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- **Business**: Find the profit-maximizing rate of oil extraction from an oil well.
- **Economics**: Would a higher minimum wage increase unemployment?
- **Political Science**: How many lawn signs would maximize a candidate's chance to win an election?

Single-Variable Optimization Overview

- Graphing functions to find Extreme Points
- Using differential calculus to find candidates for extreme points
- The first and the second derivative test to evaluate these candidates
- Optimization in Economics, Business, and Social Sciences
- Local minimums and maximums, Inflection Points

What is an Extreme Point?

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- An **Extreme Point** is where a function reaches its largest or smallest values.
- So, an extreme point can be either a max or a min.
- To find the extreme points of a function, you must first know its domain.

Review: find the domain, D, of $f(x)$

$$f(x) = \frac{\sqrt{x} + 1}{x}$$

What is an Extreme Point?

- An extreme point is the maximum or minimum value of $f(x)$ over the domain D.

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- An extreme point is the maximum or minimum value of $f(x)$ over the domain D.

$$f(x) = 3 - (x-2)^2$$

- Does the above have a maximum? A minimum? Can you answer the question by simply looking at the function?

Another way to find the max/min is to graph the function:

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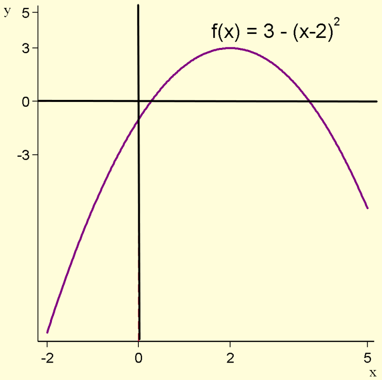
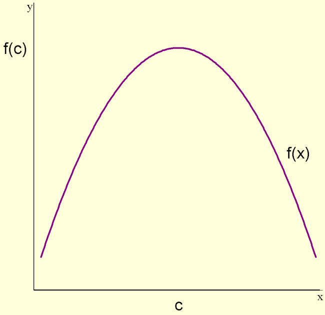


Generally, we can write:



What about the negative of $f(x)$?
$-f(x) = -3 + (x-2)^2$

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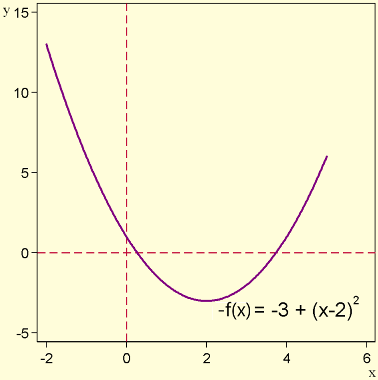

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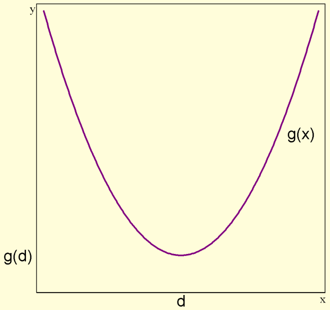
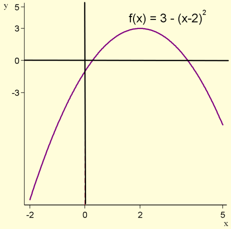
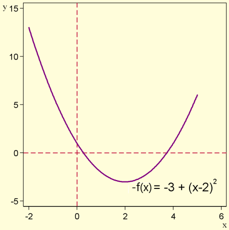


The max of $f(x)$ is equal to the min of $-f(x)$.





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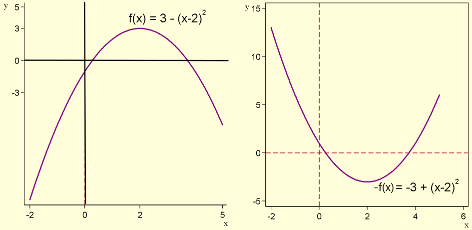

This is in general true.
If the max of $f(x)$ occurs at c, then
the min of $-f(x)$ also occurs at c.

