilities.

In this section, we have investigated the behavior of the lifetime ruin probability under Gompertz mortality, and the effects of the two financial parameters μ_w and

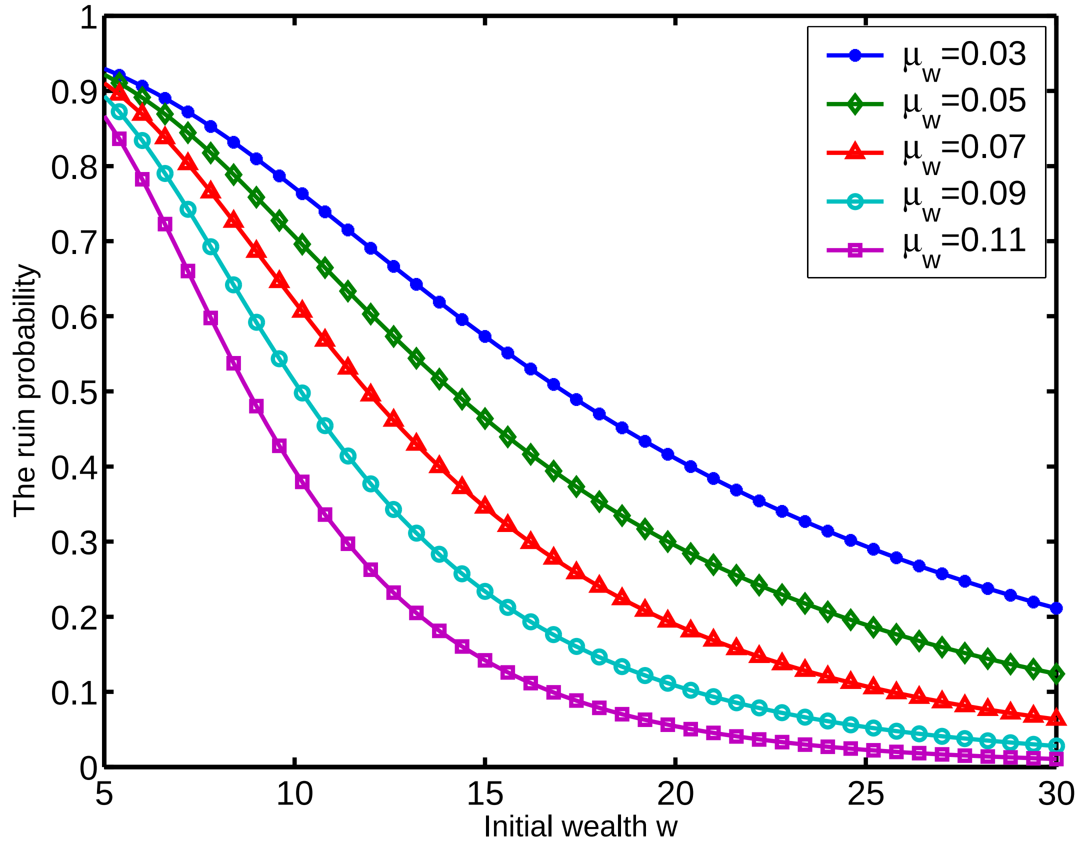
Figure 4.3: The Ruin Probability as a Function of Wealth w with Different σ_w



The figure shows the ruin probability as a function of the initial wealth w for different volatilities of the wealth process. The parameters used are $\mu_w = 0.07$, $m = 87.8$, $b = 9.5$, and $g = 1$.

σ_w for the stochastic wealth process on the ruin. In next section, we will study the effect of stochastic hazard rates on lifetime ruin probability.

Figure 4.4: The Ruin Probability as a Function of Wealth w with Different μ_w



The figure shows the ruin probability as a function of the initial wealth w for different expected returns of the wealth process. The parameters used are $\sigma_w = 0.25$, $m = 87.8$, $b = 9.5$, and $g = 1$.

4.3 Lifetime Ruin Probability under Stochastic Mortality

All the previous work done deals with Gompertz Mortality, which is also the basic assumption of most financial advisers' solutions. This assumption ignores the reality that retirees do not have a fixed mortality rate at any specific age, it may fluctuate over time. In this section, we study this randomness by assuming a stochastic process of hazard rate, which has been proven by the historical data in Section 4.1.

In order to model the stochastic process of the hazard rate, we assume its intensity Λ_{x+t} at time t for an individual aged x evolves with a log-normal distribution, a special form of the Lee-Carter model

$$d\Lambda_{x+t} = \mu_\lambda \Lambda_{x+t} dt + \sigma_\lambda \Lambda_{x+t} dB_t^\lambda, \quad \Lambda_x = \lambda, \tag{4.12}$$

where μ_λ and σ_λ are the drift and volatility coefficients, B_t^λ is the Brownian motion driving this process. This is a classical GBM, which is bounded away from zero in finite time. Note the probability that the individual is alive at time t, provided that the individual is alive at time $s < t$, is given by ${}_tp_x \ / \ {}_sp_x$. The stochastic wealth process is defined by equation (4.1) which we presented back in Section 4.2.1. We define our new lifetime ruin probability under stochastic hazard rates as

$$P_S(t, w, \lambda | \mu_w, \sigma_w, \mu_\lambda, \sigma_\lambda, \rho_{w\lambda}) := Pr(\inf_{t \le s \le T_x} W_s \le 0 | W_t = w, \lambda_t = \lambda), \tag{4.13}$$

where the subscript S on P means stochastic hazard rate, and T_x is the random variable representing time of death of the portfolio holder as in Section 4.2.1.

Let $\Lambda_{x+t} = \lambda$ and $W_t = w$, we denote the ruin probability at t as

$$P_S(t, w, \lambda) = P\left[\tau < T_x | W_t = w, \Lambda_{x+t} = \lambda\right]. \tag{4.14}$$

The subscript S will be dropped on P below for simplicity to derive the PDE that P_S must satisfy. Since

$$
\begin{aligned}
P(t, w, \lambda) &= E\left[\frac{{}_{t+h}p_x}{{}_t p_x} P(t+h, W_{t+h}, \Lambda_{x+t+h}) + \left(1 - \frac{{}_{t+h}p_x}{{}_t p_x}\right) 0\right] \qquad (4.15)\\
&= E\left[\frac{{}_{t+h}p_x}{{}_t p_x} P(t+h, W_{t+h}, \Lambda_{x+t+h})\right], \qquad (4.16)
\end{aligned}
$$

and

$$
P(t+h, W_{t+h}, \Lambda_{x+t+h}) = P(t, W_t, \Lambda_{x+t}) + \int_t^{t+h} dP, \qquad (4.17)
$$

we have

$$
E\left[\left(1 - \frac{{}_{t+h}p_x}{{}_t p_x}\right) P(t, w, \lambda)\right] = E\left[\frac{{}_{t+h}p_x}{{}_t p_x} \int_t^{t+h} dP\right]. \qquad (4.18)
$$

Applying Ito's lemma to $P(t, w, \lambda)$, we obtain

$$
\begin{aligned}
dP &= P_t + P_w dW_t + P_\lambda d\Lambda_{x+t} + P_{w\lambda} dW_t d\Lambda_{x+t} + \frac{1}{2} P_{ww} dW_t^2 + \frac{1}{2} P_{\lambda\lambda} d\,{}_x\Lambda_t^2\\
&= \mathcal{A}P dt + \sigma_w w P_w dB_t^w + \sigma_\lambda \lambda P_\lambda dB_t^\lambda, \qquad (4.19)
\end{aligned}
$$

where $\mathcal{A}P$ is a second order differential operator as below

$$
\mathcal{A}P = P_t + (\mu_w w - g) P_w + \mu_\lambda \lambda P_\lambda + \rho_{w\lambda} \sigma_\lambda \sigma_w w \lambda P_{w\lambda} + \frac{1}{2} \sigma_w^2 w^2 P_{ww} + \frac{1}{2} \sigma_\lambda^2 \lambda^2 P_{\lambda\lambda}. \qquad (4.20)
$$

