Symmetric Functions as Characters of Hyperoctahedral Group

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Abstract

In *Symmetric Group Characters as Symmetric Functions* [OZ1], the authors Orellana and Zabrocki showed that there is a non-homogenous basis of the ring of symmetric functions $\{\tilde{s}_\lambda\}$ such that when these elements are evaluated at the eigenvalues of a permutation matrix, they are the values of the irreducible characters of the symmetric group S_n. They found a formula for $\tilde{s}_\lambda$ by developing a slightly simpler basis of the symmetric functions called the induced trivial character basis $\{\tilde{h}_\lambda\}$ which represent the trivial characters induced from a subgroup to S_n.

We will be introducing an analogous idea of the S_n-irreducible character basis to the hyperoctahedral group, namely, B_n- irreducible character basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$. To do this, we have defined an intermediate basis B_n-induced trivial character basis $\{\tilde{h}^B_{(\lambda,\mu)}\}$ that is constructed in tensor square of ring of symmetric functions of form $\tilde{h}_\alpha \otimes \tilde{he}_{\beta|\eta}$ in type A. We will provide an algebraic proof that these bases have the analogous properties when evaluated at pairs of eigenvalues of permutation matrices. Moreover, a combinatorial interpretation of this calculation will be provided using fillings of pairs of diagrams in two alphabets.

We will define B_n-irreducible character basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$ by change of basis in $\{\tilde{h}^B_{(\alpha,\beta)}\}$ using Kostka coefficients and show that these bases are families of symmetric functions that are the values of the irreducible characters of the hyperoctahedral group when evaluated at pair of multi-sets of eigenvalues of permutation matrices. We will also define Φ_{B_n} is the Frobenius map on $\Lambda \otimes \Lambda$ such that $\Phi_{B_n}(\tilde{s}^B_{(\lambda,\mu)}) = s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y]$ which is used as a tool in computation of B_n-irreducible character bases.

The Kronecker product of two irreducible representations of B_n has the stability property, that is, there exists the reduced Kronecker coefficients that stabilizes in expansion of this Kronecker product for n sufficiently large. We will show an interesting result that the structure coefficients in this expansion is the same as the coefficients in expansion of the regular product of any two B_n-irreducible character basis i.e. in expansion $\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)}$.

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List of Symbols

B_n	hyperoctahedral group
$\odot_A$	binary operation on $\tilde{h}_\lambda$ and $\tilde{e}_\lambda$
$\odot_B$	binary operation on $\Lambda \otimes \Lambda$
$\mathbb{C}[\![\mathbf{x}]\!]$	formal power series
$c^{\nu}_{\lambda\mu}$	Littlewood-Richardson coefficient
$\alpha \models n$	composition α of size n
$\alpha \models_w n$	weak composition α of size n
$\triangle$	comultiplication on Λ
$\delta_{\lambda\mu}$	kronecker delta function
e_λ	elementary symmetric function indexed by λ
$e_\lambda[X]$	e_λ in variable X
$e_\lambda[\Xi_\mu]$	e_λ at eigenvalue of permutation matrix cycle type μ
F^λ	number of Young Diagram λ tableaux
$\mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}$	fillings of the left splittings of the diagram (γ, ν)
$\mathcal{F}^{\nu}_{\lambda^{(R)} \vert \mu^{(R)}}$	fillings of the right splittings of the diagram (γ, ν)
$(F^{(L)}, F^{(R)})$	pairs of fillings of tableaux
$\mathscr{F}_{S_n}$	Frobenius characteristic map on class function of S_n
$\mathscr{F}_{B_n}$	Frobenius characteristic map on class function of B_n
Φ_n	Frobenius map on Λ
Φ_{B_n}	Frobenius map on $\Lambda \otimes \Lambda$
GL_d	general linear group of dimension d
G_n	wreath product of group G
$\overline{g}^{\gamma}_{\lambda\mu}$	stable Kronecker coefficient $\overline{g}^{\gamma}_{\lambda\mu} := g^{(n-\vert\gamma\vert,\gamma)}_{(n-\vert\lambda\vert,\lambda)(n-\vert\mu\vert,\mu)}$
$\overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)}$	stable Kronecker coefficient of irreducible representations of B_n
h_λ	complete homogenous symmetric function indexed by λ
$h_\lambda[X]$	h_λ in variable X
$\tilde{h}_\lambda$	S_n-induced trivial character basis indexed by λ

$h_\lambda[\Xi_\mu]$	h_λ at eigenvalue of permutation matrix cycle type μ
$\tilde{h}_\lambda[\Xi_\mu]$	$\tilde{h}_\lambda$ at eigenvalue of permutation matrix cycle type μ
$\mathcal{H}_{\lambda,\mu}$	inner product $\tilde{h}_\lambda[\Xi_\mu] = \mathcal{H}_{\nu,\mu} := \langle h_{\lvert\mu\rvert-\lvert\nu\rvert} h_\nu, p_\mu \rangle$
$\mathcal{HE}_{(\lambda\lvert\tau),\mu}$	inner product $\tilde{he}_{(\lambda\lvert\tau)}[\Xi_\mu] = \mathcal{HE}_{(\lambda\lvert\tau),\mu} := \langle h_{\lvert\mu\rvert-\lvert\lambda\rvert-\lvert\tau\rvert} h_\lambda e_\tau, p_\mu \rangle$
$\tilde{h}^B_{(\lambda,\mu)}$	B_n-induced trivial character basis indexed by (λ,μ)
$h_\lambda[X \pm Y]$	plethystic substitution on h_λ in two alphabets
$\tilde{h}_\lambda[X + Y]$	coproduct on Λ in two alphabets
$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$	$\tilde{h}^B_{(\lambda,\mu)}$ at eigenvalues of permutation matrices indexed by (γ,ν)
$\mathbf{1}_\mu$	indicator function basis of class function of S_n
$\mathbf{1}^{B_n}_{(\gamma,\nu)}$	indicator function basis of class function of B_n
$\langle\cdot,\cdot\rangle_{\Lambda^n}$	Hall inner product on Λ^n
$\langle\cdot,\cdot\rangle_{S_n}$	S_n-character inner product (example: $\langle\chi,\psi\rangle_{S_n}$)
$\langle\cdot,\cdot\rangle_{@}$	Hall inner product on Λ (example: $\langle\tilde{s}_\lambda,\tilde{s}_\mu\rangle_{@} = \delta_{\lambda\mu}$)
$\langle\cdot,\cdot\rangle_{B_n}$	B_n-character inner product (example: $\langle\chi,\psi\rangle_{B_n}$)
$\langle\cdot,\cdot\rangle_{@_B}$	Hall inner product on $\Lambda\otimes\Lambda$ (example: $\left\langle \tilde{s}^B_{(\lambda,\mu)}, \tilde{s}^B_{(\alpha,\beta)} \right\rangle_{@_B} = \delta_{\lambda\alpha}\delta_{\mu\beta}$)
$\mathcal{I}^\lambda(\nu)$	induced trivial character indexed by λ at class ν i.e. $\mathcal{I}^\lambda(\nu) = \langle h_\lambda, p_\nu\rangle$
$\mathcal{I}^{(\lambda,\mu)}$	induced trivial characters of group B_n
$K_{\lambda\mu}$	Kostka coefficient i.e. columns-strict tableau of shape λ content μ
K_γ	conjugacy class of group S_n indexed by partitions γ
$K^{B_n}_{(\gamma,\nu)}$	conjugacy class of group B_n indexed by (γ,ν)
$*$	Kronecker product on Λ (type A)
$*_B$	Kronecker product on $\Lambda\otimes\Lambda$ (type B)
$\chi(g)$	character of the matrix representation $X(g)$
χ^λ_μ	character value indexed by λ at conjugacy class K_μ
$\chi^{(\lambda,\mu)}(\gamma,\nu)$	irreducible characters of group B_n indexed by (λ,μ) at (γ,ν)
$l(\lambda)$	length of partition λ
L_n	abelian group of linear characters of B_n
$\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_l)$	integer partition or integer composition given in sequence
$\lambda = (1^{m_1}2^{m_2}\ldots n^{m_n})$	partition given in exponential notation
$\lambda[n]$	padded partition $\lambda[n] := \big((n-\lvert\lambda\rvert),\lambda_1,\lambda_2,\ldots,\lambda_{l(\lambda)}\big)$
$\overline{\lambda}$	first row of the partition λ eliminated i.e. $\overline{\lambda} = (\lambda_2,\lambda_3,\ldots,\lambda_{l(\lambda)})$
λ/μ	skew shape partition

$\lambda \vdash n$	λ partition of n
$\pi \Vdash S$	π is a multi-set partition of the multi-set S
$\lvert\lambda\rvert$	size of partition λ
$\lvert(\lambda,\mu)\rvert$	size of pairs of partitions (λ,μ)
$\mu \subset \lambda$	partition μ contained in λ
$\lambda \uplus \mu$	disjoint union of partitions λ and μ
Λ	algebra of symmetric functions (or Sym)
Λ^n	symmetric functions homogeneous of degree n
$\Lambda[X]$	ring of symmetric function in formal variables X
m_λ	monomial symmetric function indexed by λ
$\tilde{m}(\pi)$	multiplicities of the multi-sets that occur in π
$\{\!\{1^{a_1}, 2^{a_2}, \ldots, l^{a_l}\}\!\}$	multi-set that element i is repeated a_i times
μ	multiplication function on $\Lambda \otimes \Lambda$
$[n]$	the finite set $[n] \coloneqq \{1, 2, \ldots, n\}$
$[\pm n]$	the finite set $[\pm n] \coloneqq \{\pm 1, \pm 2, \ldots, \pm n\}$
$\preceq$	graded reverse lexicographic order on (λ,μ)
$\mu \trianglerighteq \beta$	dominance ordering
p_λ	power sum symmetric function indexed by λ
$p_\lambda[X]$	p_λ in variable X
$R[X]$	polynomial ring
$R(G)$	set of all class functions on G
$R(B_n)$	set of all class functions on B_n
Sym	ring of symmetric function (or Λ)
S_n	symmetric group
S	antipode map on Λ
$s_\lambda[X \pm Y]$	plethystic substitution on s_λ in two alphabets
$S^B_{(\lambda,\mu)}[X,Y]$	Frobenius characteristic image of B_n in two alphabets
s_λ	Schur function indexed by λ
Ξ_μ	multi-set of eigenvalues of a permutation matrix cycle structure μ
$\tilde{s}_\lambda$	S_n-irreduicible character basis indexed by λ

$s_\lambda[\Xi_\mu]$	s_λ at eigenvalue of permutation matrix cycle type μ
$\tilde{s}_\lambda[\Xi_\mu]$	$\tilde{s}_\lambda$ at eigenvalue of permutation matrix cycle type μ
$\tilde{s}^B_{(\lambda,\mu)}$	B_n-irreducible character basis indexed by (λ,μ)
$\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu]$	$\tilde{s}^B_{(\lambda,\mu)}$ at eigenvalues of permutation matrices indexed by (γ,ν)
$S^{(L)} \uplus S^{(R)}$	splitting of a multi-set S in two alphabets
$sign\left(S^{(L)} \uplus S^{(R)}\right)$	sign of splitting of a multi-set S in two alphabets
$H \rtimes_\varphi K$	semi-direct product of H and K with respect to φ
T^λ	column-strict tableau shape λ
$\otimes$	tensor product
$\{u_\lambda\} \longrightarrow \{v_\mu\}$	transition coefficients of v_μ in the expansion of u_λ
ω	involution on Λ
wr	wreath product
$wt\left(F^{(L)} \uplus F^{(R)}\right)$	total weight assigned to filling $F^{(L)} \uplus F^{(R)}$
$wt^{S^{(L)} \uplus S^{(R)}}\left(F^{(L)} \uplus F^{(R)}\right)$	weight of splitting $S^{(L)} \uplus S^{(R)}$ in filling $F^{(L)} \uplus F^{(R)}$
$\mathbf{x}^\mu$	monomial weight for composition μ
$\mathbf{x}$	infinite set of variables $\mathbf{x} \coloneqq \{x_1,\ x_2,\ldots\}$
X	set of formal variables $X \coloneqq x_1 + x_2 + \cdots$
$X(g)$	matrix representation of group element g
$X_H \uparrow_H^G$	induced representation of a representation of $H \leq G$
Z_g	centralizer of group element g
z_λ	size of Z_g for $g \in S_n$ with cycle type λ

1 Preliminaries

1.1 Background and Motivation

Introduction

Symmetries have always been a fascinating topic of study in mathematics and other sciences. Group theory is an area in mathematics that studies and explores the symmetries and presents them in an organized abstract form. A group that is based on the geometrical aspect of symmetry is called the symmetry group. The symmetry group of a geometrical object consists of all transformations that keep the shape of the object unchanged under these transformations. There are many questions in physics, chemistry and other sciences which are modeled by symmetry groups. On the the combinatorial side of symmetries, the symmetric group (hereafter denoted S_n) is the group of all permutations on n elements and it is classified as a Coxeter group of type A. The symmetric group is one of the most fundamental groups in mathematics that has a very important role in algebraic combinatorics. In Cayley's theorem, it is stated that any group G is isomorphic to a subgroup of the symmetric group on G.

Fundamental questions in algebraic combinatorics describe the behavior of groups such as S_n in simple terms and in pictorial forms. The tools that are commonly used in this field are representation theory and symmetric functions. They can transform the problem in a way that allows us to use character theory and linear algebra to study and analyze these groups.

Representation theory of the symmetric group was presented with an eye on algebraic combinatorics throughout the 20th century. Some important publications to mention are in the 70's (1st edition) and in the 90's (2nd edition) of [Macdonald] by Ian G. Macdonald and in the 80's [Sagan] by Bruce E. Sagan.

The hyperoctahedral group (hereafter denoted B_n) is a "group of signed permutations" and is classified as a Coxeter group of type B. The symmetric group is a subgroup of the hyperoctahedral group i.e. $B_n \supseteq S_n$. The aim of this thesis is to prove the existence and provide an explicit construction of a new basis in the character theory of hyperoctahedral group. We will first present some history and the algebraic and combinatorial background that is used throughout this project.

Remark 1.1. In **Section 1.2** we will explain the notations and the ring of symmetric functions. Moreover, the plethystic notation and algebraic operations in symmetric functions will be studied as well.

Representation Theory of Finite Groups

A finite group is a finite set of elements equipped with binary operation that together satisfies axioms of closure, associativity and the identity property. The finite simple groups are the building block of finite groups. The classification of finite simple groups is one of the most published area in mathematics between 1960 and 1980.

Each element of a group can be assigned to an invertible matrix and the group operation can be translated to matrix multiplication. In mathematics this is called a matrix representation of a group G which is a group homomorphism $X : G \to GL_d$. The concept of representation theory of finite groups was developed by Frobenius, Burnside and Schur at around the end of 19th century. Representation theory has been used in many applications in mathematics and other sciences and this project is one of many applications of group representation as well.

A representation is made of the building blocks called irreducible representations that are the nonzero representation containing no proper sub-representation. Maschke's theorem explains how every representation is the direct sum of the irreducible representation.

A mathematical tool extensively used in representation theory of the finite group is the character of a representation. The character of a group representation is a function that assigns each group element the trace of the corresponding matrix. The reason that we study the character of a group representation is because the traces of corresponding matrices of each group element carry much of the important information about the representation in a compact form.

In 1896 Frobenius published "Über die Gruppencharactere" in which he wrote the concept of characters of finite groups. In his new studies, Frobenius didn't explicitly find the matrices corresponding to the group representations. This is due to fact that the character of a group representation is closely related to the conjugacy classes of the group and is uniquely determined (up to isomorphism) by the corresponding group character. The representation of groups which was also developed by Frobenius came to picture a year later in 1897. [Conrad, OE]

Remark 1.2. In **Section 1.4** we will explain basic topics in representation theory and character theory of finite groups.

Hyperoctahedral Group

Algebraic combinatorics is an area in mathematics that uses combinatorial method to present and solve algebraic problems or uses the algebraic methods, mainly group theory or group representations as a tool to solve combinatorial problems. Algebraic combinatorics can be viewed from two different angles. They are

1. The counting or enumerating side such as polytopes and partially ordered sets, and
2. The algebraic side involving group, representation theory and commutative algebra.

Many problems in algebraic combinatorics use representation theory and tools such as symmetric functions and Young tableaux. The ring of symmetric functions is the main family of functions which are invariant under permutation and they play a very important role in this field of study.

The hyperoctahedral group can be expressed from many different mathematical point of views.

1. In terms of a permutation group, the hyperoctahedral group is a group of all signed permutations.

2. The hyperoctahedral group is isomorphic to the wreath product of a cyclic group of order two, namely C_2, with symmetric group of order n. This is the semi-direct product of n copies of C_2 and the group S_n, i.e. $B_n := C_2^n \rtimes S_n$. The group B_n has order of $2^n n!$. [Macdonald]

3. In terms of the theory of Coxeter groups, is classified as a Coxeter group of type B.

In order to organize the representation theory for the hyperoctahedral group similar to the representations of symmetric group with the ring of symmetric functions we have many great references such as [Beck, BrBg, MRW, Stembridge, Stembridge2, Orellana].

Remark 1.3. In **Section 1.3** we will introduce the hyperoctahedral group and we study it's conjugacy classes.

Characters of Hyperoctahedral Group

In 1987, John Stembridge wrote an unpublished manuscript "The ordinary representations of B_n" [Stembridge] which includes fundamentals in representation and characters of group B_n. The following are the basic notes in characters of B_n that we will be using in this thesis.

The conjugacy classes of the hyperoctahedral group B_n are indexed by a pair of partition $(\gamma, \nu) \vdash n$ where its length is defined as $|(\gamma, \nu)| := |\gamma| + |\nu| = n$. Given $K^{B_n}_{(\gamma,\nu)}$ to be the conjugacy class of group B_n indexed at the pair of partitions (γ, ν) this class consists of all signed permutations such that

- ⋄ the parts of γ are cycles with **even** number of negative signs, and
- ⋄ the parts of ν are cycles with **odd** number of negative signs.

In the study of the character theory of group B_n it is known that the irreducible representations of B_n are also indexed by a pair of partitions (λ, μ) of total size n. An explicit formula for the character of the hyperoctahedral group in terms of Schur functions is give below.

Theorem 1.4. *[Stembridge] The irreducible characters of the group B_n are*

$$\chi^{(\lambda,\mu)} := \left(\chi^{(\lambda,\bullet)} \otimes \chi^{(\bullet,\mu)}\right) \uparrow^{Bn}_{B_{|\lambda|}\times B_{|\mu|}}$$

where $|\lambda| + |\mu| = n$ and their Frobenius characteristics image is

$$\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) = s_\lambda[X+Y] s_\mu[X-Y].$$

In the introduction of [Beck], "The combinatorics of symmetric functions and permutation enumeration of the hyperoctahedral group" in 1997, the author defined ten standard bases for the B_n-analogue of the symmetric functions. Any element of these bases can be the Frobenius characteristic image of a natural representation of B_n. One which has a very important use in this thesis and is called the induced trivial character basis of group B_n is described in the following Corollary.

Corollary 1.5. *[Beck] The induced trivial character of the group B_n are the induced characters of the trivial and parity representation i.e. $\mathcal{I}^{(\lambda,\mu)} := (1_\lambda \otimes \delta_\mu) \uparrow_{B_{|\lambda|}\times B_{|\mu|}}^{B_n}$ where $|\lambda| + |\mu| = n$ and their Frobenius characteristics image is*

$$\mathscr{F}_{B_n}(\mathcal{I}^{(\lambda,\mu)}) = h_\lambda[X+Y]h_\mu[X-Y]$$

Remark 1.6. In **Section 1.5** the characters of hyperoctahedral group and it's Frobenius characteristic map will be studied.

The S_n-Irreducible Character Basis

It is known that irreducible characters of GL_n modules are evaluations of Schur functions $\{s_\lambda\}$ at the eigenvalues of elements in GL_n. We say that the character bases values of the general linear group are symmetric functions (Schur functions). In [KT] it is shown that characters of orthogonal group O_n are "orthogonal characters" which are also a type of symmetric functions. In [OZ1] it is shown that characters of symmetric group S_n realized as permutation matrices have a collection of symmetric functions $\{\tilde{s}_\lambda\}$ called the "S_n-irreducible character basis" and they play the same role for the symmetric group as the Schur functions $\{s_\lambda\}$ do for the irreducible characters of GL_n. Given $GL_n \supseteq O_n \supseteq B_n \supseteq S_n$ our project is to find a similar character basis for the hyperoctahedral group B_n.

In [OZ1] the authors R. Orellana and M. Zabrocki showed that there is a non-homogenous basis of the ring of symmetric functions $\{\tilde{s}_\lambda\}$ such that when these elements are evaluated at the eigenvalues of a permutation matrix they are the values of the irreducible characters of the symmetric group. They found a formula for the irreducible characters by developing a slightly simpler basis of the symmetric functions called the "induced trivial character basis" $\{\tilde{h}_\lambda\}$ which represent the trivial characters induced from a subgroup to S_n.

Remark 1.7. In **Section 1.6** we will study the symmetric group character basis in [OZ1] and explain the topics which will be related and used as a tool in our thesis work.

The Main Results

The hyperoctahedral group B_n consist of elements of signed permutations that can also be realized as the group of signed permutation matrices inside of GL_n. We will be developing an analogous idea to the irreducible character basis of the symmetric group. To do this, given that the irreducible representation in the hyperoctahedral group, are indexed by pairs of partitions $(\lambda, \mu) \vdash n$ define the B_n-irreducible character basis $\{\tilde{s}^B_{(\lambda,\mu)}\} \in \Lambda \otimes \Lambda$ such that it satisfies the property that when evaluated at pairs of eigenvalues of permutation matrices it gives the irreducible character value.

B_n**-induced trivial character basis**. In order to define B_n-irreducible Character Basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$, we have adopted an analogous method to the type A solution and we have introduced an intermediate basis the B_n-induced trivial character basis $\{\tilde{h}^B_{(\lambda,\mu)}\} \in \Lambda \otimes \Lambda$ that is constructed in terms of $\tilde{h}_\alpha \otimes \tilde{h}e_{\beta|\eta}$ in type A. We will show that these bases in type B have the necessary property when evaluated at pairs of eigenvalues of permutation matrices. This is provided in the following theorem.

Theorem 1.8. *[The Main Theorem] For $n \geq 0$ and given pair of partitions $(\gamma, \nu) \vdash n$ we have*

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \left\langle h_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle.$$

Remark 1.9. In **Chapter** 2 we will introduce the B_n-induced trivial character basis $\{\tilde{h}^B_{(\lambda,\mu)}\}$ and show how it can be constructed in $\tilde{h}_\alpha \otimes \tilde{h}e_{\beta|\eta}$ in type A. We will also give a combinatorics interpretation of and tableaux for evaluating $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$. This involves fillings of pairs of tableaux shape (γ, ν) of content representing partitions (λ, μ) in two alphabets.

B_n**-irreduicble character basis**. The B_n-irreducible character basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$ are families of symmetric functions that are the values of the irreducible characters of the hyperoctahedral group when evaluated at roots of unity i.e. multi-sets of eigenvalues of permutation matrices. We will define the symmetric function $\tilde{s}^B_{(\lambda,\mu)}$ by change of basis with $\tilde{h}^B_{(\alpha,\beta)}$ using product of Kostka coefficients.

Definition 1.10. Given partitions α, β, λ and μ and non-negative integer $n \geq 2|\alpha|$ define the functions $\tilde{s}^B_{(\lambda,\mu)}$ in $\Lambda \otimes \Lambda$ that satisfies

$$\tilde{h}^B_{(\alpha,\beta)} = \sum_{\substack{(\lambda,\mu)\\ |\lambda| \leq |\alpha| \\ |\mu| = |\beta|}} K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta} \tilde{s}^B_{(\lambda,\mu)}$$

where the Kostka coefficients $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ is the coefficient of $s_{(n-|\lambda|,\lambda)}$ in $h_{(n-|\alpha|,\alpha)}$ and $K_{\mu\beta}$ is the coefficient of s_μ in h_β.

The functions B_n-irreducible character basis $\{\tilde{s}^B_{(\lambda,\mu)}\} \in \Lambda \otimes \Lambda$ must satisfy the property that it gives the irreducible character value when evaluated at pairs of eigenvalues of permutation matrices. This is given in the following proposition.

Proposition 1.11. *The function $\tilde{s}^B_{(\lambda,\mu)}$ is the 'unique' symmetric function in $\Lambda \otimes \Lambda$ of degree $|(\lambda,\mu)| := |\lambda| + |\mu|$ that for all $n \geq |\lambda| + |\mu| + \lambda_1$ satisfies*

$$\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \left\langle s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y] s_\mu[X-Y], p_\gamma[X] p_\nu[Y] \right\rangle .$$

for all pairs of partitions $(\gamma,\nu) \vdash n$ where Ξ_γ and Ξ_ν are the multi-set of eigenvalues of permutation matrices of cycle types γ and ν respectively. Equivalently we have

$$\tilde{s}^B_{(\lambda,\mu)} \left[\Xi_\gamma, \Xi_\nu\right] = \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\gamma,\nu).$$

Remark 1.12. In **Chapter 3** we will be using the algebraic and combinatoric tools in order to reach to the following results in this thesis.

1. A definition of the B_n-induced trivial character bases in terms of S_n-induced trivial character bases.
2. Proof of the existence of the B_n-irreducible character basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$ in $\Lambda \otimes \Lambda$.
3. Proof of a theorem explaining the expansion of $\{\tilde{h}^B_{(\alpha,\beta)}\}$ in $\{\tilde{s}^B_{(\lambda,\mu)}\}$ using Kostka coefficients.
4. A recursive definition for B_n-irreducible character basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$ in terms of $\{\tilde{h}^B_{(\alpha,\beta)}\}$.
5. The summary of results of combinatorial explanation of each of the transition coefficients in

$$\{\tilde{h}_\eta\}, \{\tilde{he}_{(\delta|\tau)}\} \longrightarrow \{\tilde{h}^B_{(\alpha,\beta)}\} \longrightarrow \{\tilde{s}^B_{(\lambda,\mu)}\}.$$

The last main result we have explained the following.

Kronecker coefficients in product of B_n-irreducible character bases

In [Wilson] it is shown that the stability property of the Kronecker coefficients in Kronecker product of two irreducible representations of the hyperoctahedral group can be states as, given n sufficiently large, there exists the reduced Kronecker coefficients $\overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)}$ that stabilizes in expansion as

$$S^B_{((n-|\lambda|-|\mu|,\lambda),\mu)} *_B S^B_{((n-|\alpha|-|\beta|,\alpha),\beta)} = \sum_{|(\theta,\tau)|} \overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)} S^B_{((n-|\tau|-|\theta|,\theta),\tau)}$$

where the sum is over $|(\theta,\tau)| \leq |(\lambda,\mu)| + |(\alpha,\beta)|$. We will show that the structure coefficients in this expansion is the same as the stable Kronecker coefficients in expansion of the regular product of any two B_n-irreducible character basis, that is, for pairs of partitions (λ,μ) and (α,β),

$$\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)} = \sum_{|(\theta,\tau)| \leq |(\lambda,\mu)| + |(\alpha,\beta)|} \overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)} \tilde{s}^B_{(\theta,\tau)}.$$

1.2 Ring of Symmetric Functions Λ

We begin by presenting the fundamentals and algebraic objects that will be used throughout this thesis.

Notation: Permutations, Partitions and Young Diagrams

Definition 1.13. Given a non-empty set A, a bijective function $\pi : A \longrightarrow A$ is called a permutation of A. Given the fact that a function has an inverse if and only if it is bijective, the permutation function π has a unique inverse function $\pi^{-1} : A \longrightarrow A$ that is also a permutation. The set of all permutations of set $[n] := \{1, 2, \ldots, n\}$ equipped with composition as it's binary operation is called the symmetric group on $[n]$ denoted by S_n with the order (or cardinality) of $|S_n| = n!$.

Permutation Notations. There are three common types of notations used in expressing a permutation π in S_n. In this section we are adopting some notation in Chapter 1 in [Sagan].

1. The bijection $\pi : i \longrightarrow \pi(i)$ shown by the two-line notation as

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

2. If the first line on the two-line notation is dropped, we have a one-line notation as

$$\pi = (\pi(1),\ \pi(2), \ldots, \pi(n))$$

3. The notation for a cycle is

$$\big(i\pi(i)\pi^2(i)\ldots\pi^{p-1}(i)\big)$$

   To explain this notation, let $(i\, j\, k \ldots l)$ be a cycle in π. This permutation sends i to j, j to k, $\ldots$, and l back to i. The first p such that $\pi^p(i) = i$ is how we find p in this cycle. Now if we pick any element not in the cycle containing i and repeat this process until all elements in the set $\{1, 2, \ldots, n\}$ are used, a complete cycle notation will be generated by expressing it as the composition of disjoint cycles.

Example 1.14. Given a permutation $\pi \in S_7$ such that

$$\pi(1) = 3,\ \pi(2) = 1,\ \pi(3) = 2,\ \pi(4) = 4,\ \pi(5) = 6,\ \pi(6) = 5 \text{ and } \pi(7) = 7$$

- ⋄ The two-line notation of π is $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 4 & 6 & 5 & 7 \end{pmatrix}$
- ⋄ The one-line notation of π is $\pi = (3, 1, 2, 4, 6, 5, 7)$
- ⋄ The cycle notation of π is $\pi = (132)(4)(56)(7)$

Definition 1.15. A k-cycle (or a cycle of length k) is a cycle that contains k elements. The cycle-type (or type) of a permutation π is of form $(1^{m_1}2^{m_2}\ldots n^{m_n})$, where m_k is the number of k-cycles in permutation π. A 1-cycle in a permutation is called a fixed point. The fixed points in a permutation are usually dropped from the cycle notation. An involution is a permutation which $\pi^2 = e$. A permutation is an involution if and only if all its cycles are of length 1 or 2. For example, the permutation $\sigma = (1)(2)(34)$ is an involution.

Definition 1.16. For non-negative integers $n \in \mathbb{Z}^+$ and $l \in \mathbb{Z}^+$, a partition of size n and length l is defined to be a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that λ_i is weakly decreasing, i.e. $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i \leq l$ and $\sum_{i=1}^{l} \lambda_i = n$. The size of a partition is denoted by $|\lambda| = n$ and the length of the partition is denoted by $l(\lambda) = l$. For λ a partition of n is expressed by $\lambda \vdash n$. The exponential notation for a partition is $\lambda = (1^{m_1}2^{m_2}\ldots k^{m_k})$, where m_j is the number of partitions of length j in λ. The number of partitions with cycle-type (or cycle structure) of $\lambda \vdash n$ is $\frac{n!}{z_\lambda}$ where

$$z_\lambda \coloneqq |Z_g| = 1^{m_1}m_1!2^{m_2}m_2! \cdots n^{m_n}m_n!$$

Note that $|Z_g|$ is the order of centralizer of element g in S_n that has cycle-type of λ.

Remark 1.17. There are two kinds of partitions: integer partitions and set partitions.

1. The integer partition for a non-negative integer n is written by $\lambda \vdash n$ is to write n as a sum of smaller positive integers. For example integer partition of 3 is three distinct partitions (3), (2, 1) and (1, 1, 1).
2. The set partition of a set is to divide a given set into smaller (non-empty) subsets such that every element of the set is exactly in one of these subsets. For example, the set partitions of $\{1, 2, 3\}$ are five set partitions $\{1, 2, 3\}$, $\{1, 2\} \uplus \{3\}$, $\{1, 3\} \uplus \{2\}$,$\{2, 3\} \uplus \{1\}$ and $\{1\} \uplus \{2\} \uplus \{3\}$.

Remark 1.18. For $n \geq 0$ the cycle-type of a permutation in S_n is a partition of n.

Example 1.19. Given the permutation $\pi = (132)(4)(56)(7) \in S_7$ in cycle notation,

- ◇ There are one 3-cycle, one 2-cycle and two 1-cycles in π.
- ◇ The cycle type of π is $\lambda = (3, 2, 1, 1)$ where $\lambda \vdash 7$.
- ◇ The cycle type of π in exponential notation is $(1^2 2^1 3^1)$.
- ◇ The fixed points of π are 4 and 7.
- ◇ The cycle length is $l(\lambda) = 4$ with size of $|\lambda| = 7$.

Definition 1.20. Given the non-negative integers n and l we will define the composition of size n is an ordered sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_l)$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_l = n$. A weak composition is when zero parts allowed, i.e. $\alpha_i \geq 0$.

Notations

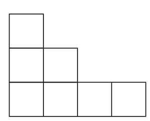
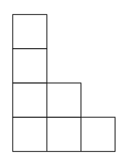
- Denote a composition α of size n by $\alpha \models n$.
- Denote a weak composition α of size n by $\alpha \models_w n$.

Example 1.21. Two examples of composition and weak composition.

- A composition for $n = 7$ is $\alpha = (1, 2, 1, 3)$.
- A weak composition for $n = 7$ is $\alpha = (1, 0, 2, 0, 0, 1, 3)$.

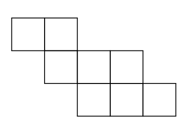
Definition 1.22. Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ to be a partition of size $n \in \mathbb{Z}^+$ and length $l \in \mathbb{Z}^+$ we can associate λ with a diagram called Young Diagram (or Ferrers Diagram) denoted F^λ. This diagram has $n = |\lambda|$ number of boxes that are in l rows that are arranged by arrays $\lambda_1, \lambda_2, \ldots, \lambda_l$ organized from bottom to top that are justified to the left. Note that this is a French notation which we will use throughout this thesis. Moreover, the conjugate partition of λ is denoted λ' which can be shown by a Young diagram that is by swapping the rows and columns and is denoted by $F^{\lambda'}$.

Example 1.23. The associated Young diagrams to the partitions $\lambda = (4, 2, 1)$ and conjugate partition $\lambda' = (3, 2, 1, 1)$ are

$F^{(4,2,1)}$:  , $F^{(3,2,1,1)}$: 

Definition 1.24. Given two partitions λ and μ with $|\mu| \leq |\lambda|$, we say that μ is contained in λ and write $\mu \subset \lambda$ whenever $\mu_i \leq \lambda_i$ for all i. If $\mu \subset \lambda$ the set-theoretic difference $\lambda - \mu$ can be written as λ/μ and pictorially associated by a diagram called a skew diagram denoted by $F^{\lambda/\mu}$.

Example 1.25. For $\lambda = (5, 4, 2)$ and $\mu = (2, 1)$ the skew diagram λ/μ is

$F^{\lambda/\mu}$: 

Definition 1.26. Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ to be a partition of size $n \in \mathbb{Z}^+$ and length $l \in \mathbb{Z}^+$ we can fill in the boxes of the Young diagram with numbers from the set $\{1, 2, \ldots, n\}$ and generate a tableau called a semi-standard or column-strict tableau T of shape λ and denoted by T^λ. This tableau is strictly increasing in columns from bottom to top and weakly increasing in the rows from left to right. The filling of the tableau can be given by the its partition type, i.e. if T^λ has the type $\alpha = (1^{\alpha_1} 2^{\alpha_2} \ldots n^{\alpha_n})$ it means that there are α_i entires are i's.

Example 1.27. Given partition $\lambda = (4, 2, 1) \vdash 7$ and the filling from elements of set $\{1, 2, 3, 4, 5, 6, 7\}$ the following column-strict tableau $T^{(4,2,1)}$ has the type $\alpha = (1^1 2^1 3^2 4^0 5^0 6^3 7^0)$.

$T^{(4,2,1)}$:

6			
2	6		
1	3	3	6

The Ring of Symmetric Functions

Definition 1.28. Let the formal power series ring defined by

$$\mathbb{C}[\![\mathbf{x}]\!] = \{\sum_{n \geq 0} a_n \mathbf{x}^n \,:\, a_n \in \mathbb{C}\}$$

where $\mathbf{x} := \{x_1, x_2, \ldots\}$ is an infinite set of variables. The formal power series $\mathbb{C}[\![\mathbf{x}]\!]$ is a ring equipped with usual ring addition and multiplication. Note that the monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_l}^{\lambda_l}$ is of degree n when $n = \sum_i \lambda_i$.

Definition 1.29. Given permutation $\sigma \in S_n$ and for a positive integer n, the function f is said to be a symmetric function if a natural action of σ on $f(\mathbf{x}) \in \mathbb{C}[\![\mathbf{x}]\!]$ will have the result

$$\sigma f(x_1,\, x_2, \ldots) = f(x_{\sigma(1)},\, x_{\sigma(2)}, \ldots)$$

Equivalently, the function f is invariant under permutation σ.

Definition 1.30. For a partition $\lambda = (\lambda_1,\, \lambda_2,\, \ldots, \lambda_l)$, the monomial symmetric function indexed by λ is defined by

$$m_\lambda = m_\lambda(\mathbf{x}) = \sum_{\lambda_i} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_l}^{\lambda_l}.$$

Given partition $\lambda \vdash n$ the monomial symmetric function $m_\lambda(\mathbf{x})$ is said to be of homogenous degree n.

Definition 1.31. The Ring of Symmetric Functions. The ring of symmetric functions is a vector space with coefficients in the ring of complex numbers that is spanned by monomial m_λ and is denoted by

$$\Lambda = \Lambda(\mathbf{x}) = \bigoplus_{n \geq 0} \mathbb{C}\text{-Span}\{m_\lambda \mid \lambda \vdash n\}$$

The Ring of Symmetric Functions is a Graded Algebra. An algebra is a vector space that the multiplication of vectors is associative and a ring structure is imposed on the space. The ring of symmetric functions Λ is a vector space and has a structure of graded algebra as we have the decomposition $\Lambda = \bigoplus_{n\geq 0} \Lambda^n$, where Λ^n is spanned by monomials m_λ of degree n, this is $\Lambda^n = \mathbb{C}\text{-Span}\{m_\lambda \mid \lambda \vdash n\}$. The ring Λ is closed under product since $f \in \Lambda^n$ and $g \in \Lambda^m$ implies $f \cdot g \in \Lambda^{n+m}$.

Fundamental Bases and Families of Symmetric Functions. There are five types of symmetric functions which are designated as fundamental bases of symmetric functions. First we will formally present the basis $\{m_\lambda\}$ and below.

Proposition 1.32. *(Proposition 4.3.3 [Sagan]) The space Λ^n has basis $\{m_\lambda : \lambda \vdash n\}$ and so has dimension equal to the number of partitions of n.*

The power sum symmetric function has an important role in building our bases. For each positive integer n, there is a single generator for each of the following three symmetric functions. This will lead us to the restatement of Theorem 4.3.7 in [Sagan].

Theorem 1.33. ***Fundamental Bases of Λ.*** *The following are bases of the ring of symmetric functions Λ.*

- ⋄ $\bigoplus_{n\geq 0} \{p_\lambda\}_{\lambda\vdash n}$ (n-th power sum symmetric function: $p_n = m_{(n)} = \sum_{i\geq 1} x_i^n$)
- ⋄ $\bigoplus_{n\geq 0} \{e_\lambda\}_{\lambda\vdash n}$ (n-th elementary symmetric function: $e_n = m_{(1^n)} = \sum_{i_1<\cdots<i_n} x_{i_1}\cdots x_{i_n}$)
- ⋄ $\bigoplus_{n\geq 0} \{h_\lambda\}_{\lambda\vdash n}$ (n-th complete homogeneous symmetric function: $h_n = \sum_{\lambda\vdash n} m_\lambda = \sum_{i_1\leq\cdots\leq i_n} x_{i_1}\cdots x_{i_n}$)

Remark 1.34. The trivial cases are given as below. Note that for the partitions indexed by zero (or empty) h_0 we use the notation $h_\bullet$ instead.

- ⋄ $p_\bullet = h_\bullet = e_\bullet = 1$
- ⋄ $p_{-n} = h_{-n} = e_{-n} = 0$ for $n > 0$.

In an abstract form, the ring of symmetric function can be expressed by

$$\Lambda = Sym \coloneqq \mathbb{Q}[p_1,\, p_2, \ldots]$$

that is, the ring of symmetric function Λ is given in the polynomial ring in variable p_i, the power sum generators, where

$$p_i(\mathbf{x}) = x_1^i + x_2^i + \cdots$$

and $deg(p_i) = i$, therefore the set $\bigoplus_{k\geq 0}\bigoplus_{\lambda\vdash k} \{p_k\}$ can be called the power sum basis. Now, given Λ^m to denote the symmetric functions of degree m for a partition $\lambda \vdash m$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$, the set $\{p_\lambda\}$ forms a basis for Λ^m and

$$p_\lambda \coloneqq p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{l(\lambda)}}.$$

Note that

$$\begin{aligned} deg(p_\lambda) &:= deg(p_{\lambda_1}, p_{\lambda_2}, \dots, p_{\lambda_{l(\lambda)}}) \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)} \end{aligned}$$

with the indices are written in weakly decreasing order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)}$.

Remark 1.35. Theorem 1.33 implies that we will have one function for each partition of n, hence this can be extended to any given partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$. To do this, we can write the functions p_λ, e_λ and h_λ multiplicatively as $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}$ and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k}$. Therefore, we say that this family of bases of symmetric functions are multiplicative.

Schur Functions

Definition 1.36. The weak composition $\mu = (\mu_1, \mu_2, ..., \mu_k)$ is said to be the content of T^λ (a generalized tableau of shape λ) when μ_i is the number of times that i appears in T^λ. The monomial weight in $\mathbb{C}[\![\mathbf{x}]\!]$ for the corresponding composition $\mu = (\mu_1, \mu_2, ..., \mu_k)$ is defined by

$$\mathbf{x}^\mu := x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_k}$$

Definition 1.37. Let T^λ be a generalized tableau of shape λ with content $\mu = (\mu_1, \mu_2, ..., \mu_k)$. Define the monomial weight on T^λ by

$$\mathbf{x}^{T^\lambda} := \prod_{(i,j) \in \lambda} x_{T_{(i,j)}} = \mathbf{x}^\mu$$

Definition 1.38. Given partition λ the corresponding Schur function is defined by

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^{T^\lambda}$$

where the sum is over all column-strict tableaux of shape λ.

Example 1.39. Given $\lambda = (2, 1)$ we can write tableaux and the weight functions as the following

Weight Function	$x_1^2 x_2$	$x_1 x_2^2$	$x_1^2 x_3$	$x_1 x_3^2$	$\dots$	$x_1 x_2 x_3$	$x_1 x_2 x_3$	$\dots$	$x_1 x_2 x_4$	$x_1 x_2 x_4$	$\dots$
Young Tableau	$\begin{array}{ll} 2 & \\ 1 & 1 \end{array}$	$\begin{array}{ll} 2 & \\ 1 & 2 \end{array}$	$\begin{array}{ll} 3 & \\ 1 & 1 \end{array}$	$\begin{array}{ll} 3 & \\ 1 & 3 \end{array}$	$\dots$	$\begin{array}{ll} 3 & \\ 1 & 2 \end{array}$	$\begin{array}{ll} 2 & \\ 1 & 3 \end{array}$	$\dots$	$\begin{array}{ll} 4 & \\ 1 & 2 \end{array}$	$\begin{array}{ll} 2 & \\ 1 & 4 \end{array}$	$\dots$

The column-strict tableau T is of shape $\lambda = (2, 1)$ can give the associated Schur function

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + 2x_1 x_2 x_3 + 2x_1 x_2 x_4 + \cdots$$

Definition 1.40. Kostka coefficients. Denote Kostka coefficients by $K_{\lambda\mu}$ which are the number of column-strict tableaux of shape λ content μ. We may define the Schur function in terms of Kostka coefficients and summing over all partitions μ of n as

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} m_\mu$$

Note that $K_{\lambda\lambda} = 1$ and that $K_{\lambda\alpha} = K_{\lambda\beta}$ if $\text{sort}(\beta) = \text{sort}(\alpha)$. Moreover, λ dominates μ is denoted by $\lambda \trianglerighteq \mu$.

Proposition 1.41. *The function $s_\lambda(\mathbf{x})$ is symmetric and the set $\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ^n.*

Theorem 1.42. ***(The Jacobi-Trudi Determinants)*** *Given $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ the Jacobi-Trudi determinants are defined by*

$$s_\lambda = det[h_{\lambda_{i-i+j}}]_{0 \leq i,j \leq k} \;\; and \;\; s_{\lambda'} = det[e_{\lambda_{i-i+j}}]_{0 \leq i,j \leq k}$$

where λ' is the conjugate tableau of λ.

Product of Schur Functions. The product of two Schur functions indexed by partitions μ and ν is expanded in Schur basis as

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

where $c^\nu_{\lambda\mu}$ is the Littlewood-Richardson coefficients. The Littlewood-Richardson rule gives us a combinatorial tool to calculate the coefficients in the above expansion.

Theorem 1.43. *(Theorem 4.9.2 in [Sagan]) Given $c^\lambda_{\mu\nu}$ to be the Littlewood-Richardson coefficients and $|\mu| + |\nu| = |\lambda|$, then*

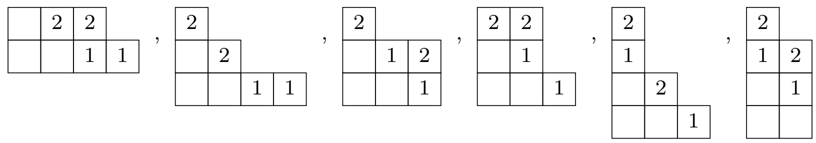
$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$$

Theorem 1.44. *(Theorem 4.9.4 [Sagan]) (Littlewood-Richardson Rule) The value of the coefficient $c^\lambda_{\mu\nu}$ is equal to the number of semi-standard tableaux T such that*

1. *T has shape λ/μ and content ν,*
2. *The row word of T, π_T is a reverse lattice permutation.*

Remark 1.45. From here on we will use a compact form of indexing partitions in symmetric functions. For example $s_{(3,1,1)}$ will be written as s_{311}.

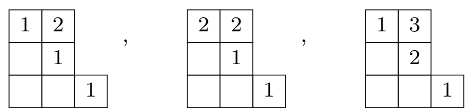
Example 1.46. Expansion of the product $s_{(2,1)} \cdot s_{(2,2)}$ in the Schur basis. By Littlewood-Richardson rule, we will list all tableaux that are lattice permutations with content $(2, 2)$, that is, two $1'$s and two $2'$s and skew shape of $\lambda/(2, 1)$ for some λ.



Therefore

$$s_{21} \cdot s_{22} = s_{43} + s_{421} + s_{331} + s_{322} + s_{3211} + s_{2221}.$$

Example 1.47. Expand $s_{(3,2,2)/(2,1)}$ in Schur basis. Using $s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}$ and $|\mu| + |\nu| = |\lambda|$ given $\lambda = (3, 2, 2)$ and $\mu = (2, 1)$ we have $|\nu| = 4$. In here we do not have the content and the shape is given as $(3, 2, 2)/(2, 1)$. We will find all possible tableaux.



Therefore

$$s_{322/21} = s_{31} + s_{22} + s_{211}.$$

Algebraic Operations

In this part we discuss two natural operations of the ring of symmetric functions Λ: product and coproduct along with antipode and involution on Λ. The content can be found in [GR, Macdonald, Zabrocki].

Definition 1.48. Multiplication function on $\Lambda \otimes \Lambda$ is the map $\mu : \Lambda \otimes \Lambda \longrightarrow \Lambda$ given by

$$\begin{aligned} \mu(p_\lambda \otimes p_\mu) &= p_\lambda p_\mu \\ &= p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{l(\lambda)}} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{l(\mu)}} \\ &= p_{\lambda \uplus \mu} \end{aligned}$$

The disjoint union partition $\lambda \uplus \mu$ is the rearrangement of sequence $(\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)}, \mu_1, \mu_2, \dots, \mu_{l(\mu)})$ in decreasing order.

Definition 1.49. Comultiplication function on Λ is the map $\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$ given by (primitive property)

$$\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$$

Remark 1.50. The comultiplication on Λ is a ring homomorphism, that is, given $f, g \in \Lambda$ and $\alpha, \beta \in \mathbb{Q}$ we have

1. $\Delta(fg) = \Delta(f)\Delta(g)$,
2. $\Delta(\alpha f + \beta g) = \alpha\Delta(f) + \beta\Delta(g)$.

Proposition 1.51. *(Proposition 2.25 [GR]) The comultiplication $\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$ has the following effect on the symmetric functions:*

i. $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$ *for every* $n \geq 1$.

ii. $\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j$ *for every* $n \in \mathbb{N}$.

iii. $\Delta(h_n) = \sum_{i+j=n} h_i \otimes h_j$ *for every* $n \in \mathbb{N}$.

iv. $\Delta(s_\lambda) = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_{\lambda/\mu}$ *for any partition* λ.

v. $\Delta(s_{\lambda/\nu}) = \sum_{\nu \subseteq \mu \subseteq \lambda} s_{\mu/\nu} \otimes s_{\lambda/\mu}$ *for any partition* λ *and* ν.

Example 1.52. Expansion of $\Delta(h_1)$ in h_λ basis is $\Delta(h_1) = h_\bullet \otimes h_1 + h_1 \otimes h_\bullet$ where $h_\bullet = 1$ is the complete homogenous symmetric function indexed by empty set. The expansion is

$$\begin{aligned}\Delta(h_{11}) &= \Delta(h_1 h_1)\\ &= \Delta(h_1)\,\Delta(h_1)\\ &= (h_\bullet \otimes h_1 + h_1 \otimes h_\bullet)(h_\bullet \otimes h_1 + h_1 \otimes h_\bullet)\\ &= h_\bullet \otimes h_{11} + 2h_1 \otimes h_1 + h_{11} \otimes h_\bullet.\end{aligned}$$

Example 1.53. Expansion of $\Delta(s_{21})$ in Schur basis is

$$\Delta(s_{21}) = s_\bullet \otimes s_{21} + s_1 \otimes s_{11} + s_1 \otimes s_2 + s_{11} \otimes s_1 + s_2 \otimes s_1 + s_{21} \otimes s_\bullet$$

where $s_\bullet = 1$ and expansion is by using Proposition iv. 1.51.

Proposition 1.54. *(Proposition 2.29 [GR]) Each of $\{e_n\}_{n=1,2,\ldots}$ and $\{h_n\}_{n=1,2,\ldots}$ are algebraically independent, and generate Λ as a polynomial algebra for any commutative ring K. The same holds for $\{p_n\}_{n=1,2,\ldots}$ when $\mathbb{Q}$ is a subring of K. Furthermore, the antipode S acts as follows:*

i. $S(p_n) = -p_n$

ii. $S(e_n) = (-1)^n h_n$

iii. $S(h_n) = (-1)^n e_n$

Example 1.55. Given $S(p_\lambda) = (-1)^{l(\lambda)} p_\lambda$ and using the preceding proposition we have

1. $S(p_{32111}) = (-1)^5 p_{32111} = -p_{32111}$

2. $S(h_{21}) = S(h_2) \cdot S(h_1) = (-1)^2 e_2 \cdot (-1)^1 e_1 = -e_{21}$

3. $S(e_{32}) = S(e_3) \cdot S(e_2) = (-1)^3 h_3 \cdot (-1)^2 h_2 = -h_{32}$

Definition 1.56. Consider the algebraic independence of the generator $\{e_n\}$ for ring of symmetric function Λ and define involution ω by a ring homomorphism on Λ.

$$\begin{aligned} \Lambda &\xrightarrow{\omega} \quad \Lambda \\ e_n &\longmapsto \quad h_n \end{aligned}$$

and it can also send $h_n \longmapsto e_n$ hence ω is an involutive automorphism of Λ, i.e. $\omega(h_\lambda) = e_\lambda$.

Remark 1.57. The antipode S on Λ (up to the sign) is the same as the involution ω, that is

$$S(f) = (-1)^n \omega(f) \text{ for } f \in \Lambda_n$$

Remark 1.58. Given λ and μ partitions satisfying $\mu \subseteq \lambda$, then the involution and the antipode action on (skew) Schur functions are given by

$$\begin{aligned} &\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'} \\ &S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda'/\mu'} \end{aligned}$$

where λ' is the conjugate partition to λ and $|\lambda/\mu|$ is the number of boxes in the skew diagram λ/μ, that is, $|\lambda/\mu| = n - k$ if $\lambda \vdash n$ and $\mu \vdash k$.

Example 1.59. Evaluate the following using $S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda'/\mu'}$.

1. $S(s_3) = (-1)^3 s_{111} = -s_{111}$

2. $S(s_{322/21}) = (-1)^4 s_{331/21} = s_{331/21} = s_{31} + s_{22} + s_{211}$

Corollary 1.60. *(Corollary 2.36 [GR]) In the Schur function basis $\{s_\lambda\}$ for Λ, the structure constants for multiplication and comultiplication are the same, that is*

$$\begin{aligned} s_\mu s_\nu &= \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \\ \Delta(s_\lambda) &= \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu \end{aligned}$$

The coefficients $c^\lambda_{\mu\nu}$ in the above expansions are the Littlewood- Richardson coefficients.

Definition 1.61. (Definition 2.41 [GR]) Define the Hall inner product on Λ to be the K-bilinear form $\langle \cdot, \cdot \rangle$ which makes s_λ an orthonormal basis, that is,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

where $\delta_{\lambda\mu} = 1$ if $\lambda = \mu$, otherwise $\delta_{\lambda\mu} = 0$. In terms of power sum basis, it can be said that power sum basis is an orthogonal set, and we can define the Hall inner product by

$$\left\langle \frac{p_\lambda}{z_\lambda}, p_\mu \right\rangle = \delta_{\lambda\mu}.$$

Example 1.62. Evaluate the following Hall inner products.

$$\begin{aligned} i. \quad \langle h_{21}, p_3 \rangle &= \langle h_2 h_1, p_3 \rangle \\ &= \left\langle \left(\frac{p_2}{2} + \frac{p_{11}}{2}\right) p_1, p_3 \right\rangle \\ &= \left\langle \frac{p_2 p_1}{2} + \frac{p_{11} p_1}{2}, p_3 \right\rangle \\ &= \left\langle \frac{p_{21}}{2} + \frac{p_{111}}{2}, p_3 \right\rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} ii. \quad \langle h_{222}, p_{222} \rangle &= \langle h_2 h_2 h_2, p_{222} \rangle \\ &= \left\langle \left(\frac{p_2}{2} + \frac{p_{11}}{2}\right)^3, p_{222} \right\rangle \\ &= \left\langle \frac{p_{222}}{8}, p_{222} \right\rangle \\ &= \frac{z_{222}}{8} \left\langle \frac{p_{222}}{z_{222}}, p_{222} \right\rangle \\ &= \frac{2^3 \cdot 3!}{8} \cdot 1 \\ &= 6 \end{aligned}$$

The following is the study of symmetric functions in two sets of variables and the Grothendieck ring of B_n. In later part of this thesis we use symmetric function notations that are obtained by plethystic substitution to encode characters where an operation $(f, g) \longmapsto f[g]$ on the ring of symmetric functions is called plethysm. The material in here can be found in [BrBg, Haiman, OZ1, Zabrocki].

Plethystic Substitution

Let $X \coloneqq x_1 + x_2 + \cdots$ and $Y \coloneqq y_1 + y_2 + \cdots$ be two sets of formal variables. Consider the ring of symmetric functions in the set of formal variable X and denote it by $\Lambda[X]$. As have know, any basis in $\Lambda[X]$ is indexed by a partition, in particular, consider the power sum symmetric function that is indexed by partition $\lambda \vdash n$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$. This can be expressed in X variable as

$$p_\lambda[X] \coloneqq p_{\lambda_1}[X] p_{\lambda_2}[X] \ldots p_{\lambda_{l(\lambda)}}[X]$$

The plethystic substitution is the evaluation of power sum symmetric function at X, this is

$$\begin{aligned} p_k[X] &= p_k(x_1 + x_2 + \cdots) \\ &= x_1^k + x_2^k + \cdots \end{aligned}$$

Now, given a symmetric functions $f \in \Lambda$, we will have plethystic substitution in alphabet X as $f[X] = f(x_1, x_2, \ldots)$. The result of plethystic substitution of a set of variables into a power sum p_k is the same as the corresponding power sum in the set of variables. It is important to note that the indeterminate are always considered as formal symbols and not as variables.

Definition 1.63. Given f a symmetric function ($f \in \Lambda$) and a formal sum of indeterminate $A = a_1 + a_2 + \cdots$, a plethystic substitution $f[A]$ is the substitution of the expression A in f. The plethystic substitution $f[A]$ is the image of f under the homomorphism mapping p_k to $p_k[A]$ and must have the following properties:

1. A plethystic substitution is additive and multiplicative. For given $f, g \in \Lambda$ and
$$(f+g)[A] = f[A] + g[A]$$
$$(fg)[A] = f[A]g[A]$$
2. A plethystic substitution of an expression A into a power sum p_k is defined by replacing all variables of the expression A by their k^{th} power and is denoted by $p_k[A]$, this is
$$p_k \longrightarrow a_1^k + a_2^k + \cdots$$
This results in the plethystic substitution being linear hence: $p_k[A_1 + A_2] = p_k[A_1] + p_k[A_2]$.
3. The plethystic substitution $f[A]$ is
$$f[A] \coloneqq \sum_{\mu} a_\mu \prod_{i=1}^{l(\mu)} p_{\mu_i}[A]$$

This expansion can be expressed in power sum in the X variable as
$$f[X] \coloneqq \sum_{\mu} a_\mu p_\mu[X].$$

Symmetric Polynomials and Plethysm. (The content in here can be found in Chapter 4 [Zabrocki]) Let $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ be the polynomial ring in variable $X_n \coloneqq x_1 + x_2 + \cdots + x_n$ over the field $\mathbb{Q}$. Now consider polynomials $f(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ such that $f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n)$ for all $\sigma \in \Lambda_n$. These polynomials are said to be symmetric and they form a ring with identity element 1 and they are closed under usual polynomial addition and multiplication. Denote this ring by

$$\Lambda[X_n] = \{f \in \mathbb{Q}[x_1, x_2, \ldots, x_n] : f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n) \text{ for all } \sigma \in \Lambda_n\}$$

If we set

$$\begin{aligned} p_k[X_n] &= p_k(x_1 + x_2 + \cdots + x_n) \\ &= x_1^k + x_2^k + \cdots + x_n^k \end{aligned}$$

then there is a linear homomorphism in the map

$$\begin{aligned} \Lambda &\longrightarrow \Lambda[X_n] \\ p_\lambda &\longmapsto p_{\lambda_1}[X_n]p_{\lambda_2}[X_n]\cdots p_{\lambda_{l(\lambda)}}[X_n] \end{aligned}$$

with a natural extension to the linear combinations of p_λ.

Example 1.64. Compute $p_2[X_4]$, $e_2[X_3]$ and $h_2[X_2]$.

$$\begin{aligned} p_2[X_4] &= p_2(x_1 + x_2 + x_3 + x_4) \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{aligned}$$

$$\begin{aligned} e_2[X_3] &= e_2[x_1 + x_2 + x_3] \\ &= \frac{1}{2}p_{11}[x_1 + x_2 + x_3] - \frac{1}{2}p_2[x_1 + x_2 + x_3] \\ &= \frac{1}{2}(x_1 + x_2 + x_3)^2 - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \\ &= x_1x_2 + x_1x_3 + x_2x_3 \end{aligned}$$

$$\begin{aligned} h_2[X_2] &= h_2[x_1 + x_2] \\ &= \frac{1}{2}p_{11}[x_1 + x_2] + \frac{1}{2}p_2[x_1 + x_2] \\ &= \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(x_1^2 + x_2^2) = x_1^2 + x_2^2 + x_1x_2 \end{aligned}$$

Remark 1.65. The power sum symmetric function $p_k \in \Lambda$ is a linear homomorphism as we have

$$p_k[cE + dF] = cp_k[E] + dp_k[F]$$

for $E, F \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ and coefficients $c, d \in \mathbb{Q}$.

Remark 1.66. We will discuss addition, subtraction and multiplication of alphabets in the algebra of symmetric functions using the plethystic notation. We know

$$p_k[X + Y] = p_k[X] + p_k[Y]$$

and

$$\Delta(p_k) = (p_k \otimes 1 + 1 \otimes p_k)$$

On the other hand, given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ we have defined

$$\begin{aligned} p_\lambda[X+Y] &= \prod_i p_{\lambda_i}[X+Y] \\ &= \prod_i (p_{\lambda_i}[X] + p_{\lambda_i}[Y]) \end{aligned}$$

The coefficient of $p_\mu[X]p_\nu[Y]$ in $p_\lambda[X+Y]$ is equal to the coefficient of $p_\mu \otimes p_\nu$ in $\Delta(p_\lambda)$ so the addition of the two sets of variables $X \coloneqq x_1 + x_2 + \cdots$ and $Y \coloneqq y_1 + y_2 + \cdots$ encodes the coproduct Δ in general terms.

Proposition 1.67. *Given $f \in \Lambda$ such that $\Delta(f)$ is given by $\Delta(f) = \sum_i f_i \otimes g_i$, then*

$$f[X+Y] = \sum_i f_i[X]g_i[Y].$$

Remark 1.68. In Proposition 1.67 the product $p_\mu[X]p_\nu[Y]$ as the basis element of Λ^{X+Y} is isomorphic to the tensor product $p_\mu \otimes p_\nu$ which is the basis element of $\Lambda \otimes \Lambda$ hence $\Lambda \otimes \Lambda \cong \Lambda^{X+Y}$.

Proposition 1.69. *For $f \in \Lambda$ such that f is homogeneous of degree k*

$$f[-X] = S(f)[X] = (-1)^k \omega(f)[X]$$

Example 1.70. For example, we can write $h_n[-X]$, $e_n[-X]$ and $s_n[-X]$ as the following.

- $h_n[-X] = S(h_n)[X] = (-1)^n \omega(h_n)[X] = (-1)^n e_n[X]$
- $e_n[-X] = S(e_n)[X] = (-1)^n \omega(e_n)[X] = (-1)^n h_n[X]$
- $s_n[-X] = S(s_n)[X] = (-1)^n \omega(s_n)[X] = (-1)^n s_{1^n}[X]$

Remark 1.71. Given f to be a homogenous function of degree n, we have

$$f[tX] = t^n f[X]$$

The indeterminate in plethystic substitution is always taken as formal symbols as they are not variables. Note that for $t = -1$ the correct equation should be as given by the antipode of the Hopf Algebra structure that is stated in Proposition 1.69, that is $f[-X] = (-1)^k \omega(f)[X]$.

Definition 1.72. (Prop.2.25 [GR]) To formulate a plethystic substitution approach to the coproduct on a single complete homogenous symmetric function, let

$$h_n[X+Y] = \sum_{k=0}^{n} h_k[X]h_{n-k}[Y]$$

as well as

$$e_n[X+Y] = \sum_{k=0}^{n} e_k[X]e_{n-k}[Y]$$

Definition 1.73. The plethystic formulas in Definition 1.72 are special cases of the following formulas

$$s_\lambda[X+Y] = \sum_{\nu\subseteq\lambda} s_\nu[X] s_{\lambda/\nu}[Y]$$

$$s_\lambda[X-Y] = \sum_{\nu\subseteq\lambda} (-1)^{|\lambda/\nu|} s_\nu[X] s_{\lambda'/\nu'}[Y]$$

$$s_{\lambda/\mu}[-Y] = (-1)^{|\lambda/\mu|} s_{\lambda'/\mu'}[Y].$$

Remark 1.74. For $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$ the last formula is expressed as $h_n[-Y] = (-1)^n e_n[Y]$.

The aim of this thesis is constructing a new character basis of the hyperoctahedral group. In this section we will present basic background about the hyperoctahedral group and it's conjugacy classes. Note that we are using definitions, propositions and theorems from [Macdonald, Stembridge, Stembridge2].

1.3 Hyperoctahedral Group B_n

The hyperoctahedral group B_n is the group of all **signed permutations**. It can be said that

- Hyperoctahedral group B_n is all bijections σ of the set $[\pm n] := \{\pm 1, \pm 2, \ldots, \pm n\}$ to itself such that for all $a \in [\pm n]$ we have $\sigma(-a) = -\sigma(a)$.
- Hyperoctahedral group B_n is the $n \times n$ signed permutation matrices.
- Hyperoctahedral group B_n is the centralizer of permutation $(12)(34)\cdots(2n-1, 2n)$ of S_{2n}.
- Hyperoctahedral group B_n as a Weyl group can be associated to the orthogonal group of odd dimensions.
- Hyperoctahedral group B_n in terms of coxeter group, is generated by τ, $\sigma_1, \cdots, \sigma_{n-1}$ and satisfying relations

  1. $\sigma_i^2 = \tau^2 = 1;$
  2. $(\sigma_i\sigma_{i+1})^3 = 1;$
  3. $(\sigma_i\tau)^2 = (\sigma_i\sigma_j)^2 = 1 \qquad \textit{where } i \neq n-1, |i-j| \geqslant 2;$
  4. $(\tau\sigma_{n-1})^4 = 1.$

The following table presents examples of group B_n for dimensions $n = 2, 3, 4$.

n	B_n	Coxeter notation	Order	Structure	Regular polytopes
2	B_2	[4]	$2^2 2! = 8$	$D_2 \cong C_2 wr S_2$	Square, Octagon
3	B_3	[4, 3]	$2^3 3! = 48$	$C_2^3 \rtimes S_3 \cong C_2 wr S_3$	Cube, Octahedron
4	B_4	[4, 3, 3]	$2^4 4! = 384$	$C_2^4 \rtimes S_4 \cong C_2 wr S_4$	Tesseract

Table 1.1: Hyperoctahedral group B_n for dimensions $n = 2, 3, 4$

A special case of a wreath product is commonly used in constructing the hyperoctahedral group.

Wreath Product, Hyperoctahedral Group B_n and Conjugacy Classes of B_n

Definition 1.75. (Appendix B in [Macdonald]) Let $G^n = G \times \ldots \times G$ be the direct product of n copies of G. For a permutation $\sigma \in S_n$, the symmetric group S_n acts on G^n by permuting the factors by $\varphi_\sigma(g_1, \ldots, g_n) = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)})$. The wreath product $G_n = GwrS_n$ is the semi-direct product of G^n with S_n, i.e. $G_n \coloneqq G^n \rtimes_\varphi S_n$, defined by this action. In fact the wreath product is a group of the ordered set of tuples $G^n \times S_n$ with the multiplication defined by $(g, \sigma)(h, \tau) = (g.\varphi_\sigma(h), \sigma\tau)$, where $g, h \in G^n$ and $\sigma, \tau \in S_n$. Note that the identity of wreath product is $e_{G_n} = (e_{G^n}, e_{S_n})$.

Remark 1.76. The elements of G_n can be expressed as permutation matrices with entries in G. The matrix corresponding to (g, σ) having entries a_{ij} in the matrix equal to

$$a_{ij} = g_i \delta_{i\sigma(j)}$$

where $g = (g_1, \ldots, g_n)$ and $\delta_{ab} = 1$ if $a = b$ otherwise $\delta_{ab} = 0$.

Remark 1.77. That the order of $G_n = GwrS_n$ is $|G|^n \cdot n!$ for $n \geq 0$ and for $n = 0$, G_0 is just a group of one element and for $n = 1$, G_1 is G. Another simple example is $S_n \cong \{e\}wrS_n$.

