

**LOCAL VOLATILITY MODEL WITH STOCHASTIC INTEREST RATE**

BING HU

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## **Abstract**

Many different models exist to describe the behaviour of asset prices and are used to value options on such an underlying asset. This report investigates the local volatility model in a stochastic interest rates framework. First, we derive the local volatility function for this model, which allows the local volatility surface to be exacted from the prices of traded call options. Next, we present numerical approaches to construct a local volatility surface based on finite difference approximation, Monte Carlo simulation and Lipschitz interpolation. Then, Monte Carlo simulation is applied to value options using the local volatility surface. Finally, a numerical implementation of the model and its results are reported and compared with real market data.

## **Dedication**

*To my husband and my daughter*

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## Abbreviations

- IVS=Implied volatility surface;
- SDE=Stochastic differential equation;
- $C_m$ :Market call option price;
- $C_{BS}$ :Black-Schole call option price;
- $K$ :Exercise or strike price;
- $T$ :Expiration date or maturity;
- $\Omega$ :Probability space;
- $\mathcal{F}_t, \mathcal{F}(t)$ : $\sigma$ -algebra;
- $\mathbb{P}$ :Objective or real measure;
- $\mathbb{Q}$ :Risk-neutral measure or martingale measure;
- $E^{\mathbb{P}}$ :Expectation under the real measure;
- $E^{\mathbb{Q}}$ :Expectation under the risk-neutral measure;
- $E^{\mathbb{Q}_F}$ :Expectation under the  $T$ -forward measure;

- $E^{\mathbb{Q}_N}$ : Expectation under the probability measure  $\mathbb{Q}_N$  associated with the numéraire  $N$ ;
- $\sigma_{imp}$ : Implied volatility;
- $\phi_F$ :  $T$ -forward probability density;
- $C(S(t), t)$ : Call option price at time  $t$  associated with asset  $S$ ;
- $D(t)$ : Discount factor at time  $t$ ;
- $r(t)$ : Domestic interest rate at time  $t$ ;
- $f(t, T)$ : Instantaneous forward rate at time  $t$  for the maturity  $T$ ;
- $F_S(t, T)$ :  $T$ -forward price at time  $t$ ;
- $M(t)$ : Money market account (or bank account) at time  $t$ ;
- $N(x)$ : Cumulative distribution function of the standard Gaussian distribution;
- $P(t, T)$ : Zero-coupon bond price at time  $t$  for the maturity  $T$ ;
- $S(t), S_t$ : The value of asset price at time  $t$ ;
- $\sigma(t), \sigma(S(t), t)$ : The volatility of the asset  $S$ ;
- $\sigma(K, T)$ : Local volatility of the asset  $S$ ;
- $V(t, T), V(t), V$ : The value of derivative at time  $t$ ;
- $W^{\mathbb{P}}(t)$ : Brownian motion under the real measure  $\mathbb{P}$ ;
- $W^{\mathbb{Q}(t)}$ : Brownian motion under the risk-neutral measure  $\mathbb{Q}$ ;
- $W^{\mathbb{Q}_F}(t)$ : Brownian motion under the  $T$ -forward measure;

- $W^{\mathbb{Q}_N}(t)$ : Brownian motion under the measure associated with the numéraire  $N$  when  $N$  is an asset;
- $\mathbf{1}_A$ : Indicator function of the set  $A$ ;

# **1 Introduction**

## **1.1 Motivation and Objectives**

In the last 40 years, derivatives have become increasingly important in finance. They are heavily used by different groups of market participants, including financial institutions, funding managers and corporations. The most popular variables underlying derivatives are the prices of traded assets. A stock option, for example, is a derivative whose value depends on the price of a stock.

The increasing use and complexity of derivatives raises the need for a framework that enables for the accurate and consistent pricing and hedging, risk management, and trading of a wide range of derivative products, including all kinds of exotic derivatives. Modern option pricing theory came into existence with the advent of the Black-Scholes model. The Black-Scholes option pricing model, developed by Black and Scholes, formalized and extended by Merton builds a cornerstone in the theory of modern quantitative finance. If Black-Scholes model holds exactly, then all option on the same underlying asset should provide the same implied volatility, i.e., a single real number that, when plugged into the Black-Scholes formula for the volatility parameter, results in a model price equal to the market option price.

Yet, in practice, the empirical investigation of the Black-Scholes model revealed statistically significant and economically relevant deviations between market prices and model prices. A con-

venient way of illustrating these discrepancies is to express the option price in terms of its implied volatility. The existence of volatility surfaces implies that the implied volatility of an option is not necessarily equal to the expected volatility of the underlying asset's rate of return.

Therefore there are several practical reasons [31] to have a smooth and well-behaved implied volatility surface(IVS):

1. market makers quote options for strike-expiry pairs which are illiquid or not listed;
2. pricing engines, which are used to price exotic options and which are based on far more realistic assumptions than Black-Scholes model, are calibrated against an observed IVS;
3. the IVS given by a listed market serves as the market of primary hedging instruments against volatility and gamma risk;
4. risk managers use stress scenarios defined on the IVS to visualize and quantify the risk inherent to option portfolios.

The obvious drawback of the Black-Scholes model has led to the development of a considerable literature on alternative option pricing models, which attempts to identify and model the financial mechanisms that give rise to volatility surfaces. The main focus on these models is that we assume the volatility of the underlying asset varies over time, either deterministically or stochastically.

Derman, Dupire and Rubinstein were firstly to model volatility as a deterministic function of time and stock price, known as local volatility model [11,4,32]. The unknown volatility function can be fitted to observed option prices to obtain an implied price process for the underlying asset. The stochastic volatility approach was motivated by empirical studies on the time series behavior of volatilities. They suggested that volatility should be viewed as a random process that satisfies

a stochastic differential equation [26,52,41]. A third explanation for implied volatility patterns that is related to the asset price process are jump-diffusion models [47]. These models incorporate discontinuous jumps in the underlying asset price. They are very useful for modeling the crash risk, which has attained considerable attention since the stock market crash of October 1987.

Local volatility models, which are widely used in the finance industry [5], have the benefit over stochastic volatility models since they are Markovian in only one factor. It avoids the problem of working in incomplete markets in comparison with stochastic volatility models and is therefore more appropriate for hedging strategies. Local volatility models also have the advantage to be calibrated on the complete implied volatility surface, and consequently local volatility models usually capture more precisely the surface of implied volatilities than stochastic volatility models. One drawback of local volatility model is that it predicts unrealistic dynamics for the asset volatility since the volatilities observed in the market are really stochastic. This is one of the main reasons of fierce criticism of local volatility models [10]. But completeness is important since it guarantee unique prices. This is the stated reason to develop the local volatility model in [4].

In original local volatility model, the interest rate is a constant (or deterministic). However, it is well-known that in reality interest rates are not constant. The risk associated with the interest rate movements motivates the interest rate model. In the constant interest rates case, the local volatility function can be obtained by using the famous Dupire's formula. But in a stochastic interest rates case, we add one more randomness in the model. "What kind of result can we get if we put local volatility model in a stochastic interest rates framework?". It has mainly been this question that motivated this work.

The *overall goal* of this work is to expose the results about local volatility expression in a

two-factor model. In the first, theoretical part of this work, we aim at deriving the local volatility function in a stochastic interest rates framework. Based on the theory developed, the objective of the second, numerical implementation of the model will be presented.

## **1.2 Structure of the Work**

This report is organized as follows.

In Chapter 2, some basic knowledge is introduced, which will serve as a reference in the later sections. An overview of financial terms and the important financial mathematical concepts are introduced in Section 2.1 and 2.2. Next, we discuss the continuous-time theory of stochastic calculus within the context of finance application. Starting with the risk-neutral pricing in Section 2.3, we state Girsanov's Theorem, which underlies change of measure. This permits a systematic treatment of risk-neutral pricing and the Fundamental Theorem of Asset Pricing. Then, Section 2.4, devoted to implied volatility, we describe Black-Scholes model and its formula. Special emphasis is put on the implied volatility concepts. In Section 2.5, an overview of research regarding local volatility model and Dupire's formula are introduced. In Section 2.6, the change of numéraire technique is reviewed as a general and powerful theoretical tool that can be used in several situation, and indeed will often be used in this report. Finally, in the remainder of this section, we introduce the dynamics of interest rates.

In Chapter 3, we expose theoretical results about the local volatility function with stochastic interest rate. We begin with the description of the two-factor model under the risk-neutral probability measure in Section 3.1. Then, in Section 3.2 we derive the local volatility expression for this model. First, we derive a PDE from the two-dimensional Fokker-Plank equation for the for-

ward probability density function. Next, we use this PDE to derive the local volatility function by differentiating European call price with respect to the strike and the maturity.

Based on the model of Chapter 3, Chapter 4 is devoted to the calibration of our local volatility function and option pricing. In Section 4.1, we present numerical approach to construct a local volatility surface based on finite difference approximation, Monte Carlo simulation and Lipschitz interpolation. Next, Monte Carlo simulation is applied to value options using the local volatility surface. Finally, one numerical experiment is presented to illustrate the effectiveness and applicability of the proposed method.

The report concludes with a short summary in Chapter 5.

## 2 Model Warm-up

For a full understanding of the contents of this report, some basic knowledge is needed. This chapter introduces the fundamental financial theory used in this report, which is essential for reading the later sections. In Section 2.1, we first introduce the basic collection of definitions and specifications concerning the financial markets in general. Section 2.2 gives a short overview of very important part in the mathematical modeling of financial processes—elementary stochastic calculus. Section 2.3 simply introduces the principles of continuous-time financial markets in a rather general framework. Special emphasis is put on the risk-neutral pricing. Section 2.4, starting with a description of Black-Scholes model, we discuss the shortcoming of Black-Scholes model and introduce the implied volatility concept. Next, in Section 2.5, an overview of local volatility model is given and Dupire’s formula is introduced. Section 2.6 is devoted to the short-rate world, which is mainly focusing on the dynamics of interest rates. Finally, in Section 2.7, the change of numéraire technique is reviewed as a general and powerful theoretical tool that can be used in several situation, and indeed will often be used in this report.

### 2.1 Financial Foundations

We start with a frictionless security market. A security market is called *frictionless*, if there are no transaction costs or taxes, no bid-ask spreads, no margin requirements, no restrictions on short

sales[34]. We know that all real market involve frictions; this assumption is made purely for simplicity. Understanding frictionless markets is a necessary step to understand market with frictions.

The simplest concept in finance is that of the *time value of money*. The idea that money available at the present time is worth more than the same amount in the future due to its potential earning capacity, i.e., \$1 today is worth more than \$1 in a year's time. This core principle of finance holds that, provided money can earn interest, any amount of money is worth more the sooner it is received. There are several types of interest. Compound interest is the main case of relevance. And compound interest comes in two forms, discretely compounded and continuously compounded.

Suppose we invest \$1 in a bank and receive  $m$  interest payments at a rate of  $r/m$  per annum. After one year, we will have

$$\left(1 + \frac{r}{m}\right)^m \quad (2.1.1)$$

If we take the limit  $m \rightarrow \infty$ , this will lead to a rate of interest that is paid continuously. Above expression becomes

$$\left(1 + \frac{r}{m}\right)^m = e^{m \log\left(1 + \frac{r}{m}\right)} \approx e^r \quad (2.1.2)$$

Similarly, after time  $t$ , it will have an amount

$$M(t) = e^{rt} \quad (2.1.3)$$

A *derivative* can be defined as a financial instrument whose value depends on the values of other underlying variables. Many different types of futures, forward contracts, swaps, options and other derivatives are actively traded on many exchanges throughout the world.

One of common derivative is *option*. The option gives the holder the right to trade asset in the future at a previously agreed price but takes away the obligation. A *call option* gives the holder the

right to buy the underlying asset by a certain date for a certain price. A *put option* gives the holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the *exercise* or *strike price*; the date in the contract is known as the *expiry* or *maturity*.

*American options* can be exercised at any time up to the expiration date. *European options* can be exercised only at maturity. Both American option and European option are known as *plain vanilla option*. The intermediary form, options that can be exercised at several specified points in time before maturity, are known as *Bermudan options*. Options with non-standard pay-offs, strikes, exercise possibilities, that consist of a combination of other options or any other non-standard conceivable structure are known as *exotic options*, e.g., lookback option, barrier option, and Asian option.

If we use  $S(t)$  and  $K$  to denote the asset price at time  $t$  and the strike respectively, the payoff of a European call option at expiry  $T$  is

$$\max(S(T) - K, 0) \quad (\text{or} \quad (S(T) - K)^+) \quad (2.1.4)$$

Similarly, the payoff of a European put option at expiry  $T$  is

$$\max(K - S(T), 0) \quad (\text{or} \quad (K - S(T))^+) \quad (2.1.5)$$

At a given time  $t$  before or at expiry, we say that a call option is *In-The-Money*(ITM) and *Out-The-Money*(OTM) if  $S(t) > K$  and  $S(t) < K$  respectively. An option is said to be *At-The-Money*(ATM) if  $S(t) = K$ .

A T-maturity *zero-coupon* bond is a contract that guarantee its holder the payment of one unit of currency at time  $T$ , with no intermediate payments. The contract value at time  $t < T$  is denoted by  $P(t, T)$ . Clearly,  $P(T, T) = 1$  for all  $T$ .

In mathematical finance, *hedge* is a very important concept. It is the replication of the contingent claims, by buying or selling other financial products(usually the underlying). By ensuring that the replicating portfolio has the same payoff as the option does, the option trader eliminates the uncertainty of making a loss(or profit).

A *portfolio* is a collection of investments held by an investment company, hedge fund, financial institution or individual. In this report, a portfolio strategy(or trading strategy) is a pair of processes  $\phi(t)$  and  $\psi(t)$  which describe respectively the number of units of security and of the bond(or cash) which we hold at time  $t$ . The processes can take positive or negative values. The value of portfolio at time  $t$  is given by

$$V(t) = \phi(t)S(t) + \psi(t)M(t), \text{ for } \forall t \in [0, T],$$

the process  $V(t)$  is called the *value process or wealth process* of the trading strategy.

Obviously, the description  $(\phi(t), \psi(t))$  is a dynamic strategy detailing the amount of each component to be held at each instant. And one particularly interesting set of strategies or portfolios is *self-financing*. A portfolio is self-financing if and only if the change of its value only depends on the change of the asset prices. In other words, all changes in the value of the portfolio are due to capital gains, as opposed to withdrawals of cash or injections of new funds.