In which $\rho_{w\lambda}$ is the correlation between dB_t^w and dB_t^λ. According to Smith (1999), wealth can buy health, and health can improve wealth accumulation. Correlations between health and wealth are much lower among retired households. It is still an open question regarding how much this correlation is. Therefore, we assume the values of $\rho_{w\lambda}$ can be either positive or negative, depending on individuals' spending habits. Dividing (4.18) by h and letting $h \to 0$, we obtain

$$\lambda P = P_t + (\mu_w w - g)P_w + \mu_\lambda \lambda P_\lambda + \rho_{w\lambda}\sigma_\lambda \sigma_w w \lambda P_{w\lambda} + \frac{1}{2}\sigma_w^2 w^2 P_{ww} + \frac{1}{2}\sigma_\lambda^2 \lambda^2 P_{\lambda\lambda}. \quad (4.21)$$

The terminal and boundary conditions for the above PDE are

$$\begin{cases} P(t,0,\lambda) = 1, \quad P(t,\infty,\lambda) = 0, \\ P(t,w,0) = f(t,w), \quad P(t,w,\infty) = 0, \\ P(\infty,w,\lambda) = 0. \end{cases} \quad (4.22)$$

The reason that at time ∞, the probability of ruin is 0 is that the probability of being alive at that time is 0 for any positive wealth. If the initial wealth is 0, the individual will immediately become ruined, and if the initial wealth is infinitely large, the individual will never become ruined during the remaining lifetime because human being's life span is finite. If the individual dies immediately, which corresponds to the condition $\lambda = \infty$, then he/she will never have a chance to become ruined. When the hazard rate is 0, the ruin probability must satisfy the following simpler PDE, which is obtained by setting $\lambda = 0$ in equation (4.21).

$$f_t + (\mu_w w - g)f_w + \frac{1}{2}\sigma_w^2 w^2 f_{ww} = 0. \quad (4.23)$$

Here we have used a new notation $f(t,w)$ to denote the ruin probability corresponding to $\lambda = 0$. The reason why $f(t,w)$ must solve this PDE is that $P(t,w,\lambda)$

is continuously differentiable with respect to all its independent variables. In consequence, the terminal and boundary conditions for $f(t,w)$ are,

$$f(t,0) = 1, \quad f(t,\infty) = 0, \quad f(\infty, w) = 0. \tag{4.24}$$

The explanations for these conditions are similar to what we have done to the ruin probability P_G. It can be observed that equation (4.23) is a special case of equation (4.5) by setting the hazard rate to be zero. Thus we can use the same numerical method as in Section 4.2.1 to solve $f(t,w)$.

4.3.1 Conditions for $P_S = P_G$

When the stochastic hazard rate breaks down to the GM mortality, the lifetime ruin P_S should collapse to P_G. To find out the conditions that must be satisfied for this scenario, we rewrite the GM mortality λ_{x+t} as

$$d\lambda_{x+t} = \frac{1}{b}\lambda_{x+t}dt. \tag{4.25}$$

We repeat the GBM for the stochastic hazard rate Λ_{x+t} here for convenience's sake

$$d\Lambda_{x+t} = \mu_\lambda \Lambda_{x+t}dt + \sigma_\lambda \Lambda_{x+t}dB_t^\lambda.$$

The obvious two conditions that must be satisfied are

$$\sigma_\lambda = 0, \quad \mu_\lambda = \frac{1}{b}. \tag{4.26}$$

The initial mortality rate must be equal, so that we have

$$\lambda = \frac{1}{b} e^{\frac{x-m}{b}}. \tag{4.27}$$

Therefore, the following conditions must be satisfied in order to make the stochastic mortality rate collapses to the GM mortality

$$\sigma_\lambda = 0, \quad \mu_\lambda = \frac{1}{b}, \quad \lambda_x = \frac{1}{b} e^{\frac{x-m}{b}}. \tag{4.28}$$

These conditions provide us criteria to match P_S and P_G in future numerical computations.

4.3.2 Numerical Methods for P_S: ADI Method

The traditional method for solving the 2-dimensional (2-D) linear partial differential equation (4.21) is the Crank-Nicolson method. This method incurs a very complicated set of equations in two dimensions, which are very expensive to solve. Instead, alternating direction implicit (ADI) method can successfully avoid this. The advantage of the ADI method is that the equations that have to be solved in each time step have a simpler structure and can be solved efficiently with the tridiagonal (banded with bandwidth 3) matrix algorithm, which significantly reduces the computational complexity. The idea behind the ADI method is to split the finite difference equations into two, one with the w-derivatives taken implicitly and the next with the λ-derivatives taken implicitly. The system of equations involved is symmetric and tridiagonal, and is typically solved using tridiagonal matrix algorithm. It can be shown that the ADI method is unconditionally stable and second order accuracy in time and space (see Burden and Faires, 2011). The upwind scheme is applied for the first order derivative with respect to wealth w to avoid big oscillations.

- Discretization of the computational domain

  The computational domain is truncated to be $(t, w, \lambda) = ([0, 50], [0, 30], [0, 10])$ due to the assumption that the maximal life span does not exceed 115 years old, the initial wealth is within 50 units of the normalized wealth, and the maximal hazard rate is less than 10. A uniform grid with equal spacing Δt, Δw and $\Delta\lambda$ are used. In details,

$$\Delta t = \frac{Tmax}{N}, \quad t(n) = (n-1)\Delta t, \quad n = 1, 2, \cdots, N+1, \tag{4.29}$$

$$\Delta w = \frac{Wmax}{I}, \quad w(i) = (i-1)\Delta w, \quad i = 1, 2, \cdots, I+1, \tag{4.30}$$

$$\Delta\lambda = \frac{\lambda max}{J}, \quad \lambda(j) = (j-1)\Delta\lambda, \quad j = 1, 2, \cdots, J+1, \tag{4.31}$$

  where N, I,J are the number of intervals divided for t, w and λ respectively. Let P^n_{ij} be the grid function which approximates $P(t, w, \lambda)$ on the grid point (t_n, w_i, λ_j). The boundary conditions on the computational boundaries $i = 1, I+1$, $j = 1, J+1$ and the terminal condition can be derived as

$$P^n_{1j} = 1, \quad P^n_{I+1,j} = 0, \quad P^n_{i1} = f^n_i, \quad P^n_{i,J+1} = 0, \quad P^{N+1}_{ij} = 0, \tag{4.32}$$

where f_i^n is the grid function approximating $f(t,w)$ on the grid point (t_n, w_i).

- Implicit discretization along w axis

To apply the ADI method, we first take time and w-derivatives implicitly and all the other derivatives explicitly (including the cross derivative). At the same time, the upwind scheme is used to discretize P_w for the stability of the algorithm. First we rewrite PDE (4.21) so that the time derivative and all derivatives with respect to w are on one side of the equation and all the other terms are on the other side of the equation

$$P_t + (\mu_w w - g)P_w + \frac{1}{2}\sigma_w^2 w^2 P_{ww} = \lambda P - \mu_\lambda \lambda P_\lambda - \rho_{w\lambda}\sigma_\lambda\sigma_w w\lambda P_{w\lambda} - \frac{1}{2}\sigma_\lambda^2\lambda^2 P_{\lambda\lambda}. \tag{4.33}$$