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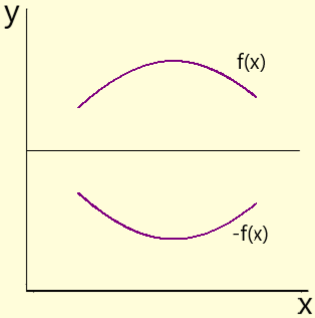

Find the Extreme Points for $g(x) = \sqrt{x-5} - 100$

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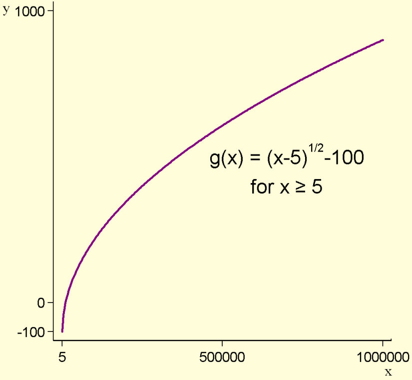

Find the Extreme Points

- Rarely are Optimization problems so easy!

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- A major challenge is to correctly model a real world problem as a function. This is the goal of the economists, political scientist, and business analyst.
- Next, we introduce calculus to help us make the most effective choice.

Differential Calculus in Optimization

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- Differential calculus can help make effective use of a resource.
- Optimization with calculus works on intervals of a function/relation that are both *continuous* and *differentiable*.

Continuous Functions

A function is **continuous** if

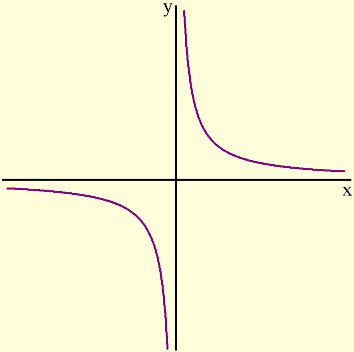
- You can draw the function from one end point to the other without lifting your pen from the paper.

Not Continuous Functions

A function is not continuous if it has

- Gaps (holes)
- Jumps

Not Continuous at $x = 0$

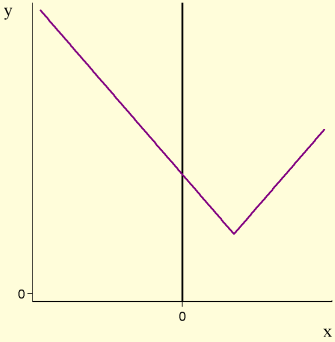


$f(x) = 1/x$

Non-Differentiable Points on a Continuous Function

Some functions are not differentiable at point(s):

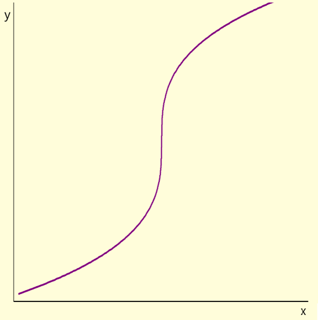
(i) $f(x) = 3 + |x - 2|$ when $x = 2$



Non-Differentiable Points on a Continuous Function

Some functions are continuous but not everywhere differentiable:

(ii) $g(x) = (x-3)^{1/3}$ if $x = 3$.



Interior of an Interval

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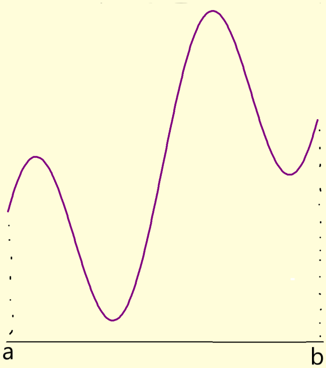
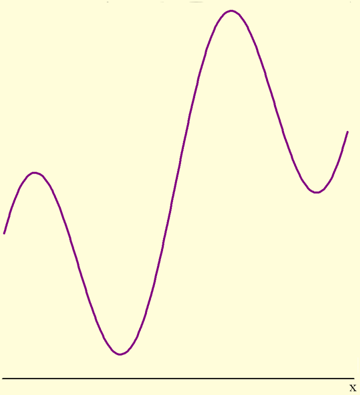
- For example, the interior of the interval $[a, b]$ is (a, b)

Interior of an Interval



The interior of the interval $[a, b]$ is (a, b)

Interior of an Interval



i) Label the end points

ii) Label the extreme points

Differential Calculus in Optimization

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- Sometimes these effective uses are called "optimal allocations."

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Restricting our analysis to continuous and differentiable intervals:

- c is an interior extreme point for the function f if and only if $f'(c) = 0$.