A financial market is called *complete* if every contingent claim can be replicated. In a complete market an option has a unique price, which can be determined by finding the cost of setting up and maintaining its replicating portfolio. An *arbitrage* opportunity would allow investors to make profits without being exposed to the risk of incurring a loss. From an economic point of view, in order for the continuous-time market model to be reasonable, it should be free of arbitrage opportunities. Unfortunately, it is very difficult to check directly if a model has any arbitrage

opportunities. However, a model is consistent with the absence of arbitrage in a complete market.

## 2.2 Elementary Stochastic Calculus

The uncertainty in the financial market is characterized by a certain probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega$  is the state space,  $\mathcal{F}$  is the  $\sigma$ -algebra representing measurable events, and  $\mathbb{P}$  is the objective(or real) measure.

A *random variable* is a real-valued function that assigns values to outcomes of a probabilistic experiment. If the value of a particular random variable,  $X$ , is known at time  $t$  it is said that  $X$  is  $\mathcal{F}_t$ -measurable, i.e., the  $\sigma$ -algebra  $\mathcal{F}_t$  represents the information available at time  $t$ .

*Filtration*  $\{\mathcal{F}_t\}_{0 \leq t \leq T}$  is a sequence of  $\sigma$ -algebra  $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots$  such that  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ . If random variable  $X_t$  is  $\mathcal{F}_t$ -measurable for every  $t$ , it is said that  $\{X_t\}$  is adapted to the filtration  $\mathcal{F}_t$ .

Let  $\mathcal{G}$  be a  $\sigma$ -algebra contained in  $\mathcal{F}$ . Then we can estimate  $X$  based on the information contained in  $\mathcal{G}$ , which is known as conditional expectation  $E[X|\mathcal{G}]$ . More specifically, if  $X \notin \mathcal{G}$ ,

$$E[X|\mathcal{G}] = \int_{\mathcal{G}} X(\omega) dP(\omega) \quad (2.2.1)$$

For a filtration  $\{\mathcal{F}\}_t$  and an adapted process  $\{X(t)\}$ , if

$$E[X(t)|\mathcal{F}(s)] = X(s), \text{ for all } 0 \leq s \leq t \quad (2.2.2)$$

we say the process  $\{X(t)\}$  is a *martingale*.

A random process  $W(t)$  is called a *standard Brownian motion* (or Wiener process) if it satisfies:

- $W(t)$  is a continuous function of  $t$ ;
- The initial point starts from  $W(0) = 0$ ;

- For each time period  $t_{k+1} - t_k$ ,  $W(t)$  has independent and normally distributed increments with mean 0 and variance  $t_{k+1} - t_k$ .

A stochastic process  $X$  is a continuous process  $\{X(t)\}_{t \geq 0}$  such that  $X(t)$  can be written as

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s), \quad (2.2.3)$$

where  $\mu(t), \sigma(t)$  are adapted processes. The differential form of (2.2.3) can be written

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \quad (2.2.4)$$

Most stochastic processes are described by stochastic differential equation(SDE), usually of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t) \quad (2.2.5)$$

We call  $\sigma(X(t), t)$  the volatility of the process  $X$  and  $\mu(X(t), t)$  the drift of  $X$ . Both of them are  $\mathcal{F}_t$ -measurable.

In mathematical finance, Itô lemma is an indispensable tool to manipulate the differential equations. If  $X$  is a stochastic process satisfying (2.2.5), and  $f(X(t), t)$  is a deterministic twice continuously differentiable function, then  $f(X(t), t)$  is also a stochastic process and is given by Itô's formula

$$\begin{aligned} df(X(t), t) &= \frac{\partial f(x, t)}{\partial t}dt + \frac{\partial f(x, t)}{\partial x}dX(t) + \frac{1}{2} \frac{\partial^2 f(x, t)}{\partial x^2}dX(t) \cdot dX(t) \\ &= \left( \frac{\partial f(x, t)}{\partial t} + \mu(X(t), t) \frac{\partial f(x, t)}{\partial x} + \frac{1}{2} \sigma^2(X(t), t) \frac{\partial^2 f(x, t)}{\partial x^2} \right) dt + \sigma(X(t), t) \frac{\partial f(x, t)}{\partial x} dW(t) \end{aligned} \quad (2.2.6)$$

If we think of  $X(t)$  as the value of an asset for which we have a stochastic differential equation, a 'model', then we can handle functions of the asset ( $f(X(t), t)$ ), and ultimately value contracts such

as options. It is then obvious why Itô's formula plays a cornerstone role in quantitative finance field.

## 2.3 Risk-Neutral Pricing

### 2.3.1 Change of Measure

Let stochastic process  $S(t)$  denote the primary traded assets price process in the market (stocks, bonds, etc.). The movement of the security prices relative to each other will be important to study, so it is convenient to normalize the price. We set

$$S^*(t) = \frac{S(t)}{N(t)},$$

where the price process  $N(t)$  is called *numéraire*. A numéraire is a price process which is almost surely positive for each  $t \in [0, T]$ . Mostly, the money market account  $M(t)$  or a zero-coupon bond  $P(t, T)$  is used as a numéraire. This explains why  $S^*(t)$  is usually called the *discounted price process*.

Two measure  $\mathbb{P}$  and  $\mathbb{Q}$  are *equivalent* if they operate on the same sample space  $\Omega$  and agree on what is possible. Formally, if  $A$  is any event in  $\Omega$ ,

$$\mathbb{P}(A) > 0 \Leftrightarrow \mathbb{Q}(A) > 0$$

We say that a probability measure  $\mathbb{Q}$  defined on  $(\Omega, \mathcal{F})$  is a *risk-neutral measure* (or equivalent martingale measure) [37] if:

1.  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$ ;
2. the discounted price process is a  $\mathbb{Q}$ -martingale.

The set of risk-neutral measures is denoted by  $\mathcal{P}$ . In our model setting, all risk-neutral measures  $\mathbb{Q} \in \mathcal{P}$  can be obtained by using *Girsanov's theorem* (or Cameron-Martin-Girsanov theorem):

**Theorem (Girsanov's theorem).**  *$W(t)$  is a  $\mathbb{P}$ -Brownian motion and  $\theta(t)$  is an adapted process satisfying the boundedness condition  $E^{\mathbb{P}}[e^{\frac{1}{2} \int_0^t \theta^2(s) ds}] < \infty$ , then there exists a risk-neutral measure  $\mathbb{Q}$  such that*

1.  $W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) + \int_0^t \theta(s) ds$  is a Brownian motion under  $\mathbb{Q}$ ;

2.  $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}(t)} = \exp\{-\int_0^t \theta(s) dW^{\mathbb{P}}(s) - \frac{1}{2} \int_0^t \theta^2(s) ds\}$ .

$Z(T) = \frac{d\mathbb{Q}}{d\mathbb{P}}$  is called Radon-Nikodym derivative. In other words, we can use  $\frac{d\mathbb{Q}}{d\mathbb{P}}$  to define a risk-neutral measure  $\mathbb{Q}$ .

Define the Radon-Nikodym derivative process

$$Z(t) = E^{\mathbb{P}}[Z(T)|\mathcal{F}(t)], 0 \leq t \leq T, \quad (2.3.1)$$

it has the following properties [12]:

- $Z(t)$  is a martingale;
- If  $Y$  is a  $\mathcal{F}(t)$ -measurable random variable, then  $E^{\mathbb{Q}}[Y] = E^{\mathbb{P}}[YZ(t)]$ ;
- If  $s$  and  $t$  satisfying  $0 \leq s \leq t \leq T$  be given, then

$$E^{\mathbb{Q}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)} E^{\mathbb{P}}[YZ(t)|\mathcal{F}(s)] \quad (2.3.2)$$

Now we present an important necessary and sufficient condition for the model to be consistent with the absence of arbitrage:

**Theorem (Fundamental Theorem of Asset Pricing).** A mathematical model of financial market admits no arbitrage if there exists a risk-neutral probability measure.

To simplify matters, the fundamental theorem of asset pricing tells us one fact: the existence of an equivalent martingale measure is equivalent to an arbitrage-free market.

### 2.3.2 Risk-Neutral Pricing Formula

Assuming that there exists at least one equivalent martingale measure for the market model, we now approach the problem of pricing derivatives. The problem of interest is to determine the time  $t$  value of the payoff  $V(t)$ . In other words: what is the fair price of claim  $V(t)$  at time  $t$  that the buyer should pay the seller in order to satisfy both parties. Usually, one might suppose that the value of a contingent claim would depend on the risk preferences and utility functions of the buyer and seller, but in many cases this is not so. It turns out that the argument of arbitrage pricing theory there is often a unique value of the claim at time  $t$ , i.e., a value that does not depend on investor's risk preferences. In the following, we are going to describe how to get this value.

**Theorem (Risk-Neutral Valuation Formula).** Let  $V_T$  be a contingent claim, i.e., an  $\mathcal{F}(T)$ -measurable random variable representing the payoff at time  $T$ . Under the risk-neutral measure  $\mathbb{Q}$ , the value of the contingent claim at time  $t$  is given by the *risk-neutral pricing formula* :

$$V(t) = N(t)E^{\mathbb{Q}}\left[\frac{V(T)}{N(T)}|\mathcal{F}(t)\right] \quad (2.3.3)$$

Therefore, if we wish to price a contingent claim, it is sufficient to find an equivalent martingale measure, but for hedging purpose we are more interested in the replicating trading strategy.

We recall the discussion of completeness in Section 2.1. The method that establishes a connection between market completeness and the number of existing equivalent martingale measure is

illustrated in the following:

**Theorem (Second Fundamental Theorem of Asset Pricing).** A financial market model is complete if and only if there exists exactly one unique risk-neutral measure.

## 2.4 Implied Volatility

### 2.4.1 The Black-Scholes Model

In the early 1970s, Black, Scholes and Merton achieved a major breakthrough in the pricing of European options [13,47]. The authors show that, under certain model assumptions, there exists a unique price for European options since the payoff can be replicated by a portfolio consisting of the underlying asset and a risk free money account, which is the mathematical equivalent of buying bonds of an institution that is assumed to never default on its obligations. Since this portfolio whose ending value is the given payoff of the options, the price of the option must be the price of construct this portfolio.

This model assumes that the price of a non-dividend paying stock  $S$  has a lognormal distribution with dynamics given by a geometric Brownian motion(GBM):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^{\mathbb{P}}(t) \quad (2.4.1)$$

where  $\mu$  is the expected return on stock,  $\sigma$  is the volatility of the stock price,  $W(t)$  is a Wiener process under the objective(or real) measure  $\mathbb{P}$ .

Under the objective probability measure  $\mathbb{P}$ , taking money market account  $M(t) = e^{rt}$  as a discount factor, then Girsanov's theorem says, that there exists a risk-neutral measure  $\mathbb{Q}$  such that

$S(t)/M(t)$  is a  $\mathbb{Q}$ -martingale. The dynamics of  $S$  under risk-neutral measure  $\mathbb{Q}$  follows the SDE

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t),$$

where  $r$  is risk-free interest rate. Under the risk-neutral measure  $\mathbb{Q}$ , we see that the expected return is replaced by the interest rate  $r$ .

Let  $C(S(t), t)$  denote the European call option price of the stock at time  $t$ . In the absence of arbitrage, applying the above replicating strategy, we obtain the Black-Scholes partial differential equation(PDE)

$$\frac{\partial C}{\partial t}(S(t), t) + rS(t)\frac{\partial C}{\partial S(t)}(S(t), t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 C}{\partial S^2(t)}(S(t), t) - rC(S(t), t) = 0 \quad (2.4.2)$$

with terminal condition

$$C(S(T), T) = (S(T) - K)^+$$

The famous Black-Scholes formula (solution to the Black-Scholes PDE (2.4.2)) for European call options on non-dividend paying stocks is as following:

$$C(S(t), t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2), \quad (2.4.3)$$

where

$$N(d) = \int_{-\infty}^d \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz,$$

$$d_1 = \frac{\ln(S(t)/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, d_2 = d_1 - \sigma\sqrt{T - t}$$

Theoretically this is a nice result, since it gives us closed formula to price plain vanilla options. It can price such options in a fast and neat way since only a few constant variables have to be considered.

Although satisfactory for computing European options, the Black-Scholes model comes up short for more complex options, such as Asian options and barrier options. For these options no analytic solution can be used so that we have to use numerical methods to solve the pricing problem( binomial trees, Monte Carlo simulation, etc.).

#### 2.4.2 The Implied Volatility

The Black-Scholes formula relates the price of an option to the current time  $t$ , the underlying stock price  $S(t)$ , the volatility of the stock  $\sigma$ , the interest rate  $r$ , the maturity date  $T$ , and the strike price  $K$ . All parameters other than the stock's volatility  $\sigma$  can be observed directly in the market. Given other parameters are known, the pricing formula relates the option price to the volatility of the underlying stock. If we can observe the price of option from market, the volatility implied by the market price can be determined by inverting the option pricing formula, i.e.,  $C_m = C_{BS}(t, S_t, K, T, r, \sigma_{imp})$ . This volatility is known as the *implied volatility*.

If Black-Scholes model holds exactly, the all option on the same underlying asset should provide the same implied volatility. In reality things are more complicated than the model of Black-Scholes assumes. Market participants have long noted that using the same constant variables for all options result in prices not compatible with the market. It seems that different options on the same underlying asset are governed by different volatilities. It is well-known that empirical implied volatilities differ systematically across strike price  $K$  and across maturity  $T$ . So the misspecified model produces the correct market prices or as Rebonato referred in [49] : “Implied volatility is the wrong number to put in the wrong formula to obtain the right price.”

Usually, there are some well-known patterns in the behavior of implied volatility as the strike

price and the maturity date of the option change. The most often quoted phenomenon testifying to the limitations of the Black-Scholes model is the smile effect: that implied volatilities vary with the strike price of the option contract. Formally, we define the *volatility smile* as follows:

For any fixed maturity date  $T$ , the function  $\sigma_{imp}^t(K, \cdot)$  against strike price  $K(K > 0)$ , is called the *volatility smile* at date  $t \in [0, T)$ .

The relationship between implied volatility and maturity date for a fixed strike option is described as *volatility term structure*:

For any fixed strike price  $K(K > 0)$ , the function  $\sigma_{imp}^t(\cdot, T)$  against maturity  $T$ , is called *term structure of implied volatility*.