At interior grid point $(t_{n+\frac{1}{2}}, w_i, \lambda_j)$, after applying the central difference quotients for λ and upwind scheme for P_w we obtain

$$\begin{aligned}
& \frac{P_{ij}^{n+1} - P_{ij}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} + (\mu_w w_i - g)\frac{P_{ij}^{n+\frac{1}{2}} - P_{i-1,j}^{n+\frac{1}{2}}}{\Delta w} + \frac{1}{2}\sigma_w^2 w_i^2 \frac{P_{i+1,j}^{n+\frac{1}{2}} + P_{i-1,j}^{n+\frac{1}{2}} - 2P_{i,j}^{n+\frac{1}{2}}}{\Delta w^2} \\
= \;& -\mu_\lambda\lambda_j \frac{P_{i,j+1}^{n+1} - P_{i,j-1}^{n+1}}{2\Delta\lambda} - \rho_{w\lambda}\sigma_\lambda\sigma_w w_i\lambda_j \frac{P_{i+1,j+1}^{n+1} + P_{i-1,j-1}^{n+1} - P_{i-1,j+1}^{n+1} - P_{i+1,j-1}^{n+1}}{4\Delta w\Delta\lambda} \\
+ \;& \lambda_j P_{i,j}^{n+\frac{1}{2}} - \frac{1}{2}\sigma_\lambda^2\lambda_j^2 \frac{P_{i,j+1}^{n+1} + P_{i,j-1}^{n+1} - 2P_{i,j}^{n+1}}{\Delta\lambda^2}, \quad \text{if } \mu_w w_i - g \le 0,
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
& \frac{P_{ij}^{n+1} - P_{ij}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} + (\mu_w w_i - g)\frac{P_{i+1,j}^{n+\frac{1}{2}} - P_{i,j}^{n+\frac{1}{2}}}{\Delta w} + \frac{1}{2}\sigma_w^2 w_i^2 \frac{P_{i+1,j}^{n+\frac{1}{2}} + P_{i-1,j}^{n+\frac{1}{2}} - 2P_{i,j}^{n+\frac{1}{2}}}{\Delta w^2} \\
= \;& -\mu_\lambda\lambda_j \frac{P_{i,j+1}^{n+1} - P_{i,j-1}^{n+1}}{2\Delta\lambda} - \rho_{w\lambda}\sigma_\lambda\sigma_w w_i\lambda_j \frac{P_{i+1,j+1}^{n+1} + P_{i-1,j-1}^{n+1} - P_{i-1,j+1}^{n+1} - P_{i+1,j-1}^{n+1}}{4\Delta w\Delta\lambda} \\
+ \;& \lambda_j P_{i,j}^{n+\frac{1}{2}} - \frac{1}{2}\sigma_\lambda^2\lambda_j^2 \frac{P_{i,j+1}^{n+1} + P_{i,j-1}^{n+1} - 2P_{i,j}^{n+1}}{\Delta\lambda^2}, \quad \text{if } \mu_w w_i - g \ge 0.
\end{aligned} \tag{4.35}$$

After collecting like terms on both sides of the equals sign, we have

$$
\begin{aligned}
& \left(\frac{2}{\Delta t}+\lambda_j-\frac{\mu_w w_i-g}{\Delta w}+\frac{\sigma_w^2 w_i^2}{\Delta w^2}\right)P_{i,j}^{n+\frac{1}{2}}-\frac{\sigma_w^2 w_i^2}{2\Delta w^2}P_{i+1,j}^{n+\frac{1}{2}}+\left(\frac{\mu_w w_i-g}{\Delta w}-\frac{\sigma_w^2 w_i^2}{2\Delta w^2}\right)P_{i-1,j}^{n+\frac{1}{2}} \\
= & \frac{\mu_\lambda \lambda_j}{2\Delta\lambda}(P_{i,j+1}^{n+1}-P_{i,j-1}^{n+1})+\frac{\rho_{w\lambda}\sigma_\lambda\sigma_w w_i\lambda_j}{4\Delta w\Delta\lambda}(P_{i+1,j+1}^{n+1}+P_{i-1,j-1}^{n+1}-P_{i-1,j+1}^{n+1}-P_{i+1,j-1}^{n+1}) \\
+ & \frac{2}{\Delta t}P_{i,j}^{n+1}+\frac{\sigma_\lambda^2\lambda_j^2}{2\Delta\lambda^2}(P_{i,j+1}^{n+1}+P_{i,j-1}^{n+1}-2P_{i,j}^{n+1}), \quad \text{if } \mu_w w_i-g\le 0,
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
& \left(\frac{2}{\Delta t}+\lambda_j+\frac{\mu_w w_i-g}{\Delta w}+\frac{\sigma_w^2 w_i^2}{\Delta w^2}\right)P_{i,j}^{n+\frac{1}{2}}-\frac{\sigma_w^2 w_i^2}{2\Delta w^2}P_{i-1,j}^{n+\frac{1}{2}}-\left(\frac{\mu_w w_i-g}{\Delta w}+\frac{\sigma_w^2 w_i^2}{2\Delta w^2}\right)P_{i+1,j}^{n+\frac{1}{2}} \\
= & \frac{\mu_\lambda \lambda_j}{2\Delta\lambda}(P_{i,j+1}^{n+1}-P_{i,j-1}^{n+1})+\frac{\rho_{w\lambda}\sigma_\lambda\sigma_w w_i\lambda_j}{4\Delta w\Delta\lambda}(P_{i+1,j+1}^{n+1}+P_{i-1,j-1}^{n+1}-P_{i-1,j+1}^{n+1}-P_{i+1,j-1}^{n+1}) \\
+ & \frac{2}{\Delta t}P_{i,j}^{n+1}+\frac{\sigma_\lambda^2\lambda_j^2}{2\Delta\lambda^2}(P_{i,j+1}^{n+1}+P_{i,j-1}^{n+1}-2P_{i,j}^{n+1}), \quad \text{if } \mu_w w_i-g\ge 0.
\end{aligned}
\tag{4.37}
$$

For each fixed value λ_j, varying i from 2 to I, we will obtain an equation system for vector $(P_{2j}^{n+\frac{1}{2}}, P_{3j}^{n+\frac{1}{2}}, \cdots, P_{I,j}^{n+\frac{1}{2}})$. Together with the boundary and terminal conditions, the discrete equations can be solved by matching from time t_n to $t_{n+\frac{1}{2}}$, starting from $j=1$. Now we have all the probabilities at time level $t_{n+\frac{1}{2}}, n=1,2,\cdots,N$.

- Implicit discretization along λ axis

Given all the probabilities at time level $t_{n+\frac{1}{2}}$, how can we get the solutions at time level t_{n+1}? To this end, we discretize time and λ-derivatives implicitly and all the other derivatives explicitly. First we rewrite the 2-D PDE (4.21) as

$$P_t + \mu_\lambda \lambda P_\lambda + \frac{1}{2}\sigma_\lambda^2 \lambda^2 P_{\lambda\lambda} = \lambda P - (\mu_w w - g)P_w - \rho_{w\lambda}\sigma_\lambda\sigma_w w\lambda P_{w\lambda} - \frac{1}{2}\sigma_w^2 w^2 P_{ww}. \tag{4.38}$$

Discretize this equation at interior point (t_n, w_i, λ_j), we have

$$\begin{aligned}
& \frac{P_{ij}^{n+\frac{1}{2}} - P_{ij}^n}{\frac{\Delta t}{2}} + \mu_\lambda \lambda_j \frac{P_{i,j+1}^n - P_{i,j-1}^n}{2\Delta\lambda} + \frac{1}{2}\sigma_\lambda^2 \lambda_j^2 \frac{P_{i,j+1}^n + P_{i,j-1}^n - 2P_{i,j}^n}{\Delta\lambda^2} \\
= \quad & \lambda_j P_{i,j}^n - (\mu_w w_i - g)\frac{P_{i+1,j}^{n+\frac{1}{2}} - P_{i-1,j}^{n+\frac{1}{2}}}{2\Delta w} - \frac{1}{2}\sigma_w^2 w_i^2 \frac{P_{i+1,j}^{n+\frac{1}{2}} + P_{i-1,j}^{n+\frac{1}{2}} - 2P_{i,j}^{n+\frac{1}{2}}}{\Delta w^2} \\
- \quad & \rho_{w\lambda}\sigma_\lambda\sigma_w w_i \lambda_j \frac{P_{i+1,j+1}^{n+\frac{1}{2}} + P_{i-1,j-1}^{n+\frac{1}{2}} - P_{i-1,j+1}^{n+\frac{1}{2}} - P_{i+1,j-1}^{n+\frac{1}{2}}}{4\Delta w \Delta\lambda}.
\end{aligned} \tag{4.39}$$