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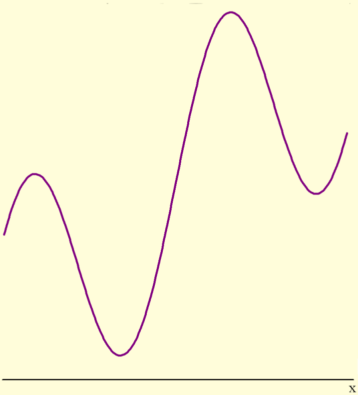
Restricting our analysis to continuous and differentiable intervals:

- c is an interior extreme point for the function f if and only if $f'(c) = 0$.
- Any x in the interior of an interval where $f'(x) = 0$ is called a **critical point**.
- For c to be an extreme point, it is necessary that $f'(c) = 0$. Otherwise, c is not an extreme point.

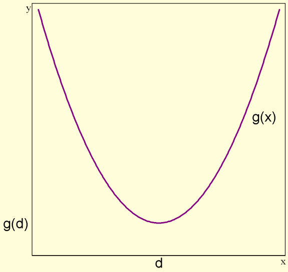
Critical Points



i) Find all critical points

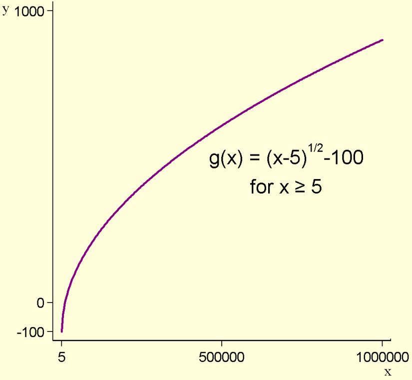
ii) Find all extreme points

Finding an interior Extreme Point



- For d to be the maximum or minimum value, the slope of g must be zero at d: $g'(d) = 0$.

Is there an interior extreme point?



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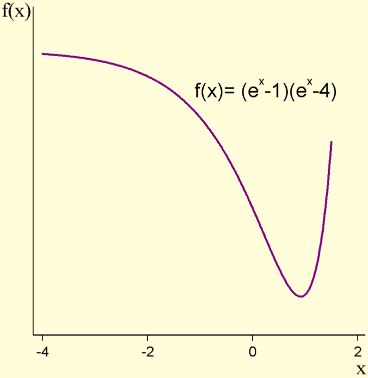
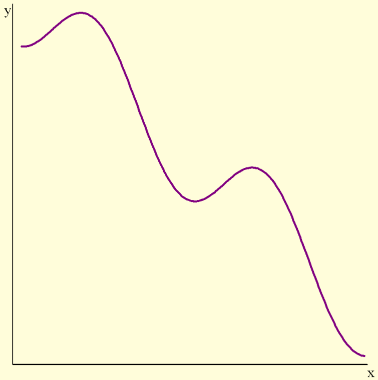


Find all critical points and Extreme Points

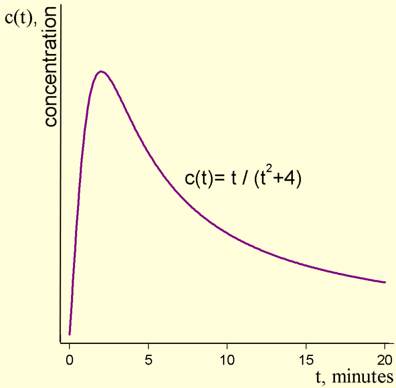


Example: the concentration of drugs in a person's bloodstream t minutes after injection is given by the function:

$$c(t) = \frac{t}{t^2 + 4}$$

Find the time after injection at which the concentration is highest.