Hyperoctahedral group and wreath product. The hyperoctahedral group is the wreath product of a cyclic group of order two, namely C_2, with symmetric group of order n, that is

$$B_n \coloneqq C_2 wr S_n$$

This means $B_n \coloneqq C_2^n \rtimes S_n$. Note that the order of B_n is $2^n \cdot n!$

Example 1.78. To construct elements of B_2, the hyperoctahedral group of order 2, let $C_2 = \{e, x\}$ and $S_2 = \{(1)(2), (12)\}$. Having S_2 act on C_2^2 by permuting the factors, $\varphi_\sigma(g_1, g_2) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)})$ for $g_i \in G$ and $\sigma \in S_2$ we have

$$\begin{aligned} B_2 &\coloneqq C_2 wr S_2 \\ &= C_2^2 \rtimes_\varphi S_2 \\ &= \{(e, e), (e, x), (x, e), (x, x)\} \rtimes_\varphi \{(1)(2), (12)\} \end{aligned}$$

The group S_2 action on C_2^2 is shown below.

Wreath Product $B_2 \coloneqq C_2 wr S_2$	S_2 Action on C_2^2 by $\varphi_\sigma(g_1,g_2) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)})$	Elements of B_2
$id = \{(e,e) \rtimes_\varphi (1)(2)\}$	$\varphi_{(1)(2)}(e,e) = (e,e)$	$id = ((e,e),(1)(2))$
$g_1 = \{(e,x)\} \rtimes_\varphi \{(1)(2)\}$	$\varphi_{(1)(2)}(e,x) = (e,x)$	$g_1 = ((e,x),(1)(2))$
$g_2 = \{(x,e)\} \rtimes_\varphi \{(1)(2)\}$	$\varphi_{(1)(2)}(x,e) = (x,e)$	$g_2 = ((x,e),(1)(2))$
$g_3 = \{(x,x)\} \rtimes_\varphi \{(1)(2)\}$	$\varphi_{(1)(2)}(x,x) = (x,x)$	$g_3 = ((x,x),(1)(2))$
$g_4 = \{(e,e)\} \rtimes_\varphi \{(12)\}$	$\varphi_{(12)}(e,e) = (e,e)$	$g_4 = ((e,e),(12))$
$g_5 = \{(e,x)\} \rtimes_\varphi \{(12)\}$	$\varphi_{(12)}(e,x) = (x,e)$	$g_5 = ((x,e),(12))$
$g_6 = \{(x,e)\} \rtimes_\varphi \{(12)\}$	$\varphi_{(12)}(x,e) = (e,x)$	$g_6 = ((e,x),(12))$
$g_7 = \{(x,x)\} \rtimes_\varphi \{(12)\}$	$\varphi_{(12)}(x,x) = (x,x)$	$g_7 = ((x,x),(12))$

Table 1.2: Constructing elements of B_2 by wreath product

Conjugacy Classes and Types in Wreath Product. (Appendix B in [Macdonald]) First we define the cycle-product and the cycle-type of an element in G_n.

Definition 1.79. Let $x = (g,\sigma) \in G_n$ where $g = (g_1,\dots,g_n) \in G^n$ and $\sigma \in S_n$. Write the permutation σ as a product of disjoint cycles as $\sigma = \sigma_1\sigma_2\dots\sigma_m$ and choose $\sigma_k = (i_1 i_2 \dots i_r)$ to be one of these cycles. Take the product of corresponding elements as $g_{i_r}g_{i_{r-1}}\cdots g_{i_1} \in G$. We call this the cycle-product of x corresponding to the cycle σ_k. Now let K_G be the set of conjugacy classes in G. For each conjugacy class $K_g \in K_G$ and an integer $r \geq 1$, let $m_r(K_g)$ to denote the number of r-cycles in $\sigma \in S_n$ whose cycle-product lies in K_g. For an element $x = (g,\sigma) \in G_n$, there is a corresponding array $(m_r(K_g))_{r\geq 1, K_g\in K_G}$ of non-negative integers such that

$$\sum_{r,K_g} r m_r(K_g) = n$$

Let $p(K_g)$ be the partition having $m_r(K_g)$ parts equal to r, for each $r \geq 1$, then $p = (p(K_g))_{K_g\in K_G}$ is a partition-valued function on K_G such that $|p| = n$, i.e. for P_n the set of all partitions of n we have

$$p \in P_n(K_G)$$

The function p is called the cycle-type of $x = (g,\sigma) \in G_n$. The cycle-type of σ in S_n is $\sigma = \bigcup_{K_g\in K_G} p(K_g)$.

Theorem 1.80. *Two Elements of G_n are conjugate if and only if they have the same cycle-type.*

Corollary 1.81. *Two elements of B_n are conjugates if and only if they have the same cycle-type.*

Example 1.82. Let $C_2 = \{1,-1\}$, the cyclic group of order two with identity and the non-identity elements as 1 and -1 respectively. The cycle structures of elements of B_2 are grouped by conjugacy classes in $+1$ and -1, i.e., it is in positive and negative cycle-products. The table below shows these classes. Note that the symbol $\bullet$ is used in place of the empty ϕ.

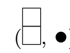
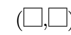
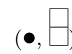
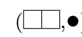
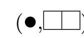
Elements of B_2 in Conjugacy Classe	Cycle-Products	Cycle-Type		Young Diagram of Cycle Structures
		Cycle-Product: +1	Cycle-Product: -1	
$id = ((1,1),(1)(2))$	(1) of cycle-product: 1 (2) of cycle-product: 1	$(1,1)$	$\bullet$	
$g_1 = ((1,-1),(1)(2))$	(1)of cycle-product: 1 (2)of cycle-product: -1	(1)	(1)	
$g_2 = ((-1,1),(1)(2))$	(1) of cycle-product: -1 (2) of cycle-product: 1			
$g_3 = ((-1,-1),(1)(2))$	(1) of cycle-product: -1 (2) of cycle-product: -1	$\bullet$	$(1,1)$	
$g_4 = ((1,1),(12))$	(12) of cycle-product: $1 \cdot 1 = 1$	(2)	$\bullet$	
$g_7 = ((-1,-1),(12))$	(12) of cycle-product: $(-1) \cdot (-1) = 1$			
$g_5 = ((-1,1),(12))$	(12) of cycle-product: $(-1) \cdot 1 = -1$	$\bullet$	(2)	
$g_6 = ((1,-1),(12))$	(12) of cycle-product: $1 \cdot (-1) = -1$			

Table 1.3: Cycle structures of elements of B_2

Pair of partitions and conjugacy classes of group B_n. The conjugacy classes of the hyperoctahedral group B_n namely $K^{B_n}_{(\gamma,\nu)}$ are indexed by a pair of partitions $(\gamma, \nu) \vdash n$ when its length is defined by $|(\gamma, \nu)| := |\gamma| + |\nu| = n$. This class consists of all signed permutations such that parts of the partition γ are the cycles with even number of negative signs and the parts of ν are the cycles with an odd number of negative signs. The group B_n is generated by τ, $\sigma_1, \cdots, \sigma_{n-1}$ and satisfying relation

1. $\sigma_i^2 = \tau^2 = 1;$
2. $(\sigma_i \sigma_{i+1})^3 = 1;$
3. $(\sigma_i \tau)^2 = (\sigma_i \sigma_j)^2 = 1 \qquad$ where $i \neq n-1, |i-j| \geqslant 2;$
4. $(\tau \sigma_{n-1})^4 = 1.$

The conjugation by σ_i and τ are as the following

- ◇ Conjugation by σ_i preserves the sign record in every cycle.
- ◇ Conjugation by τ changes the sign of two elements in some cycle.

This implies that given a conjugacy class, there should be a particular number of cycles with even number of positive elements and odd number of negative elements. Also, given an element in B_n

with cycle M_σ and conjugating by the element M_τ as

$$M_\sigma = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \pm 1 & 0 & \cdots & 0 \end{bmatrix} \text{ and } M_\tau = \begin{bmatrix} \pm 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pm 1 \end{bmatrix}$$

as $M_\tau M_\sigma M_\tau^{-1}$ could generate any sign pattern we wish to create.

Lemma 1.83. ***Order of Conjugacy Classes of B_n.*** *Let $K^{B_n}_{(\gamma,\nu)}$ be the conjugacy class of group B_n indexed at the pair of partition (γ, ν). The order of the conjugacy classes is*

$$\left|K^{B_n}_{(\gamma,\nu)}\right| = |\{\sigma \in B_n : \text{cycle-type of } \sigma \text{ is } (\gamma, \nu)\}| = \frac{2^n n!}{2^{l(\gamma)+l(\nu)} z_\gamma z_\nu}$$

Proof. The group B_n can be recognized as S_n where τ is identity. For a given element $\sigma \in B_n$ a signed permutation in the conjugacy class $K^{B_n}_{(\gamma,\nu)}$ of cycle-type of (γ, ν) with $\gamma \vdash k$ and $\nu \vdash n-k$. To calculate the order of this conjugacy class $\left|K^{B_n}_{(\gamma,\nu)}\right|$ let $(S, \alpha, \beta, \mathrm{sgn})$ be an index for our calculation. The value of $\left|K^{B_n}_{(\gamma,\nu)}\right|$ will be evaluated by multiplying all four choices evaluated below.

For S, the choices of the positive elements subset of $[n]$ of size k the number of choices is $\dfrac{n!}{k!(n-k)!}$. For α, a permutation of k elements of cycle-type γ the number choices is $\dfrac{k!}{z_\gamma}$. For β, a permutation of $n-k$ elements of cycle-type ν the number of choices is $\dfrac{(n-k)!}{z_\nu}$. For sgn, an assignment of signs for the elements of S and S^c the number of choices is $2^{n-l(\gamma)-l(\nu)}$. (Note that for a sign of a single cycle is $\mathrm{sgn}\left((a_1, \cdots, a_n)\right) = (-1)^{n-1}$.) □

Window Notation. [BB] The hyperoctahedral group is also described as the group of all signed permutations, i.e., the group of all bijections σ of the set $[\pm n] \coloneqq \{\pm 1, \pm 2, ..., \pm n\}$ to itself such that for all $a \in [\pm n]$ we have $\sigma(-a) = -\sigma(a)$. Given a signed permutation σ, an element of the hyperoctahedral group, it's window notation is denoted by $\sigma = [a_1, a_2, \ldots, a_n]$ where $\sigma(i) = a_i$ for $i = 1, 2, \ldots, n$ and $a_i \in [\pm n]$. Note that the composition is used as the operation in this group.

Example 1.84. For the window notation of the signed permutation $\sigma = [1, -4, 2, -3]$ the permutations are $\sigma(1) = 1$, $\sigma(2) = -4$, $\sigma(3) = 2$ and $\sigma(4) = -3$. A complete version that include all the signed permutations maps is shown below

$$\begin{array}{cccccccccc} & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 \\ \sigma : & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3 & -2 & 4 & -1 & 1 & -4 & 2 & -3 \end{array}$$

Signed Permutation Matrix. The hyperoctahedral group is also called the "group of signed permutation matrix". The elements of B_n can be expressed as matrices with entries in ± 1 and 0. To construct the matrix corresponding to $x = (g, \sigma)$ having entries a_{ij} in the matrix equal to

$$a_{ij} = g_i \delta_{i\sigma(j)}$$

where $g = (g_1, g_2) \in C_2^2$ for $C_2 = \{1, -1\}$ and $\delta_{ab} = 1$ if $a = b$ otherwise 0. Now, let e_i be the standard row vector with 1 in the i-th entry and 0 elsewhere.

- If $e_i M_x = e_j$ then we will put j in the i-th entry of the window notation.
- If $e_i M_x = -e_j$ then we will put $-j$ in the i-th entry of the window notation.

With this convention the composition of window notation is consistent with matrix multiplication, that is if σ is the window notation for M_x and τ is the window notation for M_y then if $e_i M_x M_y = e_j$ (respectively $e_i M_x M_y = -e_j$), then the i-th entry of $\sigma\tau$ is j (respectively $-j$).

Remark 1.85. This is right multiplication (action) of the permutation matrix so it acts on the rows.

Remark 1.86. We will use the following conventions:

- x and y are an element of wreath product $B_2 \coloneqq C_2 wr S_n$.
- σ_x and τ_y are window notation for elements x and y.
- M_x and M_y are signed permutation matrices notation for elements x and y.

Example 1.87. The signed permutation matrix corresponding to the element $x = ((-1, 1), (12))$ in B_2 evaluated as

- $a_{11} = g_1 \delta_{1\sigma(1)} = g_1 \delta_{12} = (-1) \cdot 0 = 0$
- $a_{12} = g_1 \delta_{1\sigma(2)} = g_1 \delta_{11} = (-1) \cdot 1 = -1$
- $a_{21} = g_2 \delta_{2\sigma(1)} = g_2 \delta_{22} = 1 \cdot 1 = 1$
- $a_{22} = g_2 \delta_{2\sigma(2)} = g_2 \delta_{21} = 1 \cdot 0 = 0$

and the matrix is

$$M_x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The element $x = ((-1, 1), (12))$ in window notation is $\sigma_x = [-2, 1]$ in B_2. We may also check if the right action of this signed permutation matrix M_x on the standard row vector e_1 and e_2 is consistent with the window notation.

Right Action of Signed Permutation Matrix	Standard Vectors Transformation	Window Notation $\sigma_x = [-2, 1]$
$[1, 0]M_x = [0, -1]$	$e_1 \longrightarrow -e_2$	$1 \longrightarrow -2$
$[0, 1]M_x = [1, 0]$	$e_2 \longrightarrow e_1$	$2 \longrightarrow 1$

Example 1.88. For $x = ((1, -1, 1), (12)(3))$ in window notation is $\sigma_x = [2, -1, 3]$ in B_3 and the corresponding signed permutation matrix is

$$x = ((1, -1, 1), (12)(3)) \longmapsto M_x = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 1.89. The table below illustrates the conjugacy classes and cycle-types for $n = 2$.

Conjugacy Classes in B_2		Elements in Conjugacy Classes of B_2	Elements of B_2 in Window Notation	Signed Permutation Matrix of B_2
Pair of Partitions $(\gamma, \nu) \vdash 2$	Young Diagram of Cycle Types			
$((1, 1), \bullet)$	$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \bullet)$	$id = ((1, 1), (1)(2))$	[1,2]	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$((1), (1))$	$(\square, \square)$	$g_1 = ((1, -1), (1)(2))$	[1,-2]	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
		$g_2 = ((-1, 1), (1)(2))$	[-1,2]	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$(\bullet, (1, 1))$	$(\bullet, \begin{smallmatrix}\square\\\square\end{smallmatrix})$	$g_3 = ((-1, -1), (1)(2))$	[-1,-2]	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
$((2), \bullet)$	$(\square\square, \bullet)$	$g_4 = ((1, 1), (12))$	[2,1]	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
		$g_7 = ((-1, -1), (12))$	[-2,-1]	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
$(\bullet, (2))$	$(\bullet, \square\square)$	$g_5 = ((-1, 1), (12))$	[-2,1]	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
		$g_6 = ((1, -1), (12))$	[2,-1]	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Table 1.4: Conjugacy classes in B_2

Example 1.90. Elements of B_3 in window notation and the associated signed permutation matrix grouped by cycle-types of B_3 in conjugacy classes indexed by a pair of partitions$(\gamma, \nu) \vdash 3$.

$K^{B_n}_{((1,1,1),\bullet)}$	$K^{B_n}_{((2,1),\bullet)}$	$K^{B_n}_{((3),\bullet)}$	$K^{B_n}_{((1,1),(1))}$	$K^{B_n}_{((2),(1))}$	$K^{B_n}_{((1),(1,1))}$	$K^{B_n}_{((1),(2))}$	$K^{B_n}_{(\bullet,(1,1,1))}$	$K^{B_n}_{(\bullet,(2,1))}$	$K^{B_n}_{(\bullet,(3))}$
[1,2,3] $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	[1,3,2] $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$	[2, 3, 1] $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	[1,2, -3] $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$	[2, 1, -3] $\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}$	[-1,-2, 3] $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$	[1,-3,2] $\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$	[-1,-2,-3] $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$	[-1,-3,2] $\begin{bmatrix}-1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$	[2, 3, -1] $\begin{bmatrix}0&1&0\\0&0&1\\-1&0&0\end{bmatrix}$
	[1,-3,-2] $\begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$	[2, -3, -1] $\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix}$	[1,-2,3] $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$	[-2, -1, -3] $\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}$	[1,-2,-3] $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$	[1,3,-2] $\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$		[-1,3,-2] $\begin{bmatrix}-1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$	[2, -3, 1] $\begin{bmatrix}0&1&0\\0&0&-1\\1&0&0\end{bmatrix}$
	[2, 1, 3] $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$	[-2, 3, -1] $\begin{bmatrix}0&-1&0\\0&0&1\\-1&0&0\end{bmatrix}$	[-1,2,3] $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	[3, -2, 1] $\begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}$	[-1,2,-3] $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$	[-2, 1, 3] $\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$		[-2, 1, -3] $\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix}$	[-2, 3, 1] $\begin{bmatrix}0&-1&0\\0&0&1\\1&0&0\end{bmatrix}$
	[-2, -1, 3] $\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&1\end{bmatrix}$	[-2, -3, 1] $\begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix}$		[-3, -2, -1] $\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}$		[2, -1, 3] $\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}$		[2, -1, -3] $\begin{bmatrix}0&1&0\\-1&0&0\\0&0&-1\end{bmatrix}$	[-2, -3, -1] $\begin{bmatrix}0&-1&0\\0&0&-1\\-1&0&0\end{bmatrix}$
	[3, 2, 1] $\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	[3, 1, 2] $\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$		[-1, 3, 2] $\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}$		[-3, 2, 1] $\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$		[-3, -2, 1] $\begin{bmatrix}0&0&-1\\0&-1&0\\1&0&0\end{bmatrix}$	[3, 1, -2] $\begin{bmatrix}0&0&1\\1&0&0\\0&-1&0\end{bmatrix}$
	[-3, 2, -1] $\begin{bmatrix}0&0&-1\\0&1&0\\-1&0&0\end{bmatrix}$	[3, -1, -2] $\begin{bmatrix}0&0&1\\-1&0&0\\0&-1&0\end{bmatrix}$		[-1, -3, -2] $\begin{bmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$		[3, 2, -1] $\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}$		[3, -2, -1] $\begin{bmatrix}0&0&1\\0&-1&0\\-1&0&0\end{bmatrix}$	[3, -1, 2] $\begin{bmatrix}0&0&1\\-1&0&0\\0&1&0\end{bmatrix}$
		[-3, 1, -2] $\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}$							[-3, 1, 2] $\begin{bmatrix}0&0&-1\\1&0&0\\0&1&0\end{bmatrix}$
		[-3,-1, 2] $\begin{bmatrix}0&0&-1\\-1&0&0\\0&1&0\end{bmatrix}$							[-3, -1, -2] $\begin{bmatrix}0&0&-1\\-1&0&0\\0&-1&0\end{bmatrix}$

Table 1.5: Cycle-types of B_3 in conjugacy classes

1.4 Group Representation and Character

The following notations and terminologies will be used in the study of matrix representation of an abstract group.

◇ $\mathtt{Mat}_d$: The set of all $d \times d$ matrices with entries in $\mathbb{C}$. ($\mathbb{C}$ denotes the complex numbers)

◇ Algebra: A vector space that the multiplication of vectors is associative. (A ring structure is imposed on the space)

◇ GL_d: The complex general linear group of degree d. This is a group of all $d \times d$ matrices in $\mathtt{Mat}_d$ that are invertible with respect to multiplication.

◇ X: is a matrix representation of a group G of degree d (or dimension d).

Matrix Representation of a group and Character

Definition 1.91. A matrix representation of a group G is a group homomorphism

$$X: G \to GL_d$$

Equivalently, each element of the group $g \in G$ is assigned to a matrix $X(g) \in \mathtt{Mat}_d$ such that

1. $X(e) = I_d$ (element e is the identity G and the matrix I_d is the identity matrix dimension d)

2. $X(gh) = X(g)X(h)$ for $\forall g, h \in G$.

Remark 1.92. All groups have the trivial representation. This representation is of degree 1 or 1-dimensional (linear representation). That is a map that sends every element of the group $g \in G$ to the matrix (1), i.e., the representation with $X(g) = (1)$ for $\forall g \in G$.

G-Module and Matrix Representation. The idea of G-module is to think of the matrix representations in terms of linear transformations. The following notation are used in our discussion of G-module and group algebra.

- V: the vector space $\mathbb{C}^d$ of elements with column matrix of dimension d.
- $GL(V)$: General linear group of V, the set of all invertible linear transformations of V to itself.
- If $dim\, V = d$ then two groups are isomorphism of groups $GL(V) \cong GL_d$.

Definition 1.93. Given a vector space V and a group G, it is said that V is a G-modules if there is a group homomorphism

$$\rho: G \to GL(V)$$

Equivalently, for $\forall\, \mathbf{v}, \mathbf{w} \in V$ and for $\forall\, g, h \in G$ and scalars $c, d \in \mathbb{C}$, the vector space V is a G-module if there is a multiplication, $g\mathbf{v}$, such that

1. $g\mathbf{v} \in V$ (the transformation takes takes V to itself)

2. $g(c\mathbf{v} + d\mathbf{w}) = cg(\mathbf{v}) + dg(\mathbf{w})$ (the map is linear transformation)

3. $(gh)\mathbf{v} = g(h\mathbf{v})$ (property 2 of matrix representation definition)

4. $e\mathbf{v} = \mathbf{v}$ (given identity e, properties 3 and 4 will imply all transformations are invertible)

Note that $g\mathbf{v}$ is the action of $\rho(g)$ on the vector $\mathbf{v}$.

Matrix representation and G-module. We presented two definitions of representation, matrix representation and G-module. If $dim\, V = d$ then two groups are isomorphism of groups i.e.$GL(V) \cong GL_d$. We can switch between these two as it will be described below.

- ◇ *Matrix representation to G-module*: The action of $g \in G$ on $\mathbf{v} \in V$ is defined in terms of the matrix product $X(g)\mathbf{v}$ by $g\mathbf{v} \stackrel{def}{=} X(g)\mathbf{v}$.
- ◇ *G-module to a matrix representation*: Conversely, if V is a G-module, then take any basis $\mathcal{B}$ of V and $X(g)$ will be the matrix of the linear transformation $g \in G$ in the basis $\mathcal{B}$ evaluated in usual way.

In the coming discussion, it will be shown how irreducible representations are the essential concept in representation theory. Every representation is the direct sum of the irreducible representation. We will present how representations can be built out of smaller structures and in order to do that let's introduce the concept of submodule.

Definition 1.94. Let V be a G-module. A submodule of V is a subspace of W that is closed under action of G, i.e.

$$\mathbf{w} \in W \Rightarrow g\mathbf{w} \in W \text{ for all } g \in G$$

Equivalently, W is a subset of V that is a G-module itself. Denote W is a submodule of V by $W \leq V$.

Definition 1.95. Let V be a G-module with submodules U and W. Then V is the (internal) direct sum of U and W written $V = U \oplus W$, if every $\mathbf{v} \in V$ can be written uniquely as a sum

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \quad \mathbf{u} \in U,\ \mathbf{w} \in W$$

Definition 1.96. The matrix$X$ is the direct sum of two matrices A and B written $X = A \oplus B$, if X can be expressed in block diagonal form of

$$X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Irreducible Representations. A nonzero G- module V is irreducible if it does not contain any trivial submodule, otherwise V is said to be irreducible. A matrix representation of the group G is irreducible if its corresponding module is irreducible. Maschke's theorem of a finite group shows how every G- module and the corresponding matrix representation is made of the irreducible modules.

Theorem 1.97. ***[Maschke's Theorem]*** *Let G be a finite group and let V be a non-zero G-module. Then*

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)}$$

where each $W^{(i)}$ is an irreducible G-submodule of V. Equivalently, given X to be a non-zero matrix representation of a finite group G, a similar decomposition of X to the $X^{(i)}$ irreducible *representations is*

$$X = X^{(1)} \oplus X^{(2)} \oplus \cdots \oplus X^{(k)}$$

Character of a group representation is the trace of corresponding matrix to a group element. The characters carry much of the important information about the representation in a compact form. Georg Frobenius used the character theory and developed the idea of representation theory of a finite group without having the explicit corresponding matrices of the group elements. The finite group representation can be determined by its character up to isomorphism.

Definition 1.98. Given a group G, let $X(g)$ be a matrix representation of G. The character of X is

$$\chi(g) = \mathrm{tr}X(g)$$

where tr denotes the trace of a matrix. Equivalently, the map χ is

$$\begin{array}{rccc} \chi: & G & \xrightarrow{\mathrm{tr}X} & \mathbb{C} \\ & g & \longmapsto & \mathrm{tr}X(g) \end{array}$$

Remark 1.99. Given V be a G-module, its character is the character of a matrix representation corresponding to V. As we know there are many matrix representations corresponding to a single G-module but the above definition is well-defined. To check this, consider X and Y both to be corresponding to V and by change of basis write $Y = TXT^{-1}$ for some fixed T. Therefore for all $g \in G$

$$\mathrm{tr}Y(g) = \mathrm{tr}TX(g)T^{-1} = \mathrm{tr}X(g)$$

this is true as trace is invariant under conjugation and X and Y have the same character and the definition presented above is well-defined.

Remark 1.100. Let G be an arbitrary group and X be a degree 1 representation. Then the linear character $\chi(g)$ is just a single value of $X(g)$ for a given $g \in G$. This type of characters are called linear characters.

Proposition 1.101. *(Proposition 1.8.5 [Sagan]) Let X be a matrix representation of a group G of degree d with character χ . Then*

1. $\chi(e) = d$.

2. If K is a conjugacy class of G, then for g and h in K we have $\chi(g) = \chi(h)$.

If Y is a representation of G with character ψ, then $X \cong Y$ if and only if $\chi(g) = \psi(g)$.

Definition 1.102. Let χ and ψ to be any two characters of a group G. These are functions from a group G to the complex numbers $\mathbb{C}$. The inner product of χ and ψ is defined by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

where $\overline{\psi(g)}$ is the complex conjugate of $\psi(g)$ and $\overline{\psi(g)} = \psi(g^{-1})$. Considering this inner product, the irreducible characters form an orthonormal basis for the space of class functions the following theorem holds.

Theorem 1.103. *(Theorem 1.9.3 [Sagan]) Let χ and ψ be irreducible characters of the group G. Then*

$$\langle \chi, \psi \rangle = \delta_{\chi\psi}$$

Theorem 1.104. *(Corollary 1.9.4 [Sagan]) Let X be a matrix representation of G with character χ. Suppose*

$$X \cong m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \ldots \oplus m_k X^{(k)}$$

where $X^{(i)}$ are pairwise inequivalent irreducible with characters $\chi^{(i)}$.

1. *$X \cong m_1\chi^{(1)} + m_2\chi^{(2)} + \cdots + m_k\chi^{(k)}$*
2. *$\left\langle \chi, \chi^{(j)} \right\rangle = m_j$ for all .*
3. *$\langle \chi, \chi \rangle = m_1^2 + m_2^2 + \cdots + m_k^2$*
4. *X is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*
5. *Let Y be another matrix representation of G with character ψ. Then*

$$X \cong Y \text{ if and only if } \chi(g) = \psi(g)$$

for all $g \in G$.

Definition 1.105. [Sagan] **Tensor Product.** For given vector spaces V and W, define their tensor product as the set

$$V \otimes W = \{\sum_{i,j} c_{ij}\mathbf{v}_i \otimes \mathbf{w}_j : c_{ij} \in \mathbb{C}, \mathbf{v}_i \in V, \mathbf{w}_j \in W\}$$

where in addition the following axioms must hold

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \otimes \mathbf{w} = c_1(\mathbf{v}_1 \otimes \mathbf{w}) + c_2(\mathbf{v}_2 \otimes \mathbf{w})$$
$$\mathbf{v} \otimes (d_1\mathbf{w}_1 + d_2\mathbf{w}_2) = d_1(\mathbf{v} \otimes \mathbf{w}_1) + d_2(\mathbf{v} \otimes \mathbf{w}_2)$$

for $c_i, d_j \in \mathbb{C}$.

Lemma 1.106. *(Lemma 1.7.7 [Sagan]) Suppose $A, X \in \mathtt{Mat}_{d_2}$ and $B, Y \in \mathtt{Mat}_{d_1}$. Then*

1. $(A \oplus B)(X \oplus Y) = AX \oplus BY$

2. $(A \otimes B)(X \otimes Y) = AX \otimes BY$

Example 1.107. The characters of the representation of the group of signed permutation matrix B_2.

Window Notation of Element of B_2	Signed Permutation Matrix	Character of Signed Permutation Representation of B_2
[1,2]	$X([1,2]) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\chi([1,2]) = 2$
[1,-2]	$X([1,-2]) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\chi([1,-2]) = 0$
[-1,2]	$X([-1,2]) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\chi([-1,2]) = 0$
[-1,-2]	$X([-1,-2]) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\chi([-1,-2]) = -2$
[2,1]	$X([2,1]) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\chi([2,1]) = 0$
[-2,1]	$X([-2,1]) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\chi([-2,1]) = 0$
[2,-1]	$X([2,-1]) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\chi([2,-1]) = 0$
[-2,-1]	$X([-2,-1]) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\chi([-2,-1]) = 0$

Table 1.6: Character of signed permutation matrix B_2

Class Function

We know that characters of a group representation are functions that are invariant under the conjugacy classes. This type of functions are called class functions. We will state the definitions first and present some examples.

Definition 1.108. A class function on a group G is the map

$$f : G \longrightarrow \mathbb{C}$$

such that given two elements g and h of group G that in the same conjugacy class then $f(g) = f(h)$. Let $R(G)$ be the set of all class functions on G.

1. This set is a vector space as the sum and scalar multiple of class functions are also class functions.

2. Define an indicator function as the basis of class function $R(G)$ that are evaluated to value 1 on a given conjugacy class and 0 elsewhere.

3. The dimension of this vector space is equal to the number of conjugacy classes of G.

Character Table. Let K be a conjugacy class and χ be a character of group G. Define χ_K to be the value of a given character on the given character on the given conjugacy class

$$\chi_K = \chi(g)$$

for any $g \in K$. For a group G its character table is a table whose rows are indexed by an indexing inequivalent irreducible representations or characters of G and whose columns are indexed by the conjugacy classes of the group G. The first row represents the trivial character and the first column corresponds to the conjugacy class of the identity.

Example 1.109. The character table of B_1. Elements of group is $B_1 = \{[1], [-1]\}$. Table below is the character table of B_1.

Irreducible Characters of B_1	B_1 in Conjugacy Classes	
	[1]	[-1]
	$K^{B_n}_{(\square,\bullet)}$	$K^{B_n}_{(\bullet,\square)}$
$\chi^{(\square,\bullet)}$	1	1
$\chi^{(\bullet,\square)}$	1	-1

Table 1.7: Character table of B_1

Example 1.110. The character table of B_2. Detailed evaluations are in Example 1.147

Irreducible Characters of B_2	Elements of B_2 in Conjugacy Classes				
	[1, 2]	[2, 1] [-2,-1]	[1, -2] [-1, 2]	[-1, -2]	[2, -1] [-2, 1]
	$K^{B_n}_{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}$	$K^{B_n}_{(\square\square,\bullet)}$	$K^{B_n}_{(\square,\square)}$	$K^{B_n}_{(\bullet,\begin{smallmatrix}\square\\\square\end{smallmatrix})}$	$K^{B_n}_{(\bullet,\square\square)}$
$\chi^{(\square\square,\bullet)}$	1	1	1	1	1
$\chi^{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}$	1	-1	1	1	-1
$\chi^{(\square,\square)}$	2	0	0	-2	0
$\chi^{(\bullet,\square\square)}$	1	1	-1	1	-1
$\chi^{(\bullet,\begin{smallmatrix}\square\\\square\end{smallmatrix})}$	1	-1	-1	1	1

Table 1.8: The character table of B_2

Induced Representation of G

Assume that G is a group and H is a subgroup in G. To obtain representations of G from H or representations of H from G, we can use the induction operation.

Definition 1.111. (Definition 1.12.2 [Sagan]) Let $H \leq G$ and the fixed transversal elements $t_1, \ldots, t_l$ of the left cosets of H, i.e. the group G can be written in terms of disjoint union of left cosets as $G = t_1 H \uplus t_2 H \uplus \ldots . \uplus t_{\frac{|G|}{|H|}} H$. Here $\uplus$ denotes the disjoint union. If X_H is a representation of H then the corresponding induced representation $X_H \uparrow_H^G$ assigns to each $g \in G$ the block matrix

$$X_H \uparrow_H^G (g) = \left(\overline{X}_H(t_i^{-1} g t_j)\right) = \begin{bmatrix} \overline{X}_H(t_1^{-1} g t_1) & \overline{X}_H(t_1^{-1} g t_2) & \ldots & \overline{X}_H(t_1^{-1} g t_{\frac{|G|}{|H|}}) \\ \overline{X}_H(t_2^{-1} g t_1) & \ldots \overline{X}_H(t_2^{-1} g t_2) & \ldots & \overline{X}_H(t_2^{-1} g t_{\frac{|G|}{|H|}}) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{X}_H(t_{\frac{|G|}{|H|}}^{-1} g t_1) & \overline{X}_H(t_{\frac{|G|}{|H|}}^{-1} g t_2) & \ldots & \overline{X}_H(t_{\frac{|G|}{|H|}}^{-1} g t_{\frac{|G|}{|H|}}) \end{bmatrix}$$

where $\overline{X}(x) = X_H$ if $x \in H$ otherwise 0. Let $\chi_H \uparrow_H^G$ be the character of $X_H \uparrow_H^G$ then

$$\chi_H \uparrow_H^G (g) = \sum_{i=1}^{\frac{|G|}{|H|}} \overline{\chi}(t_i^{-1} g t_i)$$

An important example in this topic that will extensively be used in this thesis is the induced trivial character. We will start with an example and explore more in later sections.

Example 1.112. Induced trivial character table of $\mathbf{B_2}$. We map the elements to have the embedding of $B_1 \times B_1$ in B_2.

$B_1 \times B_1$	$\hookrightarrow$	B_2
$([1], [1])$		$([1, 2])$
$([1], [-1])$		$([1, -2])$
$([-1], [1])$		$([-1, 2])$
$([-1], [-1])$		$([-1, -2])$

Now, let transversals in B_2 in window notation be $t_1 = [1, 2]$ and $t_2 = [2, 1]$ so the group B_2 can be written as the disjoint union of two left cosets

$$\begin{aligned} B_2 &\cong t_1 B_1 \times B_1 \uplus t_2 B_1 \times B_1 \\ &\cong [1, 2] B_1 \times B_1 \uplus [2, 1] B_1 \times B_1 \end{aligned}$$

Let $\mathcal{I}^{(\lambda,\mu)}(\gamma,\nu)$ be the induced trivial character indexed by the pair of partitions (λ,μ) at conjugacy class of (γ,ν). Because the computations are long but we would like to see how it is done, we will do the first row calculations and the rest will be listed in the table.

$$\begin{aligned}
\mathcal{I}^{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet) &= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}\uparrow_{B_1\times B_1}^{B_2}(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,2]^{-1}[1,2][1,2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,1]^{-1}[1,2][2,1])\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,2])\\
&= \chi^{(\square,\bullet)}([1])\chi^{(\square,\bullet)}([1]) + \chi^{(\square,\bullet)}([1])\chi^{(\square,\bullet)}([1])\\
&= 1\cdot 1 + 1\cdot 1\\
&= 2
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}(\square\square,\bullet) &= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}\uparrow_{B_1\times B_1}^{B_2}(\square\square,\bullet)\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,2]^{-1}[2,1][1,2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,1]^{-1}[2,1][2,1])\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,1]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,1])\\
&= 0 \qquad \text{(this is because } [2,1]\in B_2 \text{ but is not embedding of } B_1\times B_1\text{)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}(\square,\square) &= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}\uparrow_{B_1\times B_1}^{B_2}(\square,\square)\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,2]^{-1}[1,-2][1,2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,1]^{-1}[1,-2][2,1])\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,-2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([-1,2])\\
&= \chi^{(\square,\bullet)}([1])\chi^{(\square,\bullet)}([-1]) + \chi^{(\square,\bullet)}([-1])\chi^{(\square,\bullet)}([1])\\
&= 1\cdot 1 + 1\cdot 1\\
&= 2
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}(\bullet,\square\square) &= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}\uparrow_{B_1\times B_1}^{B_2}(\bullet,\square\square)\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,2]^{-1}[2,-1][1,2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,1]^{-1}[2,-1][2,1])\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,-1]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([-2,1])\\
&= 0 \qquad \text{(this is because } [2,-1],[-2,1]\in B_2 \text{ but is not embedding of } B_1\times B_1\text{)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}(\bullet,\begin{smallmatrix}\square\\\square\end{smallmatrix}) &= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}\uparrow_{B_1\times B_1}^{B_2}(\phi,\begin{smallmatrix}\square\\\square\end{smallmatrix})\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([1,2]^{-1}[-1,-2][1,2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([2,1]^{-1}[-1,-2][2,1])\\
&= \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([-1,-2]) + \chi^{(\square,\bullet)}\otimes\chi^{(\square,\bullet)}([-1,-2])\\
&= \chi^{(\square,\bullet)}([-1])\chi^{(\square,\bullet)}([-1]) + \chi^{(\square,\bullet)}([-1])\chi^{(\square,\bullet)}([-1])\\
&= 1\cdot 1 + 1\cdot 1\\
&= 2
\end{aligned}$$

The induced trivial characters in representations of the hyperoctahedral group B_2 indexed by a pair of partitions is shown in table below.

Induced Trivial Characters of B_2	Elements of B_2 in Conjugacy Classes and Cycle-Types				
	[1, 2]	[2, 1] [-2,-1]	[1, -2] [-1, 2]	[-1,-2]	[2, -1] [-2, 1]
	$K^{B_n}_{(\substack{\square\\\square},\bullet)}$	$K^{B_n}_{(\square\square,\bullet)}$	$K^{B_n}_{(\square,\square)}$	$K^{B_n}_{(\bullet,\substack{\square\\\square})}$	$K^{B_n}_{(\bullet,\square\square)}$
$\mathcal{I}^{(\substack{\square\\\square},\bullet)} = \chi^{(\square,\bullet)} \otimes \chi^{(\square,\bullet)} \uparrow^{B_2}_{B_1\times B_1}$	2	0	2	2	0
$\mathcal{I}^{(\square\square,\bullet)} = \chi^{(\square\square,\bullet)} \uparrow^{B_2}_{B_2}$	1	1	1	1	1
$\mathcal{I}^{(\square,\square)} = \chi^{(\square,\bullet)} \otimes \chi^{(\bullet,\square)} \uparrow^{B_2}_{B_1\times B_1}$	2	0	0	-2	0
$\mathcal{I}^{(\bullet,\substack{\square\\\square})} = \chi^{(\bullet,\square)} \otimes \chi^{(\bullet,\square)} \uparrow^{B_2}_{B_1\times B_1}$	2	0	-2	2	0
$\mathcal{I}^{(\bullet,\square\square)} = \chi^{(\bullet,\square\square)} \uparrow^{B_2}_{B_2}$	1	1	-1	1	-1

Table 1.9: Induced trivial characters in representations of B_2

Example 1.113. Induced Character table of $\mathbf{B_3}$

Problem 1.114. Similar computations will generate the induced character table of B_3 as shown below.

Induced Trivial Characters of B_2	Conjugacy Classes									
	$K^{B_n}_{(\substack{\square\\\square\\\square},\bullet)}$	$K^{B_n}_{(\substack{\square\\\square\square},\bullet)}$	$K^{B_n}_{(\square\square\square,\bullet)}$	$K^{B_n}_{(\substack{\square\\\square},\square)}$	$K^{B_n}_{(\square\square,\square)}$	$K^{B_n}_{(\square,\substack{\square\\\square})}$	$K^{B_n}_{(\square,\square\square)}$	$K^{B_n}_{(\bullet,\substack{\square\\\square\\\square})}$	$K^{B_n}_{(\bullet,\substack{\square\\\square\square})}$	$K^{B_n}_{(\bullet,\square\square\square)}$
$\mathcal{I}^{(\substack{\square\\\square\\\square},\bullet)} = \chi^{(\square,\bullet)} \otimes \chi^{(\square,\bullet)} \otimes \chi^{(\square,\bullet)} \uparrow^{B_3}_{B_1\times B_1\times B_1}$	6	0	0	6	0	6	0	6	0	0
$\mathcal{I}^{(\substack{\square\\\square\square},\bullet)} = \chi^{(\square\square,\bullet)}_{B_2} \otimes \chi^{(\square,\bullet)}_{B_1} \uparrow^{B_3}_{B_2\times B_1}$	3	1	0	3	1	3	1	3	1	0
$\mathcal{I}^{(\square\square\square,\bullet)} = \chi^{(\square\square\square,\bullet)}_{B_3} \uparrow^{B_3}_{B_3}$	1	1	1	1	1	1	1	1	1	1
$\mathcal{I}^{(\substack{\square\\\square},\square)} = \chi^{(\square,\bullet)} \otimes \chi^{(\square,\bullet)} \otimes \chi^{(\bullet,\square)} \uparrow^{B_3}_{B_1\times B_1\times B_1}$	6	0	0	2	0	-2	0	-6	0	0
$\mathcal{I}^{(\square\square,\square)} = \chi^{(\square\square,\bullet)}_{B_2} \otimes \chi^{(\bullet,\square)}_{B_1} \uparrow^{B_3}_{B_2\times B_1}$	3	1	0	1	-1	-1	1	-3	-1	0
$\mathcal{I}^{(\square,\substack{\square\\\square})} = \chi^{(\square,\bullet)} \otimes \chi^{(\bullet,\square)} \otimes \chi^{(\bullet,\square)} \uparrow^{B_3}_{B_1\times B_1\times B_1}$	6	0	0	-2	0	-2	0	6	0	0
$\mathcal{I}^{(\square,\square\square)} = \chi^{(\bullet,\square\square)}_{B_2} \otimes \chi^{(\square,\bullet)}_{B_1} \uparrow^{B_3}_{B_2\times B_1}$	3	1	0	-1	1	-1	-1	3	-1	0
$\mathcal{I}^{(\bullet,\substack{\square\\\square\\\square})} = \chi^{(\bullet,\square)} \otimes \chi^{(\bullet,\square)} \otimes \chi^{(\bullet,\square)} \uparrow^{B_3}_{B_1\times B_1\times B_1}$	6	0	0	-6	0	6	0	-6	0	0
$\mathcal{I}^{(\bullet,\substack{\square\\\square\square})} = \chi^{(\bullet,\square\square)}_{B_2} \otimes \chi^{(\bullet,\square)}_{B_1} \uparrow^{B_3}_{B_2\times B_1}$	3	1	0	-3	-1	3	-1	-3	1	0
$\mathcal{I}^{(\bullet,\square\square\square)} = \chi^{(\bullet,\square\square\square)}_{B_3} \uparrow^{B_3}_{B_3}$	1	1	1	-1	-1	1	-1	-1	1	-1

Table 1.10: Induced character table of B_3

1.5 B_n- Frobenius Characteristic Map

As it is discussed before, characters are invariant under the operation of conjugation, i.e. they are constant on conjugacy classes. They are class functions that map a group to a complex number. To make the connection between the character of a representation of a group to the symmetric functions, we will introduce the Frobenius characteristic map.

Frobenius Characteristic Map Type A (Symmetric Group S_n)

To begin the study of Frobenius characteristic map, let's introduce the following notation.

- $R(G) = \bigoplus_{n\geq 0} R^n(G)$: the set of all class function on G.
- $Irr(S_n)$: is the set of irreducible characters of S_n.
- $R^n = R(S_n)$: the space of class functions on S_n, i.e. $R^n = \mathbb{C}\text{-Span}\{\chi \mid \chi \in Irr(S_n)\}$.
- χ^λ_μ: is the character value indexed by partition λ at conjugacy class K_μ.

The space of class functions on S_n is isomorphic to Λ^n, the symmetric functions of degree n. In fact their dimensions are equal to the number of partition of n, i.e. $dim\, R^n = dim\, \Lambda^n = p(n)$ for $p(n)$ the number of partitions n. Recall Theorem 1.103 that the irreducible characters on S_n form an orthonormal basis with respect to the inner product defined in Definition 1.102 i.e. given χ and ψ the irreducible characters of the group G, we have

$$\langle \chi, \psi \rangle_G = \delta_{\chi\psi}$$

Definition 1.115. [Sagan]Define the Hall inner product on Λ^n by

$$\langle s_\lambda, s_\mu \rangle_{\Lambda^n} = \delta_{\lambda\mu}$$

Equivalently, s_λ is an orthonormal basis in Λ^n. It may also be given in power sum basis as

$$\left\langle p_\lambda, \frac{p_\mu}{z_\mu} \right\rangle_{\Lambda^n} = \delta_{\lambda\mu}$$

Definition 1.116. [Sagan] Given χ^λ the irreducible character indexed by partition λ, define Schur function s_λ in power sum basis in Λ^n by the generating function

$$s_\lambda = \sum_{\mu\vdash n} \chi^\lambda_\mu \frac{p_\mu}{z_\mu}$$

Remark 1.117. Combining the two preceding definitions, the irreducible character of the symmetric group S_n on conjugacy class K_μ can be evaluated by the scalar inner product

$$\langle s_\lambda, p_\mu \rangle_{\Lambda^n} = \chi^\lambda_\mu$$

Definition 1.118. The Frobenius characteristic map $\mathscr{F}_{S_n}$ defined by the map

$$\begin{array}{rccc} \mathscr{F}_{S_n}: & R^n & \longrightarrow & \Lambda^n \\ & \mathbf{1}_\mu & \longmapsto & \dfrac{p_\mu}{z_\mu} \end{array}$$

Equivalently, since $\chi^\lambda = \sum\limits_{\mu \vdash n} \chi^\lambda_\mu \mathbf{1}_\mu$ we have

$$\mathscr{F}_{S_n}(\chi^\lambda) = \sum_{\mu \vdash n} \chi^\lambda_\mu \frac{p_\mu}{z_\mu}$$

where χ^λ_μ is the value of χ^λ on conjugacy class K_μ. The map $\mathscr{F}_{S_n}$ is linear and preserves inner products $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Since

$$\begin{aligned} \left\langle \chi^\lambda, \chi^\mu \right\rangle_{S_n} &= \left\langle \mathscr{F}_{S_n}(\chi^\lambda), \mathscr{F}_{S_n}(\chi^\mu) \right\rangle_{\Lambda^n} \\ &= \langle s_\lambda, s_\mu \rangle_{\Lambda^n} \\ &= \delta_{\lambda\mu} \end{aligned}$$

Remark 1.119. Note that the Frobenius characteristic map $\mathscr{F}_{S_n}$ evaluated at χ^λ is $\mathscr{F}_{S_n}(\chi^\lambda) = s_\lambda$. Given the permutation $\sigma \in S_n$ the Frobenius characteristic map using cycle type of σ is

$$\mathscr{F}_{S_n}(\chi^\lambda) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma) p_{cyc(\sigma)}$$

Proposition 1.120. *[Sagan]The map $\mathscr{F}_{S_n}$ is an isometry between R^n and Λ^n .*

The Frobenius characteristic map $\mathscr{F} = \bigoplus\limits_{n \geq 0} \mathscr{F}_{S_n}$ is used as a tool in computation between the isomorphism of the set of class functions $R = \bigoplus\limits_{n \geq 0} R^n$ and the symmetric function $\Lambda = \bigoplus\limits_{n \geq 0} \Lambda^n$ i.e.

$$R \xrightarrow[\mathscr{F}]{\cong} \Lambda$$

On the other hand, the ring of symmetric functions Λ is a vector space and has a structure of graded algebra $\Lambda = \bigoplus\limits_{n \geq 0} \Lambda^n$, where Λ^n is spanned by monomials m_λ of degree n, i.e. $\Lambda^n = \mathbb{C}\text{-Span}\{m_\lambda \mid \lambda \vdash n\}$. The ring Λ is closed under product therefore $f \in \Lambda^n$ and $g \in \Lambda^m$ implies $fg \in \Lambda^{n+m}$. As the consequence of the isomorphism $R \cong \Lambda$, we will use the idea of induction on characters to define product on R. This is to say if χ and ψ are two characters of S_n and S_m respectively, in order to have a character in S_{n+m} define a product on R by $\chi \cdot \psi = \chi \otimes \psi \uparrow^{S_{n+m}}_{S_n \times S_m}$.

Theorem 1.121. *[Sagan] The map $\mathscr{F} : R \longrightarrow \Lambda$ is an isomorphism of algebras.*

Corollary 1.122. *For $n \geq 0$ the product of Frobenius characteristic images are Frobenius image of the character of the induced tensor product representation. That is for S_r and S_{n-r} characters χ_{S_r} and $\psi_{S_{n-r}}$,*

$$\mathscr{F}_{S_n}(\chi_{S_r} \otimes \psi_{S_{n-r}} \uparrow_{S_r \times S_{n-r}}^{S_n}) = \mathscr{F}_{S_r}(\chi_{S_r}) \cdot \mathscr{F}_{S_{n-r}}(\psi_{S_{n-r}})$$

Induced Trial Character and Frobenius Characteristic Map. The Frobenius characteristic image of the trivial character is the complete homogeneous symmetric function. Denote the trivial character as the class function such that $\chi^T(\sigma) = 1$ for all $\sigma \in S_n$ then

$$\mathscr{F}_{S_n}(\chi^T) = h_n$$

Let $\lambda \vdash n$ and denote $\mathcal{I}^\lambda(\nu)$ be the value of the character $\chi_{\lambda_1}^T \otimes \cdots \otimes \chi_{\lambda_l}^T \uparrow_{S_{\lambda_1} \times \cdots \times S_{\lambda_l}}^{S_n}$. Since

$$\begin{aligned}
\mathscr{F}_{S_n}(\mathcal{I}^\lambda) &= \mathscr{F}_{S_n}(\chi^T \uparrow_{S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_r}}^{S_n}) \\
&= \mathscr{F}_{S_{\lambda_1}}(\chi_{\lambda_1}^T) \cdots \mathscr{F}_{S_{\lambda_l}}(\chi_{\lambda_l}^T) \\
&= h_{\lambda_1} \cdots h_{\lambda_l} \\
&= h_\lambda
\end{aligned}$$

The right hand side of the third line in the above equation is product in symmetric functions for example given $n = 3$ we have

- ◇ $\mathscr{F}_{S_3}(\chi^T \uparrow_{S_3}^{S_3}) = h_3$
- ◇ $\mathscr{F}_{S_3}(\chi^T \uparrow_{S_2 \times S_1}^{S_3}) = h_{21}$
- ◇ $\mathscr{F}_{S_3}(\chi^T \uparrow_{S_1 \times S_1 \times S_1}^{S_3}) = h_{111}$

Remark 1.123. Induced Trivial Character is given by $\mathcal{I}^\lambda(\nu) = \langle h_\lambda, p_\nu \rangle$.

B_n-Frobenius Characteristic Map

The material in this section can be found in [Macdonald, Stembridge, Stembridge2, Beck, BrBg, MRW].

Conjugacy classes of group B_n. A pair of partitions denoted by $(\gamma, \nu) \vdash n$ and its length is defined as $|(\gamma, \nu)| := |\gamma| + |\nu| = n$. The conjugacy classes of the hyperoctahedral group B_n indexed by a pair of partitions $(\gamma, \nu) \vdash n$ namely $K_{(\gamma,\nu)}^{B_n}$ consists of all signed permutations such that

- ◇ parts of γ are the cycles with even number of negative signs, and
- ◇ parts of ν are the cycles with an odd number of negative signs.

We can also refer to these cycles of an element of B_n as the cycles of the underlying permutation matrix. A cycle will be called positive or negative if the number of -1's in the entries of the cycle of the matrix is even or odd respectively. The class (γ, ν) consists of those elements in B_n whose

- positive cycles have lengths of $\lambda_1, \lambda_2, \ldots, \lambda_l$,
- negative cycles have lengths of $\mu_1, \mu_2, \ldots, \mu_k$.

Irreducible representations of B_n. In 1987, John Stembridge wrote his unpublished manuscript 'The ordinary representations of B_n' [Stembridge] that includes fundamentals of the representation and character of group B_n which later on he presented a more detail version of this topic in [Stembridge2]. In the following, we will briefly go over a few topics which we will be using in our work. First, we recall 1.3 that the group B_n is generated by τ, $\sigma_1, \cdots, \sigma_{n-1}$ satisfying relations

1. $\sigma_i^2 = \tau^2 = 1$;
2. $(\sigma_i \sigma_{i+1})^3 = 1$;
3. $(\sigma_i \tau)^2 = (\sigma_i \sigma_j)^2 = 1 \qquad$ where $i \neq n-1, |i-j| \geqslant 2$;
4. $(\tau \sigma_{n-1})^4 = 1$.

Now, let $L_n = Hom(B_n, \mathbb{C}^*)$ be the abelian group of linear characters of group B_n. This group has two generators ε and δ defined by

$$\begin{aligned} \varepsilon(\sigma_i) &= -1 \qquad \text{and} \qquad & \delta(\sigma_i) &= +1 \\ \varepsilon(\tau) &= +1 & \delta(\tau) &= -1 \end{aligned}$$

therefore we can conclude that

$$L_n \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

The irreducible representations of group B_n are indexed by pairs of partitions or ordered pairs of Young diagrams (λ, μ) of total size n namely $X^{(\lambda,\mu)}$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. Moreover, we denoted the corresponding irreducible character of this representation by $\chi^{(\lambda,\mu)}$. Now, given $|\lambda| + |\mu| = n$ we define

$$X^{(\lambda,\mu)} := \left(X^{(\lambda,\bullet)} \otimes X^{(\bullet,\mu)} \right) \uparrow_{B_{|\lambda|} \times B_{|\mu|}}^{Bn}$$

where

- $X^{(\lambda,\bullet)}$ can be regarded as the extension of X^{λ} from S_n to B_n when reflection t is acting trivially (this is by setting $t \longrightarrow 1$),
- $X^{(\bullet,\mu)}$ is set equivalent to $\delta \otimes X^{(\mu,\bullet)}$.

This can be translated to the irreducible characters by $\chi^{(\lambda,\mu)} := \left(\chi^{(\lambda,\bullet)} \otimes \chi^{(\bullet,\mu)} \right) \uparrow_{B_{|\lambda|} \times B_{|\mu|}}^{Bn}$.

In Theorem 1.135 we will present an explicit formula for the character of the hyperoctahedral group in terms of Schur functions [Stembridge]. In addition, in Corollary 1.142 the Frobenius characteristic image of the induced character of group B_n is presented as well in [Beck].

Frobenius Characteristic Map of B_n. The Frobenius characteristic map for group B_n is given in the following.

Definition 1.124. Let $\Lambda_B = \bigoplus_{n\geq 0} \Lambda_{B_n}$ where Λ_{B_n} is the space

$$\Lambda_{B_n} = \Lambda_{B_n}[X,Y] \coloneqq \bigoplus_{k=0}^{n} \Lambda^k[X] \otimes \Lambda^{n-k}[Y]$$

and $\Lambda^k[X]$ denotes the space of symmetric functions of degree k in the variables $X \coloneqq x_1 + x_2 + \cdots$. Now let $R(B_n)$ be the set of class functions of group B_n over the complex numbers and let $K^{B_n}_{(\gamma,\nu)}$ be the conjugacy class of B_n indexed by the pair of partitions $(\gamma,\nu) \vdash n$. The basis of class function as indicator function is given by

$$\mathbf{1}^{B_n}_{(\gamma,\nu)}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in K^{B_n}_{(\gamma,\nu)} \\ 0 & \text{otherwise} \end{cases}$$

The $\mathbf{B_n}$**-Frobenius characteristic map** $\mathscr{F}_{B_n}$ on the class functions of the hyperoctahedral group denoted by $R(B_n)$ is defined by

$$\begin{aligned} \mathscr{F}_{B_n}: \quad R(B_n) &\longrightarrow \quad \Lambda_{B_n}[X,Y] \\ \mathbf{1}^{B_n}_{(\gamma,\nu)} &\longmapsto \frac{1}{z_\gamma} p_\gamma[X] \frac{1}{z_\nu} p_\nu[Y] \end{aligned}$$

where $z_\lambda \coloneqq 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!$ for $\lambda = (1^{m_1} 2^{m_2} \ldots n^{m_n})$.

Remark 1.125. When group B_n acts on conjugacy classes by conjugation, the stabilizer of elements in conjugacy classes will also be the centralizer of these elements in B_n. The order of centralizer in B_n is $2^{l(\gamma)} 2^{l(\nu)} z_\gamma z_\nu$ that has factors of $2^{l(\gamma)}$ and $2^{l(\nu)}$. Here in definition of $\mathscr{F}_{B_n}$ these factors are omitted in order to have a simpler image of $\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) = s_\lambda[X+Y] s_\mu[X-Y]$. In [Macdonald] the definition of B_n-Frobenius characteristic map includes factors $\frac{1}{2^{l(\gamma)}}$ and $\frac{1}{2^{l(\nu)}}$ and which would have implied $\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) = s_\lambda[\frac{X+Y}{2}] s_\mu[\frac{X-Y}{2}]$. One can choose either of these two definitions as they are isomorphic. The B_n-character inner product is given in the following.

Lemma 1.126. *The B_n-character inner product is given by*

$$\left\langle \mathbf{1}^{B_n}_{(\gamma,\nu)}, \mathbf{1}^{B_n}_{(\alpha,\beta)} \right\rangle_{B_n} = \frac{1}{2^{l(\gamma)} z_\gamma} \frac{1}{2^{l(\nu)} z_\nu} \delta_{\gamma\alpha} \delta_{\nu\beta}.$$

Proof. By Proposition 1.9.2 in [Sagan] given a group G, the inner product of χ and ψ is given by

$$\langle \chi, \psi \rangle_G = \frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1})$$

Moreover, in type B (and in type A) we have $\psi(g^{-1}) = \psi(g)$ as g and g^{-1} have the same cycle-type as they are in the same conjugacy class, hence

$$\langle \chi, \psi \rangle_{B_n} = \frac{1}{2^n n!}\sum_{\sigma\in B_n}\chi(\sigma)\psi(\sigma)$$

For $\chi = \mathbf{1}^{B_n}_{(\gamma,\nu)}$ and $\psi = \mathbf{1}^{B_n}_{(\alpha,\beta)}$ we have

$$\left\langle \mathbf{1}^{B_n}_{(\gamma,\nu)}, \mathbf{1}^{B_n}_{(\alpha,\beta)} \right\rangle_{B_n} = \frac{1}{2^n n!}\sum_{\sigma\in B_n}\mathbf{1}^{B_n}_{(\gamma,\nu)}(\sigma)\mathbf{1}^{B_n}_{(\alpha,\beta)}(\sigma)$$

This value is $\left\langle \mathbf{1}^{B_n}_{(\gamma,\nu)}, \mathbf{1}^{B_n}_{(\alpha,\beta)} \right\rangle_{B_n} = 0$ when $\gamma \neq \alpha$ or $\nu \neq \beta$ otherwise in the case where $\gamma = \alpha$ and $\nu = \beta$ given that (γ, ν)-class has $\dfrac{2^n n!}{2^{l(\gamma)}2^{l(\nu)}z_\gamma z_\nu}$ elements and we have

$$\begin{aligned}\left\langle \mathbf{1}^{B_n}_{(\gamma,\nu)}, \mathbf{1}^{B_n}_{(\alpha,\beta)} \right\rangle_{B_n} &= \frac{1}{2^n n!}\left|K^{B_n}_{(\gamma,\nu)}\right| \\ &= \frac{1}{2^n n!}\frac{2^n n!}{2^{l(\gamma)}2^{l(\nu)}z_\gamma z_\nu} \\ &= \frac{1}{2^{l(\gamma)}2^{l(\nu)}z_\gamma z_\nu}\end{aligned}$$

The proof is complete. □

We see that the Frobenius characteristic map

$$\mathscr{F}_{B_n} : \ R(B_n) \longrightarrow \bigoplus_{k=0}^{n} \Lambda^k[X] \otimes \Lambda^{n-k}[Y]$$

is a vector space isomorphism so we will define the following inner product on space $\Lambda \otimes \Lambda$ that by transferring the class function structure from $R(B_n)$. The power sums in two alphabets $p_\gamma[X]p_\nu[Y]$ are orthogonal in this inner product.

Definition 1.127. The character metric $\langle \cdot, \cdot \rangle_{B_n}$ corresponds to the metric $\langle \cdot, \cdot \rangle_{\Lambda\otimes\Lambda}$ where

$$\langle p_\gamma[X]p_\nu[Y], p_\alpha[X]p_\beta[Y] \rangle_{\Lambda\otimes\Lambda} = \frac{1}{2^{l(\gamma)}}z_\gamma \frac{1}{2^{l(\nu)}}z_\nu \delta_{\gamma\alpha}\delta_{\nu\beta}.$$

Remark 1.128. The generating function of the inner product $\langle \cdot, \cdot \rangle_{\Lambda \otimes \Lambda}$ is

$$\sum_{\lambda,\mu} \frac{1}{z_\lambda} 2^{l(\lambda)} \cdot \frac{1}{z_\mu} 2^{l(\mu)} p_\lambda[X] p_\mu[Y] p_\lambda[V] p_\mu[W] = \left(\sum_\lambda \frac{1}{z_\lambda} 2^{l(\lambda)} p_\lambda[x_i v_j] \right) \cdot \left(\sum_\mu \frac{1}{z_\mu} 2^{l(\mu)} p_\mu[y_i w_j] \right)$$
$$= \prod_{i,j} \frac{1}{(1 - x_i v_j)^2} \cdot \frac{1}{(1 - y_i w_j)^2}$$

Note that $\sum_\lambda \frac{1}{z_\lambda} p_\lambda[x] p_\lambda[v] = \prod_{i,j} \frac{1}{1 - x_i v_j}$ and the proof is complete.

Linear Characters Type B. There are four linear characters of the hyperoctahedral group B_n. We will assign values to each of these linear characters on B_n to be: 1, $(-1)^{n-l(\gamma)-l(\nu)}$, $(-1)^{l(\nu)}$ and $1^{n-l(\gamma)}$, respectively.

Character Type	Notation	Formula for $\sigma \in K^{B_n}_{(\gamma,\nu)}$	Generating Function
The trivial character of B_n	1	$1(\sigma) = 1$	$\sum t^n \mathscr{F}_{B_n}(1_n) = \prod \frac{1}{1 - x_i t} \cdot \frac{1}{1 - y_i t}$
The sign character of B_n	ε	$\varepsilon(\sigma) = (-1)^{n-l(\gamma)-l(\nu)}$	$\sum t^n \mathscr{F}_{B_n}(\varepsilon_n) = \prod (1 + x_i t)(1 + y_i t)$
The parity character of B_n	δ	$\delta(\sigma) = (-1)^{l(\nu)}$	$\sum t^n \mathscr{F}_{B_n}(\delta_n) = \prod \frac{1 - y_i t}{1 - x_i t}$
The determinant character of B_n	$\delta\varepsilon$	$\delta(\sigma)\varepsilon(\sigma) = 1^{n-l(\gamma)}$	$\sum t^n \mathscr{F}_{B_n}(\delta_n \varepsilon_n) = \prod \frac{1 + x_i t}{1 + y_i t}$

Table 1.11: Linear characters of B_n.

Now, to see this, let $X(\sigma) = [a_\sigma]$ be a 1-dimensional (linear) representation then we have $X(\sigma_i^2) = [a_{\sigma_i}]^2 = [a_{\sigma_i^2}] = 1$ but also $X(\sigma_i^2) = [a^2_{\sigma_i}]$ then it must be that $a_{\sigma_i} = \pm 1$. Similarly $a_\tau = \pm 1$ and we can have four possible cases:

Linear Character	Linear Character in Cycle-Type $(\sigma) = (\gamma, \nu)$
$a_{\sigma_i} = 1$ and $a_\tau = 1$	1
$a_{\sigma_i} = -1$ and $a_\tau = 1$	$(-1)^{n-l(\gamma)-l(\nu)}$
$a_{\sigma_i} = 1$ and $a_\tau = -1$	$(-1)^{l(\nu)}$
$a_{\sigma_i} = -1$ and $a_\tau = -1$	$1^{n-l(\gamma)}$

The B_n-Frobenius characteristic images of the linear characters in plethystic notation are as below

Character Type	Notation	Frobenius Characteristic Image
The trivial character of B_n	1	$\mathscr{F}_{B_n}(1) = h_n[X+Y]$
The sign character of B_n	ε	$\mathscr{F}_{B_n}(\varepsilon) = e_n[X+Y]$
The parity character of B_n	δ	$\mathscr{F}_{B_n}(\delta) = h_n[X-Y]$
The determinant character of B_n	$\delta\varepsilon$	$\mathscr{F}_{B_n}(\delta\varepsilon) = e_n[X-Y]$

Table 1.12: The B_n-Frobenius characteristic images of the linear characters

Theorem 1.129. *The Frobenius Characteristic map $\mathscr{F}_{B_n} : R(B_n) \longrightarrow \Lambda_{B_n}[X,Y]$ is an isomorphism of algebras.*

Proof. We need to show $\mathscr{F}_{B_n}\left((\chi \otimes \psi)\uparrow_{B_k \times B_{n-k}}^{Bn}\right) = \mathscr{F}_{B_k}(\chi)\,\mathscr{F}_{B_{n-k}}(\psi)$ where $\chi \in R(B_k)$ and $\psi \in R(B_{n-k})$. Let $\overline{p} \in \Lambda \otimes \Lambda$ such that

$$\overline{p}_n(\pi) = 2^{l(\gamma)+l(\nu)} p_\gamma[X] p_\nu[Y]$$

where $\pi \in K_{(\gamma,\nu)}^{B_n}$ and $|\gamma| = k$ and $|\nu| = n-k$ we have

$$\begin{aligned}
\mathscr{F}_{B_n}\left((\chi \otimes \psi)\uparrow_{B_k \times B_{n-k}}^{Bn}\right) &= \frac{1}{2^n n!} \sum_{\pi \in B_n} (\chi \otimes \psi)\uparrow_{B_k \times B_{n-k}}^{Bn}(\pi)\overline{p}_n(\pi) \\
&= \left\langle (\chi \otimes \psi)\uparrow_{B_k \times B_{n-k}}^{Bn}, \overline{p}_n \right\rangle_{B_k \times B_{n-k}} \\
&= \left\langle \chi \otimes \psi, \overline{p}\downarrow_{B_k \times B_{n-k}}^{Bn} \right\rangle_{B_k \times B_{n-k}} \\
&= \left\langle \chi \otimes \psi, \overline{p}_k \otimes \overline{p}_{n-k} \right\rangle_{B_k \times B_{n-k}} \\
&= \left\langle \chi, \overline{p}_k \right\rangle_{B_k} \left\langle \psi, \overline{p}_{n-k} \right\rangle_{B_{n-k}} \\
&= \left(\frac{1}{2^k k!} \sum_{\pi \in B_k} \chi(\pi)\overline{p}_k(\pi) \right) \left(\frac{1}{2^{n-k}(n-k)!} \sum_{\pi \in B_{n-k}} \psi(\pi)\overline{p}_{n-k}(\pi) \right) \\
&= \mathscr{F}_{B_k}(\chi)\,\mathscr{F}_{B_{n-k}}(\psi)
\end{aligned}$$

The third equality is by Frobenius reprocity. The proof is complete. □

Lemma 1.130. *[Stembridge] Given χ^λ to be any irreducible character of S_n. Since $B_n \longrightarrow S_n$ (as $\tau \longrightarrow 1$), we may also regard χ^λ as a B_n-character. We have*

$$\mathscr{F}_{B_n}(\chi^\lambda) = s_\lambda[X+Y].$$

Proof. We have

$$\begin{aligned}\sum_{\lambda}\mathscr{F}_{B_n}(\chi^\lambda)s_\lambda[Z] &= \sum_{\lambda,\gamma,\nu}\chi^\lambda_{(\gamma,\nu)}\frac{1}{z_\gamma}p_\gamma[X]\frac{1}{z_\nu}p_\nu[Y]s_\lambda[Z] \\ &= \sum_{\gamma,\nu}p_{(\gamma,\nu)}[Z]\frac{1}{z_\gamma}p_\gamma[X]\frac{1}{z_\nu}p_\nu[Y] \\ &= \prod_{i,j}\frac{1}{1-x_iz_j}\cdot\frac{1}{1-y_iz_j}\end{aligned}$$

Extract the coefficient of $s_\lambda[Z]$ and proof is complete. □

Definition 1.131. Define a ring homomorphism $\Theta : id \otimes S$ on $\Lambda^k[X] \otimes \Lambda^{n-k}[Y]$ as the following

Θ on Power Sum Basis	Θ on Complete Homogenous Basis	Θ on Schur Basis
$\Theta p_r[X] = p_r[X]$	$\Theta h_r[X] = h_r[X]$	$\Theta s_\lambda[Y] = s_{\lambda'}[Y]$
$\Theta p_r[Y] = -p_r[Y]$	$\Theta h_r[Y] = (-1)^r e_r[X]$	

hence, given $f \in \Lambda$ we can write $\Theta f[X] = id$ and $\Theta f[Y] = \omega f[-Y]$.

