How to Pair the Real Numbers with the Integers:
a Précis of the Proof

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Understanding is historical. To comprehend how the real numbers can be paired with the integers, one must ponder the proof. A précis, however, may convey something of its nature and scope.

Consider the real numbers ordered by size.[2] By the middle of the 20th-century, mathematicians were convinced that many real numbers must lie between any two of them. They were surprised, therefore, when Abraham Robinson showed that one could construe two real numbers as differing *infinitesimally* from each other as coherently as one's construal of the numbers themselves.[3]

No one noticed, however, that one could then construe every real number as differing infinitesimally from a smaller and from a larger one, each a *nearest neighbour* of it.

The real numbers, that is, are a succession of *nearest neighbours*.

The integers (. . . – 3, -2, -1, 0, +1, +2, +3, . . .), like the real numbers, are a succession of nearest neighbours. Were one to pair any one of the real numbers with an integer, one could then pair their smaller and their larger nearest neighbours, and then the numbers left unpaired within the smaller and the larger nearest neighbours of those just paired, and so forth, thereby pairing each of the real numbers with an integer.

As trumpeted within the titles of the proof and this précis, therefore,

The real numbers can be paired with the integers.

Q.E.D.

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[1] 'The real numbers', as commonly construed, are the positive and negative unbounded decimals (for example, 1.000000 . . ., 2.718281 . . . ['e'], or -3.141926 . . . [-'π']), and I shall by the phrase be referring to *all* of them (0.999999 . . . , for example, often excluded unjustly from less cautious accounts by other authors). By 'the pairing' of the numbers of any two kinds, I shall mean 'the pairing of each of the numbers of the first with a unique number of the second, and conversely'. Lastly, by 'the Proof', I refer to the larger essay within the Evan Wm. Cameron Collection of YorkSpace entitled 'How to Pair the Real Numbers with the Integers' (http://hdl.handle.net/10315/38097 [as of 2023 02 06]).

[2] For brevity, I shall hereafter omit the phrase 'ordered by size' when referring to the numbers of any kind, as I have within the title and opening paragraphs above.

[3] Abraham Robinson, *Non-Standard Analysis* (Amsterdam: North-Holland Publishing Company, 1966). Robinson first presented his ideas to a seminar at Princeton University in 1960.

We need no 'sets', therefore, to comprehend what little we know of mathematics, our toolbox less cluttered than many have supposed.

Addendum to the Proof

When fashioning the proof, I failed to realise, however, that one could expand the scope of its conclusion.

(a) The *numbers* of any kind, for example, if construable as an unbounded succession of nearest neighbours, can be paired with the real numbers. Were one to pair any one of them with a real number, one could then pair their smaller and their larger nearest neighbours, and then the numbers left unpaired within the smaller and the larger nearest neighbours of those just paired, and so forth, thereby pairing each of the numbers with a real number

(b) The *points of a line* can likewise be paired with the real numbers, as Georg Cantor and Richard Dedekind suggested in the late 19^{th}-century. Since no one then or thereafter could prove the conjecture, it was regarded as an 'axiom'. We now know, however, after Robinson, that each of the points of a line can be conceived to have a nearest neighbour to its left and another to its right differing from it infinitesimally, the line consisting of a succession of them. The points of a line can therefore be paired, as above, with the real numbers.

Without exception, indeed,

(c) The *things of any kind*, discernable through the ontological mist as an unbounded succession of nearest neighbours, can be paired, as above, with the real numbers.

Since any two kinds of things pairable with the real numbers can be paired with one another, the conclusion of the proof can be generalised.

> *Any* kinds of things, each construable as an unbounded succession of nearest neighbours, can be paired with one another – world without end, amen!

Q.E.D.