RESULTS ON PROXIMAL AND SEMI-PROXIMAL SPACES

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Abstract

Proximal spaces were defined by J. Bell as those topological spaces X with a compatible uniformity $\mathfrak{U}$ on which Player I has a winning strategy in the so-called proximal on $(X, \mathfrak{U})$. Nyikos defined the class of semi-proximal spaces where Player II has no winning strategy on $(X, \mathfrak{U})$ with respect to some compatible uniformity. The primary focus of this thesis is to study the relationship between the classes of semi-proximal spaces and normal spaces. Nyikos asked whether semi-proximal spaces are always normal. The main result of this thesis is the construction of two counterexamples to this question. We also examine the characterization of normality in subspaces of products of ordinals, relating it to the class of semi-proximal spaces in finite power of ω_1. In addition, we introduce a strengthening of these classes by restricting the proximal game to totally bounded uniformities. We study connections between the proximal game, the Galvin game, and the Gruenhage game. Further, we explore the relationship between semi-proximality and other convergence properties.

This dissertation is dedicated to my loving mother, my late father, and Dr.Lutfi Kalantan.

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Chapter 1

Introduction

In 2014, Jocelyn Bell introduced the notion of proximal spaces using uniformities to address problems related to uniform box products. J. Bell defines a proximal space as a topological space X for which there exists a compatible uniformity such that Player I has a winning strategy in a specific two-player game of infinite length. The class of proximal spaces leads to powerful results, including a result which is due independently to M. E. Rudin [40] and S.P. Gul'ko [18], that Σ-products of metrizable spaces are normal. In a related development, Nyikos introduced the class of semi-proximal spaces, where Player II has no winning strategy [36].

This thesis has six chapters in addition to this introductory chapter. Chapter 2 provides preliminary definitions and basic results used throughout the thesis, along with a history of the theory of uniform spaces. Chapter 3 provides the definitions of the class of proximal and semi-proximal spaces. In addition, we introduce a strengthening of this class by restricting the proximal game to totally bounded uniformities. These properties exhibit significance even in the case of discrete spaces. In chapter 4, we establish connections between the proximal game and a set-theoretic game introduced and investigated by Ulam, Galvin, and others [19, 41], as well as the Gruenhage game [9, 8]. In Chapter 5, we study the relationship between the classes of semi-proximal spaces and normal spaces. It was motivated by Nyikos' question of whether semi-proximal spaces are always normal. We prove that the answer to the question is affirmative for pseudocompact spaces and construct two counterexamples, resolving the long-standing problem. We also have shown that the De Caux Dowker space can be constructed, under additional set-theoretic assumptions, to be not semi-proximal, which gives the first (consistent) example of a first countable, locally compact normal space that is not semi-proximal. The other motivation was characterizing normality in subspaces of products of ordinals. We establish that normal subspaces of finite powers of ω_1 are semi-proximal and provide a counterexample for the converse. Furthermore, we demonstrate that normality and semi-proximality are equivalent in finite products of subspaces of ω_1, which extends a result in [2]. Finally, in Chapter 6, we study the relationship between semi-proximality and r-Ramsey spaces. Chapter 7 contains a list of open questions inspired by the results presented in the thesis.

Chapter 2

Basic Notations and Definitions

This chapter provides the topological, set-theoretical, and game-theoretical preliminaries essential for this dissertation. We will use standard topological and set-theoretic notation such as in [12] and [33].

2.1 Set Theory

Ordinals and cardinals are defined as $0 = \emptyset$, $\omega = \{0, 1, 2, \ldots\}$, ω_1 is the smallest uncountable ordinal, and $\alpha + 1 = \alpha \cup \{\alpha\}$. The order on ordinals and cardinals is given by $\alpha < \beta$ if and only if $\alpha \in \beta$. For every ordinal α, we call the ordinal $\alpha \cup \{\alpha\}$ the successor of α, and denote it by $\alpha + 1$. We will call α **a successor ordinal** if $\alpha = \beta + 1 = \beta \cup \{\beta\}$ for some ordinal β, and we will call α a **limit ordinal** if $\alpha \neq 0$ and α is not a successor ordinal. Let **Lim** be the set of all limit ordinals in ω_1 and **Succ** be the set of all successor ordinals in ω_1. If κ is a limit ordinal and $C \subseteq \kappa$, then C is **unbounded** in κ if for any $\alpha < \kappa$, there is some $\beta \in C$ such that $\alpha < \beta$. We say that C is

closed in κ if for every limit ordinal $\beta < \kappa$ such that $C \cap \beta$ is unbounded in β, we have that $\beta \in C$. A subset C of ω_1 is said to be **club** if it is closed and unbounded set in ω_1, and a subset S of ω_1 is said to be **stationary** in ω_1 if $S \cap C \neq \emptyset$ for every club C.

Lemma 2.1.1 (Fodor's Lemma) *If α is a regular uncountable cardinal, S is a stationary subset of α and $f : S \to \alpha$ with $f(\beta) < \beta$ for any $\beta \in S$, then there is γ and a stationary S' such that $f(\beta) = \gamma$ for any $\beta \in S'$.*

For a set X, a sequence is a function $f \in X^\omega$, denoted $f = \{f(0), f(1), \ldots\}$. Let $f \restriction n = \{f(0), f(1), \ldots, f(n-1)\}$. A finite sequence is a function $s \in X^{<\omega}$, denoted $t = \{t(0), \ldots, t(|t|-1)\}$. For $s, t \in X^{<\omega}$, let $s \frown t$ be **the concatenation of** s and t such that $(s \frown t)(i) = s(i)$ for $i < |s|$, and $(s \frown t)(i + |s|) = t(i)$ for $i < |t|$. We write $s \supset t$ if s end-extends t.

Given a set X, $|X|$ and $\mathcal{P}(X)$ denote, respectively, the cardinality and the power set of X. Let $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$, $[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\}$, $[X]^{\leq\kappa} = \{A \subseteq X : |X| \leq \kappa\}$, for some cardinal κ. Sometimes we write $[X]^\omega$ or $[X]^{\leq\omega_1}$ instead of $[X]^{\aleph_0}$ or $[X]^{\leq\aleph_1}$, respectively. A family $A \subseteq [\omega]^\omega$ is almost disjoint if $a \cap b$ is finite for every $a, b \in A$.

Definition 2.1.2 *[27] The principle $\diamondsuit^*$ is the statement that there exists a sequence $\{\mathcal{A}_\alpha : \alpha \in Lim(\omega_1)\}$, where $\mathcal{A}_\alpha$ is a countable subset of $\mathcal{P}(\alpha)$ such that for every subset X of ω_1, $\{\alpha \in \omega_1 : X \cap \alpha \in \mathcal{A}_\alpha\}$ is a club.*

Definition 2.1.3 *[37] The principle $\clubsuit$ is the statement that there exists $\{C_\alpha : \alpha \in Lim(\omega_1)\}$, where $C_\alpha \subset \alpha$ has order type ω, and $\sup(C_\alpha) = \alpha$ such that for every uncountable subset A of ω_1, there is a stationary set S such that $E_\alpha \subset A$, for all $\alpha \in S$.*

2.2 Topology

By a **space** we mean a non-empty topological space. It is denoted by $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is the topology on X. However, if $\mathcal{O}$ is clear, we simply write X instead of $(X, \mathcal{O})$. We say a space X is

- A **Hausdorff space** if for every $x \neq y$ in X, there exist two disjoint open subsets U and V of X such that $x \in U$ and $y \in V$.

- A **regular space** if for every closed $A \subset X$ and $x \in X$ such that $x \notin A$, there exist two disjoint open subsets U and V of X such that $x \in U$ and $A \subset V$.

- A **completely regular space (Tychonoff)** if for every closed $A \subset X$ and $x \in X$ such that x is not in A, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$, for all $a \in A$.

- A **normal space** if for every two disjoint closed subsets A and B of X, there exist two disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$.

A topological space X is **Fréchet-Urysohn** if, whenever a point x is in the closure of a subset A, there is a sequence from A converging to x. Arkhangel'skii in [1] defined a point $x \in X$ to be an α_i**-point**, for $i = 1, 2, 3, 4$ if for each family $\{S_n : n \in \omega\}$ of sequences converging to x, there is a sequence S converging to x such that:

- α_1**:** $S \setminus S_n$ is finite for all $n \in \omega$,

- α_2**:** $S \cap S_n \neq \emptyset$ for all $n \in \omega$,

- α_3**:** $|S \cap S_n| = \omega$ for infinitely many $n \in \omega$,

- α_4**:** $S \cap S_n \neq \emptyset$ for infinitely many $n \in \omega$.

A space is called an α_i-space if each point is an α_i-point. It is clear that $\alpha_1 \implies \alpha_2 \implies \alpha_3 \implies \alpha_4$. We say that a space X is α_i-Fréchet if X is both Fréchet-Urysohn and α_i. It is worth noting that every countable family of infinite sets has an infinite pairwise disjoint refinement. Therefore, we can define the concept of an α_i-point using a pairwise disjoint family of sequences converging to x. The same holds for the following charachtrization of α_2-Fréchet.

Proposition 2.2.1 *[42] A space X is α_2-Fréchet if and only if for every $x \in X$ and for any sequence (pairwise disjoint sequence) $(A_n : n \in \omega)$ with $x \in \overline{A}_n$ for all n, there is $x_n \in A_n$ such that $(x_n : n \in \omega)$ converges to x.*

A topological space X is said to be **scattered** if every non-empty subset $A \subset X$ has an isolated point relative to A. We put $X^{(0)} = X$ and

$$X^{(1)} = \{x \in X : x \ \text{ is not isolated in } X\}.$$

If α is any ordinal number and if $X^{(\beta)}$ is already defined for all ordinals $\beta < \alpha$ then we put

$$X^{(\alpha)} = (X^{(\beta)})^{(1)} \ \text{ if } \alpha = \beta + 1 \ \text{ and } \beta \ \text{ is an ordinal, and}$$

$$X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)} \ \text{ if } \alpha \ \text{ is a limit ordinal.}$$

The scattered height of X is the smallest ordinal α such that $X^{(\alpha+1)} = \emptyset$.

Recall that a cover $\mathcal{U}$ of a space X is a subset of the power set $\mathcal{P}(X)$ such that $\bigcup\mathcal{U} = X$. In addition, we call $\mathcal{U}$ an **open cover** of X if each of its elements is an open set in X. A cover $\mathcal{V}$ **refines** a cover $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$ and here we call $\mathcal{V}$ a **refinement** of $\mathcal{U}$. If in addition, each element of $\mathcal{V}$ is open, it will be called an **open refinement**. If $\mathcal{U}$ is an open cover of a space X and $x \in X$, then the star of $\mathcal{U}$ around x is the set $\mathrm{St}(x,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$. If $A \subset X$, then the star of $\mathcal{U}$ around A is the set $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$. A cover $\mathcal{V}$ **star-refines** a cover $\mathcal{U}$ if $\{\mathrm{St}(V, \mathcal{V}) : V \in \mathcal{V}\}$ refines $\mathcal{U}$. An open cover $\mathcal{U}$ of X is **normal** if there exists a sequence of open covers $\{\mathcal{V}_n : n \in \omega\}$ such that $\mathcal{V}_0 = \mathcal{U}$ and $\mathcal{V}_n$ is a star refinement of $\mathcal{V}_{n-1}$. An open cover $\{U_\alpha : \alpha \in I\}$ is **shrinkable** if there is an open cover $\{V_\alpha : \alpha \in I\}$ such that $\overline{V}_\alpha \subseteq U_\alpha$, for all α.

Let X be a topological space. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is **discrete** if every point has a neighborhood that intersects only one element of $\mathcal{A}$. We say $\mathcal{A}$ is **locally finite** if every $x \in X$ has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$. A space is **paracompact** if every open cover has a locally finite open refinement. A space is **countably paracompact** if every countable open cover has a locally finite refinement.

Theorem 2.2.2 *[47] If X is a paracompact scattered space, then X is zero-dimensional (i.e. it has a clopen base), and every open cover has a disjoint open refinement.*

A Tychonoff space X is **pseudocompact** if every continuous real-valued function defined on X is bounded [22] .

Theorem 2.2.3 *[45, 10] For a Tychonoff space X, the following are equivalent*

(1) X is pseudocompact

(2) every locally finite open cover of X has a finite subcover.

(3) For every countable family $\{G_n : n \in \omega\}$ of open subsets of X which has the finite intersection property, the intersection $\bigcap_{n\in\omega} \overline{G}_n \neq \emptyset$.

2.2.1 Uniformities and Uniform Spaces

André Weil introduced the concept of a uniform space in 1938 to extend the idea of metric spaces to spaces that might not have a defined metric [52]. This concept enables discussions about uniform behaviour similar to those in metric spaces, such as uniform continuity, but without depending on specific metrics. The concept of **uniform space** is a set X together with a structure called uniformity defined on it. Uniformity can be defined using collections of covers, collections of pseudometrics, or collections of entourages.

Equivalent Definitions of Uniformities

In 1936, Weil announced his results on uniform spaces using the term "regular" defined in terms of collections of covers [51]. He used this term to establish numerous concepts on uniform spaces, such as Cauchy families, completeness, total boundedness, and compactness. Weil presented significant theorems establishing him as the founder of uniform spaces. It is worth mentioning that Tukey also defined uniformity with respect to a collection of covers in 1940 [48], a concept that was extensively used by Isbell in [26].

Definition 2.2.4 *A* **covering uniformity** μ *on a set* X *is a family of covers of* X *satisfying the following properties:*

(1) If $\mathcal{V} \in \mu$ *and* $\mathcal{V}$ *refines a cover* $\mathcal{U}$ *of* X*, then* $\mathcal{U} \in \mu$*.*

(2) For every $\mathcal{U}, \mathcal{V} \in \mu$ *there is* $\mathcal{W} \in \mu$ *that refines both* $\mathcal{U}$ *and* $\mathcal{V}$*.*

(3) For every $\mathcal{U} \in \mu$ *there is* $\mathcal{V} \in \mu$ *that star-refines* $\mathcal{U}$*.*

(4) If $x \neq y$ *in* X*, there is* $\mathcal{U} \in \mu$ *with no member contains both* x *and* y*.*

Bourbaki established the second method in 1940 [7], defined uniformity using families of pseudometrics on a set X, providing a significant tool that Gillman and Jerison employed in [15].

Definition 2.2.5 *A* **pseudometric uniformity** *on a set* X *is defined to be a non-empty family* P *of pseudometrics on* X *satisfying the following:*

(1) If σ *is a pseudometric on the set* X *and for every* $\epsilon > 0$ *there exists a* $\rho \in P$ *and* $\delta > 0$ *such that* $\sigma(x, y) < \epsilon$ *whenever* $\rho(x, y) < \delta$*, then* $\sigma \in P$*.*

(2) If $\rho_1, \rho_2 \in P$*, then* $\max(\rho_1, \rho_2) \in P$*.*

(3) If $x \neq y$ *in* X*, there exists a* $\rho \in P$ *such that* $\rho(x, y) > 0$*.*

Now we delve into the most popular definition established in 1938 by Weil in his booklet [52] where he employed uniform neighbourhoods of the diagonal instead of working with systems of covers. This definition was given in its present form by Bourbaki [7]. Let X be a set and Δ be the diagonal of the Cartesian product $X \times X$. The element $U \subset X \times X$ is called **entourage** if $\Delta \subset U$ and $\{\langle y, x\rangle : \langle x, y\rangle \in U\} \subset U$. For an entourage U, define

$U \circ U = \{\langle x, z\rangle : \text{there exists } y \in X \text{ such that } \langle x, y\rangle, \langle y, z\rangle \in U\}$. For $n > 0$, define $nU \subset X \times X$ inductively by $1U = U$ and $nU = (n-1)U \circ U$. Let $\mathcal{D}_X$ denotes the family of all entourages and define $U[x] = \{y \in X : \langle x, y\rangle \in U\}$.

Definition 2.2.6 *A* **diagonal uniformity** *on a set X is a subfamily $\mathfrak{U}$ of $\mathcal{D}_X$ satisfying the following conditions:*

(1) If $U \in \mathfrak{U}$ and $U \subset V \in \mathcal{D}_X$, then $V \in \mathfrak{U}$.

(2) If $V, U \in \mathfrak{U}$, then $V \cap U \in \mathfrak{U}$.

(3) For every $U \in \mathfrak{U}$ there exists a $V \in \mathfrak{U}$ such that $2V \subset U$.

(4) $\bigcap \mathfrak{U} = \Delta$.

The following proposition establishes that all three definitions of uniformity are equivalent:

Proposition 2.2.7 *On a set X, we have the following:*

(a) For a pseudometric uniformity P, the family of all entourages containing a set of the form $\{\langle x, y\rangle : \rho(x, y) < 1/2^n\}$, where $\rho \in P$ and a positive integer n, is a diagonal uniformity.

(b) For a diagonal uniformity $\mathfrak{U}$, the family of all covers $\mathcal{A}$ of the set X such that there exists $U \in \mathfrak{U}$ with $\{U[x] : x \in X\}$ refines $\mathcal{A}$ is a covering uniformity.

(c) For a covering uniformity μ, the family of all pseudometrics ρ such that for every $\epsilon > 0$ there exists $\mathcal{A} \in \mu$ with $\mathcal{A}$ refines $\mathcal{U}_{\rho,\epsilon} = \{U_{\rho,\epsilon}[x] : x \in X\}$, where $U_{\rho,\epsilon}[x] = \{y \in X : \rho(x, y) < \epsilon\}$, is a pseudometric uniformity.

Three different but equivalent definitions of uniformities lead to three different but equivalent approaches to introduce new concepts and to state and prove new results. Unless specified otherwise, our definitions are based on diagonal uniformities, represented by $\mathfrak{U}$.

Uniformity Bases

A family $\mathfrak{B}$ is **a base** for a uniformity $\mathfrak{U}$ if, for every $V \in \mathfrak{U}$, there exists a $W \in \mathfrak{B}$ such that $W \subset V$. A family $\mathfrak{B} \subset \mathcal{D}_X$ is a base for some uniformity and is called **a uniformity base** if and only if its sets satisfy the following:

(1) For any $V, U \in \mathfrak{B}$, there exists a $W \in \mathfrak{B}$ such that $W \subset V \cap U$.

(2) For every $U \in \mathfrak{B}$ there exists a $V \in \mathfrak{B}$ such that $2V \subset U$.

(3) $\bigcap \mathfrak{B} = \Delta$.

Uniform Topologies and Uniform Subspaces

Weil proves that any uniformity on a set X induces a topology on X while different uniformities may produce the same topology and, in the class of compact spaces, all uniformities are equivalent [52]. For a uniformity or a uniformity base $\mathfrak{U}$ on a set X, the collection

$$\mathcal{O} = \{G \subset X : \text{for all } x \in G \text{ there exists } U \in \mathfrak{U} \text{ such that } U[x] \subset G\}$$

forms a topology on X called **the topology induced by** $\mathfrak{U}$. Furthermore, the topology induced by $\mathfrak{U}$ on the set $X \times X$ is the Tychonoff topology, and the family of all members of $\mathfrak{U}$ which are open with respect to this topology is a base for $\mathfrak{U}$. A topology of a space X can be induced by a uniformity if and only if X is Tychonoff [52]. Different uniformities may produce the same topology and, in the class of compact spaces, all uniformities are equivalent. The finest uniformity of all compatible uniformities on a Tychonoff space X exists and is called **the universal uniformity** on X. The family of all normal covers of a completely regular space X is a covering uniformity that generates the universal uniformity on the space X. In a paracompact space, the universal uniformity is generated by the family of all open covers.

For a uniform space $(X, \mathfrak{U})$ and a subset A of X, **the uniformity induced on** A **as a subspace of** $(X, \mathfrak{U})$ is $\mathfrak{U}_A = \{U \cap (A \times A) : U \in \mathfrak{U}\} \subset \mathcal{D}_A$. The pair $(A, \mathfrak{U}_A)$ is called **a uniform subspace** of $(X, \mathfrak{U})$. The topology $\mathcal{O}_A$ which is induced on A by $\mathfrak{U}_A$ coincides with the topology of the subspace of X, where X has the topology induced by $\mathfrak{U}$.