After collecting like terms on both sides, we have

$$\begin{aligned}
& \left(\frac{2}{\Delta t} + \lambda_j + \frac{\sigma_\lambda^2 \lambda_j^2}{d\lambda^2}\right)P_{i,j}^n + \left(\frac{\mu_\lambda \lambda_j}{2\Delta\lambda} - \frac{\sigma_\lambda^2\lambda_j^2}{2\Delta\lambda^2}\right)P_{i,j-1}^n - \left(\frac{\mu_\lambda\lambda_j}{2\Delta\lambda} + \frac{\sigma_\lambda^2\lambda_j^2}{2\Delta\lambda^2}\right)P_{i,j+1}^n \\
= \quad & \frac{2}{\Delta t}P_{i,j}^{n+\frac{1}{2}} + \frac{\mu_w w_i - g}{2\Delta w}\left(P_{i+1,j}^{n+\frac{1}{2}} - P_{i-1,j}^{n+\frac{1}{2}}\right) + \frac{\sigma_w^2 w_i^2}{2\Delta w^2}\left(P_{i+1,j}^{n+\frac{1}{2}} + P_{i-1,j}^{n+\frac{1}{2}} - 2P_{i,j}^{n+\frac{1}{2}}\right) \\
+ \quad & \frac{\rho_{w\lambda}\sigma_\lambda\sigma_w w_i\lambda_j}{4\Delta w\Delta\lambda}\left(P_{i+1,j+1}^{n+\frac{1}{2}} + P_{i-1,j-1}^{n+\frac{1}{2}} - P_{i-1,j+1}^{n+\frac{1}{2}} - P_{i+1,j-1}^{n+\frac{1}{2}}\right).
\end{aligned} \tag{4.40}$$

Therefore, for each fixed w_i, varying j from 2 to J, we will obtain an equation system for vector $(P_{i2}^n, P_{i3}^n, \cdots, P_{i,J}^n)$. Again with the terminal and boundary conditions, we can solve this system and acquire all the values for P_S at time level t_{n+1}. Repeating the above ADI split from $n = 1$ to $n = N$, we will obtain the ruin probabilities at all grid points.

4.3.3 Numerical Results

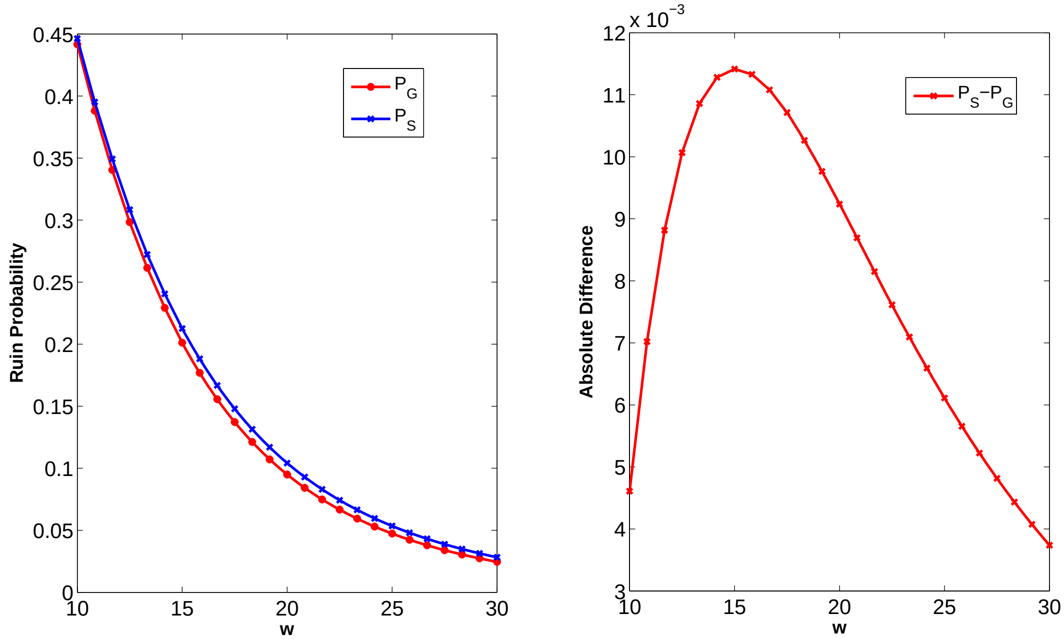
The results of the convergence analysis from Section 4.5 give us confidence that our numerical results are reliable. To further verify our algorithm, we plot the ruin probabilities P_S and P_G when the conditions $\sigma_\lambda = 0$, $\mu_\lambda = \frac{1}{b}$, and $\lambda_x = \frac{1}{b}e^{\frac{x-m}{b}}$ are satisfied for $\mu_w = 0.07$, $\sigma_w = 0.2$, $m = 87.8$, $b = 9.5$, $x = 65$, and $g = 1$ in Figure 4.5. From section 4.3.1, we know that P_S should collapse to P_G under the above conditions. We observe from this figure that the absolute difference of the two ruin probabilities is less than 1.2 percent. This is a good agreement considering round-off errors and truncation errors for both P_S and P_G.

To compare the numerical results for the ruin probabilities under stochastic and Gompertz mortality rates, we fit the mortality parameters for Canadian cohorts born in 1900 and 1920 to the historical data first. To this end, we write equation (4.12) in discrete form

$$\frac{\lambda_{t+1} - \lambda_t}{\lambda_t} = \mu_\lambda + \sigma_\lambda(dB^\lambda_{t+1} - dB^\lambda_t). \tag{4.41}$$

Therefore, the drift μ_λ can be estimated by using the average of $\frac{\lambda_{t+1}-\lambda_t}{\lambda_t}$ due to the fact that the standard deviation of dB^λ_t is 0, and σ_λ can be estimated by the standard deviation of $\frac{\lambda_{t+1}-\lambda_t}{\lambda_t}$. The parameters estimated for cohorts born in 1900 and 1920 are listed in Table 4.3 for males, females and the total population. Although the mortality rate for females is less than that for males at any given

Figure 4.5: Numerical Test for the Match of P_S and P_G



The figure shows the ruin probability P_G for $m = 87.8$, $b = 9.5$, and P_S for $\mu_\lambda = \frac{1}{b}$, $\sigma_\lambda = 0$, and $\lambda_x = \frac{1}{b}e^{\frac{x-m}{b}}$. The market parameters used are $\mu_w = 0.07$ and $\sigma_w = 0.20$. It can be seen that P_S and P_G agree very well.

age, the drift and variance which measure the derivative of the mortality rate are much greater for females than that for males. This is consistent with Milevsky and Young (2006) in which $(m, b) = (88.18, 10.5)$ for males and $(m, b) = (92.63, 8.78)$ for females due to the fact that smaller b means higher μ_λ.

Table 4.3: Parameter Estimation for Stochastic Mortality

Parameters	Cohorts	Females	Males	Total
Drift	1900 (Age 65-95)	0.0886	0.0733	0.0771
Variance	1900 (Age 65-95)	0.0337	0.0297	0.0291
Drift	1920 (Age 65-88)	0.0949	0.0759	0.0816
Variance	1920 (Age 65-88)	0.0402	0.0387	0.0357

Source: Calculations by author from Berkeley human mortality database. We estimate parameters using data starting from age 65 because we care about the retirement period only.

To estimate parameters (m, b) for GM mortality $\lambda_{x+t} = \frac{1}{b} e^{\frac{x+t-m}{b}}$, we have known that $b = \frac{1}{\mu_\lambda}$ from the analysis of section 4.3.1. Therefore, we only need to estimate parameter m. To that end, we rewrite the GM mortality as $m = x + t - b \ln(b\lambda_{x+t})$, so that m can be estimated as the average of this data. The parameters estimated for cohorts born in 1900 and 1920 are summarized in Table 4.4.

