Topics in Fomin-Kirillov Algebra

Sirous Homayouni

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Abstract

We introduce the notion of being 'z-star' for homogeneous polynomials. By proving a theorem plus developing a conjecture we state that, with a graded lexicographic monomial ordering, the reduced Gröbner basis, for the ideal I generated by the relations of Fomin-Kirilov algebra $FK(n)$, consists of 'z-star' polynomials.

For general n, we find the character of the Fomin-Kirillov algebra, for some finite usual degrees with general set partition degree. We find the decomposition of this character in irreducible characters where this decomposition stabilizes at some enough big n.

We develop a quotient of $FK(n)$, denoted by $\overline{FK}_{C_n}(n)$, by making the quotient of the free algebra generated by the edges of an n-cycle, compared to the associated complete graph, where the ideal is generated by the relations of $FK(n)$ except for letting the missing edges equal to zero and keeping only the edges of the polygon. We find the character map of algebra $\overline{FK}_{C_n}(n)$ and prove that the dimension of it equals the Lucas Number L_n and its Hilbert series is q-Lucas polynomial.

We consider the commutative quotient of Fomin-Kirilov algebra, denoted by $FK^c(n)$ and find its Gröbner basis.

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Chapter 1

Introduction

1.1 Background and Motivation

Symmetries are very important in technology and science, especially in physics. In theoretical physics any conservation law is due to a symmetry. For example conservation of linear momentum is due to symmetry of physical systems under translation in space, and conservation of energy is due to symmetry of physical system under translation in time. Therefore symmetry is an important area of research in mathematics.

Elementary symmetric polynomials

$$e_k(x_1, x_2, \cdots, x_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} x_{j_1} x_{j_2} \cdots x_{j_k}$$

are basic building block for symmetric polynomials, in the sense that any symmetric polynomial can be written uniquely as a polynomial in elementary symmetric polynomials. We say that the elementary symmetric polynomials e_i for $i = 1, \cdots, n$ are free generators for the ring of symmetric polynomials $Z[x_1, \cdots, x_n]^{S_n}$.

There are other bases for the ring of symmetric polynomials like Schur polynomials. Schur polynomials s_λ, indexed by partition λ of n, $l(\lambda) \leq n$, generalize the elementary symmetric polynomials . They form a linear basis for the space of all symmetric polynomials. Therefore there are unique coefficients $c^{\nu}_{\lambda,\mu}$ such that $s_\lambda s_\mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} s_\nu$. These coefficients are called Littlewood-Richardson coefficients and it is important that they are non-negative integers and can be calculated combinatorially by the so called Littlewood-Richardson rule.

There is a linear basis for $Z[x_1, x_2, \cdots, x_n]$ called Schubert polynomials denoted by $\mathfrak{S}_w$, indexed by the permutation w. The reason Schubert polynomials are indexed by permutation w is that they are 'stable' in the following sense. For embedding of S_n into S_{n+1} where the permutation w' obtained from w under this embedding, is such that w' fixes $n + 1$ and is the same as w elsewhere, we have $\mathfrak{S}_w = \mathfrak{S}_{w'}$. So in fact we index the Schubert polynomials by the permutations in S_∞. Schubert polynomials were introduced by Lascoux and Schützenberger in 1982 and are named after Hermann Schubert.

There is an explicit combinatorial interpretation of the Schubert polynomials $\mathfrak{S}_w$ in terms of the reduced decomposition of the permutation w due to C. Billey, W. Jockusch and R.P. Stanley [4], that we quote in here.

$$\mathfrak{S}_w = \sum_{a \in R(w)} \sum_{(i_1, \cdots, i_p) \in K(a)} x_{i_1} x_{i_2} \cdots x_{i_p}, \tag{1.1}$$

where $R(w)$ denotes the set of reduced decompositions of w, defined as follows. Let adjacent transposition $s_i = (i, i+1) \in S_n,\ 1 \leq i \leq n-1$. Then a reduced decomposition of $\omega \in S_n$ is a sequence $(a_1, \cdots, a_p),\ 1 \leq a_i \leq n-1$ s.t. $\omega = s_{a_1} \cdots s_{a_p}$ where p is the length, the number of inversions in ω, i.e., $p = l(\omega) = |\{(ij) : i < j, w(i) > w(j)\}|$. Also for $a = (a_1, \cdots, a_p) \in R(\omega)$, we call the sequence $(i_1, \cdots, i_p) \in P^p$ (where $P = \{1, 2, \cdots\}$) to be a-compatible if

$$\begin{cases} i_1 \leq i_2 \leq \cdots \leq i_p \\ i_j \leq a_j \text{ for } 1 \leq j \leq p \\ i_j < i_{j+1} \text{ if } a_j < a_{j+1}. \end{cases}$$

The (finite) set of all "a-compatible" sequences is denoted by $K(a)$ [4].