The Uniform Continuity

Weil developed uniform structures to establish a concept of uniform continuity. Suppose we have two uniform spaces $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$. A mapping $f : X \to Y$ is called **uniformly continuous with respect to the uniformities** $\mathfrak{U}$ **and** $\mathfrak{V}$ if for every $V \in \mathfrak{V}$ there is a $U \in \mathfrak{U}$ such that $\langle f(x), f(y)\rangle \in V$ for all $\langle x, y\rangle \in U$. Note that every f is a continuous mapping of X to Y with respect to the topology induced by $\mathfrak{U}$ and $\mathfrak{V}$. It is worth noting that every metric on a set X generates a diagonal uniformity on the set X. We call the uniformity generated by ρ, *the uniformity induced by the metric* ρ and denoted by $\mathfrak{U}_\rho$. For metric spaces (X, ρ) and (Y, σ), a mapping f from X to Y is uniformly continuous with respect to $\mathfrak{U}_\rho$ and $\mathfrak{V}_\sigma$ if and only if f is uniformly continuous with respect to ρ and σ.

Totally Bounded Uniformities

A uniformity $\mathfrak{U}$ on a set X is **totally bounded** if for every $V \in \mathfrak{U}$, there exists a finite set $A \subset X$ such that $\{V[x] : x \in A\}$ covers X.

Theorem 2.2.8 *[52] If X is a totally bounded uniform space, then for every $A \subset X$, the space $(A, \mathfrak{U}_A)$ is totally bounded.*

Corollary 2.2.9 *Every pseudocompact uniform space is totally bounded.*

Proof. Let $(X, \mathfrak{U})$ be a pseudocompact uniform space. Let $U \in \mathfrak{U}$ and let $V \in \mathfrak{U}$ be open such that $V \subseteq U$. Then, $\mathcal{V} = \{V[x] : x \in X\}$ has a locally finite open refinement $\mathcal{A}$ [4]. By Theorem 2.2.3, $\mathcal{A}$ has a finite subcover $\{A_m : m \leq n\}$. For every $m \leq n$, there exists $x_m \in X$ such that $A_m \subseteq U[x_m]$. Therefore, $\{U[x_m] : m \leq n\}$ covers X. □

2.3 Game Theory

Since the 1920s, mathematicians have been actively exploring mathematical games, initiated by Sierpinski in 1924 [43], followed by contributions from Hurewicz in 1925 [25], and Banach and Kuratowski in 1929 [3]. As noted by Ulam, Mazur began formulating inquiries about infinite games around 1928. Subsequently, topological games have been extensively studied to investigate the properties of topological spaces. We will consider games with only two players, where each $n \in \omega$ corresponds to one inning. In all the games, every play results in a win for one of the players. However, we do not permit draws where neither player wins. The two players will be referred to as Player I and Player II. In each game described, Player I will take the first action in every inning, while Player II will respond to the first player's actions.

A game, as defined by Gale and Stewart [14], is a tuple $\langle M, W\rangle$, where M is the set of moves for the game, $W \subset M^\omega$ is the set of winning playthroughs or victories for the first player, and M^ω is the set of all possible playthroughs of the game. If Player I has a winning strategy in a game on X, then X satisfies a certain topological property. A **strategy** of a game $G = \langle M, W\rangle$ for a player is a function from $M^{<\omega}$, whose last move was made by the opponent to M. Without loss of generality, we may assume the strategy is defined for the opponent's partial play only because the strategy uniquely determines the omitted moves of the player. An **attack** for the game G is a function from ω to M. Given a strategy τ for the first player and an attack $\langle a_0, a_1, \ldots\rangle$ by the second player, the **result** of the game G is $\langle \tau(\emptyset), a_0, \tau(\langle a_0\rangle), a_1, \tau(\langle a_0, a_1\rangle), \ldots\rangle$ and called **a play of the game**. We say that a strategy is winning if, for every attack by the opponent, the result of the game is a victory for that player. To show that a winning strategy exists for a player, we typically begin by defining it and showing that it is legal (according to the rules of the game). Then, we consider an arbitrary legal attack and prove that the result of the game is a victory for that player. To show that a winning strategy does not exist for a player, we often consider an arbitrary legal strategy and use it to define a legal counter-strategy for the opponent. If we can prove that the result of the game for that strategy and counter-strategy is a victory for the opponent, then a winning strategy does not exist for that player.

In some cases, a player can win a game on X without having complete information about the previous moves of the game. This is called a limited information strategy. A k**-Markov** strategy is a function $\sigma : M^{\leq k} \times \omega \to M$ that only considers the opponent's previous k moves and the inning number. A k**-stationary** strategy is a function $\sigma : M^{\leq k} \to M$ that only considers the opponent's previous k moves. Note that some authors refer to stationary strategies as tactical strategies. A k**-coding strategy** for a player is a strategy that depends only on the opponent's previous k moves and the player's previous k moves. The k is usually omitted when $k = 1$. When such a strategy exists, X has a stronger property than the one implied by a perfect information strategy.

Chapter 3

Proximal and Semi-proximal Spaces

Jocelyn Bell introduced the notion of proximal spaces using uniformities to address problems related to the uniform box product topology. This topology is finer than the Tychonoff topology but coarser than the box topology [5]. It was introduced by Scott Williams during the 9th Prague International Topological Symposium in 2001, where he asked whether the countable uniform box product of a compact space is normal. This question is still open, along with Bell's question of whether the countable uniform box product of $\omega_1 + 1$ is normal [4]. However, there are some results regarding uniform box products in proximal spaces. The authors in [20] proved that the uniform power of a compact proximal space is proximal. Nyikos gave an example of a non-compact proximal space such that the uniform box product of X is not proximal [36].

3.1 Definitions and Basic Results

J. Bell introduced the class of proximal spaces defined in term of a two-player game played on uniform spaces. Suppose $(X, \mathfrak{U})$ is a uniform space, where $\mathfrak{U}$ is either a uniformity or a uniformity base. The proximal game is defined on $(X, \mathfrak{U})$ as follows. Player I chooses elements of $\mathfrak{U}$ while Player II chooses points in X. In the first inning, Player I chooses an entourage $U_0 \in \mathfrak{U}$, and Player II chooses $x_0 \in X$. In inning 1, Player I chooses an entourage $U_1 \subseteq U_0$ in $\mathfrak{U}$ and Player II chooses $x_1 \in U_0[x_0]$. In general, if $x_0, \ldots, x_n$ are the first n choices of Player II, Player I chooses $U_{n+1} \in \mathfrak{U}$ such that $U_{n+1} \subseteq U_n$ and Player II chooses $x_{n+1} \in U_n[x_n]$. Player I wins the proximal game on $(X, \mathfrak{U})$ if either $\bigcap_{n\in\omega} U_n[x_n] = \emptyset$ or the sequence $(x_n : n \in \omega)$ converges in X. A play of the game is a sequence $(U_0, x_0, U_1, x_1, \ldots)$ where each $U_n \in \mathfrak{U}$ and $x_n \in X$ are chosen according to the rules of the game.

More precisely, a strategy for Player I is a map $\tau : X^{<\omega} \to \mathfrak{U}$. A τ-sequence of the strategy τ is a sequence $(x_n : n \in \omega)$ such that $x_{n+1} \in \tau(x_0, \ldots, x_{n-1})[x_n]$. Note that $\tau(x_0, \ldots, x_n)$ represents the entourage Player I would choose if $(x_0, \ldots, x_n)$ have been the first n choices for player II, and a τ-sequence represents the result of a game that has been played. A strategy τ is winning if Player I wins every τ-sequence of the game.

Definition 3.1.1 *A uniform space* $(X, \mathfrak{U})$ *is* **proximal** *if Player I has a winning strategy in the proximal game on* $(X, \mathfrak{U})$*, denoted by* $(I \uparrow P(X, \mathfrak{U}))$*. A topological space* X *is* **proximal** *if there exists a compatible uniformity* $\mathfrak{U}$ *on* X *such that* $(X, \mathfrak{U})$ *is proximal.*

A strategy for Player II is a map $\sigma : \mathfrak{U}^{<\omega} \to X$. A σ-sequence of the strategy σ is a sequence $(U_n : n \in \omega)$ such that $\sigma(U_0, \ldots, U_n) \in U_n[\sigma(U_0, \ldots, U_{n-1})]$. Note that $\sigma(U_0, \ldots, U_n)$ is the point Player II would choose if $(U_0, \ldots, U_n)$ have been the first n choices for Player I, and a σ-sequence represents the result of a game that has been played. A strategy σ is winning if for any sequence of entourages $(U_n : n \in \omega)$,

1. the sequence $\sigma(U_0), \sigma(U_0, U_1), \ldots$ does not converge, and
2. $\bigcap_{n\in\omega} U_n[\sigma(U_0, \ldots, U_n)] \neq \emptyset$.

Naturally, in any mathematical game, one Player having no winning strategy is a weaker notion than the other Player having a winning strategy, and Nyikos defined the corresponding class for the proximal game as the class of semi-proximal spaces in [36].

Definition 3.1.2 *A uniform space* $(X, \mathfrak{U})$ *is* **semi-proximal** *if Player II does not have a winning strategy in the proximal game on* $(X, \mathfrak{U})$*, denoted by (*$II \not\uparrow P(X, \mathfrak{U})$*). A topological space* X *is* **semi-proximal** *if there is a compatible uniformity* $\mathfrak{U}$ *on* X *such that* $(X, \mathfrak{U})$ *is semi-proximal.*

Note that Player II does not have a winning strategy in the proximal game on a uniform space $(X, \mathfrak{U})$ means that for every strategy σ for Player II, there exists a counter-strategy τ_σ which wins for Player I. This does not necessarily imply that Player I wins every τ_σ-sequence of the game. Only that sequence corresponding to the game played with these two strategies needs to be won by Player I.

Numerous contributions have emerged in the fields built upon Bell and Nyikos's definitions. Bell proved that every metrizable space is proximal and that every proximal space is collectionwise normal and countably paracompact. In addition, she proved that proximality is preserved under countable products and Σ-products [4]. Nyikos observed in [36] that every Corson compact space is proximal, and Clontz and Gruenhage later showed the converse to be true for compact Hausdorff spaces [9]. On the other hand, semi-proximality is not preserved under the Tychonoff product by the following two theorems in [20].

Theorem 3.1.3 *Every subspace of ω_1 is semi-proximal.*

Theorem 3.1.4 *If $A, B \subset \omega_1$, then the following are equivalent:*

1. *$A \times B$ is normal.*

2. *Either A or B is not stationary, or $A \cap B$ is stationary.*

3. *$A \times B$ is semi-proximal.*

Theorem 3.1.5 *[20] If X is semi-proximal and Y is proximal, then the product $X \times Y$ is semi-proximal.*

Theorem 3.1.6 *For a uniform space $(X, \mathfrak{U})$ and a closed uniform subspace $(C, \mathfrak{U}_C)$, we have,*

(1) $I \uparrow P(X, \mathfrak{U}) \implies I \uparrow P(C, \mathfrak{U}_C)$.

(2) $II \not\uparrow P(X, \mathfrak{U}) \implies II \not\uparrow P(C, \mathfrak{U}_C)$.

Proof. (1) was proved by J. Bell in [4]. To prove (2), let $(X, \mathfrak{U}_C)$ be a closed uniform subspace of a semi-proximal space $(X, \mathfrak{U})$. Let σ be a strategy for Player II in the proximal game on $(C, \mathfrak{U}_C)$. Let c_n be the choice of Player II using σ in inning n. For any finite sequence of entourages $U_0, \ldots, U_n$ in $\mathfrak{U}$, define $\sigma'(U_0, \ldots, U_n) = \sigma(U_0 \cap C^2, \ldots, U_n \cap C^2)$. Since $(X, \mathfrak{U})$ is semi-proximal, there exists a counter-strategy $\tau_{\sigma'}$ for Player I that defeats σ'. Define τ_σ a counter-strategy for Player I as follows. For a finite sequence $(c_0, \ldots, c_n)$ in C, let $\tau_\sigma(c_0, \ldots, c_n) = \tau_{\sigma'}(c_0, \ldots, c_n) \cap C^2$. Hence, τ_σ defeats σ. Indeed, if $\bigcap_{n<\omega} \tau_\sigma(c_0, \ldots, c_n)[c_n] \neq \emptyset$, then there is $c \in X$ such that (c_n) converges to c. Since C is closed, $c \in C$. □

Proposition 3.1.7 *If X is a disjoint union of clopen semi-proximal subspaces, then X is semi-proximal.*

Proof. Let X be a topological space that can be written as a disjoint union of clopen semi-proximal subspaces, so called the topological sum and denoted by $X = \bigoplus_{i \in I} X_i$. Then, for each $i \in I$, there is a compatible uniformity $\mathfrak{U}_i$ such that $(X_i, \mathfrak{U}_i)$ is semi-proximal. Consider the uniformity $\mathfrak{U}$ which has entourages of the form $\bigoplus_{i \in I} U_i$, where $U_i \in \mathfrak{U}_i$. Then, $\mathfrak{U}$ is a compatible uniformity on X. Let σ be any strategy for Player II in the proximal game on $(X, \mathfrak{U})$. We will define a counter-strategy τ_σ for Player I that defeats σ. In inning 0, define $\tau_\sigma(\emptyset) = X^2$. Then there exists $j \in I$ such that $\tau_\sigma(\emptyset)[x_0] = X_j$. Define a strategy σ_j for Player II as follows. For a finite sequence $(V_0, \ldots, V_n)$ in $\mathfrak{U}_j$, let $\sigma_j(V_0, \ldots, V_n) = \sigma(U_0, \ldots, U_n)$ such that $U_k = (\bigoplus_{i \neq j} X_i) \oplus V_k$, for $k \leq n$. Since $(X_j, \mathfrak{U}_j)$ is semi-proximal, there exists a counter-strategy τ_{σ_j} for Player I that defeats σ_j. In inning $n > 0$, if $x_0, \ldots, x_n$ are the element chosen by Player II using σ, define $\tau_\sigma(x_0, \ldots, x_n) = (\bigoplus_{i \neq j} X_i) \oplus \tau_{\sigma_j}(x_0, \ldots, x_n)$. Therefore, if $\bigcap \tau_\sigma(x_0, \ldots, x_n)[x_n] \neq \emptyset$, then $\bigcap \tau_{\sigma_j}(x_0, \ldots, x_n)[x_n] \neq \emptyset$. Thus, $(x_n : n \in \omega)$ is convergent in X. □

3.1.1 Equivalent definitions to the proximal game

Definition 3.1.8 *[4] Suppose $(X, \mathfrak{U})$ is a uniform space. The k-proximal game on $(X, \mathfrak{U})$, where k is a positive integer, proceeds as the proximal game on $(X, \mathfrak{U})$, except that Player II is allowed to pick points at stage n from $kU_{n-1}[x_{n-1}]$, where U_{n-1} is Player I's choice and x_{n-1} is Player II's choice from the previous round. Player I wins the k-proximal game if either the sequence $(x_n : n \in \omega)$ is convergent or $\bigcap_{n\in\omega} kU_n[x_n] = \emptyset$.*

J. Bell proved that a uniform space $(X, \mathfrak{U})$ is proximal if and only if Player I has a winning strategy in the k-proximal game on $(X, \mathfrak{U})$ for a positive integer k. This statement is also applicable for semi-proximal by the following. For the sake of completeness, we will prove the following characterization for semi-proximal spaces.

Proposition 3.1.9 *For a positive integer k, $(X, \mathfrak{U})$ is a semi-proximal if and only if Player II has no winning strategy in the k-proximal game on $(X, \mathfrak{U})$.*

Proof. Assume $(X, \mathfrak{U})$ is semi-proximal. Let k be a positive integer and σ be a strategy in the k-proximal game on $(X, \mathfrak{U})$. Define a strategy σ' for Player II in the proximal game as follows. For every finite sequence $(U_0, \ldots, U_n)$ in $\mathfrak{U}$, define $\sigma'(U_0, \ldots, U_n) = \sigma(V_0, \ldots, V_n)$, where $kV_0 \subseteq U_0$ and $kV_m \subseteq V_{m-1} \cap U_m$, for $0 < m \leq n$. Since σ' is not winning, a counter-strategy $\tau_{\sigma'}$ exists for Player I in the proximal game defeating σ'. Now, for every finite admissible sequence $(x_0, \ldots, x_n)$ chosen by Player II using σ, define $\tau_\sigma(x_0, \ldots, x_n)$ such that $k\tau_\sigma(x_0, \ldots, x_n) \subseteq \tau_{\sigma'}(x_0, \ldots, x_n)$. Therefore, τ_σ defeats σ. Hence, σ is not winning. The converse is true since $kU \supset U$ for any positive integer k and entourage U. □

Recall that uniformities have an equivalent formulation in terms of families of covers (Proposition 2.2.7). For example, let X be a Tychonoff space and consider the Stone-Čech compactification βX of X. Then, βX has a unique uniformity which is determined by the family of finite open covers. Hence, the subuniformity $\mathfrak{U}_{\beta X}$ on X corresponds to the family of finite open covers of X. For a zero-dimensional space X, $\mathfrak{U}_{\beta X}$ is corresponds to the family of clopen partitions.

Definition 3.1.10 *For a uniform space* $(X, \mathfrak{U})$*, if* $\mathcal{A}$ *is the corresponding family of covers corresponding to the uniformity* $\mathfrak{U}$ *then the proximal game on* $(X, \mathfrak{U})$ *is equivalently described as follows. In inning 0, Player I chooses a cover* $A_0 \in \mathcal{A}$ *and Player II chooses* $x_0 \in X$*. In inning* $n+1$*, Player I chooses a* $A_{n+1} \in \mathcal{A}$ *that refines* A_n *and Player II chooses* $x_{n+1} \in St(x_n, A_n)$*. We say that Player I wins the game if either* $\bigcap_{n\in\omega} St(x_n, A_n) = \emptyset$ *or the sequence* $(x_n : n \in \omega)$ *converges.*

3.2 Strengthening of Proximal and Semi-proximal Spaces

Our motivation in this work was to consider whether anything interesting could arise from considering the proximal game on uniformities with additional properties. For example, those arising from compactifications. A uniformity $\mathfrak{U}$ compatible with the topology on a space X is totally bounded if and only if there is a compactification such that $\mathfrak{U}$ is the subuniformity inherited from the compactification [12]. By requiring the uniformity witnessing the proximality or semi-proximality to be totally bounded, we obtain:

Definition 3.2.1 *A space X is* **totally proximal** *if there is a compatible totally bounded uniformity $\mathcal{U}$ on X such that $(X,\mathcal{U})$ is proximal. A space X is* **totally semi-proximal** *if there is a compatible totally bounded uniformity $\mathcal{U}$ on X such that $(X,\mathcal{U})$ is semi-proximal.*

Since the totally bounded uniformity $\mathcal{U}_{\beta X}$ inherited from the Stone Čech compactification is the finest totally bounded uniformity on X compatible with the discrete topology,ghhbbbbbb we have:

Proposition 3.2.2 *For a discrete space X:*

1. *X is totally proximal $\iff$ $I \uparrow P(X, \mathfrak{U}_{\beta X})$.*
2. *X is totally semi-proximal $\iff$ $II \not\uparrow P(X, \mathfrak{U}_{\beta X})$.*

Note that every totally proximal (totally semi-proximal) space is proximal (semi-proximal). By Theorem 2.2.9, pseudocompact proximal spaces (semi-proximal) are totally proximal (totally semi-proximal). Every discrete space is metrizable and hence proximal, but discrete spaces of measurable cardinality are not even totally semi-proximal by Corollary 4.1.4.

Theorem 3.2.3 *Every second countable metrizable space is totally proximal.*

Proof. Let X be a second countable metrizable space. Then X has a metrizable compactification Y. Let ρ be a metric generating the topology of Y. Let $\mathfrak{U}$ be the uniformity inherited from the metric ρ on Y. Therefore, $\mathfrak{U}$ is totally bounded. For every n, let $U_n = \{\langle x, y\rangle \in Y^2 : \rho(x, y) < 1/2^n\}$. Then U_n is an element of $\mathfrak{U}$. Consider the totally bounded uniformity $\mathfrak{V}$ that induced on X as a subspace of $(Y, \mathfrak{U})$. We define a winning strategy τ in the proximal game on $(X, \mathfrak{V})$ as follows. For every admissible finite sequence of points $x_1, x_2, \ldots, x_n$ from X of length n, let $\tau(x_0, x_1, \ldots, x_n) = U_{n+1} \cap X^2$. This is a Markov winning strategy for Player I. A proximal sequence $(x_n : n \in \omega)$ defined from this strategy is Cauchy. So either the sequence converges or $\bigcap_{n \in \omega} U_n[x_n] = \emptyset$. Therefore, $(X, \mathfrak{V})$ is totally proximal. □

Note that totally proximality and totally semi-proximality are closed hereditary. Indeed, if $(X, \mathfrak{U})$ is a totally proximal (totally semi-proximal) space and C is a closed subset of X. Then, by Theorem 2.2.8, $(C, \mathfrak{U}_C)$ is totally bounded uniform subspace. Thus, by Theorem 3.1.6, $(C, \mathfrak{U}_C)$ is totally proximal (totally semi-proximal).

