# ALGEBRA STRUCTURE ON SET PARTITIONS

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# Abstract

The partition algebra is an algebra with a basis of set partitions diagrams. Its subalgebra includes diagram algebras such as the uniform block permutations and the group algebra of the symmetric group.

We connect the Hopf algebra of uniform block permutations to the diagram algebra known as the party algebra. This is done by describing a new basis of the partition algebra and looking at the relationship to the basis given for the Hopf algebra of uniform block permutations.

The product and coproduct of the Hopf algebra of uniform block permutations are the generalization of the product of the Malvenuto-Reutenauer Hopf algebra of permutations. We connect the product of the uniform block permutations with the bases of the partition algebra. The centralizer algebra has an internal product and we define an external product on the partition algebra. This algebra contains the algebra of uniform block permutations and the algebra of permutations.

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# Chapter 1

# Introduction

The representation theory of the symmetric groups was first studied by Frobenius and Schur, in 1906, when they coauthored two papers [FS06b], [FS06a] which extended representation theory in several new directions. In 1900, Frobenius [Fro00] and Young [You01] introduced new combinatorial methods to compute the characters of the symmetric group, $S_k$, by constructing a map from the class function to degree $k$ homogeneous symmetric polynomials. A year later in his dissertation [Sch01] Schur, who was Frobenius' doctoral student, used the combinatorial method to classify the polynomial representations of the general linear groups, $GL_n(\mathbb{C})$, and followed it by a paper [Sch27] which related representations of the symmetric groups with representations of the general linear groups.

For a survey on the origins of representation theory refer to [Cur92]. Curtis describes the development of modular representation theory focusing on the work of Frobenius, Schur and Brauer.

Today we refer to this relation as the Schur-Weyl duality between $GL_n$ and $S_k$. It shows that if $GL_n$ acts on an $n$ dimensional vector space $V$ then the commutator algebra of the diagonal action of $GL_n$ on the tensor product space $V^{\otimes k}$ is the symmetric group algebra $\mathbb{C}S_k$ acting on the positions of the tensors. Which means, if $V = \mathbb{C}^n$, then

$$A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = Av_1 \otimes Av_2 \otimes \cdots \otimes Av_k, \qquad \text{for } A \in GL_n,$$

and $S_k$ acts on $V^{\otimes k}$ by permuting tensor coordinates. The implication being that there is a decomposition

$$V^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} S^\lambda \otimes W^\lambda,$$

where $\lambda$ ranges over all partitions of $k$ with at most $n$ parts, $S^\lambda$ is an irreducible $S_k$ module and $W^\lambda$ are irreducible representations of $GL_n$.

Restricting the action of $GL_n(\mathbb{C})$ on $V^{\otimes k}$ to the orthogonal group, $O_n(\mathbb{C})$, Brauer [Bra37] showed in 1937 that the centeralizer algebra of the orthogonal group has a basis with a combinatorial description. This is now called Brauer algebras, $B_k(\mathbb{C})$. Weyl [Wey46] used the duality to analyze the representations and invariants for the general linear and symmetric groups, along with the sympletic and orthogonal groups. Weyl describes a basis of the centralizer algebra as a pair of rows containing equal number of symbols that represent males and females. For the symmetric group, those pairings being all possible matching that contain one male and one female, and in the case of the Brauer algebra, the pairing is made "without discrimination of sex".

The centralizer algebra of the permutation representation for the symmetric group has a combinatorial basis given by the set partitions. This algebra is know as the partition algebra, $\mathsf{P}_k(n)$. In the early 1990s, the partition algebra appeared, independently, in the works of Martin [Mar90], [Mar91], [Mar94], [Mar96] and Jones [Jon94]. Both Martin and Jones study the partition algebra as a generalization of the Temperley-Lieb algebra and the Potts model in statistical mechanics. Martin describes the structure of the algebra both implicitly [Mar90], [Mar91] and explicitly [Mar94]. While Jones shows explicitly the Schur-Weyl duality between the partition algebras and the symmetric group $S_n$, such that $S_n$ is identified with the set of all permutation matrices. He showed that the algebra is generated by a quotient of a subalgebra of the Brauer algebra and the action of the symmetric group by permuting the tensor product factor.

Given that the symmetric group $S_n$ is a subgroup of the general linear and orthogonal group, the partition algebra contains the Brauer algebra and the symmetric group. That is, we have the containments,

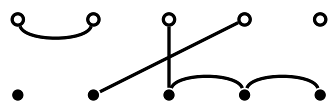
$$S_n \subseteq O_n(\mathbb{C}) \subseteq GL_n(\mathbb{C})$$
$$\mathsf{P}_k(n) \supseteq B_k(n) \supseteq \mathbb{C}S_k.$$

Following the analogy of Weyl, the partition algebra would be gatherings of males and females where the number or sex of people in the group wouldn't matter.

These algebras are represented by partition diagrams, which we refer to as diagrams. A diagram is a graph on two rows of $k$ vertices, one above the other, where each edge is incident to two distinct vertices, and there is at most one edge between any two vertices. The connected components of a diagram partition the $2k$ vertices into $l$ subsets, $1 \leq l \leq 2k$.

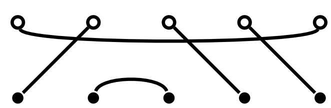
We define an equivalence relation $\equiv$ on partition diagrams by saying that two diagrams are equivalent if they determine the same partition of the $2k$ vertices.

A partition algebra diagram has no restriction on how the vertices are connected. The number of partition diagrams with $\ell$ connected components is the Stirling number $S(2k, \ell)$. Therefore the total number of partition diagrams is the Bell number $B_{2k} = \sum_{\ell=1}^{2k} S(2k, \ell)$.



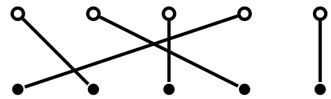
Partition algebra

A Brauer algebra diagram is represented as a graph where only two vertices are connected. Therefore the total number of Brauer diagrams is the $(2k-1)!! = (2k-1)(2k-3)(2k-5)\ldots 1$.



Brauer algebra

A symmetric group diagram is represented as a graph where only two vertices are connected and the vertices are one from the top row and one from the bottom row. Therefore the total number of symmetric group diagrams is the $k!$.

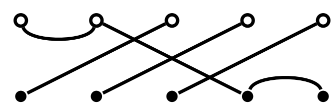


Symmetric group

The duality between the orthogonal group and the Brauer algebra was used by Ram [Ram95] to derive a Frobenius formula for the Brauer algebra, moreover, the symmetric functions were used to determine a recursive Murnaghan-Nakayama rule for Brauer algebra characters. Halverson [Hal01] later showed the analogues of the Frobenius formula and the Murnaghan-Nakayama rule for the characters of the partition algebras. The partition algebras $\mathsf{P}_k(n)$ are algebras of diagrams where a combinatorial rule defines the product and the structure coefficients of the algebra depend polynomially on a parameter. The algebras $\mathsf{P}_k(n)$ have two distinguished bases, namely, the diagram basis $d_\pi$ and the orbit basis $x_\pi$, where $\pi$ is in the set of set partitions of $\{1, 2, \ldots, 2k\}$.

In [Tan97], the author generalizes the results of Jones, studying the centralizer of the unitary reflection group $G(m, p, n)$ when it acts diagonally in the the tensor space $V^{\otimes k}$. He gave a combinatorial description of a basis of all matrices acting on $V^{\otimes k}$ that commute with $G(m, p, n)$. Kosuda [Kos06a], [Kos06b] further studies these algebras for the case $G(m, 1, n)$ under the condition $n \geq k$ and $m > k$, constructing a complete set of irreducible representations. An algebra generated by the symmetric group $S_k$ and another operator of type $G(m, p, n)$ is a subalgebra of the partition algebra known as the party algebra.

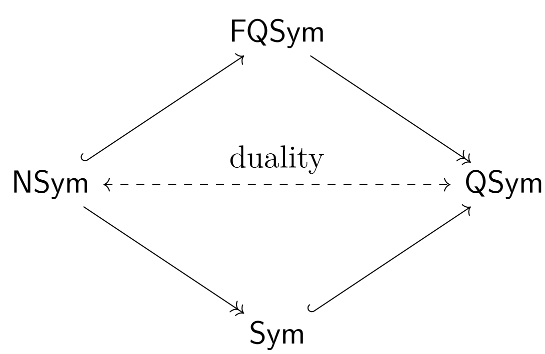
In [Kos00], Kosuda describes the party algebras as the linear span of diagrams that are a decomposition of two parties into small groups, two rows where each party contains $k$ members and the combined members in the parties split into groups such that each consists of the same number of members from either party. Given that each groups consists of equal members from each party, this algebra is also known as the uniform block permutation algebra. The diagram below shows a collection of small groups where the edges indicate the connections where the members of the top row that are in the same group as members of the bottom row.



Uniform block permutation

The uniform block permutation algebra are diagrams where the connected vertices contain equal number of elements from both the top and bottom rows of diagram.

Malvenuto and Reutenauer [MR95] introduce the Hopf algebra of permutations FQSym which gives a noncommutative lift of the quasisymmetric function. They look at quasisymmetric functions QSym and non-commutative symmetric functions NSym and unify them by showing QSym is Hopf dual to NSym.




Malvenuto and Reutenauer give a product and coproduct on a basis, which was later referred to as the fundamental basis. A new basis of the Hopf algebra was introduced by Aguiar and Sottile [AS05], the monomial basis is related to the fundamental basis by Möbius inversion on the left weak order on the symmetric groups. These basis are analogous to the monomial basis and the fundamental basis of QSym.

The classical Schur-Weyl duality states that the symmetric group algebra can be recovered from the diagonal action of $GL_n$ on $V^{\otimes k}$. The permutations are linear endomorphisms of the tensor algebra

$$T(V) := \bigoplus_{k \geq 0} V^{\otimes k}$$

where $\mathbb{C}S_k \cong \mathrm{End}_{GL_n}(V^{\otimes k})$. Given that the tensor algebra is a Hopf algebra, the convolution product can be formed from any two linear endomorphisms,

$$T(V) \xrightarrow{\Delta} T(V) \otimes T(V) \xrightarrow{\sigma \times \nu} T(V) \otimes T(V) \xrightarrow{m} T(V),$$

where $\sigma \in S_p$ and $\nu \in S_q$, and $\Delta$ and $m$ are the product and coproduct of the tensor algebra. Given that the maps commute with the action of $GL_n$, the convolution of $\sigma$ and $\nu$ belongs to $\mathrm{End}_{GL_n}(V^{\otimes k})$, where $k = p + q$. Therefore, there exists an element $\sigma * \nu \in \mathbb{C}S_k$ whose right action equals the convolution of $\sigma$ and $\nu$. This product is that of the algebra of permutations of Malvenuto and Reutenauer.

Aguiar and Orellana [AO08] apply the same argument to define a convolution product on the direct sum of the centralizer algebras $\mathrm{End}_G(V^{\otimes k})$, starting from a linear action of a group $G$ on a vector space $V$. For the complex reflection group $G(r, 1, m)$, the monomial representation is a certain linear action of this group on an $m-$dimensional space $V$. When $m \geq 2k$ and $r > k$, a result of Tanabe identifies the centralizer of $G(r,1,m)$ action on $V^{\otimes k}$ with the monoid algebra of uniform block permutations. And thus, the convolution product is defined on the space $\bigoplus_n \mathbb{C}\mathsf{P}_n$. They provide an explicit description of an operation similar to that for the convolution product of permutations. Moreover, they define a compatible coproduct.

In [AO08], the authors define the graded Hopf algebra of the uniform block permutations showing that it contains the Hopf algebra of permutations of Malvenuto and Reutenauer [MR95] and the Hopf algebra of symmetric functions in non-commuting variables [GS00], [GS01], [RS06].

We connect the Hopf algebra of uniform block permutations to the diagram algebra described by Kosuda and Tanabe. This is done by describing the product and coproduct on the diagram basis and looking at the relationship to the basis given by Aguiar and Orellana. We connect the product of the uniform block permutations, which is analog to the fundamental basis of the Malvenuto-Reutenauer Hopf algebra, with the diagram and orbit bases of the partition algebra. The centralizer algebra has an internal product and we connect this to the external product on the Hopf algebra by showing the internal product is in morphism with the external product. The product and coproduct of the uniform block permutations are the generalization of the Malvenuto-Reutenauer product.

We should note that there was a recent paper in the literature that introduce the Hopf algebra of partition diagrams [Cam23]. The product introduced in this paper is non-commutative, however, the coproduct defined in that paper does not restrict to the coproduct on the Malvenuto-Reutenauer Hopf algebra or the Hopf algebra of uniform block permutations.

In Chapter 2, we describe the change of basis between the fundamental basis and a new basis we refer to as the diagram basis of the Hopf algebra of permutations of Malvenuto and Reutenauer (*Section* 2.3). Moreover, we define product and coproduct of the diagram basis (*Section* 2.3.1), and internal product of diagram basis of the Hopf algebra of permutations of Malvenuto and Reutenauer (*Section* 2.5).

In Chapter 3, we define the external product on the diagram basis and show that it implies a formula for the external product on the orbit basis (*Section* 3.5). We then show that these products are compatible with the internal product of the diagram and orbit bases of the partition algebra given in the literature.

In Chapter 4, we describe the change of basis between the fundamental basis and the diagram basis of Hopf algebra of uniform block permutations (*Section* 4.3.1). In addition, we provide a definition of the product and coproduct of the diagram basis (*Section* 4.3.2-4.3.3) and the coarser fundamental basis (*Section* 4.5) of the Hopf algebra of uniform block permutations.

# Chapter 2

# The Hopf Algebra of Permutations

## Introduction

The Malvenuto-Reutenauer Hopf algebra of permutations $\mathsf{MR}$ was introduced by Malvenuto in her thesis [Mal94] and in her work with Reutenauer [MR95]. It provided a linear basis $\{\mathcal{F}_u \mid u \in S_n, n \geq 0\}$ indexed by permutations in the symmetric group $S_n$. It is a self dual graded algebra that is non-commutative and non-cocommutative. In [DHT02], the vector space is called the algebra of free quasi-symmetric functions.

In [AS05], the authors introduce a new basis $\{\mathcal{M}_u \mid u \in S_n, n \geq 0\}$ for $\mathsf{MR}$. The bases $\mathcal{F}_u$ and $\mathcal{M}_u$ are related by Möbius inversion on the weak order on the symmetric groups, and are analogous to the fundamental and monomial bases of the quasi-symmetric functions $\mathsf{QSym}$ which are related via Möbius inversion on their index sets, the Boolean posets $\mathcal{Q}_n$. Thus, the bases of $\mathsf{MR}$ are referred to as the fundamental and monomial bases.

In this chapter, we define a diagram basis $\{\mathcal{D}_u \mid u \in S_n, n \geq 0\}$ for $\mathsf{MR}$ related to the fundamental basis by a partial order on the set $S_n$. Moreover, we provide a combinatorial description of the product and coproduct of $\mathsf{MR}$ with respect to the diagram basis $\mathcal{D}_u$.

## 2.1 Notation

In this section, we will define some combinatorial objects and establish notation used in the chapter.

For a sequence $(v_1, \ldots, v_p)$ of distinct integers, let its *standard permutation* $u = \mathrm{st}(v_1, \ldots, v_p) \in S_p$ be the permutation defined by

$$u_i < u_j \iff v_i < v_j \qquad \text{for all } 1 \leq i < j \leq p.$$

Let $v = (v_1, v_2, \ldots, v_p)$ be a sequence of distinct integers in $S_p$, then the *reverse permutation*

$$REV(v) = (v_p, \ldots, v_2, v_1). \tag{2.1}$$

For a sequence $(v_1, \ldots, v_p)$ of distinct integers,

$$REV(st(v_1, \ldots, v_p)) = st(REV(v_1, \ldots, v_p)). \tag{2.2}$$

For a permutation $u \in S_n$, the *descent* set is the subset of $[n-1] := \{1, 2, \ldots, n-1\}$

which records the positions where there is a decrease when reading from left to right,

$$\mathrm{Des}(u) := \{p \in [n-1] \mid u_p > u_{p+1}\}.$$

The set of *global descents* of $u$ is the set

$$\mathrm{GDes}(u) := \{p \in [n-1] \mid \forall i \leq p \text{ and } j \geq p+1,\ u_i > u_j\}.$$

For example, $\mathrm{st}(94781) = 52341$ and $\mathrm{Des}(52341) = \{1, 4\}$.

Let $\mathrm{Inv}(u)$ be the set of *inversions* of a permutation $u \in S_n$,

$$\mathrm{Inv}(u) := \{(i, j) \in [n] \times [n] \mid i < j \text{ and } u_i > u_j\}. \tag{2.3}$$

For permutations $u$ and $v$ in $S_n$, the *left weak order* on $S_n$ is defined by

$$u \leq_L v \quad \text{if} \quad \mathrm{Inv}(u) \subseteq \mathrm{Inv}(v). \tag{2.4}$$

The left weak order has another characterization

$$u \leq_L v \Longleftrightarrow \exists w \in S_n \text{ such that } v = wu \text{ and } \ell(v) = \ell(w) + \ell(u), \tag{2.5}$$

where $\ell(u)$ is the number of inversions of $u$. The cover relations of the left weak order are the permutations found by exchanging consecutive entries in the permutation, $u_i$ and $u_j$ such that $u_j = u_i + 1$.

For every nonnegative integer $n$, the *right weak order* (also called the right permutohedron order) is a partial order on the symmetric group $S_n$. Let $u$ and $v$ be permutations in $S_n$, we define the right weak order on $S_n$ as

$$u \leq_R v \text{ if } \operatorname{Inv}(u^{-1}) \subseteq \operatorname{Inv}(v^{-1}). \tag{2.6}$$

The right weak order has another characterization

$$u \leq_R v \Longleftrightarrow \exists w \in S_n \text{ such that } v = uw \text{ and } \ell(v) = \ell(w) + \ell(u), \tag{2.7}$$

where $\ell(u)$ is the number of inversions of $u$. The cover relations of the right weak order are the permutations found by exchanging adjacent $u_i$ and $u_{i+1}$ entries of the permutation.

Right and left weak orders are isomorphic via the map $v \mapsto v^{-1}$. That is,

$$u \leq_L v \iff u^{-1} \leq_R v^{-1}. \tag{2.8}$$

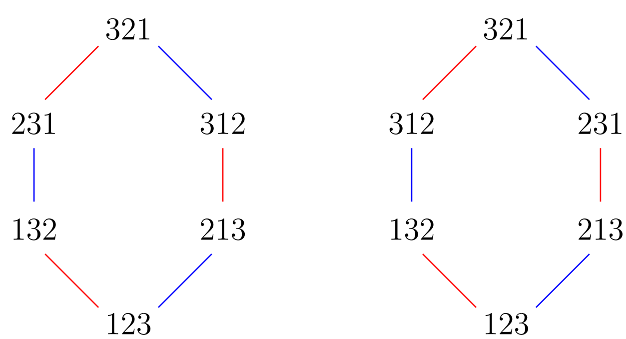



Figure 2.1: The left and right weak order of $S_3$

The identity permutation $1_n = (1, 2, \ldots, n)$ is the minimum element of $S_n$ and

$\omega_n = (n, \ldots, 2, 1)$ is the maximum in both left and right weak orders.

Given $p, q \geq 0$, we consider the product $S_p \times S_q$ to be a subgroup of $S_{p+q}$, where $S_p$ permutes $\{1, \ldots, p\}$ and $S_q$ permutes $\{p+1, \ldots, p+q\}$. For $u \in S_p$ and $v \in S_q$, write $u|v_{\uparrow_p}$ for the permutation in $S_{p+q}$ corresponding to $(u, v) \in S_p \times S_q$ under this embedding. Adding $p$ to every entry in the permutation $v$ and placing it to the right of the permutation $u$ we obtain a permutation in $S_{p+q}$, called the *concatenation* of $u$ and $v$, that is

$$u|v_{\uparrow_p} = u_1, \ldots, u_p, v_1 + p, \ldots, v_q + p.$$

The concatenation product of permutations $u = 312$ and $v = 21$ is

$$u|v_{\uparrow_3} = 31254.$$

Let $S^{(p,q)}$ be

$$\{\zeta \in S_{p+q} \mid \zeta \text{ has at most one descent, at position } p\},$$

that is, for permutations $\zeta \in S^{(p,q)}$ we have

$$\zeta_1 < \cdots < \zeta_p,$$

and

$$\zeta_{p+1} < \cdots < \zeta_{p+q}.$$

This is the collection of minimal (in length) representatives of left cosets of $S_p \times S_q$

in $S_{p+q}$, sometimes referred to as $(p,q)$-*shuffles* [AS05]. For $u \in S_p$ and $v \in S_q$,

$$sh_{u,v} = \{(u|v_{\uparrow_p}) \cdot \zeta^{-1} : \zeta \in S^{(p,q)}\} \subseteq S_{p+q}.$$

**Example 2.1.1.** Shuffling the permutations 312 $\in S_3$ and 21 $\in S_2$ we obtain the following permutations in $sh_{312,21}$.

$$31254,\ 31524,\ 31542,\ 35124,\ 35142,\ 35412,\ 53124,\ 53142,\ 53412,\ 54312 \qquad (2.9)$$

## 2.2 The Malvenuto-Reutenauer Hopf algebra of permutations

Here we review the Malvenuto-Reutenauer Hopf algebra of permutations from [Mal94], [MR95] and [PR95]. Giving the product and coproduct of the fundamental basis. Moreover, we look at the combinatorial descriptions of the product and coproduct with respect to the monomial basis.