An example of Schubert polynomial is $\mathfrak{S}_{(2431)} = x_1 x_2^2 x_3 + x_1^2 x_2 x_3$ (see Appendix A.1). As another example, if w is the transposition $(n, n+1)$, then $\mathfrak{S}_w = x_1 + \cdots + x_n$ (see Appendix A.3).

In particular all Schur polynomials (in a finite number of variables) are Schubert polynomials but Schubert polynomials form a bigger set of polynomials and they form a linear basis for $Z[x_1, x_2, \cdots, x_n]$.

Since the Schubert polynomials form a linear basis, any element of $Z[x_1, x_2, \cdots, x_n]$ can be written uniquely as a linear combination of Schubert polynomials. Therefore the product of any two Schubert polynomials $\mathfrak{S}_\beta$ and $\mathfrak{S}_\gamma$ is again an element of $Z[x_1, x_2, \cdots, x_n]$ and can be written as a linear combination of Schubert polynomials. Hence there are unique coefficients $c^\alpha_{\beta\gamma}$ such that $\mathfrak{S}_\beta \mathfrak{S}_\gamma = \sum_\alpha c^\alpha_{\beta\gamma} \mathfrak{S}_\alpha$, where $\alpha,\ \beta, \gamma$ are are elements of S_∞. These coefficients $c^\alpha_{\beta\gamma}$ are called the structure constants of Schubert polynomials and can be considered as a generalization of Littlewood-Richardson coefficients for Schur polynomials.

There does not exist a combinatorial rule for calculating structure constants of Schubert polynomials in the same vein as the Littlewood-Richardson rule and yet the Littlewood-Richardson coefficients are subset of Schubert structure constants and so mathematicians believe that such a combinatorial rule should exist and are working to find it.

This was the motivation for the introduction of the Fomin-Kirillov algebra (FK) in 1997 [14]. They wanted to study the structure constants of Schubert polynomials. We will explain the relationship with the Schubert polynomials below.

Fomin and Kirillov made their non-commutative algebra $FK(n)$ on generators $x_{ij} = -x_{ji}$, where $1 \leq i < j \leq n$, that satisfy the following relations [14]:

$$\begin{aligned} (i) &: x_{ij}^2 = 0,\ 1 \leq i < j \leq n; \\ (ii) &: x_{ij}x_{kl} - x_{kl}x_{ij} = 0;\ \text{for distinct } i,j,k,l \text{ such that } 1 \leq i,j,k < l \leq n; \\ (iii) &: x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0;\ \text{if } 1 \leq i < j < k \leq n; \\ (iii') &: x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk} = 0,\ \text{if } 1 \leq i < j < k \leq n. \end{aligned} \tag{1.2}$$

The 'free algebra' $F\langle x_{ij} : 1 \leq i < j \leq n\rangle$ generated by the generators x_{ij} (or in general any set of generators) is an algebra whose underlying vector space has a basis consisting of all the words on the generators x_{ij} (or in general any set of generators), with no relation among the generators. This algebra is equipped with concatenation of the words (monomials) as the multiplication operation and the linear extension of it to polynomials. Then $FK(n)$ is isomorphic to the free algebra modulo the ideal generated by the polynomials on the right hand side of the relations in Equation (1.2), that we call them the set of 2-terms in the sequel, i.e.,

$$\begin{aligned}
&(i): x_{ij}^2,\ 1 \leq i < j \leq n;\\
&(ii): x_{ij}x_{kl} - x_{kl}x_{ij};\ \text{for distinct } i,j,k,l \text{ such that } 1 \leq i,j,k < l \leq n;\\
&(iii): x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij};\ \text{if } 1 \leq i < j < k \leq n;\\
&(iii'): x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk},\ \text{if } 1 \leq i < j < k \leq n.
\end{aligned} \tag{1.3}$$