Chapter 4

The Proximal Game and Its Connections to Other Games

This chapter explores the relationships between the proximal game and two known games in set theory and topological dynamics: the Galvin game and the Gruenhage game.

4.1 The Proximal and the Galvin Games

In [26], J. R. Isbell mentions a useful rule of thumb in the theory of uniform spaces: *All counterexamples are discrete.* Therefore, understanding what happens in discrete spaces is essential. We show that, in the discrete space, the proximal game with respect to totally bounded uniformities is just the Galvin game.

The Galvin game, also known as the weak Ulam game in [19], was defined by Fred Galvin and introduced in [41]. For an infinite set X, define the

Galvin game as follows. Player I starts by playing a finite partition $\mathcal{A}_0$ of X, and then Player II chooses an element A_0 from $\mathcal{A}_0$ with $|A_0| > 1$. In each subsequent inning, $n > 0$, Player I plays a finite partition $\mathcal{A}_n$ of X refining $\mathcal{A}_{n-1}$ (or equivalently a finite partition of the set chosen by Player II in the previous inning), and Player II responds by picking an element $A_n \in \mathcal{A}_n$ with $A_n \subseteq A_{n-1}$ and $|A_n| > 1$. Player I wins the play of the game if the intersection of the sets $\{A_n : n < \omega\}$ is empty. If Player I has a winning strategy on a set X, we denote this as I $\uparrow G(X)$. Similarly, if Player II has no winning strategy in the Galvin game on X, we denote II $\not\uparrow G(X)$.

For a discrete space X, the family of covers corresponding to the uniformity $\mathfrak{U}_{\beta X}$ is the family of finite partitions of X. Thus, we have the following characterizations for a discrete space X.

Theorem 4.1.1 *For a discrete space X,*

(1) $I \uparrow G(X) \iff I \uparrow P(X,\mathcal{U}_{\beta X})$.

(2) $II \uparrow G(X) \iff II \uparrow P(X,\mathcal{U}_{\beta X})$.

Proof. To prove (1), let X be an infinite set where Player I has a wining strategy τ' in the Galvin game. We inductively construct a winning strategy τ for Player I in the proximal game on $(X,\mathcal{U}_{\beta X})$ using Definition 3.1.10. In inning 0, let $\tau(\emptyset) = \tau'(\emptyset)$. In inning $n+1$, let $(\tau(\emptyset), x_0, \ldots, \tau(x_0, \ldots, x_{n-1}), x_n)$ be a partial play of the proximal game where $\tau(x_0, \ldots, x_k) = \tau'(A_0, \ldots, A_k)$ and $A_k = \mathrm{St}(x_k, \tau(x_0, \ldots, x_{k-1}))$, for all $k < n$. Let $A_n = \mathrm{St}(x_n, \tau(x_0, \ldots, x_{n-1}))$. Therefore, $(\tau'(\emptyset), A_0, \ldots, \tau'(A_0, \ldots, A_{n-1}), A_n)$ is a partial play of the game in the Galvin game. Now, define $\tau(x_0, \ldots, x_n) = \tau'(A_0, \ldots, A_n)$. Since τ' is winning, then $\bigcap_{n<\omega} A_n = \emptyset$. Thus, τ is winning for Player I.

Now, let τ be the winning strategy for Player I in the proximal game on $(X,\mathcal{U}_{\beta X})$. Define a winning strategy τ' for Player I in the Galvin game on X inductively as follows. In inning 0, define $\tau'(\emptyset) = \tau(\emptyset)$, and Player II chooses $A_0 \in \tau'(\emptyset)$. If A_0 is finite, Player I wins the game by partitioning A_0 into singletons. In inning $n+1$, let $(\tau'(\emptyset), A_0, \ldots, \tau'(A_0, \ldots, A_{n-1}), A_n)$ be a partial play of the Galvin game such that $\tau'(A_0, \ldots, A_k) = \tau(x_0, \ldots, x_k)$, for every $k < n$, where $x_m \in A_m$, for all $m \leq k$. Let $x_n \in A_n \setminus \{x_k : k < n\}$. Thus, $(\tau(\emptyset), x_0, \ldots, \tau(x_0, \ldots, x_{n-1}), x_n)$ is a partial play of the game in the proximal game where Player I uses τ. Define $\tau'(A_0, \ldots, A_n) = \tau(x_0, \ldots, x_n)$. Since τ is winning and since $(x_n : n \in \omega)$ is a non-trivial sequence,

$$\emptyset = \bigcap_{n<\omega} \mathrm{St}(x_n, \tau(x_0, \ldots, x_{n-1})) = \bigcap_{n<\omega} A_n.$$

To prove (2), let σ' be a winning strategy for Player II in the Galvin game on X. We inductively construct a winning strategy σ for Player II in the proximal game. Since σ' is winning, the choices using σ' are infinite. Let $\mathcal{A}_0$ be a finite partition of X, define $\sigma(\mathcal{A}_0) = x_0 \in \sigma'(\mathcal{A}_0)$. Assume that $(\mathcal{A}_0, x_0, \ldots, x_{n-1}, \mathcal{A}_n)$ is a partial play of the proximal game where Player II uses σ and $x_k = \sigma(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}) \in \sigma'(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1})$, for all $k < n$. Now, define $\sigma(\mathcal{A}_0, \ldots, \mathcal{A}_n) = x_n \in \sigma'(\mathcal{A}_0, \ldots \mathcal{A}_n) \setminus \{x_k : k < n\}$. Since σ' is winning, then $\emptyset \neq \bigcap_{n<\omega} \sigma'(\mathcal{A}_0, \ldots \mathcal{A}_n) = \bigcap_{n<\omega} \mathrm{St}(x_n, \mathcal{A}_n)$. Since X is discrete, then $(x_n : n \in \omega)$ is not convergent. Thus, σ is winning.

Assume that Player II has a winning strategy σ in the proximal game on $(X,\mathcal{U}_{\beta X})$. We inductively construct a winning strategy σ' for Player II in the Galvin game on X. Let $\mathcal{A}_0$ be a finite partition of X, define $\sigma'(\mathcal{A}_0) = \mathrm{St}(\sigma(\mathcal{A}_0), \mathcal{A}_0)$. Assume that $(\mathcal{A}_0, A_0, \ldots, A_{n-1}, \mathcal{A}_n)$ is a partial play of the

Galvin game where Player II uses σ and $A_k = \mathrm{St}(\sigma(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}), \mathcal{A}_{k-1})$, for all $k < n$. Now, define $\sigma'(\mathcal{A}_0, \ldots, \mathcal{A}_n) = \mathrm{St}(\sigma(\mathcal{A}_0, \ldots, \mathcal{A}_n), \mathcal{A}_n)$. Since σ is winning, $\emptyset \neq \bigcap_{n<\omega} \mathrm{St}(\sigma(\mathcal{A}_0, \ldots, \mathcal{A}_n), \mathcal{A}_n)$. Therefore, σ' is winning. $\square$

The following theorems are known results about the Galvin game by F. Galvin and R. Laver, respectively. So we obtain the corresponding corollary for the proximal game played on a totally bounded uniformity using Theorem 4.1.1.

Theorem 4.1.2 *[41] For an ordinal κ,*

(a) $I \uparrow G(\kappa) \iff \kappa$ is countable.

(b) $II \not\uparrow G(\kappa)$ for $\kappa \leq 2^\omega$.

(c) if κ is measurable, then $II \uparrow G(\kappa)$.

It is worth mentioning that the proof of (b) was an inspiration for Example 5.2.2. In fact, we can get (b) as a corollary to a result in Section 5.2.1.

Theorem 4.1.3 *[41] If it is consistent that there is a measurable cardinal, then it is consistent that $II \uparrow G(\omega_2)$.*

Corollary 4.1.4 *For a discrete space X,*

1. *$I \uparrow (X, \mathcal{U}_{\beta X}) \iff X$ is countable.*
2. *if $|X| \leq 2^\omega$, then X is totally semi-proximal but not totally proximal.*
3. *If X of cardinality greater than the first measurable cardinal, then X is not totally semi-proximal.*
4. *If it is consistent that there is a measurable cardinal, then it is consistent that $(\omega_2, \mathfrak{U}_\beta)$ is not totally semi-proximal.*

4.2 The Proximal and the Gruenhage Games

The relationship between the proximal game and the Gruenhage game has been investigated in [4, 8, 36]. We aim to highlight both foundational findings and establish results with respect to total proximality and total semi-proximality.

Gruenhage defined this game in his doctoral dissertation to define a class of spaces generalizing first-countability [16]. For a topological space X, the Gruenhage game at a point $x \in X$ is as follows. In inning 0, Player I chooses an open set V_0 with $x \in V_0$, and Player II chooses $x_0 \in V_0$. In inning $n > 1$, Player I chooses an open set $V_n \subseteq V_{n-1}$ with $x \in V_n$, and Player II chooses $x_n \in V_n$. A strategy for Player I is winning if the sequence $(x_n : n \in \omega)$ converges to x. If Player I has a winning strategy at all $x \in X$, denoted by $(\text{I} \uparrow Gr(X, x))$, then the space X is called a W-space. It is called a w-space if Player II does not have a winning strategy at each point $x \in X$, denoted by $(\text{II} \not\uparrow Gr(X, x))$.

Theorem 4.2.1 *[4] Every proximal space is a W-space.*

Theorem 4.2.2 *[36] Every semi-proximal space is a w-space.*

Nyikos proved that every w-space (W-space) with a single non-isolated point is semi-proximal (proximal) with respect to the universal uniformity. He claimed that these results can be extended to scattered paracompact spaces. We will prove this claim and study this class in terms of total proximal and total semi-proximal spaces. First, we have the following lemma.

Lemma 4.2.3 *If X is a paracompact scattered space with height α, then there is a clopen cover $\mathcal{W}$ such that for every $W \in \mathcal{W}$ there is $\beta < \alpha$ with $|W \cap X^{(\beta)}| = 1$.*

Proof. By Theorem 2.2.2, X is zero dimensional. Let $\mathcal{B}$ be a clopen base. Let $x \in X$. Then, there exists $\beta_x < \alpha$ such that $x \in X^{(\beta_x)} \setminus X^{(\beta_x+1)}$. Now let $\mathcal{W}_x = \{B \in \mathcal{B} : B \text{ is clopen and } B \cap X^{(\beta_x)} = \{x\}\}$, then $\mathcal{W} = \bigcup_{x \in X} \mathcal{W}_x$ is a clopen cover of X. □

Theorem 4.2.4 *[36] Every scattered paracompact W-space is proximal.*

Proof. Let X be a scattered paracompact W-space with height α. It is trivial if $\alpha = 0$ since $X = \emptyset$. For $\alpha = 1$, the space is discrete. Therefore, it is proximal since it is metrizable. If $\alpha > 0$, assume it is true for every $\beta < \alpha$. Let $\mathfrak{U}$ be the universal uniformity on the space X. We will define a winning strategy τ for Player I in the proximal game on $(X, \mathfrak{U})$ using Definition 3.1.10. In inning $n = 0$, consider a clopen cover $\mathcal{W}$ that satisfies Lemma 4.2.3. Define $\tau(\emptyset) = \mathcal{W}_0$, where $\mathcal{W}_0$ is a disjoint open refinement of $\mathcal{W}$. Note that if α is a limit ordinal, then W is a paracompact scattered with height $< \alpha$ and hence proximal by the inductive hypothesis, for all $W \in \mathcal{W}_0$. Therefore, X is proximal since it is a disjoint union of clopen proximal spaces. Otherwise, there is $\beta < \alpha$ such that $\alpha = \beta + 1$ and $|W \cap X^{(\beta)}| \leq 1$, for all $W \in \mathcal{W}_0$. Let $x_0 \in X$ be the choice of Player II and let $W_0 = \text{St}(x_0, \mathcal{W}_0)$. If $W_0 \cap X^{(\beta)} = \emptyset$, then the rest of the game would be inside a clopen paracompact scattered W_0 with height $< \alpha$ and hence proximal by the inductive hypothesis. Otherwise, there exists $z \in X$ such that $W_0 \cap X^{(\beta)} = \{z\}$. Since W_0 is a W-space, there is a winning strategy τ_z for Player I in the Gruenhage game on W_0 at z where

Player I plays basic clopen sets. Now, we will define τ in inning $n \geq 1$ using τ_z. In inning 1, let $\tau(x_0) = \{\tau_z(x_0)\} \cup \{W_0 \setminus \tau_z(x_0)\} \cup \tau(\emptyset) \setminus \{W_0\}$. In inning $n+1$, let $(\tau(\emptyset), x_0, \ldots, \tau(x_0, \ldots, x_{n-1}), x_n)$ be a partial play of the proximal game where $\tau(x_0, \ldots, x_{k-1})$ is a disjoint open refinement of $\tau(x_0, \ldots, x_{k-2})$ containing $\tau_z(x_0, \ldots, x_{k-1})$, for all $k \leq n$. Let $G_{k+1} = \tau_z(x_0, \ldots, x_k)$, for all $k \leq n$. Define $\tau(x_0, \ldots, x_n) = \{G_{n+1}\} \cup \{G_n \setminus G_{n+1}\} \cup \tau(x_0, \ldots, x_{n-1}) \setminus \{G_n\}$. If there exists n such that Player II chooses $x_{n+1} \in G_n \setminus G_{n+1}$, then the rest of the game would be inside a clopen scattered subspace $G_n \setminus G_{n+1}$ with height $< \alpha$. Hence, it is proximal by the inductive hypothesis. Otherwise, $x_{n+1} \in G_{n+1}$, for all n. Since τ_z is winning, the sequence $(x_n : n \in \omega)$ converges to z. Hence, X is proximal. □

Note that not every scattered paracompact W-space is totally proximal. For example, ω_1 with the discrete topology is a scattered paracompact W-space that is not totally proximal by Corollary 4.1.4. Otherwise, we have the following partial converse of Theorem 4.2.1 with respect to total proximality.

Theorem 4.2.5 *If X is a W-space with a single non-isolated point p, then X is totally proximal if and only if every neighbourhood of p is co-countable.*

Proof. Assume that X is a W-space with a single non-isolated point p such that $X \setminus G$ is countable for every $G \in N_p$, where N_p is the set of all clopen neighbourhoods of p since X is zero-dimensional. For every $G \in N_p$, define a finite clopen partition $\mathcal{W}_G = \{G, X \setminus G\}$. Let $\mathfrak{U}$ be the uniformity inherited from the Stone-Čech compactification of X. We will define a winning strategy τ for Player I by Definition 3.1.10. In inning 0, define $\tau(\emptyset) = \mathcal{W}_G$, where $G \in N_p$. If Player II ever chooses inside the complement of the neighbourhood of p, then the rest of the game would be inside a totally proximal space since it is a countable discrete space by Corollary 4.1.4. So, assume this is not the case for all innings (i.e. Player II always chooses inside a neighbourhood of p). Since G is a W-space, there is a winning strategy τ_p in the Gruenhage game on X where Player I plays basic clopen sets containing p. In inning $n+1$, let $(\tau(\emptyset), x_0, \ldots, \tau(x_0, \ldots, x_{n-1}), x_n)$ be a partial play of the game where Player I uses τ, and $\tau(x_0, \ldots, x_{k-1})$ refines $\mathcal{W}_{\tau_p(x_0,\ldots,x_{k-1})}$, for all $k \leq n$. Define $\tau(x_0, \ldots, x_n)$ to be the common clopen refinement of $\tau(x_0, \ldots, x_{n-1})$ and $\mathcal{W}_{\tau_p(x_0,\ldots,x_n)}$. Since τ_p is winning, $(x_n : n \in \omega)$ converges to p. Thus, X is totally proximal. The converse is true since $X \setminus G$ is a closed subspace of a totally proximal space, for all $G \in N_p$. □

Corollary 4.2.6 *If X is a W-space with finitely many non-isolated points, then X is totally proximal if and only if every open neighbourhood containing the isolated points is co-countable.*

Proof. Let $P = \{p_i : i < k\}$ be the set of the non-isolated points in X. Assume X is a W-space such that G is co-countable for every neighbourhood G containing P. Let N_{p_i} be the clopen base at p_i. Then, there exists a finite clopen partition $\{G_i : i < k\} \cup \{X \setminus (\bigcup_{i<k} G_i)\}$, where $G_i \in N_{p_i}$, and $G_i \cap G_j = \emptyset$ for all $i \neq j$ in k. Note that, for each $i < k$, G_i is a W-space with a single non-isolated point; hence, it is totally proximal by Theorem 4.2.5. Moreover, $X \setminus (\bigcup_{i<k} G_i)$ is a totally proximal space since it is a countable discrete space by Corollary 4.1.4. Thus, X is a disjoint union of clopen totally proximal subspaces. Hence, X is totally proximal. The converse is true since the complement of a neighbourhood of P is a closed subspace of a totally proximal space. □

The proof of the following theorem is analogous to the proof of Theorem 4.2.4, but we will present it for the sake of completeness.

Theorem 4.2.7 *[36] Every scattered paracompact w-space is semi-proximal.*

Proof. Let X be a scattered paracompact w-space with height α. It is trivial if $\alpha = 0$ since $X = \emptyset$. For $\alpha = 1$, the space is discrete. Thus, it is semi-proximal since it is metrizable. If $\alpha > 0$, assume the theorem is true for every $\beta < \alpha$. Let $\mathfrak{U}$ be the universal uniformity on the space X. Let σ be a strategy for Player II in the proximal game on $(X, \mathfrak{U})$. We will define a counter-strategy τ_σ for Player I to defeat σ using Definition 3.1.10. In inning 0, consider a clopen cover $\mathcal{W}$ that satisfies Lemma 4.2.3. Define $\tau_\sigma(\emptyset) = \mathcal{W}_0$, where $\mathcal{W}_0$ is a disjoint open refinement of $\mathcal{W}$. Note that if α is a limit ordinal, then W is a paracompact scattered with height $< \alpha$; hence semi-proximal, for all $W \in \mathcal{W}_0$. Therefore, X is semi-proximal since it is a disjoint union of clopen semi-proximal spaces. Otherwise, there is $\beta < \alpha$ such that $\alpha = \beta + 1$ and $|W \cap X^{(\beta)}| \leq 1$, for all $W \in \mathcal{W}_0$. Let $x_0 = \sigma(\mathcal{W}_0)$ and let $W_0 = \mathrm{St}(x_0, \mathcal{W}_0)$. If $W_0 \cap X^{(\beta)} = \emptyset$, then the rest of the game would be inside a clopen paracompact scattered w-space W_0 with height $< \alpha$ and hence semi-proximal by the inductive hypothesis. Otherwise, there exists $z \in X$ such that $W_0 \cap X^{(\beta)} = \{z\}$. In inning 1, since W_0 is a w-space, Player II does not have any winning strategy in the Gruenhage game at z. Fix a strategy σ_z for Player II in the Gruenhage game on W_0 at z. Since σ_z is not winning, there is a counter winning strategy τ_{σ_z} for Player I that defeats σ_z where Player I plays basic clopen sets containing z. Let $\tau_\sigma(x_0) = \{\tau_{\sigma_z}(x_0)\} \cup \{W_0 \setminus \tau_{\sigma_z}(x_0)\} \cup \tau_\sigma(\emptyset) \setminus \{W_0\}$. For $n \geq 1$, in inning $n+1$, let $(\tau_\sigma(\emptyset), x_0, \ldots, \tau_\sigma(x_0, \ldots, x_{n-1}), x_n)$ be a partial play of the proximal game where Player I uses τ_σ and Player II uses σ, and $\tau_\sigma(x_0, \ldots, x_{k-1})$ is a disjoint open refinement of $\tau_\sigma(x_0, \ldots, x_{k-2})$ containing $\tau_{\sigma_z}(x_0, \ldots, x_{k-1})$, for all $k \leq n$. Let $G_{k+1} = \tau_{\sigma_z}(x_0, \ldots, x_k)$, for all $k \leq n$. Now, define $\tau_\sigma(x_0, \ldots, x_n) = \{G_{n+1}\} \cup \{G_n \setminus G_{n+1}\} \cup \tau_\sigma(x_0, \ldots, x_{n-1}) \setminus \{G_n\}$. Let Player II chooses x_{n+1} using σ. If $x_{n+1} \in G_n \setminus G_{n+1}$, then the rest of the game would be inside a clopen scattered subspace $G_n \setminus G_{n+1}$ with height $< \alpha$. Hence, it is semi-proximal by the inductive hypothesis. Therefore, Player I would find a way to defeat σ. Otherwise, $x_{n+1} \in G_{n+1}$, for all n. Since τ_{σ_z} is winning, the sequence $(x_n : n \in \omega)$ converges to z. Therefore, τ_σ defeats σ; hence, X is semi-proximal. □

By Corollary 4.1.4, if κ is a measurable cardinal, then κ with the discrete topology is a scattered paracompact w-space that is not totally semi-proximal. Otherwise, we have the following partial converse of Theorem 4.2.2 with respect to total semi-proximality.