Lemma 1.132. *[Stembridge] If $f[X,Y] = \mathscr{F}_{B_n}(\chi)$, then $\mathscr{F}_{B_n}(\delta\chi) = \Theta f[X,Y]$.*

Proof. By linearity we only need to check for $\chi = 1_{(\gamma,\nu)}$ then

$$\begin{aligned}f[X,Y] &= \mathscr{F}_{B_n}(\chi) \\ &= \frac{1}{z_\gamma}p_\gamma[X]\frac{1}{z_\nu}p_\nu[Y]\end{aligned}$$

Now we have

$$\begin{aligned}\mathscr{F}_{B_n}(\delta\chi) &= \frac{1}{z_\gamma}p_\gamma[X]\frac{(-1)^{l(\nu)}}{z_\nu}p_\nu[Y] \\ &= \Theta f[X,Y]\end{aligned}$$

this is true as by definition $\delta(\sigma) = (-1)^{l(\nu)}$ for $\sigma \in K^{B_n}_{(\gamma,\nu)}$ and the proof is complete. □

Definition 1.133. Define $s_\lambda[X-Y]$ as

$$\begin{aligned}s_\lambda[X-Y] &= \Theta s_\lambda[X,Y] \\ &= \mathscr{F}_{B_n}(\delta\chi^\lambda) \\ &= \sum_\mu s_\mu[X]\cdot(-1)^{|\lambda/\mu|}s_{\lambda/\mu}[Y]\end{aligned}$$

Remark 1.134. We can explain Definition 1.133 by

$$\sum_{\lambda} s_\lambda[X-Y]s_\lambda[Z] = \Theta\left(\prod_{i,j}\frac{1}{1-x_iz_j}\cdot\frac{1}{1-y_iz_j}\right)$$
$$= \prod_{i,j}\frac{1-y_iz_j}{1-x_iz_j}.$$

Theorem 1.135. *[Stembridge] The irreducible characters of B_n are $\chi^{(\lambda,\mu)} \coloneqq \left(\chi^{(\lambda,\bullet)}\otimes\chi^{(\bullet,\mu)}\right)\uparrow_{B_{|\lambda|}\times B_{|\mu|}}^{Bn}$ where $|\lambda|+|\mu|=n$ and their Frobenius characteristics image is*

$$\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) = S^{B_n}_{(\lambda,\mu)}[X,Y] \coloneqq s_\lambda[X+Y]s_\mu[X-Y]$$

where

$$s_\lambda[X+Y] = \sum_{\gamma\subset\lambda} s_\gamma[X]s_{\lambda/\gamma}[Y]$$
$$s_\mu[X-Y] = \sum_{\nu\subset\mu} s_\nu[X](-1)^{|\mu/\nu|}s_{(\mu/\nu)'}[Y]$$

and that the prime indicates the conjugate partition.

Proof. Note that by Lemma 1.130 we have

$$\mathscr{F}_{B_n}(\chi^{(\lambda,\bullet)}) = s_\lambda[X+Y]$$

and by Lemma 1.132 we have

$$\mathscr{F}_{B_n}(\chi^{(\bullet,\mu)}) = \mathscr{F}_{B_n}(\delta\chi^{(\mu,\bullet)})$$
$$= s_\mu[X-Y]$$

The functions $s_\lambda[X+Y]s_\mu[X-Y]$ are clearly characters, hence we only need to verify the orthogonality relations. Now consider

$$\sum_{\lambda,\mu} s_\lambda[X+Y]s_\lambda[V+W]s_\mu[X-Y]s_\mu[V-W]$$
$$= \left(\prod_{i,j}\frac{1}{1-x_iv_j}\cdot\frac{1}{1-y_iv_j}\cdot\frac{1}{1-x_iw_j}\cdot\frac{1}{1-y_iw_j}\right)\cdot\left(\Theta_V\prod_{i,j}\frac{1-y_iv_j}{1-x_iv_j}\cdot\frac{1-y_iw_j}{1-x_iw_j}\right)$$
$$= \prod_{i,j}\frac{1}{(1-x_iv_j)^2}\cdot\frac{1}{(1-y_iw_j)^2}$$

Therefore $s_\lambda[X+Y]s_\mu[X-Y]$ are by 1.127 orthogonal with respect to metric $\langle\cdot,\cdot\rangle_{\Lambda\otimes\Lambda}$. □

Corollary 1.136. *Irreducible Characters can also be evaluated by B_n-character inner product by*

$$\chi^{(\lambda,\mu)}_{(\nu,\gamma)} = \langle s_\lambda[X+Y]s_\mu[X-Y], p_\nu[X]p_\gamma[Y]\rangle .$$

Remark 1.137. The character at the conjugacy class of $K^{B_n}_{(\gamma,\upsilon)}$ and indexed by a pair of partitions is $\chi^{(\lambda,\mu)}_{(\gamma,\upsilon)}$ where $(\gamma,\nu) \vdash n$. By Theorem 1.135 we have shown that the Frobenius characteristic map on the irreducible character of group B_n is given by

$$\begin{aligned}\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) &:= \sum_{(\gamma,\nu)\vdash n} \chi^{(\lambda,\mu)}_{(\gamma,\nu)} \frac{p_\gamma[X]}{z_\gamma}\frac{p_\nu[Y]}{z_\nu} \\ &= s_\lambda[X+Y]s_\mu[X-Y].\end{aligned}$$

Remark 1.138. The maps $\chi \to \varepsilon\chi$ and $\chi \to \delta\chi$ define automorphisms of $\Lambda[X]\otimes\Lambda[Y]$ that is

$$\begin{array}{ccc} s_\lambda[X+Y] & \longrightarrow & s_{\lambda'}[X+Y] \\ s_\lambda[X-Y] & \underset{\varepsilon}{\longrightarrow} & s_{\lambda'}[X-Y] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} s_\lambda[X+Y] & \longrightarrow & \Theta s_\lambda[X+Y]{=}s_\lambda[X-Y] \\ s_\lambda[X-Y] & \underset{\delta}{\longrightarrow} & \Theta s_\lambda[X-Y]{=}s_\lambda[X+Y] \end{array}$$

Remark 1.139. The character expansion is a consequence of of ordinary linear representation, for example

Character Type	Notation	Indexing Linear Character in B_3
The trivial character of B_n	1	$(\square\square\square, \bullet)$
The sign character of B_n	ε	$(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \bullet)$
The parity character of B_n	δ	$(\bullet, \square\square\square)$
The determinant character of B_n	$\delta\varepsilon$	$(\bullet, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix})$

Induced Trivial Character and Character Table of B_n

In the introduction of [Beck] 'The combinatorics of symmetric functions and permutation enumeration of the hyperoctahedral group' in 1997, the author defined ten standard bases for the B_n-analogue of the symmetric functions. Any element of these bases can be the Frobenius characteristic image of a natural representation of B_n. One which has a very important use in this thesis and is called the induced trivial character basis of group B_n is described in Corollary 1.142.

Remark 1.140. Consider a basis of $\Lambda_{B_n} := \bigoplus_{k=0}^{n} \Lambda^k[X]\Lambda^{n-k}[Y]$ as the following given spanning set

$$\{h_\lambda[X+Y]h_\mu[X-Y]\}_{(\lambda,\mu)\vdash n}$$

This basis is the Frobenius characteristic image of a natural representation of B_n. Particularly, the Frobenius characteristic image of the induced trivial character of B_n as the plethysm of the symmetric functions

$$h_\lambda[X+Y]h_\mu[X-Y].$$

This is expressed as a corollary to Theorem 1.135.

Definition 1.141. Let $(\lambda,\mu)\vdash n$ and let $B_{|\lambda|}=B_{\lambda_1}\times\cdots\times B_{\lambda_k}$. Define

- I_λ: The trivial representation on $B_{|\lambda|}$, and
- Δ_λ: The parity representation on $B_{|\lambda|}$.

These are the restriction of the identity representation and parity representations from B_n to $B_{|\lambda|}$.

Corollary 1.142. *[Beck] Given I_λ and Δ_μ to be the trivial and parity representation on $B_{|\lambda|}$ and $B_{|\mu|}$ respectively and let $(I_\lambda\otimes\Delta_\mu)\uparrow^{Bn}_{B_{|\lambda|}\times B_{|\mu|}}$ be the induced representation and let $\mathcal{I}^{(\lambda,\mu)}:=(1_\lambda\otimes\delta_\mu)\uparrow^{Bn}_{B_{|\lambda|}\times B_{|\mu|}}$ be the induced character that is indexed by pair of partitions $(\lambda,\mu)\vdash n$. The Frobenius characteristic image is given by*

$$\mathscr{F}_{B_n}(\mathcal{I}^{(\lambda,\mu)})=h_\lambda[X+Y]h_\mu[X-Y].$$

Remark 1.143. The induced trivial character of B_n evaluated at conjugacy classes $K^{B_n}_{(\gamma,\nu)}$ can also be expressed using the B_n-character inner product

$$\begin{aligned}\mathcal{I}^{(\lambda,\mu)}(\gamma,\nu)&=\langle h_\lambda[X+Y]h_\mu[X-Y],p_\gamma[X]p_\nu[Y]\rangle\\&=\left\langle\mathscr{F}_{B_n}(\mathcal{I}^{(\lambda,\mu)}),p_\gamma[X]p_\nu[Y]\right\rangle\end{aligned}$$

Definition 1.144. The B_n-sum-difference complete homogeneous basis is denoted by

$$H^B_{(\lambda,\mu)}[X,Y]:=h_\lambda[X+Y]h_\mu[X-Y].$$

Given $h_r[-X]=S(h_r)[X]=(-1)^r\omega(h_r)[X]=(-1)^r e_r[X]$ the expansion on a single part r is given by

- $h_r[X+Y]=\sum_{i=0}^{r}h_i[X]h_{r-i}[Y]$
- $h_r[X-Y]=\sum_{i=0}^{r}(-1)^{r-i}h_i[X]e_{r-i}[Y]$

These expansions are homogenous symmetric function evaluated at a sum and difference of two alphabets respectively. Note that for a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{l(\lambda)})$ we have

$$h_\lambda[X \pm Y] = h_{\lambda_1}[X \pm Y]h_{\lambda_2}[X \pm Y] \cdots h_{\lambda_{l(\lambda)}}[X \pm Y].$$

Note that

$$\begin{aligned} h_r[X+Y] &= s_r[X+Y] \\ &= \sum_{\gamma \subseteq r} s_\gamma[X] s_{(r)/\gamma}[Y] \\ &= \sum_{i=0}^{r} h_i[X] h_{r-i}[Y] \end{aligned}$$

Example 1.145. Write the induced trivial character indexed by a pair of partitions$((1,1),\bullet) \vdash 2$ in terms of irreducible characters. To do this we may use $h_{11} = s_2 + s_{11}$ therefore

$$\begin{aligned} \mathcal{I}^{(11,\bullet)}(\gamma,\nu) &= \langle h_{11}[X+Y], p_\gamma[X]p_\nu[Y]\rangle \\ &= \langle s_{11}[X+Y] + s_2[X+Y], p_\gamma[X]p_\nu[Y]\rangle \\ &= \langle s_{11}[X+Y], p_\gamma[X]p_\nu[Y]\rangle + \langle s_2[X+Y],\, p_\gamma[X]p_\nu[Y]\rangle \\ &= \chi^{(11,\bullet)}(\nu,\gamma) + \chi^{(2,\bullet)}(\nu,\gamma) \end{aligned}$$

Frobenius Characteristic Map and Character Table of B_n. The Frobenius characteristic map on type B can be used to evaluate characters of hyperoctahedral group B_n. Recall

$$\begin{aligned} \mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) &:= \sum_{(\gamma,\nu)\vdash n} \chi^{(\lambda,\mu)}_{(\gamma,\nu)} \frac{p_\gamma[X]}{z_\gamma} \frac{p_\nu[Y]}{z_\nu} \\ &= s_\lambda[X+Y] s_\mu[X-Y]. \end{aligned}$$

where $\chi^{(\lambda,\mu)} \in R(B_n)$ the space of class functions on B_n. Using plethystic substitution

$$p_k \longrightarrow p_k \otimes 1 + 1 \otimes p_k$$

write $\Delta(p_k) \longrightarrow p_k[X+Y] = p_k[X] + p_k[Y]$. Note that in $s_\mu[X-Y]$ the antipode $p_k[X-Y] = p_k[X] - p_k[Y]$ (This is $S(p_k) = -p_k$)

Example 1.146. Evaluate character table of B_1.

The elements of group B_1 are $B_1 = [1], [-1]$. The rows are evaluated using the Frobenius characteristic map of type B as plethystic substitution that is explained above. By matching the coefficients in conjugacy classes the character table in generated.

	Irreducible Characters of B_1	$(\square, \bullet)$	$(\bullet, \square)$		Symmetric Function
Row 1	$\chi^{(\square,\bullet)}$	1	1	$\overset{\longleftrightarrow}{\mathscr{F}_{B_n}}$	$1.p_1[X] + 1.p_1[Y] = S^B_{(\square,\bullet)}$
Row 2	$\chi^{(\bullet,\square)}$	1	-1		$1.p_1[X] - 1.p_1[Y] = S^B_{(\bullet,\square)}$

Irreducible Characters of B_1	Conjugacy Classes of B_1	
	[1]	[-1]
	$K_{(\square,\bullet)}$	$K_{(\bullet,\square)}$
$\chi^{(\square,\bullet)}$	1	1
$\chi^{(\bullet,\square)}$	1	-1

Table 1.13: Character table of B_1

Example 1.147. Evaluate character table of B_2.

By he Jacobi-Trudi determinant $s_\lambda = det[h_{\lambda_{i-i+j}}]_{0\leq i,j\leq k}$ and $h_n = \sum\limits_{\lambda\vdash n} \dfrac{p_\lambda}{z_\lambda}$ we have

$$s_{\square} = \sum_{\lambda\vdash 1} \frac{p_\lambda}{z_\lambda} = p_1$$
$$s_{\square\square} = \sum_{\lambda\vdash 1} \frac{p_\lambda}{z_\lambda} = \frac{p_2}{z_2} + \frac{p_{11}}{z_{11}} = \frac{p_2}{2} + \frac{p_{11}}{2}$$
$$s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \sum_{\lambda\vdash 2} (-1)^{|\lambda|+l(\lambda)} \frac{p_\lambda}{z_\lambda} = (-1)^{2+1}\frac{p_2}{z_2} + (-1)^{2+2}\frac{p_{11}}{z_{11}} = -\frac{p_2}{2} + \frac{p_{11}}{2}$$

To compute $\chi^{(\lambda,\mu)}$, use Frobenius characteristic map and the plethystic substitution, we have:

Row 1: The coefficients of the following expansion in p basis correspond to the first row of the character table. Using substitutions

$$\begin{aligned}
S^B_{(\square\square,\bullet)}[X,Y] &= s_{(\square\square)}[X+Y]s_\bullet[X-Y] \\
&= \frac{p_2[X+Y]}{2} + \frac{(p_1[X+Y])^2}{2} \\
&= \frac{p_2[X]+p_2[Y]}{2} + \frac{(p_1[X]+p_1[Y])^2}{2} \\
&= \frac{p_{11}[X]}{2} + \frac{p_2[X]}{2} + \frac{p_1[X]p_1[Y]}{1} + \frac{p_{11}[Y]}{2} + \frac{p_2[Y]}{2}
\end{aligned}$$

The computations for the remaining rows are done is similar fashion.

Row 2:

$$\begin{aligned}
S^B_{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\bullet)}[X,Y] &= s_{(\begin{smallmatrix}\square\\\square\end{smallmatrix})}[X+Y]s_\bullet[X-Y] \\
&= -\frac{p_2[X]+p_2[Y]}{2} + \frac{(p_1[X]+p_1[Y])^2}{2} \\
&= \frac{p_{11}[X]}{2} - \frac{p_2[X]}{2} + \frac{p_1[X]p_1[Y]}{1} + \frac{p_{11}[Y]}{2} - \frac{p_2[Y]}{2}
\end{aligned}$$

Row 3:

$$\begin{aligned}
S^B_{(\square,\square)}[X,Y] &= s_{(\square)}[X+Y]s_{(\square)}[X-Y] \\
&= (p_1[X]+p_1[Y])(p_1[X]-p_1[Y]) \\
&= 2.\frac{p_{11}[X]}{2} - 2.\frac{p_{11}[Y]}{2}
\end{aligned}$$

Row 4:

$$\begin{aligned} S^B_{(\bullet,\square\!\square)}[X,Y] &= s_\bullet[X+Y]s_{(\square\!\square)}[X-Y] \\ &= \frac{p_2[X]-p_2[Y]}{2} + \frac{(p_1[X]-p_1[Y])^2}{2} \\ &= \frac{p_{11}[X]}{2} + \frac{p_2[X]}{2} - \frac{p_1[X]p_1[Y]}{1} + \frac{p_{11}[Y]}{2} - \frac{p_2[Y]}{2} \end{aligned}$$

Row 5:

$$\begin{aligned} S^B_{(\bullet,\substack{\square\\\square})} &= s_\bullet[X+Y]s_{(\substack{\square\\\square})}[X-Y] \\ &= -\frac{p_2[X]-p_2[Y]}{2} + \frac{(p_1[X]-p_1[Y])^2}{2} \\ &= \frac{p_{11}[X]}{2} - \frac{p_2[X]}{2} - \frac{p_1[X]p_1[Y]}{1} + \frac{p_{11}[Y]}{2} + \frac{p_2[Y]}{2} \end{aligned}$$

The character table of B_2 is

Irreducible Characters of B_2	Elements of B_2 in Conjugacy Classes				
	[1, 2]	[2, 1] [-2,-1]	[1, -2] [-1, 2]	[-1, -2]	[2, -1] [-2, 1]
	$K_{(\substack{\square\\\square},\bullet)}$	$K_{(\square\!\square,\bullet)}$	$K_{(\square,\square)}$	$K_{(\bullet,\substack{\square\\\square})}$	$K_{(\bullet,\square\!\square)}$
$\chi^{(\square\!\square,\bullet)}$	1	1	1	1	1
$\chi^{(\substack{\square\\\square},\bullet)}$	1	-1	1	1	-1
$\chi^{(\square,\square)}$	2	0	0	-2	0
$\chi^{(\bullet,\square\!\square)}$	1	1	-1	1	-1
$\chi^{(\bullet,\substack{\square\\\square})}$	1	-1	-1	1	1

Table 1.14: Character table of B_2

Example 1.148. Evaluate the character table of B_3.

Similar to the last example, we can calculate the table entries. For example in row 6:

$S^B_{(\ydiagram{1},\ydiagram{1,1})}$	$s_{(\ydiagram{1})}$	$s_{(\ydiagram{1,1})}$
	$p_1 \to p_1[X] + p_1[Y]$	$p_1 \to p_1[X] - p_1[Y]$
		$p_2 \to p_2[X] - p_2[Y]$

$$
\begin{aligned}
S^B_{(\ydiagram{1},\ydiagram{1,1})}[X,Y] &= s_{(\ydiagram{1})}[X+Y]s_{(\ydiagram{1,1})}[X-Y] \\
&= (p_1[X]+p_1[Y])(-\frac{p_2[X]-p_2[Y]}{2} + \frac{(p_1[X]-p_1[Y])^2}{2}) \\
&= 3\frac{p_1^3[X]}{6} - 1\frac{p_2[X]p_1[X]}{2} + 0\frac{p_3[X]}{3} - 1\frac{p_1^2[X]p_1[Y]}{2} - 1\frac{p_2[X]_1[Y]}{2} - \\
&\quad - 1\frac{p_1[X]p_1^2[Y]}{2} + 1\frac{p_1[X]p_2[Y]}{2} + 3\frac{p_1^3[Y]}{6} + 1\frac{p_2[Y]p_1[Y]}{2} + 0\frac{p_3[Y]}{3}
\end{aligned}
$$

The character table of B_3 can be presented as below.

B_3 Character Table	Cycle-Types of Conjugacy Classes of B_3									
	$K_{(\ydiagram{1,1,1},\bullet)}$	$K_{(\ydiagram{2,1},\bullet)}$	$K_{(\ydiagram{3},\bullet)}$	$K_{(\ydiagram{1,1},\ydiagram{1})}$	$K_{(\ydiagram{2},\ydiagram{1})}$	$K_{(\ydiagram{1},\ydiagram{1,1})}$	$K_{(\ydiagram{1},\ydiagram{2})}$	$K_{(\bullet,\ydiagram{1,1,1})}$	$K_{(\bullet,\ydiagram{2,1})}$	$K_{(\bullet,\ydiagram{3})}$
$\chi^{(\ydiagram{1,1,1},\bullet)}$	1	-1	1	1	-1	1	-1	1	-1	1
$\chi^{(\ydiagram{2,1},\bullet)}$	2	0	-1	2	0	2	0	2	0	-1
$\chi^{(\ydiagram{3},\bullet)}$	1	1	1	1	1	1	1	1	1	1
$\chi^{(\ydiagram{1,1},\ydiagram{1})}$	3	-1	0	1	1	-1	-1	-3	1	0
$\chi^{(\ydiagram{2},\ydiagram{1})}$	3	1	0	1	-1	-1	1	-3	-1	0
$\chi^{(\ydiagram{1},\ydiagram{1,1})}$	3	-1	0	-1	-1	-1	1	3	1	0
$\chi^{(\ydiagram{1},\ydiagram{2})}$	3	1	0	-1	1	-1	-1	3	-1	0
$\chi^{(\bullet,\ydiagram{1,1,1})}$	1	-1	1	-1	1	1	1	-1	-1	-1
$\chi^{(\bullet,\ydiagram{2,1})}$	2	0	-1	-2	0	2	0	-2	0	1
$\chi^{(\bullet,\ydiagram{3})}$	1	1	1	-1	-1	1	-1	-1	1	-1

Table 1.15: Character table of B_3

Eigenvalues of a Permutation Matrix.

Theorem 1.149. *Eigenvalues of a permutation matrix is the roots of unity.*

Proof. Let A be a permutation matrix, then the equation $A^{k+1} = A$ for some $k \in \mathbb{Z}$ always holds. To find the eigenvalues of A, let $Av = tv$. Left multiply the matrix A to this equation, we have $A(Av) = A(tv)$ therefore $A^2v = A(tv) = t(Av) = t(tv) = t^2v$. We can conclude that for $k \in \mathbb{Z}$, $A^{k+1}v = t^{k+1}v$ and as $A^{k+1} = A$, we have $Av = t^{k+1}v$. This implies $t^{k+1} = t$. Since zero is not a solution therefore: $t^k = 1$. (i.e. the roots of unity) □

Definition 1.150. For an integer $k \geq 0$, define

$$\Xi_k := 1, e^{\frac{2\pi i}{k}}, , e^{\frac{4\pi i}{k}}, e^{\frac{6\pi i}{}}, \ldots, e^{\frac{2(k-1)\pi i}{k}}$$

This notation represents the eigenvalues of a permutation matrix of a k-cycle.

Example 1.151. Eigenvalues of a Permutation Matrix. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A has eigenvalues $\{1, -1, i, -i\}$ that are solutions of 4th roots of unity $\lambda^4 - 1 = 0$. We can write

$$\Xi_4 := 1, i, -1, -i$$

Example 1.152. Eigenvalues of a Signed Permutation Matrix. Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B has eigenvalues that are solutions of $\lambda^4 + 1 = 0$. They are 4 complex roots of $\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$. These are eigenvalues of signed permutation matrix

$$\widetilde{\Xi}_4 := e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}} \quad , e^{\frac{7\pi i}{4}}$$

Example 1.153. Eigenvalues of signed permutation matrix and cycle-types of B_3 in conjugacy classes.

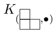
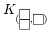
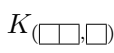
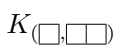
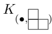
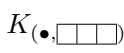
Cycle-Types of B_3	K 	K 	K 	K 	K 	K 	K 	K 	K 	K 
Eigenvalues $X = x_1, x_2, x_3$ $Y = y_1, y_2, y_3$	$1,1,1$ $0,0,0$	$1,1,-1$ $0,0,0$	$1,\zeta,\zeta^2$ $0,\ 0,0$	$1,1,0$ $-1,0,0$	$1,-1,0$ $-1,0,0$	$1,0,0$ $1,-1,0$	$1,0,0$ $i,-i,0$	$0,0,0$ $-1,-1,-1$	$0,0,0$ $-1,i,-i$	$0,0,0$ $-1,-\zeta,-\zeta^2$
Characteristic Polynomials	$(\lambda-1)^3$	$(\lambda+1)(\lambda-1)^2$	λ^3-1	$(\lambda+1)(\lambda-1)^2$	$(\lambda-1)(\lambda+1)^2$	$(\lambda-1)(\lambda+1)^2$	$(\lambda-1)(\lambda^2+1)$	$(\lambda+1)^3$	$(\lambda+1)(\lambda^2+1)$	λ^3+1
	[1,2,3] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	[1,3,2] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	[2, 3, 1] $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	[1,2, -3] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[2, 1, -3] $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[-1,-2, 3] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	[1,-3,2] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	[-1,-2,-3] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[-1,-3,2] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	[2, 3, -1] $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$
		[1,-3,-2] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	[2, -3, -1] $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$	[1,-2,3] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	[-2, -1, -3] $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[1,-2,-3] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[1,3,-2] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$		[-1,3,-2] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	[2, -3, 1] $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$
		[2, 1, 3] $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	[-2, 3, -1] $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$	[-1,2,3] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	[3, -2, 1] $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	[-1,2,-3] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[-2, 1, 3] $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		[-2, 1, -3] $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[-2, 3, 1] $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
		[-2, -1, 3] $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	[-2, -3, 1] $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$		[-3, -2, -1] $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$		[2, -1, 3] $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		[2, -1, -3] $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	[-2, -3, -1] $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$
		[3, 2, 1] $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	[3, 1, 2] $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$		[-1, 3, 2] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$		[-3, 2, 1] $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$		[-3, -2, 1] $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	[3, 1, -2] $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$
		[-3, 2, -1] $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	[3, -1, -2] $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$		[-1, -3, -2] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$		[3, 2, -1] $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$		[3, -2, -1] $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	[3, -1, 2] $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
			[-3, 1, -2] $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$							[-3, 1, 2] $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
			[-3,-1, 2] $\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$							[-3, -1, -2] $\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

Table 1.16: Eigenvalues and cycle-types of B_3 in conjugacy classes

Remark 1.154. In Example 3.61 we have included the eigenvalues of signed permutation matrix and their transformed version to eigenvalues of permutation matrix on each cycle-type of group B_2.

1.6 Symmetric Group Irreducible Character Basis $\{\tilde{s}_\lambda\}$

Notation

We will provide the notation and combinatorial objects that will be used here and the rest of the thesis. The material in this chapter are due to the results in [OZ1, OZ1v2, OZ2, OZ3]. We will state and explain important propositions and theorems which are used in solving the thesis questions.

Definition 1.155. A multi-set is a set that can have repeated elements and it will be denoted by $\{\!\{b_1, b_2, \ldots, b_r\}\!\}$. We can also use exponential notation to show a multi-set that is $\{\!\{1^{a_1}, 2^{a_2}, \ldots, l^{a_l}\}\!\}$ which represents that element i is repeated a_i times.

Definition 1.156. A set partition of a set S is a set of subsets $\{S_1, S_2, \ldots, S_l\}$ where $S_i \subseteq S$ and $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq l$ and $S_1 \cup S_2 \cup \cdots \cup S_l = S$.

Definition 1.157. A multi-set partition $\pi = \{\!\{S_1, S_2, \ldots, S_l\}\!\}$ of a multi-set S has a similar structure of the set partition but with a difference that S_i can be a multi-set and any two sets S_i and S_j can have non-empty intersection. Two or more elements of a multi-set can even be equal. We denote the length of a multi-set by $l(\pi) = l$. For π is a multi-set partition of the multi-set S we use the notation $\pi \Vdash S$.

Definition 1.158. Given a multi-set partition π the notation $\tilde{m}(\pi)$ represents the partition of $l(\pi)$ consisting of the multiplicities of the multi-sets that occur in π.

Example 1.159. Let $S = \{\!\{1^4, 2^3, 3^1\}\!\}$ be a multi-set and let $\pi = \{\!\{\, \{\!\{1,2,3\}\!\}, \{\!\{1,2\}\!\}, \{\!\{1,2\}\!\}, \{\!\{1\}\!\} \,\}\!\} \vdash \vdash \{\!\{1^4, 2^3, 3^1\}\!\}$ to be a multi-set partition of S. We have the multiplicities of this multi-set given by $\tilde{m}(\pi) = (2,1,1) \vdash 4$. As we see $\{\!\{1\}\!\}$ and $\{\!\{1,2,3\}\!\}$ occurs 1 time and $\{\!\{1,2\}\!\}$ occurs 2 times. Note that we will be using a compact notation for multi-set partition, this is, for example the multi-set partition $\pi = \{\!\{\, \{\!\{1,2,3\}\!\}, \{\!\{1,2\}\!\}, \{\!\{1,2\}\!\}, \{\!\{1\}\!\} \,\}\!\}$ will be written as $\pi = \{\!\{123 \mid 12 \mid 12 \mid 1\}\!\}$.

Definition 1.160. Given non-negative integers n and l and positive integers α_i a composition of size n is an ordered sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_l = n$. We denote this by $\alpha \vDash n$. A weak composition is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ where $\alpha_i \geq 0$ (zeros are allowed) and we denote it by $\alpha \vDash_w n$. The length of both compositions is denoted by $l(\alpha) := l$.

Example 1.161. For example $\alpha_1 = (2,2,1)$ and $\alpha_2 = (1,2,1,1)$ are both composition of size 5 i.e. $\alpha_1, \alpha_2 \vDash 5$ and sequences $\beta_1 = (0,2,0)$, $\beta_2 = (1,0,1)$ and $\beta_3 = (0,1,1)$ are all examples of weak compositions of of size 2 i.e. $\beta_1, \beta_2, \beta_3 \vDash_w 2$.

Symmetric Functions at Roots of Unity

In this part we will explain the evaluation of symmetric functions at the eigenvalues of the permutation matrix.

Definition 1.162. Given an integer $k > 0$ define the set of roots of unity by

$$\Xi_k := 1, e^{\frac{2\pi i}{k}}, e^{\frac{4\pi i}{k}}, e^{\frac{6\pi i}{k}}, \ldots, e^{\frac{2(k-1)\pi i}{k}}$$

where Ξ_k represents the eigenvalues of a permutation matrix of a k-cycle. For a partition μ denote

$$\Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu_{l(\mu)}}$$

to be the multi-set of eigenvalues of a permutation matrix with cycle structure μ. For a symmetric function $f \in \Lambda$ denote $f[\Xi_\mu]$ to be the evaluation of f at this multi-set of eigenvalues. Recall that this is done by replacing the power sum generator p_k in f with $x_1^k + x_2^k + \cdots + x_{|\mu|}^k$ and replacing the indeterminate variables by x_i with the values in the multi-set Ξ_μ .

Example 1.163. Evaluate $p_6[\Xi_{(3,2)}]$.

Given a permutation $\sigma = (134)(25)$ in S_5 with cycle structure $\mu = (3, 2)$ the multi-set of eigenvalues of a permutation matrix with cycle structure μ is written as $\Xi_{(3,2)} := \Xi_3, \Xi_2$. Note that $\Xi_3 = \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ and $\Xi_2 = \{1, -1\}$. Recall that for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{l(\mu)})$ we have $p_k[\Xi_\mu] = p_k[\Xi_{\mu_1}] + p_k[\Xi_{\mu_2}] + \cdots + p_k[\Xi_{\mu_{l(\mu)}}]$ therefore

$$\begin{aligned} p_6[\Xi_{(3,2)}] &= p_6[\Xi_3] + p_6[\Xi_2] \\ &= 1^6 + e^{6(\frac{2\pi i}{3})} + e^{6(\frac{4\pi i}{3})} + 1^6 + (-1)^6 \\ &= 5 \end{aligned}$$

Remark 1.164. [Lascoux] The evaluation of the power sum, the complete homogenous and the elementary symmetric functions at eigenvalues of roots of unity can be computed directly by the following given $n > 0$.

$$p_n[\Xi_d] = \begin{cases} d & \text{if } d|n \\ 0 & \text{otherwise} \end{cases}, \qquad h_n[\Xi_d] = \begin{cases} 1 & \text{if } d|n \\ 0 & \text{otherwise} \end{cases}, \qquad e_n[\Xi_d] = \begin{cases} (-1)^{d-1} & \text{if } n = d \\ 0 & \text{otherwise} \end{cases}$$

Remark 1.165. We will be using a compact notation for the multi-set of eigenvalues of a permutation matrix with cycle structure μ. For example $\Xi_{(3,2)}$ will be as Ξ_{32}.

Example 1.166. Evaluate $s_2[\Xi_{21}]$.

The expansion in power sum basis is

$$\begin{aligned} s_2[\Xi_{21}] &= \frac{1}{2}p_{11}[\Xi_{21}] + \frac{1}{2}p_2[\Xi_{21}] \\ &= \frac{1}{2}(p_1[\Xi_2] + p_1[\Xi_1])^2 + \frac{1}{2}(p_2[\Xi_2] + p_2[\Xi_1]) \\ &= \frac{1}{2}(0+1)^2 + \frac{1}{2}(2+1) \\ &= 2 \end{aligned}$$

Induced Trivial Character Basis of Symmetric Group $\{\tilde{h}_\lambda\}$

Definition 1.167. (Definition 4 [OZ1]) We will define symmetric functions elements $\tilde{h}_\nu$ in equation

$$h_\lambda = \sum_{\pi \Vdash \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}\}\!\}} \tilde{h}_{\widetilde{m}(\pi)}$$

This is a recursive definition for calculating the $\tilde{h}_\lambda$ basis and some computations are shown in Example 1.168.

Example 1.168. Use the formula $h_\lambda = \sum_{\pi \Vdash \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}\}\!\}} \tilde{h}_{\widetilde{m}(\pi)}$ to express $\tilde{h}_\mu$ in terms of h_λ for $n = 1, 2, 3, 4$.

First express h_λ in terms of $\tilde{h}_\mu$ in terms for $n = 1, 2, 3, 4$.

Partition λ	Exponential Notation	Multi-set Partitions π	$\widetilde{m}(\pi)$: Multiplicity in π	h_λ in terms of $\tilde{h}_{\widetilde{m}(\pi)}$
$\lambda = (1)$	$\lambda = (1^1)$	$\{\!\{1\}\!\}$	(1)	$h_1 = \tilde{h}_1$
$\lambda = (2)$	$\lambda = (1^2)$	$\{\!\{11\}\!\}$,$\{\!\{1 \mid 1\}\!\}$	(1), (2)	$h_2 = \tilde{h}_2 + \tilde{h}_1$
$\lambda = (1,1)$	$\lambda = (1^1 2^1)$	$\{\!\{12\}\!\}$,$\{\!\{1 \mid 2\}\!\}$	(1), $(1,1)$	$h_{11} = \tilde{h}_{11} + \tilde{h}_1$
$\lambda = (3)$	$\lambda = (1^3)$	$\{\!\{111\}\!\}$,$\{\!\{1 \mid 11\}\!\}$, $\{\!\{1 \mid 1 \mid 1\}\!\}$	(1), $(1,1)$,(3)	$h_3 = \tilde{h}_1 + \tilde{h}_{11} + \tilde{h}_3$
$\lambda = (2,1)$	$\lambda = (1^2 2^1)$	$\{\!\{112\}\!\}$,$\{\!\{1 \mid 12\}\!\}$, $\{\!\{2 \mid 11\}\!\}$, $\{\!\{1 \mid 1 \mid 2\}\!\}$	(1), $(1,1)$, $(1,1)$, $(2,1)$	$h_{21} = \tilde{h}_1 + 2\tilde{h}_{11} + \tilde{h}_{21}$
$\lambda = (1,1,1)$	$\lambda = (1^1 2^1 3^1)$	$\{\!\{123\}\!\}$,$\{\!\{1 \mid 23\}\!\}$, $\{\!\{2 \mid 13\}\!\}$, $\{\!\{3 \mid 12\}\!\}$, $\{\!\{1 \mid 2 \mid 3\}\!\}$	(1), $(1,1)$, $(1,1)$, $(1,1)$, $(1,1,1)$	$h_{111} = \tilde{h}_1 + 3\tilde{h}_{11} + \tilde{h}_{111}$

Table 1.17: Expression h_λ in terms of $\tilde{h}_\mu$ for $n = 1, 2, 3, 4$.

Now we can recursively find $\tilde{h}_\mu$'s with respect to h_λ's.

Partition λ	$\tilde{h}_\lambda$ in terms of h_μ
$\lambda = (1)$	$\tilde{h}_1 = h_1$
$\lambda = (2)$	$\tilde{h}_2 = h_2 - h_1$
$\lambda = (1,1)$	$\tilde{h}_{11} = h_{11} - h_1$
$\lambda = (3)$	$\tilde{h}_3 = h_3 - h_{11}$
$\lambda = (2,1)$	$\tilde{h}_{21} = h_{21} - 2h_{11} + h_1$
$\lambda = (1,1,1)$	$\tilde{h}_{111} = h_{111} - 3h_{11} + 2h_1$

Table 1.18: Expression $\tilde{h}_\lambda$ in terms of h_μ for $n = 1, 2, 3, 4$.

The existence and construction of the induced trivial character basis of symmetric functions namely $\{\tilde{h}_\lambda\}$ is a fundamental part of the discussion in this section. At first, we will state the theorem below.

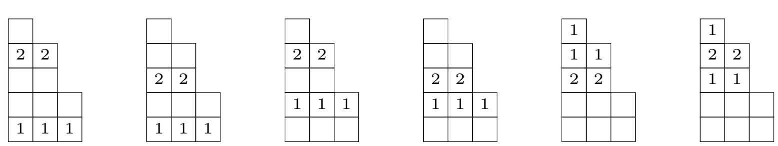
Theorem 1.169. *(Theorem 3 [OZ1]) For all partitions ν and μ, let $\mathcal{H}_{\nu,\mu} := \langle h_{|\mu|-|\nu|}h_\nu, p_\mu \rangle$. We have the evaluation*

$$h_\lambda[\Xi_\mu] = \sum_{\pi \Vdash \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}\}\!\}} \mathcal{H}_{\widetilde{m}(\pi),\mu}.$$

Note that $\mathcal{H}_{\lambda\mu} := 0$ if $|\mu| - |\nu| < 0$.

Proposition 1.170. *(Proposition 28 [OZ1]) For partitions λ and μ, $\mathcal{H}_{\lambda,\mu}$ is equal to the number of ways that some of the cells of the diagram of μ can be filled with the labels $\{1, 2, \ldots, l(\lambda)\}$ such that the whole row is given the same label and in total λ_j cells are labeled with the integer j for $1 \leq j \leq l(\lambda)$.*

Example 1.171. In this example diagrams of shape $\mu = (3, 3, 2, 2, 1)$ are labeled by labels of three 1's and two 2's such that each row has all same given label and exactly total of five cells are labeled. The diagrams are



We can conclude $\mathcal{H}_{32,33221} = 6$.

Remark 1.172. A multi-set partition $\pi = \{\!\!\{S_1 \mid S_2 \mid \ldots \mid S_l\}\!\!\}$ of a multi-set S does not necessary have an order on its parts. To be consistent in labelling the column-strict tableaux, we will choose to have the multi-set that occurs the most first and also between the ones that occurring the same number of time we choose lexicographic order given the elements of the set are in increasing order.

Example 1.173. Given a multi-set $S = \{\!\!\{1^{13}, 2^9, 3^2\}\!\!\}$ and we order a multi-set partition

$$\pi = \{\!\!\{\ 12 \mid 12 \mid 12 \mid 111 \mid 111 \mid 1223 \mid 1223 \mid 2 \mid 2 \mid 11\ \}\!\!\} \Vdash \{\!\!\{1^{13}, 2^9, 3^2\}\!\!\}$$

and we have $\tilde{m}(\pi) = (3, 2, 2, 2, 1) = (3, 2^3, 1) \vdash 10$.

Theorem 1.174. *(Theorem 3 or 37 [OZ1]) For all partitions λ,*

$$h_\lambda[\Xi_\mu] = \sum_{\pi \Vdash \{\!\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}\}\!\!\}} \mathcal{H}_{\tilde{m}(\pi),\mu}$$

The function $\tilde{h}_\lambda$ is a symmetric function that the evaluations at the eigenvalues of a permutation matrix are the values of the induced trivial module. The symmetric functions $\{\tilde{h}_\lambda\}$ are a basis of symmetric functions and we will call them the 'induced trivial character basis'.

Proposition 1.175. *(Equation 6 [OZ1]) For all partitions λ and μ,*

$$\tilde{h}_\lambda[\Xi_\mu] = \mathcal{H}_{\lambda,\mu} = \left\langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \right\rangle.$$

Definition 1.176. (Definition 33 [OZ1]) Given the partition μ define $\mathcal{T}_{\lambda,\mu}$ to be the set of filling of some of the cells of the partition μ with content λ such that any number of labels can go into the same cell and all cells in the same row must have the same multi-set of labels.

Proposition 1.177. *(Proposition 35 [OZ1]) For partitions λ and μ,*

$$\sum_{\pi \Vdash \{\!\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}\}\!\!\}} \mathcal{H}_{\tilde{m}(\pi),\mu} = |\mathcal{T}_{\lambda,\mu}|\,.$$

Example 1.178. Evaluate $h_{21}[\Xi_{321}]$.

By Theorem 1.174, given $\lambda = (2, 1)$ and $\mu = (3, 2, 1)$ we need to find all multi-set partitions π's that are multi-set partitions of the multi-set $S = \{\!\!\{1^2, 2\}\!\!\}$ i.e. to find $\pi \Vdash \{\!\!\{1^2, 2\}\!\!\} = S$.

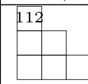
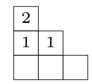
Multi-set Partition	Multiplicities in Multi-set Partition	$\mathcal{H}_{\tilde{m}(\pi_i),321}$	Fillings in $\mathcal{T}_{(2,1),(3,2,1)}$
$\pi_1 = \{\!\{112\}\!\}$	$\tilde{m}(\pi_1) = (1)$	$\mathcal{H}_{1,321} = \langle h_5 h_1, p_{321}\rangle = 1$	
$\pi_2 = \{\!\{11 \mid 2\}\!\}$	$\tilde{m}(\pi_2) = (1,1)$	$\mathcal{H}_{11,321} = \langle h_4 h_{11}, p_{321}\rangle = 0$	
$\pi_3 = \{\!\{1 \mid 12\}\!\}$	$\tilde{m}(\pi_3) = (1,1)$	$\mathcal{H}_{11,321} = \langle h_4 h_{11}, p_{321}\rangle = 0$	
$\pi_4 = \{\!\{1 \mid 1 \mid 2\}\!\}$	$\tilde{m}(\pi_4) = (2,1)$	$\mathcal{H}_{21,321} = \langle h_3 h_{21}, p_{321}\rangle = 1$	

Table 1.19: Evaluate $h_{21}[\Xi_{321}]$ by Theorem 1.174

By Proposition 1.177 in 4th column we have

$$h_{21}[\Xi_{321}] = \sum_{\pi \Vdash \{\!\{1^2,2\}\!\}} \mathcal{H}_{\widetilde{m}(\pi),3211} = |\mathcal{T}_{21,321}| = 2.$$

We may also compute directly by Theorem 1.169 in 3rd column as

$$\begin{aligned} h_{21}[\Xi_{321}] &= \sum_{\pi \Vdash \{\!\{1^2,2^1\}\!\}} \mathcal{H}_{\widetilde{m}(\pi),(3,2,1)} \\ &= \mathcal{H}_{1,321} + \mathcal{H}_{11,321} + \mathcal{H}_{11,321} + \mathcal{H}_{21,321} \\ &= \tilde{h}_1[\Xi_{321}] + \tilde{h}_{11}[\Xi_{321}] + \tilde{h}_{11}[\Xi_{321}] + \tilde{h}_{21}[\Xi_{321}] \\ &= \langle h_5 h_1, p_{321}\rangle + \langle h_4 h_{11}, p_{321}\rangle + \langle h_4 h_{11}, p_{321}\rangle + \langle h_3 h_{21}, p_{321}\rangle \\ &= 1 + 0 + +0 + 1 \\ &= 2 \end{aligned}$$

Remark 1.179. Note that in Example 1.168 we have also found the expansion $h_{21} = \tilde{h}_1 + 2\tilde{h}_{11} + \tilde{h}_{21}$.

Elementary Symmetric Functions Evaluated at Roots of Unity

In this section we will show the evaluation of elementary symmetric function at Ξ_μ. Note that the material in this section are due to the results in [OZ1, OZ1v2, OZ2] which we are following them closely.

Let a subset $S = \{i_1, i_2, \ldots, i_{|S|}\} \subseteq \{1, 2, \ldots, l(\mu)\}$ and let a sub-partition $\mu_S = (\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_{|S|}})$. Given $e_n[\Xi_d] = (-1)^{d-1}\delta_{dn}$ we will write analogous alphabet addition formula to h_n as

$$e_n[\Xi_\mu] = \sum_{\substack{\alpha \vDash_w n \\ l(\alpha)=l(\mu)}} \prod_{i=1}^{l(\mu)} e_{\alpha_i}[\Xi_{\mu_i}]$$
$$= \sum_{S:|\mu_S|=n} \prod_{i \in S} e_{\mu_i}[\Xi_{\mu_i}]$$
$$= \sum_{S:|\mu_S|=n} (-1)^{n+|S|}$$

where the sum is over all subsets $S = \{i_1, i_2, \ldots, i_{|S|}\} \subseteq \{1, 2, \ldots, l(\mu)\}$ such that $|\mu_S| = n$.

Definition 1.180. (Definition 5.5 [OZ2]) Define the set $\bar{\mathcal{C}}_{\lambda,\mu}$ to be $(S^{(1)}, S^{(2)}, \ldots, S^{(l(\lambda))})$ that is the set of sequences where each $S^{(i)}$ is a subset such that $|\mu_{S^{(i)}}| = \lambda_i$.

Proposition 1.181. *(Proposition 5.6 [OZ2]) Given partitions λ and μ,*

$$e_\lambda[\Xi_\mu] = \sum_{S^{(*)} \in \bar{\mathcal{C}}_{\lambda,\mu}} (-1)^{|\lambda| + \left|S^{(*)}\right|}$$

where $\left|S^{(*)}\right| = \sum_{i=1}^{l(\lambda)} \left|S^{(i)}\right|$.

Lemma 1.182. *[OZ2] Define $\mathcal{HE}_{(\lambda|\tau),\mu} := \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda e_\tau, p_\mu \right\rangle$ where λ, τ and μ are partitions. This is also equal to*

$$\mathcal{HE}_{(\lambda|\tau),\mu} = \sum_{\gamma^{(*)},\nu^{(*)}} sgn(\nu^{(*)}) \prod_{i=1}^{\mu_1} \binom{m_i(\mu)}{m_i(\gamma^{(1)}), \ldots, m_i(\gamma^{(l(\lambda))}), m_i(\nu^{(1)}), \ldots, m_i(\nu^{(l(\tau))})}$$

where this sum is over all sequences of partitions $\gamma^{(*)} = (\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(l(\lambda))})$ where $\gamma^{(j)} \vdash \lambda_j$ and $\nu^{(*)} = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(l(\tau))})$ where $\nu^{(j)} \vdash \tau_j$ and

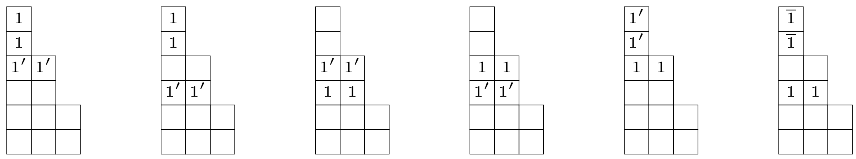
$$sgn(\nu^{(*)})) = (-1)^{\sum_i \left|\nu^{(i)}\right| + l(\nu^{(i)})}$$

Proposition 1.183. *(Proposition 5.8 [OZ2]) For partitions λ, τ and μ, let $\mathcal{F}^\mu_{\lambda,\tau}$ be the fillings of the diagram for the partition μ with λ_i labels i and τ_j labels i' such that all cells in a row are filled with the same label. For $F \in \mathcal{F}^\mu_{\lambda,\tau}$, the weight of the fillings, $wt(F)$ is equal to -1 raised to the number of cells filled with primed labels plus the number of rows occupied by the primed labels. Then*

$$\mathcal{HE}_{(\lambda|\tau),\mu} = \sum_{F \in \mathcal{F}^\mu_{\lambda,\tau}} wt(F)$$

Example 1.184. Evaluate $\mathcal{HE}_{(2|2),332211}$.

In here the prime notation is used for the sets with an odd number of barred entries and non-prime ones are used of the sets with an even number of barred entries. The diagrams below are all possible fillings of diagram $\mu = (3, 3, 2, 2, 1, 1)$ with two entries 1's and two entries of $1'$'s in such a way that rows are all labelled by the same labels.



The weights of each of the above filling is (-1) raised to the number of cells occupied with primed labels (which is 2) plus the number of rows occupied by the primed labels. We can see that the first four fillings have the weight of -1 and the last two are of weight 1. Using $\mathcal{HE}_{(\lambda|\tau),\mu} = \sum_{F \in \mathcal{F}^{\mu}_{\lambda,\tau}} wt(F)$, the sum of the weights are the desired evaluation

$$\mathcal{HE}_{(2|2),332211} = -2$$

Set Partition of a Multi-set. Recall that a set partition of a set S is a set of subsets $\{S_1, S_2, \ldots, S_l\}$ where $S_i \subseteq S$ and $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq l$ and $S_1 \cup S_2 \cup \cdots \cup S_l = S$. A multi-set partition $\pi = \{\!\{S_1, S_2, \ldots, S_l\}\!\}$ of a multi-set S has a similar structure of the set partition but with a difference that S_i can be a multi-set and any two sets S_i and S_j can have non-empty intersection. For π is a multi-set partition of the multi-set S we use the notation $\pi \Vdash S$. In addition, to represent the partition of $l(\pi)$ consisting of the multiplicities which occur in multi-set π we are using notation $\tilde{m}(\pi)$. For example, given $S = \{\!\{1^4, 2^3, 3^1\}\!\}$ and a multi-set partition $\pi = \{\!\{123 \mid 12 \mid 12 \mid 1\}\!\} \Vdash \{\!\{1^4, 2^3, 3^1\}\!\}$ of S, we have $\tilde{m}(\pi) = (2, 1, 1) = (2, 1^2) \vdash 4$.

Definition 1.185. (Section 5.1 in [OZ2]) Let $\pi = \{\!\{P^{(1)}, P^{(2)}, \ldots, P^{(l(\pi))}\}\!\}$ to be a set partition of a multi-set and denoted by $\pi \vdash \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_{l(\lambda)}}\}\!\}$ where $P^{(1)} \uplus P^{(2)} \uplus \ldots \uplus P^{(l(\pi))} = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_{l(\lambda)}}\}\!\}$ and each $P^{(i)}$ are sets with no repetitions. Define $\tilde{m}(\pi_e)$ to be a partition representing the multiplicities of the set with an even number of elements and $\tilde{m}(\pi_o)$ to be a partition that represents the multiplicities of a set with an odd number of elements and $\pi = \pi_e \uplus \pi_o$.

Example 1.186. Let $\lambda = (5, 3, 3, 2, 2)$ and

$$\pi = \{\!\{\ \{1, 2, 5\}, \{1, 2\}, \{1, 2\}, \{1, 3\}, \{1, 3\}, \{3, 4\}, \{4\}, \{5\}\ \}\!\}$$

that is a set partition of the multi-set $\{\!\{1^5, 2^3, 3^3, 4^2, 5^2\}\!\}$. The corresponding multiplicities are partitions $\tilde{m}(\pi) = (2, 2, 1, 1, 1, 1)$, $\tilde{m}(\pi_e) = (2, 2, 1)$ and $\tilde{m}(\pi_o) = (1, 1, 1)$. Note that $\tilde{m}(\pi_e) \uplus \tilde{m}(\pi_o) = \tilde{m}(\pi)$.

Remark 1.187. We will use a compact notation for a set partition of a multi-set. For example we will express the multi-set π in the preceding example by

$$\pi = \{\!\{\, 1,2,5 \rrbracket\, 12 \rrbracket\, 12 \rrbracket\, 13 \rrbracket\, 13 \rrbracket\, 34 \rrbracket\, 4 \rrbracket\, 5 \,\}\!\}$$

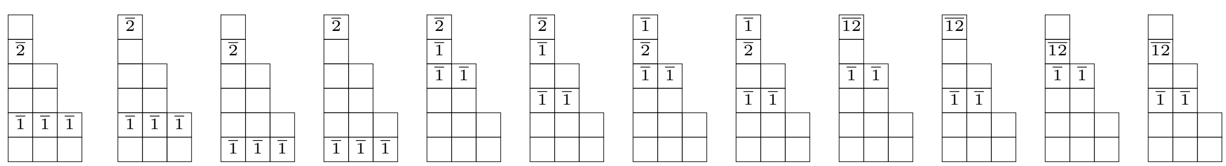
Corollary 1.188. *(Equation (61) in [OZ2]) Proposition 1.183 can be extended as*

$$\sum_{\pi \vdash \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_{l(\lambda)}}\}\!\}} \mathcal{HE}_{(\tilde{m}(\pi_e)|\tilde{m}(\pi_o)),\mu} = \sum_{F \in \bar{\mathcal{P}}_{\lambda,\mu}} wt(F)$$

Definition 1.189. (Definition 5.13 [OZ2]) For partitions λ and μ, let $\bar{\mathcal{T}}_{\lambda,\mu}$ be the fillings of some of the cells of the diagram of the partition μ with subsets of $\{\bar{1}, \bar{2}, \ldots, \bar{l(\lambda)}\}$ such that the total content of the filling is $\{\!\{\bar{1}^{\lambda_1}, \bar{2}^{\lambda_2}, \ldots, \bar{l}(\lambda)^{\lambda_{l(\lambda)}}\}\!\}$ and such that all cells in the same row have the same subset of entries. We will define the weight of one of these fillings to be -1 raised to the power of the size of λ plus the number of rows whose cells are occupied by a set of odd size (this is also equal to the number of cells plus the number of rows occupied by the sets of odd size).

Remark 1.190. From here on we will using barred entries to distinguish this type of multi-sets as it will be helpful in our discussion for type B multi-set partitions.

Example 1.191. The following diagrams are 12 elements of $\bar{\mathcal{T}}_{(3,1),(3,3,2,2,1,1)}$.



The four fillings are diagrams of weight $+1$ and the other eight are of weight -1.

Corollary 1.192. *(Corollary 5.16 [OZ2]) For partitions λ and μ,*

$$e_\lambda[\Xi_\mu] = \sum_{\pi \vdash \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_{l(\lambda)}}\}\!\}} \mathcal{HE}_{(\tilde{m}(\pi_e)|\tilde{m}(\pi_o)),\mu}$$

Example 1.193. Given the multi-set $S = \{\!\{\bar{1}^3, \bar{2}^2\}\!\}$ there are only three set partitions of S as the following

$$\{\!\{\, \bar{1} \rrbracket\, \bar{1} \rrbracket\, \bar{1} \rrbracket\, \bar{2} \rrbracket\, \bar{2} \,\}\!\},\ \{\!\{\, \bar{1} \rrbracket\, \bar{1} \rrbracket\, \bar{1}\bar{2} \rrbracket\, \bar{2} \,\}\!\},\ \{\!\{\, \bar{1} \rrbracket\, \bar{1}\bar{2} \rrbracket\, \bar{1}\bar{2} \,\}\!\}$$

By Corollary 1.192 we have

$$e_{32}[\Xi_\mu] = \mathcal{HE}_{(\bullet|32),\mu} + \mathcal{HE}_{(1|21),\mu} + \mathcal{HE}_{(2|1),\mu}$$

Proposition 1.194. *(Remark 5.18 [OZ2])The definition of the expression $\mathcal{HE}_{(\lambda|\tau),\mu}$ implies that we can define symmetric functions namely $\tilde{he}_{(\lambda|\tau)}$ such that their evaluation at the eigenvalues of roots of unity is*

$$\tilde{he}_{(\lambda|\tau)}[\Xi_\mu] = \mathcal{HE}_{(\lambda|\tau),\mu} := \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda e_\tau, p_\mu \right\rangle.$$

Proof. Given $e_\tau = \sum_{\alpha \vdash |\tau|} a_{\tau\alpha} h_\alpha$ define $\tilde{he}_{(\lambda|\tau)}$ by

$$\tilde{he}_{(\lambda|\tau)} := \sum_{\alpha \vdash |\tau|} a_{\tau\alpha} \tilde{h}_{\lambda \uplus \alpha}$$

Since $\tilde{h}_{\lambda\uplus\alpha}[\Xi_\mu] = \left\langle h_{|\mu|-|\lambda|-|\alpha|} h_\lambda h_\alpha, p_\mu \right\rangle$ we have

$$\begin{aligned}
\tilde{he}_{(\lambda|\tau)}[\Xi_\mu] &= \sum_{\alpha \vdash |\tau|} a_{\tau\alpha} \tilde{h}_{\lambda\uplus\alpha}[\Xi_\mu] \\
&= \sum_{\alpha \vdash |\tau|} a_{\tau\alpha} \left\langle h_{|\mu|-|\lambda|-|\alpha|} h_\lambda h_\alpha, p_\mu \right\rangle \\
&= \sum_{\alpha \vdash |\tau|} \left\langle h_{|\mu|-|\lambda|-|\alpha|} h_\lambda a_{\tau\alpha} h_\alpha, p_\mu \right\rangle \\
&= \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda \left(\sum_{\alpha \vdash |\tau|} a_{\tau\alpha} h_\alpha \right), p_\mu \right\rangle \\
&= \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda e_\tau, p_\mu \right\rangle
\end{aligned}$$

The proof is complete. □

Proposition 1.195. *Given the symmetric function $\tilde{e}_\tau$ defined above its evaluation at the eigenvalues of roots of unity is*

$$\tilde{e}_\tau[\Xi_\mu] = \mathcal{E}_{\tau,\mu} := \left\langle h_{|\mu|-|\tau|} e_\tau, p_\mu \right\rangle.$$

Proof. The proof is similar to the proof of Proposition 1.194 when partition λ is an empty partition. □

We will define a binary operation on the S_n-induced trivial character basis $\{\tilde{h}_\lambda\}$. This gives a multiplicative tool which will help in our future evaluation of the thesis in type B.

Definition 1.196. Define a linear binary operation $\odot_A : \Lambda \times \Lambda \longrightarrow \Lambda$ on the induced trivial characters of the symmetric group (type A) such that given

$$f = \sum_\alpha a_\alpha \tilde{h}_\alpha \quad \text{and} \quad g = \sum_\beta b_\beta \tilde{h}_\beta$$

where $f, g \in \Lambda$ and $a_\alpha, b_\beta \in \mathbb{Q}$, then

$$f \odot_A g := \sum_{\alpha,\beta} a_\alpha b_\beta \tilde{h}_{\alpha\uplus\beta}$$

The operation $\uplus$ is as defined in Definition 2.7 and the binary operation $\odot_{\!A}$ is commutative and associative where $\tilde{h}_\bullet \odot_{\!A} \tilde{h}_\bullet := \tilde{h}_\bullet = 1$.

Example 1.197. The operation $\odot_{\!A}$ is demonstrated in the following two small examples.

1. $\tilde{h}_1 \odot_{\!A} -\tilde{h}_{22} = -\tilde{h}_{221}$
2. $-\tilde{h}_{21} \odot_{\!A} (-2\tilde{h}_{32}) \odot_{\!A} \tilde{h}_{11} = 2\tilde{h}_{322111}$

Remark 1.198. By Proposition 1.175 we have $\tilde{h}_\lambda[\Xi_\mu] = \mathcal{H}_{\lambda,\mu} := \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle$. The binary operation $\odot_{\!A}$ can be used between $\tilde{h}_\lambda$ and $\tilde{h}_\tau$ (in type A) and the evaluation at the eigenvalues of roots of unity is

$$\left(\tilde{h}_\lambda \odot_{\!A} \tilde{h}_\tau\right)[\Xi_\mu] := \tilde{h}_{(\lambda \uplus \tau)}[\Xi_\mu] = \mathcal{H}_{(\lambda \uplus \tau),\mu} = \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda h_\tau, p_\mu \right\rangle.$$

Remark 1.199. In Proposition 1.194, given partitions λ, τ and μ, we have defined the symmetric functions $\widetilde{he}_{(\lambda|\tau)}$ evaluated at the eigenvalues of roots of unity Ξ_μ. Since $\widetilde{he}_{(\lambda|\mu)} = \tilde{h}_\lambda \odot_{\!A} \tilde{e}_\mu$ we have

$$\left(\tilde{h}_\lambda \odot_{\!A} \tilde{e}_\tau\right)[\Xi_\mu] := \widetilde{he}_{(\lambda|\tau)}[\Xi_\mu] = \mathcal{HE}_{(\lambda|\tau),\mu} := \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda e_\tau, p_\mu \right\rangle.$$

Remark 1.200. Recall that in a special case where λ is an empty partition in Proposition 1.194 we let $\widetilde{he}_{(\bullet|\tau)} = \tilde{e}_\tau$ and we introduced the symmetric function in type A defined by

$$\tilde{e}_\tau := \sum_{\alpha \vdash |\tau|} a_{\tau\alpha} \tilde{h}_\alpha.$$

Moreover, this resulted in Proposition 1.195 where given the symmetric function $\tilde{e}_\tau$ defined above its evaluation at the eigenvalues of roots of unity is

$$\tilde{e}_\tau[\Xi_\mu] = \mathcal{E}_{\tau,\mu} := \left\langle h_{|\mu|-|\tau|} e_\tau, p_\mu \right\rangle.$$

Now in the following lemma we can extend the binary operation $\odot_{\!A}$ for the symmetric function $\tilde{e}_\tau$.

Lemma 1.201. *Given the binary operation $\odot_{\!A} : \Lambda \times \Lambda \longrightarrow \Lambda$ on the induced trivial characters of the symmetric group (type A) and $f = \sum_\alpha a_\alpha \tilde{e}_\alpha$ and $g = \sum_\beta b_\beta \tilde{e}_\beta$ where $f, g \in \Lambda$ and $a_\alpha, b_\beta \in \mathbb{Q}$, then*

$$f \odot_{\!A} g = \sum_{\alpha,\beta} a_\alpha b_\beta \tilde{e}_{\alpha \uplus \beta}.$$

Proof. Given $\tilde{e}_\tau := \sum_{\alpha \vdash |\tau|} c_{\tau\alpha} \tilde{h}_\alpha$ we have

$$\begin{aligned}
f \odot_A g &= \left(\sum_\alpha a_\alpha \tilde{e}_\alpha \right) \odot_A \left(\sum_\beta b_\beta \tilde{e}_\beta \right) \\
&= \sum_{\alpha,\beta} a_\alpha b_\beta \left(\tilde{e}_\alpha \odot_A \tilde{e}_\beta \right) \\
&= \sum_{\alpha,\beta} a_\alpha b_\beta \left(\sum_{\lambda \vdash |\alpha|} c_{\alpha\lambda} \tilde{h}_\lambda \odot_A \sum_{\mu \vdash |\beta|} d_{\beta\mu} \tilde{h}_\mu \right) \\
&= \sum_{\alpha,\beta} a_\alpha b_\beta \left(\sum_{\substack{\lambda \vdash |\alpha| \\ \mu \vdash |\beta|}} c_{\alpha\lambda} d_{\beta\mu} \left(\tilde{h}_\lambda \odot_A \tilde{h}_\mu \right) \right) \\
&= \sum_{\alpha,\beta} a_\alpha b_\beta \left(\sum_{\substack{\lambda \vdash |\alpha| \\ \mu \vdash |\beta|}} c_{\alpha\lambda} d_{\beta\mu} \tilde{h}_{\lambda \uplus \mu} \right) \\
&= \sum_{\alpha,\beta} a_\alpha b_\beta \tilde{e}_{\alpha \uplus \beta}
\end{aligned}$$

The proof is complete. □

The following in an example of calculating $\tilde{he}_{(\lambda|\tau)}[\Xi_\mu]$. In order to save time and being accurate we have used the program SageMath.

Example 1.202. Evaluate $\tilde{he}_{11|2}[\Xi_{32211}]$. We have $\tilde{e}_2 = e_2 = h_{11} - h_2$ therefore $\tilde{he}_{\bullet|2} = \tilde{h}_{11} - \tilde{h}_2$ and we have $\tilde{he}_{11|2} = \tilde{h}_{11} \odot_A \tilde{e}_2$

$$\begin{aligned}
\tilde{he}_{11|2}[\Xi_{32211}] &= \tilde{h}_{1111}[\Xi_{32211}] - \tilde{h}_{211}[\Xi_{32211}] \\
&= 0 - 4 \\
&= -4
\end{aligned}$$

Example 1.203. Find $\tilde{he}_{(\lambda|\tau)}$ and $\tilde{e}_\tau$ for $n = 0, 1, 2, 3$ in terms of elementary symmetric functions given

$$e_\lambda = \sum_{\pi \vdash \{\!\{ 1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_{l(\lambda)}} \}\!\}} \tilde{he}_{(\tilde{m}(\pi_e) | \tilde{m}(\pi_o))}$$

The set partitions and the corresponding multiplicities are given in the table below. We may write e_λ's using above formula.