The basic shapes about volatility smile and term structure are given in Figure 2.1

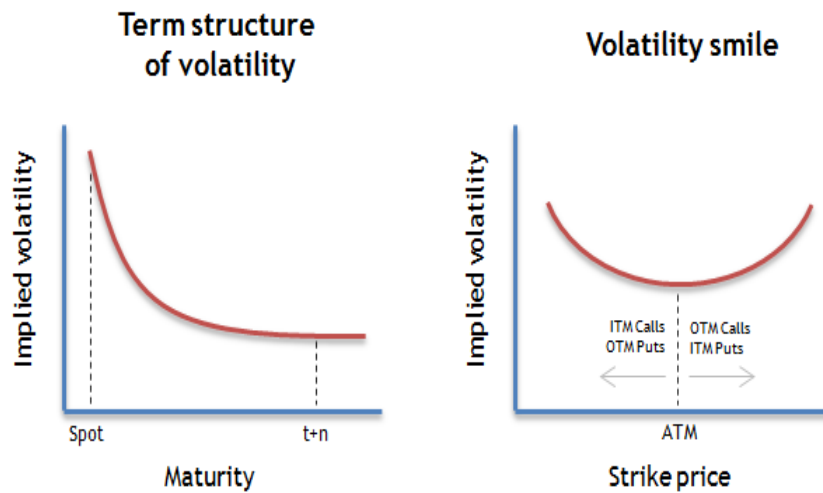


Figure 2.1: Volatility term structure and volatility smile

Volatility surfaces combine the volatility smile with the term structure of volatility to tabulate the implied volatility appropriate for market consistent pricing of an option with any strike price and any maturity. Formally, we define it as follows:

For any time  $t \in [0, T]$ , the function  $\sigma_{imp}^t : (0, \infty) \times (t, T) \rightarrow R_+$ , which assigns each strike price and maturity date tuple  $(K, T)$  its implied volatility  $\sigma_{imp}^t(K, T)$  is referred to as the *implied volatility surface*.

The basic shape about volatility surface are given in Figure 2.2.

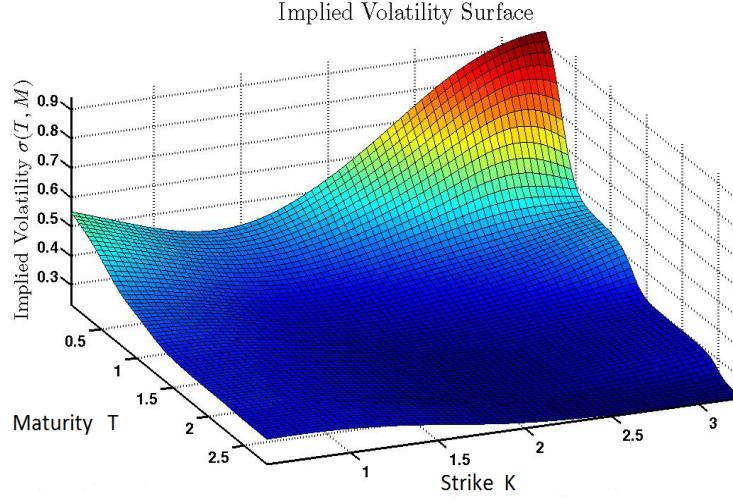


Figure 2.2: Basic shape of implied volatility surface

In general, the volatility surface  $\sigma_{imp}^t(K, T)$  will be a stochastic quantity with three variables,  $t, K$  and  $T$ , and for each outcome, in the underlying sample space, the dependence upon these variable will be different:

- For a fixed time  $t$ ,  $\sigma_{imp}^t(K, T)$  is a function of  $K$  and  $T$  providing implied volatilities or equivalently, the market prices, at the fixed time  $t$  for options of all possible strike prices and maturity dates.
- For a fixed strike  $K$  and a fixed maturity  $T$ ,  $\sigma_{imp}^t(K, T)$  (as a function of  $t$ ) will be a scalar stochastic process. This process gives the implied volatilities or equivalently, the market prices of the option with fixed strike  $K$  and fixed maturity  $T$ .

Empirically it is often advantageous to reexpress the volatility surface in terms of moneyness. When quoting options prices, one of common measure of moneyness is :

$$M = \frac{K}{S(t)}$$

### 2.4.3 Modeling Implied Volatility

Over decades, different methods have been proposed to make adjustments for the Black-Scholes model to make it more accurately describe market, in particular the volatility smiles. The search for an option pricing model which is theoretically consistent with the observable implied volatility patterns has brought on three different modeling approaches: local volatility models [4,11,32,50], stochastic volatility processes [26,41,52] and jump-diffusion processes [47].

The local volatility model assumes the volatility is a deterministic function of the asset price and time. It came into existence when Dupire showed in [4]. Calibration of this type of model requires determining the local volatility such that model prices agree with observable option prices. The stochastic volatility approach was first introduced by Hull and White in [26]. In stochastic volatility model, the volatility itself is a process that satisfies a stochastic differential equation. The most famous stochastic volatility models are the Heston model [52] and the SABR model [41]. Jump-diffusion processes have been used in finance to capture discontinuous behavior in asset pricing. They were first introduced by Merton [47]. When jumps occur, the price process is no longer continuous. Jumps have proved to be particularly useful for modeling the crash risk.

Although stochastic volatility and jump-diffusion models captures dynamics of volatility that is missing from Black-Scholes model, they introduce some additional non-traded sources of risk besides the risk of underlying assets, the completeness of the market is no longer maintained. Local

volatility model can maintain the market completeness since the number of random sources equals the number of stochastic traded asset( $r_t$  is deterministic). Completeness is of the highest value: it allows for arbitrage pricing and hedging [4]. Thus the deterministic local volatility model is very popular and attractive in financial practice.

## 2.5 Local Volatility Model

Local volatility model, which is widely used in the finance industry, is the subject of this report. Within the local volatility framework, the dynamics of the stock price under the risk-neutral measure  $\mathbb{Q}$  are given by

$$dS(t) = r(t)S(t)dt + \sigma(S(t), t)S(t)dW^{\mathbb{Q}}(t) \quad (2.5.1)$$

where the volatility is now a deterministic function of time and the asset price,  $r_t$  denotes the continuously compounded short rate.

In 1994, Dupire showed that we could find the local volatility  $\sigma(S(t), t)$  from the market prices of options [4]. The central part of the derivation is the relationship among the surface of European call prices, transition density and the Kolmogorov equation.

Assume that current time is  $t_0$ , according to the Risk-Neutral Valuation Formula (2.3.3) in Section 2.3.2, the price at time  $t_0$  of a European call option

$$\begin{aligned} C(S(t_0), t_0) &= e^{-\int_{t_0}^T r(s)ds} E^{\mathbb{Q}}[(S(T) - K)^+ | \mathcal{F}(t_0)] \\ &= e^{-\int_{t_0}^T r(s)ds} \int_K^{\infty} (x - K)\phi(x, T)dx \end{aligned} \quad (2.5.2)$$

where  $\phi(x, T)$  is the risk-neutral probability density of the underlying asset at maturity.

Since  $\phi(x, T)$  is a probability density function, its time evolution will be described by the for-

ward Kolmogorov(or Fokker-Planck) equation

$$\frac{\partial \phi(x, T)}{\partial T} = -\frac{\partial}{\partial x}[r(T)x\phi(x, T)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x, T)x^2\phi(x, T)] \quad (2.5.3)$$

Differentiating (2.5.2) twice with respect to  $K$  yields:

$$\begin{aligned} \frac{\partial C}{\partial K} &= -e^{-\int_{t_0}^T r(s)ds} \int_K^\infty \phi(x, T)dx \\ \frac{\partial^2 C}{\partial K^2} &= e^{-\int_{t_0}^T r(s)ds} \phi(K, T) \end{aligned} \quad (2.5.4)$$

Differentiating (2.5.2) with respect to  $T$  and applying (2.5.3), given call price  $C(K, T)$  at all strikes  $K$  and maturities  $T$ , the local volatility  $S(T) = K$  can be determined by *Dupire's formula*

$$\sigma^2(K, T) = 2\frac{\frac{\partial C}{\partial T} + r_T K \frac{\partial C}{\partial K}}{K^2 \frac{\partial^2 C}{\partial K^2}}, \quad (2.5.5)$$

where  $C$  is market option price.

As we know that the market often quotes options in terms of implied volatilities  $\sigma_{imp}$  instead of option prices. Consequently, the local volatility can be expressed as a function of implied volatility. As the implied volatility of an option with price  $C(K, T)$  is defined through the Black-Scholes formula ( $C_m = C_{BS}$ ), the derivatives of call prices in (2.5.5) can be computed through the chain rule, and this leads to the following equation [50]

$$\sigma^2(K, T) = \frac{\sigma_{imp}^2 + 2\sigma_{imp}T\left(\frac{\partial \sigma_{imp}}{\partial T} + r_T K \frac{\partial \sigma_{imp}}{\partial K}\right)}{\left(1 - \frac{Ky}{\sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial K}\right)^2 + K\sigma_{imp}T\left(\frac{\partial \sigma_{imp}}{\partial K} - \frac{1}{4}K\sigma_{imp}T\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2 + K\frac{\partial^2 \sigma_{imp}}{\partial K^2}\right)}, \quad (2.5.6)$$

where  $y = \ln(K/S_0) - \int_{t_0}^T r(s)ds$ ,  $\sigma_{imp}$  is the implied volatility.

If the dynamics of the stock price under the risk-neutral measure  $\mathbb{Q}$  is given by

$$dS(t) = (r(t) - q(t))S(t)dt + \sigma(S(t), t)S(t)dW^{\mathbb{Q}}(t), \quad (2.5.7)$$

where the volatility is now a deterministic function of time and the asset price,  $r(t)$  and  $q(t)$  denotes interest rate and dividend respectively.

The corresponding Dupire's formula is

$$\sigma^2(K, T) = 2 \frac{\frac{\partial C}{\partial T} + [r(T) - q(T)] K \frac{\partial C}{\partial K} + q(T) C}{K^2 \frac{\partial^2 C}{\partial K^2}} \quad (2.5.8)$$

Yet, in practice, the implied volatility (or option contracts) is a unknown continuous function of strike and maturity. To obtain a continuous local volatility surface, the implied volatility surface should be at least  $C^1$  in the  $T$  direction and  $C^2$  in the strike or moneyness direction. In general, a  $(C_T^n, C_K^m)$  implied volatility surface will produce a  $(C_T^{n-1}, C_K^{m-2})$  local volatility surface.

There are many interpolation and extrapolation techniques to be used in constructing a volatility surface (cubic splines [17], thin plate splines [50], radius basis function [24], interpolation based on fully implicit finite difference method [20], etc.). In some situations we need to perform interpolation in time. One common approach is to perform linear interpolation in variance. A variant of it, denoted "total variance interpolation", is described in [35]. In addition, various parametric or semi-parametric representations of the volatility surface have been considered in the literatures. A recent overview was given in [31]. A popular polynomial parameterization was suggested in [3], which proposed that the implied volatility surface is modeled as a quadratic function of the moneyness and maturity. Practically, it is very difficult to define a single parametric function for the entire surface. In 1999, a typical approach –the *Stochastic Volatility Inspired* (SVI) parametrization was devised at Merrill Lynch. The essence of this practitioner designed parametrization is that each time slice of the implied volatility surface is calibrated to observed option separately [22,23], such that in the logarithmic coordinates the implied variance curve is a hyperbola, and additional constraints are imposed that ensure no vertical and horizontal spread arbitrage opportunities.

After we obtain a smooth implied volatility surface from market prices of options, we can then numerically take derivatives to obtain the local volatility surface.

## 2.6 Interest Rate Model

The interest rate market is where the price of money is set-how much does it cost to have money tomorrow, money in a year, money in ten years? Previously we made the modeling assumption that the cost of money is constant(or deterministic), but this isn't actually so. The price of money over a term depends not only on the length of the term, but also on the moment-to-moment random fluctuations of the interest rate market. In this way, it raises the need for a framework of interest rate model.

Interest rate models can be used to model the dynamics of the yield curve, which is vital in pricing and hedging of fixed-income securities. In this report, we focus on the short-rate world. Models for the interest rate  $r(t)$  are sometimes called *short-rate models* because  $r(t)$  is the interest rate for short-time borrowing.

The simplest models for fixed income markets begin with a stochastic differential equation for the interest rate, e.g.,

$$dr(t) = \beta(r(t), t)dt + \gamma(r(t), t)dW^{\mathbb{Q}}(t), \quad (2.6.1)$$

where  $W^{\mathbb{Q}}(t)$  is a Brownian motion under a risk-neutral probability measure  $\mathbb{Q}$ .

In these models, one begins with a risk-neutral measure  $\mathbb{Q}$  and uses the risk-neutral pricing formula to price all assets. This guarantees that discounted asset prices are martingales under the risk-neutral measure, and hence there is no arbitrage.

When the interest rate is determined by only one stochastic differential equation, as is the case in this report, the model is said to have *one-factor*. The most classical short-time rate models are Hull-White model and C-I-R(Cox,Ingersoll and Ross) model [7].

We define the discount process

$$D(t) = e^{-\int_0^t r(s)ds} \quad (2.6.2)$$

and note that

$$dD(t) = -r(t)D(t)dt \quad (2.6.3)$$

A *zero-coupon bond* is a contract promising to pay one dollar at a fixed maturity date  $T$ . The risk-neutral pricing formula (2.3.3) says that the discounted price of this bond should be a martingale under the risk-neutral measure  $\mathbb{Q}$ . In other words, for  $0 \leq t \leq T$ , the price of the bond  $P(t, T)$  should satisfy

$$D(t)P(t, T) = E^{\mathbb{Q}}[D(T)|\mathcal{F}(t)] \quad (2.6.4)$$

(Note that  $P(T, T) = 1$ .) This gives us the *zero-coupon bond pricing formula*

$$P(t, T) = D(t)E^{\mathbb{Q}}[D(T)|\mathcal{F}(t)] = E^{\mathbb{Q}}[e^{-\int_t^T r(s)ds}|\mathcal{F}(t)], \quad (2.6.5)$$

which we take as a definition.

Since  $r$  is given by stochastic differential equation, it is a Markov process, i.e., the only relevant part of the path of  $r$  before time  $t$  is its value at time  $t$ . Then the bond price  $P(t, T)$  must be a function of time  $t$  and  $r(t)$ , i.e.,

$$P(t, T) = f(r(t), t),$$

for some functions  $f(r, t)$  of the dummy variables  $t$  and  $r$ .

To find the partial differential equation for the unknown function  $f(r, t)$ , we only need to find a martingale. In this case, the martingale is  $D(t)P(t, T) = D(t)f(r(t), t)$ . Its differentiation is

$$\begin{aligned} d(D(t)f(r(t), t)) &= f(r(t), t)d(D(t)) + D(t)df(r(t), t) \\ &= D(t)\left[-rf + \frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial r} + \frac{1}{2}\gamma^2\frac{\partial^2 f}{\partial r^2}\right]dt + D(t)\gamma\frac{\partial f}{\partial r}dW^{\mathbb{Q}}(t) \end{aligned}$$

Setting the  $dt$  term equal to zero, we obtain the partial differential equation

$$\frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial r} + \frac{1}{2} \gamma^2 \frac{\partial^2 f}{\partial r^2} = r f(r, t), \quad (2.6.6)$$

with the terminal condition

$$f(r, T) = 1 \quad \text{for all } r.$$

Vasicek Model [36], which is the subject of this report, is a special case of Hull-White model (Hull-White model is varying the time parameter in Vasicek model). The evolution of the interest rate in Vasicek model is given by

$$dr(t) = a(b - r(t))dt + \sigma dW_r^{\mathbb{Q}}(t), r(0) = r_0 \quad (2.6.7)$$

where  $a, b, \sigma$  are constant, and  $W_r^{\mathbb{Q}}(t)$  is a Brownian motion under risk-neutral measure  $\mathbb{Q}$ .