Theorem 4.2.8 *If X is a w-space with a single non-isolated point p, then X is totally semi-proximal if and only if $X \setminus G$ is totally semi-proximal for every neighbourhood G of p.*

Proof. Assume that X is a w-space with a single non-isolated point p such that $X \setminus G$ is totally semi-proximal for every $G \in N_p$, where N_p is the set of all clopen neighbourhoods of p. For every $G \in N_p$, define a finite clopen partition $\mathcal{W}_G = \{G, X \setminus G\}$. Let $\mathfrak{U}$ be the uniformity inherited from the Stone-Čech compactification of X. Let σ be a strategy for Player II in the proximal game on $(X, \mathfrak{U})$. We will define a counter-strategy τ_σ for Player I to defeat σ using Definition 3.1.10. In inning 0, define $\tau_\sigma(\emptyset) = \mathcal{W}_G$, where $G \in N_p$.

If Player II ever chooses inside the complement of the neighbourhood of p, then the rest of the game would be inside a totally semi-proximal space. So, assume this is not the case for all innings (i.e. Player II always chooses inside a neighbourhood of p using σ). Since G is a w-space, Player II does not have a winning strategy in the Gruenhage game on G at p. Fix a strategy σ_p for Player II on G at p. Since σ_p is not winning, there is a counter strategy τ_{σ_p} for Player I that defeats σ_p where Player I plays basic clopen sets containing p. In inning $n+1$, let $(\tau_\sigma(\emptyset), x_0, \ldots, \tau_\sigma(x_0, \ldots, x_{n-1}), x_n)$ be a partial play of the game where Player I uses τ_σ and Player II uses σ, and $\tau_\sigma(x_0, \ldots, x_{k-1})$ refines $\mathcal{W}_{\tau_{\sigma_p}(x_0,\ldots,x_{k-1})}$, for all $k \leq n$. Define $\tau_\sigma(x_0, \ldots, x_n)$ to be the common clopen refinement of $\tau_\sigma(x_0, \ldots, x_{n-1})$ and $\mathcal{W}_{\tau_{\sigma_p}(x_0,\ldots,x_n)}$. Since τ_{σ_p} is winning against σ_p, $(x_n : n \in \omega)$ converges to p. Thus, X is totally semi-proximal. The converse is true since $X \setminus G$ is a closed subspace of a totally semi-proximal space for every $G \in N_p$. □

Corollary 4.2.9 *If X is a w-space with finitely many non-isolated points, then X is totally semi-proximal if and only if the complement of every open set containing the isolated points is totally semi-proximal.*

Proof. Let $P = \{p_i : i < k\}$ be the set of all non-isolated points in X. Assume X is a w-space such that $X \setminus G$ is totally semi-proximal for every neighbourhood G of P. Let $\mathfrak{U}$ be the uniformity inherited from the Stone-Čech compactification of X. Let N_{p_i} be the set of all clopen neighbourhoods of p_i. Thus, there exists a finite clopen partition $\{G_i : i < k\} \cup \{X \setminus (\bigcup_{i<k} G_i)\}$, where $G_i \in N_{p_i}$, and $G_i \cap G_j = \emptyset$ for all $i \neq j$ in k. Note that, for each $i < k$, G_i is a w-space with a single non-isolated point; hence, it is totally semi-proximal with respect to the uniformity inherited from the Stone-Čech compactification of X, by Theorem 4.2.8. Moreover, $X \setminus (\bigcup_{i<k} G_i)$ is a totally semi-proximal space since $\bigcup_{i<k} G_i$ is a neighbourhood of P. Hence, X is totally semi-proximal since it is a disjoint union of clopen totally semi-proximal subspaces. The converse is true since the complement of a neighbourhood of P is a closed subspace of a totally semi-proximal space. □

4.2.1 Limited information strategies

The relationship between the proximal and the Gruenhage games in situations where Player I has a winning limited information strategy was explored in [8]. For a topological space X and for a compatible uniform space $(X, \mathfrak{U})$, Clontz proved that if Player I has a $2k$-Markov ($2k$-stationary) winning strategy in the proximal game on $(X, \mathfrak{U})$, then Player I has a k-Markov (k-stationary) winning strategy in the Gruenhage game at all $x \in X$. We examine the relationship between these games in cases where Player II does not have a winning limited information strategy. We prove that if Player II has no winning Markov and 2-coding strategies in the proximal game on $(X, \mathfrak{U})$, then Player II has no Markov and coding winning strategies in the Gruenhage game at all $x \in X$, respectively. Sharma characterized w-spaces as α_2-Fréchet, which was used in the proof of Theorem 4.2.2.

Theorem 4.2.10 *[42] A space X is a w-space if and only if it is α_2-Fréchet.*

We show that this holds for Markov and coding strategies. For a uniform space $(X, \mathfrak{U})$, we prove that Player II having no Markov or 2-coding winning strategies in the proximal game on $(X, \mathfrak{U})$ implies α_2-Fréchet. We also prove that the space is Fréchet if Player II has no coding winning strategy in the proximal game on $(X, \mathfrak{U})$ while it is still open whether it implies α_2.

Theorem 4.2.11 *For a topological space X,*

1. *X is α_2-Fréchet $\iff$ $II \underset{\text{Markov}}{\not\uparrow} Gr(X, x)$, for all $x \in X$.*
2. *X is α_2-Fréchet $\iff$ $II \underset{\text{coding}}{\not\uparrow} Gr(X, x)$, for all $x \in X$.*

Proof. To prove (1), let $x \in X$ and let $(A_n : n \in \omega)$ be such that $x \in \overline{A}_n$, for all n. Define a Markov strategy σ for Player II in the Gruenhage game on X at x as follows. For every basic open set G containing x and for every $n \in \omega$, define $\sigma(G, n) \in A_n \cap G$. Since σ is not winning, $(\sigma(G, n) : n \in \omega)$ converges to x. Thus, X is α_2-Fréchet by Proposition 2.2.1. The other direction follows directly from Theorem 4.2.10.

To prove (2) Let $x \in X$ and let $(A_n : n \in \omega)$ be such that $x \in \overline{A}_n$, for all n. Assume, without loss of generality, $(A_n : n \in \omega)$ is pairwise disjoint. Define a coding strategy σ for Player II in the Gruenhage game on X at x as follows. Let G_0 be a basic open set containing x. Define $\sigma(G_0) \in A_0 \cap G_0$. Suppose we have defined x_n to be the choice of Player II using σ such that there is k_n where $x_n \in A_{k_n}$. In inning $n+1$, let G_{n+1} be a basic open set containing x, define $\sigma(x_n, G_{n+1}) \in A_{k_{n+1}} \cap G_{n+1}$. Since σ is not wining, $(\sigma(x_n, G_{n+1}) : n \in \omega)$ converges to x. Thus, X is α_2-Fréchet by Proposition 2.2.1. The other direction follows from Theorem 4.2.10. □

Theorem 4.2.12 *For a uniform space $(X, \mathfrak{U})$,*

1. *$II \underset{\text{coding}}{\not\uparrow} P(X, \mathfrak{U}) \implies X$ is Fréchet.*
2. *$II \underset{\text{2-coding}}{\not\uparrow} P(X, \mathfrak{U}) \implies X$ is α_2-Fréchet.*

3. $\mathrm{II} \underset{\text{Markov}}{\not\uparrow} P(X,\mathfrak{U}) \implies X$ *is* α_2*-Fréchet.*

Proof. (3) was proved in [36]. To prove (1), suppose that X is not Fréchet, then there exists $A \subseteq X$, and there exists $p \in \overline{A}$ such that there is no sequence of elements of A converges to p. Let $U_0 \in \mathfrak{U}$, then define $\sigma(U_0) \in A \cap U_0[p]$. For each $n \geq 1$, suppose we have defined $\sigma(x_{n-1}, U_n)$ such that it is either p or an element of $A \cap U_n[p]$. Let $U_{n+1} \subseteq U_n$ and let $x_n = \sigma(x_{n-1}, U_n)$. If $x_n \in A \cap U_n[p]$, define $\sigma(x_n, U_{n+1}) = p$. Otherwise, define $\sigma(x_n, U_{n+1})$ to be an element of $A \cap U_{n+1}[p]$. Then the sequence $(\sigma(x_n, U_{n+1}) : n \in \omega)$ cannot be convergent and $p \in \bigcap_{n\in\omega} U_n[\sigma(x_n, U_n)]$. Thus, σ is winning.
To prove (2), suppose that X is not α_2-Fréchet. Thus, there exists $x \in X$, and there exists a pairwise disjoint $\{A_n : n \in \omega\}$ with $x \in \overline{A}_n$, for all n, such that there is no sequence converging to x that intersect each A_n. Let $U_0 \in \mathfrak{U}$, define $\sigma(U_0) \in A_0 \cap U_0[x]$. For $U_1 \subseteq U_0$, define $\sigma(U_0, x_0, U_1) = x$. For $n \geq 2$, let $(x_{n-2}, U_{n-1}, x_{n-1}, U_n)$ be a partial play of the game where Player II using σ such that there exists $k_n \in \omega$ where either $x_{n-2} \in A_{k_n}$ and $x_{n-1} = x$, or $x_{n-1} \in A_{k_n} \cap U_{n-1}[x]$ and $x_{n-2} = x$. If $x_{n-1} \in A_{k_n} \cap U_{n-1}[x]$, then define $x_{n+1} = \sigma(x_{n-2}, U_{n-1}, x_{n-1}, U_n) = x$. Otherwise, define x_{n+1} to be an element of $A_{k_n+1} \cap U_{n+1}[x]$. Then the sequence $(x_n : n \in \omega)$ cannot be convergent and $x \in \bigcap_{n\in\omega} U_n[x_n]$. Thus, σ is winning. □

Theorem 4.2.11 along with Theorem 4.2.12 established the following:

Corollary 4.2.13 *Let $k \in \omega$. For a uniform space $(X, \mathfrak{U})$ and for all $x \in X$,*

1. $\mathrm{II} \underset{\text{Markov}}{\not\uparrow} P(X,\mathfrak{U}) \implies \mathrm{II} \underset{\text{Markov}}{\not\uparrow} Gr(X,x)$.
2. $\mathrm{II} \underset{\text{2-coding}}{\not\uparrow} P(X,\mathfrak{U}) \implies \mathrm{II} \underset{\text{2-coding}}{\not\uparrow} Gr(X,x)$.

Chapter 5

Semi-proximal Spaces and Normality

We consider the relationship between normality and semi-proximality. In Section 5.1, we give a consistent example of a first countable locally compact Dowker space that is not semi-proximal. Section 5.2 provides two ZFC examples of semi-proximal non-normal spaces that answer a question of Nyikos (Problem 13, [36]). Section 5.3 discusses the relationship between the class of semi-proximal and normal spaces in subspaces of finite powers of ω_1. We show that normality and semi-proximality are equivalent in the finite product of subspaces of ω_1. In addition, we prove that normal subspaces of finite powers of ω_1 are semi-proximal, but the converse fails as one of our counterexamples is a subspace of $(\omega+1)\times\omega_1$. In Section 5.4, we prove that every semi-proximal is normal for pseudocompact spaces.

5.1 A not semi-proximal Dowker space

We consider some other closely related questions to Nyikos question concerning more broadly the relationship between the class of normal spaces and the class of semi-proximal spaces. For any semi-proximal space X, $X \times (\omega + 1)$ is semi-proximal by Theorem 3.1.5. Recall Dowker's theorem:

Theorem 5.1.1 (Dowker's Theorem,[40]) *The following are equivalent for a normal space X:*

1. *If $D_0 \supset D_1 \supset \ldots$ is a decreasing sequence of closed sets in X with $\bigcap_{n\in\omega} D_n = \emptyset$, then for each $n \in \omega$, there is an open $U_n \supset D_n$ with $\bigcap_{n\in\omega} U_n = \emptyset$.*

2. *X is countably paracompact.*

3. *$X \times Y$ is normal for all infinite compact metric space Y.*

Therefore, a counter-example to Nyikos's question could be obtained by constructing a semi-proximal Dowker space. However, it is still open whether there exists a semi-proximal Dowker space. Answering this question would help to demonstrate the relationship between normality and countably paracompactness in semi-proximal spaces. Nevertheless, since semi-proximal spaces are Fréchet, we can deduce that not every normal space is semi-proximal. For example, consider $\omega_1 + 1$ or any other normal non-Fréchet space. However, the question is more interesting for the class of Fréchet spaces. We present a consistent example of first countable, locally compact (hence Fréchet and much more) Dowker space that is not semi-proximal. We do not know of any other first countable or even Fréchet normal space that is not semi-proximal.

We describe a de Caux type Dowker space constructed by enhancing $\clubsuit$ to a principle we denote $\clubsuit^*$.

Definition 5.1.2 *The principle $\clubsuit^*$ is the statement that there is a sequence $\{C_\alpha : \alpha \in Lim\}$, where $C_\alpha \subset \alpha$, has order type ω, and $\sup(C_\alpha) = \alpha$ such that for every uncountable subset A of ω_1, $\{\alpha \in \omega_1 : |A \cap C_\alpha| = \omega\}$ contains a club.*

Lemma 5.1.3 $\diamondsuit^* \implies \clubsuit^*$.

Proof. Let $\{\mathcal{A}_\alpha : \alpha \in \omega_1\}$ be a $\diamondsuit^*$ sequence. For $\alpha \in \mathrm{Lim}$, consider an increasing sequence C_α that is cofinal in α and satisfying $C_\alpha \cap A$ is infinite for all $A \in \mathcal{A}_\alpha$ that are cofinal in α. Let $X \in [\omega_1]^{\omega_1}$, then $C_1 = \{\alpha : X \cap \alpha \in \mathcal{A}_\alpha\}$ is a club in ω_1. Note that $C_2 = \{\alpha \in \mathrm{Lim} : X \cap \alpha \text{ is cofinal in } \alpha\}$ is also a club. Hence, $|(X \cap \alpha) \cap C_\alpha| = |X \cap C_\alpha| = \omega$, for every $\alpha \in C_1 \cap C_2$. Then, $\{C_\alpha : \alpha \in \mathrm{Lim}\}$ will be a $\clubsuit^*$ sequence. $\square$

Our space is just one of the standard de Caux spaces constructed from our $\clubsuit^*$ sequence $\{C_\alpha : \alpha \in \omega_1\}$: it has as its underlying set $X = \omega_1 \times \omega$, and the topology is defined so that

1. $C_\alpha \times \{n-1\}$ converges to $\langle \alpha, n \rangle$ for each limit ordinal $\alpha \in \omega_1$.

2. $\langle \alpha, n \rangle$ is isolated for all successor ordinals α.

3. $(\alpha + 1) \times \omega$ is clopen for all $\alpha \in \omega_1$.

4. The space is first countable and locally compact.

We follow [50] to construct such a space. We will define a topology on X as follows. If $n = 0$ or $\alpha \in \text{Succ}$, define the local base $\mathcal{B}(\alpha, n) = \{\{\langle\alpha, n\rangle\}\}$. Assume that $\alpha \in \text{Lim}$ and $\mathcal{B}(\beta, n)$ has been defined, for all $\beta < \alpha$ and $n \in \omega$, so that if τ_α is the topology on $\alpha \times \omega$ having $\bigcup\{\mathcal{B}(\beta, n) : \beta < \alpha, n \in \omega\}$ as a base, then

(1) τ_α is locally compact and first countable.

(2) For every $B \in \mathcal{B}(\beta, n)$, $B \subset (\beta \times n) \cup \{\langle\beta, n\rangle\}$.

Now, we need to define a countable local base at $\langle\alpha, n\rangle$, for all $n \geq 1$. If $n = 1$, let

$$\mathcal{B}(\alpha, 1) = \{\{\langle\alpha, 1\rangle\} \cup \{\langle\beta, 0\rangle : \beta \in C_\alpha \setminus F\} : F \subset C_\alpha \text{ finite}\}.$$

Assume we have defined a countable local base at $\langle\alpha, k\rangle$, for all $k < n$ such that each element in the local base is compact and inside $(\alpha \times k) \cup \{\langle\alpha, k\rangle\}$. Note that $C_\alpha \times \{n-1\}$ is a closed discrete set in $\alpha \times \omega$. Since τ_α is countable and regular, there exists a discrete separation $\{U_{\beta,n-1} : \beta \in C_\alpha\}$ of $C_\alpha \times \{n-1\}$, where each $U_{\beta,n-1}$ is a compact open set containing $\langle\beta, n-1\rangle$. Then, define the local base at $\langle\alpha, n\rangle$ by

$$\mathcal{B}(\alpha, n) = \{\{\langle\alpha, n\rangle\} \cup \{\bigcup\{U_{\beta,n-1} : \beta \in C_\alpha \setminus F\}\} : F \text{ is finite subset of } C_\alpha\}.$$

Clearly, $\mathcal{B}(\alpha, n)$ is a countable base at $\langle\alpha, n\rangle$, and each element in $\mathcal{B}(\alpha, n)$ is compact since every open neighbourhood of $\langle\alpha, n\rangle$ contains a tail of each member. The final topology on X is generated by the base

$$\bigcup\{\mathcal{B}(\alpha, n) : \alpha \in \omega_1, n \in \omega\}.$$

Note that this space is first countable and locally compact.

The usual de Caux space that constructed using ♣ satisfies for every $k \in \omega$ and an uncountable A subset of $(\omega_1 \times \{k\})$, $\overline{A}(n) = \{\alpha : \langle \alpha, n \rangle \in A\}$ is stationary for all $n > k$. Moreover, it satisfies that for a club set C and for an open set containing $C \times \{k\}$, $U \cap (\omega_1 \times \{n\})$ is co-countable, for all $n < k$. The ♣* sequence $\{C_\alpha : \alpha \in \mathrm{Lim}(\omega_1)\}$ and the convergence property of the sequence $C_\alpha \times \{n-1\}$ to $\langle \alpha, n \rangle$ are used to prove the following lemmas, as well as to demonstrate that the space is not semi-proximal.

Lemma 5.1.4 *For $A \in [\omega_1 \times \{k\}]^{\omega_1}$, $\overline{A}(n)$ contains a club for all $n > k$.*

Proof. We will prove it by induction on $n > k$, so it is sufficient to prove that if $A \in [\omega_1 \times \{n\}]^{\omega_1}$, then $\overline{A}(n+1)$ contains a club. Let $n > k$ and $A \in [\omega_1 \times \{n\}]^{\omega_1}$. Then $\{\alpha : |(C_\alpha \times \{n\}) \cap A| = \omega\}$ contains a club C. Hence, $\langle \alpha, n+1 \rangle \in \overline{A}$, for all $\alpha \in C$. Thus, $\overline{A}(n+1)$ contains a club. □

If A and B are uncountable subsets of X, then there exist $i, j \in \omega$ such that $A \cap (\omega_1 \times \{i\})$ and $B \cap (\omega_1 \times \{j\})$ are uncountable. By Lemma 5.1.4, $\overline{A}(n)$ and $\overline{B}(n)$ contains clubs for all $n > \max\{i, j\}$. Since $\overline{A}(n) \cap \overline{B}(n)$ contains a club, $\overline{A} \cap \overline{B} \cap (\omega_1 \times \{n\})$ is uncountable for all $n > \max\{i, j\}$. As a result, if H and K are disjoint closed subsets of X, then one of them must be countable.