The Malvenuto-Reutenauer Hopf algebra $\mathsf{MR} = \oplus_{n\geq 0}\mathsf{MR}_n$ is a graded vector space over $\mathbb{Q}$ with basis of the graded component of degree $n$ indexed by the elements of the symmetric group $S_n$. The subspace $\mathsf{MR}_n$ is the graded component of degree $n$ and is spanned by the bases $\{\mathcal{F}_u \mid u \in S_n\}$ and $\{\mathcal{M}_u \mid u \in S_n\}$.

### 2.2.1 The fundamental basis of MR

In [MR95], the authors define the fundamental basis $\mathcal{F}_u$ of $\mathsf{MR}$, where $u \in S_n$ for $n \geq 0$. The product of this basis is obtained by shuffling the permutation, that is, for $u \in S_p$ and $v \in S_q$,

$$\begin{aligned} \mathcal{F}_u \times \mathcal{F}_v &= \sum_{\zeta \in S^{(p,q)}} \mathcal{F}_{(u|v_{\uparrow_p}) \cdot \zeta^{-1}} \\ &= \sum_{\alpha \in sh_{u,v}} \mathcal{F}_\alpha, \end{aligned} \tag{2.10}$$

where $u|v_{\uparrow_p} \in S_{p+q}$ is obtained by adding $p$ to the entries in $v$ and joining the two permutations.

For instance, the product $\mathcal{F}_{312} \times \mathcal{F}_{21}$ will be indexed by elements in the set in Equation (2.9).

Moreover, the algebra $\mathsf{MR}$ is also a graded coalgebra where the coproduct is obtained by splitting a permutation in all positions into a left and a right permutation and standardizing. That is,

$$\Delta(\mathcal{F}_u) = \sum_{p=0}^{n} \mathcal{F}_{\mathrm{st}(u_1,\ldots,u_p)} \otimes \mathcal{F}_{\mathrm{st}(u_{p+1},\ldots,u_n)}, \tag{2.11}$$

where $u \in S_n$.

**Example 2.2.1.** For $u = 35142$,

$$\Delta(\mathcal{F}_u) = 1 \otimes \mathcal{F}_{35142} + \mathcal{F}_1 \otimes \mathcal{F}_{4132} + \mathcal{F}_{12} \otimes \mathcal{F}_{132} + \mathcal{F}_{231} \otimes \mathcal{F}_{21} + \mathcal{F}_{2413} \otimes \mathcal{F}_1 + \mathcal{F}_{35142} \otimes 1.$$

### 2.2.2 The monomial basis of MR

Aguiar and Sottile [AS05] introduced the monimial basis of the Hopf algebra MR of permutations. The subspace $\mathsf{MR}_n$ has as basis $\{\mathcal{M}_u \mid u \in S_n\}$ indexed by permutations in all symmetric group $S_n$. They also give combinatorial descriptions of the product and coproduct of MR with respect to the monomial basis $\mathcal{M}_u$.

The monomial basis of the Malvenuto-Reutenauer Hopf algebra is defined by

$$\mathcal{F}_u = \sum_{u \leq_L v} \mathcal{M}_v. \tag{2.12}$$

The change of basis between the monomial basis and the fundamental basis is related by Möbius inversion on the left weak order on the symmetric groups. For $u \in S_n$, where $n \geq 0$,

$$\mathcal{M}_u = \sum_{u \leq_L v} \mu_{S_n}(u, v) \cdot \mathcal{F}_v, \tag{2.13}$$

where $u \leq_L v$ is the left weak order in $S_n$ and $\mu_{S_n}$ is the Möbius function of this partial order.

Although we don't use the product of the monomial basis, we define it below to give a better a context of the basis. The product of the monomial basis of MR has structure constants that count special ways of shuffling two permutations, based on conditions that involve the left weak order. For $u \in S_p$ and $v \in S_q$ and $w \in S_{p+q}$, define $A^w_{u,v} \subseteq S^{(p,q)}$ to be those $\zeta \in S^{(p,q)}$ satisfying

1. $(u|v_{\uparrow_p}) \cdot \zeta^{-1} \leq_L w$, and
2. if $u \leq_L u'$ and $v \leq_L v'$ satisfying $(u'|v'_{\uparrow_p}) \cdot \zeta^{-1} \leq_L w$, then $u = u'$ and $v = v'$.

Set $\alpha_{u,v}^{w} := \#A_{u,v}^{w}$. Then for any $u \in S_p$ and $v \in S_q$, we have

$$\mathcal{M}_u \times \mathcal{M}_v = \sum_{w \in S_{p+q}} \alpha_{u,v}^{w} \mathcal{M}_w. \tag{2.14}$$

Let $\overline{\mathrm{GDes}}(u)$ denote $\mathrm{GDes}(u) \cup \{0, n\}$. The coproduct of the monomial basis of MR is obtained by splitting a permutation at the global descents.

The coproduct of MR in terms of the monomial basis is define by

$$\Delta\left(\mathcal{M}_u\right) = \sum_{p \in \overline{\mathrm{GDes}}(u)} \mathcal{M}_{\mathrm{st}(u_1,\ldots,u_p)} \otimes \mathcal{M}_{\mathrm{st}(u_{p+1},\ldots,u_n)}, \tag{2.15}$$

where $u \in S_n$.

### 2.2.3 Self-duality of MR

The Malvenuto-Reutenauer Hopf algebra of permutations MR is self-dual, this appears in [Mal94] and [MR95]. Poirier and Reutenauer [PR95] showed that the elements of the dual basis of the fundamental freely generate $(\mathsf{MR})^*$. Duchamp, Hivert, and Thibon [DHT02] dualize the resulting linear basis, giving a basis for the space of primitive elements. Aguair and Sottile [AS05] provide a proof of self-duality, and investigate its combinatorial implications, particularly when expressed in terms of the monomial basis.

The duality of the fundamental and monomial bases of MR as defined below appears in [Section 9, [AS05]].

Let $\{\mathcal{F}_u^* \mid u \in S_n\}$ and $\{\mathcal{M}_u^* \mid u \in S_n\}$ be the bases of $(\mathsf{MR})^*$ dual to the fundamental and monomial bases of MR, respectively. The map

$$\Theta : (\mathsf{MR})^* \to \mathsf{MR}, \qquad \mathcal{F}_u^* \mapsto \mathcal{F}_{u^{-1}}$$

is an isomorphism of Hopf algebras. On the monomial basis it is given by

$$\Theta(\mathcal{M}_u^*) = \sum_v \theta(u,v)\mathcal{M}_v,$$

where $\theta(u,v) := \#\{x \in S_n \mid x \leq_L u, x^{-1} \leq_L v\}$.

The product of the fundamental basis of MR (2.10) can be written as

$$\mathcal{F}_u \times \mathcal{F}_v = \sum_{w \in S_{p+q}} \#\{\zeta \in S^{(p,q)} \mid (u|v_{\uparrow_p}) \cdot \zeta^{-1} = w\}\mathcal{F}_w, \tag{2.16}$$

where $u \in S_p$ and $v \in S_q$. Therefore, the dual coproduct of MR is

$$\Delta(\mathcal{F}_w^*) = \sum_{p+q=n} \sum_{u \in S_p, v \in S_q} \#\{\zeta \in S^{(p,q)} \mid (u|v_{\uparrow_p}) \cdot \zeta^{-1} = w\}\mathcal{F}_u^* \otimes \mathcal{F}_v^*. \tag{2.17}$$

where $w \in S_{p+q}$.

For $u \in S_p$,

$$\mathcal{F}_u = \sum_{u \leq_L v} \mathcal{M}_v,$$

thus we have

$$\mathcal{M}_u^* = \sum_{v \leq_L u} \mathcal{F}_v^*. \tag{2.18}$$

The coproduct of the monomial basis of $\mathsf{MR}$ (2.15) can be rewritten as

$$\Delta(\mathcal{M}_w) = \sum_{p+q=n} \sum_{\substack{u \in S_p, v \in S_q \\ \zeta_{p,q} \cdot (u|v_{\uparrow_p}) = w}} \mathcal{M}_u \otimes \mathcal{M}_v. \tag{2.19}$$

where $\zeta_{p,q}$ is the permutation of maximal length in $S^{(p,q)}$, so that

$$\zeta_{p,q} := (q+1, q+2, \ldots, q+p, 1, 2, \ldots, q).$$

Therefore, the dual product of $\mathsf{MR}$ is

$$\begin{aligned} \mathcal{M}_u^* \times \mathcal{M}_v^* &= \mathcal{M}^*_{\zeta_{p,q} \cdot (u|v_{\uparrow_p})} \\ &= \mathcal{M}^*_{u_{\uparrow_q}|v}, \end{aligned} \tag{2.20}$$

where $u \in S_p$ and $v \in S_q$.

## 2.3 The diagram basis of $\mathsf{MR}$

An external product is a monoid product if the product of two basis elements is a single basis element. In this section, we define a basis of the Malvenuto-Reutenauer Hopf algebra $\mathsf{MR}$ which has a monoid product and thus we refer to it as the diagram basis $\{\mathcal{D}_u \mid u \in S_n\}$. We give combinatorial descriptions of the product and coproduct of $\mathsf{MR}$ with respect to the diagram basis $\mathcal{D}_u$.

In [Duc+11], the authors define a multiplicative basis $\Lambda^\sigma$, and although they do not explicitly define $\Lambda^{\sigma^{-1}}$ it is the diagram basis we have defined.

This basis has appeared in the literature prior to this thesis as the $Z$-basis in [AO08] and implicitly as $\Lambda^\sigma$ in [Duc+11]. Moreover, we will show that it is a transformation of the dual monomial basis of $\mathsf{MR}$ as well. One of the new things that we will give is a coproduct formula on this basis. The formula for the coproduct basis should be equivalent to the product of the monomial basis however when it takes the coproduct form its different (the same up to a transformation).

For $n \geq 0$ and $u \in S_n$, the diagram basis is defined with respect to the fundamental basis using the following relation,

$$\mathcal{D}_u = \sum_{u \leq_R v} \mathcal{F}_v \tag{2.21}$$

where the sum is over permutations $v$ which are greater than or equal to $u$ in the right weak order. By Möbius inversion,

$$\mathcal{F}_u = \sum_{u \leq_R v} \mu^R_{S_n}(u, v) \cdot \mathcal{D}_v. \tag{2.22}$$

**Example 2.3.1.** The diagram basis $\mathcal{D}_{132}$ in terms of the fundamental is

$$\mathcal{D}_{132} = \mathcal{F}_{132} + \mathcal{F}_{312} + \mathcal{F}_{321}.$$

The fundamental basis $\mathcal{F}_{132}$ in terms of the diagram is

$$\mathcal{F}_{132} = \mathcal{D}_{132} - \mathcal{D}_{312}.$$

### 2.3.1 The product and coproduct of the diagram basis of MR

We give an external product and coproduct of MR in terms of its the diagram basis.

The product of the diagram bases is obtained by concatenating the corresponding permutations. This is proven in Theorem 2.4.6 given below. For $u \in S_p$ and $v \in S_q$, we have

$$\mathcal{D}_u \times \mathcal{D}_v = \mathcal{D}_{u|v_{\uparrow_p}}, \tag{2.23}$$

where $u|v_{\uparrow_p}$ denotes the concatenation of $u$ and $v$, as usual.

**Example 2.3.2.** For $u = 3412$ and $v = 231$,

$$\mathcal{D}_{3412} \times \mathcal{D}_{231} = \mathcal{D}_{3412675}.$$

For a permuatation $u \in S_n$, let $u|_S = u_{i_1}, u_{i_2}, \ldots, u_{i_n}$ for $S = \{i_1 < i_2 < \ldots < i_{|S|}\}$. As shown in Theorem 2.3.6 below, the coproduct of MR on the diagram basis is

$$\begin{aligned}\Delta(\mathcal{D}_u) = \sum_{S \in 2^{[n]}} \chi(\exists\, w \in S_n \ : \ u \leq_R w,\ u|_S = w_1, \ldots, w_{|S|} \text{ and} \\ u|_{S^c} = w_{|S|+1}, \ldots, w_n) \cdot \mathcal{D}_{\mathrm{st}(u|_S)} \otimes \mathcal{D}_{\mathrm{st}(u|_{S^c})} \\ = \sum_{\substack{S \in 2^{[n]} \\ u \leq_R (u|_S)|(u|_{S^c})}} \mathcal{D}_{\mathrm{st}(u|_S)} \otimes \mathcal{D}_{\mathrm{st}(u|_{S^c})}\end{aligned} \tag{2.24}$$

where the sum is over all $S \in 2^{[n]}$ such that there is a permutation $w$ with $u \leq_R w$ with $u|_S = w_1, \ldots, w_{|S|}$ and $u|_{S^c} = w_{|S|+1}, \ldots, w_n$.

Let $S^{(p,q)}$ be the set of minimal (in length) representative of (left) cosets $S_p \times S_q$

in the symmetric group $S_{p+q}$. The following map is a bijection defined in [AS05],

$$\begin{array}{rcccc} \lambda_{p,q} & : & S^{(p,q)} \times S_p \times S_q & \longrightarrow & S_{p+q} \\ & & (\zeta, u, v) & \longmapsto & \zeta \circ (u|v_{\uparrow_p}) \end{array} \tag{2.25}$$

where $\ell(\zeta) + \ell(u) + \ell(v) = \ell(\zeta \circ (u|v_{\uparrow_p}))$.

We note that for every $\zeta \in S^{(k,n-k)}$, if $u \in S_n$, then $u \circ \zeta = (u|_S)|(u|_{S^c})$ where $S = \{\zeta_1 < \zeta_2 < \cdots < \zeta_k\}$ and $S^c = \{\zeta_{k+1} < \zeta_{k+2} < \cdots < \zeta_n\}$. Thus we see the origin of the terms in Equation (2.24), which are appearing from a permutation $\zeta \in S^{(k,n-k)}$ such that $u \leq_R u \circ \zeta = (u|_S)|(u|_{S^c})$.

**Lemma 2.3.3.** *For fixed $0 < k < n$ and $u \in S_n$, there is a bijection between the set*

$$A_{u,k} := \{(S, \bar{u}, \tilde{u}) \in 2^{[n]} \times S_k \times S_{n-k} : |S| = k,\ u \leq_R (u|_S)|(u|_{S^c}), \\ st(u|_S) \leq_R \bar{u},\ st(u|_{S^c}) \leq_R \tilde{u}\}$$

*and the set*

$$B_u := \{v : u \leq_R v\}\ .$$

*This bijection $\Phi_{u,k} : B_u \to A_{u,k}$ has the property that if*

$$\Phi_{u,k}(v) = (S, \bar{u}, \tilde{u})$$

*then $\overline{u} = st(v_1, \ldots, v_k)$ and $\tilde{u} = st(v_{k+1}, \ldots, v_n)$.*

*Proof.* Fix a permutation $u \in S_n$ and an integer $k$ such that $0 < k < n$. We will show that there is a bijection between the sets $A_{u,k}$ and $B_u$.

Take an element $v \in B_u$, then $v = u \circ w$ for some permutation $w \in S_n$ with $\ell(u)+\ell(w) = \ell(v)$. The map, $\lambda_{k,n-k}$ defined in Equation (2.25) allows us to decompose $w$ into three components, and construct the triples $(S, \bar{u}, \tilde{u})$. Each step of this construction will be bijective.

Let $\lambda^{-1}_{k,n-k}(w) = (\zeta, x, y) \in S^{(k,n-k)} \times S_k \times S_{n-k}$. The set $S = \{\zeta_1 < \zeta_2 < \cdots < \zeta_k\}$, and then $v = u \circ w = u \circ \zeta \circ (x|y_{\uparrow k})$. Using the note above Lemma 2.3.3 we replace $u \circ \zeta$ with $(u|_S)|(u|_{S^c})$, so $v = (u|_S)|(u|_{S^c}) \circ (x|y_{\uparrow_k})$. Then set $\bar{u} = \mathrm{st}(u|_S) \circ x = \mathrm{st}(v_1, \ldots, v_k)$ and $\tilde{u} = \mathrm{st}(u|_{S^c}) \circ y = \mathrm{st}(v_{k+1}, \ldots, v_n)$.

We have that the lengths of the permutations are $\ell(v) = \ell(u) + \ell(w) = \ell(u) + \ell(\zeta) + \ell(x) + \ell(y) = \ell((u|_S)|(u|_{S^c})) + \ell(x) + \ell(y) = \ell((u|_S)|(u|_{S^c})) + \ell(x|y_{\uparrow_k})$. But now when $x|y_{\uparrow_k}$ acts on $(u|_S)|(u|_{S^c})$ on the right, the effect is that $x$ permutes only the entries of $u|_S$ and $y$ permutes only the entries of $u|_{S^c}$. Hence $\ell(\bar{u}) = \ell(\mathrm{st}(u|_S) \circ x) = \ell(\mathrm{st}(u|_S)) + \ell(x)$ and $\ell(\tilde{u}) = \ell(\mathrm{st}(u|_{S^c}) \circ y) = \ell(\mathrm{st}(u|_{S^c})) + \ell(y)$ and so we can conclude that $\mathrm{st}(u|_S) \leq_R \bar{u}$ and $\mathrm{st}(u|_{S^c}) \leq_R \tilde{u}$.

So now given $(S, \bar{u}, \tilde{u})$ we can recover $v$ since $v_1$ through $v_k$ is a permutation of $S$ whose standardization is $\bar{u}$ while $v_{k+1}$ through $v_n$ is a permutation of $S^c$ whose standardization is $\tilde{u}$. ☐

**Example 2.3.4.** For $u = 35124$, the following set are the permutations which are greater than $u$ in right weak order:

$$\{35124, 35142, 35214, 35241, 35412, 35421, 53124,$$
$$53142, 53214, 53241, 53412, 53421, 54312, 54321\}.$$

In the table below we list the bijection between the elements $v \geq_R u$ and the triples $(S, \overline{u}, \tilde{u})$ for those elements $v$ such that $v = (u|_S)|(u|_{S^c})$. There is one term in the table below for each term in the expression for $\Delta(\mathcal{D}_u)$.

| $(S, \overline{u}, \tilde{u})$ | $u\|_S$ | $u\|_{S^c}$ | $v$ |
|---|---|---|---|
| $(\emptyset$, -, 35124) | | 35124 | 35124 |
| $(\{1\}$, 1, 4123) | 3 | 5124 | 35124 |
| $(\{2\}$, 1, 3124) | 5 | 3124 | 53124 |
| $(\{1,2\}$, 12, 123) | 35 | 124 | 35124 |
| $(\{2,5\}$, 21, 312) | 54 | 312 | 54312 |
| $(\{1,2,3\}$, 231, 12) | 351 | 24 | 35124 |
| $(\{1,2,4\}$, 231, 12) | 352 | 14 | 35214 |
| $(\{1,2,5\}$, 132, 12) | 354 | 12 | 35412 |
| $(\{1,2,3,4\}$, 3412, 1) | 3512 | 4 | 35124 |
| $(\{1,2,3,5\}$, 2413, 1) | 3514 | 2 | 35142 |
| $(\{1,2,4,5\}$, 2413, 1) | 3524 | 1 | 35241 |
| $(\{1,2,3,4,5\}$, 35124, -) | 35124 | | 35124 |

**Example 2.3.5.** Again let $u = 35124$. Consider a single permutation $v = 53241 \geq u$ and take $k = 2$. To calculate $\Phi_{35124,2}(v)$ we note that $v = u \circ 21453$ and $\lambda^{-1}_{2,3}(21453) = (12345, 21, 231)$. This implies that $S = \{1, 2\}$. Following the proof to compute $\overline{u}$ and $\tilde{u}$, we calculate $(u|_S)|(u|_{S^c}) = 35124 \circ 12345$ and $\overline{u} = \mathrm{st}(35) \circ 21 = 21$ and $\tilde{u} = \mathrm{st}(124) \circ 231 = 231$, so

$$\Phi_{35124,2}(v) = (\{1, 2\}, 21, 231) \ .$$

In light of the relationship between the diagram basis and the dual monomial basis that we will show in Lemma 2.4.4, we should expect that the coproduct on the diagram basis should be similar to the product on the monomial basis. However, it seems in the dual basis the coproduct can be formulated to take a much simpler form compared to Equation (2.14).