This dynamics has some peculiarities that make the model attractive. The equation is linear and can be solved explicitly, the distribution of the short rate is Gaussian, and both the expressions and the distributions of several quantities related to the interest-rate world are easily obtainable.

Integrating (2.6.7), we obtain, for each  $s < t$ ,

$$r(t) = r(s)e^{-a(t-s)} + b(1 - e^{-a(t-s)}) + \sigma_r \int_s^t e^{-a(t-u)} dW_r^{\mathbb{Q}}(u), \quad (2.6.8)$$

so that  $r(t)$  conditional on  $\mathcal{F}(s)$  is normally distributed with mean and variance given respectively by

$$\begin{aligned} E[r(t)|\mathcal{F}(s)] &= r(s)e^{-a(t-s)} + b(1 - e^{-a(t-s)}) \\ Var[r(t)|\mathcal{F}(s)] &= \frac{\sigma_r^2}{2a} [1 - e^{-2a(t-s)}]. \end{aligned} \quad (2.6.9)$$

As a consequence of (2.6.9), the short rate  $r(t)$  is mean reverting, since the expected rate tends, for  $t$  going to infinity, to the value  $b$ .

For the zero-coupon bond with Vasicek interest rate model, the corresponding partial differential equation (2.6.6) for the zero-coupon bond price becomes

$$\frac{\partial f}{\partial t} + (a(b - r(t)))\frac{\partial f}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 f}{\partial r^2} = rf(r, t) \quad (2.6.10)$$

with the same terminal condition as (2.6.6).

After solving the partial differential equation (2.6.10)(more details about it can be found in [12]), we have an explicit formula for the price of a zero-coupon bond as a function of the interest rate in the Vasicek model

$$P(t, T) = e^{-A(t, T)r(t) + D(t, T)}, \quad (2.6.11)$$

where  $A(t, T)$  and  $D(t, T)$  are given by

$$A(t, T) = \frac{1 - e^{-a(T-t)}}{a} \quad (2.6.12)$$

and

$$D(t, T) = \left[b - \frac{\sigma^2}{2a^2}\right] [A(t, T) - (T - t)] - \frac{\sigma^2 A^2(t, T)}{4a} \quad (2.6.13)$$

## 2.7 Change of Numéraire

### 2.7.1 Numéraire

We recall the discussion of numéraire in Section 2.3. In the following, we shall see that sometimes it is convenient to change the numéraire because of modeling considerations. A model can be complicated or simple, depending on the choice of the numéraire for the model. In principle, we can take any positively priced asset as a numéraire and denominate all other assets in terms of the chosen numéraire. Associated with each numéraire, we shall have a risk-neutral measure. When making this association, we shall only take non-dividend-paying assets as numéraires.

The numéraires are considered in this report are:

- Domestic money market account ,  $M(t) = e^{\int_0^t r(s)ds}$ . The associated risk-neutral measure is our regular risk-neutral measure  $\mathbb{Q}$ .
- A zero-coupon bond maturing at time  $T$ ,  $P(t, T)$ . We denote the associated risk-neutral measure by  $\mathbb{Q}_F$ . It is usually called the  $T$ -forward measure.

The following proposition [7] provides a fundamental tool for the pricing of derivatives.

**Proposition 2.7.1** Assume there exists a numeraire  $N$  and a probability measure  $\mathbb{Q}^N$ , equivalent to  $\mathbb{P}$ , such that the price of any traded asset  $S$  relative to  $N$  is a martingale under  $\mathbb{Q}^N$ , i.e.,

$$\frac{S(t)}{N(t)} = E^N \left[ \frac{S(T)}{N(T)} \middle| \mathcal{F}(t) \right], 0 \leq t \leq T \quad (2.7.1)$$

Let  $U$  be an arbitrary numeraire. Then there exists a probability measure  $\mathbb{Q}^U$ , equivalent to  $\mathbb{P}$ , such that the price of any attainable claim  $Y$  normalized by  $U$  is a martingale under  $\mathbb{Q}^U$ , i.e.,

$$\frac{S(t)}{U(t)} = E^U \left[ \frac{S(T)}{U(T)} \middle| \mathcal{F}(t) \right], 0 \leq t \leq T \quad (2.7.2)$$

Moreover, the Radon-Nikodym derivative defining the measure  $\mathbb{Q}^U$  is given

$$\frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} = \frac{U(T)N(0)}{U(0)N(T)} \quad (2.7.3)$$

The derivation of (2.7.3) is outlined as follows. By the definition of  $\mathbb{Q}^N$ , we know that for any tradable asset price  $X$ ,

$$E^U \left[ \frac{X(T)}{U(T)} \right] = E^N \left[ \frac{X(T)}{N(T)} \cdot \frac{N(0)}{U(0)} \right] \quad (2.7.4)$$

(Both being equal to  $X(0)/U(0)$ ). By the definition of Radon-Nikodym derivative, we also know that for all  $X$

$$E^U \left[ \frac{X(T)}{U(T)} \right] = E^N \left[ \frac{X(T)}{U(T)} \cdot \frac{d\mathbb{Q}^U}{d\mathbb{Q}^N} \right] \quad (2.7.5)$$

By comparing the right-hand sides of the last two equalities from the arbitrariness of  $X$  we obtain (2.7.3).

There are basically two facts on the change of numeraire technique one should consider in practice.

**FACT ONE.** The price of any asset divided by a numeraire is a martingale under the measure associated with that measure.

**FACT TWO.** The time  $-t$  risk-neutral price

$$Price_t = E^{\mathbb{Q}} \left[ M(t) \frac{Payoff(T)}{M(T)} | \mathcal{F}(t) \right]$$

is invariant by change of numeraire: If  $N$  is any other numeraire, we have

$$Price_t = E^{\mathbb{Q}^N} \left[ N(t) \frac{Payoff(T)}{N(T)} | \mathcal{F}(t) \right]$$

The above result establishes a connection between two risk-neutral measures. The following useful result shows that we can change risk-neutral measures from one to another by changing of numeraire.

### 2.7.2 Forward Measure

We now apply the above ideas to zero-coupon bond. We recall the discussion of Section 2.6. Consider a zero-coupon bond that pays 1 unit of currency at maturity  $T$ . The value of this bond at time  $t \in [0, T]$  is given by (2.6.5).

A *forward contract* that delivers one share of this asset at time  $T$  in exchange for  $K$  has a time  $T$  payoff  $S(T) - K$ . According to the risk-neutral pricing formula, the value of this contract at

earlier times  $t$  is

$$V(t) = \frac{1}{D(t)} E^{\mathbb{Q}}[D(T)(S(T) - K)|\mathcal{F}(t)], \quad (2.7.6)$$

where  $D(t) = e^{-\int_0^t r(s)ds}$ .

Because  $D(t)S(t)$  is a martingale under  $\mathbb{Q}$ , this reduces to

$$V(t) = S(t) - \frac{K}{D(t)} E^{\mathbb{Q}}[D(T)|\mathcal{F}(t)] = S(t) - KP(t, T). \quad (2.7.7)$$

The  $T$ -forward price  $F_S(t, T)$  at time  $t$  of an asset is the value of  $K$  that causes the value of the forward contract in (2.7.7) to be zero:

$$F_S(t, T) = \frac{S(t)}{P(t, T)} \quad (2.7.8)$$

A zero-coupon bond is an asset, and therefore the discounted bond price  $D(t)P(t, T)$  must be a martingale under the risk-neutral measure  $\mathbb{Q}$ . According (2.7.3), for a fixed maturity date  $T$ , we can define the  $T$ -forward measure  $\mathbb{Q}_F$  by

$$\frac{d\mathbb{Q}_F}{d\mathbb{Q}} = \frac{D(T)P(T, T)}{P(0, T)} \quad (2.7.9)$$

Furthermore, under  $T$ -forward measure, all assets denominated in units of the zero-coupon bond maturing at time  $T$  are martingales. In other words,

*$T$ -forward prices are martingales under the  $T$ -forward measure  $\mathbb{Q}_F$ .*

The reason to introduce the  $T$ -forward measure is that it often simplifies the risk-neutral pricing formula. According to that formula, the value at time  $t$  of a contract that pays  $V(T)$  at a later time  $T$  is

$$V(t) = \frac{1}{D(t)} E^{\mathbb{Q}}[D(T)V(T)|\mathcal{F}(t)]. \quad (2.7.10)$$

The computation of the right-hand side of this formula requires that we know something about the dependence between the discount factor  $D(T)$  and the payoff  $V(T)$  of the derivative security.

In particular, this can be difficult to model when the derivative security depends on the interest rate.

However, according to **FACT TWO** in previous section, we have

$$E^{\mathbb{Q}_F} \left[ \frac{V(T)}{P(T, T)} | \mathcal{F}(t) \right] = \frac{V(t)}{P(t, T)} \quad (2.7.11)$$

This gives us the simple pricing formula

$$V(t) = P(t, T) E^{\mathbb{Q}_F} [V(T) | \mathcal{F}(t)] \quad (2.7.12)$$

Therefore, if we can find a simple model for the evolution of assets under the  $T$ -forward measure, we can use (2.7.12), in which we only need to estimate  $V(T)$ , instead of using (2.7.10), which requires us to estimate  $D(T)V(T)$ .

### 3 Modeling Local Volatility with Stochastic Interest Rate

#### 3.1 Model set-up

In original local volatility model, we know that the interest rate is a constant (or deterministic function of  $t$ ). While for short-dated options (less than 1 year), assuming constant interest rates does not lead to significant mispricing, for long-dated options the effect of interest rate volatility becomes increasingly pronounced with increasing maturity and can become as important as that of the asset price volatility.

In this section, we consider a two-factor model where the asset volatility is a deterministic function of both time and the asset price. From Section 2.5, this function is known as ‘local volatility’.

Let  $t_0$  denote the current time. In our model, the asset price is governed by the following dynamics

$$dS(t) = r(t)S(t)dt + \sigma(S(t), t)S(t)dW_S^{\mathbb{Q}}(t), S(t_0) = S_0, \quad (3.1.1)$$

where the volatility of asset price is a local volatility  $\sigma(S(t), t)$  and the interest rates  $r(t)$  follows a Vasicek model

$$dr(t) = a(b - r(t))dt + \sigma_r dW_r^{\mathbb{Q}}(t), r(t_0) = r_0, \quad (3.1.2)$$

where  $a, b, \sigma_r$  are constant, and  $W_r^{\mathbb{Q}}(t), W_S^{\mathbb{Q}}(t)$  are Brownian motions with correlation

$$\text{corr}(W_r^{\mathbb{Q}}(t), W_S^{\mathbb{Q}}(t)) = \rho, (|\rho| < 1)$$

Furthermore, applying Itô's formula to (2.6.11), we have the dynamics of zero-coupon bond in Vasicek model

$$dP(t, T) = r(t)P(t, T)dt - \sigma_r A(t, T)P(t, T)dW_r^{\mathbb{Q}}(t), \quad (3.1.3)$$

where  $A(t, T)$  is defined by (2.6.12).

### 3.2 Forward PDE

From (2.7.12), we know that, under the assumption of absence of arbitrage opportunities and the  $T$ -forward measure  $\mathbb{Q}_F$ , the present value  $V(S(t_0), r(t_0), t_0)$  of a derivative that its payoff  $V(S(T), r(T), T)$  is given by

$$\begin{aligned} V(S(t_0), r(t_0), t_0) &= P(t_0, T)E^{\mathbb{Q}_F}[V(S(T), r(T), T)|\mathcal{F}_{t_0}] \\ &= P(t_0, T) \int_R \int_R V(x, y, T) \phi_F(x, y, T) dx dy, \end{aligned} \quad (3.2.1)$$

where  $\phi_F(x, y, T)$  denotes the  $T$ -forward measure probability density. More accurately the density function should be written as  $\phi_F(x, y, T; S(t_0), r(t_0), t_0)$ , since it is the transition probability density function of going from state  $(t_0, S(t_0), r(t_0))$  to  $(T, x, y)$ . But  $t_0, S(t_0), r(t_0)$  are considered to be given constants, for briefly it is written as  $\phi_F(x, y, T)$ .

According Section 2.3, any derivative price  $V(x, y, t)$ , discounted by  $D(t) = e^{-\int_{t_0}^t r(s)ds}$  must

be a martingale in the risk-neutral  $\mathbb{Q}$ . Hence, applying Itô formula, we have

$$\begin{aligned}
& d(D(t)V(x, y, t)) \\
&= D(t) \left( \frac{\partial V}{\partial t} - r(t)V + r(t)S(t) \frac{\partial V}{\partial x} + (a(b - r(t))) \frac{\partial V}{\partial y} \right. \\
&\quad \left. + \frac{1}{2} \sigma^2(S(t), t) S^2(t) \frac{\partial^2 V}{\partial x^2} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 V}{\partial y^2} + \rho \sigma(S(t), t) S(t) \sigma_r \frac{\partial^2 V}{\partial x \partial y} \right) dt \\
&\quad + D(t) \left( S(t) \sigma(S(t), t) \frac{\partial V}{\partial x} dW_S^{\mathbb{Q}}(t) + \sigma_r \frac{\partial V}{\partial y} dW_r^{\mathbb{Q}}(t) \right)
\end{aligned} \tag{3.2.2}$$

and so setting the drift to zero gives the PDE for  $V$ :

$$\begin{aligned}
\frac{\partial V}{\partial T} &= r(T)V - r(T)S(T) \frac{\partial V}{\partial x} - (a(b - r(T))) \frac{\partial V}{\partial y} \\
&\quad - \frac{1}{2} \sigma^2(S(T), T) S^2(T) \frac{\partial^2 V}{\partial x^2} - \frac{1}{2} \sigma_r^2 \frac{\partial^2 V}{\partial y^2} - \rho \sigma(S(T), T) S(T) \sigma_r \frac{\partial^2 V}{\partial x \partial y}
\end{aligned} \tag{3.2.3}$$

To derive the forward PDE for  $\phi_F$ , we note that the left-hand side of (3.2.1) is independent of  $T$ .