Claim 5.1.5 *X is normal.*

Proof. Suppose that H and K are disjoint closed subsets of X. Then one of them, say K, is countable. Thus, there is α with $K \subset (\alpha+1) \times \omega$. Since K and $H' = H \cap ((\alpha+1) \times \omega)$ are disjoint closed subset of a countable regular clopen subspace $(\alpha+1) \times \omega$, there exists two disjoint open set $U \supset K$ and $V' \supset H$ in $(\alpha+1) \times \omega$. Now let $V = V' \cup ((\alpha, \omega_1) \times \omega)$, then U and V are disjoint open set containing K and H respectively. □

Claim 5.1.6 *X is not countably paracompact.*

Proof. let $D_n = \omega_1 \times (\omega \setminus n)$, then $(D_n : n \in \omega)$ is a decreasing sequence of closed sets in X with $\bigcap_{n\in\omega} D_n = \emptyset$. Let $U_n \supset D_n$ be open for all $n \in \omega$. Then, $X \setminus U_n$ and D_n are disjoint closed subsets of X, for all $n \in \omega$. Therefore, $X \setminus U_n$ is countable. Thus, $\bigcap\{U_n : n \in \omega\} \neq \emptyset$. By Theorem 5.1.1, X is not countably paracompact. □

Lemma 5.1.7 *For $n \in \omega$, if S is a stationary subset of ω_1 and U is any open neighbourhood of $S \times \{n+1\}$, then $U \cap (\omega_1 \times \{n\})$ is co-countable.*

Proof. Let S be a stationary subset of ω_1, $n \in \omega$, and U be any open neighbourhood of $S \times \{n+1\}$. Suppose $A = (\omega_1 \times \{n\}) \setminus U$ is uncountable. Since $A(n) \in [\omega_1]^{\omega_1}$, there exists a club C such that $A(n) \cap C_\alpha$ is infinite for all $\alpha \in C$. But for every $\alpha \in C \cap S$, $\langle \alpha, n+1 \rangle \in U$. Hence, $(C_\alpha \times \{n\}) \setminus U$ is finite, contradiction. Thus, $U \cap (\omega_1 \times \{n\})$ is co-countable. □

Lemma 5.1.8 *If S is a stationary subset of ω_1, then for every $n \in \omega$ and every U open neighbourhood of $S \times \{n\}$, $U \cap (\omega_1 \times \{k\})$ is co-countable, for all $k < n$.*

Proof. Let S be a stationary subset of ω_1. We will proceed by induction on n. If $n = 1$, let U be any open neighbourhood of $S \times \{1\}$. Then $U \cap (\omega_1 \times \{0\})$ is co-countable, by Lemma 5.1.7. Assume it is true for n, and let U be any open neighbourhood of $S \times \{n+1\}$. Then, by Lemma 5.1.7, $U \cap (\omega_1 \times \{n\})$ is co-countable. Thus, there exists $\eta \in \omega_1$ such that $(\omega_1 \setminus \eta) \times \{n\} \subset U$. Since $\omega_1 \setminus \eta$ is stationary then, by inductive hypothesis, $U \cap (\omega_1 \times \{k\})$ is co-countable, for all $k < n$ and this concludes the proof. □

Lemma 5.1.9 *For every uniformity $\mathfrak{U}$ on X, $U \in \mathfrak{U}$, $n \in \omega$, and S a stationary subset of ω_1, there exist $\beta \in S$ and a stationary set $S' \subset S$ such that $S' \times \{n+1\} \subset U[\langle \beta, n\rangle]$.*

Proof. Let $\mathfrak{U}$ be a uniformity on X, $U \in \mathfrak{U}$, $n \in \omega$ and S be a stationary subset of ω_1. Since $S \times \{n\}$ is uncountable, $\overline{S \times \{n\}}(k)$ contains a club C_k for all $k > n$, by Lemma 5.1.4. Let $A_\alpha = U[\langle \alpha, n+1\rangle] \cap (S \times \{n\})$, for each $\alpha \in C_{n+1}$. Then $A_\alpha \neq \emptyset$, for each $\alpha \in C_{n+1}$. Define $f : C_{n+1} \to S$ by $f(\alpha) = \min(A_\alpha(n))$. Then by Fodor's Lemma 2.1.1, there exist $\beta \in S$ and a stationary subset $S' \subset S$ such that $\beta = f(\alpha)$, for all $\alpha \in S'$. Hence $S' \times \{n+1\} \subset U[\langle \beta, n\rangle]$. □

Claim 5.1.10 *X is not semi-proximal.*

Proof. Let $\mathfrak{U}$ be any uniformity on X. We will define a winning strategy σ for Player II in the proximal game on $(X, \mathfrak{U})$. In inning 0, Player I plays $U_0 \in \mathfrak{U}$. Then, by Lemma 5.1.9, there exist $\beta_0 \in \omega_1$ and a stationary set $S_0 \subset \omega_1$ such that $S_0 \times \{1\} \subset U[\langle \beta_0, 0\rangle]$. Let $\sigma(U_0) = \langle \beta_0, 0\rangle$. In inning $n+1$, suppose that $\sigma(U_0, \ldots, U_n)$ has been defined so that there are S_n and β_n such that $\sigma(U_0, \ldots, U_n) = \langle \beta_n, n\rangle$ and $S_n \times \{n+1\} \subset U[\langle \beta_n, n\rangle]$. Let $U_{n+1} \subset U_n$. Since S_n is stationary, there exist $\beta_{n+1} \in S_n \setminus (\beta_n + 1)$ and a stationary set $S_{n+1} \subset S_n$ such that $S_{n+1} \times \{n+2\} \subset U_{n+1}[\langle \beta_{n+1}, n+1\rangle]$. Define $\sigma(U_0, \ldots, U_{n+1}) = \langle \beta_{n+1}, n+1\rangle$. Note that $\langle \beta_{n+1}, n+1\rangle \in U_n[\langle \beta_n, n\rangle]$ since $\beta_{n+1} \in S_n$. Therefore, $(\langle \beta_n, n\rangle : n \in \omega)$ is not convergent in X since the sets $W_n = \omega_1 \times n$ form an increasing open cover of X and $\langle \beta_k, k\rangle \notin W_n$, for all $k \geq n$. Since $S_n \times \{n+1\} \subset U_n[\langle \beta_n, n\rangle]$ for all n, $U_n[\langle \beta_n, n\rangle] \cap (\omega_1 \times \{0\})$ is co-countable by Lemma 5.1.8. Thus $\bigcap_{n\in\omega} U_n[\langle \beta_n, n\rangle] \neq \emptyset$. □

5.2 Semi-proximal not Normal Spaces

To address Nyikos's question whether semi-proximality implies normality, we will introduce two counterexamples. The initial counterexample will be a Ψ-space over an almost disjoint family of branches in $2^{<\omega}$, while the subsequent one will be a subspace of $(\omega+1)\times\omega_1$.

5.2.1 A semi-proximal non-normal Ψ-space

A Ψ-space over an almost disjoint family of branches in $2^{<\omega}$ defined as follows. Let $Z \subseteq 2^\omega$ and for each $z \in Z$, let $a_z = \{z \upharpoonright n : n \in \omega\} \subseteq 2^{<\omega}$. Then, $A_Z = \{a_z : z \in Z\}$ is an almost disjoint family of branches in $2^{<\omega}$. Let $\Psi(A_Z) = 2^{<\omega} \cup A_Z$ with the points of $2^{<\omega}$ isolated and a local base at each a_z is of the form $\{a_z\} \cup (a_z \setminus F)$ where F is finite. For $s \in 2^{<\omega}$, let $[s] = \{t \in 2^{<\omega} : t \supset s\}$.

The example will be provided after proving the following theorem. For $i = 0, 1$ and $k \in \omega$, define $A^i_k = \bigcup\{[s] : s \in 2^{k+1}, s(k) = i\} \cup \{a_z \in A_Z : z(k) = i\}$, then A^i_k is clopen, and for a finite subset $F \subset A_Z$, define the following:

1. If $F = \{a\}$, define $\mathcal{A}_{F,k} = \{\{a\} \cup (a \setminus 2^{\leq k})\}$.

2. If $|F| > 1$, fix $k_F \geq k$ so that $\{\{a\} \cup (a \setminus 2^{\leq k_F}) : a \in F\}$ is pairwise disjoint, and define $\mathcal{A}_{F,k} = \{\{a\} \cup (a \setminus 2^{\leq k_F}) : a \in F\}$.

3. $\mathcal{A}_k = \{A^0_k, A^1_k\} \cup \{\{s\} : s \in 2^{\leq k}\}$.

4. $\mathcal{A}_k \sim F = \{A^0_k \setminus (\bigcup \mathcal{A}_{F,k}), A^1_k \setminus (\bigcup \mathcal{A}_{F,k})\} \cup \mathcal{A}_{F,k} \cup \{\{s\} : s \in 2^{\leq k}\}$, then $\mathcal{A}_k \sim F$ is a clopen partition of $\Psi(A_Z)$.

Theorem 5.2.1 *If $Z \subset 2^\omega$ contains no copy of the Cantor set, then $\Psi(A_Z)$ is totally semi-proximal.*

Proof. Let $Z \subset 2^\omega$ contain no copy of the Cantor set. Consider the uniformity $\mathfrak{U}$ induced on $\Psi(A_Z)$ as a subspace of the Stone-Čech compactification on $\Psi(A_Z)$. Let σ be a strategy for Player II in the proximal game on $(\Psi(A_Z), \mathfrak{U})$. We define a counter-strategy for Player I, called **the plain strategy**, using Definition 3.1.10. The plain strategy for Player I is to play $\mathcal{A}_k$ at inning k. A finite modification of the plain strategy for Player I is to play partitions of the form $\mathcal{A}_k \sim F$. Note that for each k, both $\mathcal{A}_k$ and $\mathcal{A}_k \sim F$ are clopen partitions of the Ψ-space. Hence, Player I is free to play partitions of these forms in a play of the proximal game with respect to $\mathfrak{U}$.

We will now define, for each $g \in 2^\omega$, plays of the game P_g where Player II uses the strategy σ, so that if Player II wins each of these plays of the game, then a copy of the Cantor set would be embedded in Z. We first define a play of the game for the constant 0 function, $\langle\overline{0}\rangle$:

Let $F_{\langle\overline{0}\rangle\restriction k+1} = \emptyset$, for all $k \geq 0$. Note that $\mathcal{A}_k \sim F_{\langle\overline{0}\rangle\restriction k+1} = \mathcal{A}_k$, for all $k \geq 0$. In inning 0, Player I chooses $\mathcal{A}_0$, and Player II chooses $\sigma(\mathcal{A}_0) = x_{\langle 0\rangle}$, which gives an initial play of the game denoted by $P_{\langle 0\rangle}$. Extend it to a full play of the game where Player I uses the unmodified plain strategy and Player II uses σ to obtain $P_{\langle\overline{0}\rangle} = P_{\langle 0\rangle} \frown (\mathcal{A}_1, x_{\langle 00\rangle}, \ldots, \mathcal{A}_k, x_{\langle\overline{0}\rangle\restriction k+1}, \ldots)$. If either

1. there exists $k \geq 0$ such that $x_{\langle\overline{0}\rangle\restriction k+1} \in 2^{\leq k}$, or

2. $\bigcap_{k\in\omega} \mathrm{St}(x_{\langle\overline{0}\rangle\restriction k+1}, \mathcal{A}_k) = \emptyset$

then σ is defeated. Indeed, if (1) occurs, then Player II is forced to play $x_{\langle\overline{0}\rangle\restriction k+1}$ in each subsequent inning and so picks a convergent sequence. So we assume that (1) and (2) fail. Then for each k there is $i_{\langle\overline{0}\rangle\restriction k+1}$ such that $x_{\langle\overline{0}\rangle\restriction k+1} \in A_n^{i_{\langle\overline{0}\rangle\restriction k+1}}$. Since the plain strategy was employed, there is $z_{\langle 0\rangle}$ such that $\bigcap_{k\in\omega} \mathrm{St}(x_{\langle\overline{0}\rangle\restriction k+1}, \mathcal{A}_k) = \bigcap_{k\in\omega} A_k^{i_{\langle\overline{0}\rangle\restriction k+1}} = \{a_{z_{\langle 0\rangle}}\}$.

Now we use $a_{z_{\langle 0\rangle}}$ to define another play of the game corresponding to the branch $\langle 1\overline{0}\rangle = (1, 0, 0, \ldots)$ in 2^ω. Let $F_{\langle 1\rangle} = F_{\langle 1\overline{0}\rangle\restriction n+1} = \{a_{z_{\langle 0\rangle}}\}$, for all $n > 0$. In inning 0, Player I uses the plain strategy modified by $F_{\langle 1\rangle}$ and chooses $\mathcal{A}_0 \sim F_{\langle 1\rangle}$ while Player II chooses $x_{\langle 1\rangle}$. It gives an initial play of the game $P_{\langle 1\rangle} = (\mathcal{A}_0 \sim F_{\langle 1\rangle}, x_{\langle 1\rangle})$. We extend it to a full play of the game with Player I using the plain strategy only modified by $F_{\langle 1\rangle}$ and Player II using σ:

$$P_{\langle 1\overline{0}\rangle} = P_{\langle 1\rangle} \frown (\mathcal{A}_1 \sim F_{\langle 10\rangle}, x_{\langle 10\rangle}, \ldots, \mathcal{A}_k \sim F_{\langle 1\overline{0}\rangle\restriction k+1}, x_{\langle 1\overline{0}\rangle\restriction k+1}, \ldots).$$

If either

1. there exists k such that $x_{\langle 1\overline{0}\rangle\restriction k+1} \in 2^{\leq k}$, or

2. there exists k such that $(x_{\langle 1\overline{0}\rangle\restriction k+1}) \in (a_{z_{\langle 0\rangle}} \setminus 2^{\leq k}) \cup \{a_{z_{\langle 0\rangle}}\}$, or

3. $\bigcap_{k\in\omega} \mathrm{St}(x_{\langle 1\overline{0}\rangle\restriction k+1}, \mathcal{A}_k \sim F_{\langle 1\overline{0}\rangle\restriction k+1}) = \emptyset$,

then σ is defeated. Indeed, if (1) or (3) occurs, this is clear and if (2) occurs, then Player II is forced to play the rest of the game inside the convergent sequence $a_{z_{\langle 0\rangle}} \cup \{a_{z_{\langle 0\rangle}}\}$ and so Player I can play to defeat σ (to produce either an eventually constant sequence or one that converges to $a_{z_{\langle 0\rangle}}$). Otherwise, there exists $z_{\langle 1\rangle} \in Z$ such that

$$\bigcap_{k\in\omega} \mathrm{St}(x_{\langle 1\overline{0}\rangle\restriction k+1}, \mathcal{A}_k \sim F_{\langle 1\overline{0}\rangle\restriction k+1}) = \bigcap_{k\in\omega} (A_k^{i_{\langle 1\overline{0}\rangle\restriction k+1}} \setminus \bigcup \mathcal{A}_{F_{\langle 1\rangle},k}) = \{a_{z_{\langle 1\rangle}}\}.$$

Note that $z_{\langle 1\rangle} \neq z_{\langle 0\rangle}$, so there exists a minimum $m_\emptyset \in \omega$ such that, in inning $m_\emptyset$, $i_{\langle\overline{0}\rangle\restriction m_\emptyset+1} \neq i_{\langle 1\overline{0}\rangle)\restriction m_\emptyset+1}$ which means that $z_{\langle 0\rangle}(m_\emptyset) \neq z_{\langle 1\rangle}(m_\emptyset)$ and $z_{\langle 0\rangle} \restriction m_\emptyset = z_{\langle 1\rangle} \restriction m_\emptyset$.

Let $n > 1$ and assume we have defined P_s, a_{z_s} and F_s, for every $s \in 2^{\leq n}$, $m_t \in \omega$ for every $t \in 2^{<n}$, and we have also defined $F_{s \frown \bar{0} \restriction k}$ for all $k > |s|$ and $i_{s \frown \langle \bar{0} \rangle \restriction k}$ for all k such that:

1. $P_s = (\mathcal{A}_0 \sim F_{s\restriction 1}, x_{s\restriction 1}, \ldots, \mathcal{A}_{n-1} \sim F_{s\restriction n}, x_{s\restriction n})$ which is a partial play of the game at inning $n-1$. If s extends t, then P_s extends P_t.

2. $P_{s \frown \langle \bar{0} \rangle} = P_s \frown (\mathcal{A}_n \sim F_s, x_{s \frown \langle \bar{0} \rangle \restriction n+1}, \mathcal{A}_{n+1} \sim F_s, x_{s \frown \langle \bar{0} \rangle \restriction n+2}, \ldots)$ is a branch of a play of the game corresponding to $s \frown \langle \bar{0} \rangle$.

3. $F_s = \{a_{z_t} : t \subset s\}$. And for $k > |s|$, $F_{s \frown \bar{0} \restriction k} = F_s$.

4. $i_{s \frown \langle \bar{0} \rangle \restriction k+1} \in \{0, 1\}$ such that $x_{s \frown \langle \bar{0} \rangle \restriction k+1} \in (A_k^{i_{s \frown \langle \bar{0} \rangle \restriction k+1}} \setminus \bigcup \mathcal{A}_{F_{s \frown \bar{0} \restriction k+1}, k})$.

5. $\{a_{z_s}\} = \bigcap_{k \in \omega} (A_k^{i_{s \frown \langle \bar{0} \rangle \restriction k+1}} \setminus \bigcup \mathcal{A}_{F_{s \frown \bar{0} \restriction k+1}, k})$.

6. m_t satisfies the following:

   (a) $z_{t \frown 0}(m_t) \neq z_{t \frown 1}(m_t)$, and $z_{t \frown 0} \restriction m_t = z_{t \frown 1} \restriction m_t$.

   (b) $z_r \restriction m_r = z_t \restriction m_r$ if $t \in [r] \cap 2^{<n}$.

   (c) $z_r \restriction m_{\Delta(r,t)} = z_t \restriction m_{\Delta(r,t)}$ and $z_r(m_{\Delta(r,t)}) \neq z_t(m_{\Delta(r,t)})$ if r and t are not comparable in $2^{<n}$, where $\Delta(r, t)$ is the maximum in $2^{<n}$ which both r and t extend.

Now, for every $s \in 2^n$, define $F_{s \frown 0} = F_s$, and $F_{s \frown 0 \frown \langle \bar{0} \rangle \restriction k+1} = F_s$, for all $k > n$. In inning n, let Player I use the plain strategy modified by F_s and Player II uses σ, then the partial play of the game is $P_{s \frown 0} = P_s \frown (\mathcal{A}_n \sim F_s, x_{s \frown 0})$. We extend it to a full play of the game with the plain strategy only modified by F_s, $P_{s \frown \langle 0 \bar{0} \rangle} = P_{s \frown 0} \frown (\mathcal{A}_{n+1} \sim F_s, x_{s \frown \langle \bar{0} \rangle \restriction n+2} \ldots)$. Therefore, the play of the game, corresponds to $s \frown 0$, is equal to the one that corresponds to s, $P_{s\frown\langle\bar{0}\rangle}$. So, let $z_{s\frown 0} = z_s$.

Define $F_{s\frown 1} = F_s \cup \{a_{z_s}\}$ and $F_{s\frown\langle 1\bar{0}\rangle\restriction k+1} = F_{s\frown 1}$, for $k > n$. In inning n, let Player I use the plain strategy modified by $F_{s\frown 1\restriction k+1}$ at inning $k \leq n$ against σ, then the partial play of the game is $P_{s\frown 1} = P_s \frown (\mathcal{A}_n \sim F_{s\frown 1}, x_{s\frown 1})$. We extend it to a full play of the game with the plain strategy modified by $F_{s\frown 1}$ at inning $k > n$, $P_{s\frown\langle 1\bar{0}\rangle} = P_{s\frown 1} \frown (\mathcal{A}_{n+1} \sim F_{s\frown 1}, x_{s\frown\langle 1\bar{0}\rangle\restriction n+2}, \ldots)$. As in the base case of the construction, if either

1. there exists k such that $x_{s\frown\langle 1\bar{0}\rangle\restriction k+1} \in 2^{\leq k}$,
2. there exists $a \in F_{s\frown 1}$ such that $x_{s\frown\langle 1\bar{0}\rangle\restriction k+1} \in (a \setminus 2^{\leq k_{F_{s\frown 1}}}) \cup \{a\}$, or
3. $\bigcap_{k\in\omega}$ $\mathrm{St}(x_{s\frown\langle 1\bar{0}\rangle\restriction k+1}, \mathcal{A}_k \sim F_{s\frown\langle 1\bar{0}\rangle\restriction k+1}) = \emptyset$

Then, there is a play of the game where the strategy σ is defeated. Otherwise, for each $s \in 2^n$ there exists $z_{s\frown 1} \in Z$ such that

$$\bigcap_{k\in\omega} \mathrm{St}(x_{s\frown\langle 1\bar{0}\rangle\restriction k+1}, \mathcal{A}_k \sim F_{s\frown\langle 1\bar{0}\rangle\restriction k+1}) = \{a_{z_{s\frown 1}}\}.$$

Since $a_{z_s} \in F_{s\frown 1}$, then $z_{s\frown 0} = z_s \neq z_{s\frown 1}$, so there exists a minimum m_s such that in inning m_s, $i_{s\frown\langle\bar{0}\rangle\restriction m_s+1} \neq i_{s\frown\langle 1\bar{0}\rangle\restriction m_s+1}$. Hence, $z_s(m_s) \neq z_{s\frown 1}(m_s)$ and $z_s \restriction m_s = z_{s\frown 1} \restriction m_s$. Then, the elements of $\{z_s : s \in 2^n\}$ are distinct. To see that, let $r \neq s$ in 2^n. Thus, there is $k < n$ and $t \in 2^k$ such that $\Delta(s, r) = t$. Then, $s(k) \neq r(k)$. Assume without loss of generality that $s(k) = 0$ and $r(k) = 1$. Then, $z_s(m_t) = z_{t\frown 0}(m_t) \neq z_{t\frown 1}(m_t) = z_r(m_t)$.