Partition	Set Partitions	$(\tilde{m}(\pi_e) \mid \tilde{m}(\pi_o))$	e_λ
$\lambda = ()$	$\pi_1 = \{\!\{\}\!\}$	$(\bullet \mid \bullet)$	$e_\bullet = \widetilde{he}_{(\bullet\mid\bullet)}$
$\lambda = (1)$	$\pi_2 = \{\!\{\bar{1}\}\!\}$	$(\bullet \mid 1)$	$e_1 = \widetilde{he}_{(\bullet\mid 1)}$
$\lambda = (2)$	$\pi_3 = \{\!\{\bar{1} \rrbracket \llbracket \bar{1}\}\!\}$	$(\bullet \mid 2)$	$e_2 = \widetilde{he}_{(\bullet\mid 2)}$
$\lambda = (1,1)$	$\pi_4 = \{\!\{\bar{1}\bar{2}\}\!\}$	$(1 \mid \bullet)$	$e_{11} = \widetilde{he}_{(1\mid\bullet)} + \widetilde{he}_{(\bullet\mid 11)}$
	$\pi_5 = \{\!\{\bar{1} \rrbracket \llbracket \bar{2}\}\!\}$	$(\bullet \mid 11)$	
$\lambda = (3)$	$\pi_6 = \{\!\{\bar{1} \rrbracket \llbracket \bar{1} \rrbracket \llbracket \bar{1}\}\!\}$	$(\bullet \mid 3)$	$e_3 = \widetilde{he}_{(\bullet\mid 3)}$
$\lambda = (2,1)$	$\pi_7 = \{\!\{\bar{1} \rrbracket \llbracket \bar{1}\bar{2}\}\!\}$	$(1 \mid 1)$	$e_{21} = \widetilde{he}_{(1\mid 1)} + \widetilde{he}_{(\bullet\mid 21)}$
	$\pi_8 = \{\!\{\bar{1} \rrbracket \llbracket \bar{1} \rrbracket \llbracket \bar{2}\}\!\}$	$(\bullet \mid 21)$	
$\lambda = (1,1,1)$	$\pi_9 = \{\!\{\bar{1}\bar{2}\bar{3}\}\!\}$	$(\bullet \mid 1)$	$e_{111} = \widetilde{he}_{(\bullet\mid 1)} + 3\widetilde{he}_{(1\mid 1)} + 3\widetilde{he}_{(\bullet\mid 111)}$
	$\pi_{10} = \{\!\{\bar{1}\bar{2} \rrbracket \llbracket \bar{3}\}\!\}$	$(1 \mid 1)$	
	$\pi_{11} = \{\!\{\bar{1}\bar{3} \rrbracket \llbracket \bar{2}\}\!\}$	$(1 \mid 1)$	
	$\pi_{12} = \{\!\{\bar{1} \rrbracket \llbracket \bar{2}\bar{3}\}\!\}$	$(1 \mid 1)$	
	$\pi_{13} = \{\!\{\bar{1} \rrbracket \llbracket \bar{2} \rrbracket \llbracket \bar{3}\}\!\}$	$(\bullet \mid 111)$	

Table 1.20: Elementary symmetric functions in terms of $\widetilde{he}_{(\lambda\mid\tau)}$ for $n = 0, 1, 2, 3$.

Note that we can write $\widetilde{he}_{(\tilde{m}(\pi_e)\mid\bullet)} = \tilde{h}_{\tilde{m}(\pi_e)}$ so $\widetilde{he}_{(1\mid\bullet)} = \tilde{h}_1 = h_1 = e_1$. For the function of the form $\widetilde{he}_{(\bullet\mid\tilde{m}(\pi_o))} = \tilde{e}_{\tilde{m}(\pi_o)}$ we have

$$\tilde{e}_\tau[\Xi_\mu] = \mathcal{E}_{\tau,\mu} := \left\langle h_{|\mu|-|\tau|} e_\tau, p_\mu \right\rangle$$

Therefore we can use expansion of elementary symmetric functions in terms of homogenous symmetric functions and evaluate $\widetilde{he}_{(\bullet\mid\tilde{m}(\pi_o))}$ in terms of elementary symmetric functions. As an example

$$\begin{aligned}
\widetilde{he}_{(\bullet\mid 3)} &= \tilde{e}_3 \\
&= \tilde{h}_3 - 2\tilde{h}_{21} + \tilde{h}_{111} \\
&= h_3 - h_{11} - 2(h_{21} - 2h_{11} + h_1) + 2h_1 - 3h_{11} + h_{111} \\
&= h_3 - 2h_{21} + h_{111} \\
&= e_3
\end{aligned}$$

The results are organized in the table below.

$(\bullet \mid \tau)$	$\tilde{e}_\tau$ in terms of e_λ
$(\bullet \mid \bullet)$	$\tilde{e}_\bullet = e_\bullet$
$(\bullet \mid 1)$	$\tilde{e}_1 = e_1$
$(\bullet \mid 2)$	$\tilde{e}_2 = e_2$
$(\bullet \mid 11)$	$\tilde{e}_{11} = e_{11} - e_1$
$(\bullet \mid 3)$	$\tilde{e}_3 = e_3$
$(\bullet \mid 21)$	$\tilde{e}_{21} = e_{21} - e_{11} + e_1$
$(\bullet \mid 111)$	$\tilde{e}_{111} = e_{111} - 3e_{11} + 2e_1$

Table 1.21: $\tilde{e}_\tau$ in terms of e_λ for $n = 0, 1, 2, 3$.

The expansion of e_λ in $\tilde{e}_\tau$ is obtained recursively and it is presented below.

$(\bullet \mid \lambda)$	e_λ in terms of $\tilde{e}_\tau$
$(\bullet \mid \bullet)$	$e_\bullet = \tilde{e}_\bullet$
$(\bullet \mid 1)$	$e_1 = \tilde{e}_1$
$(\bullet \mid 2)$	$e_2 = \tilde{e}_2$
$(\bullet \mid 11)$	$e_{11} = \tilde{e}_{11} - \tilde{e}_1$
$(\bullet \mid 3)$	$e_3 = \tilde{e}_3$
$(\bullet \mid 21)$	$e_{21} = \tilde{e}_{21} + \tilde{e}_{11} - 2\tilde{e}_1$
$(\bullet \mid 111)$	$e_{111} = \tilde{e}_{111} + 3\tilde{e}_{11} - 5\tilde{e}_1$

Table 1.22: e_λ in terms of $\tilde{e}_\tau$ for $n = 0, 1, 2, 3$.

For evaluating expressions of the general form $\widetilde{he}_{(\lambda|\tau)}$ use $\widetilde{he}_{(\lambda|\tau)} = \tilde{h}_\lambda \odot_A \tilde{e}_\tau$. For example

$$\begin{aligned}
\widetilde{he}_{2|2} &= \tilde{h}_2 \odot_A \tilde{e}_2 \\
&= \tilde{h}_2 \odot_A (\tilde{h}_{11} - \tilde{h}_2) \\
&= \tilde{h}_{211} - \tilde{h}_{22} \\
&= -e_{22} + e_{211} - e_{21} - e_2 - e_{111} + 3e_{11} - 2e_1
\end{aligned}$$

Some other functions are

$\widetilde{he}_{(\lambda|\tau)}$	$\widetilde{he}_{(\lambda|\tau)}$ in terms of e_λ
$(1 \mid 1)$	$\widetilde{he}_{1|1} = e_{11}$
$(1 \mid 2)$	$\widetilde{he}_{1|2} = e_{21} - e_{11} + e_1$
$(2 \mid 2)$	$\widetilde{he}_{2|2} = -e_{22} + e_{211} - e_{21} - e_2 - e_{111} + 3e_{11} - 2e_1$

Table 1.23: $\widetilde{he}_{(\lambda|\tau)}$ in terms of e_λ

Remark 1.204. Functions $\widetilde{he}_{(\lambda|\mu)}$ is not a basis. These elements span Λ but are not linearly independent because simply there are too many. For example the degree 1 elements of Λ must have dimension 1 but there are at lease two elements $\widetilde{he}_{1|\bullet}$ and $\widetilde{he}_{\bullet|1}$ of degree 1 as $\widetilde{he}_{1|\bullet} = \tilde{h}_1 \odot_A \tilde{e}_\bullet = \tilde{h}_1 = h_1$ and $\widetilde{he}_{\bullet|1} = \tilde{h}_\bullet \odot_A \tilde{e}_1 = \tilde{e}_1 = h_1$ we have $\widetilde{he}_{1|\bullet} = \widetilde{he}_{\bullet|1}$ and therefore they can not be linearly independent.

Symmetric Group Irreducible Character Basis $\{\tilde{s}_\lambda\}$

This section is due to the results from section 3 and 4 in [OZ1, OZ3]. In the proceeding section we introduced and studied the construction of the induced trivial character basis of symmetric group namely $\{\tilde{h}_\lambda\}$. The evaluation $\tilde{h}_\lambda[\Xi_\mu]$ is the value of the character of the trivial representation in the induction $1 \uparrow^{S_{|\mu|}}_{S_{|\mu|-|\lambda|}\times S_{\lambda_1}\times S_{\lambda_2}\times\cdots\times S_{\lambda_{l(\lambda)}}}$. In Proposition 1.175 we proved that given partitions λ and μ the evaluation of basis $\tilde{h}_\lambda$ at the multi-set of eigenvalues of a permutation matrix with cycle structure μ is

$$\tilde{h}_\lambda[\Xi_\mu] = \mathcal{H}_{\lambda,\mu} = \left\langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \right\rangle$$

That is, the function $\tilde{h}_\lambda$ is a symmetric function that the evaluations at the eigenvalues of a permutation matrix are the values of the induced trivial module.

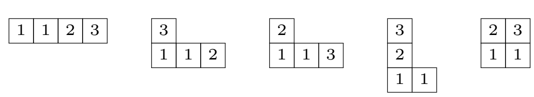
Now, in this part of our project the main goal is to introduce the irreducible character basis of symmetric group $\{\tilde{s}_\lambda\}$. These families of symmetric functions are the values of the characters of symmetric group when they are evaluated at roots of unity i.e. the eigenvalues of a permutation matrix. We will define the symmetric functions $\tilde{s}_\lambda$ by the change of basis with $\tilde{h}_\mu$ using Kostka coefficients denoted by $K_{\lambda\mu}$.

Definition 1.205. (Equation (7) [OZ1]) Given $n \geq 2|\mu|$ define the elements of $\tilde{s}_\lambda$ the basis that satisfies

$$\tilde{h}_\mu = \sum_{|\lambda| \leq |\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \tilde{s}_\lambda.$$

Equivalently the coefficient of $\tilde{s}_\lambda$ in $\tilde{h}_\mu$ is $\sum_\mu K_{\gamma\mu}$ where the sum is over partitions γ such that γ/λ is a horizontal strip of size $|\mu| - |\lambda|$.

Example 1.206. For $n = 4$ and $\mu = (1,1)$, then $(n - |\mu|, \mu) = (2,1,1)$. We have four column-strict tableaux with entries $\{\!\{1^2, 2, 3\}\!\}$.



$$\tilde{h}_{11} = \tilde{s}_\bullet + 2\tilde{s}_1 + \tilde{s}_{11} + \tilde{s}_2$$

Remark 1.207. Given $n \geq 2|\lambda|$ we can express $\tilde{s}_\lambda$ in terms of $\tilde{h}_\mu$ by the inverse of Kostka coefficients as

$$\tilde{s}_\lambda = \sum_{|\mu| \leq |\lambda|} K^{-1}_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \tilde{h}_\mu.$$

Remark 1.208. Since the Kostka coefficient $K_{\lambda\mu}$ is also the number of column-strict tableaux of shape λ and content μ the value of $K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}$ is independent of n when n is sufficiently large. That is

$$K_{(m-|\lambda|,\lambda)(m-|\mu|,\mu)} = K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}$$

for $m \geq n$. For example, given $\lambda = (2)$ and $\mu = (1,1,1)$, we can see

$$K_{(6-|\lambda|,\lambda)(6-|\mu|,\mu)} = K_{(7-|\lambda|,\lambda)(7-|\mu|,\mu)} = K_{(20-|\lambda|,\lambda)(20-|\mu|,\mu)}.$$

There are three tableaux for each of these three Kostka coefficients. The only change that will occur if n increases is in the first row of their corresponding tableaux and it is where the number of 1's is increasing. This is the stability property.

We will now present the following proposition which is important for proving Lemma 1.210.

Proposition 1.209. *(Proposition 6 [OZ1]) Let $f, g \in \Lambda$ be symmetric functions of degree less than or equal to some positive integer n. Assume that $f[\Xi_\gamma] = g[\Xi_\gamma]$ for all partitions γ such that $|\gamma| \leq n$ (respectively, $|\gamma| \geq n$), then $f = g$ as elements of Λ.*

Lemma 1.210. *(Equation 10 [OZ1]) Functions $\tilde{s}_\lambda$ are the unique symmetric functions of inhomogeneous degree $|\lambda|$ which are the characters of the symmetric group when evaluated at the roots of unity. That is*

$$\tilde{s}_\lambda[\Xi_\gamma] = \left\langle s_{(n-|\lambda|,\lambda)}, p_\gamma \right\rangle.$$

Proof. We have

$$\begin{aligned}
\tilde{s}_\lambda[\Xi_\gamma] &= \sum_{|\mu|\leq|\lambda|} K^{-1}_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \tilde{h}_\mu[\Xi_\gamma] \\
&= \sum_{|\mu|\leq|\lambda|} K^{-1}_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \left\langle h_{n-|\mu|} h_\mu, p_\gamma \right\rangle \\
&= \left\langle \sum_{|\mu|\leq|\lambda|} K^{-1}_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} h_{n-|\mu|} h_\mu, p_\gamma \right\rangle \\
&= \left\langle s_{(n-|\lambda|,\lambda)}, p_\gamma \right\rangle.
\end{aligned}$$

If $n \geq |\lambda| + \lambda_1$ then the value of the $\left\langle s_{(n-|\lambda|,\lambda)}, p_\gamma \right\rangle$ is the irreducible character of the symmetric group indexed by $(n - |\lambda|, \lambda)$ evaluated at an element of cycle type γ. That is

$$\chi^{(n-|\lambda|,\lambda)}(\gamma) = \left\langle s_{(n-|\lambda|,\lambda)}, p_\gamma \right\rangle.$$

The proof is complete. ∎

The basis $\tilde{s}_\lambda$ is the character of the irreducible representation of the defining representation of the symmetric group (that is a permutation matrix). They have the same role as Schur functions are the characters in general linear group. For this reason the basis $\tilde{s}_\lambda$ are given the name the irreducible character basis. We have the following theorem that characterizes the symmetric function $\tilde{s}_\lambda$.

Theorem 1.211. *(Theorem 1 - Part 1 [OZ1]) For a fixed partition λ, $\tilde{s}_\lambda$ is the unique symmetric function with the property that for all $n \geq |\lambda| + \lambda_1$ and for all partitions γ of n,*

$$\tilde{s}_\lambda(x_1, x_2, \ldots, x_n) = \chi^{(n-|\lambda|,\lambda)}(\gamma)$$

where $x_1, x_2, \ldots, x_n$ are the eigenvalues of a permutation matrix of cycle type structure γ and $\chi^{(n-|\lambda|,\lambda)}(\gamma)$ are the values of the irreducible characters of the symmetric group.

Proof. By Lemma 1.210 we have $\tilde{s}_\lambda[\Xi_\gamma] = \langle s_{(n-|\lambda|,\lambda)}, p_\gamma \rangle$ for all $n \geq |\lambda| + \lambda_1$ and by Proposition 1.209 the only function with this property should be equivalent to the function $\tilde{s}_\lambda$ therefore it is a unique symmetric function. □

Products of Induced Trivial Characters. Given multi-set S and a set T the restriction of S to T is the multi-set $S\mid_T = \{\!\{v \in S : v \in T\}\!\}$. We can also define the restriction of a multi-set partition to the content T by $\pi\mid_T = \{\!\{S\mid_T : S \in \pi\}\!\}$. Denote $\pi\#\tau$ to be a set of multi-set partitions that appears in product. Let π and τ be multi-set partitions on the disjoint sets S and T.

$$\pi\#\tau = \{\theta : \theta \Vdash S \cup T, \theta\mid_{S=\pi}, \theta\mid_{T=\tau}\}$$

Proposition 1.212. *(Proposition 3.4 [OZ2]) For multi-set partitions $\pi \Vdash S$ and $\tau \Vdash T$ where the multi-sets S and T are disjoint,*

$$\tilde{h}_{\tilde{m}(\pi)}\tilde{h}_{\tilde{m}(\tau)} = \sum_{\theta\in\pi\#\tau} \tilde{h}_{\tilde{m}(\theta)}$$

Example 1.213. Evaluate $\tilde{h}_{21}\tilde{h}_1$.

we have $\pi = \{\!\{1 \mid 1 \mid 2\}\!\}$ and $\tau = \{\!\{3\}\!\}$. Then we have

$$\pi\#\tau = \{\!\{1 \mid 1 \mid 2 \mid 3\}\!\}, \{\!\{1 \mid 13 \mid 2\}\!\}, \{\!\{1 \mid 1 \mid 23\}\!\}$$

Therefore

$$\tilde{h}_{21}\tilde{h}_1 = \tilde{h}_{211} + \tilde{h}_{111} + \tilde{h}_{21}.$$

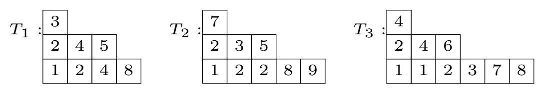
Change of Basis $\{h_\lambda\} \to \{\tilde{s}_\alpha\}$ and $\{e_\mu\} \to \{\tilde{s}_\alpha\}$

Given two bases $\{u_\lambda\}_\lambda$ and $\{v_\lambda\}_\lambda$ denote $\{u_\lambda\} \longrightarrow \{v_\mu\}$ to be the transition coefficients of v_μ in the expansion of u_λ in terms of v-basis. At first we will provide a combinatorial method that explains the coefficients of the irreducible character expansion of a complete homogenous symmetric function i.e. the transition coefficients in $\{h_\lambda\} \longrightarrow \{\tilde{s}_\mu\}$. In this method we need to combine the notion of multi-set partition of a multi-set and column-strict tableau.

In order to work with column-strict tableaux on sets or multi-sets we have chosen a total ordering of lexicographic when we are reading the entries of the multi-set in increasing order.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)})$ denote $\overline{\lambda} = (\lambda_2, \dots, \lambda_{l(\lambda)})$ to represent the partition that the first part is removed. For a tableau T, denote $shape(T)$ as the partition of the outer shape of the tableau so $\overline{shape(T)}$ is a digram with the first row removed.

Example 1.214. The following tableaux T_1, T_2 and T_3 have the same partition (or shape of diagram) when the their first rows are removed, i.e. $\overline{shape(T_1)} = \overline{shape(T_2)} = \overline{shape(T_3)}$. The partition corresponding to this shape is $\overline{shape(T_i)} = (3, 1)$.

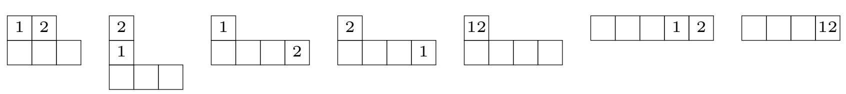


Theorem 1.215. *(Theorem 5 [OZ1v2]) For a partition μ, choose $m \geq |\mu|$, then*

$$h_\mu = \sum_T \tilde{s}_{\overline{shape(T)}}$$

where $\overline{\lambda} = (\lambda_2, \dots, \lambda_{l(\lambda)})$ and the sum is over all column strict tableaux with m blank cells in the first row and the rest of the cells filled with multi-sets of labels such that the total content of the tableau is $\{\!\{1^{\mu_1}, 2^{\mu_2}, \dots, l^{\mu_l}\}\!\}$.

Example 1.216. To find the expansion of h_{11} in the irreducible character basis, consider the following 7 column-strict tableaux that the entires are multi-sets with total content of the tableau $\{\!\{1, 2\}\!\}$. We may also take $m = 3$ empty entries.



Therefore we have

$$h_{11} = 2\tilde{s}_\bullet + 3\tilde{s}_1 + \tilde{s}_{11} + \tilde{s}_2.$$

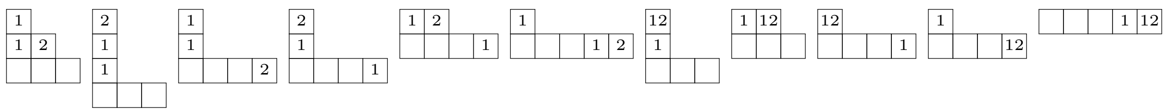
Note that in Example 1.216 the multi-sets of labels such that the total content of the tableau is $\{\!\{1, 2\}\!\}$ is from multi-set partitions $\{\!\{12\}\!\}$ and $\{\!\{1 \mid 2\}\!\}$.

Theorem 1.217. *(Theorem 8 [OZ1v2]) For a partition μ, choose $m \geq |\mu|$, then*

$$e_\mu = \sum_T \tilde{s}_{\overline{shape(T)}}$$

where the sum is over tableaux that are weakly increasing in rows and columns and that have m blank cells in the first row and sets as labels of the tableaux (no repeated values in the sets allowed) such that the content of the tableau is $\{\!\{1^{\mu_1}, 2^{\mu_2}, \ldots, l^{\mu_l}\}\!\}$. A set is allowed to appear multiple times in the same column if and only if the set has an odd number of entries. A set is allowed to appear multiple times in the same row if and only if the set has an even number of entries.

Example 1.218. Expansion of e_{21} in the irreducible character basis for $m = 3$.



We have

$$e_{21} = \tilde{s}_{\bullet} + 3\tilde{s}_1 + 2\tilde{s}_2 + 3\tilde{s}_{11} + \tilde{s}_{21} + \tilde{s}_{111}.$$

Frobenius Map Φ_n of Type A

In the last sections we studied how to construct elements $\tilde{s}_\lambda$ and $\tilde{h}_\lambda$ and the fact that they are the characters of symmetric group as permutation matrix when evaluated at eigenvalues of roots of unity. This can be written in the following theorem.

Theorem 1.219. *(Theorem 10 [OZ3]) For a partition λ, the symmetric functions $\tilde{s}_\lambda$ and $\tilde{h}_\lambda$ have the property that for a positive integer $n \geq |\lambda| + \lambda_1$ and $\mu \vdash n$,*

$$\tilde{s}_\lambda[\Xi_\mu] = \left\langle s_{(n-|\lambda|,\lambda)}, p_\mu \right\rangle = \chi^{(n-|\lambda|,\lambda)}(\mu) \quad \textit{and} \quad \tilde{h}_\lambda[\Xi_\mu] = \left\langle h_{(n-|\lambda|,\lambda)}, p_\mu \right\rangle.$$

The Frobenius map is a linear isomorphism from the class functions of the symmetric group to the ring of symmetric functions. The characters of the symmetric group are a class functions and in the previous sections we have introduced and expressed the characters of the symmetric group as symmetric functions ($\tilde{s}_\lambda$ and $\tilde{h}_\lambda$). We will define the Frobenius map on symmetric functions as in the following definition.

Definition 1.220. (Equation 12 [OZ3]) Given $\Phi_n : \Lambda \longrightarrow \Lambda$ define the Frobenius map by

$$\Phi_n(f) = \sum_{\mu \vdash n} f[\Xi_\mu] \frac{p_\mu}{z_\mu}$$

The map Φ_n is from the ring of symmetric functions to the subspace of symmetric functions of degree n. Given two functions $f, g \in \Lambda$ we have $\Phi_n(fg) = \Phi_n(f) * \Phi_n(g)$. We can restate Theorem 1.219 and write the following lemma.

Lemma 1.221. *(Equation 13 [OZ3]) Given $\Phi_n : \Lambda \longrightarrow \Lambda$ there exist functions $\tilde{s}_\lambda, \tilde{h}_\lambda \in \Lambda$ such that*

$$\Phi_n(\tilde{s}_\lambda) = s_{(n-|\lambda|,\lambda)} \quad \textit{and} \quad \Phi_n(\tilde{h}_\lambda) = h_{(n-|\lambda|,\lambda)}.$$

An Algorithm to Compute Functions $\tilde{s}_\lambda \in \Lambda$. The goal is to find a function f in $\Phi_n(f) = s_{(n-|\lambda|,\lambda)}$ and when it does, it means $f = \tilde{s}_\lambda$.

- ⋄ Step 1: Guess $f = \tilde{s}_\lambda$.
- ⋄ Step 2: Calculate $\Phi_n(f)$.
- ⋄ Step 3: Correct guess by subtracting off the leading term.
- ⋄ Step 4: Repeat Step 2 until it is 'right' in that $\Phi_n(f) = s_{(n-|\lambda|,\lambda)}$.

Example 1.222. Use $\Phi_{10}(\tilde{s}_{21})$ and find $\tilde{s}_{21}$. Given $n = 10$ let $\Phi_{S_n}(\tilde{s}_{21}) = s_{721}$ where $\tilde{s}_{21} = s_{21} +$ lower terms. The following calculation are done in SageMath. At first we will make a guess $f = s_{21}$ and execute the algorithm steps.

$$\begin{aligned}
&\text{Step 1:} && \Phi_{10}(s_{21}) = s_{721} + 2s_{811} + 2s_{82} + 3s_{91} + s_{10} \\
&\text{Step 2:} && \Phi_{10}(s_{21} - 2s_{11}) = s_{721} + 2s_{82} + s_{91} + s_{10} \\
&\text{Step 3:} && \Phi_{10}(s_{21} - 2s_{11} - 2s_2) = s_{721} - 3s_{91} - 3s_{10} \\
&\text{Step 4:} && \Phi_{10}(s_{21} - 2s_{11} - 2s_2 + 3s_1) = s_{721}
\end{aligned}$$

By Lemma 1.221 and Lemma 1.210 the unique function $\tilde{s}_{21}$ that is

$$\tilde{s}_{21} = s_{21} - 2s_{11} - 2s_2 + 3s_1$$

Example 1.223. Values of $\tilde{s}_\lambda$ in terms of Schur functions for partitions of $\lambda \vdash n$ for $n = 0, 1, 2, 3, 4$.
The following calculation are done in SageMath.

Partition λ	$\tilde{s}_\lambda$ in terms of s_μ
$\lambda = ()$	$\tilde{s}_\bullet = s_\bullet = 1$
$\lambda = (1)$	$\tilde{s}_1 = -s_\bullet + s_1$
$\lambda = (2)$	$\tilde{s}_2 = -2s_1 + s_2$
$\lambda = (1,1)$	$\tilde{s}_{11} = s_\bullet - s_1 + s_{11}$
$\lambda = (3)$	$\tilde{s}_3 = s_1 - s_{11} - 2s_2 + s_3$
$\lambda = (2,1)$	$\tilde{s}_{21} = 3s_1 - 2s_{11} - 2s_2 + s_{21}$
$\lambda = (1,1,1)$	$\tilde{s}_{111} = -s_\bullet + s_1 - s_{11} + s_{111}$
$\lambda = (4)$	$\tilde{s}_4 = 2s_{11} + s_2 - s_{21} - 2s_3 + s_4$
$\lambda = (3,1)$	$\tilde{s}_{31} = -3s_1 + 3s_{11} - s_{111} + 5s_2 - 3s_{21} - 2s_3 + s_{31}$
$\lambda = (2,2)$	$\tilde{s}_{22} = -s_1 + 4s_{11} + 2s_2 - 2s_{21} - s_{22} - s_3$
$\lambda = (2,1,1)$	$\tilde{s}_{211} = -4s_1 + 3s_{11} - 2s_{111} + 3s_2 - 2s_{21} + s_{211}$
$\lambda = (1,1,1,1)$	$\tilde{s}_{1111} = s_\bullet - s_1 + s_{11} - s_{111} + s_{1111}$

Table 1.24: $\tilde{s}_\lambda$ in terms of s_μ

Character Tables of Symmetric Groups and $\{\tilde{s}_\lambda\}$ Basis. In [Macdonald] the ring of symmetric function is defined as

$$\Lambda = \overleftarrow{\lim_{n\to\infty}}\ \mathbb{Q}[x_1,\ldots,x_n]^{S_n}$$

Equivalently we can say, if $n > m$ then by setting $x_{m+1} = \cdots = x_n = 0$ we have

$$\mathbb{Q}[x_1,\ldots,x_n]^{S_n} \longrightarrow \mathbb{Q}[x_1,\ldots,x_m]^{S_m}.$$

As we know there is one character of S_n for all partitions of n. The character table of groups S_1, S_2 and S_3 including the irreducible character bases are provided below. Note that $\xi = e^{\frac{2\pi i}{3}}$.

Eigenvalues:	1	$\tilde{s}_\lambda(\mathbf{x})$
S_1	$K_{(1)}$	
$\chi^{\yng(1)}$	1	$\tilde{s}_\bullet(x_1) = 1$

Table 1.25: $\tilde{s}_\lambda(\mathbf{x})$ in S_1

Eigenvalues:	1,1	1,-1	$\tilde{s}_\lambda(\mathbf{x})$
S_2	$K_{(1,1)}$	$K_{(2)}$	
$\chi^{\yng(2)}$	1	1	$\tilde{s}_\bullet(x_1,x_2) = 1$
$\chi^{\yng(1,1)}$	1	-1	$\tilde{s}_1(x_1,x_2) = -1 + x_1 + x_2$

Table 1.26: $\tilde{s}_\lambda(\mathbf{x})$ in S_2

Eigenvalues:	1,1,1	1,-1,1	1,ξ,ξ^2	$\tilde{s}_\lambda(\mathbf{x})$
S_3	$K_{(1,1,1)}$	$K_{(2,1)}$	$K_{(3)}$	
$\chi^{\yng(3)}$	1	1	1	$\tilde{s}_\bullet(x_1,x_2,x_3) = 1$
$\chi^{\yng(2,1)}$	2	0	-1	$\tilde{s}_1(x_1,x_2,x_3) = -1 + x_1 + x_2 + x_3$
$\chi^{\yng(1,1,1)}$	1	-1	-1	$\tilde{s}_{11}(x_1,x_2,x_3) = 1 - (x_1 + x_2 + x_3) + x_1x_2 + x_1x_3 + x_2x_3$

Table 1.27: $\tilde{s}_\lambda(\mathbf{x})$ in S_3

Example 1.224. The transition coefficients $\{\tilde{h}_\gamma\} \longrightarrow \{\tilde{s}_\mu\}$ given in Definition 1.205 by equation

$$\tilde{h}_\mu = \sum_{|\lambda| \leqslant |\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \tilde{s}_\lambda$$

where we used Kostka coefficients and for $\{\tilde{s}_\gamma\} \longrightarrow \{\tilde{h}_\mu\}$ the inverse of Kostka coefficients is applied. The following table shows the transition coefficients in $\{\tilde{h}_\mu\} \longrightarrow \{\tilde{s}_\lambda\}$ for n=1, 2, 3.

$\{\tilde{h}_\mu\} \longrightarrow \{\tilde{s}_\lambda\}$	$\tilde{s}_\bullet$	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_{11}$	$\tilde{s}_3$	$\tilde{s}_{21}$	$\tilde{s}_{111}$
$\tilde{h}_\bullet$	1						
$\tilde{h}_1$	1	1					
$\tilde{h}_2$	1	1	1				
$\tilde{h}_{11}$	1	2	1	1			
$\tilde{h}_3$	1	1	1		1		
$\tilde{h}_{21}$	1	2	2	1	1	1	
$\tilde{h}_{111}$	1	3	3	3	1	2	1

Table 1.28: Transition coefficients in $\{\tilde{h}_\mu\} \longrightarrow \{\tilde{s}_\lambda\}$

Example 1.225. Recall in Definition 1.167 the equation

$$h_\lambda = \sum_{\pi \Vdash \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \cdots, l^{\lambda_l}\}\!\}} \tilde{h}_{\widetilde{m}(\pi)}$$

provided us a formula to do this change of basis and to find the transition coefficients of $\{h_\lambda\} \longrightarrow \{\tilde{h}_\gamma\}$. Using SageMath the following table shows the transition coefficients in $\{h_\lambda\} \longrightarrow \{\tilde{h}_\mu\}$ for $n = 1, 2, 3$.

$\{h_\lambda\} \longrightarrow \{\tilde{h}_\mu\}$	$\tilde{h}_\bullet$	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{h}_{11}$	$\tilde{h}_3$	$\tilde{h}_{21}$	$\tilde{h}_{111}$
$h_\bullet$	1						
h_1		1					
h_2		1	1				
h_{11}		1		1			
h_3		1		1	1		
h_{21}		1		2		1	
h_{111}		1		3			1

Table 1.29: Transition coefficients in $\{h_\lambda\} \longrightarrow \{\tilde{h}_\mu\}$

Summary of Results

We will demonstrate the summary of results of this section with some examples below.

$\{h_\lambda\}$	$\longrightarrow$ $\{\tilde{h}_\gamma\}$	$\longrightarrow$ $\{\tilde{s}_\mu\}$
Content Multi-Set **Example:** we have h_{21} of content: $\{\!\{1^2, 2^1\}\!\}$	Multi-Set Partitions **Example:** We have $h_{21} = \tilde{h}_1 + 2\tilde{h}_{11} + \tilde{h}_{21}$ The corresponding multi-set partitions: $\{\!\{112\}\!\}$,$\{\!\{1 \mid 12\}\!\}$, $\{\!\{2 \mid 11\}\!\}$,$\{\!\{1 \mid 1 \mid 2\}\!\}$	Column-Strict Tableaux **Example:** We have $\tilde{h}_{11} = \tilde{s}_\bullet + 2\tilde{s}_1 + \tilde{s}_{11} + \tilde{s}_2$ The corresponding column-strict tableaux: $\begin{array}{\|c\|c\|c\|c\|}\hline 1&1&2&3\\\hline\end{array}$, $\begin{array}{lll} 3 & & \\ 1&1&2 \end{array}$, $\begin{array}{lll} 2 & & \\ 1&1&3 \end{array}$, $\begin{array}{ll} 3 & \\ 2 & \\ 1&1 \end{array}$, $\begin{array}{ll} 2&3 \\ 1&1 \end{array}$
Content Multi-Set **Example:** we have e_{21} of content: $\{\!\{\bar{1}^2, \bar{2}^1\}\!\}$	Set Partitions **Example:** We have $e_{21} = \tilde{he}_{(1\|1)} + \tilde{he}_{(\bullet\|21)}$ The corresponding set partitions: $\{\!\{\bar{1} \rrbracket \bar{1}\bar{2}\}\!\}$, $\{\!\{\bar{1} \rrbracket \bar{1} \rrbracket \bar{2}\}\!\}$	

Table 1.30: Summary of results $\{h_\lambda\} \longrightarrow \{\tilde{h}_\gamma\} \longrightarrow \{\tilde{s}_\mu\}$

2 Hyperoctahedral Group Induced Trivial Character Basis

This chapter is where the first new results of the thesis will be presented. This result is in Theorem 2.24 (The Main Theorem) which establishes the fact these new bases, namely, the B_n-induced trivial character basis have the required property. These bases they play a similar intermediate role to the corresponding bases of Λ for the symmetric group and this will help us to construct the irreducible character basis of the hyperoctahedral group in Chapter 3. The structure of this chapter is outlined as the following.

1. In section 2.1 we will introduce new notation, operations and we will define the B_n-*induced trivial character basis* denote it by $\{\tilde{h}^B_{(\lambda,\mu)}\}$. This basis resides in the tensor square of the ring of symmetric function, i.e. $\{\tilde{h}^B_{(\lambda,\mu)}\} \in \Lambda \otimes \Lambda$ and we will construct them in the expansion of tensor squares in the induced trivial character basis of type A, i.e. a linear combination of $\tilde{h}_\alpha \otimes \tilde{he}_{\beta|\eta}$ for partitions α, β and η. In Theorem 2.24 (The Main Theorem) we will prove that this basis have indeed the required property needed in order to be the B_n analogue of the induced trivial character basis. This is for $n \geq 0$ and given a pair of partitions $(\gamma,\nu) \vdash n$ we have
$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] = \left\langle h_{(|\gamma|+|\nu|-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle.$$

2. In Example 2.26 we will provide the table of the transition coefficients in $\{\tilde{h}^B_{(\lambda,\mu)}\} \longrightarrow \{\tilde{h}_\alpha \otimes \tilde{he}_{\beta|\eta}\}$ that is the coefficients of $\tilde{h}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}_\alpha \otimes \tilde{he}_{\beta|\eta}$ for $n = 1, 2, 3$.

3. In section 2.2 we will provide the combinatorics and tableaux on B_n-induced trivial character basis. At first some notations such as multi-set in two alphabets: the unbarred and barred entries is introduced and the splittings of these multi-sets will be defined in Definition 2.41. In Lemma 2.52 given all notations, multi-sets and operations defined in Definition 2.41 we have
$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] = \sum_{\lambda^{(L)}\uplus\mu^{(L)}} \sum_{\lambda^{(R)}|\mu^{(R)}} (-1)^{|\mu^{(R)}|} \tilde{h}_{\lambda^{(L)}\uplus\mu^{(L)}}[\Xi_\gamma] \cdot \tilde{he}_{\lambda^{(R)}|\mu^{(R)}}[\Xi_\nu].$$

4. In Proposition 2.53 we will give a combinatorial interpretation of Theorem 2.24 (The Main Theorem) that is, given all notations, multi-sets and operations in Definition 2.41 and Definition 2.51 the value of $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu]$ is equal to sum of signed weights of all possible splittings $wt^{S^{(L)}\uplus S^{(R)}}\left(F^{(L)} \uplus F^{(R)}\right)$ that is
$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] = \sum_{S^{(L)}\uplus S^{(R)}} wt^{S^{(L)}\uplus S^{(R)}}\left(F^{(L)} \uplus F^{(R)}\right).$$

5. Finally, in Corollary 2.54 we will present a very interesting and compact formula that gives the value of $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu]$ using r, the number of rows occupied by barred entries in the diagram

corresponding to partition ν

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \sum_{S^{(L)} \uplus S^{(R)}} \sum_{F^{(L)} \uplus F^{(R)}} (-1)^r.$$

6. The combinatorics methods presented in section 2.2 will be used in evaluations $\tilde{h}^B_{\bullet,1}[\Xi_{11}, \Xi_1]$ in Example 2.55 and $\tilde{h}^B_{1,2}[\Xi_{21}, \Xi_{211}]$ in Example 2.56.

2.1 Induced Trivial Character of Hyperoctahedral Group $\{\tilde{h}^B_{(\lambda,\mu)}\}$

In this section we will present a definition of the induced trivial character basis of the hyperoctahedral group. This function resides in the tensor square of the ring of symmetric function $\Lambda \otimes \Lambda$ and we will call it the *B_n-induced trivial character basis* and denote it by $\tilde{h}^B_{(\lambda,\mu)}$. This basis will be introduced and given as the expansion in tensor square in the induced trivial character basis of type A, i.e. a linear combination of $\tilde{h}_\alpha \otimes \tilde{h}e_{\beta|\eta}$ for partitions α, β and η . At first we will review and set up our basic notation and definitions and we will present two lemmas and The Main Theorem.

2.1.1 B_n - Induced Trivial Character

Recall that a pair of partition is denoted $(\lambda, \mu) \vdash n$ when its length is defined as $|(\lambda, \mu)| \coloneqq |\lambda| + |\mu| = n$. The conjugacy classes of the hyperoctahedral group B_n are indexed by a pair of partitions $(\gamma, \nu) \vdash n$ consist of all signed permutations such that parts of the partition γ are the cycles with even number of negative signs and the parts of ν are the cycles with an odd number of negative signs. In the study of the character theory of group B_n it is known that the irreducible representations of B_n are also indexed by a pair of partitions (λ, μ) of total size n. We have stated in Theorem 1.135 an explicit formula for the character of the hyperoctahedral group in terms of Schur functions [Stembridge]. In addition, in Corollary 1.142 the Frobenius characteristic image of the induced character of group B_n is presented as well [Beck].

Although we have already discussed the following concepts in more detail section in 5.2.1, we will provide and review them again as they are important in our study leading up to The Main Theorem.

Definition 2.1. Let $\Lambda_B = \bigoplus_{n \geqslant 0} \Lambda_{B_n}$ where Λ_{B_n} is the space

$$\Lambda_{B_n} = \Lambda_{B_n}[X, Y] \coloneqq \bigoplus_{k=0}^{n} \Lambda^k[X] \otimes \Lambda^{n-k}[Y]$$

and $\Lambda^k[X]$ denotes the space of symmetric functions of degree k in the variables $X \coloneqq x_1 + x_2 + \cdots$. Now let $R(B_n)$ be the set of class functions of group B_n over the complex numbers and let $K^{B_n}_{(\gamma,\nu)}$

be the conjugacy class of B_n indexed by the pair of partitions $(\gamma, \nu) \vdash n$. The basis of class function as indicator function is given by

$$\mathbf{1}^{B_n}_{(\gamma,\nu)}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in K^{B_n}_{(\gamma,\nu)} \\ 0 & \text{otherwise} \end{cases}$$

The $\mathbf{B_n}$**-Frobenius characteristic map** is defined by

$$\begin{aligned} \mathscr{F}_{B_n}: \quad R(B_n) &\longrightarrow \quad \Lambda_{B_n}[X,Y] \\ \mathbf{1}^{B_n}_{(\gamma,\nu)} &\longmapsto \frac{1}{z_\gamma} p_\gamma[X] \frac{1}{z_\nu} p_\nu[Y] \end{aligned}$$

where $z_\lambda \coloneqq 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!$ for $\lambda = (1^{m_1} 2^{m_2} \ldots n^{m_n})$.

Theorem 2.2. *[Stembridge] The irreducible characters of the group B_n are the induced characters of the irreducible and parity representation i.e. $\chi^{(\lambda,\mu)} \coloneqq \left(\chi^{(\lambda,\bullet)} \otimes \chi^{(\bullet,\mu)}\right) \uparrow^{Bn}_{B_{|\lambda|} \times B_{|\mu|}}$ where $|\lambda| + |\mu| = n$ and their Frobenius characteristics image is*

$$\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) = s_\lambda[X+Y] s_\mu[X-Y].$$

Remark 2.3. Consider a basis of $\Lambda_{B_n} \coloneqq \bigoplus_{k=0}^{n} \Lambda^k[X]\Lambda^{n-k}[Y]$ as the following given spanning set

$$\{h_\lambda[X+Y] h_\mu[X-Y]\}_{(\lambda,\mu) \vdash n}$$

This basis is the Frobenius characteristic image of a natural representation of B_n. Particularly, the Frobenius characteristic image of the induced trivial character of B_n as the plethysm of the symmetric functions $h_\lambda[X+Y] h_\mu[X-Y]$. This is expressed as a corollary to the preceding theorem.

Corollary 2.4. *[Beck] Given I_λ and Δ_μ to be the trivial and parity representation on $B_{|\lambda|}$ and $B_{|\mu|}$ respectively and $(I_\lambda \otimes \Delta_\mu) \uparrow^{Bn}_{B_{|\lambda|} \times B_{|\mu|}}$ to be the induced representation and $\mathcal{I}^{(\lambda,\mu)} \coloneqq (1_\lambda \otimes \delta_\mu) \uparrow^{B_n}_{B_{|\lambda|} \times B_{|\mu|}}$ to be the induced character that is indexed by pair of partitions $(\lambda, \mu) \vdash n$. The Frobenius characteristic image is given by*

$$\mathscr{F}_{B_n}(\mathcal{I}^{(\lambda,\mu)}) = h_\lambda[X+Y] h_\mu[X-Y].$$

Remark 2.5. The induced trivial character of B_n evaluated at conjugacy classes $K^{B_n}_{(\gamma,\nu)}$ can also be expressed using the B_n-character inner product

$$\begin{aligned} \mathcal{I}^{(\lambda,\mu)}(\gamma,\nu) &= \langle h_\lambda[X+Y] h_\mu[X-Y], p_\gamma[X] p_\nu[Y] \rangle \\ &= \left\langle \mathscr{F}_{B_n}(\mathcal{I}^{(\lambda,\mu)}), p_\gamma[X] p_\nu[Y] \right\rangle \end{aligned}$$

Definition 2.6. The B_n-sum-difference complete homogeneous basis is denoted by

$$H^B_{(\lambda,\mu)}[X,Y] := h_\lambda[X+Y]h_\mu[X-Y].$$

Given $h_r[-X] = S(h_r)[X] = (-1)^r\omega(h_r)[X] = (-1)^r e_r[X]$ the expansion on a single part r is given by

- ⋄ $h_r[X+Y] = \sum_{i=0}^{r} h_i[X]h_{r-i}[Y]$
- ⋄ $h_r[X-Y] = \sum_{i=0}^{r} (-1)^{r-i}h_i[X]e_{r-i}[Y]$

These expansions are homogenous symmetric function evaluated at a sum and difference of two alphabets respectively. Note that for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$ we have

$$h_\lambda[X \pm Y] = h_{\lambda_1}[X \pm Y]h_{\lambda_2}[X \pm Y] \cdots h_{\lambda_{l(\lambda)}}[X \pm Y].$$

Definition 2.7. Define the operation $\uplus$ on partitions λ and μ to be a concatenation of these partitions as lists and then sort the entries in (weakly) decreasing order, that is, for $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_k)$ we have $\lambda \uplus \mu = sort(\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_k)$. This operation is commutative and associative with $\tilde{h}_{\bullet\uplus\bullet} = \tilde{h}_\bullet = 1$.

Example 2.8. Given $\lambda = (3, 2, 2, 1)$ and $\mu = (4, 1)$ we have $\lambda \uplus \mu = (4, 3, 2, 2, 1, 1)$.

In 1.6 we defined a binary operation on the S_n-induced trivial character basis $\{\tilde{h}_\lambda\}$. This operation is used in our study and proofs in this part of the thesis. In this page we will recall this in the following.

Definition 2.9. (Definition 1.196) a linear binary operation $\odot_A : \Lambda \times \Lambda \longrightarrow \Lambda$ on the induced trivial characters of the symmetric group (type A) such that given

$$f = \sum_\alpha a_\alpha \tilde{h}_\alpha \text{ and } g = \sum_\beta b_\beta \tilde{h}_\beta$$

where $f, g \in \Lambda$ and $a_\alpha, b_\beta \in \mathbb{Q}$, then

$$f \odot_A g := \sum_{\alpha,\beta} a_\alpha b_\beta \tilde{h}_{\alpha\uplus\beta}$$

The operation $\uplus$ is as defined in Definition 2.7 and the binary operation $\odot_A$ is commutative and associative where $\tilde{h}_\bullet \odot_A \tilde{h}_\bullet := \tilde{h}_\bullet = 1$.

Example 2.10. For example $-\tilde{h}_{21} \odot_A 2\tilde{h}_{32} \odot_A \tilde{h}_{11} = -2\tilde{h}_{322111}$

Remark 2.11. (Remarks 1.198 and 1.199) Given all partitions λ, τ and μ since $\tilde{he}_{(\lambda|\mu)} = \tilde{h}_\lambda \odot_A \tilde{e}_\mu$ the evaluations at the eigenvalues of roots of unity are

$$\left(\tilde{h}_\lambda \odot_A \tilde{h}_\tau\right)[\Xi_\mu] := \tilde{h}_{(\lambda \uplus \tau)}[\Xi_\mu] = \mathcal{H}_{(\lambda \uplus \tau),\mu} = \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda h_\tau, p_\mu \right\rangle$$

and

$$\left(\tilde{h}_\lambda \odot_A \tilde{e}_\tau\right)[\Xi_\mu] := \tilde{he}_{(\lambda|\tau)}[\Xi_\mu] = \mathcal{HE}_{(\lambda|\tau),\mu} := \left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda e_\tau, p_\mu \right\rangle.$$

The binary operation $\odot_A$ can be extended for the symmetric function $\tilde{e}_\tau$.

Lemma 2.12. *(Lemma 1.201) Given the binary operation $\odot_A : \Lambda \times \Lambda \longrightarrow \Lambda$ on the induced trivial characters of the symmetric group (type A) and*

$$f = \sum_\alpha a_\alpha \tilde{e}_\alpha \quad and \quad g = \sum_\beta b_\beta \tilde{e}_\beta$$

where $f, g \in \Lambda$ and $a_\alpha, b_\beta \in \mathbb{Q}$, then

$$f \odot_A g = \sum_{\alpha,\beta} a_\alpha b_\beta \tilde{e}_{\alpha \uplus \beta}.$$

Example 2.13. Using SageMath, to evaluate $\tilde{he}_{11|2}[\Xi_{32211}]$ since $\tilde{e}_2 = e_2 = h_{11} - h_2$ we have $\tilde{he}_{\bullet|2} = \tilde{h}_{11} - \tilde{h}_2$ and $\tilde{he}_{11|2} = \tilde{h}_{11} \odot_A \tilde{e}_2$ therefore

$$\begin{aligned}
\tilde{he}_{11|2}[\Xi_{32211}] &= \tilde{h}_{1111}[\Xi_{32211}] - \tilde{h}_{211}[\Xi_{32211}] \\
&= 0 - 4 \\
&= -4
\end{aligned}$$

Now, in order to make the connection and move forward from our work in type A induced trivial character basis and the operation $\odot_A$ to the corresponding ideas in type B which is our main concentration, we will present the following definition.

Definition 2.14. Given $f_1 \otimes g_1 \in \Lambda \otimes \Lambda$ and $f_2 \otimes g_2 \in \Lambda \otimes \Lambda$ the linear binary operation $\odot_B$: $(\Lambda \otimes \Lambda) \times (\Lambda \otimes \Lambda) \longrightarrow \Lambda \otimes \Lambda$ on the tensor square of symmetric functions is defined as

$$(f_1 \otimes g_1) \odot_B (f_2 \otimes g_2) := (f_1 \odot_A f_2) \otimes (g_1 \odot_A g_2).$$

This operation is commutative and associative and we can extend this linearly. Furthermore, as we know the tensor square can be displayed in two variables and we may show this product in $\Lambda \otimes \Lambda$ as

$$(f_1[X] g_1[Y]) \odot_B (f_2[X] g_2[Y]) := (f_1[X] \odot_A f_2[X]) (g_1[Y] \odot_A g_2[Y]).$$

Example 2.15. An example of using the binary operation $\odot_B$ as an operation in type B is presented below.

$$\left(\tilde{h}_1[X]\tilde{h}_3[Y]\right) \odot_B \left(\tilde{h}_{21}[X]\tilde{h}_{11}[Y]\right) = \left(\tilde{h}_1[X] \odot_A \tilde{h}_{21}[X]\right)\left(\tilde{h}_3[Y] \odot_A \tilde{h}_{11}[Y]\right)$$
$$= \tilde{h}_{211}[X]\tilde{h}_{311}[Y]$$

2.1.2 Expressing $\tilde{h}^B_{(\lambda,\mu)}$ (Type B) in terms of $\tilde{h}_\alpha \otimes \tilde{he}_{\beta|\eta}$ (Type A)

In this section we will define the B_n-induced trivial character bases in terms of S_n-induced trivial character bases. It means we assume that the existence of these functions in type A i.e. S_n-induced trivial character bases in [OZ1] that is discussed in section 1.6 and we will introduce a method to construct the analogous in type B.

Definition 2.16. The B_n-induced trivial character bases that have a pair of partitions that are of form $((m), \bullet)$ or $(\bullet, (n))$ are considered as generators and they expand in form of

$$\tilde{h}^B_{(m,\bullet)} := \sum_{i=0}^{m} \tilde{h}_i \otimes \tilde{h}_{m-i}$$

and

$$\tilde{h}^B_{(\bullet,n)} := \sum_{i=0}^{n} (-1)^{n-i}\tilde{h}_i \otimes \tilde{e}_{n-i}$$

respectively, where the expansions are in tensor square of S_n-induced trivial character bases. For any other basis $\tilde{h}^B_{(\lambda,\mu)}$ that are generally indexed by pair of partitions (λ, μ) they can be constructed using the preceding generators of type B by

$$\tilde{h}^B_{(\lambda,\mu)} := \prod_i^{\odot_B} \tilde{h}^B_{(\lambda_i,\bullet)} \odot_B \prod_j^{\odot_B} \tilde{h}^B_{(\bullet,\mu_j)}.$$

Remark 2.17. In Proposition 4.5 in [OZ2] it is shown that

$$\tilde{h}_\lambda[X+Y] = \sum_{|\alpha|+|\beta|=\lambda} \tilde{h}_\alpha[X]\tilde{h}_\beta[Y]$$

In Example 3.68 we have explored the expansion of the basis $\tilde{h}^B_{(\lambda,\mu)}$ in terms of

$$\tilde{H}^B_{(\lambda,\mu)} := \tilde{h}_\lambda[X+Y]\tilde{h}_\mu[X-Y] \quad .$$

The following lemma will allow us to use the binary operation $\odot_B$ in the construction of B_n-induced trivial character bases in terms of S_n-induced trivial character bases as it is defined in Definition 2.16.

Lemma 2.18. *Assume that a basis* $\{\tilde{h}^B_{(\lambda,\mu)}\} \in \Lambda \otimes \Lambda$ *up to degree* n *is known and is well defined. By Definition 2.14 and Definition 2.16 the linear binary operation* $\odot_B : (\Lambda \otimes \Lambda) \times (\Lambda \otimes \Lambda) \longrightarrow \Lambda \otimes \Lambda$ *on the induced trivial characters of the hyperoctahedral group will be given by*

$$\tilde{h}^B_{(\lambda,\mu)}[X,Y] \odot_B \tilde{h}^B_{(\gamma,\nu)}[X,Y] = \tilde{h}^B_{(\lambda\uplus\gamma,\mu\uplus\nu)}[X,Y].$$

Proof. The binary operation $\odot_B$ is commutative and associative with $\tilde{h}^B_{(\bullet,\bullet)} = 1$. This product rule holds as we can show in the following.

$$\begin{aligned}
\tilde{h}^B_{(\lambda,\mu)} \odot_B \tilde{h}^B_{(\gamma,\nu)} &= \left(\prod_i^{\odot_B} \tilde{h}^B_{(\lambda_i,\bullet)} \odot_B \prod_j^{\odot_B} \tilde{h}^B_{(\bullet,\mu_j)} \right) \odot_B \left(\prod_k^{\odot_B} \tilde{h}^B_{(\gamma_k,\bullet)} \odot_B \prod_t^{\odot_B} \tilde{h}^B_{(\bullet,\nu_t)} \right) \\
&= \left(\prod_i^{\odot_B} \tilde{h}^B_{(\lambda_i,\bullet)} \odot_B \prod_k^{\odot_B} \tilde{h}^B_{(\gamma_k,\bullet)} \right) \odot_B \left(\prod_j^{\odot_B} \tilde{h}^B_{(\bullet,\mu_j)} \odot_B \prod_t^{\odot_B} \tilde{h}^B_{(\bullet,\nu_t)} \right) \\
&= \tilde{h}^B_{(\lambda\uplus\gamma,\bullet)} \odot_B \tilde{h}^B_{(\bullet,\mu\uplus\nu)} \\
&= \tilde{h}^B_{(\lambda\uplus\gamma,\mu\uplus\nu)}
\end{aligned}$$

Note that in here as we assumed that a basis $\{\tilde{h}^B_{(\lambda,\mu)}\}$ up to degree n, therefore we must have that if $max\,|(\lambda,\mu)| := |\lambda| + |\mu| = r$ then $max\,|(\gamma,\nu)| := |\gamma| + |\nu| = n - r$. ☐

Example 2.19. An example of Lemma 2.18 is

$$\tilde{h}^B_{21,1}[X,Y] \odot_B \tilde{h}^B_{1,3}[X,Y] = \tilde{h}^B_{211,31}[X,Y].$$

We will present the following two lemmas which are specific cases of The Main Theorem that will be presented at the end. This theorem shows the important relation between induced trivial character bases of the type B and type A.

Lemma 2.20. *Given integers* $m, n \geq 0$ *and given a pair of partitions* $(\gamma,\nu) \vdash n$ *for* $n \geq m$ *we have*

$$\tilde{h}^B_{(m,\bullet)}[\Xi_\gamma, \Xi_\nu] = \left\langle h_{(|\gamma|+|\nu|-m)}[X+Y] h_m[X+Y], p_\gamma[X] p_\nu[Y] \right\rangle$$

Proof. We have defined the induced trivial character basis of type B in terms of type A for the pair of partitions that are of form $((m),\bullet)$ as

$$\tilde{h}^B_{(m,\bullet)} = \sum_{i=0}^{m} \tilde{h}_i \otimes \tilde{h}_{m-i}.$$

Given $\tilde{h}_\alpha \otimes \tilde{h}_\beta \in \Lambda \otimes \Lambda$ we can write the evaluations at the roots of unity in type A by $\tilde{h}_\alpha[\Xi_\gamma] = \langle h_{|\gamma|-|\alpha|} h_\alpha, p_\gamma \rangle$ therefore the left hand side of the equation is

$$\begin{aligned} \tilde{h}^B_{(m,\bullet)}[\Xi_\gamma, \Xi_\nu] =& \sum_{i=0}^{m} \tilde{h}_i[\Xi_\gamma] \tilde{h}_{m-i}[\Xi_\nu] \\ =& \sum_{i=0}^{m} \langle h_{|\gamma|-i} h_i, p_\gamma \rangle \langle h_{|\nu|-m+i} h_{m-i}, p_\gamma \rangle \end{aligned}$$

The right hand side of the equation can be expanded as

$$\begin{aligned} &\langle h_{(|\gamma|+|\nu|-m)}[X+Y] h_m[X+Y], p_\gamma[X] p_\nu[Y] \rangle \\ &\quad = \left\langle \left(\sum_{j=0}^{|\gamma|+|\nu|-m} h_j[X] h_{|\gamma|+|\nu|-m-j}[Y] \right) \left(\sum_{i=0}^{m} h_i[X] h_{m-i}[Y] \right), p_\gamma[X] p_\nu[Y] \right\rangle \\ &\quad = \left\langle \sum_{j=0}^{|\gamma|+|\nu|-m} \sum_{i=0}^{m} h_j[X] h_i[X] h_{|\gamma|+|\nu|-m-j}[Y] h_{m-i}[Y], p_\gamma[X] p_\nu[Y] \right\rangle \\ &\quad = \sum_{j=0}^{|\gamma|+|\nu|-m} \sum_{i=0}^{m} \langle h_j[X] h_i[X] h_{|\gamma|+|\nu|-m-j}[Y] h_{m-i}[Y], p_\gamma[X] p_\nu[Y] \rangle \\ &\quad = \sum_{j=0}^{|\gamma|+|\nu|-m} \sum_{i=0}^{m} \langle h_j[X] h_i[X], p_\gamma[X] \rangle \langle h_{|\gamma|+|\nu|-m-j}[Y] h_{m-i}[Y], p_\nu[Y] \rangle \end{aligned}$$

Now switch the sum over i and j and notice that the only term that is non-zero if for $j = |\gamma| - i$. This mean that the terms that survive in the right hand side of the equation will be

$$\begin{aligned} =& \sum_{i=0}^{m} \langle h_{|\gamma|-i} h_i, p_\gamma \rangle \langle h_{|\nu|-m+i} h_{m-i}, p_\nu \rangle \\ =& \sum_{i=0}^{m} \tilde{h}_i[\Xi_\gamma] \tilde{h}_{m-i}[\Xi_\nu] \\ =& \tilde{h}^B_{(m,\bullet)}[\Xi_\gamma, \Xi_\nu] \end{aligned}$$

The proof is complete. □

Lemma 2.21. *Let $n \geq 0$ and given a pair of partitions $(\gamma, \nu) \vdash n$ for $n \geq m$ we have*

$$\tilde{h}^B_{(\bullet,m)}[\Xi_\gamma, \Xi_\nu] = \langle h_{(|\gamma|+|\nu|-m)}[X+Y] h_m[X-Y], p_\gamma[X] p_\nu[Y] \rangle$$

Proof. We have

$$
\begin{aligned}
&\left\langle h_{(|\gamma|+|\nu|-m)}[X+Y]h_m[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\qquad = \left\langle (\sum_{j=0}^{|\gamma|+|\nu|-m} h_j[X]h_{|\gamma|+|\nu|-m-j}[Y])(\sum_{i=0}^{m}(-1)^{m-i}h_i[X]e_{m-i}[Y]), p_\gamma[X]p_\nu[Y]\right\rangle \\
&\qquad = \left\langle \sum_{j=0}^{|\gamma|+|\nu|-m}\sum_{i=0}^{m}(-1)^{m-i}h_j[X]h_i[X]h_{|\gamma|+|\nu|-m-j}[Y]e_{m-i}[Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\qquad = \sum_{j=0}^{|\gamma|+|\nu|-m}\sum_{i=0}^{m}(-1)^{m-i}\left\langle h_j[X]h_i[X]h_{|\gamma|+|\nu|-m-j}[Y]e_{m-i}[Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\qquad = \sum_{j=0}^{|\gamma|+|\nu|-m}\sum_{i=0}^{m}\langle h_j[X]h_i[X], p_\gamma[X]\rangle\left\langle h_{|\gamma|+|\nu|-m-j}[Y]e_{m-i}[Y], p_\nu[Y]\right\rangle \\
&\qquad = \sum_{i=0}^{m}(-1)^{m-i}\left\langle h_{|\gamma|-i}h_i, p_\gamma\right\rangle\left\langle h_{|\nu|-m+i}e_{m-i}, p_\nu\right\rangle \\
&\qquad = \sum_{i=0}^{m}(-1)^{m-i}\tilde{h}_i[\Xi_\gamma]\tilde{e}_{m-i}[\Xi_\nu] \\
&\qquad = \tilde{h}^B_{(\bullet,m)}[\Xi_\gamma,\Xi_\nu]
\end{aligned}
$$

The proof is complete. ☐

Remark 2.22. Given $\tilde{h}_\alpha$ and $\tilde{h}_\beta$ of type A the usual product in $\cdot : \Lambda \times \Lambda \longrightarrow \Lambda$ is not the same as the binary operation $\odot_A : \Lambda \times \Lambda \longrightarrow \Lambda$. Functions $\tilde{h}_\alpha$ of type A are not multiplicative but the binary operation $\odot_A$ gives us a tool to concatenate their partitions. This is used in order to prove our Main Theorem (Theorem 2.24). We can compare them in examples below.

1. $\tilde{h}_1 \cdot \tilde{h}_2 = \tilde{h}_{11} + \tilde{h}_{21}$

2. $\tilde{h}_1 \odot_A \tilde{h}_2 = \tilde{h}_{21}$

Remark 2.23. The following theorem is showing that $\tilde{h}^B_{(\lambda,\mu)}$ has the property we need in order to be the B_n analogue of the induced trivial character basis. That is, we have elements of $\tilde{h}^B_{(\lambda,\mu)}$ of the ring $\Lambda \otimes \Lambda$ that have the property that they evaluate at the eigenvalues of the permutation matrices to the values of the induced trivial characters.

Note that we will present a generalization of the last two proofs of the lemmas so we will be keeping and including these lemmas to serve as a guide for the the more general result.