After differentiating both sides with respect to  $T$ ,

$$0 = \frac{\partial P}{\partial T} \int_R \int_R V(x, y, T) \phi_F(x, y, T) dx dy + P(t_0, T) \int_R \int_R \left( V(x, y, T) \frac{\partial \phi_F}{\partial T} + \phi_F \frac{\partial V}{\partial T} \right) dx dy \tag{3.2.4}$$

Applying (3.2.3) in (3.2.4), it gives us

$$\begin{aligned}
0 &= \frac{\partial P}{\partial T} \int_R \int_R V(x, y, T) \phi_F(x, y, T) dx dy + P(t_0, T) \int_R \int_R \left[ V(x, y, T) \frac{\partial \phi_F}{\partial T} \right. \\
&\quad \left. + \phi_F \left( r(T)V - r(T)S(T) \frac{\partial V}{\partial x} - (a(b - r(T))) \frac{\partial V}{\partial y} \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \sigma^2(S(T), T) S^2(T) \frac{\partial^2 V}{\partial x^2} - \frac{1}{2} \sigma_r^2 \frac{\partial^2 V}{\partial y^2} - \rho \sigma(S(T), T) S(T) \sigma_r \frac{\partial^2 V}{\partial x \partial y} \right) \right] dx dy
\end{aligned} \tag{3.2.5}$$

We define *the forward short rate* at time  $t_0$  for investing with maturity  $T$  to be

$$\begin{aligned}
f(t_0, T) &= - \lim_{\delta \rightarrow 0} \frac{\ln P(t_0, T + \delta) - \ln P(t_0, T)}{\delta} \\
&= - \frac{\partial}{\partial T} \ln P(t_0, T) \\
&= - \frac{1}{P(t_0, T)} \frac{\partial P}{\partial T}
\end{aligned} \tag{3.2.6}$$

Intergration by parts for the second term in (3.2.5) and using (3.2.6), we get

$$0 = \int_R \int_R V \left[ \frac{\partial \phi_F}{\partial T} + (r(T) - f(t_0, T))\phi_F + \frac{\partial [r(T)S(T)\phi_F]}{\partial x} + \frac{\partial [a(b - r(T))\phi_F]}{\partial y} \right. \\ \left. - \frac{1}{2} \frac{\partial^2 [\sigma^2(S(T), T)S^2(T)\phi_F]}{\partial x^2} - \frac{1}{2} \frac{\partial^2 [\sigma_r^2 \phi_F]}{\partial y^2} - \frac{\partial^2 [\rho\sigma(S(T), T)S(T)\sigma_r \phi_F]}{\partial x \partial y} \right] dx dy \\ + \text{boundary terms} \quad (3.2.7)$$

The boundary terms can be ignored since the drift and volatility terms of  $S(t)$  and  $r(t)$  are well behaved everywhere(including infinity). The above equation holds for all payoffs  $V(x, y, t)$ , which gives us

$$\frac{\partial \phi_F}{\partial T} + (r(T) - f(t_0, T))\phi_F + \frac{\partial [r(T)S(T)\phi_F]}{\partial x} + \frac{\partial [a(b - r(T))\phi_F]}{\partial y} \\ - \frac{1}{2} \frac{\partial^2 [\sigma^2(S(T), T)S^2(T)\phi_F]}{\partial x^2} - \frac{1}{2} \frac{\partial^2 [\sigma_r^2 \phi_F]}{\partial y^2} - \frac{\partial^2 [\rho\sigma(S(T), T)S(T)\sigma_r \phi_F]}{\partial x \partial y} = 0 \quad (3.2.8)$$

The above equation is also called Fokker-Planck(Kolmogorov) equation for the forward probability density function  $\phi_F$ . (3.2.8) is a forward PDE since it is solved forward in time with the initial condition at time  $t = t_0$  given by  $\phi_F(x, y, t) = \delta(x - x_0, y - y_0)$ , where  $\delta$  is the Dirac delta function and  $x_0, y_0$  correspond to the values at time  $t = t_0$  of the asset price, interest rate respectively.

Let function  $g_F(x, T)$  denote the integral of  $\phi_F(x, y, T)$  over the whole range of  $y$

$$g_F(x, T) = \int_{-\infty}^{+\infty} \phi_F(x, y, T) dy \quad (3.2.9)$$

Differentiating both sides with respect to  $T$ , we get

$$\frac{\partial g_F}{\partial T} = \int_{-\infty}^{+\infty} \frac{\partial \phi_F(x, y, T)}{\partial T} dy \quad (3.2.10)$$

Making the realistic assumptions that  $\lim_{y \rightarrow \pm\infty} \phi_F(x, y, T) = 0$  and the partial derivatives of  $\phi_F$  with respect to  $x, y$  also tend to zero when  $y$  tends to infinity. After integrating (3.2.8) with

respect to  $y$ , we obtain the following PDE

$$0 = \frac{\partial g_F}{\partial T} + \int (r(T) - f(t_0, T)) \phi_F dy + \frac{\partial}{\partial x} \left( \int r(T) S(T) \phi_F dy \right) - \frac{1}{2} \frac{\partial^2 [\sigma^2(S(T), T) S^2(T) g_F]}{\partial x^2} \quad (3.2.11)$$

### 3.3 Dupire-Like Formula Derivation

In this section, we derive the expression of the local volatility function in our model by using the same method in [4]. Let  $C$  denote the European call price, from (3.2.1), we know

$$\begin{aligned} C &= P(t_0, T) E^{\mathbb{Q}_F} [(S(T) - K)^+ | \mathcal{F}_{t_0}] \\ &= P(t_0, T) \int \int_K^\infty (x - K) \phi_F(x, y, T) dx dy \end{aligned} \quad (3.3.1)$$

Differentiating (3.3.1) twice with respect to  $K$ , we obtain

$$\begin{aligned} \frac{\partial C}{\partial K} &= P(t_0, T) \int \left( - (K - K) \phi_F(K, y, T) - \int_K^\infty \phi_F(x, y, T) dx \right) dy \\ &= -P(t_0, T) \int \int_K^\infty \phi_F(x, y, T) dx dy \\ &= -P(t_0, T) E^{\mathbb{Q}_F} [\mathbf{1}_{\{S(T) > K\}}] \\ \frac{\partial^2 C}{\partial K^2} &= -P(t_0, T) \frac{\partial}{\partial K} \left( \int \int_K^\infty \phi_F(x, y, T) dx dy \right) \\ &= P(t_0, T) \int \phi_F(K, y, T) dy \\ &= P(t_0, T) g_F(K, T) \end{aligned} \quad (3.3.2)$$

Differentiating (3.3.1) with respect to the maturity  $T$ , we get

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{\partial P}{\partial T} \int \int_K^\infty (x - K) \phi_F(x, y, T) dx dy + P(t_0, T) \int \int_K^\infty (x - K) \frac{\partial \phi_F(x, y, T)}{\partial T} dx dy \\ &= -f(t_0, T) C + P(t_0, T) \int_K^\infty (x - K) \frac{\partial g_F}{\partial T} dx \end{aligned} \quad (3.3.3)$$

For the second term of (3.3.3), using (3.2.11) leads to

$$\begin{aligned}
& \int_K^\infty (x - K) \frac{\partial g_F}{\partial T} dx \\
&= \int_K^{+\infty} (x - K) \left\{ \frac{1}{2} \frac{\partial^2 [\sigma^2(S(T), T) S^2(T) g_F]}{\partial x^2} - \int (r(T) - f(t_0, T)) \phi_F dy - \frac{\partial}{\partial x} \left( \int r(T) S(T) \phi_F dy \right) \right\} dx \\
&= \int \int_K^{+\infty} (x - K) (r(T) - f(t_0, T)) \phi_F dx dy + \frac{1}{2} \int_K^{+\infty} (x - K) \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} [\sigma^2(S(T), T) S^2(T) g_F] \right) dx \\
&\quad - \int_K^{+\infty} (x - K) \frac{\partial}{\partial x} \left( \int r(T) S(T) \phi_F dy \right) dx \\
&= - \int \int_K^{+\infty} (x - K) r(T) \phi_F dx dy + f(t_0, T) \int \int_K^{+\infty} (x - K) \phi_F dx dy \\
&\quad + \frac{1}{2} \left( \left[ (x - K) \frac{\partial}{\partial x} [\sigma^2(S(T), T) S^2(T) g_F] \right] \Big|_{x=K}^{x=+\infty} - \int_K^{+\infty} \frac{\partial}{\partial x} [\sigma^2(S(T), T) S^2(T) g_F] dx \right) \\
&\quad - \left( \left[ (x - K) \int r(T) S(T) \phi_F dy \right] \Big|_{x=K}^{x=+\infty} - \int_K^{+\infty} \int r(T) S(T) \phi_F dx dy \right) \\
&= \frac{f(t_0, T) C}{P(t_0, T)} - \int \int_K^{+\infty} r(T) S(T) \phi_F dx dy + \int \int_K^{+\infty} r(T) K \phi_F dx dy \\
&\quad - \frac{1}{2} \int_K^{+\infty} \frac{\partial}{\partial x} [\sigma^2(S(T), T) S^2(T) g_F] dx + \int \int_K^{+\infty} r(T) S(T) \phi_F dx dy \\
&= \frac{f(t_0, T) C}{P(t_0, T)} + \int \int_K^{+\infty} r(T) K \phi_F dx dy - \frac{1}{2} [\sigma^2(S(T), T) S^2(T) g_F] \Big|_{x=K}^{x=+\infty} \\
&= \frac{f(t_0, T) C}{P(t_0, T)} + \int \int_K^{+\infty} r(T) K \phi_F dx dy + \frac{1}{2} \cdot \frac{1}{P(t_0, T)} \sigma^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2} \\
&= \frac{f(t_0, T) C}{P(t_0, T)} + E[r(T) K \mathbf{1}_{S(T) > K}] + \frac{1}{2} \cdot \frac{1}{P(t_0, T)} \sigma^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2}
\end{aligned}$$

Therefore, we obtain the following Dupire-Like Formula, which the expression of local volatility  $\sigma^2(K, T)$  in terms of call price  $C$

$$\sigma^2(K, T) = 2 \frac{\frac{\partial C}{\partial T} - P(t_0, T) E^{\mathbb{Q}_F} [r(T) K \mathbf{1}_{S(T) > K}]}{K^2 \frac{\partial^2 C}{\partial K^2}} \quad (3.3.4)$$

In general, if we take dividend into consideration, our the asset price is governed by the following dynamics

$$dS(t) = (r(t) - q(t)) S(t) dt + \sigma(S(t), t) S(t) dW_S^{\mathbb{Q}}(t), S(t_0) = S_0, \quad (3.3.5)$$

where  $q(t)$  is a deterministic function of  $t$  and the interest rates  $r(t)$  follows a Vasicek model

$$dr(t) = a(b - r(t))dt + \sigma_r dW_r^{\mathbb{Q}}(t), r(t_0) = r_0 \quad (3.3.6)$$

Our local volatility  $\sigma^2(K, T)$  in terms of call prices  $C$  is

$$\sigma^2(K, T) = 2 \frac{\frac{\partial C}{\partial T} - P(t_0, T) E^{\mathbb{Q}_F} [(r(T)K - q(T)S(T)) 1_{S(T) > K}]}{K^2 \frac{\partial^2 C}{\partial K^2}} \quad (3.3.7)$$

Comparing with Dupire's formula (2.5.5), we notice that our Dupire-Like formula (3.3.4) is not easily applicable for calibration over the market since there seems no immediate way to link the expectation term with European call option prices or other liquid products. We will present numerical method to calculate local volatility in Chapter 4.

Furthermore, when assuming deterministic interest rates, our Dupire-Like formula (3.3.4) reduces to the simple Dupire's formula (2.5.5).

## 4 Numerical Approaches

Although we get the local volatility function from (3.3.4), there is still the matter that the call price is not a known continuous function of strike and maturity, but only known at certain points. Therefore, before the local volatility can be used to price derivatives, a procedure to obtain the local volatility surface must be devised. In our model, the construction of local volatility surface for the two-factor model with local volatility can be decomposed in three steps: (i) Using finite differences approach to get approximating derivative values  $\frac{\partial C(K,T)}{\partial T}$ ,  $\frac{\partial^2 C(K,T)}{\partial K^2}$ ; (ii) The expectation  $E^{\mathbb{Q}_F} [r(T)K1_{S(T)>K}]$  is approximated by using Monte Carlo simulations up to a fixed time  $t = T$ ; (iii) After these two steps, Lipschitz interpolation technique is used to construct local volatility surface. Then, Monte Carlo simulation is applied to value options using the local volatility surface. Finally, a numerical implementation of the model and its result are reported.

### 4.1 Finite Differences Approximation for Derivatives

It is well-known that the implied volatility(or market option price) is not a known continuous function of strike and maturity, but only known at certain points. Therefore, before we use finite differences method to compute derivatives, we need to use interpolation to construct an IVS.

As we discussed in Section 2.5, there are many ways to build a volatility surface. In our work, the volatility surface is built using the following procedures:

1. We generate a one-dimensional interpolation for each input maturity  $T_i$  using Fourier interpolation;

2. For each maturity  $T \in [T_i, T_{i+1}]$  and each strike  $K$ , we calculate the implied volatility

$$\sigma_{imp}(K, T) \text{ so that } \sigma_{imp}^2(K, T)T \text{ is a linear interpolation of } \sigma_{imp}^2(K, T_i)T_i \text{ and } \sigma_{imp}^2(K, T_{i+1})T_{i+1}.$$

The idea of Fourier interpolation is that we can use the series of sine and cosine functions to represent arbitrary functions. In our case, we use the discrete trigonometric approximation and interpolate the data since our implied volatilities data set is discrete.

Suppose that a collection of  $2m$  paired data points  $\{(x_i, y_i)\}_{i=0}^{2m-1}$  is given, with the first elements in the pairs equally partitioning a closed interval. For convenience, we assume that the interval is  $[-\pi, \pi]$ ,

$$x_j = -\pi + \left(\frac{j}{m}\right)\pi, \text{ for each } j = 0, 1, \dots, 2m - 1.$$

If it is not in  $[-\pi, \pi]$ , a simple linear transformation could be used to transform the data into this form [44].