Fomin and Kirillov defined elements $\theta_1, \cdots, \theta_n$ inside $FK(n)$, that they called them Dunkl elements, by the formula $\theta_j = -\sum_{1 \leq i < j} x_{ij} + \sum_{j < k \leq n} x_{jk}$. They proved that these elements commute pairwise, showing that $FK(n)$ contains a sub-algebra isomorphic to the polynomial ring $Z[x_1, x_2, \cdots, x_n]$. They then realized the Schubert polynomials inside of $FK(n)$ as evaluation at the Dunkl elements $\mathfrak{S}_w(\theta_1, \cdots, \theta_n)$ [14]. They formulated their 'non-negativity conjecture' that says for any $w \in S_n$, $\mathfrak{S}_w(\theta_1, \cdots, \theta_n)$ can be written as a linear combination of monomials in generators of $FK(n)$, x_{ij}, for $i < j$, with non-negative integers coefficients [14].

If the 'non-negativity conjecture' is true, then the expression of the evaluation $\mathfrak{S}_w(\theta_1, \cdots, \theta_n)$ gives a combinatorial rule for the structure constants of Schubert polynomials. According to this rule for $\mathfrak{S}_u\mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w$, we will have the following structure constant c_{uv}^w [14]:

$$c_{uv}^w = \langle \text{coefficient of } w \text{ in } \mathfrak{S}_u(\theta)v \rangle \tag{1.4}$$

where the action of $\mathfrak{S}_u(\theta)$ on $v \in S_n$ is the Bruhat representation action defined by

$$x_{ij}v = \begin{cases} vs_{ij}, & \text{if } l(vs_{ij}) = l(v) + 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.5}$$

where s_{ij} denotes the transposition of i and j, and $l(v)$ denotes the length of a permutation v, i.e., the number of inversions in v.

Aside from the above original motivation for introducing $FK(n)$, this algebra has been studied widely for both its combinatorial and algebraic aspects [17], [5], [1], [3] [19]. In spite of this, there are still important questions regarding the structure of this algebra yet to be addressed. Among these questions are some graphical aspects and the dimension of this algebra. While the dimensions of $FK(n)$ for $n = 1, 2, 3, 4, 5$ are known, the dimension for $n \geq 6$ is not known, nor is it even known if this algebra is finite or infinite dimensional in this case [5].

In continuation of the research listed above, my research is another attempt to understand the structure of $FK(n)$ by finding the forms of the elements of Gröbner basis for the ideal I associated for $FK(n)$ and to study some interesting quotients of the algebra.

1.2 Targets of my work

One of the targets of this thesis is to study the Gröbner basis (with some specific monomial ordering) associated to the defining ideal of $FK(n)$ and prove that it consists of z-star polynomials, to be defined later in the below.

Another target is to study the character of some components of $FK(n)$ as a graded algebra, with different degrees; usual degree, permutation degree, set partition degree.

Another target is the study of a sub-algebra of $FK(n)$ associated to the sub-graph n-cycle of the complete graph on n vertexes.

To explain the above targets and why the Gröbner basis is important to us and why the structure of a Gröbner basis of the associated ideal for $FK(n)$ is the focus of my work, I should add the following:

As mentioned above, $FK(n)$ was defined as the quotient algebra of the free algebra generated by the generators x_{ij} over the ideal generated by the generators in Equation (1.3). The ideal may have different set of generators and the most useful ones are called Gröbner bases (GB). Gröbner basis is a useful notion in studying properties of algebras, especially quotient algebras, for which $FK(n)$ is an example.

As the history of Gröbner basis, this notion of Gröbner basis for an ideal originally was developed in 1965 by Bruno Buchberger, together with an algorithm known as 'Buchberger's algorithm', to compute it for commutative polynomials and commutative ideals [6], [7], but very soon after it was extended to the non-commutative setting for which $FK(n)$ is an example. In non-commutative case [19], the algorithm is more complexed and a little less known. Here, since $FK(n)$ is non-commutative, we will need to adapt the non-commutative algorithms as developed in [19].

Gröbner basis has many application in computer science, especially in computer algebra. The computation of a Gröbner basis is a main practical tool for solving systems of polynomial equations. It can be seen as a multivariate, nonlinear generalization of both Euclid's algorithm for computing polynomial greatest common divisors, and Gaussian elimination for linear systems.