**Theorem 2.3.6.** *Let $u \in S_n$, then*

$$\Delta(\mathcal{D}_u) = \sum_{\substack{S \in 2^{[n]} \\ u \leq_R (u|_S)|(u|_{S^c})}} \mathcal{D}_{st(u|_S)} \otimes \mathcal{D}_{st(u|_{S^c})}$$

*where the sum is over all $u \leq_R v$ such that $u|_S = v_1, \ldots, v_{|S|}$ and $u|_{S^c} = v_{|S|+1}, \ldots, v_n$.*

*Proof.* Take $u \in S_n$, on the one hand we have

$$\sum_{\substack{S \in 2^{[n]} \\ u \leq_R (u|_S)|(u|_{S^c})}} \mathcal{D}_{\mathrm{st}(u|_S)} \otimes \mathcal{D}_{\mathrm{st}(u|_{S^c})} = \sum_{\substack{S \in 2^{[n]} \\ u \leq_R (u|_S)|(u|_{S^c})}} \sum_{\substack{\mathrm{st}(u|_S) \leq_R \overline{u} \\ \mathrm{st}(u|_{S^c}) \leq_R \tilde{u}}} \mathcal{F}_{\overline{u}} \otimes \mathcal{F}_{\tilde{u}} \; .$$

On the other hand, we have

$$\begin{aligned} \Delta(\mathcal{D}_u) &= \sum_{u \leq_R v} \Delta\left(\mathcal{F}_v\right) \\ &= \sum_{u \leq_R v} \sum_{k=0}^{n} \mathcal{F}_{\mathrm{st}(v_1,\ldots,v_k)} \otimes \mathcal{F}_{\mathrm{st}(v_{k+1},\ldots,v_n)}. \end{aligned}$$

Thus, by Lemma 2.3.3,

$$\Delta(\mathcal{D}_u) = \sum_{\substack{S \in 2^{[n]} \\ u \leq_R (u|_S)|(u|_{S^c})}} \mathcal{D}_{\mathrm{st}(u|_S)} \otimes \mathcal{D}_{\mathrm{st}(u|_{S^c})} \; . \qquad \square$$

**Example 2.3.7.** Let $u = 231$, then $u \leq_R v = \{231, 321\}$.

For $S = \emptyset$, $u|_S = \emptyset$ and $u|_{S^c} = 231$, and conversely for $S = \{1, 2, 3\}$, $u|_S = 231$ and $u|_{S^c} = \emptyset$. The table below show the remaining elements of $S \in 2^{[3]}$. Note that the table has empty blocks because $u|_S \neq v_1, \ldots, v_{|S|}$. For each subset we list all the right weak order if the first $k$ matches $u|_S$ then the last $n-k$ matches $u|_{S^c}$.

| $S$ | $u\|_S$ | | $v_1, \ldots, v_{\|S\|}$ | $v_{\|S\|+1}, \ldots, v_n$ |
|---|---|---|---|---|
| $\{1\}$ | 2 | $v = 231$ | 2 | 31 |
| | | $v = 321$ | | |
| $\{2\}$ | 3 | $v = 231$ | | |
| | | $v = 321$ | 3 | 21 |
| $\{3\}$ | 1 | $v = 231$ | | |
| | | $v = 321$ | | |
| $\{1, 2\}$ | 23 | $v = 231$ | 23 | 1 |
| | | $v = 321$ | | |
| $\{1, 3\}$ | 21 | $v = 231$ | | |
| | | $v = 321$ | | |
| $\{2, 3\}$ | 31 | $v = 231$ | | |
| | | $v = 321$ | | |

The coproduct of $\mathcal{D}_u$ is obtained by standardizing the permutations, so

$$\Delta(\mathcal{D}_{231}) = \mathcal{D}_{231} \otimes 1 + 2\mathcal{D}_1 \otimes \mathcal{D}_{21} + \mathcal{D}_{12} \otimes \mathcal{D}_1 + 1 \otimes \mathcal{D}_{231}.$$

**Lemma 2.3.8.** *If $u, u' \in S_p$ and $v, v' \in S_q$ are permutations where $u \leq_R u'$ and*

$v \leq_R v'$, *then* $u|v_{\uparrow_p} \leq_R u'|v'_{\uparrow_p}$.

*Proof.* The permutations $u|v_{\uparrow_p}$ and $u'|v'_{\uparrow_p}$ belong to the subgroup $S_p \times S_q$ of $S_{p+q}$. Since $u \leq_R u'$ and $v \leq_R v'$, there exist $\alpha \in S_p$ and $\beta \in S_q$ such that $u' = u\alpha$ and $v' = v\beta$ with $\ell(u') = \ell(u) + \ell(\alpha)$ and $\ell(v') = \ell(v) + \ell(\beta)$. Then $u'|v'_{\uparrow_p} = (u\alpha)|(v\beta)_{\uparrow_p} = \left(u|v_{\uparrow_p}\right)\left(\alpha|\beta_{\uparrow_p}\right)$, with $\ell\left(u'|v'_{\uparrow_p}\right) = \ell\left(u'\right) + \ell\left(v'\right) = \ell\left(u|v_{\uparrow_p}\right) + \ell\left(\alpha|\beta_{\uparrow_p}\right)$. □

The following result is clearly true by linearity to fundamental basis, however the concatentation product on the diagram basis is much simpler than the shuffle product of the fundamental basis. Hence we might want to define the Malvenuto-Reutenauer algebra with the concatenation product and start with the diagram basis. In this case, it would be useful to check the compatibility of the product and coproduct and so for completeness we include the following result.

**Theorem 2.3.9.** *For* $u \in S_p$ *and* $v \in S_q$,

$$\Delta(\mathcal{D}_u \times \mathcal{D}_v) = \Delta\left(\mathcal{D}_u\right) \times \Delta\left(\mathcal{D}_v\right).$$

*Proof.* Let $u \in S_p$ and $v \in S_q$,

$$\begin{aligned}
\Delta\left(\mathcal{D}_u\right) \times \Delta\left(\mathcal{D}_v\right) &= \sum_{u \leq_R w} \sum_{v \leq_R x} \sum_{S \in 2^{[p]}} \sum_{T \in 2^{[q]}} \left(\mathcal{D}_{st(u|_S)} \otimes \mathcal{D}_{st(u|_{S^c})}\right) \times \left(\mathcal{D}_{st(v|_T)} \otimes \mathcal{D}_{st(v|_{T^c})}\right) \\
&= \sum_{u \leq_R w} \sum_{v \leq_R x} \sum_{S \in 2^{[p]}} \sum_{T \in 2^{[q]}} \left(\mathcal{D}_{st(u|_S)} \times \mathcal{D}_{st(v|_T)}\right) \otimes \left(\mathcal{D}_{st(u|_{S^c})} \times \mathcal{D}_{st(v|_{T^c})}\right) \\
&= \sum_{u \leq_R w} \sum_{v \leq_R x} \sum_{S \in 2^{[p]}} \sum_{T \in 2^{[q]}} \mathcal{D}_{st(u|_S)|st(v|_T)_{\uparrow_{|S|}}} \otimes \mathcal{D}_{st(u|_{S^c})|st(v|_{T^c})_{\uparrow_{p-|S|}}} \\
&= \sum_{u \leq_R w} \sum_{v \leq_R x} \sum_{S \in 2^{[p]}} \sum_{T \in 2^{[q]}} \mathcal{D}_{st\left((u|_S)|(v|_T)_{\uparrow_{|S|}}\right)} \otimes \mathcal{D}_{st\left((u|_{S^c})|(v|_{T^c})_{\uparrow_{p-|S|}}\right)}
\end{aligned}$$

where $u|_S = w_1, \ldots, w_{|S|}$, $u|_{S^c} = w_{|s|+1}, \ldots, w_p$, $v|_T = x_1, \ldots, x_{|T|}$ and $v|_{T^c} = x_{|T|+1}, \ldots, x_q$. By Lemma 2.3.8,

$$u \leq_R w \text{ and } v \leq_R x \Longrightarrow u|v_{\uparrow_p} \leq_R w|x_{\uparrow_p}.$$

Therefore,

$$\sum_{u|v_{\uparrow_p} \leq_R w|x_{\uparrow_p}} \sum_{S \in 2^{[p+q]}} \mathcal{D}_{st\left(\left(u|v_{\uparrow_p}\right)|_S\right)} \otimes \mathcal{D}_{st\left(\left(u|v_{\uparrow_p}\right)|_{S^c}\right)} = \Delta\left(\mathcal{D}_u\right) \times \Delta\left(\mathcal{D}_v\right)$$

where $\left(u|v_{\uparrow_p}\right)|_S = \left(w|x_{\uparrow_p}\right)_1, \ldots, \left(w|x_{\uparrow_p}\right)_{|S|}$ and $\left(u|v_{\uparrow_p}\right)|_{S^c} = \left(w|x_{\uparrow_p}\right)_{|S|+1}, \ldots, \left(w|x_{\uparrow_p}\right)_{p+q}$. □

## 2.4 The dual monomial basis and the diagram basis of MR

In this section we look at the self-duality of MR in terms of the diagram basis.

The Hopf algebra of MR is self-dual under the map

$$\Theta : (\mathsf{MR})^* \to \mathsf{MR},$$
$$\mathcal{F}_u^* \mapsto \mathcal{F}_{u^{-1}} .$$

The map is an isomorphism of Hopf algebras.

Let $\omega_n$ be the maximum element of the symmetric group $S_n$, that is it is the permutation such that

$$\omega_n(i) = n + 1 - i.$$

The map $REV : \mathsf{MR} \to \mathsf{MR}$ is given by

$$REV(\mathcal{F}_u) = \mathcal{F}_{u_p,u_{p-1},\ldots,u_2,u_1} = \mathcal{F}_{u\cdot\omega_p},$$

where $u \in S_p$. The properties that we use of this map are well known, but it is difficult to find them organized in a single reference so we include the following proofs for the sake of completeness.

**Lemma 2.4.1.** *The map REV is an algebra homomorphism.*

*Proof.* For $u \in S_p$ and $v \in S_q$,

$$\begin{aligned} REV(\mathcal{F}_u) \times REV(\mathcal{F}_v) &= \mathcal{F}_{u\cdot\omega_p} \times \mathcal{F}_{v\cdot\omega_q} \\ &= \sum_{\zeta\in S^{(p,q)}} \mathcal{F}_{\left(u\cdot\omega_p|v\cdot\omega_{q_{\uparrow p}}\right)\cdot\zeta^{-1}}, \end{aligned}$$

where the sum is over all shuffle of the permutations $u_p < \cdots < u_1$ and $v_q + p < \cdots < v_1 + p$, and thus

$$\begin{aligned} \sum_{\zeta\in S^{(p,q)}} \mathcal{F}_{\left(u\cdot\omega_p|v\cdot\omega_{q_{\uparrow p}}\right)\cdot\zeta^{-1}} &= \sum_{\zeta\in S^{(p,q)}} \mathcal{F}_{\left(\left(u|v_{\uparrow p}\right)\cdot\zeta^{-1}\right)\cdot\omega_{p+q}} \\ &= REV(\mathcal{F}_u \times \mathcal{F}_v). \end{aligned}$$

Hence, $REV$ is an algebra homomorphism. □

**Lemma 2.4.2.** *For any $u \in S_p$, we have $REV(\mathcal{G}_u) = \mathcal{G}_{p+1-u_1,p+1-u_2,\ldots,p+1-u_n} = \mathcal{G}_{\omega_p \cdot u}$.*

*Proof.* Let $u \in S_p$,

$$\begin{aligned} REV(\mathcal{G}_u) &= REV(\mathcal{F}_{u^{-1}}) \\ &= \mathcal{F}_{u^{-1}\cdot\omega_p} \\ &= \mathcal{G}_{(u^{-1}\cdot\omega_p)^{-1}} \\ &= \mathcal{G}_{(\omega_p)^{-1}\cdot(u^{-1})^{-1}} \\ &= \mathcal{G}_{\omega_p\cdot u}. \end{aligned}$$

□

**Lemma 2.4.3.** *Let $u, v \in S_p$, then*

$$\begin{aligned} u \leq_L v &\iff v \cdot \omega_p \leq_L u \cdot \omega_p \\ u \leq_R v &\iff v \cdot \omega_p \leq_R u \cdot \omega_p, \end{aligned} \tag{2.26}$$

*and*

$$\begin{aligned} u \leq_L v &\iff \omega_p \cdot v \leq_L \omega_p \cdot u \\ u \leq_R v &\iff \omega_p \cdot v \leq_R \omega_p \cdot u. \end{aligned} \tag{2.27}$$

*Proof.* We define the left weak order on $S_p$ (2.4) as

$$u \leq_L v \ \text{ if } \ \mathrm{Inv}(u) \subseteq \mathrm{Inv}(v),$$

and the right weak order on $S_p$ (2.6) as

$$u \leq_R v \ \text{ if } \ \mathrm{Inv}(u^{-1}) \subseteq \mathrm{Inv}(v^{-1}).$$

Then

$$\begin{aligned} u \leq_L v &\iff \mathrm{Inv}(u) \subseteq \mathrm{Inv}(v) \\ &\iff \mathrm{Inv}(u \cdot \omega_p) \supseteq \mathrm{Inv}(v \cdot \omega_p) \\ &\iff u \cdot \omega_p \geq_L v \cdot \omega_p. \end{aligned}$$

Similarly,

$$\begin{aligned} u \leq_R v &\iff \mathrm{Inv}(u^{-1}) \subseteq \mathrm{Inv}(v^{-1}) \\ &\iff \mathrm{Inv}(u^{-1} \cdot \omega_p) \supseteq \mathrm{Inv}(v^{-1} \cdot \omega_p) \\ &\iff \omega_p \cdot u \geq_R \omega_p \cdot v \end{aligned}$$

Furthermore,

$$\begin{aligned} u \leq_L v &\iff \mathrm{Inv}(u) \subseteq \mathrm{Inv}(v) \\ &\iff \mathrm{Inv}(\omega_p \cdot u) \supseteq \mathrm{Inv}(\omega_p \cdot v) \\ &\iff \omega_p \cdot u \geq_L \omega_p \cdot v, \end{aligned}$$

and similarly,

$$\begin{aligned}
u \leq_R v &\iff \mathrm{Inv}(u^{-1}) \subseteq \mathrm{Inv}(^{-1}) \\
&\iff \mathrm{Inv}(\omega_p \cdot u^{-1}) \supseteq \mathrm{Inv}(\omega_p \cdot v^{-1}) \\
&\iff u \cdot \omega_p \geq_R v \cdot \omega_p.
\end{aligned}$$

□

We now define the diagram basis in term of the dual monomial basis. Let $\{\mathcal{M}^*_u \mid u \in \mathcal{S}_n\}$ be the basis of $\mathsf{MR}^*$ dual to the monomial basis of $\mathsf{MR}$.

**Lemma 2.4.4.** *Let $u \in S_p$, then*

$$\mathcal{D}_u = REV\left(\mathcal{M}^*_{\omega_p \cdot u^{-1}}\right).$$

*Proof.* By definition, $\mathcal{D}_u = \sum_{u \leq_R v} \mathcal{F}_v$, hence

$$\begin{aligned}
\mathcal{D}_u &= \sum_{u \leq_R v} \mathcal{G}_{v^{-1}} \\
&= \sum_{u \leq_R v^{-1}} \mathcal{G}_v \\
&= \sum_{u^{-1} \leq_L v} \mathcal{G}_v && \text{by Equation (2.8)} \\
&= REV\left(\sum_{u^{-1} \leq_L v} \mathcal{G}_{\omega_p \cdot v}\right) && \text{by Lemma 2.4.2} \\
&= REV\left(\sum_{u^{-1} \leq_L \omega_p \cdot v} \mathcal{G}_v\right)
\end{aligned}$$

$$= REV\left(\sum_{\omega_p \cdot u^{-1} \geq_L v} \mathcal{G}_v\right) \qquad \text{by Equation (2.27)}$$

$$= REV\left(\mathcal{M}^*_{\omega_p \cdot u^{-1}}\right) \qquad \text{by Equation (2.18).}$$

□

The following lemma seems to be known however, we provide a proof for completeness.

**Lemma 2.4.5.** *For $u \in S_p$ and $v \in S_q$,*

$$(\omega_p \cdot u)_{\uparrow_q} | \omega_q \cdot v = \omega_{p+q} \cdot \left(u | v_{\uparrow_p}\right). \tag{2.28}$$

*Proof.* For $i \in \{1, 2, \ldots, p\}$,

$$\left((\omega_p \cdot u)_{\uparrow_q} | \omega_q \cdot v\right)(i) = \omega_p \cdot u(i) + q = p + q + 1 - u(i),$$

while $$\left(\omega_{p+q} \cdot \left(u | v_{\uparrow_p}\right)\right)(i) = p + q + 1 - u(i).$$

For $i \in \{p+1, p+2, \ldots, p+q\}$,

$$\left((\omega_p \cdot u)_{\uparrow_q} | \omega_q \cdot v\right)(i) = q + 1 - v(i-p),$$

while $$\left(\omega_{p+q} \cdot \left(u | v_{\uparrow_p}\right)\right)(i) = p + q + 1 - (v(i-p) + p) = q + 1 - v(i-p).$$

Thus, we have shown (2.28) holds. □

**Theorem 2.4.6.** *For $u \in S_p$ and $v \in S_q$, we have*

$$\mathcal{D}_u \times \mathcal{D}_v = \mathcal{D}_{u|v_{\uparrow p}}.$$

*Proof.* Expand the product $\mathcal{D}_u \times \mathcal{D}_v$ in terms of the dual monomial basis to obtain

$$\begin{aligned}
\mathcal{D}_u \times \mathcal{D}_v &= REV\left(\mathcal{M}^*_{\omega_p \cdot u^{-1}}\right) \times REV\left(\mathcal{M}^*_{\omega_q \cdot v^{-1}}\right) \\
&= REV\left(\mathcal{M}^*_{\omega_p \cdot u^{-1}} \times \mathcal{M}^*_{\omega_q \cdot v^{-1}}\right) \\
&= REV\left(\mathcal{M}^*_{(\omega_p \cdot u^{-1})_{\uparrow q} | \omega_q \cdot v^{-1}}\right).
\end{aligned}$$

By Lemma 2.4.5, we have

$$REV\left(\mathcal{M}^*_{(\omega_p \cdot u^{-1})_{\uparrow q} | \omega_q \cdot v^{-1}}\right) = REV\left(\mathcal{M}^*_{\omega_{p+q} \cdot \left(u^{-1} | v^{-1}{}_{\uparrow p}\right)}\right).$$

Expressing $\mathcal{D}_{u|v_{\uparrow p}}$ in terms of the dual monomial basis gives

$$\mathcal{D}_{u|v_{\uparrow p}} = REV\left(\mathcal{M}^*_{\omega_{p+q} \cdot \left(u|v_{\uparrow p}\right)^{-1}}\right).$$

Therefore, by the inverse property of permutation

$$u^{-1} | v^{-1}{}_{\uparrow p} = \left(u | v_{\uparrow p}\right)^{-1},$$

we have $\mathcal{D}_u \times \mathcal{D}_v = \mathcal{D}_{u|v_{\uparrow p}}$. □

The lemma below shows how $REV$ behave with respect to coproduct.

**Lemma 2.4.7.** *For the coproduct* $\Delta : \mathsf{MR} \to \mathsf{MR} \otimes \mathsf{MR}$*, $REV$ is a coalgebra antimorphism. That is,*

$$\Delta \circ REV = REV \otimes REV \circ \tau \circ \Delta$$

*where* $\tau(f \otimes g) = g \otimes f$.