Concentration of drugs in a person's bloodstream



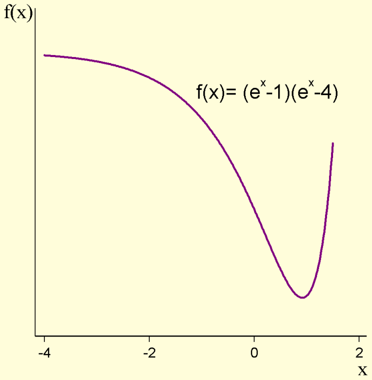
Example

$$f(x) = (e^x - 1)(e^x - 4)$$

Find:

1. the values of x for which $f(x) = 0$
2. $f'(x)$
3. $\lim_{x \to -\infty} f(x)$

Find the Extrema



New Video on Testing for Extreme Points

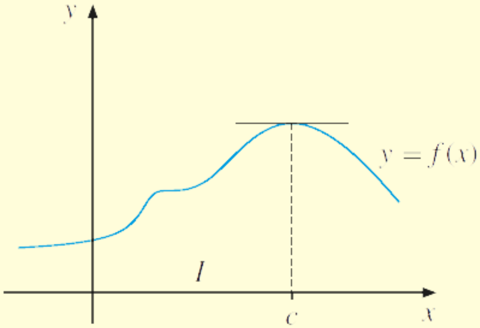
Simple Tests for Extreme Points

There are simple two tests for extreme points:

1. First-Derivative Test for Extrema
2. Extrema for Concave and Convex Functions (second-order conditions)

First-Derivative Test for Extrema

Suppose the function $f(x)$ is differentiable in an interval I that includes c.



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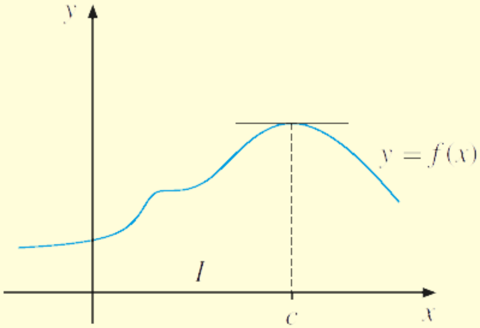
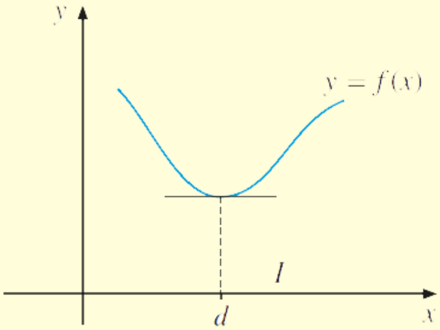


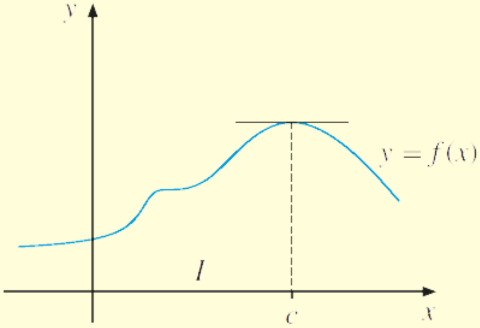
First-Derivative Test for Extrema

Suppose the function $f(x)$ is differentiable in an interval I that includes d.



Second-Order Conditions

Suppose that f is a function defined in an interval I and that c is a critical point for f in the interior of I.



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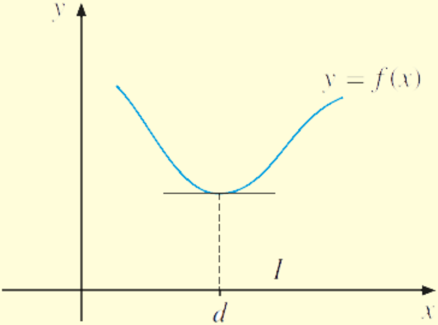


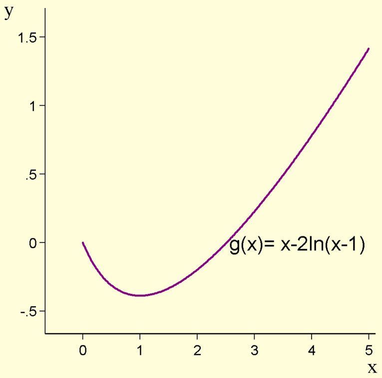
We will return to these simple tests again later when we study local minimums and maximums.

1. First-Derivative Test for Extrema
2. Extrema for Concave and Convex Functions (second-order conditions)

Example

$$g(x) = x - 2\ln(x+1)$$

1. Fnid $g'(x)$ and $g''(x)$.
2. Find all extreme points.
3. Sketch the graph.



y
1.5
1
.5
0
-.5
0
1
2
3
4
5
x
g(x)= x-2ln(x-1)