Gröbner basis of the defining ideal of a quotient algebra can be used to deduce many important properties of the quotient space such as dimension when the quotient is finite. The thing is that the set of the words on generators of a quotient algebra that are not divisible by any leading monomial of the elements of the associated Gröbner basis form a linear basis for the quotient algebra, and therefore helps to find the dimension of the quotient algebra. This is why a Gröbner basis is important to us and why the structure of a Gröbner basis of the associated ideal for $FK(n)$ is the focus of my work.

Since Fomin-Kirillov algebra is a non-commutative algebra, i.e., for a and b generators of the algebra we have in general $ab \neq ba$, the order of generators in a monomial does matter, so we need to take care of this point when defining our monomial ordering.

The monomial ordering we use in this work is called graded lexicographic ordering (glex). In plain language it is an ordering similar to the ordering of the words in a dictionary and it is defined as follows.

Definition 1.2.1. *(see Def. 2.1.3) For monomials M_1 and M_2 we say*

$$M_1 <_{glex} M_2 \textit{ if } \begin{cases} degM_1 < degM_2, \textit{ or} \\ degM_1 = degM_2 \textit{ and } M_1 <_{lex} M_2, \end{cases}$$

where the lexicographic ordering (lex) of monomials that we use in this work is defined by first introducing a variable ordering by

$$x_{ij} > x_{kl} \textit{ if } \begin{cases} j < l, \textit{ or} \\ \\ j = l \textit{ and } i > k, \end{cases}$$

then with this variable ordering, the following rule completes the definition of our lexicographic monomial ordering. For monomials M_1 and M_2 of the same usual degree d, $M_1 <_{lex} M_2$ if the first variable of M_1 is less than the first variable of M_2. if the first k variables happen to be the same, then compare the $k+1$ st variables.

By a reduced Gröbner basis we mean that the coefficients of all the leading monomials of the elements are 1, and no monomial appearing in one element is divisible by the leading monomial of another element. A reduced Gröbner basis is unique, given a fixed monomial order.

We need to mention the definition of a polynomial called 'S-polynomial' described in non-commutative Buchberger theory [19].

Definition 1.2.2. *(see Def. 2.1.8) The S-polynomial between two polynomials*
For $g_i, g_j \in K\langle x_{ab} | 1 \leq a < b \leq n\rangle$ (for characteristic zero field K), let the leading monomial of g_i and g_j be $LM(g_i) = w_i$ and $LM(g_j) = w_j$ respectively, and $n_i = l(w_i)$, $n_j = l(w_j)$, where l stands for "length of". Then for a given positive integer d such that

max$(n_i, n_j) \leq d \leq n_i + n_j$*, we define the S-polynomial of* g_i *and* g_j *of degree* d *by*

$$S(g_i, g_j, d) = \begin{cases} g_i y - \frac{LC(g_i)}{LC(g_j)} u.g_j \;\; \textit{if} \; \exists\, u, v, y, \; \textit{s.t.}\; \omega_i = uv, \; \omega_j = vy \; \textit{and} \; l(v) = n_i + n_j - d, \\ 0 \quad \textit{otherwise}. \end{cases}$$

$$\overbrace{\underbrace{-\,-\,-\,-}_{u}\underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-}_{v}}^{w_i} \qquad\qquad LM(g_i) = w_i = u.v$$

$$\underbrace{\overbrace{-\,-\,-\,-\,-\,-\,-\,-\,-\,-}^{v}\overbrace{-\,-\,-}^{y}}_{w_j} \qquad LM(g_j) = w_j = v.y$$

$$\underbrace{\leftarrow -\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,- \rightarrow}_{d}$$

It is clear that in general $S(g_i, g_j, d) \neq S(g_j, g_i, d)$. It is also clear that the S-polynomial of two homogeneous polynomials is a homogeneous polynomial.

Computer calculations already have shown that, under our special lexicographical monomial ordering defined in Definition 1.2.1, the elements of the reduced Gröbner basis of the ideal of FK follow a graphical pattern, independent of the degree, that we interpret it as being 'z-star'. The definition of 'z-star' is stated below.

Definition 1.2.3. *A monomial is called a 'z-star' monomial if for a fixed z, all of the variables of the monomial are of the form* $x_{\alpha z}$*, where* $1 \leq \alpha < z$*. A homogeneous polynomial wherein all monomials are 'z-star' with the same z, is called a 'z-star' polynomial.*

Example 1.2.4. $x_{23}x_{13}x_{23} + x_{13}x_{23}x_{13}$ *is a 3-star polynomial, while* $x_{23}x_{12}x_{23} + x_{13}x_{23}x_{13}$ *is not a z-star polynomial.*

Materials in this thesis are organized as follow:

Non-commutative Buchberger criterion. In chapter 2, after reviewing Buchberger theory we adapt the commutative Buchberger criterion from Cox [8] to prove our following non-commutative Buchberger criterion.