*Proof.* Let $u \in S_p$, then

$$\begin{aligned}
REV \otimes REV \circ \tau \circ \Delta(\mathcal{F}_u) &= REV \otimes REV \circ \tau \left( \sum_{i=0}^{p} \mathcal{F}_{st(u_1,\ldots,u_i)} \otimes \mathcal{F}_{st(u_{i+1},\ldots,u_p)} \right) \\
&= REV \otimes REV \left( \sum_{i=0}^{p} \mathcal{F}_{st(u_{i+1},\ldots,u_p)} \otimes \mathcal{F}_{st(u_1,\ldots,u_i)} \right) \\
&= \sum_{i=0}^{p} \; REV \left( \mathcal{F}_{st(u_{i+1},\ldots,u_p)} \right) \otimes REV \left( \mathcal{F}_{st(u_1,\ldots,u_i)} \right) \\
&= \sum_{i=0}^{p} \mathcal{F}_{st(u_p,\ldots,u_{i+1})} \otimes \mathcal{F}_{st(u_i,\ldots,u_1)} \\
&= \Delta \left( REV(\mathcal{F}_u) \right).
\end{aligned}$$

□

## 2.5 The internal product of the diagram basis of MR

The diagram basis gives us an internal product which is defined by the composition of permutations. The internal product is a morphism with respect to the external product. This is true of the dual-monomial basis as well, however, in terms of the diagram basis the Hopf algebra has a monoid product $\mathcal{D}_u \times \mathcal{D}_v = \mathcal{D}_w$ and $w \in S_p \times S_q \subseteq S_{p+q}$, where $S_p$ in $S_p \times S_q$ permutes $\{1, 2, \ldots, p\}$ and $S_q$ in $S_p \times S_q$ permutes $\{p+1, p+2, \ldots, p+q\}$. Therefore it is natural to define the internal product on the diagram basis

$$\mathcal{D}_u \cdot \mathcal{D}_v = \mathcal{D}_{u \circ v}$$

which is also a monoid product.

**Example 2.5.1.** For $u = 3412$ and $v = 2314$,

$$\mathcal{D}_{3412} \cdot \mathcal{D}_{2314} = \mathcal{D}_{4132}.$$

The internal and external products of the diagram basis satisfy a distributive property.

**Proposition 2.5.2.** *Let $u, w \in S_p$ and $v, x \in S_q$, then*

$$(\mathcal{D}_u \times \mathcal{D}_v) \cdot (\mathcal{D}_w \times \mathcal{D}_x) = (\mathcal{D}_u \cdot \mathcal{D}_w) \times (\mathcal{D}_v \cdot \mathcal{D}_x).$$

*Proof.* For $u, w \in S_p$ and $v, x \in S_q$,

$$\begin{aligned}(\mathcal{D}_u \times \mathcal{D}_v) \cdot (\mathcal{D}_w \times \mathcal{D}_x) &= \mathcal{D}_{u|v_{\uparrow_p}} \cdot \mathcal{D}_{w|x_{\uparrow_p}} \\ &= \mathcal{D}_{\left(u|v_{\uparrow_p}\right) \circ \left(w|x_{\uparrow_p}\right)}.\end{aligned}$$

Now for each $1 \leq i \leq p+q$, we compute $\left(u|v_{\uparrow_p}\right)\left(\left(w|x_{\uparrow_p}\right)(i)\right)$ in order to simplify the expression for $\left(u|v_{\uparrow_p}\right) \circ \left(w|x_{\uparrow_p}\right)$.

For $i \in \{1, \ldots, p\}$, $\left(u|v_{\uparrow_p}\right)\left(\left(w|x_{\uparrow_p}\right)(i)\right) = u(w(i)) = (u \circ w)(i)$. Now for $i \in \{p+1, \ldots, p+q\}$, $\left(u|v_{\uparrow_p}\right)\left(\left(w|x_{\uparrow_p}\right)(i)\right) = v_{\uparrow_p}(x_{\uparrow_p}(i))$. Thus $\left(u|v_{\uparrow_p}\right) \circ \left(w|x_{\uparrow_p}\right) = (u \circ w) \,|\, (v \circ x)_{\uparrow_p}$ and so

$$\begin{aligned}
\mathcal{D}_{\left(u|v_{\uparrow_p}\right)\circ\left(w|x_{\uparrow_p}\right)} &= \mathcal{D}_{(u\circ w)|(v\circ x)_{\uparrow_p}} \\
&= \mathcal{D}_{u\circ w} \times \mathcal{D}_{v\circ x} \\
&= (\mathcal{D}_u \cdot \mathcal{D}_w) \times (\mathcal{D}_v \cdot \mathcal{D}_x).
\end{aligned}$$

□

We have computed the internal product on the fundamental basis and the monomial basis and found that the internal product that we have defined here does not have a form for which we are able to conjecture a formula. We provide an example here for refernce.

**Example 2.5.3.** The internal products of the fundamental basis indexed by permutations in $S_2$,

$$\begin{aligned}
&\mathcal{F}_{12} \cdot \mathcal{F}_{12} = 2\mathcal{F}_{12} \\
&\mathcal{F}_{12} \cdot \mathcal{F}_{21} = -\mathcal{F}_{12} \\
&\mathcal{F}_{21} \cdot \mathcal{F}_{12} = -\mathcal{F}_{12} \\
&\mathcal{F}_{21} \cdot \mathcal{F}_{21} = \mathcal{F}_{12} + \mathcal{F}_{21}.
\end{aligned}$$

The internal products of the monomial basis indexed by permutations in $S_2$,

$$\begin{aligned}
\mathcal{M}_{12} \cdot \mathcal{M}_{12} &= 5\mathcal{M}_{12} + 6\mathcal{M}_{21} \\
\mathcal{M}_{12} \cdot \mathcal{M}_{21} &= -2\mathcal{M}_{12} - 3\mathcal{M}_{21} \\
\mathcal{M}_{21} \cdot \mathcal{M}_{12} &= -2\mathcal{M}_{12} - 3\mathcal{M}_{21} \\
\mathcal{M}_{21} \cdot \mathcal{M}_{21} &= \mathcal{M}_{12} + 2\mathcal{M}_{21}.
\end{aligned}$$

Our motivation for defining the internal product on the diagram basis is Proposition 2.5.2. If we were to define the internal product on the fundamental basis as a composition product (that is, $\mathcal{F}_u \odot \mathcal{F}_v = \mathcal{F}_{u \circ v}$) the internal and external products are not distributive as shown in the Example below.

**Example 2.5.4.** Let $u = 1$ and $v = 1$, then

$$\begin{aligned}
(\mathcal{F}_1 \odot \mathcal{F}_1) \times (\mathcal{F}_1 \odot \mathcal{F}_1) &= \mathcal{F}_1 \times \mathcal{F}_1 \\
&= \mathcal{F}_{12} + \mathcal{F}_{21},
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{F}_1 \times \mathcal{F}_1) \odot (\mathcal{F}_1 \times \mathcal{F}_1) &= (\mathcal{F}_{12} + \mathcal{F}_{21}) \odot (\mathcal{F}_{12} + \mathcal{F}_{21}) \\
&= 2\mathcal{F}_{12} + 2\mathcal{F}_{21}.
\end{aligned}$$

Thus, $(\mathcal{F}_1 \odot \mathcal{F}_1) \times (\mathcal{F}_1 \odot \mathcal{F}_1) \neq (\mathcal{F}_1 \times \mathcal{F}_1) \odot (\mathcal{F}_1 \times \mathcal{F}_1)$.

# Chapter 3

# Algebra Structure of Set Partitions

## Introduction

The centralizer algebra of the permutation representation for the symmetric group has a combinatorial basis given by the set partitions. This algebra is known as the partition algebra, $\mathsf{P}_k(n)$.

In the early 90's, the partition algebra appeared in the works of Martin [Mar90], [Mar91], [Mar94], [Mar96] and Jones [Jon94] independently. Both Martin and Jones study the partition algebra as a generalization of the Temperley-Lieb algebra and the Potts model in statistical mechanics. Martin shows the algebra both implicitly [Mar90], [Mar91] and explicitly [Mar94]. While Jones shows explicitly the Schur-Weyl duality between the partition algebras and the symmetric group. Halverson [Hal01] later showed the analogues of the Frobenius formula and the Murnaghan-Nakayama rule for the characters of the partition algebras.

The representation theory of partition algebras have been studied in the works of Martin [Mar94], [Mar96], Martin and Saleur [MS93], Doran and Wales [DIW00], Halverson and Ram [HR05], and Martin and Woodcock [MW98].

The Schur-Weyl duality was one of the motivations for the introduction of the algebra in [Jon94], and it appears independently in [Mar94] as the partition algebra $\mathsf{P}_k(n)$. In this chapter, we define an external product on the diagram and orbit bases of the partition algebra.

## 3.1 Notation

In this section, we will define some basic combinatorial objects and establish notation used in the chapter.

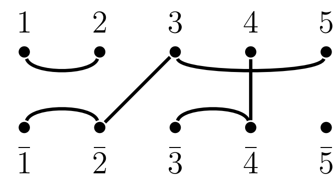
A set of subsets $\{S_1, S_2, \cdots, S_\ell\}$ with $S_i \subset S$ for $1 \leq i \leq \ell$, $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq \ell$ and $S_1 \cup S_2 \cup \cdots \cup S_\ell = S$ is a *set partition* of a set $S$. We denote the *length* of a set partition as $\ell(\pi) = \ell$ and use the notation $\pi \vdash S$ to indicate that $\pi$ is a set partition of the set $S$.

Let $\Pi_{[k]\cup[\overline{k}]}$, where $k \in \mathbb{Z}^+$, be the set of set partitions of $\{1, 2, \ldots, k, \overline{1}, \overline{2}, \ldots \overline{k}\}$. A set partition of $[k] \cup [\overline{k}]$ is a collection of non-empty disjoint subsets of $[k] \cup [\overline{k}]$, called *blocks*, whose union is $[k] \cup [\overline{k}]$. We will use vertical lines to separate the blocks in a set partition. For $\pi \in \Pi_{[k]\cup[\overline{k}]}$, let $|\pi|$ equal the number of blocks of $\pi$.

For $\pi \in \Pi_{[k]\cup[\overline{k}]}$, the *diagram* of $\pi$ has two rows of $k$ vertices each, with the top vertices indexed by $1, 2, \ldots, k$ and the bottom vertices indexed by $\bar{1}, \bar{2}, \ldots, \bar{k}$ from left to right. Edges are drawn to connect the vertices if they lie within the same block of

$\pi$. The way the edges are drawn is immaterial, what matters is that the connected components of the diagram correspond to the blocks of the set partition $\pi$.

**Example 3.1.1.** The set partition $\{1, 2|3, 5, \bar{1}, \bar{2}|4, \bar{3}, \bar{4}|\bar{5}\} \in \Pi_{[5]\cup[\bar{5}]}$ corresponds to the diagram given below.

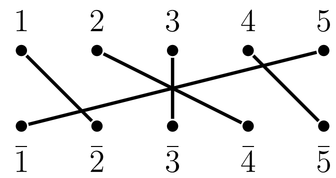



A diagram is a graph of the vertices and edges and the set partition of $S$ is the set of sets whose union is $S$. We equate the two notions and sometimes refer to the diagram or set partition interchangeably.

A set partition where all the blocks are of size two, with one element in $[k]$ and one element in $[\overline{k}]$ is known as a *set partition permutation.* When this set partition is represented as a diagram it is referred to as a *permutation diagram.*

The diagram of set partition permutation is a two-line notation for permutations, where the top row represents the ordered unbarred entries $1, 2, \ldots, k$ and the bottom row lists, under each element of $[k]$, the element of $[\overline{k}]$ that is contained in the same block. The one-line notation for a permutation is a compressed form for the two line notation where the first line is omitted because it is implicitly understood. If you have a permutation $\sigma \in S_k$ then the corresponding set partition permutation is $\mathbb{D}(\sigma) = \{\sigma(1), \overline{1}|\sigma(2), \overline{2}|\ldots|\sigma(k), \overline{k}\}$.

**Example 3.1.2.** A permutation diagram $\pi = \{5, \overline{1}|1, \overline{2}|3, \overline{3}|2, \overline{4}|4, \overline{5}\}$ can be represented as the following two-line notation



$$\begin{pmatrix} 5 & 1 & 3 & 2 & 4 \\ \\ \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \end{pmatrix}$$

sorting the bottom line will display the permutation on the top line. The permutation on the top line gives the permutation corresponding to the set partition $\pi$ as 51324.

For a set partition $\pi$ of set $X$ and $T$ a subset of $X$, then $\pi$ restricted to $T$ will be denoted by $\pi|_T = \{S \cap T : S \in \pi\}$. Let $\pi$ and $\tau$ be set partitions on disjoint sets $S$ and $T$. The *smash product* of $\pi$ and $\tau$ will be

$$\pi\#\tau = \{\theta : \theta \vdash S \cup T, \theta|_S = \pi, \theta|_T = \tau\}$$

where $\theta \in \pi\#\tau$ means that $\theta$ is of the form

$$\theta = \{S_{i_1}, \ldots, S_{i_{\ell(\pi)-k}}, T_{j_1}, \ldots, T_{j_{\ell(\tau)-k}}, S_{i'_1} \cup T_{j'_1}, \ldots, S_{i'_k} \cup T_{j'_k}\}$$

where

$$\{i_1, i_2, \ldots, i_{\ell(\pi)-k}, i'_1, i'_2, \ldots, i'_k\} = \{1, 2, \ldots, \ell(\pi)\}$$

and

$$\{j_1, j_2, \ldots, j_{\ell(\tau)-k}, j'_1, j'_2, \ldots, j'_k\} = \{1, 2, \ldots, \ell(\tau)\}.$$

**Example 3.1.3.** For $\pi = \{1, \overline{3}|2, 3, \overline{1}, \overline{2}\}$ and $\tau = \{1, \overline{2}|2, \overline{1}\}$,

$$\pi\#\tau_{\uparrow_3} = \{\{1, \overline{3}|2, 3, \overline{1}, \overline{2}|4, \overline{5}|5, \overline{4}\}, \{1, 4, \overline{3}, \overline{5}|2, 3, \overline{1}, \overline{2}|5, \overline{4}\}, \{1, \overline{3}|2, 3, 4, \overline{1}, \overline{2}, \overline{5}|5, \overline{4}\},$$

$\{1, 5, \overline{3}, \overline{4} | 2, 3, \overline{1}, \overline{2} | 4, \overline{5}\}, \{1, \overline{3} | 2, 3, 5, \overline{1}, \overline{2}, \overline{4} | 4, \overline{5}\}, \{1, 4, \overline{3}, \overline{5} | 2, 3, 5, \overline{1}, \overline{2}, \overline{4}\},$

$\{1, 5, \overline{3}, \overline{4} | 2, 3, 4, \overline{1}, \overline{2}, \overline{5}\}\}.$

## 3.2 Partition Algebra

Let $k$ be a non negative integer and $n$ be a complex number. The partition algebras $\mathsf{P}_k(n)$ are algebras whose bases are indexed by diagrams where a combinatorial rule defines the product and the structure coefficients of the algebra depend polynomially on a parameter. The algebras $\mathsf{P}_k(n)$ have two distinguished bases defined in literature, namely, the diagram basis $d_\pi$ and the orbit basis $x_\pi$, where $\pi \in \Pi_{[k]\cup[\overline{k}]}$.

In [Mar94], Martin defined an internal product in the diagram basis of the partition algebra. In [BH19a], Benkart and Halverson prove a rule for internal product in the orbit basis that was originally stated by Halverson and Ram in unpublished notes.

### 3.2.1 The diagram basis of $\mathsf{P}_k(n)$

The *diagram product* of $\pi$ and $\tau$ in $\Pi_{[k]\cup[\overline{k}]}$, denoted by $\pi * \tau$, is the concatenation of $\pi$ and $\tau$; that is placing $\pi$ above $\tau$, identifying the vertices in the bottom row of $\pi$ with those in the top row of $\tau$, concatenating the edges, and deleting all connected components that lie entirely in the middle row of the joined diagrams. The concatenated diagram often has an extra parameter $n^c$, where $c$ is the number of connected components that lie entirely in the middle row.

Let $\sigma$ and $\tau$ be permutations in $S_k$, then $\sigma \circ \tau$ is an element of $S_k$ and is the permutation such that for each $i \in [k]$ is sent to $\sigma \circ \tau(i) = \sigma(\tau(i))$. This is a product of permutations known as the *composition*. The composition of $\sigma$ and $\tau$ is a permutation of $S_k$.

*Remark*: We have chosen to associate a permutation in $\sigma \in S_k$ with a diagram $\mathbb{D}(\sigma)$ for the reason that the diagram product of two set partition permutations is the product of their corresponding diagrams. That is, for $\sigma$ and $\tau$ in $S_k$,

$$\mathbb{D}(\sigma) * \mathbb{D}(\tau) = \mathbb{D}(\sigma \circ \tau).$$

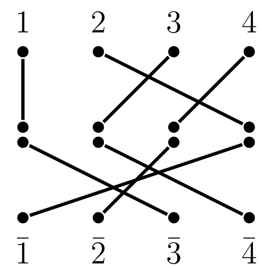
When the product of two diagram basis elements are indexed by set partition permutations, the product is equivalent to the composition of the permutations. Composition of two permutations, $\sigma \circ \tau$, is the function that maps any element $i$ of the set to $\sigma(\tau(i))$. This is similar to reading the product of the diagram by going from the barred entries of the bottom diagram to the unbarred entries on the top diagram. Note that, for product of set partition permutations, there will be no connected components that lie entirely in the middle row.

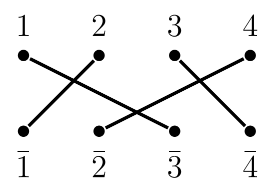
**Proposition 3.2.1.** *For $\sigma \in S_k$ with a diagram $\mathbb{D}(\sigma)$ and $\tau \in S_k$ with a diagram $\mathbb{D}(\tau)$,*

$$\mathbb{D}(\sigma) * \mathbb{D}(\tau) = \mathbb{D}(\sigma \circ \tau).$$

*Proof.* Let $\mathbb{D}(\sigma)$ and $\mathbb{D}(\tau)$ be permutation diagrams of $\sigma$ and $\tau$ in $S_k$. Multiplication of the two permutation diagrams is accomplished by placing $\mathbb{D}(\sigma)$ above $\mathbb{D}(\tau)$, identifying the bottom row of $\mathbb{D}(\sigma)$ with the top row of $\mathbb{D}(\tau)$ and concatening the edges will give us a middle row. $i$ in the bottom row is connected to $\tau(i)$ in the middle row. $j$ in the middle row is connected to $\sigma(j)$ in the top row. Therefore, $\tau(i)$ in the middle row is connected to $\sigma(\tau(i))$. There is a path from $i$ in the bottom row to $\tau(i)$ in the middle row to $\sigma(\tau(i))$ in the top row. Now delete the middle row and there is an edge from $i$ in the bottom row to $\sigma(\tau(i))$ in the top row. $\square$

**Example 3.2.2.** Set partition $\mu = \mathbb{D}(u) = \{1, \bar{1}|3, \bar{2}|4, \bar{3}|2, \bar{4}\}$ and $\nu = \mathbb{D}(v) = \{4, \bar{1}|3, \bar{2}|1, \bar{3}|2, \bar{4}\}$ can be represented using the permutations $u = 1342$ an $v = 4312$, respectively. The product of the diagrams of $\mu$ and $\nu$

$\mathbb{D}(u) * \mathbb{D}(v) =$ 


$\mathbb{D}(u \circ v) =$ 


thus,

$$u \circ v = \begin{pmatrix} 1 & 3 & 4 & 2 \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{pmatrix} \circ \begin{pmatrix} 4 & 3 & 1 & 2 \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 & 3 \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} \end{pmatrix}.$$

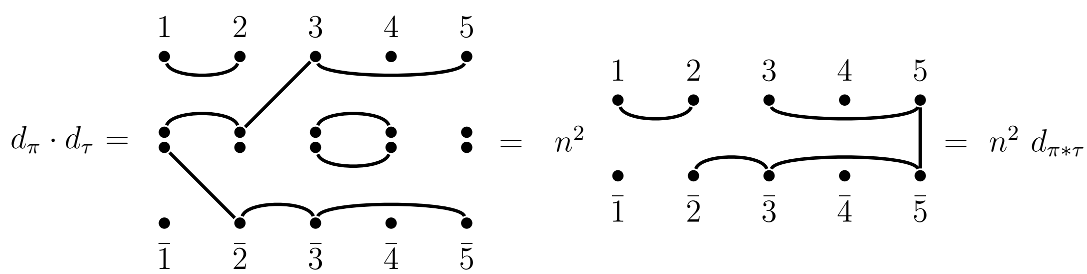
### 3.2.2 The internal product on the diagram basis

The product of the diagram basis of the partition algebra is obtained by placing the one diagram, $d_\pi$, above the other, $d_\tau$, aligning the bottom row of $d_\pi$ with the top row of $d_\tau$ to form a three row diagram, concatenating the edges of the diagrams and deleting all the connected components that lie entirely in the middle row and multiplying by a factor of $n$ for each middle row component. For set partitions $\pi, \tau \in \Pi_{[k]\cup[\bar{k}]}$, where $k$ is a positive integer, the internal product of diagram basis is given by

$$d_\pi \cdot d_\tau = n^{[\pi * \tau]} d_{\pi * \tau}, \tag{3.1}$$

where $\pi * \tau$ is the set partition obtained by the diagram product of $\pi$ and $\tau$, and $[\pi * \tau]$ is the number of blocks removed from the middle of the product $d_\pi \cdot d_\tau$.