The goal in the discrete case is to determine the trigonometric polynomial  $S_n(x)$  that will minimize

$$Error = \sum_{i=0}^{2m-1} [y_i - S_n(x_i)]^2,$$

where  $S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} [a_k \cos kx + b_k \sin kx]$

The optimal constants that minimize the least square sum are

$$a_k = \frac{1}{m} \sum_{i=0}^{2m-1} y_i \cos kx_i, \text{ for each } k = 0, 1, \dots, n$$

and

$$b_k = \frac{1}{m} \sum_{i=0}^{2m-1} y_i \sin kx_i \text{ for each } k = 1, \dots, n - 1$$

In Black-Scholes formula,  $C(K, T)$  depends on time only through  $\sigma_{imp}^2(K, T)T$ . Moreover,  $C(K, T)$  is an increasing function of  $\sigma_{imp}^2(K, T)T$ . Thus if  $\sigma_{imp}^2(K, T)T$  is an increasing function of  $T$ , our interpolation in the time domain ensures that no-arbitrage condition  $C(K, T_i) < C(K, T_{i+1})$  holds. After the implied volatility surface has been calculated for all maturities up to the last input maturity, we can calculate the associated call prices  $C(K, T)$ . Then we can use finite difference approach to approximate derivatives in our Dupire-Like formula.

The finite difference approach is based on the simple idea of approximation each partial derivative by a difference quotient. As we know, under suitable continuity and differentiability hypotheses, Taylor's theorem states that a function  $f(x)$  may be represented as

$$f(x + h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \dots \quad (4.1.1)$$

If we neglect the terms of order  $h^2$  and higher, we get

$$f'(x) \approx \frac{f(x + h) - f(x)}{h}.$$

This is the *forward* approximation for the first order derivative. Another alternative way to approximate first-order derivative is that  $f(x)$  can be represented as

$$f(x - h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(x) + \dots \quad (4.1.2)$$

The *backward* approximation for the first order derivative is defined by

$$f'(x) \approx \frac{f(x) - f(x - h)}{h}.$$

In both cases we get a truncation error of order  $O(h)$ . In this report, we use a better approximation, which can be obtained by subtracting (4.1.2) from (4.1.1) and rearranging:

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h}. \quad (4.1.3)$$

This is called *central* approximation. By adding (4.1.2) and (4.1.1), which yields

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad (4.1.4)$$

In order to apply the above ideas to (3.3.4), first we interpolate given discrete implied volatilities to get an IVS and calculate call option  $C(K, T)$  for all  $K$  and  $T$ ; then  $\frac{\partial C}{\partial T}$ ,  $\frac{\partial^2 C}{\partial K^2}$  can be approximated by (4.1.3) and (4.1.4), which are

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{C(K, T + \Delta T) - C(K, T - \Delta T)}{2 \Delta T}, \\ \frac{\partial^2 C}{\partial K^2} &= \frac{C(K + \Delta K, T) - 2C(K, T) + C(K - \Delta K, T)}{\Delta K^2}. \end{aligned} \quad (4.1.5)$$

**Algorithm 4.1:**

- **Input:**  $T_i, K_i, \sigma_{imp}(K_i, T_i), S_0, r_0$  and  $\Delta K, \Delta T$ .

1. For each  $T_i$ , for any  $K$ , calculate the implied volatility  $\sigma_{imp}(K, T_i)$  from Fourier approximation;
2. For each  $K_i$ , for any  $T \in [T_i, T_{i+1}]$ , calculate the implied volatility  $\sigma_{imp}(K_i, T)$  such that  $\sigma_{imp}^2(K_i, T)T$  is a linear interpolation of  $\sigma_{imp}^2(K_i, T_i)T_i$  and  $\sigma_{imp}^2(K_i, T_{i+1})T_{i+1}$
3. For each  $K_i$  and  $T_i$ , calculate  $C(K_i, T_i), C(K_i, T_i \pm \Delta T)$  and  $C(K_i \pm \Delta K, T_i)$ ;
4. Calculate  $\frac{\partial C}{\partial T}, \frac{\partial^2 C}{\partial K^2}$  from (4.1.5).

- **Output:** IVS picture and

$$\left. \frac{\partial C}{\partial T} \right|_{K=K_i, T=T_i}, \left. \frac{\partial^2 C}{\partial K^2} \right|_{K=K_i, T=T_i}$$

## 4.2 Monte Carlo Simulation

Monte Carlo simulation is an important tool in computational finance. It contains a broad class of computational algorithms based on statistical sampling and analyzing the outputs gives the estimate of a quantity of our interest. The technique of Monte Carlo Simulation works by replicating the outcomes of a stochastic process using of random numbers. As the number of simulations increases, the results, represented by the average, converge to the analytically correct solution.

In this section, the expectation in (3.3.4),  $E^{\mathbb{Q}^F} [r(T)K1_{\{S(T)>K\}}]$  is approximated by using Monte Carlo simulations. From original Dupire's idea, we have to use implied volatilities (or observed market option price) to calibrate local volatility. Therefore, to calculate numerically this expectation we have to simulate asset price  $S(t)$  based on implied volatilities and interest rate  $r(t)$  up to time  $T$  starting from the initial market prices  $S_0$  and initial interest rate  $r_0$  respectively. Since the expectation is expressed under the  $T$ -forward measure, we have to use the dynamics of  $S(t), r(t)$  under this measure.

For a fixed pair  $(K, T)$ , the dynamics of the process  $r(t)$  and  $S(t)$  can also be expressed in terms of two independent Brownian motion  $\widetilde{W}_r^{\mathbb{Q}}$  and  $\widetilde{W}_S^{\mathbb{Q}}$  as follows (Cholesky decomposition):

$$\begin{cases} dr(t) = a(b - r(t))dt + \sigma_r d\widetilde{W}_r^{\mathbb{Q}}(t), \\ dS(t) = r(t)S(t)dt + \sigma_{imp}(K, T)S(t)[\rho d\widetilde{W}_r^{\mathbb{Q}}(t) + \sqrt{1 - \rho^2} d\widetilde{W}_S^{\mathbb{Q}}(t)]. \end{cases} \quad (4.2.1)$$

where

$$\begin{aligned} dW_r^{\mathbb{Q}}(t) &= d\widetilde{W}_r^{\mathbb{Q}}(t), \\ dW_S^{\mathbb{Q}}(t) &= \rho d\widetilde{W}_r^{\mathbb{Q}}(t) + \sqrt{1 - \rho^2} d\widetilde{W}_S^{\mathbb{Q}}(t). \end{aligned} \quad (4.2.2)$$

This decomposition makes it easier to perform a measure transformation. Recall in Section 2.7,

the  $T$ -forward measure  $\mathbb{Q}_F$  is defined by the Radon-Nikodym derivative

$$\begin{aligned}\frac{d\mathbb{Q}_F}{d\mathbb{Q}} &= \frac{D(T)}{P(t_0, T)} \\ &= \exp \left\{ -\frac{\sigma_r}{a} \int_{t_0}^T [1 - e^{-a(T-u)}] d\widetilde{W}_r^{\mathbb{Q}}(u) - \int_{t_0}^T \frac{\sigma_r^2}{2a^2} [1 - e^{-a(T-u)}]^2 du \right\}\end{aligned}\quad (4.2.3)$$

The Girsanov theorem in Section 2.1 implies that two processes  $\widetilde{W}_r^{\mathbb{Q}}$  and  $\widetilde{W}_S^{\mathbb{Q}}$  defined by

$$\begin{aligned}dW_r^{\mathbb{Q}_F}(t) &= d\widetilde{W}_r^{\mathbb{Q}}(t) + \frac{\sigma_r [1 - e^{-a(T-t)}]}{a} dt, \\ dW_S^{\mathbb{Q}_F}(t) &= d\widetilde{W}_S^{\mathbb{Q}}(t).\end{aligned}\quad (4.2.4)$$

are two independent Brownian motion under the measure  $\mathbb{Q}_F$ .

Remember in Section 2.5,  $A(t, T) = \frac{1 - e^{-a(T-t)}}{a}$ . Therefore the dynamics of  $r(t)$  and  $S(t)$  under  $\mathbb{Q}_F$  are given by

$$\begin{cases} dr(t) = [a(b - r(t)) - \sigma_r^2 A(t, T)] dt + \sigma_r dW_r^{\mathbb{Q}_F}(t), \\ dS(t) = [r(t) - \sigma_{imp}(K, T) \sigma_r \rho A(t, T)] S(t) dt + \sigma_{imp}(K, T) S(t) [\rho dW_r^{\mathbb{Q}_F}(t) + \sqrt{1 - \rho^2} dW_S^{\mathbb{Q}_F}(t)]. \end{cases}\quad (4.2.5)$$

On each small interval,

$$\begin{cases} r(t_{i+1}) = r(t_i) + [a(b - r(t_i)) - \sigma_r^2 A(t_i, T)] (t_{i+1} - t_i) + \sigma_r \sqrt{t_{i+1} - t_i} \epsilon_1, \\ S(t_{i+1}) = S(t_i) [1 + r(t_i) - \sigma_{imp}(K, T) \sigma_r \rho A(t_i, T) \rho (t_{i+1} - t_i) + \sigma_{imp}(K, T) \sqrt{t_{i+1} - t_i} \epsilon_2]. \end{cases}\quad (4.2.6)$$

The idea of the Monte Carlo method is to simulate  $n$  times the stochastic variables  $S(t)$  and  $r(t)$  up to time  $T$ , by using Euler discretisations. The expectation is approximated by:

$$E^{\mathbb{Q}_F} [r(T) K 1_{\{S(T) > K\}}] \cong \frac{1}{n} \sum_{i=1}^n r^i(T) K 1_{\{S^i(T) > K\}}, \quad (4.2.7)$$

where  $i$  corresponds to the  $i^{\text{th}}$ -simulation  $i = 1, \dots, n$ .

**Algorithm 4.2:**

- **Input:**  $T_i, K_i, \sigma_{imp}(K_i, T_i), S_0, r_0, a, b, \sigma_r, \rho$ , time step size  $dt$  and simulation number  $Nsim$ .

1. For  $n = 1, \dots, Nsim$ , do the following Steps:

Step 1. For  $T_i$ , calculate time step number  $N = T_i/dt$ ;

Step 2. Generate two correlated random numbers  $\epsilon_1, \epsilon_2$ ;

Step 3. For  $j = 1, \dots, N$ , calculate  $r(T)$  and  $S(T)$  by (4.2.6);

2. Calculate Expectation by (4.2.7);

3. Repeat steps 1-2 until calculate expectation (4.2.7) for all  $K_i$  and  $T_i$ ;

- **Output:**  $E^{\mathbb{Q}^F} [r(T_i)K_i 1_{\{S(T_i) > K_i\}}]$  for every  $T_i, K_i$ .

### 4.3 Constructing Local Volatility Surface by Using Lipschitz Interpolation

The local volatilities that we calculate from our Dupire-Like formula are still a discrete set of strike prices and maturities. Before the local volatility can be used to price derivatives, we need to fit a local volatility surface.

In original local volatility model, local volatility surface  $\sigma(S(t), t)$  is arbitrage-free if  $\sigma(S(t), t)$  is greater than zero and  $\sigma(S(t), t)S(t)$  is Lipschitz continuous [8]. This property motivates us to use Lipschitz interpolation to construct local volatility surface [14,15]. The underlying interpolation method assumes that the data are generated by a continuous function  $f$  (Lipschitz continuous), which implements a method of reliable multivariate interpolation of scattered data.

Let  $m$  denote the dimensionality of the space, and  $N$  denote the size of the data set. Assume that we are given a data set  $\mathcal{D} = \{(x^n, y^n)\}_{n=1}^N, x^n \in R^m, y^n \in R$ . We also assume that  $y^n$  are the

values of some function  $f(x^n) = y^n$ , which is unknown to us and which we want to approximate with  $g, g \approx f$ . Thus we look for an interpolant  $g : R^m \rightarrow R$ , such that  $g(x^n) = y^n, n = 1, \dots, N$ .

We shall work in the space of continuous functions with the supremum norm, i.e.,  $\mathcal{V} = C(X), X \subset R^m, X$  is compact. We shall assume that  $f$  is bounded and Lipschitz continuous,

$$\exists M > 0, \forall x, z \in X \subset R^m, |f(x) - f(z)| \leq Md(x, z),$$

where  $M$  is Lipschitz constant and  $d(x, z)$  is a distance function.

We denote the class of functions whose Lipschitz constant is smaller or equal to  $M$  by  $Lip(M)$ . Given information about the Lipschitz constant, or its estimate, our goal is to find an interpolant  $g$ , which is the best possible approximation regardless of how inconvenient the unknown function  $f$  is, even in the worst case scenario. That is, we solve the following problem:

$$g = \arg \inf_{g \in C(X)} \max_{f \in Lip(M)} \|f - g\|_{C(X)} \quad (4.3.1)$$

such that

$$g(x^n) = y^n, n = 1, \dots, N$$

Now we construct an optimal central interpolant to functions from  $Lip(M)$ . First we need to identify tight upper and lower bounds on the values of  $f$ , and then take their half-sum as the solution. From the Lipschitz condition it follows that

$$\forall x, x^n \in X, |f(x) - y^n| \leq Md(x, x^n) \quad (4.3.2)$$

which yields

$$\max_n (y^n - Md(x, x^n)) \leq f(x) \leq \min_n (y^n + Md(x, x^n)) \quad (4.3.3)$$

Denote the upper and lower approximations to  $f$  by  $H^{upper}$  and  $H^{lower}$ ,

$$H^{upper}(x) = \min_n (y^n + Md(x, x^n)), H^{lower}(x) = \max_n (y^n - Md(x, x^n)),$$

and let

$$g(x) = \frac{1}{2}(H^{upper}(x) + H^{lower}(x)), \forall x \in X. \quad (4.3.4)$$

Since function  $d(\cdot, x^n) \in Lip(1)$ , it directly follows that  $H^{upper}(x), H^{lower}(x) \in Lip(M)$ . Then  $g \in Lip(M)$ , i.e.,  $g$  is the solution to the interpolation problem (4.3.1).

Since  $\min_n \{a_n\} - \max_j \{b_j\} = \min_n \min_j \{a_n - b_j\}$ , and

$$\begin{aligned} H^{upper}(x) - H^{lower}(x) &= \min_n \min_j \{y^n + Md(x, x^n) - (y^j - Md(x, x^j))\} \\ &= \min_n \{y^n + Md(x, x^n) - (y^n - Md(x, x^n))\} \\ &= 2M \min_n d(x, x^n) \end{aligned} \quad (4.3.5)$$

Thus maximal error of interpolation is given as

$$\max_{f \in Lip(M)} \|f - g\|_{C(X)} = M \max_{x \in X} \min_{n=1, \dots, N} d(x, x^n),$$

where the distance  $d(\cdot)$  is either Euclidean ( $l_2$ -norm) or Chebyshev-distance ( $l_\infty$ -norm), or any  $l_p$ -norm,  $p \geq 1$ . More details about how to construct optimal  $g$  can be found in [1,2,54].

So far we assume that the Lipschitz constant of  $f$  is provided (e.g., as part of the interpretation of the data). We emphasize that only in this case one can obtain finite error bounds and construct an optimal interpolant.