If there is an $f \in 2^\omega$ such that $\bigcap_{k\in\omega}(A_k^{i_{f\restriction k+1}} \setminus \bigcup \mathcal{A}_{F_{f\restriction k+1},k}) = \emptyset$, then the play of the game, corresponding to that f, is the one when Player I defeats σ.

Otherwise, for each $f \in 2^\omega$ there is a unique z_f such that

$$\bigcap_{k\in\omega}(A_k^{i_{f\restriction k+1}} \setminus \bigcup \mathcal{A}_{F_{f\restriction k+1},k}) = \{a_{z_f}\}.$$

Hence, the mapping $G : 2^\omega \to Z$ is one-to-one, where $G(f) = z_f$, for each $f \in 2^\omega$. It is also continuous. Indeed, let V be a basic open set in Z, then there exist n and $s \in 2^n$ such that $V = \{f \in 2^\omega : f \supset s\} \cap Z$. Note that $z_f(k) = i_{f\restriction k+1}$, for all $f \in G^{-1}(V)$. Then, for every $k < n$, $s(k) = i_{f\restriction k+1}$ for all $f \in G^{-1}(V)$. Therefore, there exists $t_s \in 2^n$ such that $s(k) = i_{t_s\restriction k+1}$. Hence, $G^{-1}(V) = \{f \in 2^\omega : f \supset t_s\}$ which is a basic open set in 2^ω. Then Z contains a Cantor set which is a contradiction. □

Jones' Lemma, as stated by Hodel [23], asserts that every closed discrete subset of a separable normal space has a cardinality less than the continuum. Therefore, if Z has size continuum then the corresponding Ψ-space is not normal.

Example 5.2.2 *There is a semi-proximal not normal Ψ-space.*

Proof. Let $Z \subseteq 2^\omega$ be any set of size continuum that does not contain a copy of the Cantor set, for example, a Bernstein set. A Bernstein set is a subset of the real line that meets every uncountable closed subset of the real line but contains none of them [35]. Hence, $\Psi(A_Z)$ is semi-proximal, by Theorem 5.2.1. Since Z has size continuum, A_Z is a closed discrete subset of $\Psi(A_Z)$ of size continuum. Also, note that $2^{<\omega}$ is a countable dense subset of $\Psi(A_Z)$. Therefore, $\Psi(A_Z)$ is not normal by Jones' Lemma. □

It is worth mentioning that the proof of Theorem 5.2.1 was inspired by Galvin's original proof that Player II does not have a winning strategy in the Galvin game on $\kappa \leq 2^\omega$. We can now obtain that result as a corollary to Theorem 5.2.1.

Corollary 5.2.3 *$II \not\uparrow G(\kappa)$ for $\kappa \leq 2^\omega$.*

Proof. Identify κ with a subset $Z = \{z_\alpha : \alpha \in \kappa\}$ of the Cantor set that contains no copies of the Cantor set. For example, in the case $\kappa = 2^\omega$, let Z be a Bernstein set. Let σ be a strategy for Player II in the Galvin game on κ. Consider the uniformity $\mathfrak{U}$ on $\Psi(A_Z)$ inherited from the Stone-Čech compactification of $\Psi(A_Z)$. For a finite clopen partition $\mathcal{A}$ of $\Psi(A_Z)$, define $\mathcal{A}' = \{\{\alpha \in \kappa : a_\alpha \in U \cap A_Z\} : U \in \mathcal{A}\}$, where $a_\alpha = a_{z_\alpha}$ is the element of the almost disjoint family A_Z. Then, $\mathcal{A}'$ is a partition of κ.

Define a strategy σ' for Player II in the proximal game on $(\Psi(A_Z), \mathfrak{U})$ as follows: For every finite sequence of finite clopen partitions $\mathcal{A}_0, \ldots, \mathcal{A}_n$ of $\Psi(A_Z)$, let $\sigma'(\mathcal{A}_0, \ldots, \mathcal{A}_n) = a_{\alpha_n}$ such that $\alpha_n \in \sigma(\mathcal{A}'_0, \ldots, \mathcal{A}'_n)$. Since $(\Psi(A_Z), \mathfrak{U})$ is semi-proximal, by Theorem 5.2.1, there is a counter-strategy $\tau_{\sigma'}$ for Player I defeating σ'. Define a counter-strategy τ_σ for Player I in the Galvin game on κ as follows. In inning 0, let $\tau_\sigma(\emptyset) = \mathcal{A}'_0$, where $\mathcal{A}_0 = \tau_{\sigma'}(\emptyset)$. In inning $n+1$, let $(\tau_\sigma(\emptyset), A_0, \ldots, \tau_\sigma(A_0, \ldots, A_{n-1}), A_n)$ be a partial play of the Galvin game such that $\tau_\sigma(A_0, \ldots, A_k) = \tau_{\sigma'}(a_{\alpha_0}, \ldots, a_{\alpha_k})$, for every $k < n$, where $\alpha_m \in A_m$, for all $m \leq k$. If A_n is finite, then Player I can defeat it by partitioning the set into singletons. Otherwise, assume the choices of Player II are always infinite and let $\alpha_n \in A_n \setminus \{\alpha_k : k < n\}$. Define $\tau_\sigma(A_0, \ldots, A_n) = \mathcal{A}'_{n+1}$, where $\mathcal{A}_{n+1} = \tau_{\sigma'}(a_{\alpha_0}, \ldots, a_{\alpha_{n-1}}, a_{\alpha_n})$. Since $(a_{\alpha_n} : n \in \omega)$ does not converge in $\Psi(A_Z)$ as it is a non eventually constant sequence in the discrete closed set A_Z, $\bigcap_{n\in\omega} \mathrm{St}(a_{\alpha_n}, \tau_{\sigma'}(a_{\alpha_0}, \ldots, a_{\alpha_{n-1}})) = \emptyset$. Hence, $\bigcap_{n\in\omega} \sigma(\mathcal{A}'_0, \ldots, \mathcal{A}'_n) = \emptyset$. Thus, σ is not winning. □

5.2.2 A semi-proximal non-normal subspace of ω_1^2

We construct an example of a semi-proximal non-normal subspace of the space $(\omega+1)\times\omega_1$. This example is also not countably paracompact. Indeed, the authors, in [30], proved that, for all $\alpha\in\omega_1$, if X is a countably paracompact subspace of $\alpha\times\omega_1$, then X must be normal. We delve deeper into the relationship between semi-proximal spaces and normality in finite powers of ω_1 in Section 5.3.

The following example was first introduced by N. Kemoto in [28]. Let $A = \{a_\alpha : \alpha\in\omega_1\}$ be a family of infinite subsets of ω. Define $X_A \subseteq (\omega+1)\times\omega_1$ by $X_A = \big(\omega\times\mathrm{Lim}\big)\cup\big(\bigcup_{\alpha\in\omega_1}(a_\alpha\cup\{\omega\})\times\{\alpha+1\}\big)$.

N. Kemoto demonstrated that X_A is not normal for any family A by showing $H=\omega\times\mathrm{Lim}$ and $K=\{\omega\}\times\mathrm{Succ}$ cannot be separated. Theorem 5.2.4 below shows that there is a semi-proximal subspace of $(\omega+1)\times\omega_1$ that is neither normal nor countably paracompact.

Theorem 5.2.4 *There is a family A such that X_A is semi-proximal.*

Proof. Let $Z=\{z_\alpha:\alpha\in\omega_1\}\subseteq 2^\omega$ contain no copy of the Cantor set. For each $\alpha\in\omega_1$, let $a_\alpha=\{z_\alpha\restriction n : n\in\omega\}\subseteq 2^{<\omega}$. Then $A_Z=\{a_\alpha:\alpha\in\omega_1\}$ is an almost disjoint family of branches in $2^{<\omega}$. Enumerate $2^{<\omega}$ as $\{s_n : n\in\omega\}$ and define $X_{A_Z}\subset(\omega+1)\times\omega_1$ by

$$X_{A_Z}=\big(\omega\times\mathrm{Lim}\big)\cup\big(\{\omega\}\times\mathrm{Succ}\big)\cup\bigcup_{\alpha\in\omega_1}\{\langle n,\alpha+1\rangle : s_n\in a_\alpha\}.$$

To see that X_{A_Z} is semi-proximal, consider the uniformity $\mathfrak{V}$ inherited from the Stone-Čech compactification of X_{A_Z}. Let σ be a strategy for Player II in the proximal game on $(X_{A_Z},\mathfrak{V})$. We will define a counter-strategy τ_σ

defeating σ using Definition 3.1.10. Consider the uniformity $\mathfrak{U}$ on $\Psi(A_Z)$ inherited from the Stone-Čech compactification of $\Psi(A_Z)$.

Let $X_k = (\{k\} \times \omega_1) \cap X_{A_Z}$. For a subset U of $\Psi(A_Z)$, define

$$U' = \bigcup\{X_k : s_k \in U\} \cup \{\langle \omega, \alpha+1\rangle : a_\alpha \in U\}.$$

Note that if $\mathcal{U}$ is a finite clopen partition of $\Psi(A_Z)$, then $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$ is a finite clopen partition of X_{A_Z} and so corresponds to an entourage in $\mathfrak{V}$ so can be played by Player I in the proximal game.

For $n \in \omega$ and a finite $F \subset A_Z$, let $\alpha_F = \max\{\beta : a_\beta \in F\}$ and define the clopen partition of X_{A_Z},

$$\mathcal{U}_{n,F} = \{X_{A_Z} \cap ([0,n] \times \omega_1)), X_{A_Z} \cap ((n,\omega] \times [0,\alpha_F]), X_{A_Z} \cap ((n,\omega] \times (\alpha_F, \omega_1))\}.$$

Now consider the clopen partition $\mathcal{A}_n \sim F$ and n_F that we defined at the beginning of Section 5.2.1. Then $\mathcal{A}'_{n,F} = \{V' \cap U : V \in \mathcal{A}_n \sim F, U \in \mathcal{U}_{n_F,F}\}$ is a finite clopen partition of X_{A_Z}.

Define a strategy σ' for Player II in the proximal game on $(\Psi(A_Z), \mathfrak{U})$ as follows: For a finite sequence of clopen partitions $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of $\Psi(A_Z)$ such that $\mathcal{U}_n$ refines $\mathcal{U}_{n-1}$,

$$\sigma'(\mathcal{U}_0, \ldots, \mathcal{U}_n) = \begin{cases} a_{\alpha_n} & \text{if } \sigma(\mathcal{V}'_0, \ldots, \mathcal{V}'_n) = \langle \omega, \alpha_n + 1\rangle \\ s_{k_n} & \text{if } \sigma(\mathcal{V}'_0, \ldots, \mathcal{V}'_n) \in X_{k_n} \end{cases}$$

where $\mathcal{V}'_n$ is a finite clopen partition of X_{A_Z} and the common refinement of $\mathcal{U}'_k$, for all $k \leq n$. Since $(\Psi(A_Z), \mathfrak{U})$ is semi-proximal, then there is a counter-strategy $\tau_{\sigma'}$ for Player I defeating σ'. Recall that, in proof of Theorem 5.2.1, for every partial play of the game $(\tau_{\sigma'}(\emptyset), x_0, \ldots, \tau_{\sigma'}(x_0, \ldots, x_{n-1}), x_n)$ in the proximal game on $(\Psi(A_Z), \mathfrak{U})$ where Player I uses $\tau_{\sigma'}$ and Player II uses σ', there is a finite subset $F_n \subset A_Z$ such that $\tau_{\sigma'}(x_0, \ldots, x_n) = \mathcal{A}_n \sim F_n$, for all $n \in \omega$. Then the play of the game $(\mathcal{A}_0 \sim F_0, x_0, \ldots, \mathcal{A}_n \sim F_n, x_n, \ldots)$ resulted in one of three outcomes:

(i) at some stage of the game, $(\mathcal{A}_0 \sim F_0, x_0, \ldots, \mathcal{A}_n \sim F_n, x_n)$, there is an $s \in 2^{\leq n}$ such that $x_n = s$, or

(ii) at some stage of the game, $(\mathcal{A}_0 \sim F_0, x_0, \ldots, \mathcal{A}_n \sim F_n, x_n)$, there is an $a \in F_n$ such that $x_n \in \{a\} \cup a \setminus 2^{\leq n_{F_n}}$, or

(iii) the play of the game satisfied $\bigcap_{n\in\omega} \operatorname{St}(x_n, \mathcal{A}_n \sim F_n) = \emptyset$.

Now, we define a counter-strategy τ_σ for Player I to defeat σ. In inning 0, let $\tau_\sigma(\emptyset) = \mathcal{A}'_{0,F_0}$. In inning $n > 0$, let y_m be the element chosen by Player II using σ at inning $m < n$ and define $\tau_\sigma(y_0, \ldots, y_{n-1}) = \mathcal{V}'_n$, where $\mathcal{V}'_n$ is the finite clopen partition of X_{A_Z} and the common refinement of $\mathcal{A}'_{m,F_m}$, for all $m \leq n$. Then, we have the following cases:

Case 1: If the outcome (i) holds. Then at some stage n, $x_n = s_k$ for some k and $\{s_k\} \in \mathcal{A}_n \sim F_n$. Then, by definition of the proximal game, all subsequent choices of Player II using σ are played inside $X_k \in \mathcal{A}'_{n,F_n}$ which, being homeomorphic to a subspace of ω_1, is semi-proximal, so Player I can win the game.

Case 2: If the outcome (ii) holds. Then, there is $\alpha \in \omega_1$ such that $a = a_\alpha$. Let $V_\alpha = \{a_\alpha\} \cup (a_\alpha \setminus 2^{\leq n_{F_n}})$, then $V_\alpha \in \mathcal{A}_n \sim F_n$. If $y_n \in V'_\alpha \cap ((n_{F_n}, \omega] \times [0, \alpha_{F_n}])$, then the rest of the game will be inside a metric space which is semi-proximal. Otherwise, $y_n \in V'_\alpha \cap ((n_{F_n}, \omega]) \times (\alpha_{F_n}, \omega_1))$ and if this is the case for all $n \in \omega$, then $\bigcap_{n\in\omega} \operatorname{St}(y_n, \mathcal{V}'_n) \subset \bigcap_{n\in\omega} V'_\alpha \cap ((n_{F_n}, \omega]) \times (\alpha_{F_n}, \omega_1)) = \emptyset$.

Case 3: If the outcome (iii) holds. Let $U_n = \mathrm{St}(x_n, \mathcal{A}_n \sim F_n)$. Then consider $V_n = \mathrm{St}(y_n, \mathcal{V}_n')$, then $V_n \subset U_n' \cap U$, where $U \in \mathcal{U}_{n_{F_n}, F_n}$. We claim that $\bigcap_{n\in\omega} V_n = \emptyset$. To see that, suppose there is an element $x \in \bigcap_{n\in\omega} V_n$, then $x \in U_n'$, for all n. If there is k such that $x \in X_k$, then $X_k \subset U_n'$, for all n, and hence $s_k \in U_n$, for all n, contradiction. If there is α such that $x = \langle\omega, \alpha+1\rangle$, then $a_\alpha \in U_n$, for all n, contradiction.

Thus σ is defeated and hence X_{A_Z} is semi-proximal. □

5.3 Subspaces of finite powers of ω_1

Characterizing normality in subspaces of products of ordinals has been studied extensively (e.g., [29], [13], [46]). For example, Fleissner proved the following theorem in [13]:

Theorem 5.3.1 *The following are equivalent for a subspace X of a finite product of ordinals:*

1. *X is normal.*
2. *X is normal and strongly zero-dimensional.*
3. *X is collectionwise normal.*
4. *X is shrinking (i.e. every open cover of X is shrinkable).*

One might naturally question the general characteristics of semi-proximality in subspaces of products of ordinals. We can restrict our study to powers of ω_1 since subspaces of κ, for $\kappa > \omega_1$, are not in general Fréchet, and therefore not semi-proximal (even if normal) by Theorem 4.2.2 together with Theorem 4.2.10. So, we consider the relationship between normality and semi-proximality in subspaces of finite products of ω_1. Since every shrinking space is countably paracompact, it is worth mentioning the following corollary to Theorem 5.3.1.

Corollary 5.3.2 *Every normal subspace of finite powers of ω_1 is countably paracompact.*

Note that, as demonstrated in Section 5.2.2, not all semi-proximal subspaces of ω_1 are normal. In this section, we establish that normal subspaces of finite powers of ω_1 are semi-proximal. Furthermore, we demonstrate that normality and semi-proximality are equivalent in finite products of subspaces of ω_1. The case of the product of two subspaces was previously proven in [20], where Theorem 3.1.4 was established.

For a finite ordinal k and for $X \subseteq \omega_1^{k+1}$, fix the following notations:

1. $\Delta(X) = \{\alpha : \text{the } k\text{-tuple } (\alpha, \dots, \alpha) \in X\}$.
2. For $\alpha \in \omega_1$, $X_\alpha = \{\langle \delta_1, \dots, \delta_k \rangle \in \omega_1^k : \langle \alpha, \delta_1, \dots, \delta_k \rangle \in X\}$.
3. For $A \subset \omega_1$, let $X_A^k = X \cap (A \times \omega_1^k)$.
4. For $A, B \subseteq \omega_1$, let $X_A^{B^k} = X \cap (A \times B^k)$.

In order to prove that following theorem, recall a lemma in [13]:

Lemma 5.3.3 *Let X be a normal subspace of κ^n, where κ is an uncountable regular cardinal and $n \in \omega$. If $\Delta(X)$ is not stationary, then there is C, club in κ, such that $X \cap C^n = \emptyset$.*

Theorem 5.3.4 *A normal subspace of finite powers of ω_1 is semi-proximal.*

Proof. Let $k \in \omega$ and X be a normal subspace of ω_1^{k+1}. The statement is true if $k = 0$, by Theorem 3.1.3. Assume the theorem is valid for all $j \leq k$.

Claim 5.3.5 *For $\alpha \in \omega_1$, $X^k_{[0,\alpha]}$ is semi-proximal.*

Proof. Let $Y = X^k_{[0,\alpha]}$. We will prove that Y is semi-proximal by induction on α. For $\alpha = 0$, Y is a normal subspace of $\{0\} \times \omega_1^k$. Since $\{0\} \times \omega_1^k$ is homeomorphic to ω_1^k, Y is semi-proximal by the induction hypothesis on k. Now, assume it is true for all $\gamma < \alpha$. If α is a successor, then Y is semi-proximal since it is a disjoint union of clopen semi-proximal subspaces. If α is a limit ordinal. Note that $(\{\alpha\} \times \omega_1^k) \cap Y = \{\alpha\} \times Y_\alpha$ is closed in Y and homeomorphic to $Y_\alpha \subseteq \omega_1^k$. Since $\{\alpha\} \times Y_\alpha$ is normal, then Y_α is a normal subspace of ω_1^k. Now, we have two cases:

Case 1: $\Delta(Y_\alpha)$ is not stationary. By Lemma 5.3.3, a club C exists such that $C^k \cap Y_\alpha = \emptyset$. Note that $X^{C^k}_{[0,\alpha)}$ and $X^k_{\{\alpha\}}$ are disjoint closed subsets of Y. Given that Y is normal and strongly zero dimensional by Theorem 5.3.1, there exists a clopen set U in Y such that $X^{C^k}_{[0,\alpha)} \subseteq U$ and $X^k_{\{\alpha\}} \subseteq Y \setminus U$. Note that U is normal since it is closed in Y. We fix a strictly increasing cofinal sequence $\{\alpha_n : n \in \omega\}$ in α. Then, each $U^k_{(\alpha_{n-1},\alpha_n]}$ is semi-proximal by the inductive hypothesis on α. Since $U \subseteq X^k_{[0,\alpha)}$, it can be represented as the topological sum $\bigoplus_{n\in\omega} U^k_{(\alpha_{n-1},\alpha_n]}$, where $\alpha_{-1} = -1$. Thus, U is semi-proximal. Note that $Y \setminus U$ is also semi-proximal since it is a subset of the metrizable space $(\alpha+1) \times (\omega_1^k \setminus C^k)$. Therefore, Y is semi-proximal since it is a disjoint union of clopen semi-proximal subspaces.