Table 4.4: Parameter Estimation for Gompertz Mortality

Parameters	Cohorts	Females	Males	Total
m	1900 (Age 65-95)	84.1120	79.3560	82.1897
b	1900 (Age 65-95)	11.2914	10.6020	10.5339
m	1920 (Age 65-88)	84.9040	80.4396	84.4228
b	1920 (Age 65-88)	10.5339	13.1726	12.2540

Source: Calculations by author from Berkeley human mortality database.

4.3.3.1 Numerical Results Comparison for P_S and P_G

Now we are ready to investigate the effect of stochastic mortality under realistic parameter values using historical data. Without loss of generality, we will do these comparison using the parameters for the cohort born in 1900. It is known that the lifetime ruin probability P_S is a function of λ given any initial wealth w, and P_G is only a point at time zero. Therefore, we plot both P_S and P_G in one figure for three different initial wealth values: $w = 10$, $w = 20$, and $w = 30$.

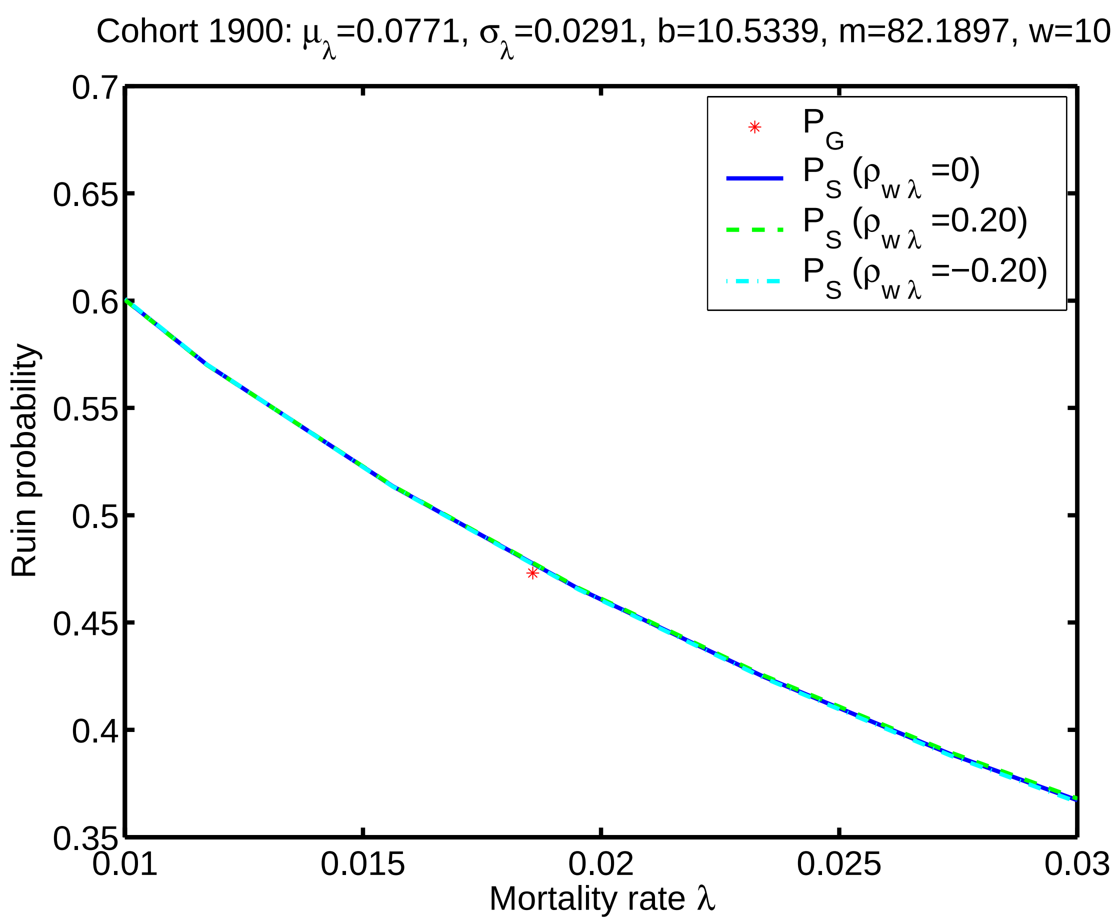
Figure 4.6 compares ruin probability P_S and P_G for initial wealth $w = 10$ for the cohort born in 1900. The parameters used for this figure are $\mu_\lambda = 0.0771$, $\sigma_\lambda = 0.0291$, $b = 10.5339$, and $m = 82.1897$. The ruin probability P_S for correlation coefficient $\rho_{w\lambda} = -0.2, 0, 0.2$ are 0.4772, 0.4775, 0.4778 respectively, and the ruin probability P_G is 0.4731. Therefore, the effect of $\rho_{w\lambda}$ for the initial wealth \$10 on P_S is trivial, and they are all greater than P_G.

Figure 4.7 displays ruin probability P_S and P_G for initial wealth $w = 20$ for the cohort born in 1900 using the same parameters as in Figure 4.7. The ruin probability P_S for correlation coefficient $\rho_{w\lambda} = -0.2, 0, 0.2$ are 0.1096, 0.1116, 0.1135 respectively. The ruin probability P_G has nothing to do with $\rho_{w\lambda}$, its value is 0.1063. We can see that the ruin probabilities P_S for three different correlation coefficients are greater than the ruin probability under GM mortality.

Figure 4.8 shows ruin probability P_S and P_G for initial wealth $w = 30$ for the cohort born in 1900 using the same parameters as in Figure 4.8. The ruin probability P_S are 0.0305, 0.0317, 0.0329 respectively for $\rho_{w\lambda} = -0.2, 0, 0.2$, and the ruin probability P_G is 0.0291.

From the above three figures, we can observe that the ruin probability under stochastic hazard rates is always greater than the ruin probability under Gompertz mortality. The effect of the correlation coefficient $\rho_{w\lambda}$ on P_S is trivial when the initial wealth is small, while the effect $\rho_{w\lambda}$ on P_S is large when the initial wealth is large.