Theorem 2.24. *[The Main Theorem] For $n \geq 0$ and given pair of partitions $(\gamma,\nu) \vdash n$ we have*

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \left\langle h_{(|\gamma|+|\nu|-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle .$$

Proof. The right hand side of the equation is

$$\begin{aligned}
&\left\langle h_{(|\gamma|+|\nu|-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
=&\Bigg\langle\Bigg(\sum_{k=0}^{|\gamma|+|\nu|-|\lambda|-|\mu|} h_k[X]h_{|\gamma|+|\nu|-|\lambda|-|\mu|-k}[Y]\Bigg)\Bigg(\sum_{i_1=0}^{\lambda_1} h_{i_1}[X]h_{\lambda_1-i_1}[Y]\Bigg)\cdots\Bigg(\sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}} h_{i_{l(\lambda)}}[X]h_{\lambda_{l(\lambda)}-i_{l(\lambda)}}[Y]\Bigg)\cdot \\
&\cdot\Bigg(\sum_{j_1=0}^{\mu_1}(-1)^{\mu_1-j_1}h_{j_1}[X]e_{\mu_1-j_1}[Y]\Bigg)\cdots\Bigg(\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{\mu_{l(\mu)}-j_{l(\mu)}}h_{j_{l(\mu)}}[X]e_{\mu_{l(\mu)}-j_{l(\mu)}}[Y]\Bigg), p_\gamma[X]p_\nu[Y]\Bigg\rangle \\
=&\Bigg\langle\sum_{k=0}^{|\gamma|+|\nu|-|\lambda|-|\mu|}\sum_{i_1=0}^{\lambda_1}\cdots\sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}}\sum_{j_1=0}^{\mu_1}\cdots\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{|\mu|-(j_1+\cdots+j_{l(\mu)})}\Big(h_k[X]h_{i_1}[X]\cdots h_{i_{l(\lambda)}}[X]h_{j_1}[X]\cdots h_{j_{l(\mu)}}[X]\Big)\cdot \\
&\cdot\Big(h_{|\gamma|+|\nu|-|\lambda|-|\mu|-k}[Y]h_{\lambda_1-i_1}[Y]\cdots h_{\lambda_{l(\lambda)}-i_{l(\lambda)}}[Y]\Big)\Big(e_{\mu_1-j_1}[Y]\cdots e_{\mu_{l(\mu)}-j_{l(\mu)}}[Y]\Big), p_\gamma[X]p_\nu[Y]\Bigg\rangle \\
=&\sum_{k=0}^{|\gamma|+|\nu|-|\lambda|-|\mu|}\sum_{i_1=0}^{\lambda_1}\cdots\sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}}\sum_{j_1=0}^{\mu_1}\cdots\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{|\mu|-(j_1+\cdots+j_{l(\mu)})}\Big\langle h_k[X]h_{i_1}[X]\cdots h_{i_{l(\lambda)}}[X]h_{j_1}[X]\cdots h_{j_{l(\mu)}}[X], p_\gamma[X]\Big\rangle\cdot \\
&\cdot\Big\langle\Big(h_{|\gamma|+|\nu|-|\lambda|-|\mu|-k}[Y]h_{\lambda_1-i_1}[Y]\cdots h_{\lambda_{l(\lambda)}-i_{l(\lambda)}}[Y]\Big)\Big(e_{\mu_1-j_1}[Y]\cdots e_{\mu_{l(\mu)}-j_{l(\mu)}}[Y]\Big), p_\nu[Y]\Big\rangle
\end{aligned}$$

Now switch the sum over k and $j_{l(\mu)}$ and notice that the only term that is non-zero the inner products is for $k = |\gamma| - \big(i_1+\cdots+i_{l(\lambda)}\big) - \big(j_1+\cdots+j_{l(\mu)}\big)$ hence the last equality is equal to

$$\begin{aligned}
&\sum_{i_1=0}^{\lambda_1}\cdots\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{|\mu|-(j_1+\cdots+j_{l(\mu)})}\Big\langle h_{|\gamma|-(i_1+\cdots+i_{l(\lambda)})-(j_1+\cdots+j_{l(\mu)})}[X]h_{i_1}[X]\cdots h_{j_{l(\mu)}}[X], p_\gamma[X]\Big\rangle\cdot \\
&\cdot\Big\langle\Big(h_{|\nu|-|\lambda|-|\mu|+(i_1+\cdots+i_{l(\lambda)})+(j_1+\cdots+j_{l(\mu)})}[Y]h_{\lambda_1-i_1}[Y]\cdots h_{\lambda_{l(\lambda)}-i_{l(\lambda)}}[Y]\Big)\Big(e_{\mu_1-j_1}[Y]\cdots e_{\mu_{l(\mu)}-j_{l(\mu)}}[Y]\Big), p_\nu[Y]\Big\rangle
\end{aligned}$$

Now since $\tilde{h}^B_{(\lambda,\mu)} := \prod_i^{\odot_B} \tilde{h}^B_{(\lambda_i,\bullet)} \odot_B \prod_j^{\odot_B} \tilde{h}^B_{(\bullet,\mu_j)}$ to evaluate the left hand side we have

$$\begin{aligned}
\tilde{h}^B_{(\lambda,\mu)} &:= \left(\tilde{h}^B_{(\lambda_1,\bullet)}\odot_B\cdots\odot_B\tilde{h}^B_{(\lambda_{l(\lambda)},\bullet)}\right)\odot_B\left(\tilde{h}^B_{(\bullet,\mu_1)}\odot_B\cdots\odot_B\tilde{h}^B_{(\bullet,\mu_{l(\mu)})}\right) \\
&= \left(\sum_{i_1=0}^{\lambda_1}\tilde{h}_{i_1}\otimes\tilde{h}_{\lambda_1-i_1}\right)\odot_B\cdots\odot_B\left(\sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}}\tilde{h}_{i_{l(\lambda)}}\otimes\tilde{h}_{\lambda_{l(\lambda)}-i_{l(\lambda)}}\right)\odot_B \\
&\quad\odot_B\left(\sum_{j_1=0}^{\mu_1}(-1)^{\mu_1-j_1}\tilde{h}_{j_1}\otimes\tilde{e}_{\mu_1-j_1}\right)\odot_B\cdots\odot_B\left(\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{\mu_{l(\mu)}-j_{l(\mu)}}\tilde{h}_{j_{l(\mu)}}\otimes\tilde{e}_{\mu_{l(\mu)}-j_{l(\mu)}}\right)
\end{aligned}$$

By Definition 2.14 we have

$$\tilde{h}^B_{(\lambda,\mu)} = \left[\sum_{i_1=0}^{\lambda_1}\cdots\sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}} \left(\tilde{h}_{i_1}\odot_A\cdots\odot_A\tilde{h}_{i_{l(\lambda)}}\right)\otimes\left(\tilde{h}_{\lambda_1-i_1}\odot_A\cdots\odot_A\tilde{h}_{\lambda_{l(\lambda)}-i_{l(\lambda)}}\right)\right]\odot_B$$
$$\odot_B\left[\sum_{j_1=0}^{\mu_1}\cdots\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{|\mu|-\left(j_1+\cdots+j_{l(\mu)}\right)}\left(\tilde{h}_{j_1}\odot_A\cdots\odot_A\tilde{h}_{j_{l(\lambda)}}\right)\otimes\left(\tilde{e}_{\mu_1-j_1}\odot_A\cdots\odot_A\tilde{e}_{\mu_{l(\mu)}-j_{l(\mu)}}\right)\right]$$
$$=\sum_{i_1=0}^{\lambda_1}\cdots\sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}}\sum_{j_1=0}^{\mu_1}\cdots\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{|\mu|-\left(j_1+\cdots+j_{l(\mu)}\right)}\left[\tilde{h}_{sort(i_1,\cdots,i_{l(\lambda)},j_1,\cdots,j_{l(\mu)})}\right]\otimes$$
$$\otimes\left[\tilde{h}_{sort(\lambda_1-i_1,\cdots,\gamma_{l(\gamma)}-i_{l(\gamma)})}\odot_A\tilde{e}_{sort(\mu_1-j_1,\cdots,\mu_{l(\mu)}-j_{l(\mu)})}\right]$$

Now the left hand side of the equation in the theorem is the evaluation of $\tilde{h}^B_{(\lambda,\mu)}$ at the eigenvalues of roots of unity i.e. $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu]$. By Remark 1.198 and Remark 1.199 we have

$$\left(\tilde{h}_\lambda\odot_A\tilde{h}_\tau\right)[\Xi_\mu]\coloneqq\tilde{h}_{(\lambda\uplus\tau)}[\Xi_\mu]=\mathcal{H}_{(\lambda\uplus\tau),\mu}\coloneqq\left\langle h_{|\mu|-|\lambda|-|\tau|}h_\lambda h_\tau,p_\mu\right\rangle$$

and

$$\left(\tilde{h}_\lambda\odot_A\tilde{e}_\tau\right)[\Xi_\mu]=\tilde{he}_{(\lambda|\tau)}[\Xi_\mu]=\mathcal{HE}_{(\lambda|\tau),\mu}\coloneqq\left\langle h_{|\mu|-|\lambda|-|\tau|}h_\lambda e_\tau,p_\mu\right\rangle.$$

Now the last equality is

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu]$$
$$=\sum_{i_1=0}^{\lambda_1}\cdots\sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}}\sum_{j_1=0}^{\mu_1}\cdots\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{|\mu|-\left(j_1+\cdots+j_{l(\mu)}\right)}\left(\tilde{h}_{sort(i_1,\cdots,i_{l(\lambda)},j_1,\cdots,j_{l(\mu)})}[\Xi_\gamma]\right)\cdot$$
$$\cdot\left(\tilde{he}_{sort(\lambda_1-i_1,\cdots,\lambda_{l(\lambda)}-i_{l(\lambda)})|sort(\mu_1-j_1,\cdots,\mu_{l(\mu)}-j_{l(\mu)})}[\Xi_\nu]\right)$$
$$=\sum_{i_1=0}^{\lambda_1}\cdots\sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}}(-1)^{|\mu|-\left(j_1+\cdots+j_{l(\mu)}\right)}\left\langle h_{|\gamma|-\left(i_1+\cdots+i_{l(\lambda)}\right)-\left(j_1+\cdots+j_{l(\mu)}\right)}[X]h_{i_1}[X]\cdots h_{j_{l(\mu)}}[X],p_\gamma[X]\right\rangle\cdot$$
$$\cdot\left\langle\left(h_{|\nu|-|\lambda|-|\mu|+(i_1+\cdots+i_{l(\lambda)})+(j_1+\cdots+j_{l(\mu)})}[Y]h_{\lambda_1-i_1}[Y]\cdots h_{\lambda_{l(\lambda)}-i_{l(\lambda)}}[Y]\right)\left(e_{\mu_1-j_1}[Y]\cdots e_{\mu_{l(\mu)}-j_{l(\mu)}}[Y]\right),p_\nu[Y]\right\rangle$$

The last equality is exactly the right hand side hence the proof is complete. □

Example 2.25. Show

$$\tilde{h}^B_{21,1}[\Xi_{11},\Xi_{21}] = \langle h_{211}[X+Y]h_1[X-Y], p_{11}[X]p_{21}[Y]\rangle$$

To evaluate the left hand side we need

$$\begin{aligned}
\tilde{h}^B_{21,1} &:= \left(\tilde{h}^B_{2,\bullet}\odot_B\tilde{h}^B_{1,\bullet}\right)\odot_B\left(\tilde{h}^B_{\bullet,1}\right)\\
&= \left(\tilde{h}_\bullet\otimes\tilde{h}_2+\tilde{h}_1\otimes\tilde{h}_1+\tilde{h}_2\otimes\tilde{h}_\bullet\right)\odot_B\left(\tilde{h}_\bullet\otimes\tilde{h}_1+\tilde{h}_1\otimes\tilde{h}_\bullet\right)\odot_B\left(-\tilde{h}_\bullet\otimes\tilde{e}_1+\tilde{h}_1\otimes\tilde{e}_\bullet\right)\\
&= -\tilde{h}_\bullet\otimes\tilde{he}_{21|1}+\tilde{h}_1\otimes\tilde{he}_{21|\bullet}-\tilde{h}_1\otimes\tilde{he}_{2|1}+\tilde{h}_{11}\otimes\tilde{he}_{2|\bullet}-\tilde{h}_1\otimes\tilde{he}_{11|1}+\tilde{h}_{11}\otimes\tilde{he}_{11|\bullet}-\\
&-\tilde{h}_{11}\otimes\tilde{he}_{1|1}+\tilde{h}_{111}\otimes\tilde{he}_{1|\bullet}-\tilde{h}_2\otimes\tilde{he}_{1|1}+\tilde{h}_{21}\otimes\tilde{he}_{1|\bullet}-\tilde{h}_{21}\otimes\tilde{he}_{\bullet|1}+\tilde{h}_{211}\otimes\tilde{he}_{\bullet|\bullet}\\
&= -\tilde{h}_\bullet\otimes\tilde{h}_{211}+\tilde{h}_{11}\otimes\tilde{h}_2-\tilde{h}_1\otimes\tilde{h}_{111}+\tilde{h}_{111}\otimes\tilde{h}_1-\tilde{h}_2\otimes\tilde{h}_{11}+\tilde{h}_{211}\otimes\tilde{h}_\bullet
\end{aligned}$$

Note that the last equation the right tensors are all changed to $\tilde{h}_\alpha$ basis. The left hand side is

$$\begin{aligned}
\tilde{h}^B_{21,1}[\Xi_{11},\Xi_{21}] &= -\tilde{h}_\bullet[\Xi_{11}]\tilde{h}_{211}[\Xi_{21}]+\tilde{h}_{11}[\Xi_{11}]\tilde{h}_2[\Xi_{21}]-\tilde{h}_1[\Xi_{11}]\tilde{h}_{111}[\Xi_{21}]+\\
&\quad+\tilde{h}_{111}[\Xi_{11}]\tilde{h}_1[\Xi_{21}]-\tilde{h}_2[\Xi_{11}]\tilde{h}_{11}[\Xi_{21}]+\tilde{h}_{211}[\Xi_{11}]\tilde{h}_\bullet[\Xi_{21}]\\
&= -\langle h_2h_\bullet,p_{11}\rangle\langle h_{-1}h_{211},p_{21}\rangle+\langle h_{11},p_{11}\rangle\langle h_1h_2,p_{21}\rangle-\langle h_1h_1,p_{11}\rangle\langle h_{111},p_{21}\rangle+\\
&\quad+\langle h_{-1}h_{111},p_{11}\rangle\langle h_2h_1,p_{21}\rangle-\langle h_2,p_{11}\rangle\langle h_1h_{11},p_{21}\rangle+\langle h_{-2}h_{211},p_{11}\rangle\langle h_{21}h_\bullet,p_{21}\rangle\\
&= -1\cdot0+2\cdot1-2\cdot0+0\cdot1-1\cdot0+0\cdot1\\
&= 2
\end{aligned}$$

The corresponding inner products form the left hand side and right hand side are the equivalent. The right hand side is evaluated below.

$$\begin{aligned}
&\langle h_{211}[X+Y]h_1[X-Y],p_{11}[X]p_{21}[Y]\rangle\\
&= \langle(h_2[X]+h_1[X]h_1[Y]+h_2[Y])(h_1[X]+h_1[Y])(h_1[X]+h_1[Y])(h_1[X]-e_1[Y]),p_{11}[X]p_{21}[Y]\rangle\\
&= \langle h_{2111}[X]+h_{211}[X]h_1[Y]-h_{21}[X]h_1[Y]e_1[Y]-h_2[X]h_{11}[Y]e_1[Y]+\\
&+h_{111}[X]h_{11}[Y]-h_1[X]h_{111}[Y]e_1[Y]+h_{111}[X]h_2[Y]-\\
&-h_1[X]h_{21}[Y]e_1[Y]+h_{11}[X]h_{21}[Y]-h_{211}[Y]e_1[Y],p_{11}[X]p_{21}[Y]\rangle\\
&= \langle h_{11}[X]h_{21}[Y],p_{11}[X]p_{21}[Y]\rangle\\
&= \langle h_{11}[X],p_{11}[X]\rangle\langle h_{21}[Y],p_{21}[Y]\rangle\\
&= 2\cdot1\\
&= 2
\end{aligned}$$

When evaluating both sides of the equation in our example the only inner products that survived (non-zero) in both is the term $\langle h_{11}[X],p_{11}[X]\rangle\langle h_{21}[Y],p_{21}[Y]\rangle$.

Transition coefficients in $\{\tilde{h}^B_{\lambda,\mu)}\}\longrightarrow\{\tilde{h}_\alpha\otimes\tilde{he}_{\beta|\eta}\}$

Example 2.26. Evaluate $\tilde{h}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}_\alpha\otimes\tilde{he}_{\beta|\eta}$ in Type A for $n=0,1,2,3$.

In Definition 2.16, we have defined the generators of the B_n-induced trivial character bases in pair of partitions that are of form $((m),\bullet)$ or $(\bullet,(n))$ in expansions

$$\tilde{h}^B_{(m,\bullet)}:=\sum_{i=0}^{m}\tilde{h}_i\otimes\tilde{h}_{m-i}\ \text{ and }\ \tilde{h}^B_{(\bullet,n)}:=\sum_{i=0}^{n}(-1)^{n-i}\tilde{h}_i\otimes\tilde{e}_{n-i}$$

where the expansions are in tensor square of S_n-induced trivial character bases. In order to be consistent, since $\tilde{h}_i=\tilde{he}_{(i|\bullet)}$ and $\tilde{e}_j=\tilde{he}_{(\bullet|j)}$ we chose to write the above expressions to have the right tensors in symmetric functions of the form $\tilde{he}_{(i|\bullet)}$ and $\tilde{he}_{(\bullet|j)}$ respectively. Recall a basis $\tilde{h}^B_{(\lambda,\mu)}$ that is generally indexed by a pair of partitions (λ,μ) it can be constructed by also Definition 2.16, that is

$$\tilde{h}^B_{(\lambda,\mu)}:=\prod_i^{\odot_B}\tilde{h}^B_{(\lambda_i,\bullet)}\odot_B\prod_j^{\odot_B}\tilde{h}^B_{(\bullet,\mu_j)}.$$

The following three examples will help us to see how to write some basic expressions. The complete table is presented on the next page. Note that the pair of partitions are shown in a more compact notation in the following evaluations.

1. For the generators indexed by a pair of partitions of form $(1,\bullet)$ the expansion of $\tilde{h}^B_{1,\bullet}$ is

$$\begin{aligned}\tilde{h}^B_{1,\bullet}&=\sum_{i=0}^{1}\tilde{h}_i\otimes\tilde{h}_{1-i}\\&=\tilde{h}_\bullet\otimes\tilde{h}_1+\tilde{h}_1\otimes\tilde{h}_\bullet\\&=\tilde{h}_\bullet\otimes\tilde{he}_{1|\bullet}+\tilde{h}_1\otimes\tilde{he}_{\bullet|\bullet}\end{aligned}$$

2. For the generators indexed by pairs of partitions of form $(\bullet,2)$ the expansion of $\tilde{h}^B_{\bullet,2}$ is

$$\begin{aligned}\tilde{h}^B_{\bullet,2}&=\sum_{i=0}^{2}(-1)^{2-i}\tilde{h}_i\otimes\tilde{e}_{2-i}\\&=\tilde{h}_\bullet\otimes\tilde{e}_2-\tilde{h}_1\otimes\tilde{e}_1+\tilde{h}_2\otimes\tilde{e}_\bullet\\&=\tilde{h}_\bullet\otimes\tilde{he}_{\bullet|2}-\tilde{h}_1\otimes\tilde{he}_{\bullet|1}+\tilde{h}_2\otimes\tilde{he}_{\bullet|\bullet}\end{aligned}$$

3. For the B_n-induced trivial character bases $\tilde{h}^B_{1,2}$ indexed by pairs of partitions $(2,1)$ we simply have the expansion

$$\begin{aligned}\tilde{h}^B_{1,2}&:=\tilde{h}^B_{1,\bullet}\odot_B\tilde{h}^B_{\bullet,2}\\&=\left(\tilde{h}_\bullet\otimes\tilde{he}_{1|\bullet}+\tilde{h}_1\otimes\tilde{he}_{\bullet|\bullet}\right)\odot_B\left(\tilde{h}_\bullet\otimes\tilde{he}_{\bullet|2}-\tilde{h}_1\otimes\tilde{he}_{\bullet|1}+\tilde{h}_2\otimes\tilde{he}_{\bullet|\bullet}\right)\\&=\tilde{h}_\bullet\otimes\tilde{he}_{1|2}-\tilde{h}_1\otimes\tilde{he}_{1|1}+\tilde{h}_2\otimes\tilde{he}_{1|\bullet}+\tilde{h}_1\otimes\tilde{he}_{\bullet|2}-\tilde{h}_{11}\otimes\tilde{he}_{\bullet|1}+\tilde{h}_{21}\otimes\tilde{he}_{\bullet|\bullet}\end{aligned}$$

We have summarized the evaluations in the table below.

n	(λ,μ)	$\tilde{h}^B_{(\lambda,\mu)}$ in Terms of $\tilde{h}_\alpha \otimes \tilde{he}_{\beta\vert\eta}$ in Type A
0	$(\bullet,\bullet)$	$\tilde{h}^B_{\bullet,\bullet} = \tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert\bullet}$
1	$(1,\bullet)$	$\tilde{h}^B_{1,\bullet} = \tilde{h}_\bullet \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(\bullet,1)$	$\tilde{h}^B_{\bullet,1} = -\tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert\bullet}$
2	$(2,\bullet)$	$\tilde{h}^B_{2,\bullet} = \tilde{h}_\bullet \otimes \tilde{he}_{2\vert\bullet} + \tilde{h}_1 \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_2 \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(11,\bullet)$	$\tilde{h}^B_{11,\bullet} = \tilde{h}_\bullet \otimes \tilde{he}_{11\vert\bullet} + 2\tilde{h}_1 \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(1,1)$	$\tilde{h}^B_{1,1} = -\tilde{h}_\bullet \otimes \tilde{he}_{1\vert 1} + \tilde{h}_1 \otimes \tilde{he}_{1\vert\bullet} - \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(\bullet,2)$	$\tilde{h}^B_{\bullet,2} = \tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert 2} - \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_2 \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(\bullet,11)$	$\tilde{h}^B_{\bullet,11} = \tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert 11} - 2\tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert\bullet}$
3	$(3,\bullet)$	$\tilde{h}^B_{3,\bullet} = \tilde{h}_\bullet \otimes \tilde{he}_{3\vert\bullet} + \tilde{h}_1 \otimes \tilde{he}_{2\vert\bullet} + \tilde{h}_2 \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_3 \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(21,\bullet)$	$\tilde{h}^B_{21,\bullet} = \tilde{h}_\bullet \otimes \tilde{he}_{21\vert\bullet} + \tilde{h}_1 \otimes \tilde{he}_{2\vert\bullet} + \tilde{h}_1 \otimes \tilde{he}_{11\vert\bullet} + \tilde{h}_{11} \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_2 \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_{21} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(111,\bullet)$	$\tilde{h}^B_{111,\bullet} = \tilde{h}_\bullet \otimes \tilde{he}_{111\vert\bullet} + 3\tilde{h}_1 \otimes \tilde{he}_{11\vert\bullet} + 3\tilde{h}_{11} \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_{111} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(11,1)$	$\tilde{h}^B_{11,1} = -\tilde{h}_\bullet \otimes \tilde{he}_{11\vert 1} + \tilde{h}_1 \otimes \tilde{he}_{11\vert\bullet} - 2\tilde{h}_1 \otimes \tilde{he}_{1\vert 1} + 2\tilde{h}_{11} \otimes \tilde{he}_{1\vert\bullet} - \tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{111} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(2,1)$	$\tilde{h}^B_{2,1} = -\tilde{h}_\bullet \otimes \tilde{he}_{2\vert 1} + \tilde{h}_1 \otimes \tilde{he}_{2\vert\bullet} - \tilde{h}_1 \otimes \tilde{he}_{1\vert 1} + \tilde{h}_{11} \otimes \tilde{he}_{1\vert\bullet} - \tilde{h}_2 \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{21} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(1,2)$	$\tilde{h}^B_{1,2} = \tilde{h}_\bullet \otimes \tilde{he}_{1\vert 2} - \tilde{h}_1 \otimes \tilde{he}_{1\vert 1} + \tilde{h}_2 \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 2} - \tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{21} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(1,11)$	$\tilde{h}^B_{1,11} = \tilde{h}_\bullet \otimes \tilde{he}_{1\vert 11} - 2\tilde{h}_1 \otimes \tilde{he}_{1\vert 1} + \tilde{h}_{11} \otimes \tilde{he}_{1\vert\bullet} + \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 11} - 2\tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{111} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(\bullet,3)$	$\tilde{h}^B_{\bullet,3} = -\tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert 3} + \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 2} - \tilde{h}_2 \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_3 \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(\bullet,21)$	$\tilde{h}^B_{\bullet,21} = -\tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert 21} + \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 2} - \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 11} - \tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert 1} - \tilde{h}_2 \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{21} \otimes \tilde{he}_{\bullet\vert\bullet}$
	$(\bullet,111)$	$\tilde{h}^B_{\bullet,111} = -\tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert 111} + 3\tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 11} - 3\tilde{h}_{11} \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_{111} \otimes \tilde{he}_{\bullet\vert\bullet}$

Table 2.1: $\tilde{h}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}_\alpha \otimes \tilde{he}_{\beta\vert\eta}$ in Type A

Example 2.27. Evaluate $\tilde{h}^B_{(\lambda,\mu)}$ in terms of $h_\alpha \otimes h_\beta$ for $n = 0, 1, 2, 3$. We used SageMath and we have the following

n	(λ,μ)	$\tilde{h}^B_{(\lambda,\mu)}$ in terms of $h_\alpha \otimes h_\beta$
0	$(\bullet,\bullet)$	$\tilde{h}^B_{\bullet,\bullet} = h_\bullet \otimes h_\bullet$
1	$(1,\bullet)$	$\tilde{h}^B_{1,\bullet} = h_\bullet \otimes h_1 + h_1 \otimes h_\bullet$
	$(\bullet,1)$	$\tilde{h}^B_{\bullet,1} = -h_\bullet \otimes h_1 + h_1 \otimes h_\bullet$
2	$(2,\bullet)$	$\tilde{h}^B_{2,\bullet} = -h_\bullet \otimes h_1 + h_\bullet \otimes h_2 - h_1 \otimes h_\bullet + h_1 \otimes h_1 + h_2 \otimes h_\bullet$
	$(11,\bullet)$	$\tilde{h}^B_{11,\bullet} = -h_\bullet \otimes h_1 + h_\bullet \otimes h_{11} - h_1 \otimes h_\bullet + 2h_1 \otimes h_1 + h_{11} \otimes h_\bullet$
	$(1,1)$	$\tilde{h}^B_{1,1} = h_\bullet \otimes h_1 - h_\bullet \otimes h_{11} - h_1 \otimes h_\bullet + h_{11} \otimes h_\bullet$
	$(\bullet,2)$	$\tilde{h}^B_{\bullet,2} = h_\bullet \otimes h_{11} - h_\bullet \otimes h_2 - h_1 \otimes h_\bullet - h_1 \otimes h_1 + h_2 \otimes h_\bullet$
	$(\bullet,11)$	$\tilde{h}^B_{\bullet,11} = -h_\bullet \otimes h_1 + h_\bullet \otimes h_{11} - h_1 \otimes h_\bullet - 2h_1 \otimes h_1 + h_{11} \otimes h_\bullet$
3	$(3,\bullet)$	$\tilde{h}^B_{3,\bullet} = -h_\bullet \otimes h_{11} + h_\bullet \otimes h_3 - 2h_1 \otimes h_1 + h_1 \otimes h_2 - h_{11} \otimes h_\bullet + h_2 \otimes h_1 + h_3 \otimes h_\bullet$
	$(21,\bullet)$	$\tilde{h}^B_{21,\bullet} = h_\bullet \otimes h_1 - 2h_\bullet \otimes h_{11} + h_\bullet \otimes h_{21} + h_1 \otimes h_\bullet - 4h_1 \otimes h_1 + h_1 \otimes h_{11} + h_1 \otimes h_2 - 2h_{11} \otimes h_\bullet + h_{11} \otimes h_1 + h_2 \otimes h_1 + h_{21} \otimes h_\bullet$
	$(111,\bullet)$	$\tilde{h}^B_{111,\bullet} = 2h_\bullet \otimes h_1 - 3h_\bullet \otimes h_{11} + h_\bullet \otimes h_{111} + 2h_1 \otimes h_\bullet - 6h_1 \otimes h_1 + 3h_1 \otimes h_{11} - 3h_{11} \otimes h_\bullet + 3h_{11} \otimes h_1 + h_{111} \otimes h_\bullet$
	$(11,1)$	$\tilde{h}^B_{11,1} = -2h_\bullet \otimes h_1 + 3h_\bullet \otimes h_{11} - h_\bullet \otimes h_{111} + 2h_1 \otimes h_\bullet - h_1 \otimes h_{11} - 3h_{11} \otimes h_\bullet + h_{11} \otimes h_1 + h_{111} \otimes h_\bullet$
	$(2,1)$	$\tilde{h}^B_{2,1} = -h_\bullet \otimes h_1 + 2h_\bullet \otimes h_{11} - h_\bullet \otimes h_{21} + h_1 \otimes h_\bullet - h_1 \otimes h_{11} + h_1 \otimes h_2 - 2h_{11} \otimes h_\bullet + h_{11} \otimes h_1 - h_2 \otimes h_1 + h_{21} \otimes h_\bullet$
	$(1,2)$	$\tilde{h}^B_{1,2} = h_\bullet \otimes h_1 - h_\bullet \otimes h_{11} + h_\bullet \otimes h_{111} - h_\bullet \otimes h_{21} + h_1 \otimes h_\bullet + h_1 \otimes h_1 - h_1 \otimes h_2 - 2h_{11} \otimes h_\bullet - h_{11} \otimes h_1 + h_2 \otimes h_1 + h_{21} \otimes h_\bullet$
	$(1,11)$	$\tilde{h}^B_{1,11} = 2h_\bullet \otimes h_1 - 3h_\bullet \otimes h_{11} + h_\bullet \otimes h_{111} + 2h_1 \otimes h_\bullet + 2h_1 \otimes h_1 - h_1 \otimes h_{11} - 3h_{11} \otimes h_\bullet - h_{11} \otimes h_1 + h_{111} \otimes h_\bullet$
	$(\bullet,3)$	$\tilde{h}^B_{\bullet,3} = -h_\bullet \otimes h_{111} + 2h_\bullet \otimes h_{21} - h_\bullet \otimes h_3 + h_1 \otimes h_1 + h_1 \otimes h_{11} - h_1 \otimes h_2 - h_{11} \otimes h_\bullet - h_2 \otimes h_1 + h_3 \otimes h_\bullet$
	$(\bullet,21)$	$\tilde{h}^B_{\bullet,21} = -h_\bullet \otimes h_1 + h_\bullet \otimes h_{11} - h_\bullet \otimes h_{111} + h_\bullet \otimes h_{21} + h_1 \otimes h_\bullet + h_1 \otimes h_1 + 2h_1 \otimes h_{11} - h_1 \otimes h_2 - 2h_{11} \otimes h_\bullet - h_{11} \otimes h_1 - h_2 \otimes h_1 + h_{21} \otimes h_\bullet$
	$(\bullet,111)$	$\tilde{h}^B_{\bullet,111} = -2h_\bullet \otimes h_1 + 3h_\bullet \otimes h_{11} - h_\bullet \otimes h_{111} + 2h_1 \otimes h_\bullet + 3h_1 \otimes h_{11} - 3h_{11} \otimes h_\bullet - 3h_{11} \otimes h_1 + h_{111} \otimes h_\bullet$

Table 2.2: $\tilde{h}^B_{(\lambda,\mu)}$ in terms of $h_\alpha \otimes h_\beta$ for $n = 0, 1, 2, 3$

2.2 Combinatorics and Tableaux on $\tilde{h}^B_{(\lambda,\mu)}$

At first we will set up our basic notation and combinatorial objects that will be used in this study.

2.2.1 Multi-Sets in Two Alphabets: The Unbarred and the Barred Entries

Recall that a set partition of a set S is a set of subsets $\{S_1, S_2, \ldots, S_l\}$ where $S_i \subseteq S$ and $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq l$ and $S_1 \cup S_2 \cup \cdots \cup S_l = S$. We also know that a multi-set is a set that repetition of elements of the set is allowed and we will denote it by $\{\!\{a_1, a_2, \ldots, a_r\}\!\}$. Another notation that is commonly used is the exponential notation $\{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}\}\!\}$ which represents a multi-set that the element i is repeated λ_i times. To represent a multi-set notation in our study of the hyperoctahedral group B_n (type B) we use the exponential notation in two alphabets namely the unbarred and barred entries as

$$S = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, ..., l^{\lambda_l}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, ..., \bar{k}^{\mu_k}\}\!\}$$

where an unbarred element i is repeated λ_i times and a barred element $\bar{j}$ is repeated μ_j times. At first, recall Definition 2.7 where partitions λ and μ are given in exponential notation.

Definition 2.28. Given partitions $\lambda = (1^{\lambda_1} 2^{\lambda_2} \ldots l^{\lambda_l})$ and $\mu = (\bar{1}^{\mu_1} \bar{2}^{\mu_2} \ldots \bar{k}^{\mu_k})$ define the operation $\uplus$ on partitions λ and μ to be a concatenation of these partitions as lists and then sort the entries in (weakly) decreasing order, that is

$$\lambda \uplus \mu = sort(\lambda_1, \lambda_2, \ldots, \lambda_l, \mu_1, \mu_2, \ldots, \mu_k).$$

Definition 2.29. Given partitions

$$\lambda^{(L)} = (1^{\lambda_1^{(L)}}, \ldots, l^{\lambda_l^{(L)}}), \lambda^{(R)} = (1^{\lambda_1^{(R)}}, \ldots, l^{\lambda_l^{(R)}}), \mu^{(L)} = (\bar{1}^{\mu_1^{(L)}}, \ldots, \bar{k}^{\mu_k^{(L)}}) \text{ and } \mu^{(R)} = (\bar{1}^{\mu_1^{(R)}}, \ldots, \bar{k}^{\mu_k^{(R)}})$$

define the operation $\uplus$ between two multi-sets in two alphabets by

$$\{\!\{1^{\lambda_1^{(L)}}, \ldots, l^{\lambda_l^{(L)}}, \bar{1}^{\mu_1^{(L)}}, \ldots, \bar{k}^{\mu_k^{(L)}}\}\!\} \uplus \{\!\{1^{\lambda_1^{(R)}}, \ldots, l^{\lambda_l^{(R)}}, \bar{1}^{\mu_1^{(R)}}, \ldots, \bar{k}^{\mu_k^{(R)}}\}\!\}$$
$$= \{\!\{1^{\lambda_1^{(L)}+\lambda_1^{(R)}}, \ldots, l^{\lambda_l^{(L)}+\lambda_l^{(R)}}, \bar{1}^{\mu_1^{(L)}+\mu_1^{(R)}}, \ldots, \bar{k}^{\mu_k^{(L)}+\mu_k^{(R)}}\}\!\}$$

Definition 2.30. Multi-Set Partition of a Multi-Set (In Two Alphabets) A multi-set partition π (in two alphabets) of a multi-set $S = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{k}^{\mu_k}\}\!\}$ is a set of subsets $\{\!\{S_1, S_2, \ldots, S_l\}\!\}$ where $S_1 \uplus S_2 \uplus \cdots \uplus S_l = S$ has a similar structure of the set partition but with a difference that S_i can be a multi-set and any two sets S_i and S_j can have non-empty intersection. Two or more elements of a multi-set may even be equal. Recall that given π a multi-set partition of the multi-set S we use the notation $\pi \Vdash S$.

Example 2.31. The multi-set partition $\pi = \{\!\!\{1 \mid 112\bar{1} \mid \bar{1}\bar{2} \mid \bar{2} \mid \bar{2}\bar{2}\bar{3}\}\!\!\}$ is a multi-set partition of the multi-set $S = \{\!\!\{1^3, 2^1, \bar{1}^2, \bar{2}^4, \bar{3}^1\}\!\!\}$.

In order to distinguish and name different types of multi-set partitions with respect to the combination of the two alphabets, unbarred and barred entries, we state the following definition.

Definition 2.32. Given a multi-set S and a multi-set partition π in two alphabets we will split π into three distinct types of multiplicities such that $\pi = \pi_e \uplus \pi_o \uplus \pi_r$ where

- ⋄ The multi-set partitions π_e are the multi-sets which contain an even number of distinct barred entries. For example $\pi_e = \{\!\!\{\ \{\!\!\{1, 2, 2\}\!\!\}, \{\!\!\{\bar{1}, \bar{2}\}\!\!\}, \{\!\!\{\bar{1}, \bar{2}\}\!\!\}, \{\!\!\{2, 3, \bar{1}, \bar{2}\}\!\!\}\ \}\!\!\}$.
- ⋄ The multi-set partitions π_o are the multi-sets which contain an odd number of distinct barred entries. For example $\pi_o = \{\!\!\{\ \{\!\!\{1, 1, 2, \bar{1}\}\!\!\}, \{\!\!\{1, 1, \bar{1}, \bar{2}, \bar{3}\}\!\!\}, \{\!\!\{\bar{1}\}\!\!\}, \{\!\!\{\bar{1}\}\!\!\}, \{\!\!\{\bar{1}\}\!\!\}, \{\!\!\{\bar{1}, \bar{2}, \bar{3}\}\!\!\}\ \}\!\!\}$.
- ⋄ The multi-set partitions π_r are the multi-sets which contain at least one repeated barred entry. For example $\pi_r = \{\!\!\{\ \{\!\!\{\bar{1}, \bar{1}\}\!\!\}, \{\!\!\{\bar{1}, \bar{1}\}\!\!\}, \{\!\!\{2, \bar{1}, \bar{1}\}\!\!\}, \{\!\!\{2, \bar{1}, \bar{1}\}\!\!\}, \{\!\!\{1, 1, 3, \bar{1}, \bar{1}, \bar{1}, \bar{2}, \bar{2}\}\!\!\}\ \}\!\!\}$.
- ⋄ Note that in this example we have $\pi = \pi_e \uplus \pi_o \uplus \pi_r \Vdash S = \{\!\!\{1^7, 2^6, 3^2, \bar{1}^{20}, \bar{2}^7, \bar{3}^2\ \}\!\!\}$

Example 2.33. Given multi-set partition $\pi = \{\!\!\{1 \mid 12\bar{1} \mid \bar{1}\bar{1} \mid \bar{1}\bar{1} \mid \bar{2}\bar{2} \mid 1\bar{1}\bar{1}\bar{2}\bar{2}\}\!\!\}$ (in our compact notation) we have $\pi_e = \{\!\!\{1\}\!\!\}$, $\pi_o = \{\!\!\{12\bar{1}\}\!\!\}$ and $\pi_r = \{\!\!\{\ 1\bar{1}\bar{1}\bar{2}\bar{2} \mid \bar{1}\bar{1} \mid \bar{1}\bar{1} \mid \bar{2}\bar{2}\ \}\!\!\}$.

Definition 2.34. To show precisely the multiplicities of each type of multi-set partitions π_e, π_o and π_r, for a given multi-set S let $\tilde{m}(\pi)$ be the number of times that S is repeated in π. We will use the list or 3-tuple $\tilde{m}(\pi) = (\tilde{m}(\pi_e), \tilde{m}(\pi_o), \tilde{m}(\pi_r))$ where $\tilde{m}(\pi)$ represents a partition of $l(\pi)$. The multiplicities $\tilde{m}(\pi_e)$, $\tilde{m}(\pi_o)$ and $\tilde{m}(\pi_r)$ are the partitions representing the multiplicities of even, odd and repeated barred entries in multi-sets that occur in π.

Example 2.35. In the example in Definition 2.32 we have $\tilde{m}(\pi) = ((2, 1, 1), (3, 1, 1, 1), (2, 2, 1))$.

Example 2.36. The multi-set partition $\pi = \{\!\!\{112 \mid 3 \mid 3\bar{2} \mid \bar{1} \mid \bar{1} \mid \bar{1}\bar{1} \mid \bar{1}\bar{2}\}\!\!\}$ consists of $\pi_e = \{\!\!\{112 \mid 3 \mid \bar{1}\bar{2}\}\!\!\}$, $\pi_o = \{\!\!\{3\bar{2} \mid \bar{1} \mid \bar{1}\}\!\!\}$ and $\pi_r = \{\!\!\{\bar{1}\bar{1}\}\!\!\}$. This is $\pi = \pi_e \uplus \pi_o \uplus \pi_r \Vdash \{\!\!\{\ 1^2, 2, 3^2, \bar{1}^5, \bar{2}^2\ \}\!\!\}$ and the multiplicities are $\tilde{m}(\pi_e) = (1, 1, 1)$, $\tilde{m}(\pi_o) = (2, 1)$ and $\tilde{m}(\pi_r) = (1)$. In here $l(\pi_e) = 3$, $l(\pi_o) = 3$, $l(\pi_r) = 1$ and $l(\pi) = 7$. The multiplicities of this multi-set partition (in our compact notation) can be expressed as the sequence $\tilde{m}(\pi) = (111, 21, 1) \vdash 7$.

In order to be consistent we establish an ordering on the multi-set partition $\pi = \{\!\!\{S_1, S_2, \ldots, S_l\}\!\!\}$ of a multi-set S with elements $\{\!\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{k}^{\mu_k}\}\!\!\}$. At first we will order the two sets of alphabets as $1 < 2 < \cdots < \bar{1} < \bar{2} < \cdots$ and a lexicographic order in multi-set partitions given the elements of the set are in increasing order.

Example 2.37. The following examples are the multi-set partitions that are ordered as we described above.

1. Multi-set partition $\pi_1 = \{\!\{1 \mid 1\bar{1} \mid \bar{1}\}\!\}$ where $\tilde{m}(\pi_1) = (1, 11, 0)$.
2. Multi-set partition $\pi_2 = \{\!\{1 \mid 1\bar{1}\bar{1} \mid \bar{1} \mid \bar{1} \mid \bar{1}\bar{1} \mid \bar{2}\}\!\}$ where $\tilde{m}(\pi_2) = (1, 21, 11)$.
3. Multi-set partition $\pi_3 = \{\!\{1 \mid 1 \mid 11\bar{1} \mid 1\bar{1} \mid \bar{1} \mid \bar{1}\bar{1} \mid \bar{2}\}\!\}$ where $\tilde{m}(\pi_3) = (2, 1111, 1)$.
4. Multi-set partition $\pi_4 = \{\!\{12 \mid 12 \mid 123 \mid 12\bar{1}\bar{1} \mid 12\bar{1}\bar{1} \mid 1\bar{2} \mid 23 \mid 2\bar{3} \mid \bar{1} \mid \bar{1} \mid \bar{1}\}\!\}$ where $\tilde{m}(\pi_4) = (211, 311, 2)$.

Definition 2.38. Set Partition of a Multi-set (In Two Alphabets) Given

$$S = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{k}^{\mu_k}\}\!\}$$

to be a multi-set define $\pi = \{\!\{P^{(1)}, P^{(2)}, \ldots, P^{(l(\pi))}\}\!\}$ (in two alphabets) to be a set partition of a multi-set S and denoted by $\pi \vdash S$ where $P^{(1)} \uplus P^{(2)} \uplus \ldots \uplus P^{(l(\pi))} = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l^{\lambda_l}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{k}^{\mu_k}\}\!\}$ and each $P^{(i)}$ are sets with no repetitions of barred entries.

Example 2.39. Given $\lambda = (4, 2, 1)$ and $\mu = (2, 2, 1, 1)$, for a pair of partitions (λ, μ) let

$$\pi = \{\!\{ \{1, 2\}, \{2, 3\}, \{1, \bar{1}\}, \{1, \bar{1}, \bar{3}\}, \{1, \bar{4}\}, \{\bar{2}\}, \{\bar{2}\} \}\!\}$$

that is a set partition of the multi-set $\{\!\{1^4, 2^2, 3, \bar{1}^2, \bar{2}^2, \bar{3}, \bar{4}\}\!\}$. The corresponding multiplicities are partitions $\tilde{m}(\pi) = (2, 2, 1, 1, 1, 1)$, $\tilde{m}(\pi_e) = (1, 1, 1)$ and $\tilde{m}(\pi_o) = (2, 1, 1)$. Note that in here $\tilde{m}(\pi_e) \uplus \tilde{m}(\pi_o) = \tilde{m}(\pi)$.

Remark 2.40. We will use a compact notation for a set partition of a multi-set. The symbol used here is different than the one for multi-set partition. For example the set partition π in the preceding example expressed as

$$\pi = \{\!\{ \, 12 \,]\![\, 23 \,]\![\, 1\bar{1} \,]\![\, 1\bar{1}\bar{3} \,]\![\, 1\bar{4} \,]\![\, \bar{2} \,]\![\, \bar{2} \,] \}\!\}$$

2.2.2 Tableaux Representing Multi-Set Partitions in Two Alphabets

In this section the goal is to explore the combinatorics using tableaux and evaluate $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$ i.e. to evaluate B_n-induced trivial character bases in a pair of partitions at eigenvalues of permutation matrices which we have previously provided an algebraic presentation in section 2.1. Given partitions λ and μ we will assign labels (a bijection) in two alphabets in multi-sets

$$S_\lambda = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}}\}\!\} \text{ and } \bar{S}_\mu = \{\!\{\bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}}\}\!\}$$

respectively. This will be used in presenting combinatorics of $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$ which is indexed by a pair of partitions (λ, μ) from the multi-set

$$S \coloneqq S_\lambda \uplus \bar{S}_\mu = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}}\}\!\}.$$

Furthermore, in order to explain the combinatorics in this section we also need to set up more notation to present the splittings of the multi-set S.

Definition 2.41. Let $S_{\lambda^{(L)}} = \{\!\{1^{\lambda_1^{(L)}}, 2^{\lambda_2^{(L)}}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}^{(L)}}\}\!\}$ and $S_{\lambda^{(R)}} = \{\!\{1^{\lambda_1^{(R)}}, 2^{\lambda_2^{(R)}}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}^{(R)}}\}\!\}$ be multi-sets such that $S_\lambda = S_{\lambda^{(L)}} \uplus S_{\lambda^{(R)}}$. The super scripts (L) and (R) are to indicate left and right side of this splittings. Similarly let $\bar{S}_{\mu^{(L)}} = \{\!\{\bar{1}^{\mu_1^{(L)}}, \bar{2}^{\mu_2^{(L)}}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}^{(L)}}\}\!\}$ and $\bar{S}_{\mu^{(R)}} = \{\!\{\bar{1}^{\mu_1^{(R)}}, \bar{2}^{\mu_2^{(R)}}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}^{(R)}}\}\!\}$ given $\bar{S}_\mu = \bar{S}_{\mu^{(L)}} \uplus \bar{S}_{\mu^{(R)}}$. Now, let the disjoint union of the unbarred and barred entries of the left side multi-set of a splitting to be denoted by $S^{(L)} := S_{\lambda^{(L)}} \uplus \bar{S}_{\mu^{(L)}}$ and let the right side of a splitting by $S^{(R)} := S_{\lambda^{(R)}} \uplus \bar{S}_{\mu^{(R)}}$. We can express the splitting of a multi-set S by

$$S^{(L)} \uplus S^{(R)} := \left(S_{\lambda^{(L)}} \uplus \bar{S}_{\mu^{(L)}}\right) \uplus \left(S_{\lambda^{(R)}} \uplus \bar{S}_{\mu^{(R)}}\right).$$

Moreover, define the partitions the $\lambda^{(L)} \uplus \mu^{(L)}$ and $\lambda^{(R)} \mid \mu^{(R)}$ by

$$\lambda^{(L)} \uplus \mu^{(L)} := sort(\lambda_1^{(L)}, \lambda_2^{(L)}, \ldots, \lambda_{l(\lambda)}^{(L)}, \mu_1^{(L)}, \mu_2^{(L)}, \ldots, \mu_{l(\mu)}^{(L)})$$

and

$$\lambda^{(R)} \mid \mu^{(R)} := sort(\lambda_1^{(R)}, \lambda_2^{(R)}, \ldots, \lambda_{l(\lambda)}^{(R)}) \mid sort(\mu_1^{(R)}, \mu_2^{(R)}, \ldots, \mu_{l(\mu)}^{(R)})$$

be the corresponding partitions to the multi-set $S^{(L)}$ and $S^{(R)}$ that are in exponential notation respectively.

Example 2.42. Show all possible splitting of the multi-set $S = \{\!\{1, \bar{1}, \bar{1}\}\!\}$. This multi-set can also be expressed in exponential notation as $S = \{\!\{1, \bar{1}^2\}\!\}$ where $S_\lambda = \{\!\{1\}\!\}$ and $\bar{S}_\mu = \{\!\{\bar{1}^2\}\!\}$. The following is all splittings of the multi-set $S = \{\!\{1, \bar{1}, \bar{1}\}\!\}$.

$S^{(L)} \uplus S^{(R)}$	$\{\!\{1,\bar{1},\bar{1}\}\!\} \uplus \{\!\{\}\!\}$	$\{\!\{1,\bar{1}\}\!\} \uplus \{\!\{\bar{1}\}\!\}$	$\{\!\{\bar{1},\bar{1}\}\!\} \uplus \{\!\{1\}\!\}$	$\{\!\{\bar{1}\}\!\} \uplus \{\!\{1,\bar{1}\}\!\}$	$\{\!\{1\}\!\} \uplus \{\!\{\bar{1},\bar{1}\}\!\}$	$\{\!\{\}\!\} \uplus \{\!\{1,\bar{1},\bar{1}\}\!\}$
$(\lambda^{(L)} \uplus \mu^{(L)}, \lambda^{(R)} \mid \mu^{(R)})$	$((2,1),(\bullet \mid \bullet))$	$((1,1),(\bullet \mid 1))$	$((2),(1 \mid \bullet))$	$((1),(1 \mid 1))$	$((1),(\bullet \mid 2))$	$(\bullet,(1 \mid 2))$

Table 2.3: Splitting of the multi-set $S = \{\!\{1, \bar{1}, \bar{1}\}\!\}$.

Remark 2.43. Given partitions λ, μ and γ we can extend Proposition 1.175 in two alphabets. Let a multi-set in two alphabets $S := S_\lambda \uplus \bar{S}_\mu = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}}\}\!\}$ as an example we can write

$$\tilde{h}_{\lambda^{(L)} \uplus \mu^{(L)}}[\Xi_\gamma] = \mathcal{H}_{\lambda^{(L)} \uplus \mu^{(L)}, \gamma} := \left\langle h_{|\gamma| - |\lambda^{(L)} \uplus \mu^{(L)}|} h_{\lambda^{(L)} \uplus \mu^{(L)}}, p_\gamma \right\rangle$$

where the value of $\mathcal{H}_{\lambda^{(L)} \uplus \mu^{(L)}, \gamma}$ is equal to the number of ways that some of the cells of the partition γ can be filled with labels $\{1, 2, \ldots, l(\lambda), \bar{1}, \bar{2}, \ldots, \bar{l}(\mu)\}$ such that the whole row is given the same label and in total $\lambda_j^{(L)}$ cells can be filled with integers j for $1 \le j \le l(\lambda^{(L)})$ and $\mu_k^{(L)}$ cells can be filled with barred integers $\bar{k}$ for $1 \leq k \leq l(\mu^{(L)})$. Now, we let $\mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}$ be all these possible fillings of the diagram for the partition γ. For a filling of $F^{(L)} \in \mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}$ we will assign a weight of $+1$ and denote it by $wt(F^{(L)})$ and as an example we can write

$$\mathcal{H}_{\lambda^{(L)} \uplus \mu^{(L)}, \gamma} = \sum_{F^{(L)} \in \mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}} wt(F^{(L)}) = \sum_{F^{(L)} \in \mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}} 1.$$

Remark 2.44. Given partitions λ, μ and γ we can extend Proposition 1.194 in two alphabets. Let a multi-set in two alphabets $S \coloneqq S_\lambda \uplus \bar{S}_\mu = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}}\}\!\}$ we have

$$\tilde{he}_{\lambda^{(R)} | \mu^{(R)}}[\Xi_\nu] = \mathcal{HE}_{\lambda^{(R)} | \mu^{(R)}, \nu} \coloneqq \left\langle h_{|\nu| - |\lambda^{(R)}| - |\mu^{(R)}|} h_{\lambda^{(R)}} e_{\mu^{(R)}}, p_\nu \right\rangle$$

Let $\mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}$ be the fillings of the diagram for the partition ν with labels $\{1, 2, \ldots, l(\lambda), \bar{1}, \bar{2}, \ldots, \bar{l}(\mu)\}$ such that the whole row is given the same label and in total $\lambda^{(R)}_j$ cells can be filled with integers j for $1 \leq j \leq l(\lambda^{(R)})$ and $\mu^{(R)}_k$ cells can be filled with barred integers $\bar{k}$ for $1 \leq k \leq l(\mu^{(R)})$. For $F^{(R)} \in \mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}$ the weight of the fillings $wt(F^{(R)})$ is equal to $(-1)^m$ where m is the number of cells filled with barred labels plus the number of rows occupied by the barred labels in diagram ν. This gives

$$\mathcal{HE}_{\lambda^{(R)} | \mu^{(R)}, \nu} = \sum_{F^{(R)} \in \mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}} wt(F^{(R)}) = \sum_{F^{(R)} \in \mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}} (-1)^m.$$

Example 2.45. Given multi-set $S = \{\!\{1, 2^2, 3, \bar{1}^2, \bar{2}^2, \bar{3}^2\}\!\}$, for the splitting

$$S^{(L)} \uplus S^{(R)} = \{\!\{1, 2, \bar{1}, \bar{1}, \bar{2}, \bar{2}\}\!\} \uplus \{\!\{2, 3, \bar{3}, \bar{3}\}\!\}$$

we have corresponding pair of partitions $\lambda^{(L)} \uplus \mu^{(L)} = ((1,1) \uplus (2,2)) = (2,2,1,1)$ and $\lambda^{(R)} \mid \mu^{(R)} = ((1,1) \mid (2))$. Now let partitions $\gamma = (3,2,2,1,1)$ and $\nu = (2,1,1)$. We will evaluate two separate values $\tilde{h}_{\lambda^{(L)} \uplus \mu^{(L)}}[\Xi_\gamma]$ and $\tilde{he}_{\lambda^{(R)} | \mu^{(R)}}[\Xi_\nu]$. At first, using SageMath, evaluate

$$\tilde{h}_{\lambda^{(L)} \uplus \mu^{(L)}}[\Xi_\gamma] = \tilde{h}_{2211}[\Xi_{32211}] = 4$$

This is number of tableaux (with weight of 1) that some cells of diagram of shape $(3,2,2,1,1)$ can be filled with labels $\{1, 2, \bar{1}, \bar{2}\}$ such that the whole row is given the same label and in total one cell is labeled by 1, one cell is labeled by 2, two cells are labeled by $\bar{1}$ and two cells are labeled by $\bar{2}$. The diagrams are

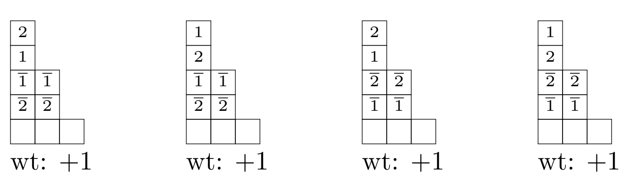


Table 2.4: Fillings of tableaux for $\tilde{h}_{2211}[\Xi_{32211}]$

Now we evaluate (and using SageMath)

$$\begin{aligned}
\widetilde{he}_{\lambda^{(R)}|\mu^{(R)}}[\Xi_\nu] &= \widetilde{he}_{11|2}[\Xi_{211}] \\
&= \left(\tilde{h}_{11} \odot_A \tilde{e}_2\right)[\Xi_{211}] \\
&= \left(\tilde{h}_{11} \odot_A (\tilde{h}_{11} - \tilde{h}_2)\right)[\Xi_{211}] \\
&= \left(\tilde{h}_{1111} - \tilde{h}_{211}\right)[\Xi_{211}] \\
&= -2
\end{aligned}$$

This is number of tableaux with weights that some cells of diagram of shape $(2,1,1)$ can be filled with labels $\{2,3,\bar{3}\}$ such that the whole row is given the same label and in total one cell is labeled by 2, one cell is labeled by 3, two cells are labeled by $\bar{3}$. These filling have a weight that is calculated as -1 raised to the number of cell that are occupied by barred labels plus the number of rows occupied by barred labels i.e. the weight of $(-1)^{2+1} = -1$ for each diagram. This is

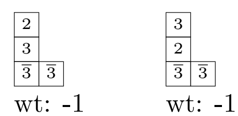


Table 2.5: Fillings of tableaux for $\widetilde{he}_{11|2}[\Xi_{211}]$

Finally we have

$$\tilde{h}_{2211}[\Xi_{32211}]\widetilde{he}_{11|2}[\Xi_{211}] = 4(-2) = -8$$

This can be shown in 8 diagrams of pair of tableaux each of weight -1 as the following.

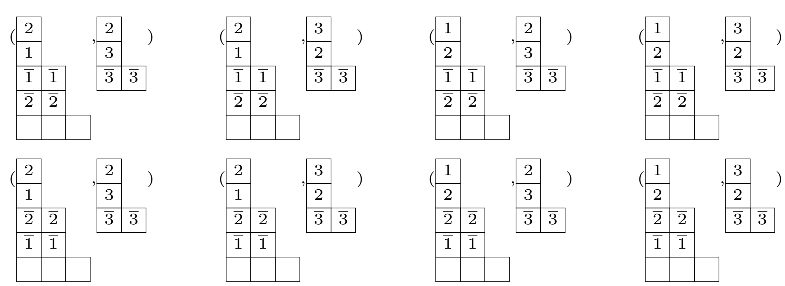


Table 2.6: Fillings of tableaux for $\tilde{h}_{2211}[\Xi_{32211}]\widetilde{he}_{11|2}[\Xi_{211}]$

In order to build our tools for combinatorial evaluations, we will define the following.

Definition 2.46. Given a splitting $S^{(L)} \uplus S^{(R)}$ of a multi-set

$$S \coloneqq S_\lambda \uplus \bar{S}_\mu = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}}\}\!\}$$

define a sign of this splitting $S^{(L)} \uplus S^{(R)}$ by

$$sign\left(S^{(L)} \uplus S^{(R)}\right) = (-1)^p$$

where p is the number of barred entries in the multi-set of the right side of the splitting $S^{(R)} \coloneqq S_{\lambda^{(R)}} \uplus \bar{S}_{\mu^{(R)}}$.

Example 2.47. The following are examples a sign of some of the splittings.

1. For splitting $\{\!\{\bar{1}\}\!\} \uplus \{\!\{1,1\}\!\}$ we have $sign\left(\{\!\{\bar{1}\}\!\} \uplus \{\!\{1,1\}\!\}\right) = 1$.
2. For splitting $\{\!\{1,\bar{1}\}\!\} \uplus \{\!\{\bar{1}\}\!\}$ we have $sign\left(\{\!\{1,\bar{1}\}\!\} \uplus \{\!\{\bar{1}\}\!\}\right) = -1$.
3. For splitting $\{\!\{\}\!\} \uplus \{\!\{1,\bar{1},\bar{1}\}\!\}$ we have $sign\left(\{\!\{\}\!\} \uplus \{\!\{1,\bar{1},\bar{1}\}\!\}\right) = 1$.

Definition 2.48. Let $\mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}$ and $\mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}$ be all possible fillings of the diagram (γ,ν) of the left and the right splittings described in Remark 2.43 and Remark 2.44 respectively. Define all possible pairs (multiplication principle) by having disjoint union of all these possible fillings $\mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}$ and $\mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}$ from the left and the right diagrams denoted by

$$\mathcal{F}^{(\gamma,\nu)}_{\left(\lambda^{(L)} \uplus \mu^{(L)}, \lambda^{(R)} | \mu^{(R)}\right)} \coloneqq \mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}} \uplus \mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}$$

where operation $\uplus$ indicated a disjoint union between the pairs of diagrams. Furthermore given a filling $F^{(L)} \in \mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}$ and a filling $F^{(R)} \in \mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}$ we may define the pair of fillings of diagrams diagram (γ,ν) by $F^{(L)} \uplus F^{(R)} \in \mathcal{F}^{(\gamma,\nu)}_{\left(\lambda^{(L)} \uplus \mu^{(L)}, \lambda^{(R)} | \mu^{(R)}\right)}$.

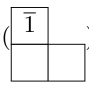
Definition 2.49. Given a filling $F^{(L)} \uplus F^{(R)}$ for it's left filling $F^{(L)} \in \mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}$ we will assign a weight of $+1$ and denote it by $wt(F^{(L)})$ and for the right filling $F^{(R)} \in \mathcal{F}^{\gamma}_{\lambda^{(L)} | \mu^{(L)}}$ assign the weight of the fillings $wt(F^{(R)})$ to be equal to $(-1)^m$ where m is the number of cells filled with barred labels plus the number of rows occupied by the barred labels in partition ν in $F^{(R)}$. Given a fixed splitting $S^{(L)} \uplus S^{(R)}$ let

$$F^{(L)} \uplus F^{(R)} \in \mathcal{F}^{(\gamma,\nu)}_{\left(\lambda^{(L)} \uplus \mu^{(L)}, \lambda^{(R)} | \mu^{(R)}\right)}$$

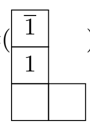
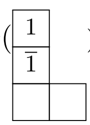
to be all possible filling of diagram (γ,ν) then by multiplication principle the total weight assigned to this filling $F^{(L)} \uplus F^{(R)}$ is

$$wt\left(F^{(L)} \uplus F^{(R)}\right) = \left(\sum_{F^{(L)} \in \mathcal{F}^{\gamma}_{\lambda^{(L)} \uplus \mu^{(L)}}} wt(F^{(L)})\right)\left(\sum_{F^{(R)} \in \mathcal{F}^{\nu}_{\lambda^{(R)} | \mu^{(R)}}} wt(F^{(R)})\right)$$
$$= \sum_{F^{(L)}} \sum_{F^{(R)}} wt(F^{(L)}) wt(F^{(R)})$$
$$= \sum_{F^{(L)} \uplus F^{(R)}} 1 \cdot (-1)^m$$
$$= \sum_{F^{(L)} \uplus F^{(R)}} (-1)^m$$

Example 2.50. Given splitting $S^{(L)} \uplus S^{(R)} = \{\!\{\bar{1}\}\!\} \uplus \{\!\{1, \bar{1}\}\!\}$ and $(\gamma, \nu) = ((2,1),(2,1,1))$ evaluate $wt\left(F^{(L)} \uplus F^{(R)}\right)$. At first we can show the weight of all possible fillings of the left splitting $\{\!\{\bar{1}\}\!\}$ is

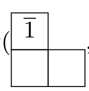
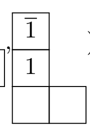
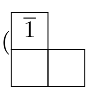
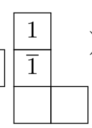
$wt(F^{(L)}) = wt($  $) = +1$

and weight of all possible fillings of the right splitting $\{\!\{1, \bar{1}\}\!\}$ is

$wt(F^{(R)}) = wt($  $) + wt($  $)$

$$= (-1)^{1+1} + (-1)^{1+1} = 2$$

By Definition 2.49 having pairs of fillings of the diagrams is the following disjoint unions, we can evaluate the weight of this splitting.