Case 2: $\Delta(Y_\alpha)$ is stationary. Let $\mathfrak{U}$ be the uniformity induced on Y as a subspace of the compact space $(\alpha + 1) \times (\omega_1 + 1)^k$. Therefore, we will play the open cover version of the proximal game, where Player I will play finite clopen partitions of the space. Let σ be a strategy for Player II in the proximal game on $(Y, \mathfrak{U})$. Let $\mathcal{M}$ be a countable elementary submodel of a large enough portion of the universe containing σ and $\Delta(Y_\alpha)$ such that $\lambda = \mathcal{M} \cap \omega_1 \in \Delta(Y_\alpha)$. This can be done since $\Delta(Y_\alpha)$ is stationary. Let $\eta < \alpha$, $\gamma \in \omega_1$, and $A \subseteq (k+1) \setminus \{0\}$ and define

$$X_{\eta,\gamma,A} = X \cap ((\eta, \alpha] \times \prod_{i \in A} [0, \gamma] \times \prod_{i \notin A} (\gamma, \omega_1)),$$

and consider the following finite partition of Y:

$$\mathcal{U}_{\eta,\gamma} = \left\{ X^k_{[0,\eta]} \right\} \cup \left\{ X_{\eta,\gamma,A} : A \subseteq (k+1) \setminus \{0\} \right\}$$

Let $\{\alpha_n : n \in \omega\}$ and $\{\lambda_n : n \in \omega\}$ be strictly increasing sequences that converge to α and λ, respectively. For $n \in \omega$, let $x_n = \langle \beta_n, \delta^n_1, \ldots, \delta^n_k \rangle \in Y$ be the point chosen by Player II during the nth inning following the strategy σ. We define a winning counter-strategy τ_σ for Player I. In inning $n = 0, 1$, let $\eta_n = \alpha_n$ and $\gamma_n = \lambda_n$. Define $\tau_\sigma(\emptyset) = \mathcal{U}_0$ and $\tau_\sigma(x_0) = \mathcal{U}_1$, where $\mathcal{U}_0 = \mathcal{U}_{\eta_0,\gamma_0}$ and $\mathcal{U}_1$ is the common refinement of $\mathcal{U}_{\eta_0,\gamma_0}$ and $\mathcal{U}_{\eta_1,\gamma_1}$. For $n \in \omega$, assume we have chosen η_{n+1} in inning $n+1$ such that $\eta_n < \eta_{n+1} < \alpha$ and γ_{n+1} such that $\gamma_n < \gamma_{n+1} < \omega_1$, and assume we have defined $\tau_\sigma(x_0, \ldots, x_n) = \mathcal{U}_{n+1}$, where $\mathcal{U}_{n+1}$ is the common refinement of $\mathcal{U}_{\eta_0,\gamma_0}, \ldots, \mathcal{U}_{\eta_{n+1},\gamma_{n+1}}$. Note that, $x_{n+1} \in U_n$, for some $U_n \in \mathcal{U}_n$. If U_n is not a subset of $X_{\eta_n,\gamma_n,\emptyset} = X \cap ((\eta_n, \alpha] \times (\gamma_n, \omega_1)^k)$, then the remainder of the game would be conducted within a semi-proximal space. Consequently, there exists a play of the game where Player I defeats σ. Otherwise, let $\eta_{n+2} < \alpha$ such that $\eta_{n+2} > \max\{\eta_{n+1}, \beta_{n+1}, \alpha_{n+2}\}$ and let $\gamma_{n+2} \in \mathcal{M}$ such that $\gamma_{n+2} > \max(\{\gamma_{n+1}, \lambda_{n+2}\} \cup \{\delta^{n+1}_1, \ldots, \delta^{n+1}_k\})$, and define $\tau_\sigma(x_0, \ldots, x_{n+1}) = \mathcal{U}_{n+2}$ such that $\mathcal{U}_{n+2}$ is the common clopen refinement of $\mathcal{U}_{\eta_0,\gamma_0}, \ldots, \mathcal{U}_{\eta_{n+2},\gamma_{n+2}}$. If this is the case for all $n \in \omega$, then we get $\{\langle \beta_n, \delta_1^n, \ldots, \delta_k^n \rangle \colon n \in \omega\}$ converges to $\langle \alpha, \lambda, \ldots, \lambda \rangle \in Y$ and hence τ_σ defeats σ. Thus, Y is semi-proximal. $\square$

Now, consider two cases for $X \subset \omega_1^{k+1}$:

Case 1: if $\Delta(X)$ is not stationary, then by Lemma 5.3.3, there exists a club $C \subset \omega_1$ such that $X \cap C^{k+1} = \emptyset$. Let $V_i = \{\langle \beta_0, \ldots, \beta_k \rangle \in \omega_1^{k+1} : \beta_i \notin C\}$, for every $i \leq k+1$. By Claim 5.3.5, $X^k_{(\alpha,\alpha^+]}$ is a clopen semi-proximal subspace, where $\alpha^+ = \min(C \setminus \alpha)$, for each $\alpha \in C$. Thus, each $V_i \cap X$ is semi-proximal since it can be written as the topological sum $\bigoplus_{\alpha \in C} X^k_{(\alpha,\alpha^+]}$. By Theorem 5.3.1, X is normal and strongly zero-dimensional. Thus, the finite open cover $\{V_i \cap X : 0 < i \leq k+1\}$ of X has a disjoint clopen refinement $\{W_i : 0 < i \leq k+1\}$. Since each W_i is semi-proximal, X is semi-proximal.

Case 2: $\Delta(X)$ is stationary. Let $\mathfrak{U}$ be the uniformity induced on X as a subspace of $(\omega_1+1)^{k+1}$. Therefore, we will play the open cover version of the proximal game, where Player I will play finite clopen partitions of the space. Let σ be a strategy for Player II in the proximal game on $(X, \mathfrak{U})$, and let $\mathcal{M}$ be a countable elementary submodel of a large enough portion of the universe containing σ and $\Delta(X)$. Let $\lambda = \mathcal{M} \cap \omega_1 \in \Delta(X)$ and $\{\lambda_n : n \in \omega\}$ be a strictly increasing sequence that converges to λ. For every $\gamma \in \omega_1$ and $A \subseteq (k+1)$ define

$$X_{\gamma,A} = X \cap \Big(\prod_{i \in A} [0,\gamma] \times \prod_{i \notin A} (\gamma, \omega_1) \Big),$$

then $\mathcal{U}_\gamma = \Big\{ X_{\gamma,A} : A \subseteq (k+1) \Big\}$ is a finite partition of X.

Let $x_n = \langle \delta_0^n, \ldots, \delta_k^n \rangle$ be the point chosen by Player II using σ. We will define a winning counter-strategy τ_σ for Player I. In inning $n = 0, 1$, let $\gamma_n = \lambda_n$. Define $\tau_\sigma(\emptyset) = \mathcal{U}_0$ and $\tau_\sigma(x_0) = \mathcal{U}_1$, where $\mathcal{U}_0 = \mathcal{U}_{\gamma_0}$ and $\mathcal{U}_1$ is the common refinement of $\mathcal{U}_{\gamma_0}$ and $\mathcal{U}_{\gamma_1}$. Let $n > 0$ and assume that, in inning $n+1$, we have chosen γ_{n+1} such that $\gamma_n < \gamma_{n+1} < \omega_1$. Assume we have defined $\tau_\sigma(x_0, \ldots, x_n) = \mathcal{U}_{n+1}$, where $\mathcal{U}_{n+1}$ is the common refinement of $\mathcal{U}_{\gamma_0}, \ldots, \mathcal{U}_{\gamma_{n+1}}$. Note that, $x_{n+1} \in U_n$, for some $U_n \in \mathcal{U}_n$. If U_n is not a subset of $X_{\gamma_n, \emptyset} = (\gamma_n, \omega_1)^{k+1} \cap X$, then the rest of the game would be played inside a clopen semi-proximal space. Thus, there is a subsequent play of the game where Player I would find a way to defeat σ. Otherwise, let $\gamma_{n+2} \in \mathcal{M}$ such that $\gamma_{n+2} > \max(\{\gamma_{n+1}, \lambda_{n+2}\} \cup \{\delta_0^{n+1}, \ldots, \delta_k^{n+1}\})$. Define $\tau_\sigma(x_0, \ldots, x_{n+1}) = \mathcal{U}_{n+2}$ such that $\mathcal{U}_{n+2}$ is the common clopen refinement of $\mathcal{U}_{\gamma_0}, \ldots, \mathcal{U}_{\gamma_{n+2}}$. If this is the case for all $n \in \omega$, then $\{\langle \delta_0^n, \ldots, \delta_k^n \rangle : n \in \omega\}$ converges to $\langle \lambda, \ldots, \lambda \rangle \in X$ and hence τ_σ defeats σ. Thus, X is semi-proximal. $\square$

Now, we will prove that normality and semi-proximality are equivalent for the product of finitely many subspaces of ω_1. To prove this, we need some preliminary results.

Lemma 5.3.6 *If $X = \prod_{k<n} A_k$ is a finite product of non-empty subspaces of ω_1 such that X is semi-proximal and $\bigcap_{k<n} A_k$ is not stationary, then there is $k < n$ such that A_k is not stationary.*

Proof. We will establish the proof by induction on n. The cases $n = 0, 1$ are straightforward, and the case where $n = 2$ is given by Theorem 3.1.4. Assume the statement holds true for n and suppose that $\{A_k : k \leq n\}$ is such that its product $X = \prod_{k \leq n} A_k$ is semi-proximal and $\bigcap_{k \leq n} A_k$ is not stationary. For each $i \leq n$, define

$$X_i = \{\langle \alpha_0, \ldots, \alpha_n \rangle \in X : \exists \beta \text{ such that } \alpha_k = \beta, \forall k \neq i\}.$$

For each i, X_i is a closed subspace of X and is therefore semi-proximal. Note that each X_i is homeomorphic to the product of the two sets, $\left(\bigcap_{k \neq i} A_k\right) \times A_i$. Now suppose that A_k is stationary for all $k \leq n$. According to Theorem 3.1.4, $\bigcap_{k \neq i} A_k$ is not stationary for all $i \leq n$. By our inductive hypothesis, this implies that $\prod_{k \neq i} A_k$ is not semi-proximal for all i. Consequently, X_i is not semi-proximal, contradiction. Therefore, there must exist $k \leq n$ for which A_k is not stationary. $\square$

In [38] Przymusiński proved the following theorem characterizing normality of products with a metric factor.

Theorem 5.3.7 *Let $\mathcal{B}$ be a base for a metrizable space M. The product space $X \times M$ is normal if and only if X is normal and for every family $\{F_B : B \in \mathcal{B}\}$ of closed subsets of X such that if $B \subset B'$ then $F_B \subset F_{B'}$ and for all $y \in M$, $\bigcap\{F_B : y \in B\} = \emptyset$, there exists a family $\{U_B : B \in \mathcal{B}\}$ of open subsets of X such that $F_B \subset U_B$ and for all $y \in M$, $\bigcap\{U_B : y \in B\} = \emptyset$.*

We now show that it follows from Prymusiński's theorem that if X is normal and countably paracompact, then its product with any countable metrizable space is normal.

Corollary 5.3.8 *The product of a countably paracompact normal space with a countable metric space is normal.*

Proof. Let X be a normal and countably paracompact space and let Y be a countable metric space. It is straightforward to show that any countable metrizable space Y has a base $\mathcal{B} = \bigcup_{y \in Y} \mathcal{B}_y$ such that, for all $y \in Y$, $\mathcal{B}_y$ is a decreasing local neighborhood base at y, and $\mathcal{B}_y \cap \mathcal{B}_z = \emptyset$, for $z \neq y$ in Y.

So we fix such a base and let $\{F_B : B \in \mathcal{B}\}$ be a family of closed subsets of X which satisfies the condition in Theorem 5.3.7. Note that by the monotonicity property of the family of F_B's we have that for every $y \in Y$, $\{F_B : B \in \mathcal{B}_y\}$ is a decreasing sequence of closed subsets of X. Since $\mathcal{B}_y$ is a local base at y we have $\bigcap\{F_B : B \in \mathcal{B}_y\} \subseteq \bigcap\{F_B : B \in \mathcal{B},\ y \in B\} = \emptyset$. Since X is countably paracompact, there exists a family of open sets $\{U_B : B \in \mathcal{B}_y\}$ in X such that $\bigcap\{U_B : B \in \mathcal{B}_y\} = \emptyset$ and $F_B \subset U_B$, for all $B \in \mathcal{B}_y$. Since $\mathcal{B}_y$ and $\mathcal{B}_z$ are disjoint for all $y \neq z$, the family $\{U_B : B \in \mathcal{B}\}$ is well defined. To show that $X \times Y$ is normal, it remains to prove that $\{U_B : B \in \mathcal{B}\}$ satisfies the conclusion of Theorem 5.3.7. To see that, let $y \in Y$ then we have $\bigcap\{U_B : y \in B\} \subset \bigcap\{U_B : B \in \mathcal{B}_y\} = \emptyset$. □

Theorem 5.3.9 *If $X = \prod_{k<n} A_k$ is a finite product of non-empty subspaces of ω_1. Then X is semi-proximal if and only if X is normal.*

Proof. We have already shown in Theorem 5.3.4 that any normal subspace of ω_1^n is semi-proximal, so we only need to show sufficiency.
We will prove by induction on n that for any family of n many subspaces of ω_1, if the product is semi-proximal, then the product is normal. The base case, $n = 2$, is given by Theorem 3.1.4. Now assume it is valid for n and suppose that $\{A_k : k \leq n\}$ is such that its product $X = \prod_{k \leq n} A_k$ is semi-proximal. Then we have two cases:

Case 1: If there exists $k_0 \leq n$ such that A_{k_0} is not stationary, then there exists a club C such that $A_{k_0} = \bigoplus_{\gamma \in C}(\gamma, \gamma^+] \cap A_{k_0}$, where $\gamma^+ = \min C \setminus \gamma$.

Then $X = \bigoplus_{\gamma \in C} X_\gamma$, where

$$X_\gamma = \Big((\gamma, \gamma^+] \cap A_{k_0} \Big) \times \prod_{k \le n, k \ne k_0} A_k.$$

Since $\prod_{k \le n, k \ne k_0} A_k$ is semi-proximal, it is normal by induction, and hence it is countably paracompact by Corollary 5.3.2. Therefore, X_γ is normal by Corollary 5.3.8, for all $\gamma \in C$. Thus, X is normal.

Case 2: If A_k is stationary for all $k \le n$, then $\Delta(X) = \bigcap_{k \le n} A_k$ is stationary as stated in Lemma 5.3.6. Let H and K be two disjoint closed subsets of X. Given that $\Delta(X)$ is stationary, there exists an $\gamma \in \Delta(X)$ such that either H or K is a subset of $X^\gamma = \bigoplus_{k \le n} Y_k$, where

$$Y_k = (\gamma + 1) \times \prod_{j<k} (A_j \setminus \gamma + 1) \times \prod_{k<m \le n} A_m.$$

To prove this, suppose neither H nor K is a subset of X^γ, for all $\gamma \in \omega$. Let $\mathcal{M}$ be a countable elementary submodel such that $A_k \in \mathcal{M}$ for all $k \le n$. We define $\alpha = \mathcal{M} \cap \omega_1 \in \Delta(X)$. Consider $(\alpha_i : i \in \omega)$, an increasing sequence that converges to α. Then there exists $(\langle x_i^0, \ldots, x_i^n \rangle : i \in \omega) \subset H$ and $(\langle y_i^0, \ldots, y_i^n \rangle : i \in \omega) \subset K$ such that for each $i \in \omega$ $x_i^k < y_i^k < \alpha_i < x_{i+1}^k$, for all $k \le n$. Then, $\langle x_i^0, \ldots, x_i^n \rangle < \langle y_i^0, \ldots, y_i^n \rangle < \langle \alpha_i, \ldots, \alpha_i \rangle < \langle x_{i+1}^0, \ldots, x_{i+1}^n \rangle$. Hence $(\langle x_i^0, \ldots, x_i^n \rangle : i \in \omega)$ and $(\langle y_i^0, \ldots, y_i^n \rangle : i \in \omega)$ would converge to $\langle \alpha, \ldots, \alpha \rangle \in H \cap K$, which contradicts our initial assumption.

So, suppose $K \subset X'$. Note that $\prod_{j<k} (A_j \setminus \alpha + 1) \times \prod_{k<m \le n} A_m$ is semi-proximal since it is homeomorphic to a closed subspace of X. By induction, it is normal, and according to Corollary 5.3.2, it is countably paracompact. By Corollary 5.3.8, Y_k is normal for all $k \le n$. Therefore X' is normal. Thus, there exist two disjoint open sets U' and V in X' with $(H \cap X') \subseteq U'$

and $K \subseteq V$. Since X' is clopen in X, U' and V are open in X. Let $U = U' \cup \prod_{k \le n}(A_k \setminus (\alpha + 1))$, then U and V are disjoint open subsets in X which separate H and K. Thus, X is normal. □

The following characterization of countable paracompactness must be known, although it seems it has not been explicitly stated before. This characterization can be easily derived from Corollary 5.3.8.

Theorem 5.3.10 *A normal space is countably paracompact if and only if its product with any countable metrizable space is normal.*

5.4 Pseudocompact Spaces

When Nyikos asked whether semi-proximality implies normality, he suggested that every semi-proximal countably compact space is normal. However, we prove that it is valid for pseudocompact. Upon establishing the following theorem, we derive the result for countably compact spaces as a corollary since pseudocompact is a weaker property than countably compact for Tychonoff spaces [22].

Theorem 5.4.1 *Every semi-proximal pseudocompact space is normal.*

Proof. Let $(X, \mathfrak{U})$ be a pseudocompact uniform space that is not normal. Note that $\mathfrak{U}$ is a totally bounded uniformity by Corollary 2.2.9. This, together with (3) in Theorem 2.2.3, would help to define a winning strategy for Player II in a non-normal pseudocompact space. Since X is not normal, there exist two disjoint closed subsets, H and K that cannot be separated by disjoint open sets. We will define a winning strategy σ for Player II in the 2-proximal game on $(X, \mathfrak{U})$ by Definition 3.1.8. Let $U_0 \in \mathfrak{U}$.

Claim 5.4.2 *There exists $x \in H \cap U_0[K]$ such that for every $U \subseteq U_0$ in $\mathfrak{U}$, $K \cap U[H] \cap 2U_0[x] \neq \emptyset$.*

Proof. Suppose not, then for every $x \in H \cap U_0[K]$, there is $U_x \subseteq U_0$ such that $K \cap U_x[H] \cap 2U_0[x] = \emptyset$. Since $\mathfrak{U}$ is totally bounded, consider a finite set $F \subset H \cap U_0[K]$ such that $\{U_0[x] : x \in F\}$ covers $H \cap U_0[K]$. Let $U = \bigcap_{x \in F} U_x$. Since H and K cannot be separated, then $H \cap U[K] \neq \emptyset$. Then, there exist $z \in H$ and $y \in K$ such that $z \in U[y] \subseteq U_0[y]$. Since there exists $x \in F$ such that $z \in U_0[x]$, $y \in 2U_0[x]$. Thus, $y \in K \cap U_x[H] \cap 2U_0[x]$ which is a contradiction. □

Therefore, define $\sigma(U_0) \in H \cap U_0[K]$ satisfying Claim 5.4.2. At stage $n+1$, assume we have defined σ for all finite sequences $(U_0, \ldots, U_n)$ such that:

1. If n is even, then $\sigma(U_0, \ldots, U_n) \in H \cap U_n[K]$ and for all $U \subseteq U_n$, $K \cap U[H] \cap 2U_n[\sigma(U_0, \ldots, U_n)] \neq \emptyset$.

2. If n is odd, then $\sigma(U_0, \ldots, U_n) \in K \cap U_n[H]$ and for all $U \subseteq U_n$, $H \cap U[K] \cap 2U_n[\sigma(U_0, \ldots, U_n)] \neq \emptyset$.

Let $U_0, \ldots, U_n, U_{n+1}$ be a sequence of entourages of length $n+1$. If n is odd, then by the inductive hypothesis, $H \cap U_{n+1}[K] \cap 2U_n[\sigma(U_0, \ldots, U_n)] \neq \emptyset$.