**Example 3.2.3.** For $\pi = \{1, 2|3, 5, \bar{1}, \bar{2}|4|\bar{3}, \bar{4}|\bar{5}\}$ and $\tau = \{1, \bar{2}, \bar{3}, \bar{5}|2|3, 4|5|\bar{1}|\bar{4}\}$ in $\Pi_{[5]\cup[\bar{5}]}$,




## 3.3 The orbit basis of $\mathsf{P}_k(n)$

In the Schur-Weyl duality, $S_n$ acts on the basis $v_{i_1} \otimes \cdots \otimes v_{i_k}$ to form orbits. The orbit basis of the partition algebra are the linear transformations which take a single $S_n$-orbit and maps it to a single $S_n$-orbit. The action on $V_n^{\otimes k}$ by $\pi \cdot v_i = v_{\pi_i}$, for $\pi \in S_n$, forms a basis for $\mathrm{End}_{S_k}(V_n^{\otimes k})$ that are constant on the $S_k$-orbits. These orbits decompose $[k] \cup [\bar{k}]$ into subsets that correspond to set partitions of the orbit basis in $\mathsf{P}_k(n)$ [HR05]. The product of these transformations is more complex than the product rule for the diagram basis but the rule is described and proven in [BH19b] (Corollary 4.12).

### 3.3.1 Change of basis

The change of basis between the diagram basis of partition algebra and the orbit basis is described in Halverson and Ram [HR05]. A good reference is Benkart and Halverson [BH19a] who give additional details and exposition using the Möbius function of set partition lattice.

For $k \in \mathbb{Z}^+$ and set partitions $\pi$ and $\rho$ in $\Pi_{[k]\cup[\bar{k}]}$, we say $\rho$ is coarser than $\pi$, $\pi \preceq \rho$, if every block of $\pi$ is contained in a block of $\rho$. The diagram basis element $d_\pi$ is the sum of all orbit basis elements $x_\rho$ for which $\rho$ is coarser than $\pi$,

$$d_\pi = \sum_{\pi \preceq \rho} x_\rho. \tag{3.2}$$

And the inverse of the above equation uses the Möbius function [Sta10, section 5.1, p.7] of the set partition lattice. If $\pi \preceq \rho$ and $\rho$ consists of $\ell$ blocks such that the $i$th block of $\rho$ is the union of $b_i$ blocks of $\pi$ then the Möbius function of this relation is

$$\mu_{2k}(\pi, \rho) = \prod_{i=1}^{\ell} (-1)^{b_i - 1}(b_i - 1)!.$$

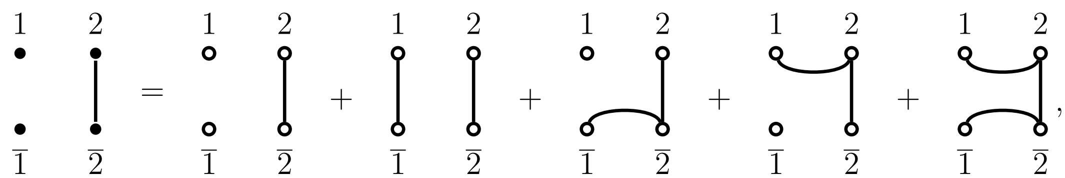
The diagram and orbit basis then satisfy

$$x_\pi = \sum_{\pi \preceq \rho} \mu_{2k}(\pi, \rho) d_\rho, \tag{3.3}$$

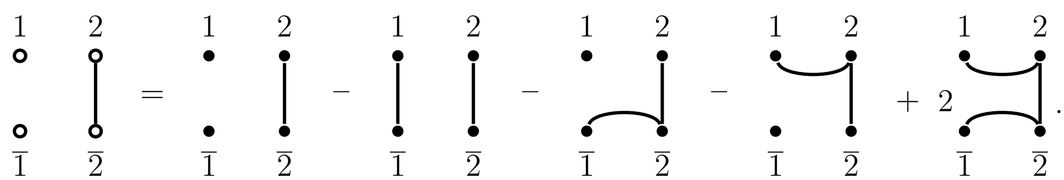
where $\mu_{2k}$ is the Möbius function as defined above.

In this thesis, we will represent diagrams of the diagram basis using filled black vertices and those of the orbit basis using the unfilled black vertices.

**Example 3.3.1.** For $\pi = \{1|2,\bar{2}|\bar{1}\} \in \Pi_{[2]\cup[\bar{2}]}$, the expansion of the diagram $d_\pi$ in terms of the orbit basis is expressed as



and the expansion of the orbit $x_\pi$ in terms of the diagram basis is expressed as



## 3.3.2 The internal product on the orbit basis

In this section, we will state the multiplication rule of the orbit basis of the partition algebra given in [HR05], [BH19a].

For $\pi, \tau \in \Pi_{[k]\cup[\bar{k}]}$, we say that $\pi * \tau$ exactly matches in the middle if the set partition that $\pi$ induces on its bottom row equals the set partition that $\tau$ induces on the top row when the bars are ignored.

For $\ell, m \in \mathbb{Z}_{\geq 0}$,

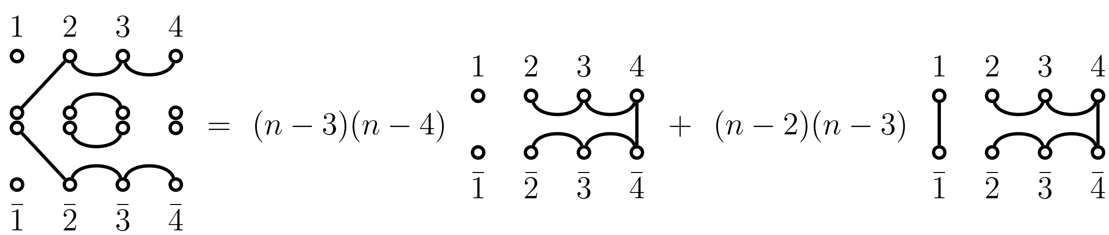
$$(m)_\ell = \begin{cases} m(m-1)\cdots(m-\ell+1) = \frac{m!}{(m-\ell)!} & \text{if } \ell \leq m \\ 1 & \text{if } \ell = 0 \\ 0 & \text{if } \ell > m. \end{cases}$$

The internal product in $\mathsf{P}_k(n)$ in terms of the orbit basis $\{x_\pi\}_{\pi \in \Pi_{[k]\cup[\bar{k}]}}$ is given by

$$x_\pi \cdot x_\tau = \begin{cases} \sum_\rho (n - |\rho|)_{[\pi * \tau]} x_\rho, & \text{if } \pi * \tau \text{ exactly matches in the middle,} \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

where the sum is over those coarsenings $\rho$ of $\pi * \tau$ obtained by connecting blocks that lie entirely in the top row of $\pi$ to blocks that lie entirely in the bottom row of $\tau$.

**Example 3.3.2.** Suppose $k = 4$, $n \geq 4$, $\pi = \{1|2, 3, 4, \bar{1}|\bar{2}, \bar{3}|\bar{4}\}$ and $\tau = \{1, \bar{2}, \bar{3}, \bar{4}|2, 3|4|$
$\bar{1}\}$ in $\Pi_{[4]\cup[\bar{4}]}$, so that $[\pi * \tau] = 2$. Then the product $x_\pi \cdot x_\tau$ is the expression



## 3.4 The partition algebra as a centralizer algebra

Let $n \in \mathbb{Z}_{>0}$ and $V$ denote the $n-$dimensional permutation representation of the symmetric group $S_n$. That is,

$$V = \text{span}_{\mathbb{C}}\{v_i \mid i = 1, \ldots, n\},$$

where $\sigma \cdot v_i = v_{\sigma_i}$, for $\sigma \in S_n$. Alternatively we may view $V$ as the restriction of the natural module of $GL_n(\mathbb{C})$ to $S_n$, viewing $S_n$ as $n \times n$ permutation matrices. Let $S_n$ act diagonally on the basis of simple tensors in $V^{\otimes k}$

$$\sigma \cdot v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} = v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots \otimes v_{\sigma(i_k)}.$$

For $x_\pi \in \mathsf{P}_k(n)$ and for all $i_1, i_2, \ldots i_k \in [k]$ and $\bar{i}_1, \bar{i}_2, \ldots \bar{i}_k \in [\bar{k}]$, we define

$$(x_\pi)^{i_1,i_2,\ldots i_k}_{\bar{i}_1,\bar{i}_2,\ldots \bar{i}_k} = \begin{cases} 1, & \text{if } i_r = i_s \text{ if and only if } r \text{ and } s \text{ are in the same block of } \pi, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Given that the orbit basis and diagram basis in the partition algebra are related by the refinement relation (3.2) [BH19b], as a consequence we have

$$(d_\pi)^{i_1,i_2,\ldots i_k}_{\bar{i}_1,\bar{i}_2,\ldots \bar{i}_k} = \begin{cases} 1, & \text{if } i_r = i_s \text{ when } r \text{ and } s \text{ are in the same block of } \pi, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Looking at $(d_\pi)^{i_1,i_2,\ldots i_k}_{\bar{i}_1,\bar{i}_2,\ldots \bar{i}_k}$ as the diagram $d$ with vertices labeled by the values $i_1, i_2, \ldots, i_k$

and $\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_k$, we have the formula

$$d_\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sum_{1 \le \bar{i}_1, \bar{i}_2, \ldots \bar{i}_k \le n} (d_\pi)^{i_1, i_2, \ldots i_k}_{\bar{i}_1, \bar{i}_2, \ldots \bar{i}_k} v_{\bar{i}_1} \otimes v_{\bar{i}_2} \otimes \cdots \otimes v_{\bar{i}_k}$$

which defines actions $\Phi_k : \mathsf{P}_k(n) \to \mathrm{End}(V^{\otimes k})$ [HR05].

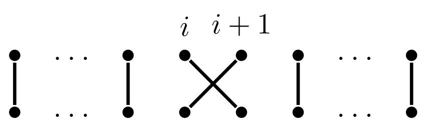
The partition algebra $\mathsf{P}_k(n)$, for $n \in \mathbb{C}$, is the asssociative algebra over $\mathbb{C}$ with basis $\Pi_{[k] \cup [\bar{k}]}$,

$$\begin{aligned} \mathsf{P}_k(n) := \mathbb{C}\mathsf{P}_k(n) &= \mathrm{span}_{\mathbb{C}}\{d_\pi \mid \pi \in \Pi_{[k]\cup[\bar{k}]}\} \\ &= \mathrm{span}_{\mathbb{C}}\{x_\pi \mid \pi \in \Pi_{[k]\cup[\bar{k}]}\}, \end{aligned}$$

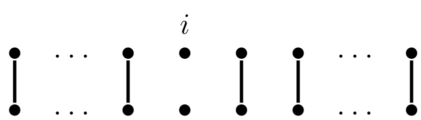
under the multiplication in (3.1) and (3.4). The diagrams that we draw each represent a linear transformation that is an element in $\mathrm{End}(V^{\otimes k})$.

### 3.4.1 Generators and relations

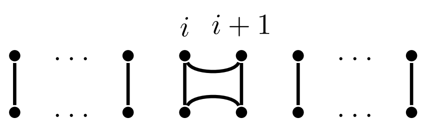
The partition algebra $\mathsf{P}_k(n)$ is prestented by the generators

$\mathfrak{s}_i =$ 

$1 \le i \le k-1,$

$\mathfrak{p}_i =$ 

$1 \le i \le k,$

$\mathfrak{b}_i =$ 

$1 \le i \le k-1.$

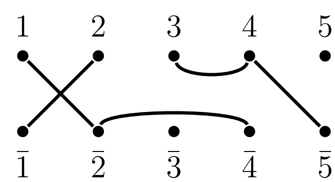
and the relations given in [HR05] [Theorem 1.11]

$$\mathfrak{p}_i^2 = \mathfrak{p}_i, \qquad \mathfrak{p}_i\mathfrak{b}_i\mathfrak{p}_i = \mathfrak{p}_i, \qquad \mathfrak{p}_i\mathfrak{p}_j = \mathfrak{p}_j\mathfrak{p}_i, \qquad\qquad \text{for } |i-j| > \frac{1}{2},$$
$$\mathfrak{s}_i^2 = 1, \qquad \mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1}, \qquad \mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{s}_i, \qquad\qquad \text{for } |i-j| > 1,$$

and

$$\mathfrak{s}_i\mathfrak{p}_i\mathfrak{p}_{i+1} = \mathfrak{p}_i\mathfrak{p}_{i+1}\mathfrak{s}_i = \mathfrak{p}_i\mathfrak{p}_{i+1}, \qquad \mathfrak{s}_i\mathfrak{b}_i = \mathfrak{b}_i\mathfrak{s}_i = \mathfrak{b}_i,$$
$$\mathfrak{s}_i\mathfrak{p}_i\mathfrak{s}_i = \mathfrak{p}_{i+1}, \qquad \mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{b}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{b}_{i+1} \qquad \mathfrak{s}_i\mathfrak{p}_j = \mathfrak{p}_j\mathfrak{s}_i, \qquad \text{for } j \neq i, i+1.$$

Every set partition $d \in \mathsf{P}_k$ can be written as a product of the generators. For example,



$= \mathfrak{s}_1\mathfrak{b}_3\mathfrak{s}_4\mathfrak{p}_3\mathfrak{p}_4\mathfrak{b}_3\mathfrak{b}_2\mathfrak{p}_3.$

### 3.4.2 Schur-Weyl Duality

The partition algebra $\mathsf{P}_k(n)$ and the symmetric group $S_n$ are in Schur-Weyl duality on the k-fold tensor product $V_n^{\otimes k}$ of the $n$-dimensional permutation module $V_n$ of the symmetric group $S_n$. When $n \geq 2k$, the partition algebra is isomorphic to the centralizer algebra of the symmetric group on $V_n^{\otimes k}$ (refer to the following references [HR05] and [Jon94]).

The actions of $\mathsf{P}_k(n)$ and $S_n$ on $V_n^{\otimes k}$ commute, that is, there is a surjective algebra homomorphism

$$\mathsf{P}_k(n) \to \mathrm{End}_{S_n}(V_n^{\otimes k}),$$

and when $n \geq 2k$,

$$\mathsf{P}_k(n) \cong \mathrm{End}_{S_n}(V_n^{\otimes k}),$$

the centralizer of $S_n$ on $V_n^{\otimes k}$.

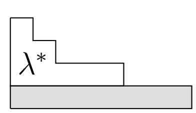
For $n \geq 2k$, the decomposition of $V_n^{\otimes k}$ as a $(\mathsf{P}_k(n), \mathbb{C}S_n)$-bimodule is given by

$$V_n^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_{k,n}} \mathsf{P}_k^\lambda \otimes S_n^\lambda,$$

where $\Lambda_{k,n}$ indexes the irreducible $S_n$ modules that appear as constituents of $V_n^{\otimes k}$ [HR05, Theorem 3.22]. For irreducible $S_n$ modules indexed by partitions of $n$, $\Lambda_{k,n} \subseteq \{\lambda \vdash n\}$ so it can be shown by induction on $k$ [HR05], [BH19a], [HJ20], that

$$\Lambda_{k,n} = \{\lambda \vdash n | 0 \leq |\lambda^*| \leq k\},$$

where if $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ is an integer partition of $n$ then $\lambda^* = [\lambda_2, \ldots, \lambda_\ell]$ is the partition $\lambda$ with its first part removed.

$\lambda =$  . (3.7)


This gives us two ways to index the irreducible $\mathsf{P}_k(n)-$modules. One is from the basic construction (explained in [HR05]) $\Lambda^{\mathsf{P}_k(n)} = \{\mu \vdash m \mid 0 \leq m \leq k\}$ and another from Schur–Weyl duality (explained in [HJ20]) $\Lambda_{k,n} = \{\lambda \vdash n \mid 0 \leq |\lambda^*| \leq k\}$. The two methods are in bijection, when $n \geq 2k$, by identifying $\lambda \in \Lambda_{k,n}$ with $\lambda^* \in \Lambda^{\mathsf{P}_k(n)}$.

In [Jon94] Jones introduces the centralizer algebra $\mathrm{End}_{S_n}(V^{\otimes k})$ where the permutation representation of the symmetric group acts on a vector space $V$ and it acts diagonally on the tensor product $V^{\otimes k}$ by restriction. Martin [Mar94] introduces this algebra independently as the partition algebra $\mathsf{P}_k(n)$.

## 3.5 The external products of $\mathsf{P}_k(n)$

In this section, we will define an external product of the partition algebra, which is the concatenation product on the diagram basis and the smash product on the orbit basis.

### 3.5.1 The external product on the diagram basis

Let $\pi$ be a set partition in $\Pi_{[k]\cup[\bar{k}]}$ and $\gamma$ be a set partition in $\Pi_{[l]\cup[\bar{l}]}$, where $k$ and $l$ are non-negative integers, the embedding

$$\Phi_{k,l} : \mathsf{P}_k(n) \otimes \mathsf{P}_l(n) \hookrightarrow \mathsf{P}_{k+l}(n)$$

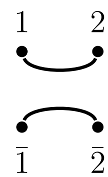
is defined by mapping

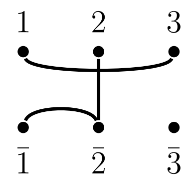
$$\Phi_{k,l}(d_\pi \otimes d_\gamma) = d_{\pi|\gamma_{\uparrow_k}} \tag{3.8}$$

where $\gamma_{\uparrow_k}$ is obtained by adding $k$ to the entries in $\gamma$ and $\pi|\gamma_{\uparrow_k} \in \Pi_{[k+l]\cup[\overline{k+l}]}$ is obtained by joining the set partitions $\pi$ and $\gamma_{\uparrow_k}$. That is, we define an external product on the partition algebra

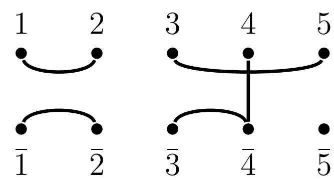
$$\Phi_{k,l}(d_\pi \otimes d_\gamma) = d_\pi \times d_\gamma = d_{\pi|\gamma_{\uparrow_k}}. \tag{3.9}$$

**Example 3.5.1.** The set partitions $\pi = \{1, 2|\bar{1}, \bar{2}\} \in \Pi_{[4]\cup[\bar{4}]}$ and $\gamma = \{1, 3|2, \bar{1}, \bar{2}|\bar{3}\} \in \Pi_{[3]\cup[\bar{3}]}$ correspond to the diagrams $d_\pi \in \mathsf{P}_2(n)$ and $d_\gamma \in \mathsf{P}_3(n)$ shown below

$d_\pi =$ 


$d_\gamma =$ 


then $d_{\{1,2|\bar{1},\bar{2}\}} \times d_{\{1,3|2,\bar{1},\bar{2}|\bar{3}\}} = d_{\{1,2|3,5|4,\bar{3},\bar{4}|\bar{1},\bar{2}|\bar{5}\}}$ shown below

$d_{\pi|\gamma_{\uparrow_2}} =$ 


**Proposition 3.5.2.** *The function $\Phi_{k,l}$ is a homomorphism, that is*

$$\Phi_{k,l}\left(d_\pi \otimes d_\gamma\right) \cdot \Phi_{k,l}\left(d_{\pi'} \otimes d_{\gamma'}\right) = \Phi_{k,l}\left(\left(d_\pi \cdot d_{\pi'}\right) \otimes \left(d_\gamma \cdot d_{\gamma'}\right)\right),$$

*where $d_\pi \cdot d_{\pi'}$ is a product in $\mathsf{P}_k(n)$, $d_\gamma \cdot d_{\gamma'}$ is a product in $\mathsf{P}_l(n)$ and $\Phi_{k,l}\left(d_\pi \otimes d_\gamma\right) \cdot \Phi_{k,l}\left(d_{\pi'} \otimes d_{\gamma'}\right)$ is a product in $\mathsf{P}_{k+l}(n)$.*

*Proof.* For set partitions $\pi, \pi' \in \Pi_{[k]\cup[\bar{k}]}$ and $\gamma, \gamma' \in \Pi_{[l]\cup[\bar{l}]}$, we have

$$\Phi_{k,l}\left(d_\pi \otimes d_\gamma\right) \cdot \Phi_{k,l}\left(d_{\pi'} \otimes d_{\gamma'}\right) = d_{\pi|\gamma_{\uparrow_k}} \cdot d_{\pi'|\gamma'_{\uparrow_k}}$$

$$= n^{[\pi|\gamma_{\uparrow_k} * \pi'|\gamma'_{\uparrow_k}]} d_{\pi|\gamma_{\uparrow_k} * \pi'|\gamma'_{\uparrow_k}}$$

and

$$\begin{aligned}
\Phi_{k,l}\left((d_\pi \cdot d_{\pi'}) \otimes (d_\gamma \cdot d_{\gamma'})\right) &= \Phi_{k,l}\left(n^{[\pi*\pi']} d_{\pi*\pi'} \otimes n^{[\gamma*\gamma']} d_{\gamma*\gamma'}\right) \\
&= n^{[\pi*\pi']} n^{[\gamma*\gamma']} \Phi_{k,l}\left(d_{\pi*\pi'} \otimes d_{\gamma*\gamma'}\right) \\
&= n^{[\pi*\pi']} n^{[\gamma*\gamma']} d_{\pi*\pi'|(\gamma*\gamma')_{\uparrow_k}}.
\end{aligned}$$

Since the barrier of $\pi$ and $\gamma$, and similarly $\pi'$ and $\gamma'$, can't be passed $\pi|\gamma_{\uparrow_k} * \pi'|\gamma'_{\uparrow_k}$ is equivalent $\pi * \pi'|\left(\gamma * \gamma'\right)_{\uparrow_k}$. And given that the connected components that lie entirely in the middle row of the joined diagrams are a union of the connected components indexed by $\{1, 2, \ldots, k\}$ and the connected components that are indexed by $\{k+1, k+2, \ldots, k+l\}$, then

$$[\pi|\gamma_{\uparrow_k} * \pi'|\gamma'_{\uparrow_k}] = [\pi * \pi'] + [\gamma * \gamma'].$$

Thus, $\Phi_{k,l}\left(d_\pi \otimes d_\gamma\right) \cdot \Phi_{k,l}\left(d_{\pi'} \otimes d_{\gamma'}\right) = \Phi_{k,l}\left((d_\pi \cdot d_{\pi'}) \otimes (d_\gamma \cdot d_{\gamma'})\right)$, and therefore $\Phi_{k,l}$ is a homomorphism from $\mathsf{P}_k(n) \otimes \mathsf{P}_l(n)$ to $\mathsf{P}_{k+l}(n)$. $\square$

### 3.5.2 The external product on the orbit basis

Now that the external product on the diagram basis is defined in Equation (3.8), we would like to know what this product is on the orbit basis of the partition algebra. However, before we determine the product on the orbit basis, let us look at the following lemma which shows a bijection between the terms that arises in two expressions.