$wt\left(F^{(L)} \uplus F^{(R)}\right) = wt($  ,  $) + wt($   $) = (-1)^{1+1} + (-1)^{1+1} = 2$

Definition 2.51. Given a fixed splitting $S^{(L)} \uplus S^{(R)}$ of a multi-set

$$S \coloneqq S_\lambda \uplus \bar{S}_\mu = \{\!\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, l(\lambda)^{\lambda_{l(\lambda)}}, \bar{1}^{\mu_1}, \bar{2}^{\mu_2}, \ldots, \bar{l}(\mu)^{\mu_{l(\mu)}}\}\!\}$$

let $sign\left(S^{(L)} \uplus S^{(R)}\right) = (-1)^p$. Let $F^{(L)} \uplus F^{(R)} \in \mathcal{F}^{(\gamma,\nu)}_{\left(\lambda^{(L)} \uplus \mu^{(L)}, \lambda^{(R)} | \mu^{(R)}\right)}$ to be all possible fillings of the splitting $S^{(L)} \uplus S^{(R)}$ of pair of diagrams (γ, ν) then define

$$wt^{S^{(L)} \uplus S^{(R)}}\left(F^{(L)} \uplus F^{(R)}\right) = sign\left(S^{(L)} \uplus S^{(R)}\right) \cdot wt\left(F^{(L)} \uplus F^{(R)}\right)$$
$$= (-1)^p \cdot \left(\sum_{F^{(L)} \uplus F^{(R)}} (-1)^m\right)$$
$$= \sum_{F^{(L)} \uplus F^{(R)}} (-1)^{m+p}$$

Since m is the number of cells filled with barred labels plus the number of rows occupied by the barred labels in diagram ν and p is the number of barred entries in the multi-set of the right side of the splitting we have $m = p + r$ where r is the number of rows occupied by the barred labels in diagram ν. This means $(-1)^{m+p} = (-1)^{p+r+p} = (-1)^r$ and we have

$$wt^{S^{(L)} \uplus S^{(R)}}\left(F^{(L)} \uplus F^{(R)}\right) = \sum_{F^{(L)} \uplus F^{(R)}} (-1)^r.$$

Lemma 2.52. *Given all notations, multi-sets and operations defined in Definition 2.41 we have*

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \sum_{\lambda^{(L)} \uplus \mu^{(L)}} \sum_{\lambda^{(R)} | \mu^{(R)}} (-1)^{|\mu^{(R)}|} \tilde{h}_{\lambda^{(L)} \uplus \mu^{(L)}}[\Xi_\gamma] \cdot \tilde{h\!e}_{\lambda^{(R)}|\mu^{(R)}}[\Xi_\nu].$$

Proof. Recall in a proof of Theorem 2.24 in one of the steps we had

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \sum_{i_1=0}^{\lambda_1} \cdots \sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}} \sum_{j_1=0}^{\mu_1} \cdots \sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}} (-1)^{|\mu| - \left(j_1 + \cdots + j_{l(\mu)}\right)} \left(\tilde{h}_{sort(i_1, \cdots, i_{l(\lambda)}, j_1, \cdots, j_{l(\mu)})}[\Xi_\gamma]\right) \cdot$$
$$\cdot \left(\tilde{h\!e}_{sort(\lambda_1 - i_1, \cdots, \lambda_{l(\lambda)} - i_{l(\lambda)}) | sort(\mu_1 - j_1, \cdots, \mu_{l(\mu)} - j_{l(\mu)})}[\Xi_\nu]\right)$$

The partition $\lambda = (\lambda_1, \cdots, \lambda_{l(\lambda)})$ that can be split into two partitions left and right as $\lambda = \lambda^{(L)} \uplus \lambda^{(R)}$ so the summations can be changed as $\sum_{i_1=0}^{\lambda_1} \cdots \sum_{i_{l(\lambda)}=0}^{\lambda_{l(\lambda)}} = \sum_{\lambda^{(L)} \uplus \lambda^{(R)}}$. Similarly for partition $\mu = (\mu_1, \ldots, \mu_{l(\mu)})$ we have $\sum_{j_1=0}^{\mu_1} \cdots \sum_{j_{l(\mu)}=0}^{\mu_{l(\mu)}} = \sum_{\mu^{(L)} \uplus \mu^{(R)}}$. The part i_k in $\tilde{h}_{sort(i_1, \cdots, i_{l(\lambda)}, j_1, \cdots, j_{l(\mu)})}$ is summed over $i_k = 0$ to $i_k = \lambda_k$. This means in $\tilde{h\!e}_{sort(\lambda_1 - i_1, \cdots, \lambda_{l(\lambda)} - i_{l(\lambda)}) | sort(\mu_1 - j_1, \cdots, \mu_{l(\mu)} - j_{l(\mu)})}$ the part $\lambda_k - i_k$ is summed from $\lambda_k - i_k = \lambda_k$ to $i_k = 0$ therefore if we assign each part the corresponding multi-set that is

Part	i_1	i_2	$\cdots$	$i_{l(\lambda)}$	j_1	j_2	$\cdots$	$j_{l(\mu)}$
Multi-Set	$\{\!\{1^{i_1}\}\!\}$	$\{\!\{2^{i_2}\}\!\}$	$\cdots$	$\{\!\{l(\lambda)^{i_{l(\lambda)}}\}\!\}$	$\{\!\{\bar{1}^{j_1}\}\!\}$	$\{\!\{\bar{2}^{j_2}\}\!\}$	$\cdots$	$\{\!\{\bar{l}(\mu)^{j_{l(\mu)}}\}\!\}$

and

Part	$\lambda_1 - i_1$	$\lambda_2 - i_2$	$\cdots$	$\lambda_{l(\lambda)} - i_{l(\lambda)}$	$\mu_1 - j_1$	$\mu_2 - j_2$	$\cdots$	$\mu_{l(\mu)} - j_{l(\mu)}$
Multi-Set	$\{\!\{1^{\lambda_1 - i_1}\}\!\}$	$\{\!\{2^{\lambda_2 - i_2}\}\!\}$	$\cdots$	$\{\!\{\lambda_{l(\lambda)}^{\lambda_{l(\lambda)} - i_{l(\lambda)}}\}\!\}$	$\{\!\{\bar{1}^{\mu_1 - j_1}\}\!\}$	$\{\!\{\bar{2}^{\mu_2 - j_2}\}\!\}$	$\cdots$	$\{\!\{\bar{l}(\mu)^{\mu_{l(\mu)} - j_{l(\mu)}}\}\!\}$

Given $\lambda^{(L)} = sort(\lambda^{(L)}_1, \lambda^{(L)}_2, \ldots, \lambda^{(L)}_{l(\lambda)})$ and $\lambda^{(R)} = sort(\lambda^{(R)}_1, \lambda^{(R)}_2, \ldots, \lambda^{(R)}_{l(\lambda)})$ we may write these partitions in exponential notation $\lambda^{(L)} = (1^{\lambda^{(L)}_1} 2^{\lambda^{(L)}_2} \ldots l(\lambda)^{\lambda^{(L)}_{l(\lambda)}})$ and $\lambda^{(R)} = (1^{\lambda^{(R)}_1} 2^{\lambda^{(R)}_2} \ldots l(\lambda)^{\lambda^{(R)}_{l(\lambda)}})$ and we have

$$\lambda = (1^{\lambda^{(L)}_1 + \lambda^{(R)}_1} 2^{\lambda^{(L)}_2 + \lambda^{(R)}_2} \ldots l(\lambda)^{\lambda^{(L)}_{l(\lambda)} + \lambda^{(R)}_{l(\lambda)}})$$

Similarly we will write for partition μ. We have

$$\begin{aligned}\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] &= \sum_{\lambda^{(L)}\uplus\lambda^{(R)}}\sum_{\mu^{(L)}\uplus\mu^{(R)}} (-1)^{|\mu|-\left(\mu^{(L)}_1+\cdots+\mu^{(L)}_{l(\mu)}\right)}\tilde{h}_{sort(\lambda^{(L)}_1,\cdots,\lambda^{(L)}_{l(\lambda)},\mu^{(L)}_1,\cdots,\mu^{(L)}_{l(\mu)})}[\Xi_\gamma]\cdot\\ &\quad\cdot\left(\tilde{he}_{sort(\lambda^{(R)}_1,\cdots,\lambda^{(R)}_{l(\lambda)})|sort(\mu^{(R)}_1,\cdots,\mu^{(R)}_{l(\mu)})}[\Xi_\nu]\right)\\ &= \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}}(-1)^{|\mu|-|\mu^{(L)}|}\tilde{h}_{\lambda^{(L)}\uplus\mu^{(L)}}[\Xi_\gamma]\cdot\tilde{he}_{\lambda^{(R)}|\mu^{(R)}}[\Xi_\nu]\\ &= \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}}(-1)^{|\mu^{(R)}|}\tilde{h}_{\lambda^{(L)}\uplus\mu^{(L)}}[\Xi_\gamma]\cdot\tilde{he}_{\lambda^{(R)}|\mu^{(R)}}[\Xi_\nu]\end{aligned}$$

This completes the proof. □

Proposition 2.53. *Given all notations, multi-sets and operations defined in Definition 2.41 and Definition 2.51 the value of $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu]$ is equal to sum of signed weights of all possible splittings $wt^{S^{(L)}\uplus S^{(R)}}\left(F^{(L)}\uplus F^{(R)}\right)$ that is*

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] = \sum_{S^{(L)}\uplus S^{(R)}} wt^{S^{(L)}\uplus S^{(R)}}\left(F^{(L)}\uplus F^{(R)}\right)$$

Proof. By Lemma 2.52 we have

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] = \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}}(-1)^{|\mu^{(R)}|}\tilde{h}_{\lambda^{(L)}\uplus\mu^{(L)}}[\Xi_\gamma]\cdot\tilde{he}_{\lambda^{(R)}|\mu^{(R)}}[\Xi_\nu].$$

On the other hand by Remark 2.43, Remark 2.44, Definition 2.49 and Definition 2.51 we have

$$\begin{aligned}\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] &= \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}}(-1)^{|\mu^{(R)}|}\tilde{h}_{\lambda^{(L)}\uplus\mu^{(L)}}[\Xi_\gamma]\cdot\tilde{he}_{\lambda^{(R)}|\mu^{(R)}}[\Xi_\nu]\\ &= \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}}(-1)^{|\mu^{(R)}|}\left(\sum_{F^{(L)}\in\mathcal{F}^\gamma_{\lambda^{(L)}\uplus\mu^{(L)}}} wt(F^{(L)})\right)\left(\sum_{F^{(R)}\in\mathcal{F}^\nu_{\lambda^{(R)}|\mu^{(R)}}} wt(F^{(R)})\right)\\ &= \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}}(-1)^p\cdot wt\left(F^{(L)}\uplus F^{(R)}\right)\\ &= \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}} sign\left(S^{(L)}\uplus S^{(R)}\right)\cdot wt\left(F^{(L)}\uplus F^{(R)}\right)\\ &= \sum_{\lambda^{(L)}\uplus\mu^{(L)}}\sum_{\lambda^{(R)}|\mu^{(R)}} wt^{S^{(L)}\uplus S^{(R)}}\left(F^{(L)}\uplus F^{(R)}\right)\\ &= \sum_{S^{(L)}\uplus S^{(R)}} wt^{S^{(L)}\uplus S^{(R)}}\left(F^{(L)}\uplus F^{(R)}\right)\end{aligned}$$

where $S^{(L)}\uplus S^{(R)} := \left(S_{\lambda^{(L)}}\uplus\bar{S}_{\mu^{(L)}}\right)\uplus\left(S_{\lambda^{(R)}}\uplus\bar{S}_{\mu^{(R)}}\right)$ and the proof is complete. □

Corollary 2.54. *Given all notations, multi-sets and operations defined in Definition 2.41 and Definition 2.51 the value of* $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$ *is equal to*

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \sum_{S^{(L)} \uplus S^{(R)}} \sum_{F^{(L)} \uplus F^{(R)}} (-1)^r$$

Proof. This is an immediate result of Proposition 2.53 and Definition 2.51. $\square$

Example 2.55. Evaluate $\tilde{h}^B_{\bullet,1}[\Xi_{11}, \Xi_1]$ using Definition 2.16 we have

$$\begin{aligned}
\tilde{h}^B_{\bullet,1}[\Xi_{11}, \Xi_1] &= \left(-\tilde{h}_\bullet \otimes \tilde{he}_{\bullet|1} + \tilde{h}_1 \otimes \tilde{he}_{\bullet|\bullet}\right)[\Xi_{11}, \Xi_1] \\
&= -\tilde{h}_\bullet[\Xi_{11}]\tilde{he}_{\bullet|1}[\Xi_1] + \tilde{h}_1[\Xi_{11}]\tilde{he}_{\bullet|\bullet}[\Xi_1] \\
&= -\left\langle h_{\underset{\square}{2}} h_\bullet, p_{11}\right\rangle \langle h_\bullet e_1, p_1\rangle + \left\langle h_{\underset{\square}{1}} h_1, p_{11}\right\rangle \left\langle h_{\underset{\square}{1}} h_\bullet e_\bullet, p_1\right\rangle \\
&= -\mathcal{H}_{\underset{\square}{2},11} \cdot \mathcal{HE}_{(\bullet|1),1} + \mathcal{H}_{\underset{\square}{1}1,11} \cdot \mathcal{HE}_{(\underset{\square}{1}|\bullet),1} \\
&= -1 + 2 \\
&= 1
\end{aligned}$$

where the empty/blank cell(s) are indicated by $\square$. Given $\tilde{h}^B_{\bullet,1}[\Xi_{11}, \Xi_1]$ the pair of partitions $(\lambda, \mu) = (\bullet, (1))$ of multi-set $S = \{\!\{\bar{1}\}\!\}$ has two splittings

$S^{(L)} \uplus S^{(R)}$	$\{\!\{\bar{1}\}\!\} \uplus \{\!\{\}\!\}$	$\{\!\{\}\!\} \uplus \{\!\{\bar{1}\}\!\}$
$\left(\lambda^{(L)} \uplus \mu^{(L)}, \lambda^{(R)} \mid \mu^{(R)}\right)$	$((1), (\bullet \mid \bullet))$	$(\bullet, (\bullet \mid 1))$

All possible diagrams are

Splitting $S^{(L)} \uplus S^{(R)}$	Sign $sign\left(S^{(L)} \uplus S^{(R)}\right)$	Evaluation $\pm\tilde{h}_{\alpha'}[\Xi_\gamma]\tilde{he}_{\beta'\mid\eta}[\Xi_\nu]$	Inner Product $\pm\mathcal{H}_{\alpha,\gamma} \cdot \mathcal{HE}_{(\beta\mid\eta),\nu}$	Diagrams and Associated Weight $\left(F^{(L)} \uplus F^{(R)}\right)$ and $wt(F^{(L)} \uplus F^{(R)})$	Signed Weight $wt^{S^{(L)} \uplus S^{(R)}}\left(F^{(L)} \uplus F^{(R)}\right)$
$\{\!\{\}\!\} \uplus \{\!\{\bar{1}\}\!\}$	-1	$-\tilde{h}_\bullet[\Xi_{11}]\tilde{he}_{\bullet\mid1}[\Xi_1]$	$-\mathcal{H}_{\underset{\square}{2},11} \cdot \mathcal{HE}_{(\bullet\mid1),1}$	$\left(\begin{array}{\|c\|}\hline \ \\ \hline \ \\ \hline\end{array}, \begin{array}{\|c\|}\hline \bar{1} \\ \hline\end{array}\right)$ wt: $+1$	$(-1)^1 = -1$
$\{\!\{\bar{1}\}\!\} \uplus \{\!\{\}\!\}$	$+1$	$\tilde{h}_1[\Xi_{11}]\tilde{he}_{\bullet\mid\bullet}[\Xi_1]$	$\mathcal{H}_{\underset{\square}{1}1,11} \cdot \mathcal{HE}_{(\underset{\square}{1}\mid\bullet),1}$	$\left(\begin{array}{\|c\|}\hline \bar{1} \\ \hline \ \\ \hline\end{array}, \begin{array}{\|c\|}\hline \ \\ \hline\end{array}\right)$ wt: $+1$ $\left(\begin{array}{\|c\|}\hline \ \\ \hline \bar{1} \\ \hline\end{array}, \begin{array}{\|c\|}\hline \ \\ \hline\end{array}\right)$ wt: $+1$	$(-1)^0 + (-1)^0 = +2$

Table 2.7: Fillings of splittings of multi-set $S = \{\!\{\bar{1}\}\!\}$ to evaluate $\tilde{h}^B_{\bullet,1}[\Xi_{11}, \Xi_1]$

As we can see the total signed weights in the right most column will give us the answer, that is

$$\tilde{h}^B_{\bullet,1}[\Xi_{11}, \Xi_1] = \sum_{S^{(L)} \uplus S^{(R)}} \sum_{F^{(L)} \uplus F^{(R)}} (-1)^r = -1 + 2 = 1.$$

Example 2.56. Evaluate $\tilde{h}^B_{1,2}[\Xi_{21},\Xi_{211}]$. By Example 2.26 we have

$$
\begin{aligned}
\tilde{h}^B_{1,2}[\Xi_{21},\Xi_{211}] &= \Big(-\tilde{h}_{11}\otimes\tilde{he}_{\bullet|1}+\tilde{h}_{1}\otimes\tilde{he}_{\bullet|2}+\tilde{h}_{21}\otimes\tilde{he}_{\bullet|\bullet}+\tilde{h}_{\bullet}\otimes\tilde{he}_{1|2}-\tilde{h}_{1}\otimes\tilde{he}_{1|1}+\tilde{h}_{2}\otimes\tilde{he}_{1|\bullet}\Big)[\Xi_{21},\Xi_{211}]\\
&= -\tilde{h}_{11}[\Xi_{21}]\tilde{he}_{\bullet|1}[\Xi_{211}]+\tilde{h}_{1}[\Xi_{21}]\tilde{he}_{\bullet|2}[\Xi_{211}]+\tilde{h}_{21}[\Xi_{21}]\tilde{he}_{\bullet|\bullet}[\Xi_{211}]+\\
&\quad+\tilde{h}_{\bullet}[\Xi_{21}]\tilde{he}_{1|2}[\Xi_{211}]-\tilde{h}_{1}[\Xi_{21}]\tilde{he}_{1|1}[\Xi_{211}]+\tilde{h}_{2}[\Xi_{21}]\tilde{he}_{1|\bullet}[\Xi_{211}]\\
&= -\Big\langle h_{\underset{\square}{1}11}[X], p_{21}[X]\Big\rangle\Big\langle h_{\underset{\square}{3}}[Y]e_1[Y], p_{211}[Y]\Big\rangle+\Big\langle h_{\underset{\square}{2}1}[X], p_{21}[X]\Big\rangle\Big\langle h_{\underset{\square}{2}}[Y]e_2[Y], p_{211}[Y]\Big\rangle+\\
&\quad+\langle h_{21}[X], p_{21}[X]\rangle\Big\langle h_{\underset{\square}{4}}[Y], p_{211}[Y]\Big\rangle+\Big\langle h_{\underset{\square}{3}}[X], p_{21}[X]\Big\rangle\Big\langle h_{\underset{\square}{1}1}[Y]e_2[Y], p_{211}[Y]\Big\rangle-\\
&\quad-\Big\langle h_{\underset{\square}{2}1}[X], p_{21}[X]\Big\rangle\Big\langle h_{\underset{\square}{2}1}[Y]e_1[Y], p_{211}[Y]\Big\rangle+\Big\langle h_{2\underset{\square}{1}}[X], p_{21}[X]\Big\rangle\Big\langle h_{\underset{\square}{3}1}[Y], p_{211}[Y]\Big\rangle\\
&= -\mathcal{H}_{\underset{\square}{1}11,21}\cdot\mathcal{HE}_{(\underset{\square}{3}|1),2111}+\mathcal{H}_{\underset{\square}{2}1,21}\cdot\mathcal{HE}_{(\underset{\square}{2}|2),2111}+\mathcal{H}_{21,21}\cdot\mathcal{HE}_{(\underset{\square}{4}|\bullet),2111}+\\
&\quad+\mathcal{H}_{\underset{\square}{3},21}\cdot\mathcal{HE}_{(\underset{\square}{1}1|2),2111}-\mathcal{H}_{\underset{\square}{2}1,21}\cdot\mathcal{HE}_{(\underset{\square}{2}1|1),2111}+\mathcal{H}_{2\underset{\square}{1},21}\cdot\mathcal{HE}_{(\underset{\square}{3}1|\bullet),2111}\\
&= -0\cdot 2+1\cdot 0+1\cdot 1+1(-2)-1\cdot 2+1\cdot 2\\
&= -1
\end{aligned}
$$

The diagrams are shown below.

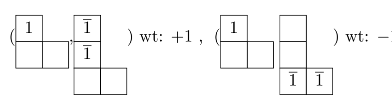
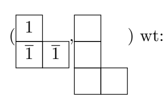
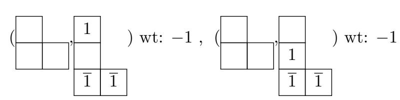
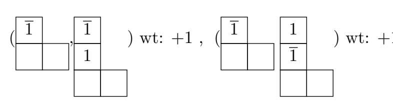
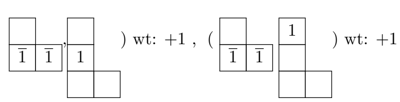
Splitting $S^{(L)}\uplus S^{(R)}$	Sign $sign\left(S^{(L)}\uplus S^{(R)}\right)$	Evaluation $\pm\tilde{h}_{\alpha'}[\Xi_\gamma]\tilde{he}_{\beta'\vert\eta}[\Xi_\nu]$	Inner Product $\pm\mathcal{H}_{\alpha,\gamma}\cdot\mathcal{HE}_{(\beta\vert\eta),\nu}$	Diagrams and Associated Weight $\left(F^{(L)}\uplus F^{(R)}\right)$ and $wt(F^{(L)}\uplus F^{(R)})$	Signed Weight $wt^{S^{(L)}\uplus S^{(R)}}\left(F^{(L)}\uplus F^{(R)}\right)$
$\{\!\{1,\bar{1}\}\!\}\uplus\{\!\{\bar{1}\}\!\}$	-1	$-\tilde{h}_{11}[\Xi_{21}]\tilde{he}_{\bullet\vert 1}[\Xi_{211}]$	$-\mathcal{H}_{\underset{\square}{1}11,21}\cdot\mathcal{HE}_{(\underset{\square}{3}\vert 1),2111}$	No Diagram	0
$\{\!\{1\}\!\}\uplus\{\!\{\bar{1},\bar{1}\}\!\}$	$+1$	$\tilde{h}_{1}[\Xi_{21}]\tilde{he}_{\bullet\vert 2}[\Xi_{211}]$	$\mathcal{H}_{\underset{\square}{2}1,21}\cdot\mathcal{HE}_{(\underset{\square}{2}\vert 2),2111}$		$(-1)^2+(-1)^1=0$
$\{\!\{1,\bar{1},\bar{1}\}\!\}\uplus\{\!\{\}\!\}$	$+1$	$\tilde{h}_{21}[\Xi_{21}]\tilde{he}_{\bullet\vert\bullet}[\Xi_{211}]$	$\mathcal{H}_{21,21}\cdot\mathcal{HE}_{(\underset{\square}{4}\vert\bullet),2111}$	 wt: +1	$(-1)^0=+1$
$\{\!\{\}\!\}\uplus\{\!\{1,\bar{1},\bar{1}\}\!\}$	$+1$	$\tilde{h}_{\bullet}[\Xi_{21}]\tilde{he}_{1\vert 2}[\Xi_{211}]$	$\mathcal{H}_{\underset{\square}{3},21}\cdot\mathcal{HE}_{(\underset{\square}{1}1\vert 2),2111}$		$(-1)^1+(-1)^1=-2$
$\{\!\{\bar{1}\}\!\}\uplus\{\!\{1,\bar{1}\}\!\}$	-1	$-\tilde{h}_{1}[\Xi_{21}]\tilde{he}_{1\vert 1}[\Xi_{211}]$	$-\mathcal{H}_{\underset{\square}{2}1,21}\cdot\mathcal{HE}_{(\underset{\square}{2}1\vert 1),2111}$		$(-1)^1+(-1)^1=-2$
$\{\!\{\bar{1},\bar{1}\}\!\}\uplus\{\!\{1\}\!\}$	$+1$	$\tilde{h}_{2}[\Xi_{21}]\tilde{he}_{1\vert\bullet}[\Xi_{211}]$	$\mathcal{H}_{2\underset{\square}{1},21}\cdot\mathcal{HE}_{(\underset{\square}{3}1\vert\bullet),2111}$		$(-1)^0+(-1)^0=+2$

Table 2.8: Fillings of diagrams to evaluate $\tilde{h}^B_{1,2}[\Xi_{21},\Xi_{211}]$.

The total signed weight is: $\tilde{h}^B_{1,2}[\Xi_{21},\Xi_{211}]=\sum_{S^{(L)}\uplus S^{(R)}}\sum_{F^{(L)}\uplus F^{(R)}}(-1)^r=0+0+1-2-2+2=-1.$

3 Irreducible Character Basis of Hyperoctahedral Group

3.1 B_n - Irreducible Character Basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$

In Chapter 2 we have shown a method of constructing the B_n-induced trivial character basis namely $\{\tilde{h}^B_{(\lambda,\mu)}\}$. These functions each expand in tensor square of symmetric function of form $\tilde{h}_\alpha \otimes \tilde{h\!\!e}_{\beta|\eta}$ in type A. In this chapter we will present a new symmetric function that was our goal in this thesis and introduce them as B_n-irreduicible character basis namely $\{\tilde{s}^B_{(\lambda,\mu)}\}$ and express them in terms of $\{\tilde{h}^B_{(\lambda,\mu)}\}$.

3.1.1 Definition of B_n - Irreducible Character Basis $\{\tilde{s}^B_{(\lambda,\mu)}\}$

In this section we will define B_n-irreducible character basis i.e. $\{\tilde{s}^B_{(\lambda,\mu)}\}$. These functions are families of symmetric functions that are the values of the irreducible characters of the hyperoctahedral group when evaluated at roots of unity i.e. multi-set of eigenvalues of permutation matrices. We will define the symmetric function $\tilde{s}^B_{(\lambda,\mu)}$ by change of basis with $\tilde{h}^B_{(\alpha,\beta)}$ using Kostka coefficients.

Definition 3.1. Given partitions α, β, λ and μ and non-negative integer $n \geq 2|\alpha|$ define the functions $\tilde{s}^B_{(\lambda,\mu)}$ in $\Lambda \otimes \Lambda$ that satisfies

$$\tilde{h}^B_{(\alpha,\beta)} = \sum_{\substack{(\lambda,\mu)\\ |\lambda|\leq|\alpha| \\ |\mu|=|\beta|}} K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta} \tilde{s}^B_{(\lambda,\mu)}$$

where the Kostka coefficients $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ is the coefficient of $s_{(n-|\lambda|,\lambda)}$ in $h_{(n-|\alpha|,\alpha)}$ and $K_{\mu\beta}$ is the coefficient of s_μ in h_β. Recall the Kostka coefficient $K_{\mu\beta}$ is the number of column strict tableaux of shape μ content β (Definition 1.40). Moreover the Kostka coefficients $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} > 0$ and $K_{\mu\beta} > 0$ if

$$(1) \quad (n-|\lambda|, \lambda) \trianglerighteq (n-|\alpha|, \alpha) \quad \text{and}$$
$$(2) \quad \mu \trianglerighteq \beta$$

Note that $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} = K_{(n-|\lambda|+1,\lambda)(n-|\alpha|+1,\alpha)}$ if $n \geq 2|\alpha|$, that is, Kostka numbers stabilizes when adding another cell in the first row.

In type A we have $h_\alpha = \sum_\lambda K_{\lambda\alpha} s_\lambda$ and $s_\lambda = \sum_\alpha K^{-1}_{\lambda\alpha} h_\alpha$ and similarly, given $n \geq 2|\alpha|$ and $n \geq 2|\lambda|$ we have $h_{(n-|\alpha|,\alpha)} = \sum_{|\lambda|\leq|\alpha|} K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} s_{(n-|\lambda|,\lambda)}$ and $s_{(n-|\lambda|,\lambda)} = \sum_{|\alpha|\leq|\lambda|} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} h_{(n-|\alpha|,\alpha)}$. In Remark 3.2 we provide a similar idea for type B.

Remark 3.2. We will show in Proposition 3.23 that the change of basis between the $\tilde{h}^B_{(\lambda,\mu)}$ and $\tilde{s}^B_{(\alpha,\beta)}$ are made by lower unitriangular matrices with respect to some order, and this change of basis is invertible. Given $n \geq 2|\lambda|$ we can express $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}^B_{(\alpha,\beta)}$ by the product of the inverse of Kostka coefficients, that is

$$\tilde{s}^B_{(\lambda,\mu)} = \sum_{\substack{(\alpha,\beta)\\ |\alpha|\leq|\lambda|\\ |\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K^{-1}_{\mu\beta} \tilde{h}^B_{(\alpha,\beta)}$$

where the inverse of Kostka coefficient $K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ is the coefficient of $h_{(n-|\alpha|,\alpha)}$ in $s_{(n-|\lambda|,\lambda)}$ and $K^{-1}_{\mu\beta}$ is the coefficient of h_β in s_μ.

The product of Kostka numbers has a combinatorial interpretation as pairs of tableaux. The product $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta}$ is equal to the number of pairs of column strict tableaux where the first is of shape $(n-|\lambda|,\lambda)$ content $(n-|\alpha|,\alpha)$ and the second is of shape μ content β. This means we can explain the combinatorial interpretation of Definition 3.1 in the following proposition.

Proposition 3.3. *Given the pair of partitions (α,β) we have*

$$\tilde{h}^B_{(\alpha,\beta)} = \sum_{(T^{(L)},T^{(R)})} \tilde{s}^B_{\left(\overline{shape(T^{(L)})},shape(T^{(R)})\right)}$$

where the summation is over all possible column strict tableaux of left and right diagrams $(T^{(L)},T^{(R)})$ such that the left tableau $T^{(L)}$ is of shape $(n-|\lambda|,\lambda)$ content $(n-|\alpha|,\alpha)$ and the right tableau $T^{(R)}$ is of shape μ content β for some pair of partitions (λ,μ). The partition $\overline{shape(T^{(L)})}$ is obtained by eliminating the first row of the partition corresponding to the diagram of left tableau $T^{(L)}$ that is $\overline{\lambda} = (\lambda_2,\lambda_3,\ldots,\lambda_{l(\lambda)})$.

Proof. By Definition 3.1 for $n \geq 2|\alpha|$ the elements of $\tilde{s}^B_{(\lambda,\mu)}$ satisfy

$$\tilde{h}^B_{(\alpha,\beta)} = \sum_{\substack{(\lambda,\mu)\\ |\lambda|\leq|\alpha|\\ |\mu|=|\beta|}} K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta} \tilde{s}^B_{(\lambda,\mu)}$$

By definition of Kostka coefficients $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ is the number of column strict tableaux of shape $(n-|\lambda|,\lambda)$ content $(n-|\alpha|,\alpha)$ in the left diagram $T^{(L)}$ and $K_{\mu\beta}$ is the number of number of column strict tableaux of shape μ content β in right diagram $T^{(R)}$. By multiplication principle the number of column strict tableaux of a pair of diagram $(T^{(L)},T^{(R)})$ in this fashion is $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta}$. Finally summing over all possible diagrams $(T^{(L)},T^{(R)})$ we have

$$\tilde{h}^B_{(\alpha,\beta)} = \sum_{(T^{(L)},T^{(R)})} \tilde{s}^B_{\left(\overline{shape(T^{(L)})},shape(T^{(R)})\right)}$$

The proof is complete. □

In Theorem 2.24 we have shown that the evaluation of basis $\tilde{h}^B_{(\lambda,\mu)}$ at multi-set of eigenvalues of permutation matrices with cycle structure $(\gamma,\nu)\vdash n$ is the value of induced trivial module, i.e.

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] = \left\langle h_{(|\gamma|+|\nu|-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle.$$

We will now show that if we evaluate the B_n-irreducible character basis i.e. function $\tilde{s}^B_{(\lambda,\mu)}$ at the multi-set of eigenvalues of permutation matrices with cycle structure $(\gamma,\nu)\vdash n$ we will have the value of irreducible character of hyperoctahedral group indexed by $((n-|\lambda|-|\mu|,\lambda),\mu)$ namely $\chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\gamma,\nu)$.

Proposition 3.4. *Functions $\tilde{s}^B_{(\lambda,\mu)}$ are the symmetric functions in $\Lambda\otimes\Lambda$ that are of (not necessarily of homogeneous) degree $|(\lambda,\mu)| \coloneqq |\lambda|+|\mu|$ and for all $n\geq|\lambda|+|\mu|+\lambda_1$ are the irreducible characters of the hyperoctahedral group indexed by $((n-|\lambda|-|\mu|,\lambda),\mu)$ in the sense that when evaluated at Ξ_γ and Ξ_ν, the multi-set of eigenvalues of permutation matrices of cycle types γ and ν respectively where $(\gamma,\nu)\vdash n$ (i.e. $|\gamma|+|\nu|=n$). That is*

$$\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] = \left\langle s_{(|\gamma|+|\nu|-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle.$$

Proof. Given $n\geq 2|\alpha|$ and $(\gamma,\nu)\vdash n$ i.e. $|\gamma|+|\nu|=n$, by Remark 3.2 and Theorem 2.24 we can express $\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu]$ as the following.

$$\begin{aligned}
\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] &= \sum_{\substack{(\alpha,\beta)\\|\alpha|\leq|\lambda|\\|\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}K^{-1}_{\mu\beta}\tilde{h}^B_{(\alpha,\beta)}[\Xi_\gamma,\Xi_\nu]\\
&= \sum_{\substack{(\alpha,\beta)\\|\alpha|\leq|\lambda|\\|\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}K^{-1}_{\mu\beta}\left\langle h_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle\\
&= \left\langle \sum_{\substack{(\alpha,\beta)\\|\alpha|\leq|\lambda|\\|\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}K^{-1}_{\mu\beta}h_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle\\
&= \left\langle s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle
\end{aligned}$$

By Theorem 2.2 and Corollary 1.136 the expression on the right hand side of the equation in Proposition 3.4 is in fact the value of irreducible character of hyperoctahedral group indexed by $((n-|\lambda|-|\mu|,\lambda),\mu)$ evaluated at the multi-set of eigenvalues Ξ_γ and Ξ_ν namely $\chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\gamma,\nu)$ equivalently

$$\tilde{s}^B_{(\lambda,\mu)}\left[\Xi_\gamma,\Xi_\nu\right] = \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\gamma,\nu)$$

The proof is complete. □

Remark 3.5. The proof of the uniqueness of the symmetric functions $\tilde{s}^B_{(\lambda,\mu)}$ will be provided in section 3.1.2 in Theorem 3.17.

Example 3.6. Express $\tilde{h}^B_{2,1}$ in elements of $\tilde{s}^B_{(\lambda,\mu)}$ basis.

By Definition 3.1 in order to express $\tilde{h}^B_{2,1}$ in $\tilde{s}^B_{(\lambda,\mu)}$ let $(\alpha,\beta) = ((2),(1))$ since $n \geq 2|\alpha| = 4$ it suffices to take $n = 4$ and the corresponding partitions for Kostka coefficients are $(n-|\alpha|,\alpha) = (2,2)$ and $\beta = (1)$. Note that in order to distinguish the contents of the two kostka coefficients $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ and $K_{\mu\beta}$ we have used unbarred and barred entries. In the table below we have shown all possible column strict tableaux which are the coefficients of $\tilde{s}^B_{(\lambda,\mu)}$ in $\tilde{h}^B_{2,1}$ expansion.

Filling of Tableaux $(F^{(L)}, F^{(R)})$	Corresponding Function $\tilde{s}^B_{\lambda,\mu}$	Kostka Coefficients $K_{(n-\lvert\lambda\rvert,\lambda)(n-\lvert\alpha\rvert,\alpha)}K_{\mu\beta}$
$\left(\begin{array}{cccc}1&1&2&2\end{array}, \bar{1}\right)$	$\tilde{s}^B_{\bullet,1}$	1
$\left(\begin{array}{ccc}2&&\\1&1&2\end{array}, \bar{1}\right)$	$\tilde{s}^B_{1,1}$	1
$\left(\begin{array}{cc}2&2\\1&1\end{array}, \bar{1}\right)$	$\tilde{s}^B_{2,1}$	1

Table 3.1: Expression $\tilde{h}^B_{2,1}$ in $\tilde{s}^B_{(\lambda,\mu)}$ basis

The expansion is

$$\tilde{h}^B_{2,1} = \tilde{s}^B_{\bullet,1} + \tilde{s}^B_{1,1} + \tilde{s}^B_{2,1}.$$

Example 3.7. Express $\tilde{h}^B_{1,11}$ in elements of $\tilde{s}^B_{(\lambda,\mu)}$ basis.

Given $(\alpha,\beta) = ((1),(1,1))$ since $n \geq 2|\alpha| = 2$ it suffices to take $n = 2$ therefore corresponding partitions of Kostka coefficients are $(n-|\alpha|,\alpha) = (1,1)$ and $\beta = (1,1)$.

Filling in Pairs of Tableaux $(F^{(L)}, F^{(R)})$	Corresponding Function $\tilde{s}^B_{\lambda,\mu}$	Kostka Coefficient $K_{(n-\lvert\lambda\rvert,\lambda)(n-\lvert\alpha\rvert,\alpha)}K_{\mu\beta}$
$\left(\begin{array}{cc}1&2\end{array}, \begin{array}{cc}\bar{1}&\bar{2}\end{array}\right)$	$\tilde{s}^B_{\bullet,2}$	1
$\left(\begin{array}{cc}1&2\end{array}, \begin{array}{c}\bar{2}\\\bar{1}\end{array}\right)$	$\tilde{s}^B_{\bullet,11}$	1
$\left(\begin{array}{c}2\\1\end{array}, \begin{array}{cc}\bar{1}&\bar{2}\end{array}\right)$	$\tilde{s}^B_{1,2}$	1
$\left(\begin{array}{c}2\\1\end{array}, \begin{array}{c}\bar{2}\\\bar{1}\end{array}\right)$	$\tilde{s}^B_{1,11}$	1

Table 3.2: Expression $\tilde{h}^B_{1,11}$ in elements of $\tilde{s}^B_{(\lambda,\mu)}$ basis

The expansion is

$$\tilde{h}^B_{1,11} = \tilde{s}^B_{\bullet,2} + \tilde{s}^B_{\bullet,11} + \tilde{s}^B_{1,2} + \tilde{s}^B_{1,11}.$$

Example 3.8. Express $\tilde{h}^B_{1,2}$ in elements of $\tilde{s}^B_{(\lambda,\mu)}$ basis.

Let $(\alpha,\beta) = ((1),(2))$ therefore since $n \geq 2|\alpha| = 2$ it suffices to take $n = 2$ and the Kostka coefficients are indexed by the partitions $(n-|\alpha|,\alpha) = (1,1)$ and $\beta = (2)$.

Filling of Tableaux $(F^{(L)}, F^{(R)})$	Corresponding Function $\tilde{s}^B_{\lambda,\mu}$	Kostka Coefficients $K_{(n-\|\lambda\|,\lambda)(n-\|\alpha\|,\alpha)}K_{\mu\beta}$
$(\boxed{1}\boxed{2}, \boxed{\bar{1}}\boxed{\bar{1}})$	$\tilde{s}^B_{\bullet,2}$	1
$(\begin{array}{\|c\|}\hline 2\\ \hline 1\\ \hline\end{array}, \boxed{\bar{1}}\boxed{\bar{1}})$	$\tilde{s}^B_{1,2}$	1

Table 3.3: Expression $\tilde{h}^B_{1,2}$ in elements of $\tilde{s}^B_{(\lambda,\mu)}$ basis

The expansion is

$$\tilde{h}^B_{1,2} = \tilde{s}^B_{\bullet,2} + \tilde{s}^B_{1,2}.$$

3.1.2 Uniqueness of Functions $\tilde{s}^B_{(\lambda,\mu)}$

In this section we will show the uniqueness of the functions $\tilde{s}^B_{(\lambda,\mu)}$ and $\tilde{h}^B_{(\lambda,\mu)}$. Before that, we will go over some background and related material in type A in [OZ1].

Uniqueness of functions $\tilde{s}_\lambda$ and $\tilde{h}_\lambda$. At first we will state the following theorem. Note that in here $F[X]$ is a polynomial ring with coefficients in field F.

Theorem 3.9. *(Factor Theorem, 23.3 Corollary, [Fraleigh]) An element $a \in F$ is a zero of $f(x) \in F[X]$ if and only if $x - a$ is a factor of $f(x)$ in $F[X]$.*

Lemma 3.10. *Let $p(x) \in F[X]$ be a polynomial of degree smaller or equal to a non-negative integer n and assume that there are values $a_1, a_2, \ldots, a_{n+1}$ with $a_i < a_{i+1}$ in F such that $p(a_i) = 0$ for all $1 \leq i \leq n+1$. Then $p(x) = 0$ i.e. $p(x)$ is a zero polynomial in $F[X]$.*

Proof. By Theorem 3.9 we have

$$(x - a_1)(x - a_2)\cdots(x - a_{n+1}) \mid p(x)$$

On the other hand polynomial $d(x) = (x - a_1)(x - a_2)\cdots(x - a_{n+1})$ is of degree $n + 1$ but, since $p(x)$ is a polynomial of degree smaller or equal to a non-negative integer n then this implies that $p(x)$ is a zero polynomial. □

Now, we will present Proposition 38 [OZ1] and after that we will provide the restatement of this proposition and continue our discussion.

Proposition 3.11. *(Proposition 38 [OZ1]) Let $f, g \in Sym$ be symmetric functions of degree less than or equal to some positive integer n. Assume that*

$$f[\Xi_\mu] = g[\Xi_\mu]$$

for all partitions μ such that $|\mu| \leq n$, then

$$f = g$$

as elements of Sym.

Proposition 3.11 (Proposition 38 in [OZ1]) says that if two symmetric functions when evaluated at the same eigenvalue of a permutation matrix are of the same value then the functions are identical. In [OZ1] an equivalent statement of this proposition says: Given a symmetric function $k = f - g$ if $k[\Xi_\mu] = 0$ for all partitions μ such that $|\mu| \leq n$ then this implies $k = 0$.

Proposition 3.12. *(Restatement of Proposition 38 [OZ1]) Given a symmetric function $f \in Sym$ of degree less than or equal to some non-negative integer n assume $f[\Xi_\mu] = 0$ for all partitions μ such that $|\mu| \leq n$ then this implies $f = 0$ as an element of Sym.*

The proof of this proposition which consists of introducing character polynomials and Lemma 39 in [OZ1]. The reason we are stating the Proposition 3.12 in type A is because we will be using it in the proof of the analogous proposition in Type B (Proposition 3.15).

Now, we will continue by presenting the following corollary that is important for us in this section.

Corollary 3.13. *(Corollary 40 [OZ1]) Let $f, g \in Sym$ be symmetric functions of degree less than or equal to some positive integer n. Assume that*

$$f[\Xi_\mu] = g[\Xi_\mu]$$

for all partitions μ such that $|\mu| \geq n$, then

$$f = g$$

as elements of Sym.

A restatement of this corollary is given in the following.

Corollary 3.14. *(Restatement of Corollary 40 [OZ1]) Given a symmetric function $f \in Sym$ of degree less than or equal to some non-negative integer n assume $f[\Xi_\mu] = 0$ for all partitions μ such that $|\mu| \geq n$ then this implies $f = 0$ as an element of Sym.*

Uniqueness of functions $\tilde{s}^B_{(\lambda,\mu)}$ and $\tilde{h}^B_{(\lambda,\mu)}$. After providing the related background in type A in [OZ1] we would like prove the following proposition (Proposition 3.15) that extends the Proposition 3.12 in type A to the symmetric function in two alphabets $f[X,Y] \in \Lambda \otimes \Lambda$ in type B.

Proposition 3.15. *Given a symmetric function in two alphabets $f[X,Y] \in \Lambda\otimes\Lambda$ with deg$(f[X,Y]) \leq n$ if $f[\Xi_\gamma, \Xi_\nu] = 0$ for all pairs of partitions (γ,ν) such that $|\gamma| + |\nu| \leq n$ then $f[X,Y] = 0$.*

Proof. Let $f[X,Y] = \sum\limits_i k^{(i)}[X]g^{(i)}[Y]$ be such that,

1. The total degree of $f[X,Y]$ in X and Y is at most n.
2. Let $f[\Xi_\gamma, \Xi_\nu] = 0$ for all $|\gamma| + |\nu| \leq n$.

We will present this proof by contradiction. Assume that there are components $k^{(i)}[X]g^{(i)}[Y]$ in $\Lambda^{n-r} \otimes \Lambda^r$ which are non-zero. We will also assume that there is a component of $g^{(i)}[Y]$ which is of maximal degree r and consequently the coefficients $k^{(i)}[X]$ will be of degree at most $n-r$. We will use this case and give a proof by minimal counter example, that is, we will show that $k^{(i)}[X]$ are zero and the component $g^{(i)}[Y]$ under these conditions don't exist.

Given a fixed partition γ such that $|\gamma| \leq n-r$ the symmetric function $f[\Xi_\gamma, Y]$ in Y is of degree r. Moreover if we evaluate these symmetric functions $f[\Xi_\gamma, Y]$ at the multi-set of eigenvalues of permutation matrices of cycle structure ν i.e. $Y = \Xi_\nu$ for every $|\nu| \leq r$ they vanish by assumption 2. Therefore by Proposition 3.12 we have that $f[\Xi_\gamma, Y] = 0$ and each of the coefficients $k^{(i)}[\Xi_\gamma] = 0$ for each $|\gamma| \leq n-r$.

On the other hand the coefficients $k^{(i)}[X]$ which are symmetric functions in X of degree at most $n-r$ that are the coefficients of $g^{(i)}[Y]$ of (maximal) degree r they vanish at all $X = \Xi_\gamma$ for all γ such that $|\gamma| \leq n-r$. By Proposition 3.12 this means that we must have $k^{(i)}[X] = 0$. This is a contradiction because we assumed that the components $k^{(i)}[X]g^{(i)}[Y]$ were non-zero and the proof is complete. ☐

Now we will provide an analogous corollary to Corollary 3.14 in type B.

Corollary 3.16. *Given a symmetric function in two alphabets $f[X,Y] \in \Lambda\otimes\Lambda$ with deg$(f[X,Y]) \leq n$ if $f[\Xi_\gamma, \Xi_\nu] = 0$ for all pairs of partitions (γ,ν) such that $|\gamma| + |\nu| \geq n$ then $f[X,Y] = 0$.*

Proof. We will show that for Case 1 and Case 2 we can reduce the conditions on this corollary to conditions of Proposition 3.15.

Case 1. If $|\gamma| \geq n$ and $|\nu| \geq n$, since $|\gamma| + |\nu| \geq n$ we may write $(\gamma,\nu) = ((n+1,\bar{\gamma}),(n+1,\bar{\nu}))$ such that $|\bar{\gamma}| + |\bar{\nu}| \leq n$. Since for all $k + k' \leq n$ we have

$$\begin{aligned}(p_k \otimes p_{k'})\,[\Xi_\gamma, \Xi_\nu] &= p_k[\Xi_\gamma] p_{k'}[\Xi_\nu] \\ &= (p_k[\Xi_{n+1} + \Xi_{\bar{\gamma}}])\,(p_{k'}[\Xi_{n+1} + \Xi_{\bar{\nu}}]) \\ &= (p_k[\Xi_{n+1}] + p_k[\Xi_{\bar{\gamma}}])\,(p_{k'}[\Xi_{n+1}] + p_{k'}[\Xi_{\bar{\nu}}]) \\ &= p_k[\Xi_{\bar{\gamma}}] p_{k'}[\Xi_{\bar{\nu}}] \\ &= (p_k \otimes p_{k'})\,[\Xi_{\bar{\gamma}}, \Xi_{\bar{\nu}}]\end{aligned}$$

The fourth equality is because $p_k[\Xi_{n+1}] = 0$ as $n + 1 \nmid k$ (given $k, k' \leq n$). Given $deg(f[X, Y]) \leq n$ and as we may write $f = \sum_{|\alpha|+|\beta|\leq n} c_{(\alpha,\beta)} p_\alpha \otimes p_\beta$ for partitions α and β, we have

$$\begin{aligned}f[\Xi_\gamma, \Xi_\nu] &= \sum_{|\alpha|+|\beta|\leq n} c_{(\alpha,\beta)} p_\alpha[\Xi_\gamma] p_\beta[\Xi_\nu] \\ &= \sum_{|\alpha|+|\beta|\leq n} c_{(\alpha,\beta)} p_\alpha[\Xi_{\bar{\gamma}}] p_\beta[\Xi_\nu] \\ &= f[\Xi_{\bar{\gamma}}, \Xi_{\bar{\nu}}]\end{aligned}$$

Now if $f[\Xi_\gamma, \Xi_\nu] = 0$ for all pairs of partitions (γ, ν), then $f[\Xi_{\bar{\gamma}}, \Xi_{\bar{\nu}}] = 0$ for all partitions $(\bar{\gamma}, \bar{\nu})$ such that $|\bar{\gamma}| + |\bar{\nu}| \leq n$. This by Proposition 3.15 implies $f[X, Y] = 0$.

Case 2. If $|\gamma| \geq n$ and $|\nu| \leq n$ since $|\gamma| + |\nu| \geq n$ we may write $(\gamma, \nu) = ((n + 1, \bar{\gamma}), \nu)$ such that $|\bar{\gamma}| + |\nu| \leq n$. Since for all $k + k' \leq n$ we have

$$\begin{aligned}(p_k \otimes p_{k'})\,[\Xi_\gamma, \Xi_\nu] &= p_k[\Xi_\gamma] p_{k'}[\Xi_\nu] \\ &= (p_k[\Xi_{n+1} + \Xi_{\bar{\gamma}}])\,p_{k'}[\Xi_\nu] \\ &= (p_k[\Xi_{n+1}] + p_k[\Xi_{\bar{\gamma}}])\,p_{k'}[\Xi_\nu] \\ &= p_k[\Xi_{\bar{\gamma}}] p_{k'}[\Xi_\nu] \\ &= (p_k \otimes p_{k'})\,[\Xi_{\bar{\gamma}}, \Xi_\nu]\end{aligned}$$

A similar argument to Case 1, will give $f[X, Y] = 0$. (The case $|\gamma| \leq n$ and $|\nu| \geq n$ has a similar proof.)

Case 3. If $|\gamma| \leq n$ and $|\nu| \leq n$, write

$$(p_k \otimes p_{k'})\,[\Xi_\gamma, \Xi_\nu] = p_k[\Xi_\gamma] p_{k'}[\Xi_\nu]$$

Now if $p_k[\Xi_\gamma] = 0$ for all given $|\gamma| \leq n$ since $k \leq n$ by Proposition 3.12 we can conclude that $p_k = 0$. Similarly, if $p_{k'}[\Xi_\nu] = 0$ for all $|\nu| \leq n$, since $k' \leq n$ gives $p_{k'} = 0$. For $f = \sum_{|\alpha|+|\beta|\leq n} c_{(\alpha,\beta)} p_\alpha \otimes p_\beta$ we have

$$f[\Xi_\gamma, \Xi_\nu] = \sum_{|\alpha|+|\beta|\leq n} c_{(\alpha,\beta)} p_\alpha[\Xi_\gamma] p_\beta[\Xi_\nu]$$

For $f[\Xi_\gamma, \Xi_\nu] = 0$, either $p_k[\Xi_\gamma] = 0$ or $p_{k'}[\Xi_\nu] = 0$ that for partitions $|\gamma|, |\nu| \leq n$ and $k, k' \leq n$ will imply $p_k = 0$ or $p_{k'} = 0$ which in turn implies $f[X, Y] = 0$ and that completes the proof. □

Theorem 3.17. *Given a pair of partitions (λ, μ) the function $\tilde{s}^B_{(\lambda,\mu)}$ is the 'unique' symmetric function in $\Lambda \otimes \Lambda$ of degree $|(\lambda, \mu)| := |\lambda| + |\mu|$ that for all $n \geq |\lambda| + |\mu| + \lambda_1$ which satisfies the equation*

$$\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \left\langle s_{(|\gamma|+|\nu|-|\lambda|-|\mu|,\lambda)}[X+Y] s_\mu[X-Y], p_\gamma[X] p_\nu[Y] \right\rangle.$$

for all partitions γ and ν and $|\gamma| + |\nu| \leq n$ (respectively $|\gamma| + |\nu| \geq n$) where Ξ_γ and Ξ_ν are multi-set of eigenvalues of permutation matrices of cycle types γ and ν respectively and $(\gamma, \nu) \vdash n$.

Proof. In order to show that the symmetric function $\tilde{s}^B_{(\lambda,\mu)}$ is unique, let $f \in \Lambda \otimes \Lambda$ be another symmetric function of degree less than or equal to some positive integer n such that

$$f[\Xi_\gamma, \Xi_\nu] = \tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$$

for all pairs of partitions $|\gamma| + |\nu| \leq n$ (respectively $|\gamma| + |\nu| \geq n$). Now let $k = f - \tilde{s}^B_{(\lambda,\mu)}$ for $k \in \Lambda \otimes \Lambda$ and the evaluation at Ξ_γ and Ξ_ν is

$$k[\Xi_\gamma, \Xi_\nu] = f[\Xi_\gamma, \Xi_\nu] - \tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = 0$$

The symmetric functions $f, \tilde{s}^B_{(\lambda,\mu)} \in \Lambda \otimes \Lambda$ are both of degree less than or equal to some positive integer n hence by Proposition 3.15 (respectively Corollary 3.14) we must have $k = 0$ i.e. $f = \tilde{s}^B_{(\lambda,\mu)}$. This means the function $\tilde{s}^B_{(\lambda,\mu)}$ that satisfies the equation in this theorem is the only function with this property and the proof is complete. □

Corollary 3.18. *Given a pair of partitions (λ, μ) the function $\tilde{h}^B_{(\lambda,\mu)}$ is the 'unique' symmetric function in $\Lambda \otimes \Lambda$ that for all $n \geq |\lambda| + |\mu| + \lambda_1$ and $(\gamma, \nu) \vdash n$ which satisfies the equation*

$$\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \left\langle h_{(|\gamma|+|\nu|-|\lambda|-|\mu|,\lambda)}[X+Y] h_\mu[X-Y], p_\gamma[X] p_\nu[Y] \right\rangle.$$

for all partitions γ and ν and $|\gamma| + |\nu| \leq n$ where Ξ_γ and Ξ_ν are multi-set of eigenvalues of permutation matrices of cycle types γ and ν respectively and $(\gamma, \nu) \vdash n$.

Proof. The proof is similar to the proof of Theorem 3.17. □

Example 3.19. Let a symmetric function in two alphabets $\tilde{h}^B_{\bullet,2} \in \Lambda \otimes \Lambda$ and $deg(\tilde{h}^B_{\bullet,2}) \leq 2$ where

$$\tilde{h}^B_{\bullet,2}[\Xi_\gamma, \Xi_\nu] = \left\langle h_{(|\gamma|+|\nu|-2,\bullet)}[X+Y] h_2[X-Y], p_\gamma[X] p_\nu[Y] \right\rangle.$$

Moreover, let a function $f \in \Lambda \otimes \Lambda$ and $deg(f) \leq 2$ be given by

$$f = -(p_\bullet \otimes p_1 + p_1 \otimes p_\bullet) + \frac{1}{2}(p_\bullet \otimes p_{11} - 2p_1 \otimes p_1 + p_{11} \otimes p_\bullet) + \frac{1}{2}(p_\bullet \otimes p_2 + p_2 \otimes p_\bullet)$$

For the function $k = \tilde{h}^B_{\bullet,2} - f$ the evaluation of k at the eigenvalues of permutation matrices of cycle structure (γ, ν) i.e. $k[\Xi_\gamma, \Xi_\nu]$ for all pairs of partitions (γ, ν) such that $|\gamma| + |\nu| \leq 2$ will result in

$\|\gamma\| + \|\nu\|$	(γ, ν)	$k[\Xi_\gamma, \Xi_\nu]$
0	$((), ())$	$k[\Xi_\bullet, \Xi_\bullet] = 0$
1	$((1), ())$	$k[\Xi_1, \Xi_\bullet] = 0$
	$((), (1))$	$k[\Xi_\bullet, \Xi_1] = 0$
2	$((2), ())$	$k[\Xi_2, \Xi_\bullet] = 0$
	$((1,1), ())$	$k[\Xi_{11}, \Xi_\bullet] = 0$
	$((1), (1))$	$k[\Xi_1, \Xi_1] = 0$
	$((), (1,1))$	$k[\Xi_\bullet, \Xi_{11}] = 0$
	$((), (2))$	$k[\Xi_\bullet, \Xi_2] = 0$

that is

$$\begin{aligned} k[\Xi_\gamma, \Xi_\nu] &= \tilde{h}^B_{\bullet,2}[\Xi_\gamma, \Xi_\nu] - f[\Xi_\gamma, \Xi_\nu] \\ &= 0 \otimes 0 \end{aligned}$$

By Proposition 3.15 this implies k in $\Lambda \otimes \Lambda$ is zero hence the two given symmetric functions are equal and

$$f = \tilde{h}^B_{\bullet,2}.$$

3.1.3 Transition coefficients $\{\tilde{h}^B_{(\alpha,\beta)}\} \longrightarrow \{\tilde{s}^B_{(\lambda,\mu)}\}$

At first we will set up a total ordering on our pairs of partitions.

Definition 3.20. To establish a total ordering $\preceq$ on a pair of partitions (λ, μ) we will choose the graded reverse lexicographic order where $(\lambda, \mu) \preceq (\alpha, \beta)$ if

1. $|\lambda| + |\mu| < |\alpha| + |\beta|$ or [example: $(11, \bullet) \preceq (1, 2)$]
2. $|\lambda| + |\mu| = |\alpha| + |\beta|$ and $|\mu| < |\beta|$ or [example: $(2, 1) \preceq (1, 11)$]

3. $|\lambda| + |\mu| = |\alpha| + |\beta|$ and $|\mu| = |\beta|$ and $\mu >_{lex} \beta$ or [example: $(2,2) \preceq (11,11)$]

4. $|\lambda| + |\mu| = |\alpha| + |\beta|$ and $\mu = \beta$ and $\lambda >_{lex} \alpha$. [example: $(2,11) \preceq (11,11)$]

Definition 3.21. Given partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{l(\mu)})$ partition λ dominates μ (denoted by $\lambda \trianglerighteq \mu$) if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $1 \leq i \leq max(l(\lambda), l(\mu))$.

The lexicographic order is a total order and the dominance order is a partial order which is a sub order of the lexicographic ordering.

Theorem 3.22. *The matrix of Kostka coefficient is strictly upper unitriangular, this is $K_{\lambda\mu} > 0$ if and only if $\lambda \trianglerighteq \mu$ and $K_{\lambda\mu} = 1$ if $\lambda = \mu$. [Macdonald]*

Proposition 3.23. *We have*

$$\tilde{h}^B_{(\alpha,\beta)} = \tilde{s}^B_{(\alpha,\beta)} + \sum_{(\lambda,\mu)\preceq(\alpha,\beta)} a_{(\lambda,\mu)} \tilde{s}^B_{(\lambda,\mu)}$$

where the sum is overall pairs of partitions (λ, μ) such that $(\lambda, \mu) \preceq (\alpha, \beta)$.

Proof. By Definition 3.1, given pairs of partitions (α, β) and (λ, μ) and non-negative integer $n \geq 2|\alpha|$ define the function $\tilde{s}^B_{(\lambda,\mu)}$ that satisfies

$$\tilde{h}^B_{(\alpha,\beta)} = \sum_{\substack{(\lambda,\mu) \\ |\lambda|\leq|\alpha| \\ |\mu|=|\beta|}} K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta} \tilde{s}^B_{(\lambda,\mu)}$$

The Kostka coefficients have the triangularity property, this is $K_{\lambda\mu} > 0$ if and only if $\lambda \trianglerighteq \mu$ given $\trianglerighteq$ denotes the dominance ordering. This property in here implies that

$$K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta} > 0$$

if and only if

$$\begin{aligned} &(1) \quad (n - |\lambda|, \lambda) \trianglerighteq (n - |\alpha|, \alpha) \quad \text{and} \\ &(2) \quad \mu \trianglerighteq \beta \end{aligned}$$

The condition (1) implies condition (3) and condition (4) is by properties of Kostka numbers, this is

$$\begin{aligned} &(3) \quad |\lambda| \leq |\alpha|, \quad \text{and} \\ &(4) \quad |\mu| = |\beta| \end{aligned}$$

The total order $\preceq$ defined in Definition 3.20 is compatible with conditions (1), (2), (3) and (4) since the lexicographic ordering is a total order and the dominance order is a suborder of lexicographic ordering (If $\tau <_{lex} \theta$ then $\tau \trianglelefteq \theta$ [Macdonald]). To see this in here, let $(\lambda, \mu) \preceq (\alpha, \beta)$ for a fixed (α, β) and given condition (4) $|\mu| = |\beta|$ so by the total oder $\preceq$ we must have

i. $|\lambda| + |\mu| < |\alpha| + |\beta|$ (This is simplified as $|\lambda| < |\alpha|$ that is condition (3)); or

ii. $|\lambda| + |\mu| = |\alpha| + |\beta|$ and $|\mu| < |\beta|$ (This is not possible as $|\mu| = |\beta|$) ; or

iii. $|\lambda| + |\mu| = |\alpha| + |\beta|$ and $|\mu| = |\beta|$ and $\mu >_{lex} \beta$ (Which implies $\mu \trianglerighteq \beta$ that is condition (2)) ; or

iv. $|\lambda| = |\alpha|$ and $\mu = \beta$ and $\lambda >_{lex} \alpha$. (Which implies $(n - |\lambda|, \lambda) \trianglerighteq (n - |\alpha|, \alpha)$ that is condition (2)))

On the other hand, by Theorem 3.22 the change of basis matrix is unitriangular for each matrix of the two Kostka coefficients $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ and $K_{\mu\beta}$. An important observation here is $K_{\mu\beta}$ will result in a upper unitriangular matrix but $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ will give a lower unitriangular matrix. The later is due to condition (3) $|\lambda| \leq |\alpha|$. Also, the leading term is when $K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} = 1$ and $K_{\mu\beta} = 1$ that is $(\lambda, \mu) = (\alpha, \beta)$. All other terms that are $(\lambda, \mu) \preceq (\alpha, \beta)$ will be the smaller order terms and we have

$$\tilde{h}^B_{(\alpha,\beta)} = \tilde{s}^B_{(\alpha,\beta)} + \sum_{(\lambda,\mu) \preceq (\alpha,\beta)} a_{(\lambda,\mu)} \tilde{s}^B_{(\lambda,\mu)}.$$

□

Here we are following the total ordering $\preceq$ on a pair of partitions (λ, μ) as in Definition 3.20 and we will calculate and present the matrix of the transition coefficients in $\{\tilde{h}^B_{(\alpha,\beta)}\} \longrightarrow \{\tilde{s}^B_{(\lambda,\mu)}\}$. The Kostka coefficients will be entries of this matrix and is of lower triangular form with 1's on diagonal entries. This will also make this transition easy to invert.

Corollary 3.24. *The matrix of transition coefficient* $\{\tilde{h}^B_{(\alpha,\beta)}\} \rightarrow \{\tilde{s}^B_{(\lambda,\mu)}\}$ *is lower unitriangular.*

Proof. By Proposition 3.23 we have

$$\tilde{h}^B_{(\alpha,\beta)} = \tilde{s}^B_{(\alpha,\beta)} + \sum_{(\lambda,\mu) \preceq (\alpha,\beta)} a_{(\lambda,\mu)} \tilde{s}^B_{(\lambda,\mu)}$$

For pairs of partitions that $(\alpha, \beta) \preceq (\lambda, \mu)$ we have the negation of the two conditions (1) and (2) of the proof of Proposition 3.23 therefore

$$K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta} = 0$$

This means for entries which $(\alpha, \beta) \preceq (\lambda, \mu)$ the entries are 0's. For pairs $(\lambda, \mu) = (\alpha, \beta)$ we have $\tilde{h}^B_{(\alpha,\beta)} = \tilde{s}^B_{(\alpha,\beta)}$ which implies the transition matrix to have 1's on the diagonal entries hence we conclude that the matrix of transition coefficient $\{\tilde{h}^B_{(\alpha,\beta)}\} \rightarrow \{\tilde{s}^B_{(\lambda,\mu)}\}$ is lower unitriangular. □

Example 3.25. In Proposition 3.23 and Corollary 3.24 we have arranged the columns and rows of the transition coefficients matrix according to a total order $\preceq$ defined in Definition 3.20. The functions $\tilde{s}^B_{(\lambda,\mu)}$ (or $\tilde{h}^B_{(\lambda,\mu)}$) are in a one to one correspondence between partition (λ, μ). For example we have ordering of

$$\tilde{s}^B_{1,\bullet} \preceq \tilde{s}^B_{2,1} \preceq \tilde{s}^B_{11,1} \preceq \tilde{s}^B_{1,2} \preceq \tilde{s}^B_{1,11} \preceq \tilde{s}^B_{\bullet,3}$$

Example 3.26. Evaluate the transition coefficients matrix when expressing $\tilde{h}^B_{(\alpha,\beta)}$ in $\tilde{s}^B_{(\lambda,\mu)}$ for $n = 1, 2, 3$.

We will use Definition 3.1 in order to express $\tilde{h}^B_{(\alpha,\beta)}$ in $\tilde{s}^B_{(\lambda,\mu)}$, that is, for a non-negative integer $n \geq 2\,|\alpha|$ define the function $\tilde{s}^B_{(\lambda,\mu)}$ in $\Lambda \otimes \Lambda$ that satisfies

$$\tilde{h}^B_{(\alpha,\beta)} = \sum_{\substack{(\lambda,\mu) \\ |\lambda| \leq |\alpha| \\ |\mu| = |\beta|}} K_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K_{\mu\beta} \tilde{s}^B_{(\lambda,\mu)}$$

We have shown examples of using this definition in Example 3.6, Example 3.7 and Example 3.8. The matrix of transition coefficients $\{\tilde{h}^B_{(\alpha,\beta)}\} \longrightarrow \{\tilde{s}^B_{(\lambda,\mu)}\}$ for $n = 1, 2, 3$ is presented below.

$\{\tilde{h}^B_{(\alpha,\beta)}\} \to \{\tilde{s}^B_{(\lambda,\mu)}\}$	$\tilde{s}^B_{\bullet,\bullet}$	$\tilde{s}^B_{1,\bullet}$	$\tilde{s}^B_{\bullet,1}$	$\tilde{s}^B_{2,\bullet}$	$\tilde{s}^B_{11,\bullet}$	$\tilde{s}^B_{1,1}$	$\tilde{s}^B_{\bullet,2}$	$\tilde{s}^B_{\bullet,11}$	$\tilde{s}^B_{3,\bullet}$	$\tilde{s}^B_{21,\bullet}$	$\tilde{s}^B_{111,\bullet}$	$\tilde{s}^B_{2,1}$	$\tilde{s}^B_{11,1}$	$\tilde{s}^B_{1,2}$	$\tilde{s}^B_{1,11}$	$\tilde{s}^B_{\bullet,3}$	$\tilde{s}^B_{\bullet,21}$	$\tilde{s}^B_{\bullet,111}$
$\tilde{h}^B_{\bullet,\bullet}$	1																	
$\tilde{h}^B_{1,\bullet}$	1	1																
$\tilde{h}^B_{\bullet,1}$			1															
$\tilde{h}^B_{2,\bullet}$	1	1		1														
$\tilde{h}^B_{11,\bullet}$	1	2		1	1													
$\tilde{h}^B_{1,1}$			1			1												
$\tilde{h}^B_{\bullet,2}$							1											
$\tilde{h}^B_{\bullet,11}$							1	1										
$\tilde{h}^B_{3,\bullet}$	1	1		1					1									
$\tilde{h}^B_{21,\bullet}$	1	2		2	1				1	1								
$\tilde{h}^B_{111,\bullet}$	1	3		3	3				1	2	1							
$\tilde{h}^B_{2,1}$			1			1						1						
$\tilde{h}^B_{11,1}$			1			2						1	1					
$\tilde{h}^B_{1,2}$							1							1				
$\tilde{h}^B_{1,11}$							1	1						1	1			
$\tilde{h}^B_{\bullet,3}$																1		
$\tilde{h}^B_{\bullet,21}$																1	1	
$\tilde{h}^B_{\bullet,111}$																1	2	1

Table 3.4: Transition coefficients matrix $\{\tilde{h}^B_{(\alpha,\beta)}\} \to \{\tilde{s}^B_{(\lambda,\mu)}\}$

Example 3.27. We will compute the inverse of the transition matrix from Example 3.26 which is to express $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}^B_{(\alpha,\beta)}$. This is shown in the following transition matrix for $n = 1, 2, 3$. Note that in Remark 3.2 it is stated that given $n \geq 2\,|\lambda|$ we can express $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}^B_{(\alpha,\beta)}$ by the product of the inverse of Kostka coefficients, i.e.

$$\tilde{s}^B_{(\lambda,\mu)} = \sum_{\substack{(\alpha,\beta) \\ |\alpha|\leq|\lambda| \\ |\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K^{-1}_{\mu\beta} \tilde{h}^B_{(\alpha,\beta)}$$

where the inverse of Kostka coefficient $K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)}$ is the coefficient of $h_{(n-|\alpha|,\alpha)}$ in $s_{(n-|\lambda|,\lambda)}$ and $K^{-1}_{\mu\beta}$ is the coefficient of h_β in s_μ. For example given $n \geq 2\,|\lambda| = 2$ let $n = 2$

$$\begin{aligned}
\tilde{s}^B_{1,1} &= \sum_{\substack{(\alpha,\beta) \\ |\alpha|\leq 1 \\ |\beta|=1}} K^{-1}_{(1,1)(2-|\alpha|,\alpha)} K^{-1}_{(1)\beta} \tilde{h}^B_{(\alpha,\beta)} \\
&= K^{-1}_{(1,1)(2)} K^{-1}_{(1)(1)} \tilde{h}^B_{\bullet,1} + K^{-1}_{(1,1)(1,1)} K^{-1}_{(1)(1)} \tilde{h}^B_{1,1} \\
&= (-1)(1)\tilde{h}^B_{\bullet,1} + (1)(1)\tilde{h}^B_{1,1} \\
&= \tilde{h}^B_{1,1} - \tilde{h}^B_{\bullet,1}
\end{aligned}$$

Note that to calculate the Kostka numbers above, given expansion $s_{11} = h_{11} - h_2$ the coefficient of h_2 in s_{11} is $K^{-1}_{(1,1)(2)} = -1$ and the coefficient of h_{11} is $K^{-1}_{(1,1)(1,1)} = 1$. Moreover, for expansion of $s_1 = h_1$ the coefficient of h_1 in s_1 is $K^{-1}_{(1)(1)} = 1$.

We may also find these expression of $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}^B_{(\alpha,\beta)}$ recursively, that is shown below.

$\tilde{h}^B_{(\lambda,\mu)}$ in $\tilde{s}^B_{(\alpha,\beta)}$	$\longrightarrow$	$\tilde{s}^B_{(\lambda,\mu)}$ in $\tilde{h}^B_{(\alpha,\beta)}$
$\tilde{h}^B_{\bullet,\bullet} = \tilde{s}^B_{\bullet,\bullet}$		$\tilde{s}^B_{\bullet,\bullet} = \tilde{h}^B_{\bullet,\bullet}$
$\tilde{h}^B_{1,\bullet} = \tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$		$\tilde{s}^B_{1,\bullet} = \tilde{h}^B_{1,\bullet} - \tilde{h}^B_{\bullet,\bullet}$
$\tilde{h}^B_{\bullet,1} = \tilde{s}^B_{\bullet,1}$		$\tilde{s}^B_{\bullet,1} = \tilde{h}^B_{\bullet,1}$
$\tilde{h}^B_{2,\bullet} = \tilde{s}^B_{2,\bullet} + \tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$		$\tilde{s}^B_{2,\bullet} = \tilde{h}^B_{2,\bullet} - \tilde{h}^B_{1,\bullet}$
$\tilde{h}^B_{11,\bullet} = \tilde{s}^B_{11,\bullet} + \tilde{s}^B_{2,\bullet} + 2\tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$		$\tilde{s}^B_{11,\bullet} = \tilde{h}^B_{11,\bullet} - \tilde{h}^B_{2,\bullet} - \tilde{h}^B_{1,\bullet} + \tilde{h}^B_{\bullet,\bullet}$
$\tilde{h}^B_{1,1} = \tilde{s}^B_{1,1} + \tilde{s}^B_{\bullet,1}$		$\tilde{s}^B_{1,1} = \tilde{h}^B_{1,1} - \tilde{h}^B_{\bullet,1}$
$\tilde{h}^B_{\bullet,2} = \tilde{s}^B_{\bullet,2}$		$\tilde{s}^B_{\bullet,2} = \tilde{h}^B_{\bullet,2}$
$\tilde{h}^B_{\bullet,11} = \tilde{s}^B_{\bullet,11} + \tilde{s}^B_{\bullet,2}$		$\tilde{s}^B_{\bullet,11} = \tilde{h}^B_{\bullet,11} - \tilde{h}^B_{\bullet,2}$
$\tilde{h}^B_{3,\bullet} = \tilde{s}^B_{3,\bullet} + \tilde{s}^B_{2,\bullet} + \tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$		$\tilde{s}^B_{3,\bullet} = \tilde{h}^B_{3,\bullet} - \tilde{h}^B_{2,\bullet}$
$\tilde{h}^B_{21,\bullet} = \tilde{s}^B_{21,\bullet} + \tilde{s}^B_{3,\bullet} + \tilde{s}^B_{11,\bullet} + 2\tilde{s}^B_{2,\bullet} + 2\tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$		$\tilde{s}^B_{21,\bullet} = \tilde{h}^B_{21,\bullet} - \tilde{h}^B_{3,\bullet} - \tilde{h}^B_{11,\bullet} + \tilde{h}^B_{1,\bullet}$
$\tilde{h}^B_{111,\bullet} = \tilde{s}^B_{111,\bullet} + 2\tilde{s}^B_{21,\bullet} + \tilde{s}^B_{3,\bullet} + 3\tilde{s}^B_{11,\bullet} + 3\tilde{s}^B_{2,\bullet} + 3\tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$		$\tilde{s}^B_{111,\bullet} = \tilde{h}^B_{111,\bullet} - 2\tilde{h}^B_{21,\bullet} + \tilde{h}^B_{3,\bullet} - \tilde{h}^B_{11,\bullet} + \tilde{h}^B_{2,\bullet} + \tilde{h}^B_{1,\bullet} - \tilde{h}^B_{\bullet,\bullet}$
$\tilde{h}^B_{2,1} = \tilde{s}^B_{2,1} + \tilde{s}^B_{1,1} + \tilde{s}^B_{\bullet,1}$		$\tilde{s}^B_{2,1} = \tilde{h}^B_{2,1} - \tilde{h}^B_{1,1}$
$\tilde{h}^B_{11,1} = \tilde{s}^B_{11,1} + \tilde{s}^B_{2,1} + 2\tilde{s}^B_{1,1} + \tilde{s}^B_{\bullet,1}$		$\tilde{s}^B_{11,1} = \tilde{h}^B_{11,1} - \tilde{h}^B_{2,1} - \tilde{h}^B_{1,1} + \tilde{h}^B_{\bullet,1}$
$\tilde{h}^B_{1,2} = \tilde{s}^B_{1,2} + \tilde{s}^B_{\bullet,2}$		$\tilde{s}^B_{1,2} = \tilde{h}^B_{1,2} - \tilde{h}^B_{\bullet,2}$
$\tilde{h}^B_{1,11} = \tilde{s}^B_{1,11} + \tilde{s}^B_{1,2} + \tilde{s}^B_{\bullet,11} + \tilde{s}^B_{\bullet,2}$		$\tilde{s}^B_{1,11} = \tilde{h}^B_{1,11} - \tilde{h}^B_{1,2} - \tilde{h}^B_{\bullet,11} + \tilde{h}^B_{\bullet,2}$
$\tilde{h}^B_{\bullet,3} = \tilde{s}^B_{\bullet,3}$		$\tilde{s}^B_{\bullet,3} = \tilde{h}^B_{\bullet,3}$
$\tilde{h}^B_{\bullet,21} = \tilde{s}^B_{\bullet,21} + \tilde{s}^B_{\bullet,3}$		$\tilde{s}^B_{\bullet,21} = \tilde{h}^B_{\bullet,21} - \tilde{h}^B_{\bullet,3}$
$\tilde{h}^B_{\bullet,111} = \tilde{s}^B_{\bullet,111} + 2\tilde{s}^B_{\bullet,21} + \tilde{s}^B_{\bullet,3}$		$\tilde{s}^B_{\bullet,111} = \tilde{h}^B_{\bullet,111} - 2\tilde{h}^B_{\bullet,21} + \tilde{h}^B_{\bullet,3}$

Table 3.5: Expression of $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}^B_{(\alpha,\beta)}$ recursively.

The matrix of transition coefficients $\{\tilde{s}^B_{(\lambda,\mu)}\}\to\{\tilde{h}^B_{(\alpha,\beta)}\}$ for $n=1,2,3$ is presented below.