However, a more typical situation is when the only information is the data set  $\mathcal{D}$  itself. In this case, we have to estimate the value of the Lipschitz constant to adequately choose the class of function  $Lip(M)$ . In the following, we will address the issue of estimation of the Lipschitz constant from the data set.

We call the data set  $\mathcal{D}$  compatible with the class  $Lip(M)$  if there exists a function  $f \in Lip(M)$  such that  $f(x^n) = y^n, n = 1, 2, \dots, N$ . The following result gives us the relation of compatibility of  $Lip(M)$  with the given data set:

*The data set  $\mathcal{D}$  is compatible with the class  $Lip(M)$  if and only if the following conditions hold:*

$$\forall i, j \in \{1, 2, \dots, N\}, |y^i - y^j| \leq M d(x^i, x^j)$$

Obviously,  $g$  exists only if the data are compatible with the given Lipschitz condition. Since  $M_1 > M_2$  implies  $Lip(M_2) \subset Lip(M_1)$ , compatibility with the data will not be affected by choosing a larger Lipschitz constant. Therefore, in our report, we calculate the smallest Lipschitz constant  $M$  compatible with a given data set as

$$M = \inf\{C \in R : \forall i, j = 1, 2, \dots, N, |y^i - y^j| \leq C d(x^i, x^j)\} \quad (4.3.6)$$

This implies that we choose the smallest class of Lipschitz function  $Lip(M)$  still compatible with the given data.

**Algorithm 4.3:**

• **Input:**  $T_i, K_i, \sigma_{imp}(K_i, T_i), S_0, r_0, a, b, \sigma_r$ .

1. Calculate bond price  $P(t_0, T)$  by (2.6.11);
2. Calculate local volatility  $\sigma(K, T)$  by (3.3.4);
3. Do the following steps for each  $T_i$ :

Step 1. Calculate  $y_i = \sigma(K_i, T)K_i$ ;

Step 2. Calculate Lipschitz  $M$  by (4.3.6), where  $x_i = K_i$ ;

Step 3. Calculate the local volatility  $\sigma(K, T)$  so that  $\sigma(K, T_i)K$  is a Lipschitz interpolation of  $\sigma(K_i, T_i)K_i$ ;

4. Apply the same linear interpolation technique as Algorithm 4.2 for  $T$  direction, ;

- **Output:** Local volatility  $\sigma(K_i, T_i)$  and Local Volatility Surface

#### 4.4 Price European Option

Based on our model, under forward measure, recall European call price  $C$  is given by

$$C = P(t_0, T)E^{\mathbb{Q}^F}[(S(T) - K)^+ | \mathcal{F}_{t_0}],$$

where

$$\begin{aligned} P(t, T) &= e^{-A(t, T)r(t) + D(t, T)}, \\ A(t, T) &= \frac{1 - e^{-a(T-t)}}{a} \\ D(t, T) &= \left[ b - \frac{\sigma^2}{2a^2} \right] [A(t, T) - (T - t)] - \frac{\sigma^2 A^2(t, T)}{4a} \end{aligned} \tag{4.4.1}$$

As the same method in Section 4.2(calculating expectation), we still use Monte Carlo simulations to value European option. This is a simple procedure, requiring a few programming skills. The drawback is that a large amount of simulations have to be performed to get accurate results.

In order to simulate  $S(T)$ , we divide the interval  $[t_0, T]$  into small subintervals, i.e.,  $t_0 = T_0 < T_1 < T_2 < \dots < T_n = T$ . On each interval, we simulate  $dW_S^{\mathbb{Q}^F}(t), dW_r^{\mathbb{Q}^F}(t)$  as related normal variable

$$S(T_{i+1}) = S(T_i) \left[ 1 + r(T_i) - \sigma(K, T_i) \sigma_r A(T_i, T) \rho (T_{i+1} - T_i) + \sigma(K, T_i) \sqrt{T_{i+1} - T_i} \epsilon_1 \right],$$

$$r(T_{i+1}) = r(T_i) + \left[ a(b - r(T_i)) - \sigma_r^2 A(T_i, T) \right] (T_{i+1} - T_i) + \sigma_r \sqrt{T_{i+1} - T_i} \epsilon_2.$$

(4.4.2)

where  $\sigma(K, T)$  denotes local volatility.

The expectation is approximated by :

$$E^{\mathbb{Q}_F} [(S(T) - K)^+] \cong \frac{1}{n} \sum_{i=1}^n ((S^i(T) - K)^+) \quad (4.4.3)$$

where  $i$  corresponds to the  $i$ th simulation,  $i = 1, 2, \dots, n$ .

The algorithm consists of the following steps:

1. Calculate  $P(t_0, T)$  by (4.4.1);
2. Calculate  $(S^i(T) - K)^+$  by (4.4.2) for  $i = 1, \dots, Nsim$ ;
3. Calculate  $E^{\mathbb{Q}_F} [(S(T) - K)^+]$  by (4.4.3);
4. Calculate European call option price  $C$  by (3.3.1);
5. For all given  $T, K$ , repeat steps 1-4;

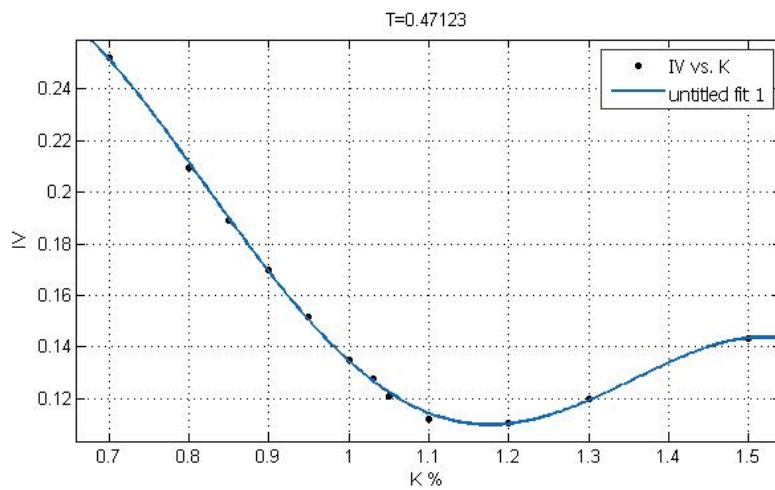
## 4.5 Numerical Experiments

In this section, we implement our method on the market implied volatilities of the S& P/TSX 60 stock market on October 21, 2013. The number of expiries is 11 with up to 12 strikes per expiry. The target data is given in Table 4.1, where the first row denotes expiries and the first column denotes strikes.

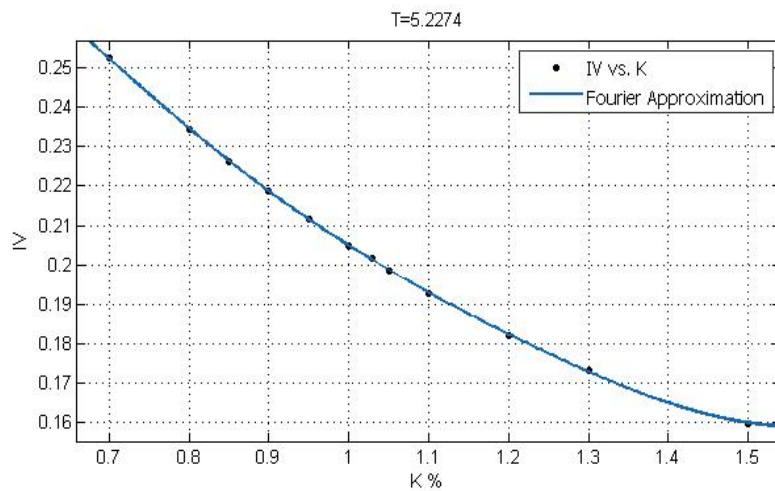
	0.0493	0.1260	0.2219	0.4712	0.7205	1.2192	2.2164	2.7151	3.2137	4.2110	5.2274
70.00%	38.45%	32.93%	28.23%	25.21%	24.17%	23.17%	23.51%	23.49%	23.71%	24.28%	25.23%
80.00%	29.76%	25.54%	22.65%	20.95%	20.53%	20.13%	20.94%	21.22%	21.58%	22.40%	23.43%
85.00%	25.15%	21.99%	19.99%	18.94%	18.83%	18.71%	19.77%	20.18%	20.62%	21.54%	22.62%
90.00%	20.55%	18.49%	17.42%	17.01%	17.21%	17.35%	18.65%	19.20%	19.71%	20.74%	21.86%
95.00%	15.94%	15.09%	14.96%	15.18%	15.68%	16.06%	17.61%	18.27%	18.85%	19.98%	21.15%
100.00%	11.64%	12.09%	12.75%	13.51%	14.26%	14.85%	16.63%	17.40%	18.05%	19.27%	20.48%
102.50%	10.37%	11.12%	11.87%	12.77%	13.61%	14.27%	16.17%	16.98%	17.66%	18.94%	20.16%
105.00%	10.38%	10.78%	11.24%	12.12%	13.02%	13.73%	15.72%	16.58%	17.30%	18.61%	19.85%
110.00%	12.30%	11.63%	10.91%	11.22%	12.04%	12.73%	14.91%	15.83%	16.60%	18.00%	19.26%
120.00%	16.98%	14.85%	12.35%	11.06%	11.18%	11.31%	13.58%	14.53%	15.39%	16.91%	18.21%
130.00%	21.18%	18.04%	14.37%	11.99%	11.43%	10.78%	12.74%	13.54%	14.44%	16.00%	17.32%
150.00%	27.08%	23.56%	18.15%	14.34%	12.96%	11.26%	12.32%	12.53%	13.34%	14.74%	15.99%

Table 4.1: Implied Volatilities on S& P/TSX 60

Two examples of fitting results using discrete least squares trigonometric polynomial of degree 2 is shown in Figure 4.1 ( $T = 0.47123$  and  $T = 5.2274$ ).



(a)  $T = 0.47123$



(b)  $T = 5.2274$

Figure 4.1: Fitting results Fourier interpolation for slice  $T = 0.47123$  and  $T = 5.2274$

The current spot  $S_0 = 735.07$ ,  $r_0 = 0.02$ , the corresponding Vasicek model parameters  $a = 0$ ,  $b = 0.0005$ ,  $\sigma_r = 0.0059$ , and the correlation number  $\rho = 0.3$ .

According to **Algorithm 4.1**, after using Fourier interpolation in strike direction and linear interpolation in maturity direction, we plot in Figure 4.2 the interpolation implied volatility surface.

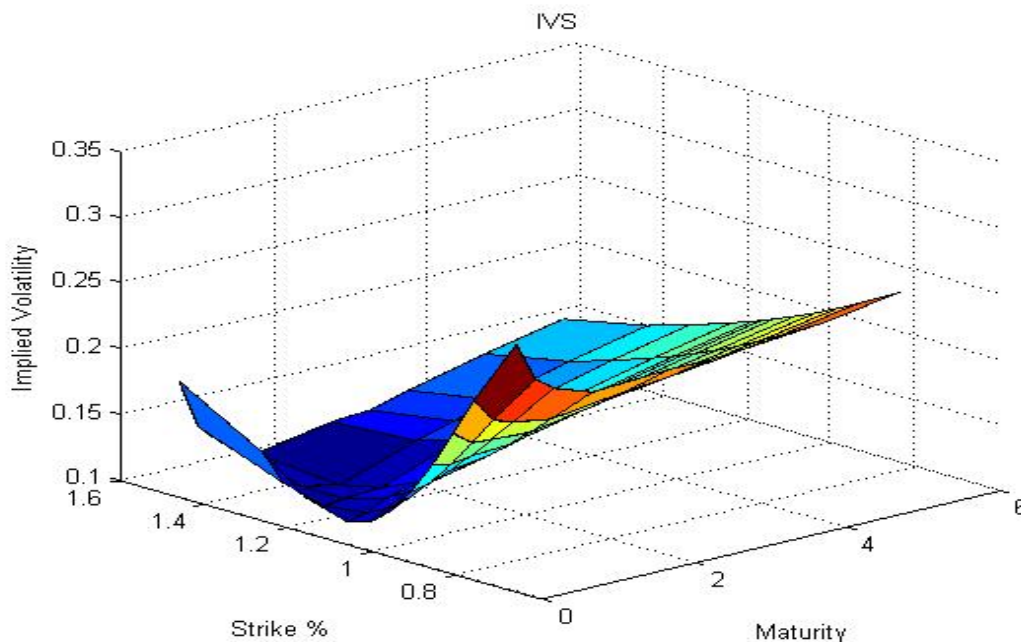
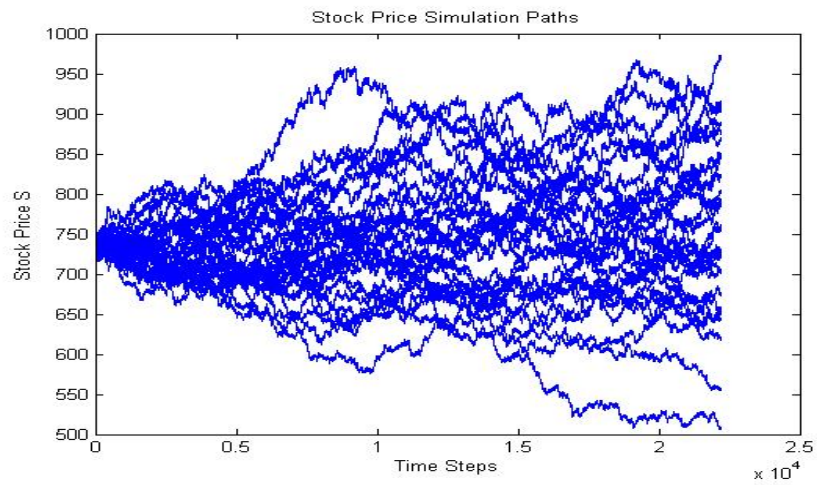


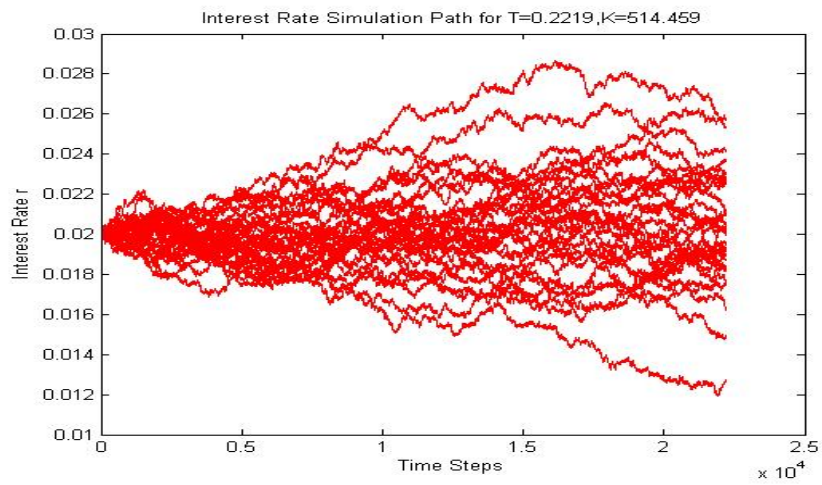
Figure 4.2: Implied Volatilities Surface

According **Algorithm 4.2**, choosing time step size  $dt = 0.0001$ , simulation number  $N_{sim} = 30$ , example simulation paths of  $S$  and  $r$  for  $T = 0.2219$ ,  $K = 70\%$  are plotted in figure 4.3.