**Lemma 3.5.3.** *For fixed* $\pi \in \Pi_{[k]\cup[\bar{k}]}$ *and* $\gamma \in \Pi_{[l]\cup[\bar{l}]}$*, where* $k$ *and* $l$ *are positive integers, there is a bijection between the set*

$$A_{\pi,\gamma} := \{(\pi', \gamma', \nu) \in \Pi_{[k]\cup[\bar{k}]} \times \Pi_{[l]\cup[\bar{l}]} \times \Pi_{[k+l]\cup[\overline{k+l}]} : \pi \preceq \pi', \gamma \preceq \gamma' \text{ and } \nu \in \pi' \# \gamma'_{\uparrow_k}\}$$

*and the set*

$$B_{\pi,\gamma} := \{\nu \in \Pi_{[k+l]\cup[\overline{k+l}]} : \pi|\gamma_{\uparrow_k} \preceq \nu\}.$$

*Proof.* We need to show that there is a bijection between the sets $A_{\pi,\gamma}$ and $B_{\pi,\gamma}$. Fix a $(\pi', \gamma', \nu) \in A_{\pi,\gamma}$ such that $\nu$ is an element in $\pi' \# \gamma'_{\uparrow_k}$. That is, by the definition of the smash product, the set partition $\nu$ when restricted to $\{1, \ldots, k, \bar{1}, \ldots, \bar{k}\}$ is the set partition $\pi'$ and when restricted to $\{k+1, \ldots, k+l, \overline{k+1}, \ldots, \overline{k+l}\}$ is the set partition $\gamma'_{\uparrow_k}$. Given that a set $S \in \pi|\gamma_{\uparrow_k}$ is either an element in $\pi$ or $\gamma_{\uparrow_k}$, we have a case where $S' \in \nu$ such that $S = S' \cap \{1, \ldots, k, \bar{1}, \ldots, \bar{k}\}$ or there is an $S' \in \nu$ such that $S = S' \cap \{k+1, \ldots, k+l, \overline{k+1}, \ldots, \overline{k+l}\}$. In both cases $S$ is contained in a part of $\nu$ therefore, $\pi|\gamma_{\uparrow_k} \preceq \nu$.

Now take $\nu \in B_{\pi,\gamma}$. It is possible to determine $\pi'$ and $\gamma'_{\uparrow_k}$ from the set partition $\nu$ that is coarser than $\pi|\gamma_{\uparrow_k}$. The set partition $\nu \succeq \pi|\gamma_{\uparrow_k}$ when restricted to $\{1, \ldots, k, \bar{1}, \ldots, \bar{k}\}$ is coarser than $\pi$ such that $\nu|_{\{1,\ldots,k,\bar{1},\ldots,\bar{k}\}} = \pi'$. And the set partition $\nu$ restricted to $\{k+1, \ldots, k+l, \overline{k+1}, \ldots, \overline{k+l}\}$ is coarser than $\gamma_{\uparrow_k}$ so $\nu|_{\{k+1,\ldots,k+l,\overline{k+1},\ldots,\overline{k+l}\}} = \gamma'_{\uparrow_k}$. Since $\nu$ is coarser than $\pi|\gamma_{\uparrow_k}$ then $\nu|_{\{1,\ldots,k,\bar{1},\ldots,\bar{k}\}}$ is coarser than $\pi$. Therefore, by the definition of the smash product $\nu \in \pi' \# \gamma'_{\uparrow_k}$. □

Using the embedding given above, we define an external product

$$x_\pi \times x_\gamma = \Phi_{k,l}(x_\pi \otimes x_\gamma). \tag{3.10}$$

**Proposition 3.5.4.** *For set partitions $\pi \in \Pi_{[k]\cup[\bar{k}]}$ and $\gamma \in \Pi_{[l]\cup[\bar{l}]}$,*

$$\Phi_{k,\ell}(x_\pi \otimes x_\gamma) = \sum_{\nu \in \pi \# \gamma_{\uparrow_k}} x_\nu, \tag{3.11}$$

*where $\gamma_{\uparrow_k}$ is obtained by adding $k$ to the entries in $\gamma$.*

*Proof.* Let $\Phi'_{k,\ell} : \mathsf{P}_k(n) \otimes \mathsf{P}_l(n) \to \mathsf{P}_{k+l}(n)$ be the bilinear product whose action on the orbit basis is defined by the sum (3.11). We show that $\Phi'_{k,\ell}$ is the same as the product $\Phi_{k,\ell}$, as defined in (3.9). This technique for proving that $\Phi'_{k,\ell} = \Phi_{k,\ell}$ is used elsewhere in the literature and the details of why it works is explained in the appendix.

Hence,

$$\begin{aligned}\Phi'_{k,\ell}(d_\pi \otimes d_\gamma) &= \sum_{\pi \preceq \pi'} \sum_{\gamma \preceq \gamma'} \Phi'_{k,\ell}(x_{\pi'} \otimes x_{\gamma'}) \\ &= \sum_{\pi \preceq \pi'} \sum_{\gamma \preceq \gamma'} \sum_{\nu \in \pi' \# \gamma'_{\uparrow_k}} x_\nu.\end{aligned}$$

And by the bijection in Lemma 3.5.3,

$$\sum_{\pi \preceq \pi'} \sum_{\gamma \preceq \gamma'} \sum_{\nu \in \pi' \# \gamma'_{\uparrow_k}} x_\nu = \sum_{\pi | \gamma_{\uparrow_k} \preceq \nu} x_\nu.$$

Hence,

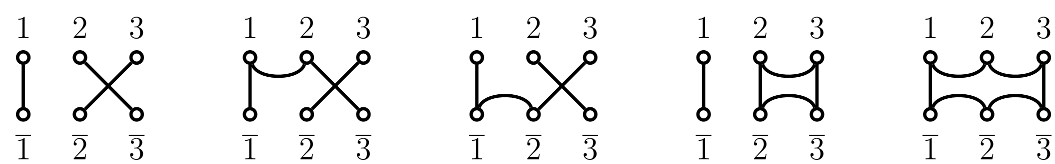
$$\begin{aligned}\Phi'_{k,\ell}(d_\pi \otimes d_\gamma) &= \sum_{\pi|\gamma_{\uparrow_k} \preceq \nu} x_\nu \\ &= d_{\pi|\gamma_{\uparrow_k}} \\ &= \Phi_{k,\ell}(d_\pi \otimes d_\gamma).\end{aligned}$$

Therefore, the product $\Phi'_{k,\ell}$ is equal to $\Phi_{k,\ell}$ so (3.11) holds. □

**Example 3.5.5.** Let $\pi = \{1, \overline{1}\}$ and $\gamma = \{1, \overline{2}|2, \overline{1}\}$, then

$$\Phi_{k,\ell}(d_{\{1,\overline{1}\}} \otimes d_{\{1,\overline{2}|2,\overline{1}\}}) = \sum_{\{1,\overline{1}\}\preceq\pi'} \sum_{\{1,\overline{2}|2,\overline{1}\}\preceq\gamma'} \Phi_{k,\ell}(x_{\pi'} \otimes x_{\gamma'})$$

where $\pi' \in \{\{1, \overline{1}\}\}$ and $\gamma' \in \{\{1, \overline{2}|2, \overline{1}\}, \{1, 2, \overline{1}, \overline{2}\}\}$. The smash product of $\pi$ and $\gamma$ is represented by the following diagrams,



all which are coarser than $\{1, \overline{1}|3, \overline{2}|2, \overline{3}\}$. Thus, the sum of orbit basis elements indexed by these diagrams is equivalent to $d_{\{1,\overline{1}|3,\overline{2}|2,\overline{3}\}}$.

# Chapter 4

# The Hopf Algebra of Uniform Block Permutations

## Introduction

In the early 90's, the partition algebra appeared in the works of Martin [Mar90], [Mar91], [Mar94], [Mar96] and Jones [Jon94] independently. Both Martin and Jones study the partition algebra as a generalization of the Temperley-Lieb algebra and the Potts model in statistical mechanics. Martin shows the algebra both implicitly [Mar90], [Mar91] and explicitly [Mar94]. While Jones shows explicitly the Schur-Weyl duality between the partition algebras and the symmetric group. Halverson [Hal01] later showed the analogous of the Frobenius formula and the Murnaghan-Nakayama rule for the characters of the partition algebras.

The results of Jones were generalized by Tanabe [Tan97] to study the centralizer of the unitary reflection group $G(m,p,n)$. Kosuda [Kos00], [Kos06a], [Kos06b] further studies these algebra for the case $G(m,1,n)$ in the endomorphism ring of the tensor space $V^{\otimes k}$, under the condition $n \geq k$ and $m > k$, constructing a complete set of irreducible representations. An algebra generated by the symmetric group $S_k$ and another operator of type $G(m,p,n)$ is a subalgebra of the partition algebra known as party algebra. This algebra is also know as the uniform block permutations [AO08].

In this chapter, we look at the uniform block permutation algebra defined by Aguiar and Orellana [AO08]. They showed that the algebra is a Hopf algebra on a basis we refer to in this chapter as the fundamental basis. We describe the change of basis between the fundamental basis and the diagram basis. Moreover, we prove a formula for the external product of the diagram basis and define the external product of the orbit basis and provide a coproduct for the diagram basis.

## 4.1 Notation

In this section, we will define some combinatorial objects and establish notation used in the chapter.

A *uniform set partition* of $[k] \cup [\overline{k}] := \{1, 2, \ldots, k, \overline{1}, \overline{2}, \ldots, \overline{k}\}$ is set partition where each set contains equal number of barred and unbarred entries,

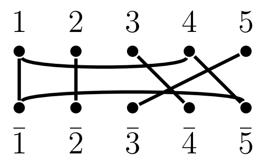
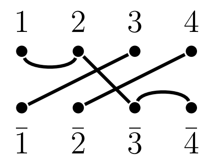
$$USP^k = \{\pi : \pi \vdash [k] \cup [\bar{k}] \ \forall s \in \pi, \ |s \cap [k]| = |s \cap [\bar{k}]|\}.$$

For example, $\pi = \{1, 3, \bar{3}, \bar{4} | 2, \bar{2} | 4, 5, \bar{1}, \bar{5}\}$ is a uniform set partition in $USP^5$.

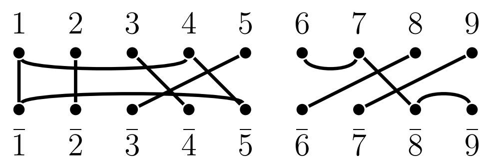
Let $UBP_k$ be the vector space of *uniform block permutations* of $[k] \cup [\overline{k}]$, where the basis is indexed by uniform set partitions.

Let $\pi \in USP^k$ and $\gamma \in USP^l$. The *shifted concatenation* of $\pi$ and $\gamma$ denoted as $\pi|\gamma_{\uparrow_k}$ is accomplished by adding $k$ every entry in the uniform set partition $\gamma$ and placing the placing it to the right of the uniform set partition $\pi$ obtaining a set partition which is an element of $USP^{k+l}$.

**Example 4.1.1.** Let $\pi = \{1,4,\overline{1},\overline{5}|2,\overline{2}|3,\overline{4}|5,\overline{3}\} \in USP^5$ and $\gamma = \{1,2,\overline{3},\overline{4}|3,\overline{1}|4,\overline{2}\} \in USP^4$.

$\pi =$  $\gamma =$ 

The concatenation of the diagrams $\pi$ and $\gamma$ is

$\pi|\gamma_{\uparrow_5} =$  .

Let $S^{(p,q)}$ denote the set of $(p,q)-$*shuffles*, that is, those permutations $\zeta \in S_{p+q}$ such that

$$\zeta_1 < \zeta_2 < \cdots < \zeta_p \text{ and } \zeta_{p+1} < \zeta_{p+2} < \cdots < \zeta_{p+q}.$$

For $\pi \in USP^p$ and $\tau \in USP^q$,

$$sh_{\pi,\tau} = \left\{ \left(\pi|\tau_{\uparrow_p}\right) * \zeta^{-1} : \zeta \in S^{(p,q)} \right\}.$$

**Example 4.1.2.** The shuffle of $\{1,\bar{2}|2,\bar{1}\}$ and $\{1,\bar{1}\}$ gives the following set partitions in $sh_{\{1,\bar{2}|2,\bar{1}\},\{1,\bar{1}\}}$

$$\{1,\bar{2}|2,\bar{1}|3,\bar{3}\},\ \{1,\bar{3}|2,\bar{1}|3,\bar{2}\},\ \{1,\bar{3}|2,\bar{2}|3,\bar{1}\}.$$

## 4.2 The fundamental basis of $\mathit{UBP}_k$

In [AO08], the authors introduce the Hopf algebra of uniform block permutations. The Hopf algebra is indexed by diagrams of the partition algebra such that in each block of the diagram, the number of elements in the top row is equal to the number of elements in the bottom row. And it contains as a Hopf subalgebra the Hopf algebra of permutations of Malvenuto and Reutenauer, which has a linear basis indexed by permutations in all symmetric groups $S_n$.

Aguiar and Orellana introduce a basis of the Hopf algebra of uniform block permutations. This basis is analogous to the fundamental basis of the Hopf algebra of permutations of Malvenuto and Reutenauer, in that the product and the coproduct agree when the element indexing the basis in the Hopf algebra of uniform block permutations is a permutation. In this thesis we refer to the basis in [AO08] as the fundamental basis of Hopf algebra of uniform block permutations.

The Hopf algebra of uniform block permutation $\mathit{UBP} := \bigoplus_{k\geq 0} \mathit{UBP}_k$ is a graded vector space over $\mathbb{Q}$ with basis of the graded component of degree $k$ indexed by the elements of the uniform set partition $\mathit{USP}^k$. The subspace $\mathit{UBP}_k$ is the graded component of degree $k$ and is spanned by the basis $\{f_\pi | \pi \in \mathit{USP}^k\}$. The space $\mathit{UBP}_0$ consists of the unique uniform block permutation of $k$, represented by the empty diagram, which we denote by 1.

### 4.2.1 The product of the fundamental basis of $UBP_k$

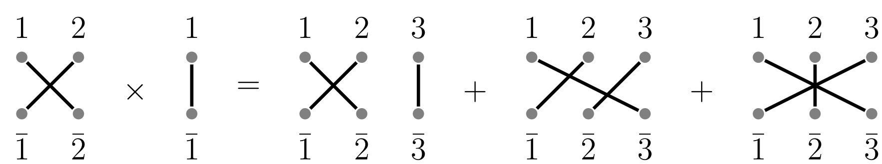
The product of the fundamental basis of the uniform block permutation, as defined in [AO08], is obtained by concatenating the two uniform set partitions and shuffling the barred entries.

$$f_\pi \times f_\tau = \sum_{\zeta \in S^{(k,l)}} f_{(\pi|\tau_{\uparrow_k})*\zeta^{-1}}, \tag{4.1}$$

where $\pi \in USP^k$, $\tau \in USP^l$ and $\pi|\tau_{\uparrow_k} \in USP^{k+l}$ is obtained by adding $k$ to the entries in $\tau$ and joining the two set partitions.

For the remainder of the thesis, diagrams of the fundamental basis will be indicated using filled grey vertices and those of the coarsening basis are indicated using unfilled grey vertices. While the diagrams of the diagram basis will be indicated using filled black vertices and those of the orbit basis are indicated using unfilled black vertices.

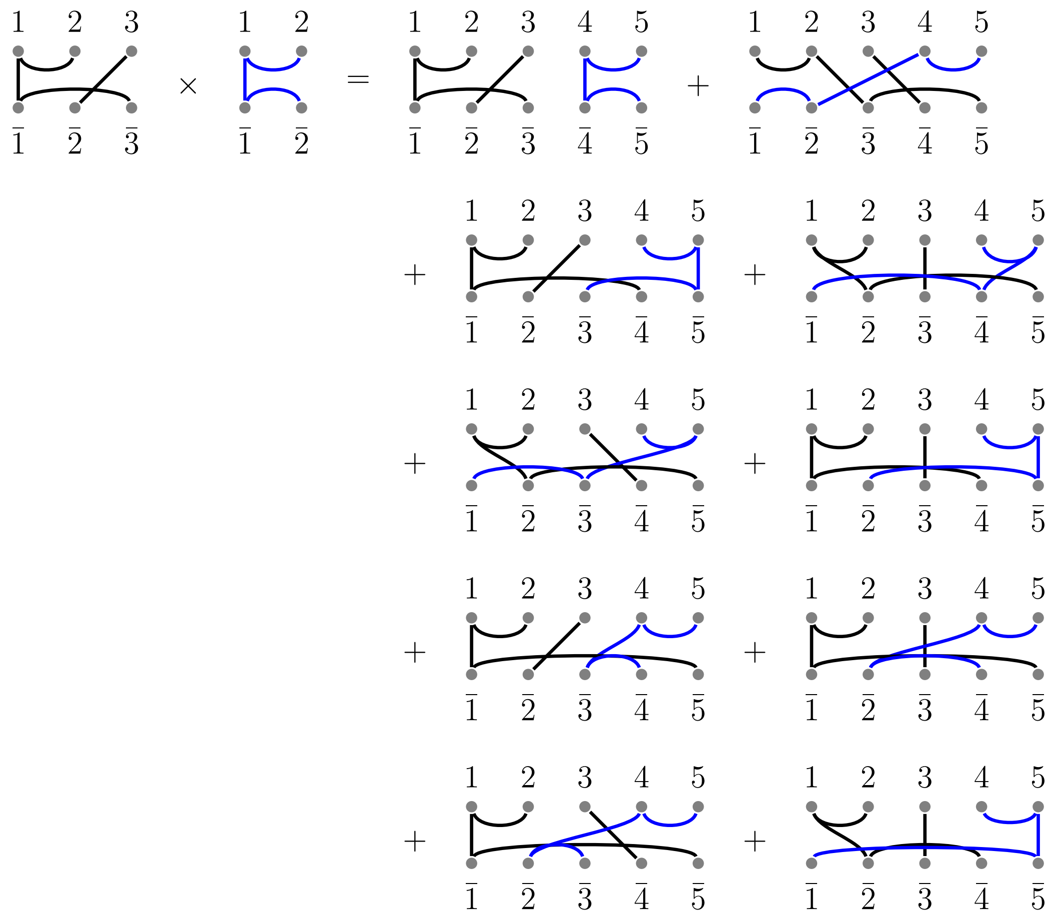
**Example 4.2.1.** The product of the fundamental basis $\{1, \overline{2}|2, \overline{1}\}$ and $\{1, \overline{1}\}$ is




which is the same as the product of the permutations from the Malvenuto-Reutenauer algebra of Equation 2.10:

$$\mathcal{F}_{21} \cdot \mathcal{F}_1 = \mathcal{F}_{213} + \mathcal{F}_{231} + \mathcal{F}_{321}.$$

**Example 4.2.2.**



### 4.2.2 The coproduct of the fundamental basis of $UBP_k$

A *breaking point* of a set partition $\mathcal{B} \vdash [k]$ is an integer $i \in \{0, 1, \ldots, k\}$ for which there exists a subset $S \subseteq \mathcal{B}$ such that

$$\bigcup_{A \in S} A = \{1, \ldots, i\} \qquad \text{and} \qquad \bigcup_{A \in \mathcal{B} \setminus S} A = \{i+1, \ldots, k\}.$$

Note that $i = 0$ and $i = k$ are breaking points of any set partition $\pi$.