$\{\tilde{s}^B_{(\lambda,\mu)}\}\to\{\tilde{h}^B_{(\alpha,\beta)}\}$	$\tilde{h}^B_{\bullet,\bullet}$	$\tilde{h}^B_{1,\bullet}$	$\tilde{h}^B_{\bullet,1}$	$\tilde{h}^B_{2,\bullet}$	$\tilde{h}^B_{11,\bullet}$	$\tilde{h}^B_{1,1}$	$\tilde{h}^B_{\bullet,2}$	$\tilde{h}^B_{\bullet,11}$	$\tilde{h}^B_{3,\bullet}$	$\tilde{h}^B_{21,\bullet}$	$\tilde{h}^B_{111,\bullet}$	$\tilde{h}^B_{2,1}$	$\tilde{h}^B_{11,1}$	$\tilde{h}^B_{1,2}$	$\tilde{h}^B_{1,11}$	$\tilde{h}^B_{\bullet,3}$	$\tilde{h}^B_{\bullet,21}$	$\tilde{h}^B_{\bullet,111}$
$\tilde{s}^B_{\bullet,\bullet}$	1																	
$\tilde{s}^B_{1,\bullet}$	−1	1																
$\tilde{s}^B_{\bullet,1}$			1															
$\tilde{s}^B_{2,\bullet}$		−1		1														
$\tilde{s}^B_{11,\bullet}$	1	−1		−1	1													
$\tilde{s}^B_{1,1}$			−1			1												
$\tilde{s}^B_{\bullet,2}$							1											
$\tilde{s}^B_{\bullet,11}$							−1	1										
$\tilde{s}^B_{3,\bullet}$				−1					1									
$\tilde{s}^B_{21,\bullet}$		1			−1				−1	1								
$\tilde{s}^B_{111,\bullet}$	−1	1		1	−1				1	−2	1							
$\tilde{s}^B_{2,1}$						−1						1						
$\tilde{s}^B_{11,1}$			1			−1						−1	1					
$\tilde{s}^B_{1,2}$							−1							1				
$\tilde{s}^B_{1,11}$							1	−1						−1	1			
$\tilde{s}^B_{\bullet,3}$																1		
$\tilde{s}^B_{\bullet,21}$																−1	1	
$\tilde{s}^B_{\bullet,111}$																1	−2	1

Table 3.6: The matrix of transition coefficients $\{\tilde{s}^B_{(\lambda,\mu)}\}\to\{\tilde{h}^B_{(\alpha,\beta)}\}$

Example 3.28. We can use Example 3.27 and Example 2.26 to express $\tilde{s}^B_{(\lambda,\mu)}$ in $\tilde{h}_\alpha\otimes\widetilde{he}_{\beta|\gamma}$ for $n=1,2$. Since the function $\widetilde{he}_{\beta|\gamma}$ is not a basis, the number of elements $\tilde{h}_\alpha\otimes\widetilde{he}_{\beta|\gamma}$ are too long to fit in a table, but the matrix is relatively sparse, we are providing a table of expressions below.

n	(λ,μ)	$\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}_\alpha\otimes\widetilde{he}_{\beta\vert\gamma}$ in type A
0	$(\bullet,\bullet)$	$\tilde{s}^B_{\bullet,\bullet}=\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet\vert\bullet}$
1	$(1,\bullet)$	$\tilde{s}^B_{1,\bullet}=-\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet\vert\bullet}+\tilde{h}_\bullet\otimes\widetilde{he}_{1\vert\bullet}+\tilde{h}_1\otimes\widetilde{he}_{\bullet\vert\bullet}$
	$(\bullet,1)$	$\tilde{s}^B_{\bullet,1}=-\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet\vert 1}+\tilde{h}_1\otimes\widetilde{he}_{\bullet\vert\bullet}$
2	$(2,\bullet)$	$\tilde{s}^B_{2,\bullet}=-\tilde{h}_\bullet\otimes\widetilde{he}_{1\vert\bullet}+\tilde{h}_1\otimes\widetilde{he}_{\bullet\vert\bullet}+\tilde{h}_\bullet\otimes\widetilde{he}_{2\vert\bullet}+\tilde{h}_1\otimes\widetilde{he}_{1\vert\bullet}+\tilde{h}_2\otimes\widetilde{he}_{\bullet\vert\bullet}$
	$(11,\bullet)$	$\tilde{s}^B_{11,\bullet}=-\tilde{h}_\bullet\otimes\widetilde{he}_{2\vert\bullet}+\tilde{h}_1\otimes\widetilde{he}_{1\vert\bullet}-\tilde{h}_2\otimes\widetilde{he}_{\bullet\vert\bullet}+\tilde{h}_\bullet\otimes\widetilde{he}_{11\vert\bullet}+\tilde{h}_{11}\otimes\widetilde{he}_{\bullet\vert\bullet}$
	$(1,1)$	$\tilde{s}^B_{1,1}=\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet\vert 1}-\tilde{h}_1\otimes\widetilde{he}_{\bullet\vert\bullet}-\tilde{h}_\bullet\otimes\widetilde{he}_{1\vert 1}+\tilde{h}_1\otimes\widetilde{he}_{1\vert\bullet}-\tilde{h}_1\otimes\widetilde{he}_{\bullet\vert 1}+\tilde{h}_{11}\otimes\widetilde{he}_{\bullet\vert\bullet}$
	$(\bullet,2)$	$\tilde{s}^B_{\bullet,2}=\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet\vert 2}-\tilde{h}_1\otimes\widetilde{he}_{\bullet\vert 1}+\tilde{h}_2\otimes\widetilde{he}_{\bullet\vert\bullet}$
	$(\bullet,11)$	$\tilde{s}^B_{\bullet,11}=-\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet\vert 2}-\tilde{h}_1\otimes\widetilde{he}_{\bullet\vert 1}-\tilde{h}_2\otimes\widetilde{he}_{\bullet\vert\bullet}+\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet\vert 11}+\tilde{h}_{11}\otimes\widetilde{he}_{\bullet\vert\bullet}$

Table 3.7: Expression $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}_\alpha\otimes\widetilde{he}_{\beta|\gamma}$ in type A

Example 3.29. By Example 3.28 and Example 1.203 we can express $\tilde{s}^B_{(\lambda,\mu)}$ in $\tilde{s}_\alpha\otimes\tilde{s}_\beta$ for $n=1,2$. As an example, the expansion of $\tilde{s}^B_{1,\bullet}$ in $\tilde{s}_\alpha\otimes\tilde{s}_\beta$ is

$$\begin{aligned}\tilde{s}^B_{1,\bullet}&=-\tilde{h}_\bullet\otimes\widetilde{he}_{\bullet|\bullet}+\tilde{h}_\bullet\otimes\widetilde{he}_{1|\bullet}+\tilde{h}_1\otimes\widetilde{he}_{\bullet|\bullet}\\&=-\tilde{h}_\bullet\otimes\tilde{h}_\bullet+\tilde{h}_\bullet\otimes\tilde{h}_1+\tilde{h}_1\otimes\tilde{h}_\bullet\\&=-\tilde{s}_\bullet\otimes\tilde{s}_\bullet+\tilde{s}_\bullet\otimes\tilde{s}_1+\tilde{s}_1\otimes\tilde{s}_\bullet\end{aligned}$$

Below is the transition matrix of $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{s}_\alpha \otimes \tilde{s}_\beta$ for $n = 1, 2$. Note that here we used SageMath for these change of basis.

$\{\tilde{s}^B_{(\lambda,\mu)}\} \to \{\tilde{s}_\alpha \otimes \tilde{s}_\beta\}$	$\tilde{s}_\bullet \otimes \tilde{s}_\bullet$	$\tilde{s}_1 \otimes \tilde{s}_\bullet$	$\tilde{s}_\bullet \otimes \tilde{s}_1$	$\tilde{s}_2 \otimes \tilde{s}_\bullet$	$\tilde{s}_{11} \otimes \tilde{s}_\bullet$	$\tilde{s}_1 \otimes \tilde{s}_1$	$\tilde{s}_\bullet \otimes \tilde{s}_2$	$\tilde{s}_\bullet \otimes \tilde{s}_{11}$
$\tilde{s}^B_{\bullet,\bullet}$	1							
$\tilde{s}^B_{1,\bullet}$	-1	1	1					
$\tilde{s}^B_{\bullet,1}$		1	-1					
$\tilde{s}^B_{2,\bullet}$	1	1	1	1		1	1	
$\tilde{s}^B_{11,\bullet}$		1	1		1	1		1
$\tilde{s}^B_{1,1}$		1	-2	1	1		-1	-2
$\tilde{s}^B_{\bullet,2}$				1		-1		1
$\tilde{s}^B_{\bullet,11}$					1	-1	1	

Table 3.8: Transition coefficient $\{\tilde{s}^B_{(\lambda,\mu)}\} \to \{\tilde{s}_\alpha \otimes \tilde{s}_\beta\}$

Example 3.30. By Example 3.29 we can change basis and express $\tilde{s}^B_{(\lambda,\mu)}$ in $s_\alpha \otimes s_\beta$ for $n = 1, 2$. Note that here we used SageMath for these change of basis.

$\{\tilde{s}^B_{(\lambda,\mu)}\} \to \{s_\alpha \otimes s_\beta\}$	$s_\bullet \otimes s_\bullet$	$s_1 \otimes s_\bullet$	$s_\bullet \otimes s_1$	$s_2 \otimes s_\bullet$	$s_{11} \otimes s_\bullet$	$s_1 \otimes s_1$	$s_\bullet \otimes s_2$	$s_\bullet \otimes s_{11}$
$\tilde{s}^B_{\bullet,\bullet}$	1							
$\tilde{s}^B_{1,\bullet}$	-1	1	1					
$\tilde{s}^B_{\bullet,1}$		1	-1					
$\tilde{s}^B_{2,\bullet}$		-2	-2	1		1	1	
$\tilde{s}^B_{11,\bullet}$	1	-1	-1		1	1		1
$\tilde{s}^B_{1,1}$		-2	2	1	1		-1	-1
$\tilde{s}^B_{\bullet,2}$		-1		1		-1		1
$\tilde{s}^B_{\bullet,11}$			-1		1	-1	1	

Table 3.9: Transition coefficient $\{\tilde{s}^B_{(\lambda,\mu)}\} \to \{s_\alpha \otimes s_\beta\}$

3.2 Frobenius Map and Stability of Kronecker Coefficients of $\tilde{s}^B_{(\lambda,\mu)}$

3.2.1 B_n-Frobenius Map Φ_{B_n}

Type A. The Frobenius characteristic map $\mathscr{F}_{S_n}$ is a linear isomorphism from the class functions of the symmetric group denoted by $R(S_n)$, to the ring of symmetric functions Λ^n, this is

$$\begin{array}{rccc} \mathscr{F}_{S_n}: & R(S_n) & \longrightarrow & \Lambda^n \\ & \mathbf{1}_\mu & \longmapsto & \dfrac{p_\mu}{z_\mu} \end{array}$$

Equivalently, since $\chi^\lambda = \sum_{\mu\vdash n} \chi^\lambda_\mu \mathbf{1}_\mu$, we have $\mathscr{F}_{S_n}(\chi^\lambda) = \sum_{\mu\vdash n} \chi^\lambda_\mu \frac{p_\mu}{z_\mu}$ where χ^λ_μ is the value of χ^λ on conjugacy class of cycle-type μ i.e. K_μ. Generally, given f to be a class function on S_n we can write

$$\mathscr{F}_{S_n}(f) = \sum_{\lambda\vdash n} f(\sigma)\frac{p_\lambda}{z_\lambda}.$$

The characters of the symmetric group are class functions which in [OZ1] were also introduced as symmetric functions $\tilde{s}_\lambda$ and $\tilde{h}_\lambda$. In Definition 1.220 (Equation 12 [OZ3]) the S_n-Frobenius map on the ring of symmetric functions in Type A, is $\Phi_n : \Lambda \longrightarrow \Lambda$ and is given by

$$\Phi_n(f) = \sum_{\mu\vdash n} f[\Xi_\mu]\frac{p_\mu}{z_\mu}.$$

Now, we will develop these ideas and relate them to discussion in Type B.

Type B. In Definition 1.124 the Frobenius characteristic map $\mathscr{F}_{B_n}$ on the class functions of the hyperoctahedral group denoted by $R(B_n)$ and letting $\Lambda_{B_n} = \Lambda_{B_n}[X,Y] := \bigoplus_{k=0}^{n} \Lambda^k[X] \otimes \Lambda^{n-k}[Y]$ is given by

$$\begin{array}{rccc} \mathscr{F}_{B_n}: & R(B_n) & \longrightarrow & \Lambda_{B_n}[X,Y] \\ & \mathbf{1}^{B_n}_{(\gamma,\nu)} & \longmapsto & \dfrac{1}{z_\gamma}p_\gamma[X]\dfrac{1}{z_\nu}p_\nu[Y] \end{array}$$

This is called the B_n-Frobenius characteristic map. Moreover, in Theorem 1.135 the Frobenius characteristic map on the irreducible character of group B_n is given by

$$\begin{aligned} \mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) &:= \sum_{(\gamma,\nu)\vdash n} \chi^{(\lambda,\mu)}(\gamma,\nu)\frac{p_\gamma[X]}{z_\gamma}\frac{p_\nu[Y]}{z_\nu} \\ &= s_\lambda[X+Y]s_\mu[X-Y]. \end{aligned}$$

The following definition is the analogue of S_n-Frobenius map in Type B.

Definition 3.31. For positive integer n, define the $\mathbf{B_n}$**-Frobenius map** $\Phi_{B_n} : \Lambda \otimes \Lambda \longrightarrow \Lambda \otimes \Lambda$ by

$$\Phi_{B_n}(f[X,Y]) = \sum_{(\gamma,\nu)\vdash n} f[\Xi_\gamma, \Xi_\nu] \frac{p_\gamma[X]}{z_\gamma} \frac{p_\nu[Y]}{z_\nu}.$$

The map Φ_{B_n} is from the tensor square of the ring of symmetric functions to the subspace of the tensor square of symmetric functions of degree n.

Lemma 3.32. *Given $\Phi_{B_n} : \Lambda \otimes \Lambda \longrightarrow \Lambda \otimes \Lambda$ for function $\tilde{s}^B_{(\lambda,\mu)} \in \Lambda \otimes \Lambda$ and for $n \geq |\lambda| + |\mu| + \lambda_1$ we have*

$$\Phi_{B_n}(\tilde{s}^B_{(\lambda,\mu)}[X,Y]) = s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y] s_\mu[X-Y].$$

Proof. By Theorem 3.17 function $\tilde{s}^B_{(\lambda,\mu)}$ is the 'unique' symmetric function that for all $n \geq |\lambda| + |\mu| + \lambda_1$ and $(\gamma,\nu) \vdash n$ satisfies the equation

$$\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \left\langle s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y] s_\mu[X-Y], p_\gamma[X] p_\nu[Y] \right\rangle$$

where Ξ_γ and Ξ_ν are eigenvalues of permutation matrices of cycle types γ and ν respectively. Moreover, the expression on the right hand side of this formula is in fact the value of irreducible character of hyperoctahedral group indexed by $((n-|\lambda|-|\mu|, \lambda), \mu)$ evaluated at an element of cycle type (γ,ν) that is

$$\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] = \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\gamma,\nu)$$

On the other hand by Definition 1.124 and by Theorem 1.135 the B_n-Frobenius characteristic map is

$$\begin{aligned}\mathscr{F}_{B_n}(\chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}) &:= \sum_{(\gamma,\nu)\vdash n} \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\gamma,\nu) \frac{p_\gamma[X]}{z_\gamma} \frac{p_\nu[Y]}{z_\nu} \\ &= s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y] s_\mu[X-Y].\end{aligned}$$

Now, by Definition 3.31 the above formula on the space $\Lambda \otimes \Lambda$ can be presented as

$$\begin{aligned}\Phi_{B_n}(\tilde{s}^B_{(\lambda,\mu)}[X,Y]) &= \sum_{(\gamma,\nu)\vdash n} \tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] \frac{p_\gamma[X]}{z_\gamma} \frac{p_\nu[Y]}{z_\nu} \\ &= s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y] s_\mu[X-Y]\end{aligned}$$

and this will complete the proof. ∎

Example 3.33. Expand $\Phi_{B_n}(\tilde{s}^B_{(\lambda,\mu)}[X,Y])$ for $\lambda = (1)$ and $\mu = (1)$ when $n = 3$. We have

$$\Phi_{B_n}(\tilde{s}^B_{1,1}[X,Y]) = s_{11}[X+Y]s_1[X-Y]$$

We will drop the X and Y's and write the equation in tensor notation, we have

$$\begin{aligned}\Phi_{B_n}(\tilde{s}^B_{1,1}) &= (s_\bullet \otimes s_{11} + s_1 \otimes s_1 + s_{11} \otimes s_\bullet)\,(-s_\bullet \otimes s_1 + s_1 \otimes s_\bullet) \\ &= -s_\bullet \otimes s_{111} - s_\bullet \otimes s_{21} - s_1 \otimes s_2 + s_2 \otimes s_1 + s_{111} \otimes s_\bullet + s_{21} \otimes s_\bullet\end{aligned}$$

The image of $\Phi_{B_n}(\tilde{s}^B_{1,1})$ is expanded in p basis as the following.

$$\begin{aligned}&\Phi_{B_n}(\tilde{s}^B_{1,1}) \\ &= -\frac{1}{2}p_\bullet \otimes p_{111} + \frac{1}{2}p_\bullet \otimes p_{21} - \frac{1}{2}p_1 \otimes p_{11} - \frac{1}{2}p_1 \otimes p_2 + \frac{1}{2}p_{11} \otimes p_1 + \frac{1}{2}p_{111} \otimes p_\bullet + \frac{1}{2}p_2 \otimes p_1 - \frac{1}{2}p_{21} \otimes p_\bullet \\ &= -3\frac{p_\bullet}{1} \otimes \frac{p_{111}}{6} + 1\frac{p_\bullet}{1} \otimes \frac{p_{21}}{2} - 1\frac{p_1}{1} \otimes \frac{p_{11}}{2} - 1\frac{p_1}{1} \otimes \frac{p_2}{2} + 1\frac{p_{11}}{2} \otimes \frac{p_1}{1} + 3\frac{p_{111}}{6} \otimes \frac{p_\bullet}{1} + 1\frac{p_2}{2} \otimes \frac{p_1}{1} - 1\frac{p_{21}}{2} \otimes \frac{p_\bullet}{1}\end{aligned}$$

By Lemma 3.32 the irreducible characters $\chi^{((1,1),(1))}_{(\gamma,\nu)}$ in each conjugacy class can be found in the coefficients of $\frac{p_\gamma}{z_\gamma} \otimes \frac{p_\nu}{z_\nu}$ in this expansion.

Corollary 3.34. *Given $\Phi_{B_n} : \Lambda \otimes \Lambda \longrightarrow \Lambda \otimes \Lambda$ for function $\tilde{h}^B_{(\lambda,\mu)} \in \Lambda \otimes \Lambda$ and for $n \geq |\lambda| + |\mu| + \lambda_1$ we have*

$$\Phi_{B_n}(\tilde{h}^B_{(\lambda,\mu)}[X,Y]) = h_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]h_\mu[X-Y].$$

Proof. Proof is similar to Lemma 3.32. □

Kronecker Product in Type B. Recall that the Kronecker product denoted by $*$ on the power sum symmetric functions is given by

$$\frac{p_\gamma}{z_\gamma} * \frac{p_\nu}{z_\nu} = \delta_{\gamma\nu}\frac{p_\gamma}{z_\gamma}$$

where the Kronecker delta function $\delta_{\lambda\mu}$ is equal to 1 if $\lambda = \mu$ and is 0 otherwise. Moreover, given symmetric functions $f, g \in \Lambda$, the Kronecker product and scalar product are bilinear

$$\langle f * g, p_\gamma \rangle = \langle f, p_\gamma \rangle \langle g, p_\gamma \rangle$$

Note that this equation can be easily verified that it holds on the p bases. This is, take $f = \frac{p_\lambda}{z_\lambda}$ and $g = \frac{p_\mu}{z_\mu}$ and we have

$$\left\langle \frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu}, p_\gamma \right\rangle = \left\langle \frac{p_\lambda}{z_\lambda}, p_\gamma \right\rangle \left\langle \frac{p_\mu}{z_\mu}, p_\gamma \right\rangle = \delta_{\lambda\gamma}\delta_{\mu\gamma}.$$

Now, we will extend this definition to Kronecker product in two alphabets that can be used in type B.

Definition 3.35. Define the Kronecker product in two alphabets $*_B : (\Lambda \otimes \Lambda) \times (\Lambda \otimes \Lambda) \longrightarrow \Lambda \otimes \Lambda$ given by

$$\left(\frac{p_\gamma}{z_\gamma} \otimes \frac{p_\nu}{z_\nu}\right) *_B \left(\frac{p_\alpha}{z_\alpha} \otimes \frac{p_\beta}{z_\beta}\right) = \delta_{\gamma\alpha}\delta_{\nu\beta}\left(\frac{p_\gamma}{z_\gamma} \otimes \frac{p_\nu}{z_\nu}\right)$$

where the $\delta_{\lambda\mu}$ is Kronecker delta function.

Lemma 3.36. *For functions f and g in $\Lambda \otimes \Lambda$ the Kronecker product $*_B$ of the B_n-Frobenius map satisfies*

$$\Phi_{B_n}(f) *_B \Phi_{B_n}(g) = \Phi_{B_n}(f \cdot g)$$

Proof. The Kronecker product $*_B$ of the B_n-Frobenius maps of functions f and g in $\Lambda \otimes \Lambda$ can be written as

$$\begin{aligned}
\Phi_{B_n}(f) *_B \Phi_{B_n}(g) &= \left(\sum_{(\gamma,\nu)\vdash n} f[\Xi_\gamma, \Xi_\nu]\frac{p_\gamma[X]}{z_\gamma[X]}\frac{p_\nu[Y]}{z_\nu[Y]}\right) *_B \left(\sum_{(\alpha,\beta)\vdash n} g[\Xi_\alpha, \Xi_\beta]\frac{p_\alpha[X]}{z_\alpha[X]}\frac{p_\beta[Y]}{z_\beta[Y]}\right)\\
&= \sum_{(\gamma,\nu)\vdash n} f[\Xi_\gamma, \Xi_\nu]g[\Xi_\gamma, \Xi_\nu]\frac{p_\gamma[X]}{z_\gamma[X]}\frac{p_\nu[Y]}{z_\nu[Y]}\\
&= \sum_{(\gamma,\nu)\vdash n} (f \cdot g)\,[\Xi_\gamma, \Xi_\nu]\frac{p_\gamma[X]}{z_\gamma[X]}\frac{p_\nu[Y]}{z_\nu[Y]}\\
&= \Phi_{B_n}(f \cdot g)
\end{aligned}$$

Note that the non-zero Kronecker products which survived from the first equality is when $\delta_{\gamma\alpha}\delta_{\nu\beta}$ are non-zero i.e. $\gamma = \alpha$ and $\nu = \beta$ simultaneously. Moreover, the product $f \cdot g$ is a point-wise product of two functions, so the proof is complete. □

3.2.2 Scalar Product on B_n-Irreducible Character Basis

Scalar Product in Type A. In Definition 1.61 we have defined Hall inner product on Λ that is the bilinear form $\langle \cdot, \cdot \rangle$ which makes functions s_λ an orthonormal basis, that is $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Furthermore by Definition 1.118 the Frobenius characteristic map is $\mathscr{F}_{S_n}(\chi^\lambda) = s_\lambda$ and is expanded as

$$\mathscr{F}_{S_n}(\chi^\lambda) = \sum_{\mu \vdash n} \chi^\lambda_\mu \frac{p_\mu}{z_\mu}$$

where χ^λ_μ is the value of χ^λ on conjugacy class of cycle-type μ i.e. K_μ. Moreover, $\mathscr{F}_{S_n}$ is a linear map that preserves inner products and

$$\begin{aligned}\left\langle \chi^\lambda, \chi^\mu \right\rangle_{S_n} &= \left\langle \mathscr{F}_{S_n}(\chi^\lambda), \mathscr{F}_{S_n}(\chi^\mu) \right\rangle_{\Lambda^n} \\ &= \langle s_\lambda, s_\mu \rangle_{\Lambda^n} \\ &= \delta_{\lambda\mu}\end{aligned}$$

Equivalently, given permutation $\sigma \in S_n$, the Frobenius characteristic map is given by

$$\mathscr{F}_{S_n}(\chi^\lambda) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma) p_{cyc(\sigma)}$$

where cycle-type of σ is denoted by $cyc(\sigma)$. In section 2.7 of [OZ3] a scalar product is defined on symmetric functions when n is sufficiently large by

$$\langle f, g \rangle_@ = \sum_{\nu \vdash n} \frac{1}{z_\nu} f[\Xi_\nu] g[\Xi_\nu] = \frac{1}{n!} \sum_{\sigma \in S_n} f[\Xi_{cyc(\sigma)}] g[\Xi_{cyc(\sigma)}]$$

Furthermore, in Proposition 11 of [OZ3] the scalar product on the S_n-irreducible character bases, where the @ subscript is to differentiate this with usual scalar product in $\langle s_\lambda, s_\mu \rangle_{\Lambda^n} = \delta_{\lambda\mu}$ is given by

$$\langle \tilde{s}_\lambda, \tilde{s}_\mu \rangle_@ = \delta_{\lambda\mu}.$$

Scalar Product in Type B. We would like to translate a transformation of the preceding idea (in type A) to type B. To do this, at first, recall Definition 1.124 that the B_n-Frobenius characteristic map is defined on the set of class functions of group B_n over the complex numbers.

$$\begin{aligned}\mathscr{F}_{B_n} : \quad R(B_n) &\longrightarrow \Lambda_{B_n}[X, Y] \\ \mathbf{1}^{B_n}_{(\gamma,\nu)} &\longmapsto \frac{1}{z_\gamma} p_\gamma[X] \frac{1}{z_\nu} p_\nu[Y]\end{aligned}$$

where $z_\lambda := 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!$ for $\lambda = (1^{m_1} 2^{m_2} \ldots n^{m_n})$. We are adopting the definition given in references [Beck, BrBg, Orellana, Stembridge].

The conjugacy class of B_n are indexed by the pair of partitions $(\gamma, \nu) \vdash n$ is denoted by $K^{B_n}_{(\gamma,\nu)}$. Let the indicator function basis of class (γ, ν) to be denoted by $\mathbf{1}^{B_n}_{(\gamma,\nu)}$ and

$$\mathbf{1}^{B_n}_{(\gamma,\nu)}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in K^{B_n}_{(\gamma,\nu)} \\ 0 & \text{otherwise} \end{cases}$$

Moreover, by Lemma 1.83 we have shown that the number of elements in the conjugacy class of cycle-type (γ, ν) in group B_n is

$$\left| K^{B_n}_{(\gamma,\nu)} \right| = \frac{2^n n!}{2^{l(\gamma)} 2^{l(\nu)} z_\gamma z_\nu}.$$

Recall that in Lemma 1.126 we have presented B_n-character inner product that is given in [Stembridge] as the following.

Lemma 3.37. *The B_n-character inner product is given by*

$$\left\langle \mathbf{1}^{B_n}_{(\gamma,\nu)}, \mathbf{1}^{B_n}_{(\alpha,\beta)} \right\rangle_{B_n} = \frac{1}{2^{l(\gamma)} z_\gamma} \frac{1}{2^{l(\nu)} z_\nu} \delta_{\gamma\alpha}\delta_{\nu\beta}.$$

Lemma 3.38. *The following property holds for B_n-Frobenius characteristic map.*

$$\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) = \frac{1}{2^n n!} \sum_{\sigma \in B_n} \chi^{(\lambda,\mu)}(\sigma) 2^{\#cyc(\sigma)} p_{even-cyc(\sigma)}[X] p_{odd-cyc(\sigma)}[Y].$$

Proof. By Lemma 1.83 the number of elements in conjugacy class in group B_n is

$$\left| K^{B_n}_{(\gamma,\nu)} \right| = \frac{2^n n!}{2^{l(\gamma)} 2^{l(\nu)} z_\gamma z_\nu}$$

then we have

$$\begin{aligned}
\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) &:= \sum_{(\gamma,\nu)\vdash n} \chi^{(\lambda,\mu)}(\gamma,\nu) \frac{p_\gamma[X]}{z_\gamma} \frac{p_\nu[Y]}{z_\nu} \\
&= \sum_{(\gamma,\nu)\vdash n} \chi^{(\lambda,\mu)}(\gamma,\nu) \frac{2^{l(\gamma)} 2^{l(\nu)} \left| K_{(\gamma,\nu)} \right|}{2^n n!} p_\gamma[X] p_\nu[Y] \\
&= \frac{1}{2^n n!} \sum_{(\gamma,\nu)\vdash n} \chi^{(\lambda,\mu)}(\gamma,\nu) 2^{l(\gamma)+l(\nu)} \left| K_{(\gamma,\nu)} \right| p_\gamma[X] p_\nu[Y] \\
&= \frac{1}{2^n n!} \sum_{\sigma \in B_n} \chi^{(\lambda,\mu)}(\sigma) 2^{\#cyc(\sigma)} p_{even-cyc(\sigma)}[X] p_{odd-cyc(\sigma)}[Y]
\end{aligned}$$

where $\#cyc(\sigma)$ is the total number of cycles in σ and the indices in $p_{even-cyc(\sigma)}$ and $p_{odd-cyc(\sigma)}$ indicate the partition representing the lengths of the cycles with an even number of negative signs and odd number of negative signs in σ respectively. □

Now, we will provide a scalar product that is defined on $\Lambda \otimes \Lambda$ that can be used on the B_n-irreducible character basis.

Definition 3.39. For n sufficiently large, for example $n \geq 2\,(deg(f), deg(g))$ and $f, g \in \Lambda \otimes \Lambda$ we have

$$\langle f, g \rangle_{@_B} = \sum_{(\theta,\tau)\vdash n} \frac{1}{2^{l(\theta)} z_\theta} \frac{1}{2^{l(\tau)} z_\tau} f[\Xi_\theta, \Xi_\tau] g[\Xi_\theta, \Xi_\tau] \quad .$$

Remark 3.40. In Definition 3.39 the correct formula should have been

$$\langle f, g\rangle_{@_B} = \sum_{(\theta,\tau)\vdash n} \frac{1}{2^{l(\theta)}z_\theta}\frac{1}{2^{l(\tau)}z_\tau} f[\Xi_\theta, \Xi_\tau] g[\overline{\Xi}_\theta, \overline{\Xi}_\tau]$$

but generally in type B (and in type A) we have $\overline{\Xi}_\theta = \Xi_\theta$ and $\overline{\Xi}_\tau = \Xi_\tau$ that this is due to the fact that complex roots of unity occur in conjugate pairs.

Example 3.41. Evaluate the following inner products.

1. $\left\langle \tilde{s}^B_{\bullet,1}, \tilde{s}^B_{1,\bullet}\right\rangle_{@_B}$
2. $\left\langle \tilde{s}^B_{\bullet,1}, \tilde{s}^B_{\bullet,1}\right\rangle_{@_B}$

Solution to 1: Let $n = 2$ and we have

$$\begin{aligned}
\left\langle \tilde{s}^B_{\bullet,1}, \tilde{s}^B_{1,\bullet}\right\rangle_{@_B} &= \sum_{(\theta,\tau)\vdash 2} \frac{1}{2^{l(\theta)}z_\theta}\frac{1}{2^{l(\tau)}z_\tau}\tilde{s}^B_{\bullet,1}[\Xi_\theta, \Xi_\tau]\tilde{s}^B_{1,\bullet}[\Xi_\theta, \Xi_\tau] \\
&= \frac{1}{2^1 z_2}\frac{1}{2^0 z_0}\tilde{s}^B_{\bullet,1}[\Xi_2, \Xi_\bullet]\tilde{s}^B_{1,\bullet}[\Xi_2, \Xi_\bullet] + \frac{1}{2^2 z_{11}}\frac{1}{2^0 z_0}\tilde{s}^B_{\bullet,1}[\Xi_{11}, \Xi_\bullet]\tilde{s}^B_{1,\bullet}[\Xi_{11}, \Xi_\bullet] \\
&\quad + \frac{1}{2^1 z_1}\frac{1}{2^1 z_1}\tilde{s}^B_{\bullet,1}[\Xi_1, \Xi_1]\tilde{s}^B_{1,\bullet}[\Xi_1, \Xi_1] + \frac{1}{2^0 z_0}\frac{1}{2^2 z_{11}}\tilde{s}^B_{\bullet,1}[\Xi_\bullet, \Xi_{11}]\tilde{s}^B_{1,\bullet}[\Xi_\bullet, \Xi_{11}] \\
&\quad + \frac{1}{2^0 z_0}\frac{1}{2^1 z_2}\tilde{s}^B_{\bullet,1}[\Xi_\bullet, \Xi_2]\tilde{s}^B_{1,\bullet}[\Xi_\bullet, \Xi_2] \\
&= \frac{1}{4}0\cdot(-1) + \frac{1}{8}2\cdot(1) + \frac{1}{4}0\cdot 1 + \frac{1}{8}(-2)\cdot 1 + \frac{1}{4}0\cdot(-1) \\
&= 0
\end{aligned}$$

Solution to 2: Let $n = 2$ and we have

$$\begin{aligned}
\left\langle \tilde{s}^B_{\bullet,1}, \tilde{s}^B_{\bullet,1}\right\rangle_{@_B} &= \sum_{(\theta,\tau)\vdash 2} \frac{1}{2^{l(\theta)}z_\theta}\frac{1}{2^{l(\tau)}z_\tau}\tilde{s}^B_{\bullet,1}[\Xi_\theta, \Xi_\tau]\tilde{s}^B_{\bullet,1}[\Xi_\theta, \Xi_\tau] \\
&= \frac{1}{2^1 z_2}\frac{1}{2^0 z_0}\tilde{s}^B_{\bullet,1}[\Xi_2, \Xi_\bullet]\tilde{s}^B_{\bullet,1}[\Xi_2, \Xi_\bullet] + \frac{1}{2^2 z_{11}}\frac{1}{2^0 z_0}\tilde{s}^B_{\bullet,1}[\Xi_{11}, \Xi_\bullet]\tilde{s}^B_{\bullet,1}[\Xi_{11}, \Xi_\bullet] \\
&\quad + \frac{1}{2^1 z_1}\frac{1}{2^1 z_1}\tilde{s}^B_{\bullet,1}[\Xi_1, \Xi_1]\tilde{s}^B_{\bullet,1}[\Xi_1, \Xi_1] + \frac{1}{2^0 z_0}\frac{1}{2^2 z_{11}}\tilde{s}^B_{\bullet,1}[\Xi_\bullet, \Xi_{11}]\tilde{s}^B_{\bullet,1}[\Xi_\bullet, \Xi_{11}] \\
&\quad + \frac{1}{2^0 z_0}\frac{1}{2^1 z_2}\tilde{s}^B_{\bullet,1}[\Xi_\bullet, \Xi_2]\tilde{s}^B_{\bullet,1}[\Xi_\bullet, \Xi_2] \\
&= \frac{1}{4}0\cdot 0 + \frac{1}{8}2\cdot(2) + \frac{1}{4}0\cdot 0 + \frac{1}{8}(-2)\cdot(-2) + \frac{1}{4}0\cdot 0 \\
&= 1
\end{aligned}$$

We will now give the B_n analogous of Proposition 11 in [OZ3] i.e. $\langle \tilde{s}_\lambda, \tilde{s}_\mu \rangle_{@} = \delta_{\lambda\mu}$, as the following.

Proposition 3.42. *For all pairs of partitions (λ, μ) and (α, β) we have*

$$\left\langle \tilde{s}^B_{(\lambda,\mu)}, \tilde{s}^B_{(\alpha,\beta)} \right\rangle_{@_B} = \delta_{\lambda\alpha}\delta_{\mu\beta}.$$

Proof. Let $n \geq |\lambda| + |\mu| + \lambda_1$ and $n \geq |\alpha| + |\beta| + \alpha_1$ so that $((n - |\lambda| - |\mu|, \lambda), \mu)$ and $((n - v\,|\alpha| - |\beta|, \alpha), \beta)$ are both partitions. We evaluate

$$\begin{aligned}
\left\langle \tilde{s}^B_{(\lambda,\mu)}, \tilde{s}^B_{(\alpha,\beta)} \right\rangle_{@_B} &= \sum_{(\theta,\tau)\vdash n} \frac{1}{2^{l(\theta)} z_\theta} \tilde{s}^B_{(\lambda,\mu)}[\Xi_\theta, \Xi_\tau] \cdot \frac{1}{2^{l(\tau)} z_\tau} \tilde{s}^B_{(\alpha,\beta)}[\Xi_\theta, \Xi_\tau] \\
&= \sum_{(\theta,\tau)\vdash n} \frac{1}{2^{l(\theta)} z_\theta} \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\theta,\tau) \cdot \frac{1}{2^{l(\tau)} z_\tau} \chi^{((n-|\alpha|-|\beta|,\alpha),\beta)}(\theta,\tau) \\
&= \sum_{(\theta,\tau)\vdash n} \frac{1}{2^{l(\theta)} z_\theta} \frac{1}{2^{l(\tau)} z_\tau} \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\theta,\tau) \cdot \chi^{((n-|\alpha|-|\beta|,\alpha),\beta)}(\theta,\tau) \\
&= \left\langle \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}, \chi^{((n-|\alpha|-|\beta|,\alpha),\beta)} \right\rangle_{B_n} \\
&= \delta_{\lambda\alpha}\delta_{\mu\beta}
\end{aligned}$$

Note that the second equality uses Lemma 3.32. The proof is complete. □

3.2.3 Stability of Kronecker coefficients in Type B

Stability of Kronecker Coefficients in Type A. In the late 1930's Murnaghan in exhibited the stability property of the Kronecker coefficients in product of two irreducible representations of the symmetric group in [Murnaghan2, Murnaghan3]. This means that the coefficients in the expansion of the Kronecker product of two Schur functions of degree n, when n is sufficiently large do not depend on the first part of the indexing partition of Schur function and they only depend on the remaining parts of the partitions.

Define the coefficient

$$\overline{g}^\gamma_{\lambda\mu} := g^{(n-|\gamma|,\gamma)}_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}$$

to denote the stable limit of these coefficients when n is sufficiently large. By work of Murnaghan in [Murnaghan2, Murnaghan3] we can deduce that there exists coefficient $\overline{g}^\gamma_{\lambda\mu}$ with properties

$$s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)} = \sum_\gamma \overline{g}^\gamma_{\lambda\mu} s_{(n-|\gamma|,\gamma)}$$

for all $n \geq 0$. Note that the coefficients $\overline{g}_{\lambda\mu}^{\gamma}$ are referred to as 'reduced' or 'stable' Kronecker coefficients and these coefficients in the expression stabilize and are independent of n when n is sufficiently large. In [BOR] this result is presented in the following theorem.

Theorem 3.43. *(Murnaghan's Theorem in [BOR]) [Murnaghan2, Murnaghan3] There exists a family of non-negative integers $\overline{g}_{\alpha\beta}^{\gamma}$ indexed by triples of partitions (α, β, γ) such that, for α and β fixed, only finitely many terms $\overline{g}_{\alpha\beta}^{\gamma}$ are nonzero, and for all $n \geq 0$,*

$$s_{(n-|\alpha|,\alpha)} * s_{(n-|\beta|,\beta)} = \sum_{\gamma} \overline{g}_{\alpha\beta}^{\gamma} s_{(n-|\gamma|,\gamma)}$$

Moreover, the coefficient $\overline{g}_{\alpha\beta}^{\gamma}$ vanishes unless the weights of the three partitions fulfill the inequalities:

$$|\alpha| \leq |\beta| + |\gamma|\,,\ |\beta| \leq |\alpha| + |\gamma| \ \text{and}\ |\gamma| \leq |\alpha| + |\beta|\,.$$

Example 3.44. Given $\lambda = (2)$ and $\mu = (1,1)$ we will evaluate the Kronecker product of $s_{(n-2,2)} * s_{(n-2,1,1)}$. It can be shown the Kronecker coefficients stabilize for $n \geq 7$. The following evaluations are performed in SageMath and is summarized in the table below. As an example in SageMath for $n = 8$ we have

`s[5,2].itensor(s[5,1,1])` which SageMath generates the answer:

`s[3, 2, 1, 1] + s[3, 3, 1] + s[4, 1, 1, 1] + 2*s[4, 2, 1] + s[4, 3] + 2*s[5, 1, 1] + s[5, 2] + s[6, 1]`

n	$s_{(n-2,2)} * s_{(n-2,1,1)} = \sum_{\gamma} \overline{g}_{(2)(1,1)}^{\gamma} s_{(n-|\gamma|,\gamma)}$
$n = 4$	$s_{22} * s_{211} = s_{211} + s_{31}$
$n = 5$	$s_{32} * s_{311} = s_{2111} + s_{221} + 2s_{311} + s_{32} + s_{41}$
$n = 6$	$s_{42} * s_{411} = s_{2211} + s_{3111} + 2s_{321} + s_{33} + 2s_{411} + s_{42} + s_{51}$
$n = 7$	$s_{52} * s_{511} = s_{3211} + s_{331} + s_{4111} + 2s_{421} + s_{43} + 2s_{511} + s_{52} + s_{61}$
$n = 8$	$s_{62} * s_{611} = s_{4211} + s_{431} + s_{5111} + 2s_{521} + s_{53} + 2s_{611} + s_{62} + s_{71}$
$n = 9$	$s_{72} * s_{711} = s_{5211} + s_{531} + s_{6111} + 2s_{621} + s_{63} + 2s_{711} + s_{72} + s_{81}$
$n = 10$	$s_{82} * s_{811} = s_{6211} + s_{631} + s_{7111} + 2s_{721} + s_{73} + 2s_{811} + s_{82} + s_{91}$

Table 3.10: Kronecker product of $s_{(n-2,2)} * s_{(n-2,1,1)}$

Extending this discussion to S_n-irredicible character basis, Theorem 7 in [OZ1] presents an interesting result, which is, when computing the regular product of any two functions $\tilde{s}_\lambda$ and $\tilde{s}_\mu$ the structure coefficients in the expansion is the same as the stable Kronecker coefficients. This means, given the coefficients $\overline{g}_{\lambda\mu}^{\gamma}$ that stabilize in the expression

$$s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)} = \sum_{|\gamma| \leq |\lambda|+|\mu|} \overline{g}_{\lambda\mu}^{\gamma} s_{(n-|\gamma|,\gamma)}$$

then coefficients $\overline{g}^{\nu}_{\lambda\mu}$ corresponds to the coefficients in product of the irreducible character basis i.e.

$$\tilde{s}_\lambda \cdot \tilde{s}_\mu = \sum_{|\gamma|\leq|\lambda|+|\mu|} \overline{g}^{\gamma}_{\lambda\mu}\tilde{s}_\gamma$$

Theorem 3.45. *(Theorem 7 [OZ1]) For partitions λ and μ,*

$$\tilde{s}_\lambda \cdot \tilde{s}_\mu = \sum_{|\gamma|\leq|\lambda|+|\mu|} \overline{g}^{\gamma}_{\lambda\mu}\tilde{s}_\gamma$$

Proof. We will provide the proof by evaluating the product of function $\tilde{s}_\lambda$ and $\tilde{s}_\mu$ at the eigenvalues of a permutation matrix. Also, recall that for symmetric functions $f, g \in \Lambda$, the Kronecker product and scalar product are bilinear

$$\langle f * g, p_\theta \rangle = \langle f, p_\theta \rangle \langle g, p_\theta \rangle$$

We have

$$\begin{aligned}
\tilde{s}_\lambda[\Xi_\theta]\tilde{s}_\mu[\Xi_\theta] &= \left\langle s_{(|\theta|-|\lambda|,\lambda)}, p_\theta \right\rangle \left\langle s_{(|\theta|-|\mu|,\mu)}, p_\theta \right\rangle \\
&= \left\langle s_{(|\theta|-|\lambda|,\lambda)} * s_{(|\theta|-|\mu|,\mu)}, p_\theta \right\rangle \\
&= \left\langle \sum_{|\gamma|\leq|\lambda|+|\mu|} \overline{g}^{\gamma}_{\lambda\mu} s_{(|\theta|-|\gamma|,\gamma)}, p_\theta \right\rangle \\
&= \sum_{|\gamma|\leq|\lambda|+|\mu|} \overline{g}^{\gamma}_{\lambda\mu} \left\langle s_{(|\theta|-|\gamma|,\gamma)}, p_\theta \right\rangle \\
&= \sum_{|\gamma|\leq|\lambda|+|\mu|} \overline{g}^{\gamma}_{\lambda\mu} \tilde{s}_\gamma[\Xi_\theta].
\end{aligned}$$

This expression is true for all θ of sufficiently large size, hence by Proposition 1.209 (Proposition 6 [OZ1]) the structure coefficients for the functions $\tilde{s}_\lambda$ are the same as the reduced Kronecker coefficients. □

Example 3.46. Evaluate $\tilde{s}_2 \cdot \tilde{s}_{11}$.

This example shows that the structure coefficients in product $\tilde{s}_2 \cdot \tilde{s}_{11}$ is the same as the stable Kronecker coefficients in product $s_{(n-2,2)} * s_{(n-2,1,1)}$ in Example 3.44 (for $n \geq 7$). Using SageMath and taking the regular multiplication, we have

`st[2]*st[1,1]` and the answer is

```
st[1] + 2*st[1, 1] + st[1, 1, 1] + st[2] + 2*st[2, 1] + st[2, 1, 1] + st[3] + st[3, 1]
```

Follow the same order as in Example 3.44 we have

$$\tilde{s}_2 \cdot \tilde{s}_{11} = \tilde{s}_{211} + \tilde{s}_{31} + \tilde{s}_{111} + 2\tilde{s}_{21} + \tilde{s}_3 + 2\tilde{s}_{11} + \tilde{s}_2 + \tilde{s}_1.$$

Stability of Kronecker coefficients in Type B. In this part, we will briefly go over the stability of kronecker coefficients in type B, i.e., the stability property of the Kronecker coefficients in product of two irreducible representations of the hyperoctahedral group. There are two references that we chose to give each a short statement here.

1. *Stable Grothendieck Rings of Wreath Product Categories, by Christopher Ryba [Ryba].*

Let $\mathcal{C} = \mathcal{R} - mod$ be the category of finite-dimensional modules over a fixed Hopf algebra $\mathcal{R}$ over k, an algebraically closed field with characteristic zero. Moreover a wreath product categories $\mathcal{W}_n(\mathcal{C}) = (\mathcal{R} wr S_n) - mod$ can be constructed that it's Grothendieck groups will have the structure of a ring. The Hopf algebra structure allows a multiplication on the Grothendieck groups (or the Grothendieck rings).

For a fixed generating sets of the Grothendieck rings, namely basic hooks, one can show the stability of structure constants in the Grothendieck rings by allocating the simple objects in $\mathcal{W}_n(\mathcal{C})$ and to define a limiting Grothendieck ring $\mathcal{G}_\infty(\mathcal{C})$ namely $\mathcal{C} = \mathcal{R} - mod$. This ring is indeed is the Grothendieck ring of wreath product Deligne category $S_t(\mathcal{C})$ which is given in [Mori]. In this paper, polynomials in basic hooks is given as an expression for the basis generated by simple objects in the wreath product categories $\mathcal{W}_n(\mathcal{C})$.

There exists a basis $X_{\overrightarrow{\lambda}}$ that is coming from the image of objects Deligne category $S_t(\mathcal{C})$ which in Theorem 9.10 in [Ryba] a generating function is given that is related to $X_{\overrightarrow{\lambda}}$ as the basis of $\mathcal{G}_\infty(\mathcal{C})$ to the collection of elements in $T_n(U) \in \mathcal{G}_\infty(\mathcal{C})$ that are indexed by positive integers and simple $\mathcal{R}$-modules.

It can also be said that this theorem gives a generating function for the family of functions, S_n-irreduicible character basis $\tilde{s}_\lambda$ in [OZ1] which it is reinterpreted in Theorem 10.4 in [Ryba]. It is concluded that $\tilde{s}_\lambda$ are fundamental objects in the asymptotic representation theory of symmetric group S_n. After a comparison of results from these generating functions, the combinatorial description in [OZ1] and character polynomials in Example 14 of Section 7 in [Macdonald] it is concluded that the functions $\tilde{s}_\lambda$ are indeed the same symmetric functions.

2. *FI_W –modules and stability criteria for representations of classical Weyl groups, by Jennifer C. H. Wilson [Wilson].*

The representation stability for a sequence of representations V_n for families of groups of G_n that include S_n and B_n was developed in [CF]. In [Wilson] the connection of representation stability to the theory of finitely generated modules was made to present Murnaghan's stability theorem for B_n (and D_n).

At first, we will explain a few notation that we need which is used in [CF], [CEF] and [Wilson].

Type A. In [CF] it is stated that, to compare representations of different values in n in symmetric groups S_n, we can identify those irreducible representations associated to partitions of n that differ only in their largest parts, i.e., two irreducible representations are identical if their Young diagram differs by boxes in the bottom row (French notation). Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$ where $\lambda \vdash |\lambda|$, denote $V(\lambda)_n$ to be the irreducible S_n-representation corresponding to the padded partition of λ, i.e. $\lambda[n] \coloneqq \big((n-|\lambda|), \lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}\big)$. Define $V(\lambda)_n$ by

$$V(\lambda)_n \coloneqq \begin{cases} V_{\lambda[n]} & \text{if } (n-|\lambda|) \geq \lambda_1 \\ 0 & \text{otherwise} \end{cases}$$

Type B. In case of hyperoctahedral group B_n we can identify any two pairs of partitions if they differ only by the largest part of the left partition. This is, for a pair of partition $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^+ \vdash |\lambda^+|$ and $\lambda^- \vdash |\lambda^-|$, define $\lambda[n] \coloneqq (\lambda^+[n-|\lambda^-|], \lambda^-)$ to be the padded pair of partition of $\lambda = (\lambda^+, \lambda^-)$ and denote $V(\lambda)_n$ (or $V(\lambda^+, \lambda^-)$) to be the irreducible B_n-representation defined by

$$V(\lambda)_n \coloneqq \begin{cases} V_{\lambda[n]} & \text{if } (n-|\lambda^+|-|\lambda^-|) \geq \lambda_1^+ \\ 0 & \text{otherwise} \end{cases}$$

It is stated in [CF] and [Wilson] that, in order for a sequence V of rational G_n-representations to be called representation stable, the coefficients of the irreducible constituents $V(\lambda)_n$ of V_n must eventually become constant when n is sufficiently large. Theorem 5.3 in [Wilson] provides an analogue of Murnaghan's stability theorem in type B.

Theorem 3.47. *(Theorem 5.3 in [Wilson]), (Murnaghan's stability theorem for B_n). For any pair of double partitions $\lambda = (\lambda^+, \lambda^-)$ and $\mu = (\mu^+, \mu^-)$ there exist nonnegative integers $g^{\nu}_{\lambda,\mu}$, independent of n , such that for all n sufficiently large:*

$$V(\lambda)_n \otimes V(\mu)_n = \bigoplus_{\nu} g^{\nu}_{\lambda,\mu} V(\nu)_n.$$

The coefficients $g^{\nu}_{\lambda,\mu}$ are nonzero for only finitely many double partitions ν.

Interpretation of Theorem 5.3 in [Wilson] in our notation

Remark 3.48. We will rewrite Theorem 3.47 (Theorem 5.3 in [Wilson]) in our notation. Let

$$S^B_{(\lambda,\mu)}[X,Y] \coloneqq s_\lambda[X+Y] s_\mu[X-Y]$$

therefore

$$S^B_{((n-|\lambda|-|\mu|,\lambda),\mu)}[X,Y] \coloneqq s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y] s_\mu[X-Y].$$

For n sufficiently large, there exists the reduced Kronecker coefficients $\overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)}$ that stabilizes in the expansion of the Kronecker product of two irreducible representations of the hyperoctahedral group, that is

$$S^B_{((n-|\lambda|-|\mu|,\lambda),\mu)} *_B S^B_{((n-|\alpha|-|\beta|,\alpha),\beta)} = \sum_{|(\theta,\tau)|} \overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)} S^B_{((n-|\tau|-|\theta|,\theta),\tau)}$$

where the sum is over $|(\theta,\tau)| \leq |(\lambda,\mu)| + |(\alpha,\beta)|$. One important note to mention is Theorem 3.47 (Theorem 5.3 in [Wilson]) the reduced Kronecker coefficient is denoted by $g^{\nu}_{\lambda,\mu}$ (without the barred sign).

Example 3.49. Given $(\lambda,\mu) = (\bullet,(1))$ and $(\alpha,\beta) = (\bullet,(1))$, it can be shown that the Kronecker coefficients stabilize for $n \geq 2$ when we evaluate the Kronecker product of $S^B_{((n-1),(1))} *_B S^B_{((n-1),(1))}$. We will use short-hand/compact notation for indices in $S^B_{((n-|\lambda|-|\mu|,\lambda),\mu)}$. At first, for $n = 1$ we have

$$\begin{aligned} S^B_{\bullet,1}[X,Y] *_B S^B_{\bullet,1}[X,Y] &= (s_\bullet[X+Y]s_1[X-Y]) *_B (s_\bullet[X+Y]s_1[X-Y]) \\ &= (-s_\bullet[X]s_1[Y] + s_1[X]s_\bullet[Y]) *_B (-s_\bullet[X]s_1[Y] + s_1[X]s_\bullet[Y]) \end{aligned}$$

We may drop the X and Y variables and use tensor notation and we have

$$\begin{aligned} S^B_{\bullet,1} *_B S^B_{\bullet,1} &= (-s_\bullet \otimes s_1 + s_1 \otimes s_\bullet) *_B (-s_\bullet \otimes s_1 + s_1 \otimes s_\bullet) \\ &= (s_\bullet * s_\bullet) \otimes (s_1 * s_1) + (s_1 * s_1) \otimes (s_\bullet * s_\bullet) \\ &= s_\bullet \otimes s_1 + s_1 \otimes s_\bullet \\ &= S^B_{1,\bullet} \end{aligned}$$

After calculations and using SageMath, the Kronecker products are shown in the following table.

n	$S^B_{((n-1),(1))} *_B S^B_{((n-1),(1))} = \sum_{\|(\theta,\tau)\|} \overline{g}^{(\theta,\tau)}_{(\bullet,(1))(\bullet,(1))} S_{((n-\|\tau\|-\|\theta\|,\theta),\tau)}$
$n=1$	$S^B_{\bullet,1} *_B S^B_{\bullet,1} = S^B_{1,\bullet}$
$n=2$	$S^B_{1,1} *_B S^B_{1,1} = S^B_{11,\bullet} + S^B_{2,\bullet} + S^B_{\bullet,11} + S^B_{\bullet,2}$
$n=3$	$S^B_{2,1} *_B S^B_{2,1} = S^B_{21,\bullet} + S^B_{3,\bullet} + S^B_{1,11} + S^B_{1,2}$
$n=4$	$S^B_{3,1} *_B S^B_{3,1} = S^B_{31,\bullet} + S^B_{4,\bullet} + S^B_{2,11} + S^B_{2,2}$

Table 3.11: Kronecker product of $S^B_{((n-1),(1))} *_B S^B_{((n-1),(1))}$

Remark 3.50. There are helpful SageMath syntaxes that we used in the computations of the last examples.

- ◇ For example to evaluate the product $s_1[X+Y]s_1[X-Y]$ we have used

`s[1].coproduct()*XmY(s[1])` that SageMath generates the answer:

```
-s[] # s[1, 1] - s[] # s[2] + s[1, 1] # s[] + s[2] # s[]
```

- ◇ In evaluating Kronecker products of the last example we have used `.itensor` syntax in SageMath.

Product (Non-Zeros)	Product (Non-Zeros) in SageMath	Answers in SageMath
$s_\bullet * s_\bullet$	`s[[]].itensor(s[[]])`	`s[]`
$s_1 * s_1$	`s[1].itensor(s[1])`	`s[1]`
$s_2 * s_2$	`s[2].itensor(s[2])`	`s[2]`
$s_{11} * s_2$	`s[1,1].itensor(s[2])`	`s[1,1]`
$s_{21} * s_{21}$	`s[2,1].itensor(s[2,1])`	`s[1,1,1]+s[2,1]+s[3]`
$s_{111} * s_{111}$	`s[1,1,1].itensor(s[1,1,1])`	`s[3]`
$s_{111} * s_{21}$	`s[1,1,1].itensor(s[2,1])`	`s[2,1]`
$s_{111} * s_3$	`s[1,1,1].itensor(s[3])`	`s[3]`
$s_{21} * s_3$	`s[2,1].itensor(s[3])`	`s[2,1]`
$s_3 * s_3$	`s[3].itensor(s[3])`	`s[3]`
$s_{31} * s_{31}$	`s[3,1].itensor(s[3,1])`	`s[2,1,1]+s[2,2]+s[3,1]+s[4]`
$s_{31} * s_4$	`s[3,1].itensor(s[4])`	`s[3,1]`

Table 3.12: SageMath Kronecker products syntaxes

The following Lemma is an important key in proving Theorem 3.52.

Lemma 3.51. *Given $f, g \in \Lambda \otimes \Lambda$ the Kronecker product $*_B$ and scalar product are bilinear, that is*

$$\langle f[X,Y] *_B g[X,Y], p_\gamma[X]p_\nu[Y]\rangle_{\Lambda\otimes\Lambda} = \langle f[X,Y], p_\gamma[X]p_\nu[Y]\rangle_{\Lambda\otimes\Lambda} \langle g[X,Y], p_\gamma[X]p_\nu[Y]\rangle_{\Lambda\otimes\Lambda}$$

Proof. We will verify that this equality holds for the basis

$$f[X,Y] = \frac{p_\alpha[X]p_\beta[Y]}{z_\alpha z_\beta} \quad \text{and} \quad g[X,Y] = \frac{p_\theta[X]p_\tau[Y]}{z_\theta z_\tau}$$

This is

$$\begin{aligned}
&\left\langle \frac{p_\alpha[X]p_\beta[Y]}{z_\alpha z_\beta} *_B \frac{p_\theta[X]p_\tau[Y]}{z_\theta z_\tau}, p_\gamma[X]p_\nu[Y] \right\rangle_{\Lambda\otimes\Lambda} \\
&\qquad = \left\langle \delta_{\alpha\theta}\delta_{\beta\tau}\frac{p_\alpha[X]p_\beta[Y]}{z_\alpha z_\beta}, p_\gamma[X]p_\nu[Y] \right\rangle_{\Lambda\otimes\Lambda} \\
&\qquad = \delta_{\alpha\theta}\delta_{\beta\tau}\left\langle \frac{p_\alpha[X]}{z_\alpha}, p_\gamma[X] \right\rangle \left\langle \frac{p_\beta[Y]}{z_\beta}, p_\nu[Y] \right\rangle \\
&\qquad = \delta_{\alpha\theta}\delta_{\beta\tau}\delta_{\alpha\gamma}\delta_{\beta\nu} \\
&\qquad = \delta_{\alpha\gamma}\delta_{\beta\nu}\delta_{\theta\gamma}\delta_{\tau\nu} \\
&\qquad = \left\langle \frac{p_\alpha[X]p_\beta[Y]}{z_\alpha z_\beta}, p_\gamma[X]p_\nu[Y] \right\rangle_{\Lambda\otimes\Lambda} \left\langle \frac{p_\theta[X]p_\tau[Y]}{z_\theta z_\tau}, p_\gamma[X]p_\nu[Y] \right\rangle_{\Lambda\otimes\Lambda}
\end{aligned}$$

Therefore this is true for arbitrary

$$f[X,Y] = \sum_{(\alpha,\beta)} a_{(\alpha,\beta)}\frac{p_\alpha[X]p_\beta[Y]}{z_\alpha z_\beta} \quad \text{and} \quad g[X,Y] = \sum_{(\theta,\tau)} b_{(\theta,\tau)}\frac{p_\theta[X]p_\tau[Y]}{z_\theta z_\tau}.$$

The proof is complete. ∎

Kronecker coefficients in product of B_n-irreducible character bases. So far we have briefly shown the stability property of the Kronecker coefficients in Kronecker product of two irreducible representations of the hyperoctahedral group. This is, for n sufficiently large, there exists the reduced Kronecker coefficients $\overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)}$ that stabilizes in expansion as

$$S^B_{((n-|\lambda|-|\mu|,\lambda),\mu)} *_B S^B_{((n-|\alpha|-|\beta|,\alpha),\beta)} = \sum_{|(\theta,\tau)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} S^B_{((n-|\tau|-|\theta|,\theta),\tau)}$$

where the sum is over $|(\theta,\tau)| \leq |(\lambda,\mu)| + |(\alpha,\beta)|$. Now, our goal is to show that the structure coefficients in this expansion is the same as the stable Kronecker coefficients in expansion of the regular product of any two B_n-irreducible character basis, that is

$$\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)} = \sum_{|(\theta,\tau)|\leq|(\lambda,\mu)|+|(\alpha,\beta)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} \tilde{s}^B_{(\theta,\tau)}.$$

Theorem 3.52. *For pairs of partitions (λ,μ) and (α,β),*

$$\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)} = \sum_{|(\theta,\tau)|\leq|(\lambda,\mu)|+|(\alpha,\beta)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} \tilde{s}^B_{(\theta,\tau)}$$

Proof. The product of functions evaluated at roots of unity is computed here. Note that we will be dropping subscript $\Lambda \otimes \Lambda$ in $\langle \cdot,\cdot \rangle_{\Lambda\otimes\Lambda}$.

$$\begin{aligned}
&\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma,\Xi_\nu] \cdot \tilde{s}^B_{(\alpha,\beta)}[\Xi_\gamma,\Xi_\nu] \\
&\quad= \left\langle s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle \cdot \left\langle s_{(n-|\alpha|-|\beta|,\alpha)}[X+Y]s_\beta[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\quad= \left\langle S^B_{((n-|\lambda|-|\mu|,\lambda),\mu)}[X,Y], p_\gamma[X]p_\nu[Y]\right\rangle \cdot \left\langle S^B_{((n-|\alpha|-|\beta|,\alpha),\beta)}[X,Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\quad= \left\langle S^B_{((n-|\lambda|-|\mu|,\lambda),\mu)}[X,Y] *_B S^B_{((n-|\alpha|-|\beta|,\alpha),\beta)}[X,Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\quad= \left\langle \sum_{|(\theta,\tau)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} S^B_{((n-|\tau|-|\theta|,\theta),\tau)}[X,Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\quad= \sum_{|(\theta,\tau)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} \left\langle S^B_{((n-|\tau|-|\theta|,\theta),\tau)}[X,Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\quad= \sum_{|(\theta,\tau)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} \left\langle s_{(n-|\tau|-|\theta|,\theta)}[X+Y]s_\tau[X-Y], p_\gamma[X]p_\nu[Y]\right\rangle \\
&\quad= \sum_{|(\theta,\tau)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} \tilde{s}^B_{(\theta,\tau)}[\Xi_\gamma,\Xi_\nu]
\end{aligned}$$

Since this expression is true for all pairs of partitions (γ,ν) sufficiently large size, by Theorem 3.17 we conclude that

$$\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)} = \sum_{|(\theta,\tau)|\leq|(\lambda,\mu)|+|(\alpha,\beta)|} \overline{g}_{(\lambda,\mu)(\alpha,\beta)}^{(\theta,\tau)} \tilde{s}^B_{(\theta,\tau)}$$

and the proof is complete. □

Example 3.53. Perform the product $\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)}$ and expand them in $\tilde{s}^B_{(\theta,\tau)}$. We will provide examples for small partitions of $|(\lambda,\mu)| := |\lambda| + |\mu| \leq 1$ and $|(\alpha,\beta)| := |\alpha| + |\beta| \leq 1$.

- ⋄ For $\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{\bullet,\bullet}$ using expansion in $s_\lambda \otimes s_\mu$ and express it in $\tilde{s}^B_{(\theta,\tau)}$ then

$$\begin{aligned}\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{\bullet,\bullet} &= s_\bullet \otimes s_\bullet \\ &= \tilde{s}^B_{\bullet,\bullet}\end{aligned}$$

- ⋄ For $\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{1,\bullet}$ the expansion by SageMath written in sum of $\tilde{s}^B_{(\theta,\tau)}$ will be

$$\begin{aligned}\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{1,\bullet} &= s_\bullet \otimes s_\bullet - 2s_\bullet \otimes s_1 + s_\bullet \otimes s_{11} + s_\bullet \otimes s_2 - 2s_1 \otimes s_\bullet + 2s_1 \otimes s_1 + s_{11} \otimes s_\bullet + s_2 \otimes s_\bullet \\ &= \tilde{s}^B_{11,\bullet} + \tilde{s}^B_{2,\bullet} + \tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}\end{aligned}$$

- ⋄ For $\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{\bullet,1}$, similarly we have

$$\begin{aligned}\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{\bullet,1} &= s_\bullet \otimes s_1 - s_\bullet \otimes s_{11} - s_\bullet \otimes s_2 - s_1 \otimes s_\bullet + s_{11} \otimes s_\bullet + s_2 \otimes s_\bullet \\ &= \tilde{s}^B_{1,1} + \tilde{s}^B_{\bullet,1}\end{aligned}$$

- ⋄ For $\tilde{s}^B_{\bullet,1} \cdot \tilde{s}^B_{\bullet,1}$, we have

$$\begin{aligned}\tilde{s}^B_{\bullet,1} \cdot \tilde{s}^B_{\bullet,1} &= s_\bullet \otimes s_{11} + s_\bullet \otimes s_2 - 2s_1 \otimes s_1 + s_{11} \otimes s_\bullet + s_2 \otimes s_\bullet \\ &= \tilde{s}^B_{\bullet,11} + \tilde{s}^B_{\bullet,2} + \tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}\end{aligned}$$

Example 3.54. After performing the following computations in SageMath in table below, for pairs of partitions such that $|(\lambda,\mu)| := |\lambda| + |\mu| \leq 1$ and $|(\alpha,\beta)| := |\alpha| + |\beta| \leq 1$ and for $n = 4$, we can confirm that the regular products and the Kronecker Products that are expanded in their respective bases have the same reduced Kronecker coefficients that match.

$\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)} = \sum_{\lvert(\theta,\tau)\rvert} \overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)} \tilde{s}^B_{(\theta,\tau)}$	$S^B_{((n-\lvert\lambda\rvert-\lvert\mu\rvert,\lambda),\mu)} *_B S^B_{((n-\lvert\alpha\rvert-\lvert\beta\rvert,\alpha),\beta)} = \sum_{\lvert(\theta,\tau)\rvert} \overline{g}^{(\theta,\tau)}_{(\lambda,\mu)(\alpha,\beta)} S^B_{((n-\lvert\tau\rvert-\lvert\theta\rvert,\theta),\tau)}$
$\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{\bullet,\bullet} = \tilde{s}^B_{\bullet,\bullet}$	$S^B_{4,\bullet} *_B S^B_{4,\bullet} = S^B_{4,\bullet}$
$\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{1,\bullet} = \tilde{s}^B_{1,\bullet}$	$S^B_{4,\bullet} *_B S^B_{31,\bullet} = S^B_{31,\bullet}$
$\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{\bullet,1} = \tilde{s}^B_{\bullet,1}$	$S^B_{4,\bullet} *_B S^B_{3,1} = S^B_{3,1}$
$\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{1,\bullet} = \tilde{s}^B_{11,\bullet} + \tilde{s}^B_{2,\bullet} + \tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$	$S^B_{31,\bullet} *_B S^B_{31,\bullet} = S^B_{211,\bullet} + S^B_{22,\bullet} + S^B_{31,\bullet} + S^B_{4,\bullet}$
$\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{\bullet,1} = \tilde{s}^B_{1,1} + \tilde{s}^B_{\bullet,1}$	$S^B_{31,\bullet} *_B S^B_{3,1} = S^B_{21,1} + S^B_{3,1}$
$\tilde{s}^B_{\bullet,1} \cdot \tilde{s}^B_{\bullet,1} = \tilde{s}^B_{\bullet,11} + \tilde{s}^B_{\bullet,2} + \tilde{s}^B_{1,\bullet} + \tilde{s}^B_{\bullet,\bullet}$	$S^B_{3,1} *_B S^B_{3,1} = S^B_{2,11} + S^B_{2,2} + S^B_{31,\bullet} + S^B_{4,\bullet}$

Table 3.13: Regular products and the Kronecker products in Type B

Example 3.55. The following table is the expansion of the regular product of B_n-irreducible character bases $\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)}$ for $|(\lambda,\mu)| \coloneqq |\lambda| + |\mu| \leq 1$ and $|(\alpha,\beta)| \coloneqq |\alpha| + |\beta| \leq 1$.