Theorem 1.2.5 (see Theorem 2.2.7)**.** *Non-commutative Buchberger criterion.*
For an ideal $I \subset K\langle x_1, \cdots, x_n\rangle$, $G^{(k)}$ is the set of elements of deg $\leq k$ in the reduced Gröbner basis w.r.t. a glex monomial order, if and only if $\langle G^{(k)}\rangle =_{\leq k} I$; all elements $g \in G^{(k)}$ are of $deg \leq k$ and no monomial of them is divisible by any other leading term, and for any positive integer $d \leq k$ and for any pair (g_i, g_j) where $g_i, g_j \in G^{(k)}$ we have $\overline{S(g_i, g_j, d)}^{G^{(k)}} = 0$.

In the statement of the above theorem the sign $=_{\leq k}$ means that the equality is valid up to degree k. The reason we use this sign is the fact that our non-commutative Buchberger criterion works on a degree by degree basis. This theorem is applied in an algorithm to find the elements of the Gröbner basis degree by degree.

Symmetry Properties of Fomin-Kirillov Algebra. In Chapter 3, we study some symmetry properties of $FK(n)$. We show that the group S_n defines an action on $FK(n)$. We also define an anti-automorphism on $FK(n)$ based on reverse polynomial defined as

Definition 1.2.6. *(see Def. 3.1.4) Reverse word. Let $w = w_1w_2\cdots w_k$ be a word of degree k with $w_j = x_{i_j}$ for some $1 \leq i_j \leq n$. We call $\overleftarrow{w} = w_kw_{k-1}\cdots w_1$ the reverse word of w [1].*

Related to the above definition, in Chapter 3, we prove the following lemma.

Lemma 1.2.7. *(see Lemma 3.1.5) Let $S = \{x_{ij} : 1 \leq i < j \leq n\}$ be the set of generators of $FK(n)$, and let w be a word on the letters in S. Then we have an anti-automorphism $\phi : FK(n) \to FK(n)$ well defined by $\phi(w+I) = \overleftarrow{w} + I$ and linear extension of it to $FK(n)$, where $\overleftarrow{w}$ is the reverse word of ω.*

In Chapter 4 we prove the following propositions according to which for any n, degree 3 and degree 4 elements of $GB(n)$, the reduced Gröbner basis w.r.t the special monomial order defined in Definition 1.2.1, associated for $FK(n)$ are z-star monomials defined in Definition 1.2.3.

Proposition 1.2.8. *(see Proposition 4.1.1) Let $T^{(3)}(n)$ be the set of degree 3 elements of $GB(n)$, the reduced Gröbner basis with respect to the ordering defined in Definition 1.2.1 for the ideal of $FK(n)$, then*

$$T^{(3)}(n) = \{x_{bz}x_{az}x_{bz} + x_{az}x_{bz}x_{az} : 1 \leq a < b < z \leq n\}, \tag{1.6}$$

i.e., $T^{(3)}(n)$ consists of z-star polynomials for $3 \leq z \leq n$.

Proposition 1.2.9. *(see Proposition 4.2.1) Let $T^{(4)}(n)$ be the set of degree 4 elements of $GB(n)$, the reduced Gröbner basis with respect to the ordering defined in Definition 1.2.1, for the ideal of $FK(n)$, then*

$$\begin{aligned} T^{(4)}(n) =\{& x_{a_3z}x_{a_2z}x_{a_1z}x_{a_3z} + x_{a_2z}x_{a_1z}x_{a_3z}x_{a_2z} + x_{a_1z}x_{a_3z}x_{a_2z}x_{a_1z} \textit{ and} \\ & x_{a_3z}x_{a_1z}x_{a_2z}x_{a_3z} + x_{a_2z}x_{a_3z}x_{a_1z}x_{a_2z} + x_{a_1z}x_{a_2z}x_{a_3z}x_{a_1z}, \textit{ where} \\ & 1 \leq a_1 < a_2 < a_3 < z \leq n\}, \end{aligned} \tag{1.7}$$

i.e., $T^{(4)}(n)$ consists of z-star polynomials for $4 \leq z \leq n$.