For a uniform set partition $\pi$, we say that $i$ is a breaking point of $\pi$ if it is the breaking point of $\{A \cap [\bar{k}] : A \in \pi\}$. Denote the set of breaking points of $\pi$ as $B(\pi)$. If $\pi$ is a permutation diagram (i.e. if all blocks of $\pi$ are of size 1), then $B(\pi) = \{0, 1, \ldots, k\}$. In terms of the diagram of $\pi$, $i \in B(\pi)$ if it is possible to put a vertical line between the first $i$ and the last $k - i$ vertices in the bottom row without intersecting any edges joining bottom vertices.

**Lemma 4.2.3** (Lemma 3.2, [AO08])**.** *If $i$ is a breaking point of $\pi$, then there exists a unique $(i, n-i)-$shuffle $\zeta \in S_n$ and unique uniform block permutations $\tau \in USP^i$ and $\gamma \in USP^{(n-i)}$ such that*

$$\pi = \zeta * \left(\tau | \gamma_{\uparrow_i}\right)$$

*Conversely, if such a decomposition exists, $i$ is a breaking point of $\pi$.*

The coproduct of fundamental basis stated below is as defined by Aguiar and Orellana in [AO08]. Let $\pi \in USP^k$, the coproduct of the fundamental basis of $UBP_k$ is

$$\Delta(f_\pi) = \sum_{i \in B(\pi)} f_\tau \otimes f_\gamma, \tag{4.2}$$

where $\tau$ and $\gamma$ are as in Lemma 4.2.3.

In [AO08], Aguiar and Orellana give an explicit definition of the product (4.1) of the fundamental basis of the uniform block permutations along with a compatiable coproduct (4.2) which turns the space into a graded Hopf algebra.

**Theorem 4.2.4** (Theorem 3.6, [AO08])**.** *The graded vector space $UBP := \bigoplus_{k \geq 0} UBP_k$ equipped with the product $\times$, coproduct $\Delta$, unit $\emptyset$, and counit $\epsilon$, is a graded connected Hopf algebra.*

## 4.3 The diagram basis of $UBP_k$

In this section we define the diagram and orbit basis of the Hopf algebra of uniform block permutations, which is an external product of the diagram and orbit bases of the partition algebra defined in [BH19a]. Furthermore, we connect the product of the uniform block permutations, which is analog to the fundamental basis, with the diagram and orbit bases of the partition algebra.

### 4.3.1 Change of basis

The uniform set partitions $USP^k$ are partially ordered by a cover relation that occurs by transposing a pair of consecutive values $\overline{i}, \overline{i+1} \in [\overline{k}]$.

For $\pi$ in $USP^k$, and let $\text{top}(\pi)$ denote the set partition $\{A \cap [k] : A \in \pi\} \vdash [k]$.

Let $\gamma \vdash [k]$ be a set partition and define

$$\mathcal{I}_\gamma = \{A \cup \overline{A} : A \in \gamma\} \in USP^k \ .$$

We will define a partial order on elements $\pi \in USP^k$ with $\text{top}(\pi) = \gamma$ where the minimal element is $\mathcal{I}_\gamma$.

We define the cover relation for $\rho \leq \pi$ if $\text{top}(\rho) = \text{top}(\pi)$ and $\pi = (\rho \setminus \{B_1, B_2\}) \cup \{C_1, C_2\}$ where

$$\max(B_1) = \max(C_1) < \max(C_2) = \max(B_2),$$

and there is some $i$ in $B_1$ such that

$$C_1 = (B_1 \setminus \{\bar{i}\}) \cup \{\overline{i+1}\},$$

$$C_2 = (B_2 \setminus \{\overline{i+1}\}) \cup \{\bar{i}\},$$

where $B_1$ are $B_2$ are blocks in the set partition $\rho$, and $C_1$ and $C_2$ are blocks in the set partition $\pi$.

**Example 4.3.1.** Let $\rho = \{1, 2, \overline{1}, \overline{2}|3, \overline{3}\}$ and if $i = 2$, then $B_1 = \{1, 2, \overline{1}, \overline{2}\}$, $B_2 = \{3, \overline{3}\}$ then $C_1 = (B_1 \setminus \{\overline{2}\}) \cup \{\overline{3}\} = \{1, 2, \overline{1}, \overline{3}\}$, $C_2 = (B_2 \setminus \{\overline{3}\}) \cup \{\overline{2}\} = \{3, \overline{2}\}$, then $\rho \leq \pi = \{1, 2, \overline{1}, \overline{3}|3, \overline{2}\}$.

The partial order we have defined on the uniform set partitions $USP^k$ is the same as the partial order defined by Aguiar and Orellana which is analogous to the right weak order on $S_k$ [AO08, Section 4.2]. Every $\pi \in USP^k$ is factorizable as $\pi = \mathcal{I}_{\text{top}(\pi)} * \mathbb{D}(w_\pi)$ where $\mathcal{I}_{\text{top}(\pi)}$ is a symmetric set partition in $USP^k$ and $w_\pi \in S_k$ is a permutation. The weak order on set partitions can equivalently be defined as $\rho \leq \pi$ if $\text{top}(\rho) = \text{top}(\pi)$ and $w_\rho \leq_R w_\pi$. If $\rho \leq \pi$, then $\pi$ will be equal to $\rho * \mathbb{D}(w_\rho^{-1}) * \mathbb{D}(w_\pi)$. That is $\pi$ is $\rho$ multiplied on the right by a permutation.

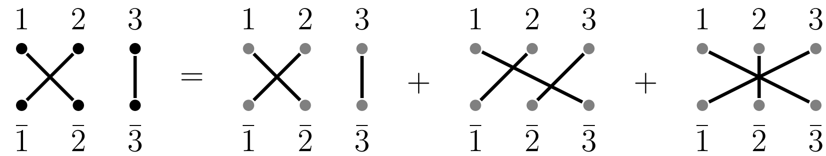
The diagram basis $\{d_\pi : \pi \in USP^k\}$ of uniform block permutation is defined by the following relation with respect to the fundamental basis,

$$d_\pi = \sum_{\pi \leq \rho} f_\rho \tag{4.3}$$

where the sum is over all permutations greater than or equal to the barred entries of

$\pi$ in the right weak order.

**Example 4.3.2.** The diagram basis $d_{\{2,\bar{1}|1,\bar{2}|3,\bar{3}\}}$ in $UBP_3$ in terms of the fundamental basis,



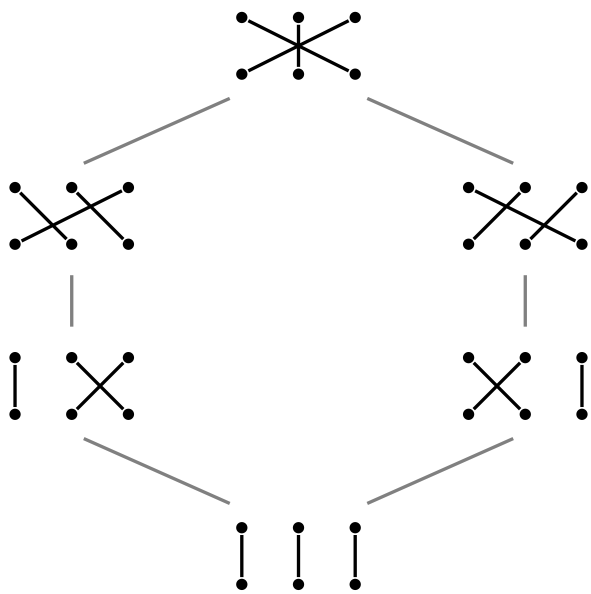


Figure 4.1: $USP^3$ with top row 1|2|3

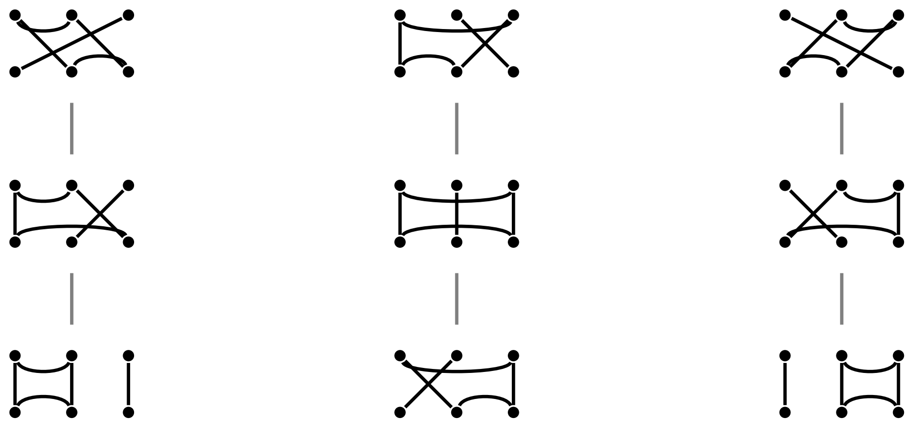


Figure 4.2: $USP^3$ with top row 12|3, 13|2 and 1|23

### 4.3.2 The product of the diagram basis of $UBP_k$

In this section, we prove a formula for the product of the diagram basis of the uniform block permutation. The product is obtained by concatenating two uniform set partitions. In [AO08], the authors define this basis in Remark 4.6 which they refer to as $Z_g$ and state without proof that its product is a concatenation. Here we provide the details of the proof of that formula.

**Lemma 4.3.3** (Lemma 4.3, [AO08])**.** *Let* $\lambda : S^{(p,q)} \times USP^p \times USP^q \to USP^{p+q}$ *be defined by*

$$\lambda(\xi, \sigma, \tau) := (\sigma|\tau_{\uparrow_p}) * \xi^{-1}.$$

*Endow each set of shuffles with the weak order. Then*

*(i)* $\lambda$ *is bijective;*

*(ii)* $\lambda^{-1}$ *is order preserving, that is,*

$$(\sigma|\tau_{\uparrow_p}) * \xi^{-1} \leq (\sigma'|\tau'_{\uparrow_p}) * \xi'^{-1} \quad \Longrightarrow \quad \sigma \leq \sigma', \tau \leq \tau', \text{ and } \xi^{-1} \leq_R \xi'^{-1}.$$

**Proposition 4.3.4.** *For any $\pi \in USP^k$ and $\gamma \in USP^l$, we have*

$$d_\pi \times d_\gamma = d_{\pi|\gamma_{\uparrow k}} \tag{4.4}$$

*where $\pi|\gamma_{\uparrow k} \in USP^{k+l}$ is obtained by adding $k$ to the entries in $\gamma$ and joining the two set partitions.*

*Proof.* Expand the product $d_\pi \times d_\gamma$ in the fundamental basis by Equation (4.3) and then use Formula (4.1) to obtain

$$\begin{aligned} d_\pi \times d_\gamma &= \sum_{\pi \leq \rho} \sum_{\gamma \leq \tau} f_\rho \times f_\tau \\ &= \sum_{\pi \leq \rho} \sum_{\gamma \leq \tau} \sum_{\zeta \in S^{(k,l)}} f_{(\rho|\tau_{\uparrow_k})*\zeta^{-1}} \ . \end{aligned}$$

By the bijection in Lemma 4.3.3,

$$\sum_{\pi \leq \rho} \sum_{\gamma \leq \tau} \sum_{\zeta \in S^{(k,l)}} f_{(\rho|\tau_{\uparrow_k})*\zeta^{-1}} = \sum_{\nu \geq \pi|\gamma_{\uparrow_k}} f_\nu,$$

and by Equation (4.3),

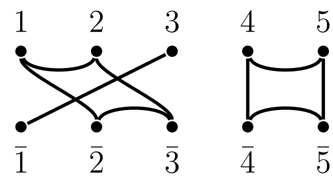
$$\sum_{\nu \geq \pi|\gamma_{\uparrow_k}} f_\nu = d_{\pi|\gamma_{\uparrow_k}}.$$

Thus, we can conclude that for uniform set partitions $\pi \in USP^k$ and $\gamma \in USP^l$

$$d_\pi \times d_\gamma = d_{\pi|\gamma_{\uparrow_k}}.$$

□

**Example 4.3.5.** For $\pi = \{1, 2, \bar{2}, \bar{3}|3, \bar{1}\} \in USP^3$ and $\gamma = \{1, 2, \bar{1}, \bar{2}\} \in USP^2$, concatenation of the diagrams

$d_\pi \times d_\gamma =$ 


### 4.3.3 The coproduct of the diagram basis of $UBP_k$

Let $\pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ be a element of $USP^k$ and $S = \{s_1 < s_2 < \cdots < s_{|S|}\}$ be a subset of set $\{1, \ldots, k\}$, where $S^c = \{s'_1 < s'_2 < \cdots < s'_{k-|S|}\}$ is the complement set of $S$. If $\pi_i \cap [\overline{k}] \subseteq \overline{S}$ or $|\pi_i \cap \overline{S}| = 0$ for all $1 \leq i \leq \ell$ then

$$\begin{aligned}\overline{\text{split}}_S(\pi) = &\{(\pi_i \cap [k]) \cup \{\overline{j} : \overline{s}_j \in \pi_i\} : \pi_i \cap [\overline{k}] \subseteq \overline{S}\} \bigcup \\ &\{(\pi_i \cap [k]) \cup \{\overline{j} + |S| : \overline{s}'_j \in \pi_i\} : \pi_i \cap [\overline{k}] \not\subseteq \overline{S}\}.\end{aligned}$$

Given a uniform set partition $\pi \in USP^k$ that satisfies this condition, let

$$\begin{aligned}T &= \bigcup_{i : \pi_i \cap [\overline{k}] \subseteq \overline{S}} \pi_i \cap [k] \\ &= \{t_1 < t_2 < \cdots < t_{|S|}\} \\ T^c &= \{t'_1 < t'_2 < \cdots < t'_{k-|S|}\},\end{aligned}$$

then

$$\text{left}_S(\pi) = \Big\{\{j : t_j \in \pi_i\} \cup \{\overline{j} : \overline{s}_j \in \pi_i\} : \pi_i \cap [\overline{k}] \subseteq \overline{S}\Big\}$$

$$\mathrm{right}_S(\pi) = \left\{ \{j : t'_j \in \pi_i\} \cup \{\overline{j} : \overline{s}'_j \in \pi_i\} : \pi_i \cap [\overline{k}] \not\subseteq \overline{S} \right\}.$$

Note that $|T| = |S|$ for uniform set partition $\pi \in \mathit{USP}^k$. Computing $\mathrm{left}_S(\pi)$ and $\mathrm{right}_S(\pi)$ is analogous to computing $\mathrm{st}(u|_S)$ and $\mathrm{st}(u|_{S^c})$ in the Malvenuto-Reutenauer algebra while $\overline{\mathrm{split}}_S(\pi)$ is analogous to the operation of computing $(u|_S)|(u|_{S^c})$.

The operation of 'split' has the effect of moving the elements of the subset in the bottom row of the diagram to the left while the top row remains unchanged. If the new diagram is greater than or equal to $\pi$ in the right weak order, then split the set partition where the left set partition contains all the elements connected to $\overline{S}$ and the remaining are on the right.

The operation of left (and right) standardizes the set partitions with barred entries that are in $S$ (respectively, not in $S$).

**Example 4.3.6.** For $\pi = \{1, 3, \overline{1}, \overline{3}|2, \overline{2}\}$ and $S = \{1, 3\}$,

$$\overline{\mathrm{split}}_S(\pi) = \{1, 3, \overline{1}, \overline{2}|2, \overline{3}\},$$

then $\mathrm{left}_S(\pi) = \{1, 2, \overline{1}, \overline{2}\}$ and $\mathrm{right}_S(\pi) = \{1, \overline{1}\}$.

**Example 4.3.7.** Let $\pi = \{5, \overline{1}|1, \overline{2}|3, \overline{3}|2, \overline{4}|4, \overline{5}\}$ be the set partition from Example 3.1.2 which corresponds to the permutation $u = 51324$. For $S = \{1, 3\}$, we have that

$$\overline{\mathrm{split}}_{\{1,3\}}(\pi) = \{5, \overline{1}|3, \overline{2}|1, \overline{3}|2, \overline{4}|4, \overline{5}\}\ ,$$

$$\mathrm{left}_{\{1,3\}}(\pi) = \{2, \overline{1}|1, \overline{2}\}$$

and

$$\mathrm{right}_{\{1,3\}}(\pi) = \{1, \overline{1}|2, \overline{2}|3, \overline{3}\}\ .$$

For the permutation $u$, we have $(u|_{\{1,3\}})|(u|_{\{2,4,5\}}) = 53124$.

For the proof of the next theorem we need the following lemma.

**Lemma 4.3.8.** *For fixed $0 < k < n$ and $\pi \in USP^n$, there is a bijection between the set*

$$A_{\pi,k} := \{(S, \alpha, \beta) \in 2^{[n]} \times USP^k \times USP^{n-k} : |S| = k, \pi \leq \overline{split}_S(\pi)$$
$$left_S(\pi) \leq \alpha,\ \ right_S(\pi) \leq \beta\}$$

*and the set*

$$B_{\pi,k} := \{\rho : \pi \leq \rho, k \in B(\rho)\}.$$

*The bijection is defined by*

$$\Phi_{\pi,k}(\rho) = (S, \alpha, \beta)$$

*where $\alpha = left_{[k]}(\rho)$ and $\beta = right_{[k]}(\rho)$.*

The proof for this lemma is similar to that of Lemma 2.3.3.

**Example 4.3.9.** Let $\pi = \{1, 2, \overline{1}, \overline{3}|3, 4, \overline{2}, \overline{4}\}$ and $k = 2$.

The set of $\rho$ such that $\rho \geq \pi$, are

$$\{\{1, 2, \overline{1}, \overline{3}|3, 4, \overline{2}, \overline{4}\}, \{1, 2, \overline{2}, \overline{3}|3, 4, \overline{1}, \overline{4}\}, \{1, 2, \overline{1}, \overline{4}|3, 4, \overline{2}, \overline{3}\},$$

$$\{1,2,\overline{2},\overline{4}|3,4,\overline{1},\overline{3}\},\{1,2,\overline{3},\overline{4}|3,4,\overline{1},\overline{2}\}\} \ .$$

Only one of these has 2 as a breakpoint so

$$B_{\pi,2} = \{\{1,2,\overline{3},\overline{4}|3,4,\overline{1},\overline{2}\}\} \ .$$

We compute for $\rho \in B_{\pi,2}$,

$$\rho = \{1,2,\overline{3},\overline{4}|3,4,\overline{1},\overline{2}\} = \pi * \{2,\overline{1}|4,\overline{2}|1,\overline{3}|3,\overline{4}\} \ .$$

Now $w = 2413$ and $\lambda_{2,2}^{-1}(2413) = (\zeta, x, y)$ with $\zeta = 2413$, $x = 12$, $y = 12$. We then set $S = \{\zeta_1, \zeta_2\} = \{2,4\}$, $\alpha = \mathrm{left}_{\{1,2\}}(\rho) = \{1,2,\overline{1},\overline{2}\}$, and $\beta = \mathrm{right}_{\{1,2\}}(\rho) = \{1,2,\overline{1},\overline{2}\}$, and hence

$$\Phi_{\pi,2}(\{1,2,\overline{3},\overline{4}|3,4,\overline{1},\overline{2}\}) = (\{2,4\},\{1,2,\overline{1},\overline{2}\},\{1,2,\overline{1},\overline{2}\}) \in A_{\pi,2} \ .$$

Calculations using the change of basis between the diagram and the fundamental basis and the definition of the coproduct in Equation (4.2) allow us to show the following formula for the coproduct on the diagram basis.

**Theorem 4.3.10.** *Let $\pi \in USP^n$, then*

$$\Delta(d_\pi) = \sum_{S \in 2^{[n]}} \chi(\overline{split}_S(\pi) \geq \pi) \ d_{left_S(\pi)} \otimes d_{right_S(\pi)} \ . \tag{4.5}$$

*Proof.* Fix $\pi \in USP^n$. On the one hand we have

$$\sum_{\substack{S \in 2^{[n]} \\ \pi \leq \overline{\text{split}}_S(\pi)}} d_{\text{left}_S(\pi)} \otimes d_{\text{right}_S(\pi)} = \sum_{\substack{S \in 2^{[n]} \\ \pi \leq \overline{\text{split}}_S(\pi)}} \sum_{\substack{\text{left}_S(\pi) \leq \alpha \\ \text{right}_S(\pi) \leq \beta}} f_\alpha \otimes f_\beta,$$

where $\alpha \in USP^{|S|}$ and $\beta \in USP^{n-|S|}$.