Figure 4.6: P_S, P_G Comparison for $w = 10$

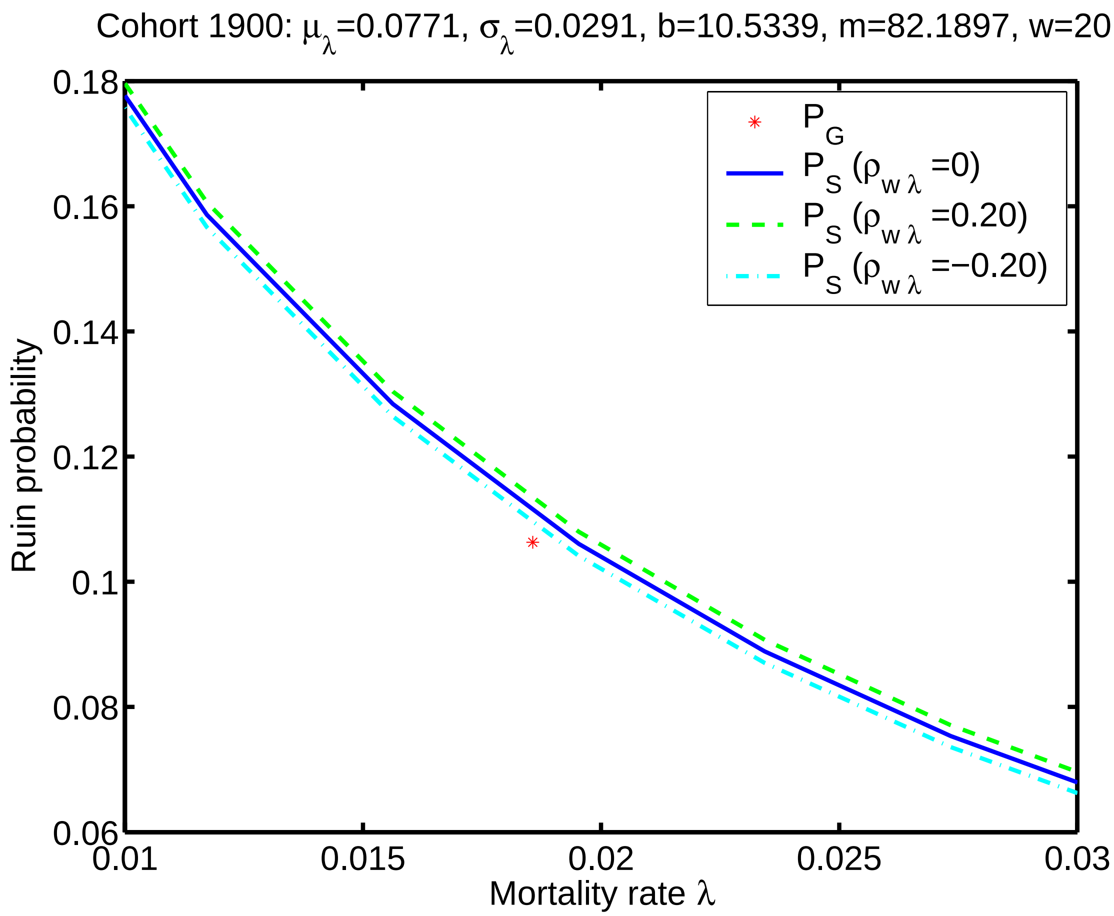


The figure compares ruin probability P_S and P_G for initial wealth $w = 10$ for the cohort born in 1900.

4.3.3.2 The Effect of the Parameters μ_λ and σ_λ

To understand the numerical results in-depth, we will do sensitivity analysis for the parameters. Since we are mainly interested in the sensitivity of the ruin probability to the parameters of the stochastic hazard rate μ_λ, σ_λ and the correlation coefficient

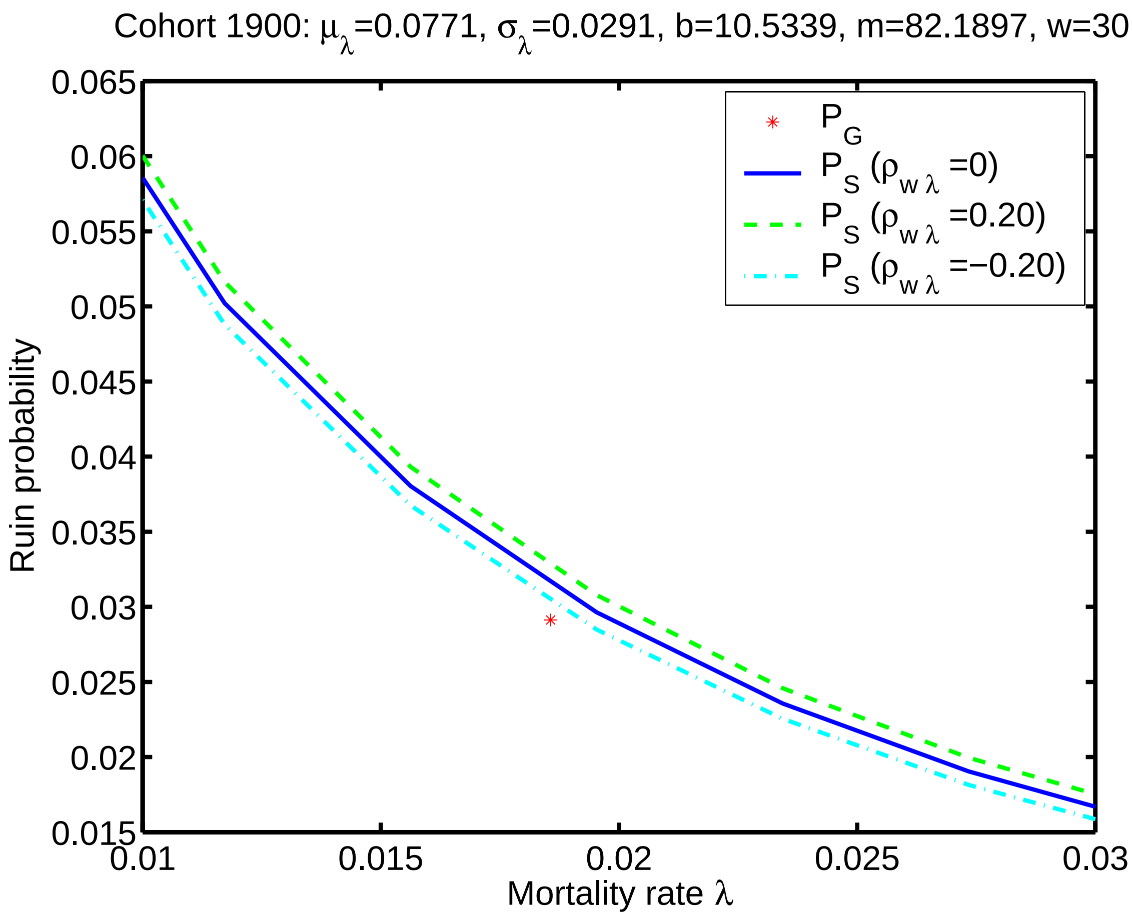
Figure 4.7: P_S, P_G Comparison for $w = 20$



The figure compares ruin probability P_S and P_G for initial wealth $w = 20$ for the cohort born in 1900.

$\rho_{w\lambda}$, we will illustrate these sensitivities one by one below. Figure 4.9 compares the ruin probability that a retiree with initial wealth of $[8, 20]$ who withdraws \$1 per annum will become ruined, where ruin is defined as wealth hitting zero within their lifetime using the numerical PDE method. When the expected hazard rate

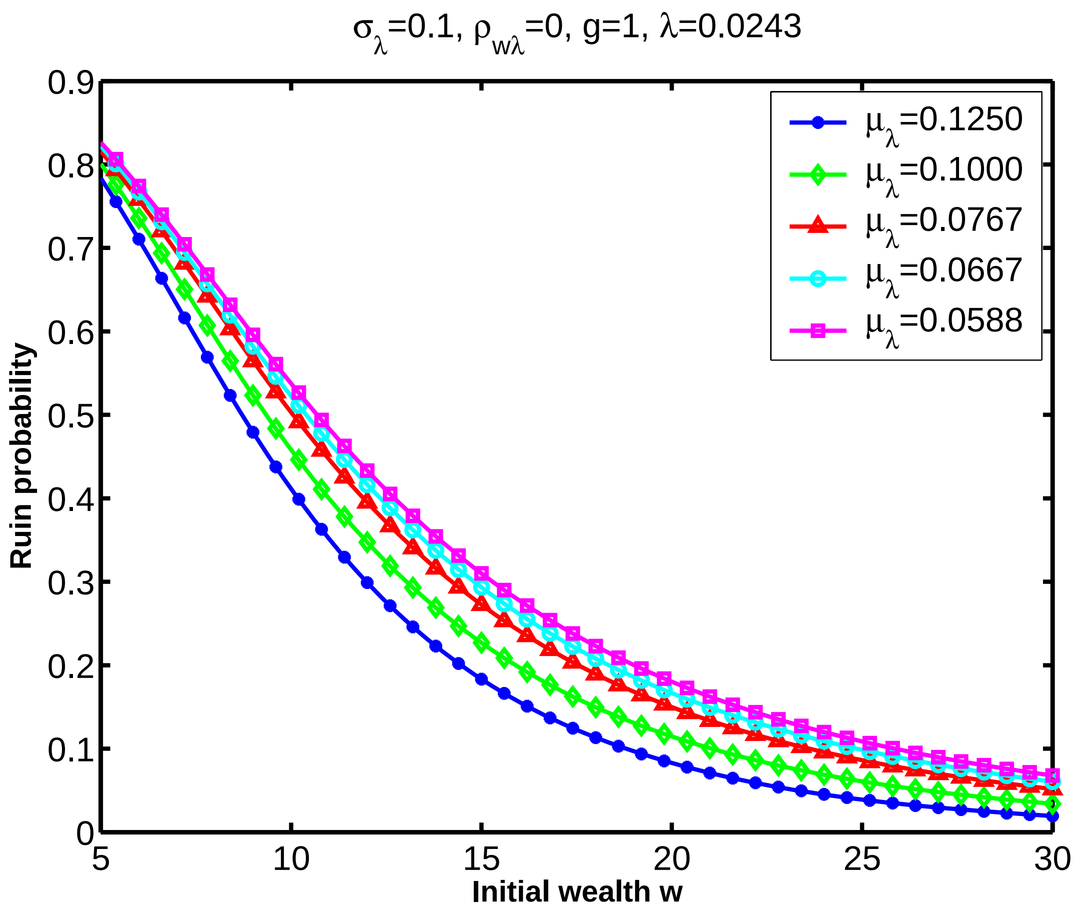
Figure 4.8: P_S, P_G Comparison for $w = 30$