Claim 5.4.3 *There exists $x \in H \cap U_{n+1}[K] \cap 2U_n[\sigma(U_0, \ldots, U_n)]$ such that $K \cap U[H] \cap 2U_{n+1}[x] \neq \emptyset$, for every $U \subseteq U_{n+1}$.*

Proof. Let $A = H \cap U_{n+1}[K] \cap 2U_n[\sigma(U_0, \ldots, U_n)]$ and suppose that for every $x \in A$, there is $U_x \subseteq U_{n+1}$ such that $K \cap U_x[H] \cap 2U_{n+1}[x] = \emptyset$. Consider a finite set $F \subset A$ such that $\{U_{n+1}[x] : x \in F\}$ covers A. Let $U = \bigcap_{x \in F} U_x$. Since $U \subseteq U_n$, then there is some $z \in H \cap U[K] \cap 2U_n[\sigma(U_0, \ldots, U_n)] \subseteq A$.

Then, there exists $y \in K$ such that $z \in U[y] \subseteq U_{n+1}[y]$. Since $z \in A$, there exists $x \in F$ such that $z \in U_{n+1}[x]$. Therefore, $y \in K \cap U_x[H] \cap 2U_{n+1}[x]$ which is a contradiction. $\square$

Define $\sigma(U_0, \ldots, U_{n+1}) \in H \cap U_{n+1}[K] \cap 2U_n[\sigma(U_0, \ldots, U_n)]$ satisfying Claim 5.4.3. Similarly, if n is even, we can define $\sigma(U_0, \ldots, U_n, U_{n+1})$ to be an element of $K \cap U_{n+1}[H] \cap 2U_n[\sigma(U_0, \ldots, U_n)]$ such that for every $U \subseteq U_{n+1}$, $H \cap U[K] \cap 2U_{n+1}[\sigma(U_0, \ldots, U_n, U_{n+1})] \neq \emptyset$.

The sequence resulting from a play of the game using σ is not convergent since Player II alternate between two disjoint closed sets. Since X is pseudocompact and since $\{int(U_n[\sigma(U_0, \ldots, U_n)]) : n \in \omega\}$ is a countable family of open sets which has the finite intersection property, then $\emptyset \neq \bigcap_{n\in\omega} \overline{U_n[\sigma(U_0, \ldots, U_n)]} \subseteq \bigcap_{n\in\omega} 2U_n[\sigma(U_0, \ldots, U_n)]$. Hence, X is not semi-proximal. $\square$

Corollary 5.4.4 *[36] Every semi-proximal countably compact is normal.*

Chapter 6

Semi-proximal Spaces and Ramsey Property

Semi-proximality is a convergence property, so it's natural to ask about its relation to other convergence properties. For example, bisequentiality and the Ramsey property. Indeed, It is known that every bisequential space is α_3 [1]. On the other hand, semi-proximality and α_2-Fréchet are equivalent in paracompact scattered spaces by Theorem 4.2.2 and Theorem 4.2.2. Under the Continuum Hypothesis (CH), we construct a Ψ-space such that its one-point compactification is α_1-Fréchet space that does not have the Ramsey property. The authors in [11] proved that compact bisequential spaces have the Ramsey property. We provide a ZFC example of a compact bisequential space that is not semi-proximal.

6.1 Definitions

Recall that for a countably infinite set X, a compact space K and a positive $r \in \omega$, we say that a function $f : [X]^r \to K$ converges to $p \in K$ if for every neighbourhood U of p, there is a finite set F such that $f([X \setminus F]^r) \subseteq U$. Once this happens for some p, we say that f is convergent [6]. Kubiś and Szeptycki introduced a convergence notion in the context of *Ramsey* [32]. Let X be a topological space and let $r \in \omega$ be positive. We shall say that X has the r**-Ramsey property** (or X is an r-Ramsey space) if for every function $f : [\omega]^r \to X$ there exist $p \in X$ and an infinite set $B \subseteq \omega$ such that $f \restriction [B]^r$ converges to p. We shall say that X has **the Ramsey property** if it has the r-Ramsey property for every positive $r \in \omega$. Every space with the r-Ramsey property has the $(r-1)$-Ramsey property, whenever $r > 1$.

6.2 Counterexamples

Given an almost disjoint family A, the Mrówka-Isbell space $\Psi(A)$ is the space $\omega \cup A$ where the points of ω are isolated and a neighbourhood base for $a \in A$ is given by sets of the form $\{a\} \cup (a \setminus n)$. The one-point compactification of $\Psi(A)$ is $K(A) = \Psi(A) \cup \{\infty\}$. Note that an open neighbourhood of ∞ is $K(A) \setminus \bigcup \mathcal{F}$, where $\mathcal{F}$ is a finite family containing finite subsets of ω or neighbourhood bases of elements in A. We say that A is a **mad** family if it is an almost disjoint family and is maximal with respect to this property. For $A \subseteq \mathcal{P}(\omega)$, let $A^\perp = \{Y \subseteq \omega : |Y \cap a| < \omega, \forall a \in A\}$. A sequence $S \subseteq \omega$ converges to ∞ if and only if $S \in A^\perp$. The ideal $I(A)$ generated by A is the set of all subsets of ω that can be covered by finitely many elements of A

together with a finite subset of ω. Given an ideal $I \subseteq \mathcal{P}(\omega)$, $I^+ = \mathcal{P}(\omega) \setminus I$. For an infinite set $X \subseteq \omega$, $\infty \in \overline{X}$ if and only if $X \in I(A)^+$. Then $K(A)$ is Fréchet if and only if it is **nowhere mad** [44] (i.e. for every $X \in I(A)^+$, there exists $Y \in A^\perp$ such that $|Y \cap X| = \omega$). Also, $K(A)$ is α_1 if and only if for every sequence $\{S_n : n \in \omega\} \subseteq A^\perp$ there is an $S \in A^\perp$ such that $S_n \setminus S$ is finite, for all n. For more on almost disjoint families and Ψ-spaces, we refer the reader to [24].

Assuming (CH), we will construct an almost disjoint family A on ω^{r+1} such that $K(A)$ is α_1-Fréchet r-Ramsey space that is not $(r+1)$-Ramsey for $r > 0$. This space is also semi-proximal by Theorem 4.2.7. In order to construct the almost disjoint family, we need to recall some notations and standard results along with some lemmas.

Recall that a pseudo-intersection of a family of sets is an infinite set S such that each element of the family contains all but a finite number of elements of S. The pseudo-intersection number $\mathfrak{p}$ is the smallest size of a family of infinite subsets of the natural numbers that has the strong finite intersection property (i.e. if the intersection over any finite subfamily is infinite) but has no pseudo-intersection. For any two functions $f, g \in \omega^\omega$, we denote by $f \leq^* g$ the statement that for all but finitely many n, $f(n) \leq g(n)$. The bounding number $\mathfrak{b}$ is the least cardinality of an unbounded set in this relation, that is, $\mathfrak{b} = \min(\{|F| : F \subseteq \omega^\omega \wedge \forall f \in \omega^\omega\, \exists g \in F \text{ such that } g \not\leq^* f\})$. Then we have $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq 2^\omega$, the proof can be found in [49].

Lemma 6.2.1 *[32] For any almost disjoint family A on a countable set I of isolated points, $K(A)$ is r-Ramsey if for every $f : [\omega]^r \to I$, there is an infinite $B \subseteq \omega$ such that $|\{a \in A : a \cap f([B]^r) \text{ is infinite}\}| < \mathfrak{b}$.*

Recall that FIN is the ideal of finite subsets of ω and, for each $n > 1$, FIN^n is the ideal on ω^n defined recursively by

$$A \in \mathrm{FIN}^n \text{ if } \{x \in \omega^{n-1} : \{j : x \frown j \in A\} \text{ is infinite}\} \in \mathrm{FIN}^{n-1}.$$

Lemma 6.2.2 *[32] For any function $f : [\omega]^n \to \omega^{n+1}$ there is a $B \subset \omega$ infinite such that $f([B]^n) \in FIN^{n+1}$.*

Theorem 6.2.3 *(CH) There is r-Ramsey α_1- Fréchet not $(r+1)$-Ramsey.*

Proof. Define a function $G : [\omega]^{r+1} \to \omega^{r+1}$ by $G(x) = \langle x(0), \dots, x(r)\rangle$ where $x = \{x(0), \dots, x(r)\}$ is the increasing enumeration of x. We will construct $A = \{a_\alpha : \alpha \in \omega_1\}$ an almost disjoint family on ω^{r+1} by defining each a_α by recursion on α. Enumerate $[\omega]^\omega = \{B_\alpha : \alpha < \omega_1\}$, $[\omega^{r+1}]^\omega = \{C_\alpha : \alpha < \omega_1\}$, $[[\omega^{r+1}]^\omega]^\omega = \{\mathcal{Z}_\alpha : \alpha < \omega_1\}$, and $(\omega^{r+1})^{[\omega]^r} = \{f_\alpha : \alpha \in \omega_1\}$.

We start by letting $\{a_n : n \in \omega\}$ be an enumeration of the disjoint family $\{\{x\} \times \omega : x \in \omega^r\}$. Note that each $a_n \in \mathrm{FIN}^{r+1}$ and any set that is almost disjoint from all the a_n is also in FIN^{r+1}. Assume we have constructed $\{X_\beta : \beta < \alpha\} \subset [\omega]^\omega$, and other three families in $[\omega^{r+1}]^\omega$, $A_\alpha = \{a_\beta : \beta < \alpha\}$, $\{Y_\beta : \beta < \alpha\}$, and $\{Z_\beta : \beta < \alpha\}$ such that:

1. For all $\omega \le \beta < \alpha$, $Y_\beta \in [C_\beta]^\omega$ and $Y_\beta \in A_\alpha^\perp$.

2. For all $\omega \le \beta < \alpha$, $Z \setminus Z_\beta$ is finite, for all $Z \in \mathcal{Z}_\beta$ and $Z_\beta \in A_\alpha^\perp$.

3. For all $\beta < \alpha$, $f_\beta([X_\beta]^r) \in \mathrm{FIN}^{r+1}$.

4. For all $\beta < \gamma < \alpha$, $a_\beta \cap a_\gamma$ is finite.

5. For all $\omega \le \beta < \gamma < \alpha$, $f_\beta([X_\beta]^r) \cap a_\gamma$ is finite.

6. For all $\omega \leq \beta < \alpha$ and $n \in \omega$, $a_\beta \setminus G([B_\beta \setminus n]^{r+1})$ is finite.

To define Y_α and Z_α, assume without loss of generality that $C_\alpha \in I(\mathcal{A}_\alpha)^+$ and $\mathcal{Z}_\alpha \subset A_\alpha^\perp$. Since $\omega^{r+1} \cup A_\alpha \cup \{\infty\}$ is first countable, it is α_1-Fréchet. Thus, there exists an infinite $Y_\alpha \subset X_\alpha$ such that $Y_\alpha \in A_\alpha$, and there is a sequence $Z_\alpha \in A_\alpha^\perp$ such that $Z \setminus Z_\alpha$ is finite, for all $Z \in \mathcal{Z}_\alpha$. Hence, Y_α and Z_α satisfies the inductive hypothesises (1) and (2), respectively. Since $f_\alpha \in (\omega^{r+1})^{[\omega]^r}$, there exists an infinite subset X_α of ω such that $f_\alpha([X_\alpha]^r)$ is an element of FIN^{r+1} by Lemma 6.2.2. Hence, X_α satisfies the inductive hypothesis (3). It remains to define a_α. Let

$$\mathcal{H} = \{f_\beta([X_\beta]^r) : \beta < \alpha\} \cup \{a_\beta : \beta < \alpha\} \cup \{Y_\beta : \beta < \alpha\} \cup \{Z_\beta : \beta < \alpha\},$$

then $\mathcal{H} \subseteq \mathrm{FIN}^{r+1}$. Note that the family $\{G([B_\alpha \setminus n]^{r+1}) \setminus H : n \in \omega, H \in \mathcal{H}\}$ has an infinite pseudo-intersection since its cardinality is less than the pseudo-intersection number $\mathfrak{p} = \omega_1$. Let a_α be any such infinite pseudo-intersection. Thus, a_α satisfies the inductive hypothesis (4)-(6). This completes the construction of $A = \{a_\alpha : \alpha \in \omega_1\}$.

To see that $K(A) = \omega^{r+1} \cup A \cup \{\infty\}$ is r-Ramsey, let $f : [\omega]^r \to \omega^{r+1}$. Then there exists α such that $f = f_\alpha$. Note that $|\{a \in A : a \cap f_\alpha([X_\alpha]^r) \text{ is infinite}\}|$ is countable, so it is less than $\mathfrak{b} = \omega_1$. Thus, $K(A)$ is r-Ramsey by Lemma 6.2.1. The space $K(A)$ is α_1. Indeed, let $\mathcal{S} = (S_n : n < \omega) \subset A^\perp$, then there exists $\alpha \in \omega_1$ such that $\mathcal{S} = \mathcal{Z}_\alpha$ and hence Z_α witnesses α_1. Also, it is Fréchet since if $C \in I(A)^+$, then there exists an $\alpha \in \omega_1$ such that $C = C_\alpha$ and hence Y_α witnesses the property. To see that the space is not $(r+1)$-Ramsey. Let B be an infinite subset of ω, then there exists $\alpha \in \omega_1$ such that $B = B_\alpha$. Indeed, if there was p such that G converges to p, then p would be ∞. Let

$n \in \omega$ and $U = K(A) \setminus ((a_\alpha \setminus n) \cup \{a_\alpha\})$, then U is an open neighbourhood of ∞, but $G([B_\alpha \setminus n]^{r+1}) \setminus U$ is infinite since $\{a_k : |a_k \cap G([B_\alpha \setminus n]^{r+1})| = \aleph_0\}$ is infinite. Hence, $K(A)$ is not $(r+1)$-Ramsey. $\square$

Now, we will demonstrate the other direction, but first, we will recall some notations and definitions. For a topological space X, a filter base $\mathcal{G}$ converges to a point $x \in X$ if, for every neighbourhood U of x, there is a $G \in \mathcal{G}$ such that $G \subseteq U$. Given a filter $\mathcal{F}$, recall that $\mathcal{F}^+$ denotes the family of all sets which intersects every element of $\mathcal{F}$ in an infinite set. And we say that $x \in \overline{\mathcal{F}}$ if $x \in \overline{F}$, for all $F \in \mathcal{F}$. E.A. Michael in [34] defined a space X to be **bisequential** if for $x \in X$ and every filter $\mathcal{F}$ in X such that $x \in \overline{\mathcal{F}}$, there exists a decreasing sequence $\mathcal{G} = \{G_n : n \in \omega\} \subseteq \mathcal{F}^+$ such that $\mathcal{G}$ converges to x. Recall that an almost disjoint family A is **$\mathbb{R}$-embeddable** if there is a one-to-one function $f : \omega \to \mathbb{Q}$ which extends to a continuous one-to-one function $F : \Psi(A) \to \mathbb{R}$, see [21]. Gruenhage and Szeptycki proved the following in [17]:

Proposition 6.2.4 *If the almost disjoint family A is $\mathbb{R}$-embeddable, then $K(A)$ is bisequential.*

Since every compact bisequential has the Ramsay property, it is enough to provide an example of a compact bisequential space X that is not semi-proximal. We will give an example of a Ψ-space over an almost disjoint family of branches in $2^{<\omega}$ as defined in Section 5.2.1.

Example 6.2.5 *There is a compact bisequential not semi-proximal space.*

Proof. Let $Z = 2^\omega$, then the one-point compactification $K(A_Z)$ is not semi-proximal. To see this, we will define a winning strategy σ in the proximal game with respect to the uniformity $\mathfrak{U}$ on $K(A_Z)$. Let $U_0 \in \mathfrak{U}$, define $\sigma(U_0) = \infty$. Now, for $U_1 \subseteq U_0$, define $\sigma(U_0, U_1) = s_1$ such that $[s_1] \subseteq U_1[\infty]$. For $n > 1$, if n is even, we define $\sigma(U_0, \ldots, U_n) = \infty$, while for odd n, we define $\sigma(U_0, \ldots, U_n) = s_n$ such that $[s_n] \subseteq U_n[\infty] \cap [s_{n-2}]$. Since $\bigcup_{n\in\omega} s_n \in 2^\omega$, $(\sigma(U_0, \ldots, U_n) : n \in \omega)$ is not convergent. Indeed, Let $a = \bigcup_{n\in\omega} s_n$ and $U = K(A_Z)\setminus(a \cup \{a\})$, then $(\sigma(U_0, \ldots, U_n) : n \in \omega)\setminus U$ is infinite. Moreover, $\infty \in \bigcap_{n\in\omega} U_n[\sigma(U_0, \ldots, U_n)]$. Therefore, $K(A_Z)$ is not semi-proximal.

Note that A_Z is $\mathbb{R}$-embeddable. To see that, consider a one-to-one function $f : 2^{<\omega} \to \mathbb{Q}$, and define $F : \Psi(A_Z) \to \mathbb{R}$ such that $F \restriction 2^{<\omega} = f$ and $F(a) = \lim_{n\to\infty} f(a \restriction n)$, for all $a \in A_Z$. Thus, F is continuous and extends f. Therefore, $K(A_Z)$ is bisequential by Proposition 6.2.4. $\square$

Chapter 7

Problems

Note that, in compact spaces, semi-proximality (proximality) and totally semi-proximality (totally proximality) are equivalent. In Section 4.2, we prove that every w-space (W-space) with finitely many non-isolated points is totally semi-proximal (totally proximal) if and only if the complement of every neighbourhood containing the isolated points is totally semi-proximal (countable). Indeed, if X is a discrete space of measurable (uncountable) cardinality, then X is a scattered paracompact w-space (W-space), but it is not totally semi-proximal (not totally proximal) by Corollary 4.1.4. This suggested the following two questions:

Question 7.0.1 *Is every compact w-space semi-proximal?*

Question 7.0.2 *Is it consistent that every scattered paracompact w-space is totally semi-proximal?*

Question 7.0.3 *Is every compact W-space proximal?*

Generally, first countability is one of the strongest convergence properties.

It is known that first countability implies both α_1-Fréchet and bisequential. In Section 6, we provide an example of a compact bisequential space that is not semi-proximal. It's important to note that this example is not first countable, prompting the following question.

Question 7.0.4 *Are first countable compact spaces semi-proximal?*

The relationship between normal and countably paracompactness in the class of semi-proximal spaces is not understood. It appears challenging to ascertain whether a Dowker space can be semi-proximal while it is evident that any example that is not Fréchet is not semi-proximal. In Section 5.1, we have a consistent example of Fréchet Dowker space that is not semi-proximal. We lack any instances of semi-proximal Dowker spaces. Ideally, we seek a ZFC example, yet we have not identified any consistent examples thus far.

Question 7.0.5 *Can there exist a semi-proximal Dowker space?*

Any uncountable subset of the real line that is a Δ-space is called a Δ-set [39]. R. Knight defined a topological space X to be a Δ-space if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X with empty intersection, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X, also with empty intersection, and such that $D_n \subseteq V_n$ for every $n \in \omega$ [31]. An uncountable set $X \subset \mathbb{R}$ is called a Q-set if every subset of X is relative G_δ. F. B. Jones proved that the well-known construction of the tangent disc topology over a set of reals X is normal and non-metrizable if and only if X is a Q-set. The same construction is also countably paracompact if and only if X is a Δ-set. R. Knight claims in [31] to have a consistent example of Δ-set, which is not a Q-set, but no experts in the field understand it. Therefore, whether, consistently, there is a Δ -set which is not a Q -set is still open. If Z is a Δ-set that is not a Q-set, then Z does not contain a Cantor set, so $\Psi(A_Z)$ is semi-proximal countably paracompact, not normal space.

Question 7.0.6 *Are semi-proximal countably paracompact spaces normal?*

N. Kemoto and others have been systematically studying separation properties of products of ordinals and their subspaces. One of the most interesting questions left open by this investigation is whether countably paracompact subspaces of ω_1^2 are normal. In [2], the authors constructed a consistent example of a subspace Z_L of ω_1^2 using a ladder system on a stationary subset of ω_1. They prove that it is countably paracompact if and only if the ladder system is thin and countably metacompact. However, we do not know whether this subspace is semi-proximal if the ladder system is thin and countably metacompact. On the other hand, we believe there is a ladder system where Z_L is not semi-proximal under $\diamond^*$. This motivated the following question:

Question 7.0.7 *Is every semi-proximal countably paracompact subspace of ω_1^2 normal?*

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