On the other hand, we have

$$\begin{aligned}
\Delta(d_\pi) &= \sum_{\pi \leq \tau} \Delta(f_\tau) \\
&= \sum_{\pi \leq \tau} \sum_{i \in B(\tau)} f_\nu \otimes f_\mu,
\end{aligned}$$

where $\nu \in USP^i$ and $\mu \in USP^{n-i}$ such that $\zeta * (\nu | \mu_{\uparrow_i}) = \tau$ for a unique $(i, n-i)$-shuffle $\zeta \in S_n$ (from Lemma 4.2.3).

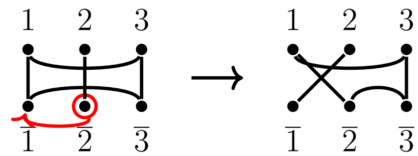
Thus, by Lemma 4.3.8,

$$\Delta(d_\pi) = \sum_{\substack{S \in 2^{[n]} \\ \pi \leq \overline{\text{split}}_S(\pi)}} d_{\text{left}_S(\pi)} \otimes d_{\text{right}_S(\pi)} \ . \qquad \square$$

**Example 4.3.11.** Given $\pi = \{1, 3, \overline{1}, \overline{3} | 2, \overline{2}\}$, moving the vertices of the element of the subset [3] we get the following diagrams. We need to consider sets $S$ that are a union of some of the parts of the bottom row of $\pi$ (otherwise $\overline{\text{split}}_S(\pi)$ is not defined).

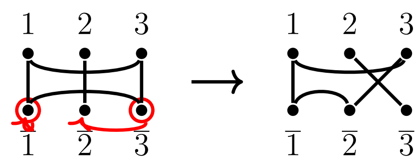
When $S = \emptyset$ and $S = \{1, 2, 3\}$, the vertices remain unmoved.

When $S = \{2\}$, the $\overline{2}$ vertex moves to the left to give us the set partition $\{1, 3, \overline{2}, \overline{3} | 2, \overline{1}\}$.




Given that $\{1,3,\overline{2},\overline{3}|2,\overline{1}\}$ is not larger in the weak order of $\pi$, the term does not appear as a term in the coproduct of the diagram basis.

When $S = \{1,3\}$, the $\overline{1},\overline{3}$ vertices moves to the left to give us the set partition $\{1,3,\overline{1},\overline{2}|2,\overline{3}\}$.




Given that $\{1,3,\overline{1},\overline{2}|2,\overline{3}\}$ is larger in the weak order of $\pi$, we split the set partition $\pi$ into two set partitions. The right is a set partition containing the sets that have elements $\overline{1}$ and $\overline{3}$, and the left is a set partition containing the reminaing sets. Thus, $\text{left}_{\{1,3\}}(\pi) = \{1,2,\overline{1},\overline{2}\}$ and $\text{right}_{\{1,3\}}(\pi) = \{1,\overline{1}\}$. And hence, by Equation (4.5),

$$\Delta(d_{\{1,3,\overline{1},\overline{3}|2,\overline{2}\}}) = d_{\{1,3,\overline{1},\overline{3}|2,\overline{2}\}} \otimes 1 + d_{\{1,2,\overline{1},\overline{2}\}} \otimes d_{\{1,\overline{1}\}} + 1 \otimes d_{\{1,3,\overline{1},\overline{3}|2,\overline{2}\}}.$$

## 4.4 The orbit basis of $UBP_k$

The uniform block permutation is a subalgebra of the partition algebra, and thus, the change of basis between the diagram and orbit bases of the uniform block permutation is defined by a coarsening relation (3.2). However, when $\pi$ is a uniform set partition then all the terms which are coarser than $\pi$ are also uniform set partitions.

The change of basis between the diagram basis and the orbit basis is

$$d_\pi = \sum_{\pi \preceq \gamma} x_\gamma \tag{4.6}$$

where the sum is over set partitions coarser than $\pi \in USP^k$.

Recall Lemma 3.5.3 which states that for fixed $\pi \in USP^k$ and $\gamma \in USP^l$, where $k$ and $l$ are positive integers, there is a bijection between $(\pi', \gamma', \nu)$ such that $\pi \preceq \pi'$, $\gamma \preceq \gamma'$ and $\nu \in \pi' \# \gamma'_{\uparrow_k}$ and set partitions $\nu' \in USP^{k+l}$ such that $\pi | \gamma_{\uparrow_k} \preceq \nu'$.

The product of $x_\pi$ and $x_\gamma$ where $\pi \in USP^k$ and $\gamma \in USP^l$ is the sum of all set partions in $USP^{k+l}$ such that when restricted to $\{1, \ldots, k\} \cup \{\overline{1}, \ldots, \overline{k}\}$ will give us $\pi$ and when restricted to $\{k+1, \ldots, k+l\} \cup \{\overline{k+1}, \ldots, \overline{k+l}\}$ will give us $\gamma_{\uparrow_k}$.

The corollary below is a consequence of Proposition 3.5.4. The proof for Corollary 4.4.1 is exactly the same as the proof of Proposition 3.5.4 however all of the terms in the expressions are uniform set partitions.

**Corollary 4.4.1.** *For $\pi \in USP^k$, $\gamma \in USP^l$, the product of the orbit basis is defined as*

$$x_\pi \times x_\gamma = \sum_{\tau \in \pi \# \gamma_{\uparrow k}} x_\tau \tag{4.7}$$

*where $\gamma_{\uparrow_k}$ is obtained by adding $k$ to the entries in $\gamma$.*

*Proof.* Let $\times' : UBP_k \otimes UBP_l \to UBP_{k+l}$ be the map whose action on the orbit basis is defined by the sum (4.7). We show that $\times'$ is the product $\times$, as defined in (4.4) using a technique that is explained in more detail in the appendix.

Therefore,

$$
\begin{aligned}
d_\rho \times' d_{\rho'} &= \sum_{\rho \preceq \pi} \sum_{\rho' \preceq \gamma} x_\pi \times' x_\gamma \\
&= \sum_{\rho \preceq \pi} \sum_{\rho' \preceq \gamma} \sum_{\tau \in \pi \# \gamma_{\uparrow_k}} x_\tau,
\end{aligned}
$$

by the definition of smash product and the bijection in Lemma 3.5.3

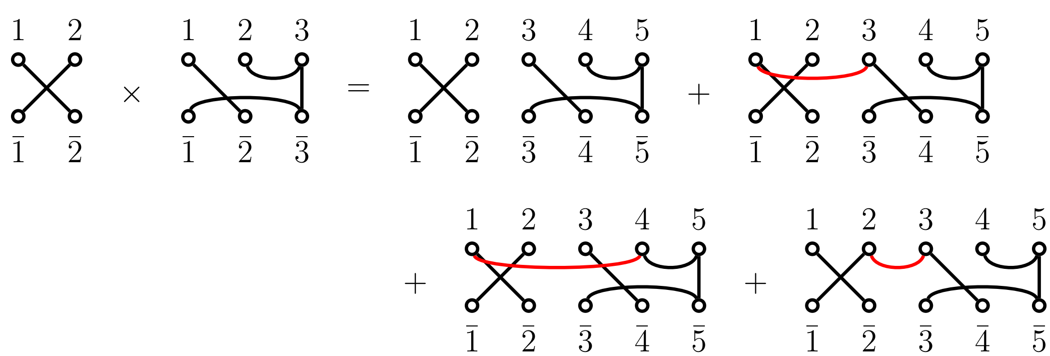
$$
\sum_{\rho \preceq \pi} \sum_{\rho' \preceq \gamma} \sum_{\tau \in \pi \# \gamma_{\uparrow_k}} x_\tau = \sum_{\rho | \rho'_{\uparrow_k} \preceq \tau} x_\tau,
$$

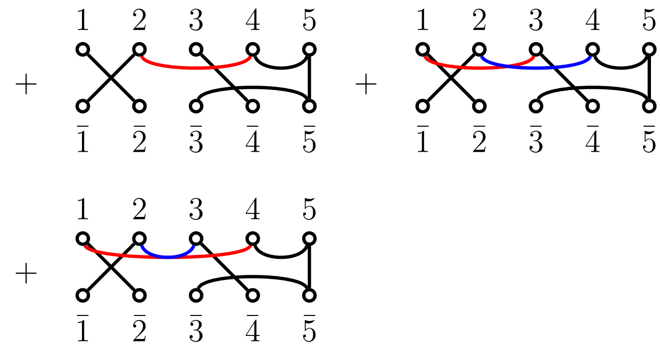
and thus,

$$
\begin{aligned}
d_\rho \times' d_{\rho'} &= \sum_{\rho | \rho'_{\uparrow_k} \preceq \tau} x_\tau \\
&= d_{\rho | \rho'_{\uparrow_k}} \\
&= d_\rho \times d_{\rho'}.
\end{aligned}
$$

Therefore, the product $\times'$ is equal to $\times$ so (4.7) holds. □

**Example 4.4.2.**






*Remark*: we do not have an explicit formula for the coproduct, but it may be computed using a change of basis with the diagram basis and the coproduct formula for the diagram basis (4.5). The experimental computations we have done indicate that the formula is complex and that the coefficients are not all non-negative. We encounter the first negative coefficient at $n = 3$. We provide an example here for reference.

**Example 4.4.3.** Let $\pi = \{1, \bar{3}|2, \bar{2}|3, \bar{1}\}$, then

$$\Delta(x_\pi) = x_\pi \otimes 1 + x_{\{1,\bar{2}|2,\bar{1}\}} \otimes x_{\{1,\bar{1}\}} - x_{\{1,2,\bar{1},\bar{2}\}} \otimes x_{\{1,\bar{1}\}} + x_{\{1,\bar{1}\}} \otimes x_{\{1,\bar{2}|2,\bar{1}\}} + 1 \otimes x_\pi.$$

## 4.5 The coarser fundamental basis of $UBP_k$

In some initial experiments we considered what would happen if we started with the fundamental basis and defined a basis using the refinement relation. The goal was to find a relation between the fundamental basis of the uniform block permutations and the partition algebra.

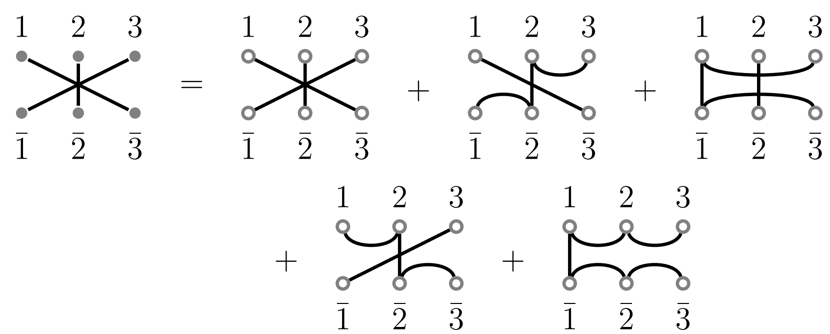
Although the relation did not connect the fundamental basis of the uniform block permutations and the partition algebra, we noticed in these experiments that the terms in the product of the fundamental basis involved shuffle and saw a similar pattern in the product of this new basis (that we refer to as the coarser fundamental basis).

A coarsening basis of the fundamental basis, $\{\tilde{f}_\pi : \pi \in USP^k, k \geq 0\}$ of $UBP_k$ can be defined by the following relation

$$f_\pi = \sum_{\pi \preceq \gamma} \tilde{f}_\gamma \tag{4.8}$$

where the sum is over all set partition $\gamma$ coarser than $\pi$, that is every block of $\pi$ is contained in a block of $\gamma$.

**Example 4.5.1.** The fundamental $f_{\{1,\bar{3}|2,\bar{2}|3,\bar{1}\}}$ in $UBP_3$ can be expressed in terms of the coarsening basis as follows



**Theorem 4.5.2.** *The product of the coarsening basis is given by*

$$\tilde{f}_\pi \times \tilde{f}_{\pi'} = \sum_{\tau \in \pi \# \pi'_{\uparrow_k}} \sum_{\zeta \in S^{(k,l)}} \tilde{f}_{\tau * \zeta^{-1}} \tag{4.9}$$

*for $\pi \in USP^k$ and $\pi' \in USP^l$, and $\pi'_{\uparrow_k}$ is obtained by adding $k$ to the entries in $\pi'$.*

*Proof.* Let $\times' : UBP_k \otimes UBP_l \to UBP_{k+l}$ be the map whose action on the coarsening basis is defined by the sum (4.9). We show that $\times'$ is the product $\times$, as defined in (4.1) using a technique that is explained in more detail in the appendix.

Hence,

$$f_\pi \times' f_{\pi'} = \sum_{\pi \preceq \gamma} \sum_{\pi' \preceq \gamma'} \tilde{f}_\gamma \times' \tilde{f}_{\gamma'}$$
$$= \sum_{\pi \preceq \gamma} \sum_{\pi' \preceq \gamma'} \sum_{\tau \in \pi \# \pi'_{\uparrow_k}} \sum_{\zeta \in S^{(k,l)}} \tilde{f}_{\tau * \zeta^{-1}}$$

By Lemma 3.5.3, we have

$$\sum_{\pi \preceq \gamma} \sum_{\pi' \preceq \gamma'} \sum_{\tau \in \pi \# \pi'_{\uparrow_k}} \sum_{\zeta \in S^{(k,l)}} \tilde{f}_{\tau * \zeta^{-1}} = \sum_{\pi | \pi'_{\uparrow_k} \preceq \tau} \sum_{\zeta \in S^{(k,l)}} \tilde{f}_{\tau * \zeta^{-1}}.$$

Therefore,

$$f_\pi \times' f_{\pi'} = \sum_{\pi | \pi'_{\uparrow_k} \preceq \tau} \sum_{\zeta \in S^{(k,l)}} \tilde{f}_{\tau * \zeta^{-1}}$$
$$= \sum_{\zeta \in S^{(k,l)}} f_{\left(\pi | \pi'_{\uparrow_k}\right) * \zeta^{-1}}$$
$$= f_\pi \times f_{\pi'}.$$

Therefore, the product $\times'$ is equal to $\times$ so (4.9) holds. □

**Example 4.5.3.** Let $\pi = \{1, \bar{2} | 2, \bar{1}\}$ and $\pi' = \{1, \bar{1}\}$, then

$$\pi \# \pi' = \{\{1, \bar{2} | 2, \bar{1} | 3, \bar{3}\},\ \{1, \bar{2} | 2, 3, \bar{1}, \bar{3}\},\ \{1, 3, \bar{2}, \bar{3} | 2, \bar{1}\}\}.$$

Therefore,

$$\begin{aligned}\tilde{f}_\pi \times \tilde{f}_{\pi'} = \tilde{f}_{\{1,\bar{2}|2,\bar{1}|3,\bar{3}\}} + \tilde{f}_{\{1,\bar{3}|2,\bar{1}|3,\bar{2}\}} + \tilde{f}_{\{1,\bar{3}|2,\bar{2}|3,\bar{1}\}} \\ + 2\tilde{f}_{\{1,3,\bar{2},\bar{3}|2,\bar{1}\}} + \tilde{f}_{\{1,3,\bar{1},\bar{3}|2,\bar{2}\}} \\ + \tilde{f}_{\{1,\bar{2}|2,3,\bar{1},\bar{3}\}} + 2\tilde{f}_{\{1,\bar{3}|2,3,\bar{1},\bar{2}\}}.\end{aligned}$$

The coproduct of the $UBP_k$ takes a simple form on the coarser fundamental basis. Surprisingly we find that the coproduct on the coarsening fundamental basis is the same as the coproduct formula on the fundamental basis. Here the formulation of the coproduct uses the notation established in Section 4.3.1.

**Theorem 4.5.4.** *Let $\pi \in USP^k$, then*

$$\Delta(\tilde{f}_\pi) = \sum_{i \in B(\pi)} \tilde{f}_{\text{left}_{[i]}(\pi)} \otimes \tilde{f}_{\text{right}_{[i]}(\pi)} \ . \tag{4.10}$$

*Proof.* Let $\Delta' : UBP_k \to UBP_i \otimes UBP_{k-i}$ be the map whose action on the coarser fundamental basis is defined by the sum (4.10). We show that $\Delta'$ is the coproduct $\Delta$, as defined in (4.2). We will do this using the dual form of the technique that is explained in the appendix for proving that two products are the same.

Therefore,

$$\begin{aligned}\Delta'(f_\pi) &= \sum_{\pi \preceq \gamma} \Delta'(\tilde{f}_\gamma) \\ &= \sum_{\pi \preceq \gamma} \sum_{i \in B(\gamma)} \tilde{f}_{\text{left}_{[i]}(\gamma)} \otimes \tilde{f}_{\text{right}_{[i]}(\gamma)}.\end{aligned} \tag{4.11}$$

Let $i$ be a breaking point of $\gamma$ then it must be that $i$ is a breaking point of $\pi$ as well because $\text{bottom}(\pi) \preceq \text{bottom}(\gamma)$. Moreover, $\text{left}_{[i]}(\pi) \preceq \text{left}_{[i]}(\gamma)$ and $\text{right}_{[i]}(\pi) \preceq \text{right}_{[i]}(\gamma)$. Since $\pi \preceq \gamma$ and if $i \in B(\pi)$ and $i \in B(\gamma)$ then the parts of $\pi$ that were joined in forming $\gamma$ must all be contained in the left or the right (with no parts from the left joined with those in the right). Then we can sum over the breaking point of $\pi$ and let $\alpha = \text{left}_{[i]}(\gamma)$ and $\beta = \text{right}_{[i]}(\gamma)$ and (4.11) is equal to

$$\begin{aligned}\sum_{i \in B(\pi)} \sum_{\text{left}_{[i]}(\gamma) \preceq \alpha} \sum_{\text{right}_{[i]}(\gamma) \preceq \beta} \tilde{f}_\alpha \otimes \tilde{f}_\beta &= \sum_{i \in B(\pi)} f_{\text{left}_{[i]}(\gamma)} \otimes f_{\text{right}_{[i]}(\gamma)} \\ &= \Delta(f_\pi).\end{aligned}$$

Therefore, the coproduct $\Delta'$ is equal to $\Delta$ so (4.10) holds. $\square$

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# Appendix A

# Appendix

The following lemma shows a technique that we use at several points in this thesis. It says that if we have two products on two different bases, then we can use a change of basis to show that they are equivalent. Throughout the thesis, we have two bases where we know the product on one and have a conjecture on the second, and we want to know these are equivalent.

Although we don't use this lemma directly we use the reasoning of the lemma repeatedly throughout the thesis. We show that it is easier to go from the known to the conjecture than it is to to prove the conjecture directly. This technique is used elsewhere in the literature (see for example, Theorem 3.1 [AS05]).

**Lemma A.0.1.** *Let $P$ be a graded algebra with two product structures $(P, *)$, and $(P, \odot)$, where $\{A_x\}$ and $\{B_x\}$ are bases of $P$. Suppose $A_x = \sum_y c_{x,y} B_y$ where $c_{x,y} \in \mathbb{Q}$, and $B_x = \sum_y c^{-1}_{x,y} A_y$, that is $c^{-1}_{x,y}$ is the entry in the inverse change of basis matrix.*

*Assume there is a product*

$$A_x * A_y = \sum_z a_{xyz} A_z. \tag{A.1}$$

*If* $B_x \odot B_y = \sum_z b_{xyz} B_z$ *and* $A_x \odot A_y = \sum_z a_{xyz} A_z$*, then*

$$B_x * B_y = \sum_z b_{xyz} B_z. \tag{A.2}$$

*Proof.* Let $\lambda^{\odot} : P \otimes P \to P$ be the map whose action on the $A_x$ basis is defined by the sum (A.1). We show that $\odot$ is the product $*$ as defined by (A.2). Therefore,

$$\begin{aligned}
B_x \odot B_y &= \sum_{x'} \sum_{y'} c_{xx'}^{-1} c_{yy'}^{-1} A_{x'} \odot A_{y'} \\
&= \sum_{x'} \sum_{x'} \sum_{z'} c_{xx'}^{-1} c_{yy'}^{-1} a_{x'y'x'} A_{z'} \\
&= \sum_{x'} \sum_{y'} c_{xx'}^{-1} c_{yy'}^{-1} A_{x'} * A_{y'} \\
&= B_x * B_y.
\end{aligned}$$

Thus, the product $\odot$ is equal to $*$. $\square$