Implementing **Algorithm 4.3** with time step size  $dt = 0.0001$ , simulation number  $N_{sim} = 5000$  and  $N_{sim} = 10000$ , we show the resulting local volatilities in Table 4.2 and Table 4.3.



(a) Stock Price Simulation Paths



(b) Interest Rate Simulation Paths

Figure 4.3: Stock Price and Interest Rate Simulation Paths for  $T = 0.2219, K = 70\%$

	0.0493	0.1260	0.2219	0.4712	0.7205	1.2192	2.2164	2.7151	3.2137	4.2110	5.2274
70.00%	56.12%	48.36%	37.84%	39.65%	37.67%	36.72%	35.20%	34.10%	35.07%	36.39%	37.05%
80.00%	41.65%	54.01%	37.24%	33.25%	30.30%	29.77%	28.87%	28.82%	29.71%	31.14%	31.79%
85.00%	57.59%	40.99%	30.33%	27.74%	25.67%	25.92%	26.09%	26.44%	27.32%	28.75%	29.27%
90.00%	32.75%	26.45%	23.04%	22.04%	21.71%	22.64%	23.77%	24.37%	25.42%	26.62%	27.52%
95.00%	18.61%	17.34%	17.28%	17.74%	18.16%	19.70%	21.66%	22.49%	23.46%	25.00%	26.14%
100.00%	12.18%	12.48%	13.34%	14.44%	15.28%	17.17%	19.46%	21.01%	22.16%	23.66%	24.48%
102.50%	11.00%	11.20%	11.80%	12.98%	13.97%	15.93%	18.40%	20.04%	21.16%	22.69%	23.78%
105.00%	11.09%	10.89%	11.18%	12.25%	13.16%	15.04%	17.78%	19.52%	20.69%	22.41%	23.17%
110.00%	14.72%	11.94%	10.48%	11.00%	11.65%	13.39%	16.23%	17.95%	19.23%	20.95%	22.00%
120.00%	49.39%	24.31%	12.11%	10.98%	10.19%	11.01%	13.83%	15.56%	17.12%	19.04%	19.84%
130.00%	48.98%	43.45%	21.96%	13.31%	10.51%	10.46%	12.36%	13.77%	15.33%	16.99%	18.02%
150.00%	61.24%	35.29%	11.56%	14.99%	16.84%	15.85%	12.43%	12.27%	13.62%	14.30%	14.39%

Table 4.2: Local Volatility on S& P/TSX 60 for Nsim=5000

	0.0493	0.1260	0.2219	0.4712	0.7205	1.2192	2.2164	2.7151	3.2137	4.2110	5.2274
70.00%	58.00%	48.42%	37.78%	39.40%	37.72%	36.56%	35.24%	33.89%	34.93%	36.51%	37.25%
80.00%	41.62%	54.54%	37.29%	33.20%	30.31%	29.59%	28.86%	28.77%	29.65%	31.11%	31.51%
85.00%	57.85%	40.93%	30.45%	27.56%	25.72%	25.93%	26.14%	26.56%	27.53%	28.78%	29.23%
90.00%	32.66%	26.44%	23.02%	22.07%	21.67%	22.70%	23.71%	24.59%	25.29%	26.74%	27.34%
95.00%	18.58%	17.37%	17.28%	17.70%	18.12%	19.70%	21.54%	22.64%	23.62%	25.17%	26.00%
100.00%	12.18%	12.51%	13.33%	14.42%	15.30%	17.11%	19.47%	20.91%	22.07%	23.69%	24.31%
102.50%	10.99%	11.18%	11.84%	13.01%	13.94%	15.89%	18.45%	19.98%	21.26%	22.72%	23.84%
105.00%	11.10%	10.89%	11.18%	12.25%	13.16%	14.98%	17.77%	19.38%	20.60%	22.36%	23.16%
110.00%	14.61%	11.94%	10.45%	11.02%	11.69%	13.34%	16.27%	18.02%	19.38%	21.13%	22.09%
120.00%	49.39%	22.42%	12.79%	10.82%	10.25%	11.17%	13.81%	15.52%	16.98%	18.94%	19.84%
130.00%	48.98%	43.45%	21.96%	13.47%	10.67%	10.44%	12.29%	13.70%	15.35%	16.99%	17.73%
150.00%	61.24%	35.29%	11.56%	14.99%	14.65%	15.57%	12.40%	12.27%	13.81%	14.35%	14.52%

Table 4.3: Local Volatility on S& P/TSX 60 for Nsim=10000

The corresponding LVS (for  $N_{sim} = 10000$  case ) are showed in Figure 4.4.

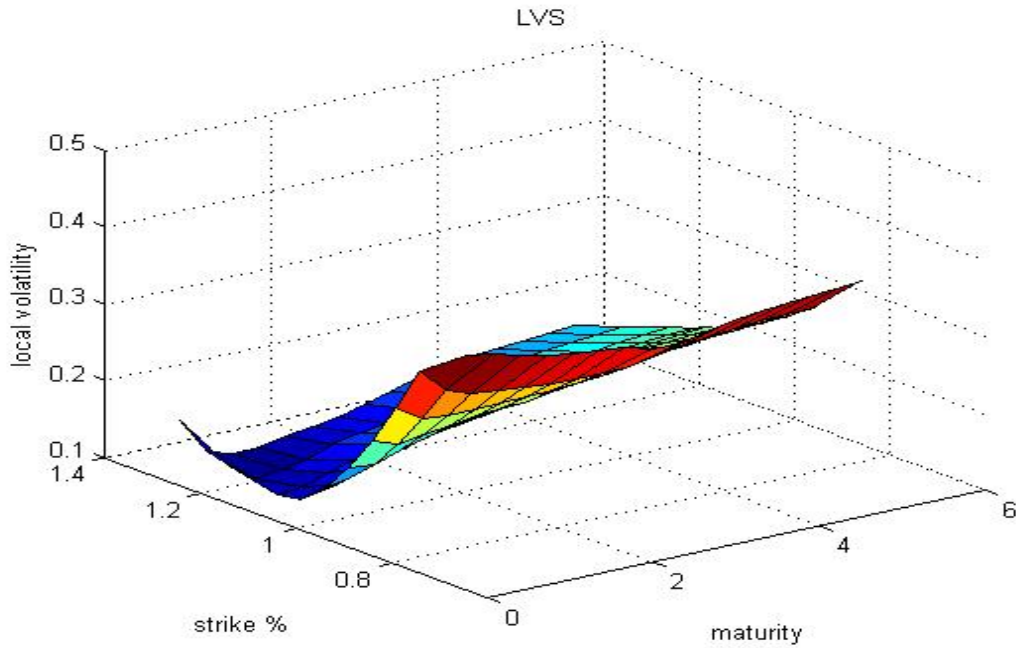


Figure 4.4: Local Volatility Surface

The European call prices calculated by Black-Schole and Monte Carlo method(based on our LVS) are given in Table 4.4 and Table 4.5.

We compared the Black-Scholes prices calculated via the input implied volatilities to those obtained via Monte Carlo simulation. The errors between Monte Carlo European call prices and the Black-Scholes prices are give in the Table 4.6.The minimum absolute error is 0,the maximum absolute error is 0.4925, and the average absolute error is 0.1664.

	0.0493	0.1260	0.2219	0.4712	0.7205	1.2192	2.2164	2.7151	3.2137	4.2110	5.2274
70.00%	221.0284	221.8375	222.8818	226.0022	229.6255	237.5540	255.8879	264.6449	273.7282	291.8440	311.1242
80.00%	147.5976	148.6177	150.0310	154.6266	159.9371	170.7823	194.0857	205.0550	215.9738	237.5156	259.6415
85.00%	110.8951	112.1619	113.9994	119.9215	126.4640	139.0374	164.9276	176.9839	188.8128	211.8838	235.3198
90.00%	74.2657	76.1302	78.8319	86.6786	94.6920	108.9863	137.2398	150.3642	162.9874	187.4859	212.0394
95.00%	38.2247	41.6768	45.9984	56.1851	65.6363	81.3031	111.4649	125.4162	138.6981	164.3667	189.9024
100.00%	7.9441	13.5164	19.2474	30.6463	40.7290	56.8274	87.9118	102.4497	116.1998	142.7066	168.9479
102.50%	0.8855	4.3104	8.9020	19.1661	28.8012	44.3957	75.4286	90.0774	103.9572	130.8742	157.3680
105.00%	0.1206	1.5852	4.4735	12.8563	21.6850	36.4631	66.9966	81.6831	95.6459	122.6330	149.2604
110.00%	0.0015	0.1356	0.6112	3.9340	9.5367	20.9963	49.1794	63.4405	77.2069	104.2555	130.9156
120.00%	0.0000	0.0035	0.0146	0.2379	1.1686	4.8663	23.3637	35.0103	47.3189	72.7654	98.5547
130.00%	0.0000	0.0003	0.0009	0.0202	0.1359	0.8370	9.6356	17.1442	26.7090	48.4601	72.0656
150.00%	0.0000	0.0000	0.0000	0.0006	0.0046	0.0335	1.4848	3.2881	7.1693	19.1582	35.6870

Table 4.4: S&P/TSX 60 Black-Scholes European Call Price (Market Price)

	0.0493	0.1260	0.2219	0.4712	0.7205	1.2192	2.2164	2.7151	3.2137	4.2110	5.2274
70.00%	221.2408	221.7404	222.4243	226.1709	229.4408	237.5848	256.0195	264.5559	273.4142	291.6440	311.0829
80.00%	147.5435	148.5420	150.1356	154.7867	159.5847	170.3790	193.7632	204.9206	215.8768	237.3126	259.4111
85.00%	110.7486	111.8422	113.8076	119.7975	126.5359	139.2356	165.1001	177.0919	188.5749	211.8062	235.1643
90.00%	74.4344	75.7289	78.5398	86.5975	94.5837	108.8470	136.7930	149.9258	162.7046	187.3740	211.5655
95.00%	38.4082	41.3561	45.9394	56.4754	65.5890	80.8605	111.7568	125.1519	138.4566	164.2594	189.8231
100.00%	7.9309	13.5367	19.2237	30.6771	40.7584	56.8588	88.0071	102.4361	116.2157	142.7211	168.9000
102.50%	0.6346	4.1119	8.6019	19.3271	28.6477	43.9930	75.3183	89.6973	103.4894	130.5694	156.8791
105.00%	0.1392	1.4963	4.0168	12.5145	21.6507	36.6264	67.0948	81.5520	95.6175	122.3670	148.9888
110.00%	0.0105	0.1458	0.5222	4.1396	9.7725	21.2295	48.9976	63.3772	77.1041	103.7982	130.9174
120.00%	0.0073	0.0475	0.0171	0.4839	1.3756	4.4747	22.9414	35.1123	47.4072	73.0552	98.4304
130.00%	0.0000	0.0007	0.0008	0.0356	0.1052	0.8136	9.7480	17.3917	26.6496	48.4638	71.9118
150.00%	0.0000	0.0000	0.0013	0.0002	0.0045	0.0300	1.5590	2.9321	6.6768	19.2066	35.9185

Table 4.5: S&P/TSX 60 Monte Carlo European Call Price(Based on LVS) Nsim=10000

	0.0493	0.1260	0.2219	0.4712	0.7205	1.2192	2.2164	2.7151	3.2137	4.2110	5.2274
70.00%	0.2124	0.0971	0.4575	0.1687	0.1847	0.0308	0.1316	0.0890	0.3140	0.2001	0.0413
80.00%	0.0541	0.0757	0.1046	0.1601	0.3524	0.4033	0.3225	0.1344	0.0969	0.2030	0.2304
85.00%	0.1465	0.3197	0.1918	0.1240	0.0720	0.1982	0.1725	0.1080	0.2379	0.0776	0.1555
90.00%	0.1687	0.4013	0.2921	0.0812	0.1083	0.1393	0.4468	0.4384	0.2828	0.1119	0.4740
95.00%	0.1835	0.3207	0.0590	0.2903	0.0473	0.4427	0.2919	0.2642	0.2415	0.1073	0.0793
100.00%	0.0132	0.0203	0.0237	0.0308	0.0294	0.0314	0.0953	0.0136	0.0159	0.0145	0.0479
102.50%	0.2509	0.1984	0.3000	0.1610	0.1535	0.4027	0.1103	0.3801	0.4677	0.3048	0.4889
105.00%	0.0186	0.0889	0.4567	0.3418	0.0344	0.1633	0.0981	0.1312	0.0283	0.2660	0.2716
110.00%	0.0090	0.0102	0.0890	0.2056	0.2359	0.2332	0.1818	0.0632	0.1028	0.4573	0.0018
120.00%	0.0073	0.0440	0.0025	0.2460	0.2071	0.3916	0.4224	0.1020	0.0884	0.2898	0.1243
130.00%	0.0000	0.0004	0.0001	0.0153	0.0306	0.0234	0.1124	0.2475	0.0595	0.0037	0.1538
150.00%	0.0000	0.0000	0.0013	0.0004	0.0001	0.0035	0.0742	0.3559	0.4925	0.0485	0.2314

Table 4.6: Price Error between Market Price and Our Model Price

## 5 Conclusion

The aim of this report was to study the local volatility model in a stochastic interest rate framework. We have derived a Dupire-Like formula that gives us a nice theoretical method to calculate local volatility with Vasicek interest rate model. Comparing with Dupire's formula, our Dupire-Like formula becomes more complicated. It depends not only on European call prices, but also a particularly complicated expectation where no closed form expression exists.

We have proposed numerical approaches for the calibration of our local volatility function based on finite difference approximation and Monte Carlo methods. Afterwards, we apply Lipschitz interpolation to construct a stable local volatility surface. Finally, the model was tested on the market data. European call prices calculated from our local volatility surface using Monte Carlo simulation closely match market prices. Our method can be used to price exotic options in a way consistent with the smile.

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