$\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)}$	$\tilde{s}^B_{\bullet,\bullet}$	$\tilde{s}^B_{1,\bullet}$	$\tilde{s}^B_{\bullet,1}$	$\tilde{s}^B_{2,\bullet}$	$\tilde{s}^B_{11,\bullet}$	$\tilde{s}^B_{1,1}$	$\tilde{s}^B_{\bullet,2}$	$\tilde{s}^B_{\bullet,11}$
$\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{\bullet,\bullet}$	1							
$\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{1,\bullet}$		1						
$\tilde{s}^B_{\bullet,\bullet} \cdot \tilde{s}^B_{\bullet,1}$			1					
$\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{1,\bullet}$	1	1		1	1			
$\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{\bullet,1}$			1			1		
$\tilde{s}^B_{\bullet,1} \cdot \tilde{s}^B_{\bullet,1}$	1	1					1	1

Table 3.14: Regular product of B_n-irreducible character bases $\tilde{s}^B_{(\lambda,\mu)} \cdot \tilde{s}^B_{(\alpha,\beta)}$

Remark 3.56. We are plugging-in eigenvalues of permutation matrices in $\tilde{s}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$ and this is due to the way these functions were defined so they are chosen such that they have this property. To see this, we have defined generators that they expand in tensor square of S_n-induced trivial character bases as

$$\tilde{h}^B_{(m,\bullet)} \coloneqq \sum_{i=0}^{m} \tilde{h}_i \otimes \tilde{h}_{m-i}$$

and

$$\tilde{h}^B_{(\bullet,n)} \coloneqq \sum_{i=0}^{n} (-1)^{n-i} \tilde{h}_i \otimes \tilde{e}_{n-i}$$

The B_n-induced trivial character basis $\tilde{h}^B_{(\lambda,\mu)}$ generally indexed by pair of partitions (λ,μ) can be constructed using the preceding generators of type B by

$$\tilde{h}^B_{(\lambda,\mu)} \coloneqq \prod_i^{\odot_B} \tilde{h}^B_{(\lambda_i,\bullet)} \odot_B \prod_j^{\odot_B} \tilde{h}^B_{(\bullet,\mu_j)}.$$

In the Main Theorem (Theorem 2.24) we have shown that functions $\tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu]$ have the property. This implies that we can write

$$\begin{aligned} \tilde{h}^B_{(\lambda,\mu)}[\Xi_\gamma, \Xi_\nu] &\coloneqq \left(\prod_i^{\odot_B} \tilde{h}^B_{(\lambda_i,\bullet)} \odot_B \prod_j^{\odot_B} \tilde{h}^B_{(\bullet,\mu_j)} \right) [\Xi_\gamma, \Xi_\nu] \\ &\coloneqq \left(\prod_i^{\odot_B} \left(\sum_{k=0}^{\lambda_i} \tilde{h}_k \otimes \tilde{h}_{\lambda_i - k} \right) \odot_B \prod_j^{\odot_B} \left(\sum_{t=0}^{\mu_j} (-1)^{\mu_j - t} \tilde{h}_t \otimes \tilde{e}_{\mu_j - t} \right) \right) [\Xi_\gamma, \Xi_\nu] \end{aligned}$$

This shows that we are plugging-in eigenvalues of permutation matrices. Moreover, since we defined function $\tilde{s}^B_{(\lambda,\mu)}$ in terms of $\tilde{h}^B_{(\alpha,\beta)}$ by

$$\tilde{s}^B_{(\lambda,\mu)} = \sum_{\substack{(\alpha,\beta)\\|\alpha|\leq|\lambda|\\|\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K^{-1}_{\mu\beta} \tilde{h}^B_{(\alpha,\beta)}$$

the eigenvalues of permutation matrices will be used here in evaluating $\tilde{s}^B_{(\lambda,\mu)}\,[\Xi_\gamma, \Xi_\nu]$ as well.

Remark 3.57. In evaluations of the character values we are not plugging-in eigenvalues of 'signed' permutation matrices in $\tilde{s}^B_{(\lambda,\mu)}\left[\widetilde{\Xi}_\gamma, \widetilde{\Xi}_\nu\right]$. We did try this and the transition coefficients were not nice when B_n-induced trivial characters $\tilde{h}^B_{(\lambda,\mu)}\left[\widetilde{\Xi}_\gamma, \widetilde{\Xi}_\nu\right]$ defined in terms of Frobenius image of induced trivial characters i.e. $h_\lambda[X+Y]h_\mu[X+Y]$.

Remark 3.58. Given $\tilde{s}_\lambda\,[\Xi_\gamma] = \left\langle s_{(n-|\lambda|,\lambda)}, p_\gamma \right\rangle$ we add a part (row) as in $s_{(n-|\lambda|,\lambda)}$. We have extended this to type B in $\tilde{s}^B_{(\lambda,\mu)}\,[\Xi_\gamma, \Xi_\nu] = \chi^{((n-|\lambda|-|\mu|,\lambda),\mu)}(\gamma,\nu)$ as well. We can say that somehow these symmetric functions encode the character in a way that is independent of n for $|\gamma| = n$ or $|\gamma|+|\nu| = n$ respectively. There are other ways possible, but this is the most natural and the answer works out relatively nicely.

3.3 Examples and Applications of B_n-Irreducible Character Basis

3.3.1 SageMath Computations of B_n-Irreducible Character Basis by Φ_{B_n}

An Algorithm to Compute Functions $\tilde{s}^B_{(\lambda,\mu)} \in \Lambda \otimes \Lambda$. The goal is to find a function $f \in \Lambda \otimes \Lambda$ that satisfies

$$\Phi_{B_n}(f[X,Y]) = s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y]$$

which by Lemma 3.32 results in $f = \tilde{s}^B_{(\lambda,\mu)}$.

1. Step 1: Guess $f = \tilde{s}^B_{(\lambda,\mu)}$.
2. Step 2: Calculate $\Phi_{B_n}(f)$.
3. Step 3: Correct guess by subtracting off the leading term.
4. Step 4: Repeat Step 2 until it is right in that $\Phi_{B_n}(f[X,Y]) = s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y]$.

Example 3.59. Use $\Phi_{B_n}(\tilde{s}^B_{1,1})$ and find $\tilde{s}^B_{1,1}$. We know

$$\Phi_{B_n}(\tilde{s}^B_{(\lambda,\mu)}[X,Y]) = s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y].$$

For partitions $\lambda = (1)$ and $\mu = (1)$ let $n = 3$ and we have

$$\Phi_{B_n}(\tilde{s}^B_{1,1}[X,Y]) = s_{11}[X+Y]s_1[X-Y]$$

where

$$\tilde{s}^B_{1,1}[X,Y] = s_1[X+Y]s_1[X-Y] + \text{lower terms}$$

In order to be efficient in computations, we will use SageMath for finding $\tilde{s}^B_{1,1}$. We will start this by defining B_n-Frobenius map Φ_{B_n}

```
s = SymmetricFunctions(QQ).s()
h = SymmetricFunctions(QQ).h()
p = SymmetricFunctions(QQ).p()


f=tensor([s([]),s([])])
def zee(la):
    return la.centralizer_size()
def frob_im_Bn( f, n ):
    # f is an element of the tensor square of Sym
    f = s.tensor_square()(f)
    return sum(c*s(ga).eval_at_permutation_roots(mu)*s(nu).eval_at_permutation_roots(la)*\
        tensor([p(mu)/zee(mu),p(la)/zee(la)]) for r in range(n+1)\
        for la in Partitions(r) for mu in Partitions(n-r)\
        for ((ga,nu), c) in f)
def XmY( f ):
    ss = s.tensor_square()
    return ss(sum(c*tensor([p(la),p(mu).antipode()]) for ((la,mu),c) in p(f).coproduct()))
```

At first we will make a guess $f[X,Y] = s_1[X+Y]s_1[X-Y]$ and let the image of the Frobenius map

$$g[X,Y] = s_{11}[X+Y]s_1[X-Y]$$

we have

```
g=s[1,1].coproduct()*XmY(s[1])
f=s[1].coproduct()*XmY(s[1])
t=g-frob_im_Bn(f,3)
print t
print f stilde[(Partition([1]),Partition([1]))] = f
```

The answers to t and f are

```
t = 2*s[]#s[2, 1] + 2*s[]#s[3] + 2*s[1]#s[1, 1] - 2*s[1, 1]#s[1] - 2*s[2, 1]#s[] - 2*s[3]#s[]
f = -s[]#s[1, 1] - s[]#s[2] + s[1, 1]#s[] + s[2]#s[]
```

This means

$$\begin{aligned}\Phi_{B_n}\left(s_1[X+Y]s_1[X-Y]\right) &= g[X,Y] - t[X,Y] \\ &= s_{11}[X+Y]s_1[X-Y] - \\ &\quad - \left(2s_\bullet \otimes s_{21} + 2s_\bullet \otimes s_3 + 2s_1 \otimes s_{11} - 2s_{11} \otimes s_1 - 2s_{21} \otimes s_\bullet - 2s_3 \otimes s_\bullet\right)\end{aligned}$$

We will correct the guess by taking the term $2s_{21} \otimes s_\bullet$ as the new leading term therefore we will subtract $2s_1 \otimes s_\bullet$ from our first guess $f = s_1[X+Y]s_1[X-Y]$ add apply Frobenius map, this is to evaluate

$$\Phi_{B_n}\left(s_1[X+Y]s_1[X-Y] - 2s_1 \otimes s_\bullet\right)$$

we have

```
g=s[1,1].coproduct()*XmY(s[1])
f=s[1].coproduct()*XmY(s[1])-2*tensor([s[1],s[[]]])
t=g-frob_im_Bn(f,3)
print t
print f stilde[(Partition([1]),Partition([1]))] = f
```

The answers to the new t and f are

```
t = 2*s[] # s[2, 1] + 2*s[] # s[3] + 2*s[1] # s[1, 1] + 2*s[1] # s[2] + 2*s[2] # s[1]
f = -s[] # s[1, 1] - s[] # s[2] - 2*s[1] # s[] + s[1, 1] # s[] + s[2] # s[]
```

This means

$$\begin{aligned}\Phi_{B_n}\left(s_1[X+Y]s_1[X-Y] - 2s_1 \otimes s_\bullet\right) &= g[X,Y] - t[X,Y] \\ &= s_{11}[X+Y]s_1[X-Y] - \\ &\quad - \left(2s_\bullet \otimes s_{21} + 2s_\bullet \otimes s_3 + 2s_1 \otimes s_{11} + 2s_1 \otimes s_2 + 2s_2 \otimes s_1\right)\end{aligned}$$

We will correct the guess by taking the term $-2s_2 \otimes s_1$ as the new leading term therefore we will add $2s_\bullet \otimes s_1$ from our second guess $f = s_1[X+Y]s_1[X-Y] - 2s_1 \otimes s_\bullet$ add apply Frobenius map, this is to evaluate

$$\Phi_{B_n}\left(s_1[X+Y]s_1[X-Y] - 2s_1 \otimes s_\bullet + 2s_\bullet \otimes s_1\right)$$

we have

```
g=s[1,1].coproduct()*XmY(s[1])
f=s[1].coproduct()*XmY(s[1])-2*tensor([s[1],s[[]]])+2*tensor([s[[]],s[1]])
t=g-frob_im_Bn(f,3)
print t
print f stilde[(Partition([1]),Partition([1]))] = f
```

The answers to the new t and f are

```
t = 0
f = 2*s[] # s[1] - s[] # s[1, 1] - s[] # s[2] - 2*s[1] # s[] + s[1, 1] # s[] + s[2] # s[]
```

We will correct the guess by taking the term $-2s_2 \otimes s_1$ as the new leading term therefore we will add $2s_\bullet \otimes s_1$ from our second guess $f = s_1[X+Y]s_1[X-Y] - 2s_1 \otimes s_\bullet$ add apply Frobenius map, this is to evaluate

$$\Phi_{B_n}\left(s_1[X+Y]s_1[X-Y] - 2s_1 \otimes s_\bullet + 2s_\bullet \otimes s_1\right)$$

This means

$$\Phi_{B_n}\left(s_1[X+Y]s_1[X-Y]-2s_1\otimes s_\bullet+2s_\bullet\otimes s_1\right)=g[X,Y]-t[X,Y]$$
$$=s_{11}[X+Y]s_1[X-Y]$$

Which implies

$$f[X,Y]=s_1[X+Y]s_1[X-Y]-2s_1\otimes s_\bullet+2s_\bullet\otimes s_1$$
$$=s_1[X+Y]s_1[X-Y]-2s_\bullet[X+Y]s_1[X-Y]$$

Finally, as we have shown in Theorem 8.14 $\tilde{s}^B_{1,1}$ is a unique function hence $f=\tilde{s}^B_{1,1}$ and

$$\tilde{s}^B_{1,1}[X,Y]=s_1[X+Y]s_1[X-Y]-2s_\bullet[X+Y]s_1[X-Y]\quad .$$

We may also in Schur basis by

`s[1].coproduct()*XmY(s[1])-2*(s[[]].coproduct()*XmY(s[1]))` which gives

`2*s[] # s[1] - s[] # s[1, 1] - s[] # s[2] - 2*s[1] # s[] + s[1, 1] # s[] + s[2] # s[]`

That is

$$\tilde{s}^B_{1,1}=2s_\bullet\otimes s_1-s_\bullet\otimes s_{11}-s_\bullet\otimes s_2-2s_1\otimes s_\bullet+s_{11}\otimes s_\bullet+s_2\otimes s_\bullet$$

Lemma 3.60. *Given $\tilde{s}^B_{(\lambda,\mu)}\in\Lambda\otimes\Lambda$ that satisfies*

$$\Phi_{B_n}(\tilde{s}^B_{(\lambda,\mu)}[X,Y])=s_{(n-|\lambda|-|\mu|,\lambda)}[X+Y]s_\mu[X-Y]$$

it can be expressed as

$$\tilde{s}^B_{(\lambda,\mu)}[X,Y]=s_\lambda[X+Y]s_\mu[X-Y]+\textit{lower terms.}$$

Proof. The following generators expand in tensor square of S_n-induced trivial character bases as

$$\tilde{h}^B_{(m,\bullet)}:=\sum_{i=0}^{m}\tilde{h}_i\otimes\tilde{h}_{m-i}\ \text{ and }\ \tilde{h}^B_{(\bullet,n)}:=\sum_{i=0}^{n}(-1)^{n-i}\tilde{h}_i\otimes\tilde{e}_{n-i}$$

To find the leading terms of maximum degree of each of these generators in terms of complete homogenous symmetric functions, define function LT that eliminates smaller terms, this can be shown in the following by

$$\begin{aligned}
LT\left(\tilde{h}^B_{(m,\bullet)}[X,Y]\right) &= LT\left(\sum_{i=0}^{m} \tilde{h}_i[X]\tilde{h}_{m-i}[Y]\right) \\
&= LT\left(\sum_{i=0}^{m} h_i[X]h_{m-i}[Y]\right) \\
&= LT\left(h_m[X+Y]\right) \\
&= h_m[X+Y]
\end{aligned}$$

Similarly we have

$$LT\left(\tilde{h}^B_{(\bullet,n)}[X,Y]\right) = h_n[X-Y]$$

As the B_n-induced trivial character basis $\tilde{h}^B_{(\lambda,\mu)}$ indexed by (λ,μ) can be constructed by generators of type B and we have

$$\begin{aligned}
\tilde{h}^B_{(\lambda,\mu)}[X,Y] &:= \left(\prod_i^{\odot_B} \tilde{h}^B_{(\lambda_i,\bullet)} \odot_B \prod_j^{\odot_B} \tilde{h}^B_{(\bullet,\mu_j)}\right)[X,Y] \\
&:= \prod_i^{\odot_B} \left(\sum_{k=0}^{\lambda_i} \tilde{h}_k[X]\tilde{h}_{\lambda_i-k}[Y]\right) \odot_B \prod_j^{\odot_B} \left(\sum_{t=0}^{\mu_j} (-1)^{\mu_j-t}\tilde{h}_t[X]\tilde{e}_{\mu_j-t}[Y]\right)
\end{aligned}$$

therefore

$$\begin{aligned}
LT\left(\tilde{h}^B_{(\lambda,\mu)}[X,Y]\right) &= LT\left(h_\lambda[X+Y]h_\mu[X-Y]\right) \\
&= h_\lambda[X+Y]h_\mu[X-Y]
\end{aligned}$$

Moreover, since function $\tilde{s}^B_{(\lambda,\mu)}$ is defined in terms of $\tilde{h}^B_{(\alpha,\beta)}$ by

$$\tilde{s}^B_{(\lambda,\mu)} = \sum_{\substack{(\alpha,\beta) \\ |\alpha|\le|\lambda| \\ |\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K^{-1}_{\mu\beta} \tilde{h}^B_{(\alpha,\beta)}$$

we have

$$\begin{aligned}
LT\left(\tilde{s}^B_{(\lambda,\mu)}\right) &= \sum_{\substack{(\alpha,\beta) \\ |\alpha|\le|\lambda| \\ |\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K^{-1}_{\mu\beta} \left(LT\left(\tilde{h}^B_{(\alpha,\beta)}\right)\right) \\
&= \sum_{\substack{(\alpha,\beta) \\ |\alpha|\le|\lambda| \\ |\beta|=|\mu|}} K^{-1}_{(n-|\lambda|,\lambda)(n-|\alpha|,\alpha)} K^{-1}_{\mu\beta} h_\alpha[X+Y]h_\beta[X-Y] \\
&= s_\lambda[X+Y]s_\mu[X-Y]
\end{aligned}$$

The proof is complete. □

3.3.2 Irreducible Character Computations by B_n-Irreducible Character Basis

Example 3.61. Irreducible character basis at eigenvalues of permutation matrices of B_2.

Elements of B_2 (Conjugacy Classes)	[1, 2]	[2, 1], [-2,-1]	[1, -2], [-2,-1]	[-1, -2]	[2, -1], [-2, 1]	
Conjugacy Classes (Cycle Types)	$K_{(\substack{\square\\\square},\phi)}$	$K_{(\square\square,\phi)}$	$K_{(\square,\square)}$	$K_{(\phi,\substack{\square\\\square})}$	$K_{(\phi,\square\square)}$	
Eigenvalues of Signed Permutation $\{(x_1,x_2),(\bar{y_1},\bar{y_2})\}$	$\{(1,1),(0,0)\}$	$\{(1,-1),(0,0)\}$	$\{(1,0),(-1,0)\}$	$\{(0,0),(-1,-1)\}$	$\{(0,0),(i,-i)\}$	
Transformed Eigenvalues $\{(x_1,x_2),(y_1,y_2)\}$	$\{(1,1),(0,0)\}$	$\{(1,-1),(0,0)\}$	$\{(1,0),(1,0)\}$	$\{(0,0),(1,1)\}$	$\{(0,0),(1,-1)\}$	
Irreducible Characters						Irreducible Character Basis
Row 1: $\chi^{(\square\square,\phi)}$	1	1	1	1	1	$\tilde{s}^B_{\bullet,\bullet}$
Row 2: $\chi^{(\substack{\square\\\square},\phi)}$	1	-1	1	1	-1	$\tilde{s}^B_{1,\bullet}$
Row 3: $\chi^{(\square,\square)}$	2	0	0	-2	0	$\tilde{s}^B_{\bullet,1}$
Row 4: $\chi^{(\phi,\square\square)}$	1	1	-1	1	-1	$\tilde{s}^B_{\bullet,2}$
Row 5: $\chi^{(\phi,\substack{\square\\\square})}$	1	-1	-1	1	1	$\tilde{s}^B_{\bullet,11}$

Table 3.15: Irreducible character basis at eigenvalues of permutation matrices of B_2

The following table shows the irreducible character basis of B_2 in two alphabets X and Y.

Irreducible Characters	$\tilde{s}^B_{(\lambda,\mu)}$ in $s_\alpha \otimes s_\beta$	$\tilde{s}^B_{(\lambda,\mu)}[X,Y]$ in Two Alphabets
Row 1: $\chi^{(\square\square,\phi)}$	$\tilde{s}^B_{\bullet,\bullet} = s_\bullet \otimes s_\bullet$	$\tilde{s}^B_{\bullet,\bullet} = 1$
Row 2: $\chi^{(\substack{\square\\\square},\phi)}$	$\tilde{s}^B_{1,\bullet} = -s_\bullet \otimes s_\bullet + s_\bullet \otimes s_1 + s_1 \otimes s_\bullet$	$\tilde{s}^B_{1,\bullet} = -1 + x_1 + x_2 + y_1 + y_2$
Row 3: $\chi^{(\square,\square)}$	$\tilde{s}^B_{\bullet,1} = -s_\bullet \otimes s_1 + s_1 \otimes s_\bullet$	$\tilde{s}^B_{\bullet,1} = x_1 + x_2 - (y_1 + y_2)$
Row 4: $\chi^{(\phi,\square\square)}$	$\tilde{s}^B_{\bullet,2} = s_\bullet \otimes s_{11} - s_1 \otimes s_\bullet - s_1 \otimes s_1 + s_2 \otimes s_\bullet$	$\tilde{s}^B_{\bullet,2} = y_1y_2 - (x_1 + x_2) - (x_1 + x_2)(y_1 + y_2) + (x_1^2 + x_1x_2 + x_2^2)$
Row 5: $\chi^{(\phi,\substack{\square\\\square})}$	$\tilde{s}^B_{\bullet,11} = -s_\bullet \otimes s_1 + s_\bullet \otimes s_2 - s_1 \otimes s_1 + s_{11} \otimes s_\bullet$	$\tilde{s}^B_{\bullet,11} = -(y_1 + y_2) + (y_1^2 + y_1y_2 + y_2^2) - (x_1 + x_2)(y_1 + y_2) + x_1x_2$

Table 3.16: Irreducible character basis of B_2 in two sets of alphabets.

When evaluating the irreducible characters, the transformed eigenvalues are plugged in in the sets of X and Y variables. For example in row 3 we have

	Elements of B_2 (Conjugacy Classes)	[1, 2]	[2, 1], [-2,-1]	[1, -2], [-2,-1]	[-1, -2]	[2, -1], [-2, 1]
	Conjugacy Classes (Cycle Types)	$K_{(\substack{\square\\\square},\phi)}$	$K_{(\square\square,\phi)}$	$K_{(\square,\square)}$	$K_{(\phi,\substack{\square\\\square})}$	$K_{(\phi,\square\square)}$
	Transformed Eigenvalues $\{(x_1,x_2),(y_1,y_2)\}$	$\{(1,1),(0,0)\}$	$\{(1,-1),(0,0)\}$	$\{(1,0),(1,0)\}$	$\{(0,0),(1,1)\}$	$\{(0,0),(1,-1)\}$
Irreducible Characters	$\tilde{s}^B_{(\lambda,\mu)}[X,Y]$ in Two Alphabets					
Row 3: $\chi^{(\square,\square)}$	$\tilde{s}^B_{\bullet,1}(x_1,x_2,y_1,y_2) = x_1 + x_2 - (y_1 + y_2)$	2	0	0	-2	0

Table 3.17: Irreducible character basis of B_2 at the eigenvalues of permutation matrices

Example 3.62. Evaluate $\left\langle \tilde{s}^B_{\bullet,1}, \tilde{s}^B_{\bullet,1} \right\rangle_{@_B}$. This example includes a detailed solution to Example 3.41 part 2. Let $n=2$ and to evaluate $\left\langle \tilde{s}^B_{\bullet,1}, \tilde{s}^B_{\bullet,1} \right\rangle_{@_B}$ we have

$$\begin{aligned}
\left\langle \tilde{s}^B_{\bullet,1}, \tilde{s}^B_{\bullet,1} \right\rangle_{@_B} &= \sum_{(\theta,\tau)\vdash 2} \frac{1}{2^{l(\theta)}z_\theta}\frac{1}{2^{l(\tau)}z_\tau}\tilde{s}^B_{\bullet,1}[\Xi_\theta,\Xi_\tau]\tilde{s}^B_{\bullet,1}[\Xi_\theta,\Xi_\tau] \\
&= \frac{1}{2^1 z_2}\frac{1}{2^0 z_0}\tilde{s}^B_{\bullet,1}[\Xi_2,\Xi_\bullet]\tilde{s}^B_{\bullet,1}[\Xi_2,\Xi_\bullet] + \frac{1}{2^2 z_{11}}\frac{1}{2^0 z_0}\tilde{s}^B_{\bullet,1}[\Xi_{11},\Xi_\bullet]\tilde{s}^B_{\bullet,1}[\Xi_{11},\Xi_\bullet] \\
&\quad + \frac{1}{2^1 z_1}\frac{1}{2^1 z_1}\tilde{s}^B_{\bullet,1}[\Xi_1,\Xi_1]\tilde{s}^B_{\bullet,1}[\Xi_1,\Xi_1] + \frac{1}{2^0 z_0}\frac{1}{2^2 z_{11}}\tilde{s}^B_{\bullet,1}[\Xi_\bullet,\Xi_{11}]\tilde{s}^B_{\bullet,1}[\Xi_\bullet,\Xi_{11}] \\
&\quad + \frac{1}{2^0 z_0}\frac{1}{2^1 z_2}\tilde{s}^B_{\bullet,1}[\Xi_\bullet,\Xi_2]\tilde{s}^B_{\bullet,1}[\Xi_\bullet,\Xi_2] \\
&= \frac{1}{4}\langle s_1[X+Y]s_1[X-Y], p_2[X]p_\bullet[Y]\rangle \langle s_1[X+Y]s_1[X-Y], p_2[X]p_\bullet[Y]\rangle \\
&\quad + \frac{1}{8}\langle s_1[X+Y]s_1[X-Y], p_{11}[X]p_\bullet[Y]\rangle \langle s_1[X+Y]s_1[X-Y], p_{11}[X]p_\bullet[Y]\rangle \\
&\quad + \frac{1}{4}\langle s_1[X+Y]s_1[X-Y], p_1[X]p_1[Y]\rangle \langle s_1[X+Y]s_1[X-Y], p_1[X]p_1[Y]\rangle \\
&\quad + \frac{1}{8}\langle s_1[X+Y]s_1[X-Y], p_\bullet[X]p_{11}[Y]\rangle \langle s_1[X+Y]s_1[X-Y], p_\bullet[X]p_{11}[Y]\rangle \\
&\quad + \frac{1}{4}\langle s_1[X+Y]s_1[X-Y], p_\bullet[X]p_2[Y]\rangle \langle s_1[X+Y]s_1[X-Y], p_\bullet[X]p_2[Y]\rangle
\end{aligned}$$

We will continue evaluations in SageMath and converting into p bases, we have

$$\begin{aligned}
&= \frac{1}{4}\langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_2[X]p_\bullet[Y]\rangle \cdot \langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_2[X]p_\bullet[Y]\rangle \\
&\quad + \frac{1}{8}\langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_{11}[X]p_\bullet[Y]\rangle \cdot \langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_{11}[X]p_\bullet[Y]\rangle \\
&\quad + \frac{1}{4}\langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_1[X]p_1[Y]\rangle \cdot \langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_1[X]p_1[Y]\rangle \\
&\quad + \frac{1}{8}\langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_\bullet[X]p_{11}[Y]\rangle \cdot \langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_\bullet[X]p_{11}[Y]\rangle \\
&\quad + \frac{1}{4}\langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_\bullet[X]p_2[Y]\rangle \cdot \langle -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y], p_\bullet[X]p_2[Y]\rangle \\
&= \frac{1}{4}0\cdot 0 + \frac{1}{8}(-2)\cdot(-2) + \frac{1}{4}0\cdot 0 + \frac{1}{8}(-2)\cdot(-2) + \frac{1}{4}0\cdot 0 \\
&= 1
\end{aligned}$$

Remark 3.63. In Example 3.62 in the the fourth equality to change basis $s_1[X+Y]s_1[X-Y]$ to power sum, we can do direct hand computation or use SageMath, this is

- ◇ The direct computation is

$$\begin{aligned}
s_1[X+Y]s_1[X-Y] &= (s_\bullet[X]s_1[Y] + s_1[X]s_\bullet[Y])(-s_\bullet[X]s_1[Y] + s_1[X]s_\bullet[Y]) \\
&= (p_\bullet[X]p_1[Y] + p_1[X]p_\bullet[Y])(-p_\bullet[X]p_1[Y] + p_1[X]p_\bullet[Y]) \\
&= -p_\bullet[X]p_{11}[Y] + p_{11}[X]p_\bullet[Y]
\end{aligned}$$

◇ The SageMath is

`p(s[1]).coproduct()*XmY(s[1])` that gives the answer:

`-p[] # p[1, 1] + p[1, 1] # p[]`

SageMath Expansion of $S^B_{(\lambda,\mu)}[X,Y] = s_\lambda[X+Y]s_\mu[X-Y]$ in $s_\alpha \otimes s_\beta$

Example 3.64. In [Stembridge] the irreducible characters of B_n are given by $(\chi^\lambda \otimes \delta\chi^\mu)\uparrow^{Bn}_{B_{|\lambda|}\times B_{|\mu|}}$ where $|\lambda| + |\mu| = n$ and their Frobenius characteristics image are $s_\lambda[X+Y]s_\mu[X-Y]$. In our plethystic notation this can be restated as: Given $\chi^{(\lambda,\mu)}$ to be the irreducible character of B_n indexed by partition $(\lambda,\mu) \vdash n$ its Frobenius characteristic map will be given by

$$\mathscr{F}_{B_n}(\chi^{(\lambda,\mu)}) = S^B_{(\lambda,\mu)}[X,Y] \coloneqq s_\lambda[X+Y]s_\mu[X-Y]$$

where

$$s_\lambda[X+Y] = \sum_{\gamma\subset\lambda} s_\gamma[X]s_{\lambda/\gamma}[Y]$$
$$s_\mu[X-Y] = \sum_{\nu\subset\mu} s_\nu[X](-1)^{|\mu/\nu|}s_{(\mu/\nu)'}[Y]$$

and the prime indicates the conjugate partition. The character at the conjugacy class of $K^{B_n}_{(\gamma,\upsilon)}$ and indexed by a pair of partitions is $\chi^{(\lambda,\mu)}(\gamma,\upsilon)$ where $(\gamma,\nu) \vdash n$. Now to illustrate calculations $s_\lambda[X+Y]$ and $s_\mu[X-Y]$ we may use SageMath. This is summarized for $|\lambda| \le 3$ and $|\mu| \le 3$ in tables below.

SageMath calculations of $s_\lambda[X+Y]$.

$s_\lambda[X+Y]$	SageMath: $s_\lambda[X+Y]$	Answers in SageMath
$s_\bullet[X+Y]$	`s[[]].coproduct()`	`s[]#s[]`
$s_1[X+Y]$	`s[1].coproduct()`	`s[]#s[1]+ s[1]#s[]`
$s_{11}[X+Y]$	`s[1,1].coproduct()`	`s[]#s[1,1]+s[1]#s[1]+s[1,1]#s[]`
$s_2[X+Y]$	`s[2].coproduct()`	`s[]#s[2]+s[1]#s[1]+s[2]#s[]`
$s_{111}[X+Y]$	`s[1,1,1].coproduct()`	`s[]#s[1,1,1]+s[1]#s[1,1]+s[1,1]#s[1]+s[1,1,1]#s[]`
$s_{21}[X+Y]$	`s[2,1].coproduct()`	`s[]#s[2,1]+s[1]#s[1,1]+s[1]#s[2]+s[1,1]#s[1]+s[2]#s[1]+s[2,1]#s[]`
$s_3[X+Y]$	`s[3].coproduct()`	`s[]#s[3]+s[1]#s[2]+s[2]#s[1]+s[3]#s[]`

Table 3.18: SageMath calculations of $s_\lambda[X+Y]$

Expansion of $s_\lambda[X+Y]$ in $s_\alpha \otimes s_\beta$ is listed below.

$s_\lambda[X+Y]$	Expansion in $s_\alpha \otimes s_\beta$
$s_\bullet[X+Y]$	$s_\bullet \otimes s_\bullet$
$s_1[X+Y]$	$s_\bullet \otimes s_1 + s_1 \otimes s_\bullet$
$s_{11}[X+Y]$	$s_\bullet \otimes s_{11} + s_1 \otimes s_1 + s_{11} \otimes s_\bullet$
$s_2[X+Y]$	$s_\bullet \otimes s_2 + s_1 \otimes s_1 + s_2 \otimes s_\bullet$
$s_{111}[X+Y]$	$s_\bullet \otimes s_{111} + s_1 \otimes s_{11} + s_{11} \otimes s_1 + s_{111} \otimes s_\bullet$
$s_{21}[X+Y]$	$s_\bullet \otimes s_{21} + s_1 \otimes s_{11} + s_1 \otimes s_2 + s_{11} \otimes s_1 + s_2 \otimes s_1 + s_{21} \otimes s_\bullet$
$s_3[X+Y]$	$s_\bullet \otimes s_3 + s_1 \otimes s_2 + s_2 \otimes s_1 + s_3 \otimes s_\bullet$

Table 3.19: Expansion of $s_\lambda[X+Y]$ in $s_\alpha \otimes s_\beta$

SageMath calculations of $s_\mu[X-Y]$.

$s_\mu[X-Y]$	Answers in SageMath
$s_\bullet[X-Y]$	`s[]#s[]`
$s_1[X-Y]$	`-s[]#s[1]+ s[1]#s[]`
$s_{11}[X-Y]$	`s[]#s[2]-s[1]#s[1]+s[1,1]#s[]`
$s_2[X-Y]$	`s[]#s[1,1]-s[1]#s[1]+s[2]#s[]`
$s_{111}[X-Y]$	`-s[]#s[3]+s[1]#s[2]-s[1,1]#s[1]+s[1,1,1]#s[]`
$s_{21}[X-Y]$	`-s[]#s[2,1]+s[1]#s[1,1]+s[1]#s[2]-s[1,1]#s[1]-s[2]#s[1]+s[2,1]#s[]`
$s_3[X-Y]$	`-s[]#s[1,1,1]+s[1]#s[1,1]-s[2]#s[1]+s[3]#s[]`

Table 3.20: SageMath calculations of $s_\mu[X-Y]$

Expansion of $s_\mu[X-Y]$ in $s_\alpha \otimes s_\beta$.

$s_\mu[X-Y]$	Expansion in $s_\alpha \otimes s_\beta$
$s_\bullet[X-Y]$	$s_\bullet \otimes s_\bullet$
$s_1[X-Y]$	$-s_\bullet \otimes s_1 + s_1 \otimes s_\bullet$
$s_{11}[X-Y]$	$s_\bullet \otimes s_2 - s_1 \otimes s_1 + s_{11} \otimes s_\bullet$
$s_2[X-Y]$	$s_\bullet \otimes s_{11} - s_1 \otimes s_1 + s_2 \otimes s_\bullet$
$s_{111}[X-Y]$	$-s_\bullet \otimes s_3 + s_1 \otimes s_2 - s_{11} \otimes s_1 + s_{111} \otimes s_\bullet$
$s_{21}[X-Y]$	$-s_\bullet \otimes s_{21} + s_1 \otimes s_{11} + s_1 \otimes s_2 - s_{11} \otimes s_1 - s_2 \otimes s_1 + s_{21} \otimes s_\bullet$
$s_3[X-Y]$	$-s_\bullet \otimes s_{111} + s_1 \otimes s_{11} - s_2 \otimes s_1 + s_3 \otimes s_\bullet$

Table 3.21: Expansion of $s_\mu[X-Y]$ in $s_\alpha \otimes s_\beta$

In table below the products $S^B_{(\lambda,\mu)}[X,Y] = s_\lambda[X+Y]s_\mu[X-Y]$ for $|\lambda| \leq 2$ and $|\mu| \leq 2$ are listed.

$S^B_{(\lambda,\mu)}[X,Y] = s_\lambda[X+Y]s_\mu[X-Y]$	Product in SageMath	Answers in SageMath
$S^B_{\bullet,\bullet}[X,Y] = s_\bullet[X+Y]s_\bullet[X-Y]$	`s[[]].coproduct()*XmY(s[[]])`	`s[]#s[]`
$S^B_{1,\bullet}[X,Y] = s_1[X+Y]s_\bullet[X-Y]$	`s[1].coproduct()*XmY(s[[]])`	`s[]#s[1]+ s[1]#s[]`
$S^B_{\bullet,1}[X,Y] = s_\bullet[X+Y]s_1[X-Y]$	`s[[]].coproduct()*XmY(s[1])`	`-s[]#s[1]+ s[1]#s[]`
$S^B_{1,1}[X,Y] = s_1[X+Y]s_1[X-Y]$	`s[1].coproduct()*XmY(s[1])`	`-s[]#s[1,1]-s[]#s[2]+s[1,1]#s[]+s[2]#s[]`
$S^B_{11,\bullet}[X,Y] = s_{11}[X+Y]s_\bullet[X-Y]$	`s[1,1].coproduct()*XmY(s[[]])`	`s[]#s[1,1]+s[1]#s[1]+s[1,1]#s[]`
$S^B_{\bullet,11}[X,Y] = s_\bullet[X+Y]s_{11}[X-Y]$	`s[[]].coproduct()*XmY(s[1,1])`	`s[]#s[2]-s[1]#s[1]+s[1,1]#s[]`
$S^B_{2,\bullet}[X,Y] = s_2[X+Y]s_\bullet[X-Y]$	`s[2].coproduct()*XmY(s[[]])`	`s[]#s[2]+s[1]#s[1]+s[2]#s[]`
$S^B_{\bullet,2}[X,Y] = s_\bullet[X+Y]s_2[X-Y]$	`s[[]].coproduct()*XmY(s[2])`	`s[]#s[1,1]-s[1]#s[1]+s[2]#s[]`
$S^B_{11,1}[X,Y] = s_{11}[X+Y]s_1[X-Y]$	`s[1,1].coproduct()*XmY(s[1])`	`-s[]#s[1,1,1]-s[]#s[2,1]-s[1]#s[2]+s[1,1,1]#s[]+s[2]#s[1]+s[2,1]#s[]`
$S^B_{1,11}[X,Y] = s_1[X+Y]s_{11}[X-Y]$	`s[1].coproduct()*XmY(s[1,1])`	`s[]#s[2,1]+s[]#s[3]-s[1]#s[1,1]+s[1,1,1]#s[]-s[2]#s[1]+s[2,1]#s[]`
$S^B_{2,1}[X,Y] = s_2[X+Y]s_1[X-Y]$	`s[2].coproduct()*XmY(s[1])`	`-s[]#s[2,1]-s[]#s[3]-s[1]#s[1,1]+s[1,1]#s[1]+s[2,1]#s[]+s[3]#s[]`
$S^B_{21,\bullet}[X,Y] = s_{21}[X+Y]s_\bullet[X-Y]$	`s[2,1].coproduct()*XmY(s[[]])`	`s[]#s[2,1]+s[1]#s[1,1]+s[1]#s[2]+s[1,1]#s[1]+s[2]#s[1]+s[2,1]#s[]`
$S^B_{1,2}[X,Y] = s_1[X+Y]s_2[X-Y]$	`s[1].coproduct()*XmY(s[2])`	`s[]#s[1,1,1]+s[]#s[2,1]-s[1]#s[2]-s[1,1]#s[1]+s[2,1]#s[]+s[3]#s[]`
$S^B_{\bullet,21}[X,Y] = s_\bullet[X+Y]s_{21}[X-Y]$	`s[[]].coproduct()*XmY(s[2,1])`	`-s[]#s[2,1]+s[1]#s[1,1]+s[1]#s[2]-s[1,1]#s[1]-s[2]#s[1]+s[2,1]#s[]`

Table 3.22: SageMath product $S^B_{(\lambda,\mu)}[X,Y] = s_\lambda[X+Y]s_\mu[X-Y]$

The products $S^B_{(\lambda,\mu)}[X,Y] = s_\lambda[X+Y]s_\mu[X-Y]$ for $|\lambda| \leq 2$ and $|\mu| \leq 2$ are listed in tensor notation.

$S^B_{(\lambda,\mu)}[X,Y] = s_\lambda[X+Y]s_\mu[X-Y]$	Expansion
$S^B_{\bullet,\bullet}[X,Y] = s_\bullet[X+Y]s_\bullet[X-Y]$	$s_\bullet \otimes s_\bullet$
$S^B_{1,\bullet}[X,Y] = s_1[X+Y]s_\bullet[X-Y]$	$s_\bullet \otimes s_1 + s_1 \otimes s_\bullet$
$S^B_{\bullet,1}[X,Y] = s_\bullet[X+Y]s_1[X-Y]$	$-s_\bullet \otimes s_1 + s_1 \otimes s_\bullet$
$S^B_{1,1}[X,Y] = s_1[X+Y]s_1[X-Y]$	$-s_\bullet \otimes s_{11} - s_\bullet \otimes s_2 + s_{11} \otimes s_\bullet + s_2 \otimes s_\bullet$
$S^B_{11,\bullet}[X,Y] = s_{11}[X+Y]s_\bullet[X-Y]$	$s_\bullet \otimes s_{11} + s_1 \otimes s_1 + s_{11} \otimes s_\bullet$
$S^B_{\bullet,11}[X,Y] = s_\bullet[X+Y]s_{11}[X-Y]$	$s_\bullet \otimes s_2 - s_1 \otimes s_1 + s_{11} \otimes s_\bullet$
$S^B_{2,\bullet}[X,Y] = s_2[X+Y]s_\bullet[X-Y]$	$s_\bullet \otimes s_2 + s_1 \otimes s_1 + s_2 \otimes s_\bullet$
$S^B_{\bullet,2}[X,Y] = s_\bullet[X+Y]s_2[X-Y]$	$s_\bullet \otimes s_{11} - s_1 \otimes s_1 + s_2 \otimes s_\bullet$
$S^B_{11,1}[X,Y] = s_{11}[X+Y]s_1[X-Y]$	$-s_\bullet \otimes s_{111} - s_\bullet \otimes s_{21} - s_1 \otimes s_2 + s_{111} \otimes s_\bullet + s_2 \otimes s_1 + s_{21} \otimes s_\bullet$
$S^B_{1,11}[X,Y] = s_1[X+Y]s_{11}[X-Y]$	$s_\bullet \otimes s_{21} + s_\bullet \otimes s_3 - s_1 \otimes s_{11} + s_{111} \otimes s_\bullet - s_2 \otimes s_1 + s_{21} \otimes s_\bullet$
$S^B_{2,1}[X,Y] = s_2[X+Y]s_1[X-Y]$	$-s_\bullet \otimes s_{21} - s_\bullet \otimes s_3 - s_1 \otimes s_{11} + s_{11} \otimes s_1 + s_{21} \otimes s_\bullet + s_3 \otimes s_\bullet$
$S^B_{21,\bullet}[X,Y] = s_{21}[X+Y]s_\bullet[X-Y]$	$s_\bullet \otimes s_{21} + s_1 \otimes s_{11} + s_1 \otimes s_2 + s_{11} \otimes s_1 + s_2 \otimes s_1 + s_{21} \otimes s_\bullet$
$S^B_{1,2}[X,Y] = s_1[X+Y]s_2[X-Y]$	$s_\bullet \otimes s_{111} + s_\bullet \otimes s_{21} - s_1 \otimes s_2 - s_{11} \otimes s_1 + s_{21} \otimes s_\bullet + s_3 \otimes s_\bullet$
$S^B_{\bullet,21}[X,Y] = s_\bullet[X+Y]s_{21}[X-Y]$	$-s_\bullet \otimes s_{21} + s_1 \otimes s_{11} + s_1 \otimes s_2 - s_{11} \otimes s_1 - s_2 \otimes s_1 + s_{21} \otimes s_\bullet$

Table 3.23: Product $S^B_{(\lambda,\mu)}[X,Y] = s_\lambda[X+Y]s_\mu[X-Y]$ expansion

Example 3.65. In Example 3.54, given $n = 4$ we have the expansion $\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{\bullet,1} = \tilde{s}^B_{1,1} + \tilde{s}^B_{\bullet,1}$. The Frobenius map is

$$\begin{aligned}
\Phi_{B_n}\left(\tilde{s}^B_{1,\bullet} \cdot \tilde{s}^B_{\bullet,1}\right) &= \Phi_{B_n}\left(\tilde{s}^B_{1,1} + \tilde{s}^B_{\bullet,1}\right) \\
&= \Phi_{B_n}\left(\tilde{s}^B_{1,1}\right) + \Phi_{B_n}\left(\tilde{s}^B_{\bullet,1}\right) \\
&= S^B_{21,1} + S^B_{3,1}
\end{aligned}$$

This can also be calculated using Lemma 3.36 and Example 3.54 as

$$\begin{aligned}\Phi_{B_n}\left(\tilde{s}^B_{1,\bullet}\cdot\tilde{s}^B_{\bullet,1}\right) &= \Phi_{B_n}\left(\tilde{s}^B_{1,\bullet}\right) *_B \Phi_{B_n}\left(\tilde{s}^B_{\bullet,1}\right)\\ &= S^B_{31,\bullet} *_B S^B_{3,1}\\ &= S^B_{21,1}+S^B_{3,1}\end{aligned}$$

Example 3.66. Expansion of $\tilde{s}^B_{(\alpha,\beta)}$ in $s_\lambda[X+Y]s_\mu[X-Y]$ for $|\alpha|+|\beta|\le 2$.

We will use our compact notation as

$$S^B_{(\lambda,\mu)}[X,Y] \coloneqq s_\lambda[X+Y]s_\mu[X-Y].$$

Calculating by SageMath we have

$\tilde{s}^B_{(\alpha,\beta)}[X,Y]$	Expansion in $S^B_{(\lambda,\mu)}[X,Y]$
$\tilde{s}^B_{\bullet,\bullet}$	$S^B_{\bullet,\bullet}$
$\tilde{s}^B_{1,\bullet}$	$S^B_{1,\bullet}-S^B_{\bullet,\bullet}$
$\tilde{s}^B_{\bullet,1}$	$S^B_{\bullet,1}$
$\tilde{s}^B_{2,\bullet}$	$S^B_{2,\bullet}-2S^B_{1,\bullet}$
$\tilde{s}^B_{11,\bullet}$	$S^B_{11,\bullet}-S^B_{1,\bullet}+S^B_{\bullet,\bullet}$
$\tilde{s}^B_{1,1}$	$S^B_{1,1}-2S^B_{\bullet,1}$
$\tilde{s}^B_{\bullet,2}$	$S^B_{\bullet,2}-\frac{1}{2}S^B_{1,\bullet}-\frac{1}{2}S^B_{\bullet,1}$
$\tilde{s}^B_{\bullet,11}$	$S^B_{\bullet,11}-\frac{1}{2}S^B_{1,\bullet}+\frac{1}{2}S^B_{\bullet,1}$

Table 3.24: Expansion of $\tilde{s}^B_{(\alpha,\beta)}$ in $S^B_{(\lambda,\mu)}[X,Y] \coloneqq s_\lambda[X+Y]s_\mu[X-Y]$.

The SageMath Codes are

`stB([.],[.])`	Expansion in `s[.].coproduct()*XmY(s[.])`
`stB([],[])`	`s[[]].coproduct()*XmY(s[[]])`
`stB([1],[])`	`s[1].coproduct()*XmY(s[[]])-s[[]].coproduct()*XmY(s[[]])`
`stB([],[1])`	`s[[]].coproduct()*XmY(s[1])`
`stB([2],[])`	`s[2].coproduct()*XmY(s[[]])-2*s[1].coproduct()*XmY(s[[]])`
`stB([1,1],[])`	`s[1,1].coproduct()*XmY(s[[]])-s[1].coproduct()*XmY(s[[]])+s[[]].coproduct()*XmY(s[[]])`
`stB([1],[1])`	`s[1].coproduct()*XmY(s[1])-2*s[[]].coproduct()*XmY(s[1])`
`stB([],[2])`	`s[[]].coproduct()*XmY(s[2])-(1/2)*s[1].coproduct()*XmY(s[[]])-(1/2)*s[[]].coproduct()*XmY(s[1])`
`stB([],[2])`	`s[[]].coproduct()*XmY(s[1,1])-(1/2)*s[1].coproduct()*XmY(s[[]])+(1/2)*s[[]].coproduct()*XmY(s[1])`

Table 3.25: Expansion of $\tilde{s}^B_{(\alpha,\beta)}$ in $S^B_{(\lambda,\mu)}[X,Y] \coloneqq s_\lambda[X+Y]s_\mu[X-Y]$ SageMath codes.

The transition coefficients matrix $\{\tilde{s}^B_{(\alpha,\beta)}\}\to\{S^B_{(\lambda,\mu)}\}$ is

$\{\tilde{s}^B_{(\alpha,\beta)}\}\to\{S^B_{(\lambda,\mu)}\}$	$S^B_{\bullet,\bullet}$	$S^B_{1,\bullet}$	$S^B_{\bullet,1}$	$S^B_{2,\bullet}$	$S^B_{11,\bullet}$	$S^B_{1,1}$	$S^B_{\bullet,2}$	$S^B_{\bullet,11}$
$\tilde{s}^B_{\bullet,\bullet}$	1							
$\tilde{s}^B_{1,\bullet}$	-1	1						
$\tilde{s}^B_{\bullet,1}$			1					
$\tilde{s}^B_{2,\bullet}$		-2		1				
$\tilde{s}^B_{11,\bullet}$	1	-1			1			
$\tilde{s}^B_{1,1}$			-2			1		
$\tilde{s}^B_{\bullet,2}$		$-\frac{1}{2}$	$-\frac{1}{2}$				1	
$\tilde{s}^B_{\bullet,11}$		$-\frac{1}{2}$	$\frac{1}{2}$					1

Table 3.26: The transition coefficients matrix $\{\tilde{s}^B_{(\alpha,\beta)}\}\to\{S^B_{(\lambda,\mu)}\}$

Example 3.67. Expansion of $\tilde{h}^B_{(\alpha,\beta)}$ in $h_\lambda[X+Y]h_\mu[X-Y]$ for $|\alpha|+|\beta|\le 2$.

We will use our compact notation in $H^B_{(\lambda,\mu)}[X,Y] := h_\lambda[X+Y]h_\mu[X-Y]$. Calculating by SageMath we have

$\tilde{h}^B_{(\alpha,\beta)}[X,Y]$	Expansion in $H^B_{(\lambda,\mu)}[X,Y]$
$\tilde{h}^B_{\bullet,\bullet}$	$H^B_{\bullet,\bullet}$
$\tilde{h}^B_{1,\bullet}$	$H^B_{1,\bullet}$
$\tilde{h}^B_{\bullet,1}$	$H^B_{\bullet,1}$
$\tilde{h}^B_{2,\bullet}$	$H^B_{2,\bullet} - H^B_{1,\bullet}$
$\tilde{h}^B_{11,\bullet}$	$H^B_{11,\bullet} - H^B_{1,\bullet}$
$\tilde{h}^B_{1,1}$	$H^B_{1,1} - H^B_{\bullet,1}$
$\tilde{h}^B_{\bullet,2}$	$H^B_{\bullet,2} - \frac{1}{2}H^B_{1,\bullet} - \frac{1}{2}H^B_{\bullet,1}$
$\tilde{h}^B_{\bullet,11}$	$H^B_{\bullet,11} - H^B_{1,\bullet}$

Table 3.27: Expansion of $\tilde{h}^B_{(\alpha,\beta)}$ in $H^B_{(\lambda,\mu)}[X,Y] := h_\lambda[X+Y]h_\mu[X-Y]$.

The transition coefficients matrix $\{\tilde{h}^B_{(\alpha,\beta)}\}\to\{H^B_{(\lambda,\mu)}\}$ is

$\{\tilde{h}^B_{(\alpha,\beta)}\}\to\{H^B_{(\lambda,\mu)}\}$	$H^B_{\bullet,\bullet}$	$H^B_{1,\bullet}$	$H^B_{\bullet,1}$	$H^B_{2,\bullet}$	$H^B_{11,\bullet}$	$H^B_{1,1}$	$H^B_{\bullet,2}$	$H^B_{\bullet,11}$
$\tilde{h}^B_{\bullet,\bullet}$	1							
$\tilde{h}^B_{1,\bullet}$		1						
$\tilde{h}^B_{\bullet,1}$			1					
$\tilde{h}^B_{2,\bullet}$		-1		1				
$\tilde{h}^B_{11,\bullet}$		-1			1			
$\tilde{h}^B_{1,1}$			-1			1		
$\tilde{h}^B_{\bullet,2}$		$-\frac{1}{2}$	$-\frac{1}{2}$				1	
$\tilde{h}^B_{\bullet,11}$		-1						1

Table 3.28: The transition coefficients matrix $\{\tilde{h}^B_{(\alpha,\beta)}\}\to\{H^B_{(\lambda,\mu)}\}$

3.3.3 Example: Expansion of $\tilde{s}^B_{(\alpha,\beta)}$ in $\tilde{s}_\lambda[X+Y]\tilde{s}_\mu[X-Y]$

Example 3.68. Expansion of $\tilde{h}^B_{(\alpha,\beta)}$ in $\tilde{h}_\lambda[X+Y]\tilde{h}_\mu[X-Y]$ for $|\alpha|+|\beta|\leq 2$. To make a compact notation, let

$$\tilde{H}^B_{(\lambda,\mu)}[X,Y] \coloneqq \tilde{h}_\lambda[X+Y]\tilde{h}_\mu[X-Y].$$

Using SageMath we have

$\tilde{h}^B_{(\alpha,\beta)}[X,Y]$	Expansion in $\tilde{H}^B_{(\lambda,\mu)}[X,Y]$
$\tilde{h}^B_{\bullet,\bullet}$	$\tilde{H}^B_{\bullet,\bullet}$
$\tilde{h}^B_{1,\bullet}$	$\tilde{H}^B_{1,\bullet}$
$\tilde{h}^B_{\bullet,1}$	$\tilde{H}^B_{\bullet,1}$
$\tilde{h}^B_{2,\bullet}$	$\tilde{H}^B_{2,\bullet}$
$\tilde{h}^B_{11,\bullet}$	$\tilde{H}^B_{11,\bullet}$
$\tilde{h}^B_{1,1}$	$\tilde{H}^B_{1,1}-\tilde{H}^B_{\bullet,1}$
$\tilde{h}^B_{\bullet,2}$	$\tilde{H}^B_{\bullet,2}-\frac{1}{2}\tilde{H}^B_{1,\bullet}+\frac{1}{2}\tilde{H}^B_{\bullet,1}$
$\tilde{h}^B_{\bullet,11}$	$\tilde{H}^B_{\bullet,11}-\tilde{H}^B_{1,\bullet}+\tilde{H}^B_{\bullet,1}$

Table 3.29: Expansion of $\tilde{h}^B_{(\alpha,\beta)}$ in $\tilde{H}^B_{(\lambda,\mu)}[X,Y] \coloneqq \tilde{h}_\lambda[X+Y]\tilde{h}_\mu[X-Y]$.

The SageMath Codes are

`htB([.],[.])`	Expansion in `ht[.].coproduct()*XmY(ht[.])`
`htB([],[])`	`ht[[]].coproduct()*XmY(ht[[]])`
`htB([1],[])`	`ht[1].coproduct()*XmY(ht[[]])`
`htB([],[1])`	`ht[[]].coproduct()*XmY(ht[1])`
`htB([2],[])`	`ht[2].coproduct()*XmY(ht[[]])`
`htB([1,1],[])`	`ht[1,1].coproduct()*XmY(ht[[]])`
`htB([1],[1])`	`ht[1].coproduct()*XmY(ht[1])-ht[[]].coproduct()*XmY(ht[1])`
`htB([],[2])`	`ht[[]].coproduct()*XmY(ht[2])-(1/2)*ht[1].coproduct()*XmY(ht[[]])+(1/2)*ht[[]].coproduct()*XmY(ht[1])`
`htB([],[1,1])`	`ht[[]].coproduct()*XmY(ht[1,1])-ht[1].coproduct()*XmY(ht[[]])+ht[[]].coproduct()*XmY(ht[1])`

Table 3.30: Expansion of $\tilde{h}^B_{(\alpha,\beta)}$ in $\tilde{h}_\lambda[X+Y]\tilde{h}_\mu[X-Y]$ SageMath codes.

The transition coefficients matrix $\{\tilde{h}^B_{(\alpha,\beta)}\}\to\{\tilde{H}^B_{(\lambda,\mu)}\}$ is

$\{\tilde{h}^B_{(\alpha,\beta)}\}\to\{\tilde{H}^B_{(\lambda,\mu)}\}$	$\tilde{H}^B_{\bullet,\bullet}$	$\tilde{H}^B_{1,\bullet}$	$\tilde{H}^B_{\bullet,1}$	$\tilde{H}^B_{2,\bullet}$	$\tilde{H}^B_{11,\bullet}$	$\tilde{H}^B_{1,1}$	$\tilde{H}^B_{\bullet,2}$	$\tilde{H}^B_{\bullet,11}$
$\tilde{h}^B_{\bullet,\bullet}$	1							
$\tilde{h}^B_{1,\bullet}$		1						
$\tilde{h}^B_{\bullet,1}$			1					
$\tilde{h}^B_{2,\bullet}$				1				
$\tilde{h}^B_{11,\bullet}$					1			
$\tilde{h}^B_{1,1}$			-1			1		
$\tilde{h}^B_{\bullet,2}$		$-\frac{1}{2}$	$\frac{1}{2}$				1	
$\tilde{h}^B_{\bullet,11}$		-1	1					1

Table 3.31: Transition coefficients matrix $\{\tilde{h}^B_{(\alpha,\beta)}\}\to\{\tilde{H}^B_{(\lambda,\mu)}\}$

Example 3.69. Expansion of $\tilde{s}^B_{(\alpha,\beta)}$ in $\tilde{s}_\lambda[X+Y]\tilde{s}_\mu[X-Y]$ for $|\alpha|+|\beta|\leq 2$. To make a compact notation, let

$$\tilde{S}^B_{(\lambda,\mu)}[X,Y] \coloneqq \tilde{s}_\lambda[X+Y]\tilde{s}_\mu[X-Y].$$

Using SageMath we have

$\tilde{s}^B_{(\alpha,\beta)}[X,Y]$	Expansion in $\tilde{S}^B_{(\lambda,\mu)}[X,Y]$
$\tilde{s}^B_{\bullet,\bullet}$	$\tilde{S}^B_{\bullet,\bullet}$
$\tilde{s}^B_{1,\bullet}$	$\tilde{S}^B_{1,\bullet}$
$\tilde{s}^B_{\bullet,1}$	$\tilde{S}^B_{\bullet,1}+\tilde{S}^B_{\bullet,\bullet}$
$\tilde{s}^B_{2,\bullet}$	$\tilde{S}^B_{2,\bullet}$
$\tilde{s}^B_{11,\bullet}$	$\tilde{S}^B_{11,\bullet}$
$\tilde{s}^B_{1,1}$	$\tilde{S}^B_{1,1}+\tilde{S}^B_{1,\bullet}-\tilde{S}^B_{\bullet,1}-\tilde{S}^B_{\bullet,\bullet}$
$\tilde{s}^B_{\bullet,2}$	$\tilde{S}^B_{\bullet,2}-\frac{1}{2}\tilde{S}^B_{1,\bullet}+\frac{3}{2}\tilde{S}^B_{\bullet,1}+\tilde{S}^B_{\bullet,\bullet}$
$\tilde{s}^B_{\bullet,11}$	$\tilde{S}^B_{\bullet,11}-\frac{1}{2}\tilde{S}^B_{1,\bullet}+\frac{3}{2}\tilde{S}^B_{\bullet,1}$

Table 3.32: Expansion of $\tilde{s}^B_{(\lambda,\mu)}$ in $\tilde{S}^B_{(\lambda,\mu)}[X,Y] \coloneqq \tilde{s}_\lambda[X+Y]\tilde{s}_\mu[X-Y]$.

The SageMath Codes are

`stB([.],[.])`	Expansion in `st[.].coproduct()*XmY(st[.])`
`stB([],[])`	`st[[]].coproduct()*XmY(st[[]])`
`stB([1],[])`	`st[1].coproduct()*XmY(st[[]])`
`stB([],[1])`	`st[[]].coproduct()*XmY(st[1])-st[[]].coproduct()*XmY(st[[]])`
`stB([2],[])`	`st[2].coproduct()*XmY(st[[]])`
`stB([1,1],[])`	`st[1,1].coproduct()*XmY(st[[]])`
`stB([1],[1])`	`st[1].coproduct()*XmY(st[1])+st[1].coproduct()*XmY(st[[]])-st[[]].coproduct()*XmY(st[1])-st[[]].coproduct()*XmY(st[[]])`
`stB([],[2])`	`st[[]].coproduct()*XmY(st[2])-(1/2)*st[1].coproduct()*XmY(st[[]])+(3/2)*st[[]].coproduct()*XmY(st[1])+st[[]].coproduct()*XmY(st[[]])`
`stB([],[1,1])`	`st[[]].coproduct()*XmY(st[1,1])-(1/2)*st[1].coproduct()*XmY(st[[]])+(3/2)*st[[]].coproduct()*XmY(st[1])`

Table 3.33: Expansion of $\tilde{s}^B_{(\alpha,\beta)}$ in $\tilde{S}^B_{(\lambda,\mu)}[X,Y] \coloneqq \tilde{s}_\lambda[X+Y]\tilde{s}_\mu[X-Y]$ SageMath codes.

The transition coefficients matrix $\{\tilde{s}^B_{(\alpha,\beta)}\}\to\{\tilde{S}^B_{(\lambda,\mu)}\}$ is

$\{\tilde{s}^B_{(\alpha,\beta)}\}\to\{\tilde{S}^B_{(\lambda,\mu)}\}$	$\tilde{S}^B_{\bullet,\bullet}$	$\tilde{S}^B_{1,\bullet}$	$\tilde{S}^B_{\bullet,1}$	$\tilde{S}^B_{2,\bullet}$	$\tilde{S}^B_{11,\bullet}$	$\tilde{S}^B_{1,1}$	$\tilde{S}^B_{\bullet,2}$	$\tilde{S}^B_{\bullet,11}$
$\tilde{s}^B_{\bullet,\bullet}$	1							
$\tilde{s}^B_{1,\bullet}$		1						
$\tilde{s}^B_{\bullet,1}$	1		1					
$\tilde{s}^B_{2,\bullet}$				1				
$\tilde{s}^B_{11,\bullet}$					1			
$\tilde{s}^B_{1,1}$	-1	1	-1			1		
$\tilde{s}^B_{\bullet,2}$	1	$-\frac{1}{2}$	$\frac{3}{2}$				1	
$\tilde{s}^B_{\bullet,11}$		$-\frac{1}{2}$	$\frac{3}{2}$					1

Table 3.34: Transition coefficients matrix $\{\tilde{s}^B_{(\alpha,\beta)}\}\to\{\tilde{S}^B_{(\lambda,\mu)}\}$

3.3.4 Summary of Thesis Results

The goal of this thesis was to find the unique basis, B_n-irreducible character basis, namely $\tilde{s}^B_{(\lambda,\mu)}$. The following table shows the change of basis the transition coefficients of constructing $\tilde{s}^B_{(\lambda,\mu)}$.

$\{h_\theta\}$, $\{e_\theta\}$ $\longrightarrow$	$\{\tilde{h}_\eta\}$, $\{\tilde{he}_{(\delta\vert\tau)}\}$ $\longrightarrow$	$\{\tilde{h}^B_{(\alpha,\beta)}\}$ $\longrightarrow$	$\{\tilde{s}^B_{(\lambda,\mu)}\}$
i. Content Multi-Set **Example:** Given $\theta = (2,1)$, We have h_{21} The content: $\{\!\{1^2, 2^1\}\!\}$	i. Multi-Set Partitions **Example:** Given $\theta = (2,1)$, We have $h_{21} = \tilde{h}_1 + 2\tilde{h}_{11} + \tilde{h}_{21}$ The multi-set partitions: $\{\!\{112\}\!\}$ $\{\!\{1 \mid 12\}\!\}$ $\{\!\{2 \mid 11\}\!\}$ $\{\!\{1 \mid 1 \mid 2\}\!\}$	Splittings of Multi-Sets **Example:** Given $(\alpha,\beta) = ((),(2))$, We have $\tilde{h}^B_{\bullet,2} = \tilde{h}_\bullet \otimes \tilde{he}_{\bullet\vert 2} - \tilde{h}_1 \otimes \tilde{he}_{\bullet\vert 1} + \tilde{h}_2 \otimes \tilde{he}_{\bullet\vert\bullet}$ The splittings of multi-set partitions: $\{\!\{\}\!\} \uplus \{\!\{\bar{1},\bar{1}\}\!\}$ $\{\!\{\bar{1}\}\!\} \uplus \{\!\{\bar{1}\}\!\}$ $\{\!\{\bar{1},\bar{1}\}\!\} \uplus \{\!\{\}\!\}$	Pairs of Column-Strict Tableaux **Example:** Given $(\alpha,\beta) = ((2),(1))$, We have $\tilde{h}^B_{2,1} = \tilde{s}^B_{\bullet,1} + \tilde{s}^B_{1,1} + \tilde{s}^B_{2,1}$ Pairs of column-strict tableaux: $\left(\begin{array}{\|c\|c\|c\|c\|}\hline 1&1&2&2\\\hline\end{array}, \begin{array}{\|c\|}\hline \bar{1}\\\hline\end{array}\right)$ $\left(\begin{array}{\|c\|cc}\cline{1-1} 2&&\\\hline 1&\multicolumn{1}{c\|}{1}&\multicolumn{1}{c\|}{2}\\\hline\end{array}, \begin{array}{\|c\|}\hline \bar{1}\\\hline\end{array}\right)$ $\left(\begin{array}{\|c\|c\|}\hline 2&2\\\hline 1&1\\\hline\end{array}, \begin{array}{\|c\|}\hline \bar{1}\\\hline\end{array}\right)$
ii. Content Multi-Set **Example:** Given $\theta = (2,1)$, We have e_{21} The content: $\{\!\{\bar{1}^2, \bar{2}^1\}\!\}$	ii. Set Partitions **Example:** Given $\theta = (2,1)$, We have $e_{21} = \tilde{he}_{(1\vert 1)} + \tilde{he}_{(\bullet\vert 21)}$ The set partitions: $\{\!\{\bar{1} \rrbracket \bar{1}\bar{2}\}\!\}$ $\{\!\{\bar{1} \rrbracket \bar{1} \rrbracket \bar{2}\}\!\}$		

Table 3.35: Summary of results $\{h_\theta\},\{e_\theta\} \longrightarrow \{\tilde{h}_\eta\},\{\tilde{he}_{(\delta\vert\tau)}\} \longrightarrow \{\tilde{h}^B_{(\alpha,\beta)}\} \longrightarrow \{\tilde{s}^B_{(\lambda,\mu)}\}$

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