The figure compares ruin probability P_S and P_G for initial wealth $w = 30$ for the cohort born in 1900.

is lower, i.e., the individual will live longer, the lifetime ruin is higher. This is intuitively pleasant since if a human being lives longer, his/her wealth has a higher chance to hit zero in his/her lifetime. Figure 4.10 displays the ruin probability as a function of mortality volatility using the numerical PDE method assuming initial

Figure 4.9: Sensitivity Analysis of μ_λ

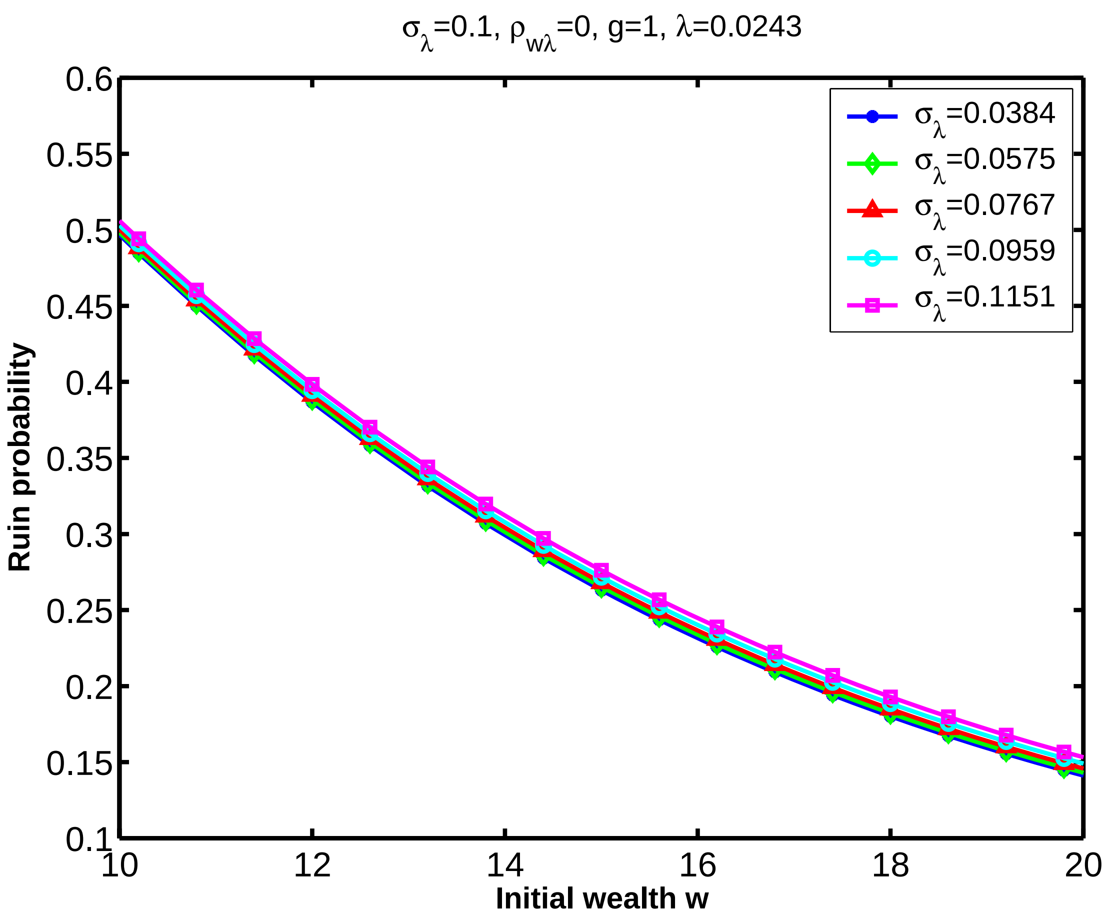


The figure shows the effect of the expected hazard rate on the ruin probability for

$\lambda = 0.0243$

wealth in domain $[8, 20]$ and withdrawal rate of \$1 per annum at time zero. When the volatility of the hazard rate is lower, which means the individual has a higher chance to live longer, the lifetime ruin is higher. Both figures show us that the ruin probability will change when the drift or dispersion coefficient changes, but

Figure 4.10: Sensitivity Analysis of σ_λ



The figure shows the effect of the volatility of the stochastic hazard rate on the ruin probability for $\lambda = 0.0243$

the changes are not very significant.

Now we move on to the numerical results for different coefficients $\rho_{w\lambda}$. Table 4.5 displays the ruin probability for different correlation coefficients $\rho_{w\lambda}$ at time $t = 0$ for $\mu_\lambda = 0.0802$ and $\sigma_\lambda = 0.0312$ for the cohort born in 1900. Firstly, we can

Table 4.5: Ruin Probability P_S for Different $\rho_{w\lambda}$ and Initial Wealth

Correlation coefficient $\rho_{w\lambda}$	$w = 10$	$w = 20$	$w = 30$	$w = 40$
0.30	0.4779	0.1145	0.0335	0.0103
0.20	0.4778	0.1135	0.0329	0.0115
0.10	0.4777	0.1126	0.0322	0.0112
0.00	0.4775	0.1116	0.0317	0.0109
-0.10	0.4774	0.1106	0.0311	0.0103
-0.20	0.4772	0.1096	0.0305	0.0100
-0.30	0.4771	0.1086	0.0299	0.0098
P_G	0.4731	0.1063	0.0291	0.0095

The table shows the effect of $\rho_{w\lambda}$ on the lifetime ruin P_S for different initial wealth w. Positive correlation increases lifetime ruin, and negative correlation decreases lifetime ruin. We have also displayed negative correlation values of $\rho_{w\lambda}$ because this parameter depends on individual's spending habits. People's health maybe deteriorated when they are richer.

observe that the ruin probability under stochastic hazard rates is higher than that of Gompertz mortality rate, which is due to a higher chance of living longer under stochastic hazard rates. Secondly, no matter whether the correlation coefficient is positive or negative, higher correlation leads to higher ruin probability. This effect is trivial when the initial wealth is small, and it is larger when the initial wealth is big. Thirdly, positive correlation between wealth and mortality increases lifetime ruin probability, and negative correlation decreases lifetime ruin probability. This is due to the fact that when wealth is lower, and hazard rate is lower, the retiree will live longer, hence the ruin probability is higher.

4.4 Concluding Remarks

In this chapter, we have studied lifetime ruin probability under some institutional assumptions for the wealth process for two different mortality models, GM mortality and stochastic mortality. This is motivated by the stochastic behavior of historical mortality data. The ruin probability under stochastic hazard rates P_S will collapse to the ruin probability P_G under deterministic Gompertz mortality when certain conditions are satisfied. The partial differential equations that the ruin probabilities must solve were derived by applying Ito's lemma, and solved numerical using different finite difference methods.

The numerical results indicate that under stochastic mortality, higher levels of mortality drift and volatility lead to a lower ruin probability during one's lifetime. For the wealth process, higher expected investment returns and lower volatility of the investment will reduce the chances of the individual to become ruined. As for the correlation coefficient between the two Brownian motions that drives the wealth process and the stochastic mortality process, higher coefficients lead to higher ruin probabilities. We also observed that the ruin probability under stochastic mortality rates is higher than the ruin probability for GM mortality.

The mortality model we have presented in this chapter is a log-normal distribution with constant drift and volatility. One extension of this model is to calibrate the mortality to be stochastic with varying drift (instead of constant) and constant volatility as proposed by Huang, Milevsky and Salisbury (it is appearing in IME) for the same wealth process and consumption. Another extension could be using the same stochastic mortality model with more complex investment models and consumption strategies. The third extension, a more interesting and relevant one, is to minimize the ruin probability when the investment and consumption strategies are optimized, a dynamic programming problem in which an HJB equation can be derived and solved such as Milevsky, Moore, and Young (2006).

4.5 Appendix: Convergence Analysis

The solutions obtained by numerical methods are usually not the exact solutions of the problem. This is due to round-off errors and truncation errors. Round-off errors arise because it is impossible to represent all real numbers exactly on a finite-state machine (which is what all practical digital computers are). Truncation errors are errors resulting from the difference of the approximate solution and the exact solution. For example, to differentiate a function, the differential element approaches zero but numerically we can only choose a finite value of the differential element. Once an error is generated, it will generally propagate through the calculation. There is an important criterion which guarantees the solution of the numerical scheme to move towards the real solution of the PDE: convergence.

In numerical analysis, the speed at which a convergent sequence approaches its limit is called the rate of convergence. Similar concepts are used for discretization methods. The solution of the discretized problem converges to the solution of the continuous problem as the grid size goes to zero, and the speed of convergence is one of the factors of the efficiency of the method.

In this section, we will carry out a convergence investigation of our numerical methods. Lots of different experiments are performed with different wealth and time step sizes. Since the analytic solution is not available in these experiments, we choose the results gained from the finest grid as our reference solution and compute the L_2 error between the reference solution and the solution obtained on the coarser grid. The convergence rate is the logarithm of two immediate quotients of the L_2 norms.

- Convergence test for $P_G(t, w)$

  To calculate the convergence rate, we perform six experiments with varying grids $\Delta w = 0.7813, 0.3906, 0.1953, 0.0977, 0.0488, 0.0244$. The time step size is set to be $\Delta t = 0.05$. The parameters for the Gompertz mortality are $m = 87.8$ and $b = 9.5$ introduced back in section 4.2.3.

  Table 4.6 is the convergence analysis for P_G by centered difference method and the upwind scheme. We see that the convergence rates for both methods are increasing as the grid is refined while the convergence rate for the first method is close to second-order and the convergence rate for the second method is about first-order. When the wealth step size is smaller, both methods converge faster to the real solution.

- Convergence test for $P_S(t, w, \lambda)$

  The closed-form expression for P_S is also not available so the results obtained from the finest grid are chosen to be our reference solution. To do convergence test for spacial variables, seven different computations with varying grids are

Table 4.6: Convergence Analysis for w (Centered Difference/Upwind)

Step size Δw	L_2 norm $\|\|P_G - P_G^*\|\|_2$	Convergence rate
0.7813	2.3000e-03/2.47e-02	-
0.3906	8.1694e-04/1.73e-02	1.49/0.51
0.1953	2.8500e-04/1.16e-02	1.52/0.58
0.0977	9.5928e-05/7.1e-03	1.57/0.71
0.0488	2.7130e-05/3.3e-03	1.82/1.11

performed. The time step size is set to be 0.02. For computational simplicity, the mixed derivative is treated explicitly. Table 4.7 shows the convergence analysis for w and λ. From this table we see that our numerical algorithm converges faster when the step sizes get smaller.

- Remarks

  From Table 4.6 and Table 4.7 we see that our algorithms converge to the exact solution when the step sizes go to zero. This gives us confidence that our numerical methods for the PDE of P_G and P_S are convergent and trustable. Hence the numerical results we have obtained in this chapter are good.

Table 4.7: Convergence Analysis for w and λ

Number of grids: (I,J)	$\|\|P_S - P_S^*\|\|_2$	Convergence rate
(96,8)	0.1502	-
(192,16)	0.1382	0.1200
(384,32)	0.1133	0.2869
(768,64)	0.0848	0.4174
(1536,128)	0.0455	0.8964

5 Conclusion

This dissertation considered two major issues in retirement planning. The first issue studied the effect of the introduction of stochastic interest rates to retirees' annuitization choices without bequest motives in an all-or-nothing framework. The second part of the dissertation examined the lifetime ruin probability for a retired individual who might run out of money before the end of his/her life under stochastic hazard rates. The main purpose of this dissertation is to provide useful information to help retired individuals to plan their finances in order to achieve a better and more comfortable retirement and a higher standard of living. If individuals choose to self-annuitize their wealth, they have the advantage of higher liquidity, but they need to be aware of the risk of becoming ruined, which depends on their initial wealth and personal mortality rate. If they choose to annuitize, they may get a better financial trade-off upon the optimal annuitization time under stochastic interest rates when they consume their annuity payout optimally with no bequests left for heirs.

5.1 Takeaway from the Annuitization Problem

In the annuitization problem, we have looked at the effect of the introduction of stochastic interest rates to a retiree's annuitization choice with no bequest motives in an all-or-nothing framework for a utility maximizer. To do so we have chosen to represent the annuity market via fixed annuities, a traditional and popular product. We also assumed that the individual only has initial wealth in the form of a lump sum cash amount, does not come pre-annuitized with a pre-existing pension or social security, and has no other lifetime income.

In the first life cycle model we assumed that the retiree would consume all his/her annuity payout after annuitization. In this setting, the optimal control problem is a free boundary problem (similar to the American option pricing problem) which can be converted to an equivalent LCP problem, and solved by the successive SOR method. We found that the individual will gain more financial advantage at any given age if he/she chooses to annuitize his/her lump sum cash amount when the interest rates are below a critical interest rate, no matter whether the force of mortality is constant or Gompertz. This is reasonable because we have assumed constant spread between the risky asset's expected return and the risk-free interest rate, and the subjective discount factor deviates considerably from current interest rate when the latter is bigger.

In the second life cycle model we assumed that the retiree will consume optimally after annuitization. This optimal consumption strategy has been obtained through dynamic programming techniques. It turns out that the individual will be better-off if he/she chooses to annuitize for any interest rate at any time after retirement. This result is consistent with existing literature in the sense that it is always optimal to annuitize with no bequest motives and loading fees. It is always better to annuitize since the annuitization value function is always greater under the optimal consumption strategy.

One natural follow up on this annuitization problem is to extend fixed annuities to other annuity products such as variable annuities, deferred annuities or joint annuities for married couples. We can also extend our research by including substantial load factors since annuities are not priced fairly in reality.

5.2 Takeaway from the Ruin Problem

In the ruin problem, we have studied the effect of stochastic hazard rates on the ruin probability, and compared it with its counterpart under GM mortality. This is motivated by the observation from historical data that the hazard rate behaves stochastically. The problem was formulated using PDE solution techniques, and was numerically solved by the ADI method. Numerical results indicate that the ruin probability for stochastic mortality rates is higher than the ruin probability for the GM mortality. For stochastic mortality, higher levels of mortality drift and volatility lead to lower ruin probability during one's lifetime. For the wealth process, higher expected investment returns and lower volatility of the investment will reduce the chances of the individual to become ruined. As for the correlation coefficient between the two Brownian motions that drives the wealth process and the stochastic mortality process, higher coefficients lead to higher ruin probabilities.

The mortality model we have presented for stochastic hazard rates is a log-normal distribution with constant drift and volatility. An alternative model is to calibrate the mortality to be stochastic with varying drift (instead of constant) and constant volatility as proposed by Huang, Milevsky and Salisbury (it is appearing in IME) for the same wealth process and consumption. An extension could be relaxing the constant consumption rate to be stochastic, or to minimize the ruin probability under optimal investment and consumption strategies with stochastic mortality.

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