In Chapter 5 we prove the following proposition, however before introducing the statement we need to mention that by $G^{(d)}(n)$ in the statement we mean the set of elements of degree $\leq d$ of $GB(n)$, the reduced Gröbner basis w.r.t the monomial order defined in 1.2.1. Also by '2-terms' we mean the set of terms in Equation (1.3).

Proposition 1.2.10. *(see Proposition 5.1.17) Let $g = \sum_{i=1}^{l} M_i = \underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$*

be a general degree d, z-star element of $G^{(d)}(n)$. Assume that all the elements of degree ≥ 3 in $G^{(d)}(n)$ are z-stars. Then $S(p,\ g,\ d+1)$, where p is a degree 2 element of the Gröbner basis, under reduction w.r.t. $G^{(d)}(n)$ reduces either to zero or a z-star element of $GB(n)$.

We also provide evidences in support of the following conjecture.

Conjecture 1.2.11. *(see Conjecture 5.2.1) Let g be an element of degree $d \geq 3$ in $G^{(d)}(n)$, the set of elements of the reduced Gröbner basis with degree $\leq d$, for the ideal associated to $FK(n)$ with respect to the ordering defined in Definition 1.2.1. Assume all of the elements of degree ≥ 3 in $G^{(d)}(n)$ are z-stars. Then for $p \in G^{(2)}(n)$, $S(g, p, d+1)$, w.r.t. $G^{(d)}(n)$ reduces to zero.*

On condition that the above conjecture holds we come up with the following Proposition.

Proposition 1.2.12. *(see Proposition 5.2.3) On condition that Conjecture 1.2.11 holds, let $G^{(d)}(n)$ be the set of elements of Gröbner basis of degree $\leq d$ for the ideal associated to $FK(n)$ with respect to the ordering defined in Definition 1.2.1. Then if all the elements of $G^{(d)}(n)$ of degree ≥ 3 are z-stars, same is true for $G^{(d+1)}(n)$.*

However Proposition 1.2.12 serves as an inductive step for Proposition 1.2.9 as the base of induction to prove the following theorem which is one of the goals of my project.

Theorem 1.2.13. *(see Theorem 5.2.4) On condition that Conjecture 1.2.11 holds, let $G^{(d)}(n)$ be the set of elements of degree $\leq d$ of $GB(n)$, the reduced Gröbner basis w.r.t. ordering defined in 1.2.1 for the ideal associated to $FK(n)$. Then all the elements of the $G^{(d)}(n)$ with degree ≥ 3 are z-star polynomials for $3 \leq z \leq n$.*

This result would yield important information about the elements of Gröbner basis. We hope that this condition of being 'z-star' on the elements of Gröbner basis could put a limitation on the maximum degree d of the elements of $FK(n)$ basis and/or gives a relation between maximum degree of $FK(n)$ basis and n and thereby provides a closer step from which one can investigate the problem of finding the dimension of $FK(n)$ for $n \geq 6$.

In Chapter 6 we study different degrees defined for $FK(n)$. We find the character of $FK(n)$ under symmetric group action for usual degree 2 with different set partition types. We prove the following proposition and its associated corollary;

Proposition 1.2.14. *(see Proposition 6.4.5) Let $FK^{(1)}(n)$ be degree one part of $FK(n)$. Then for $n \geq 3$, we have*

$$\begin{aligned} char_{S_n}[FK^{(1)}(n)] &= \chi_{n-1,1} + \chi_{n-2,1,1}. \\ F_{S_n}(char_{S_n}[FK^{(1)}(n)]) &= s_{n-1,1} + s_{n-2,1,1}. \end{aligned} \tag{1.8}$$

For Fomin-Kirillov algebra of general n with usual degree 2 and set partition degree type $q_2^2 q_1^{n-4}$ we found the character in the following proposition;

Proposition 1.2.15. *(see Proposition 6.5.2) Let $\sigma \in S_n$ is of type $(1^{a_1}, 2^{a_2}, \cdots, n^{a_n})$. Then the value of character of $FK^{(2)}_{q_2^2 q_1^{n-4}}(n)$ at σ is*

$$char_{S_n}[FK^{(2)}_{q_2^2 q_1^{n-4}}(n)](\sigma) = 3\binom{a_1}{4} + 3\binom{a_2}{2} - \binom{a_1}{2}a_2 - a_4. \tag{1.9}$$

In the following proposition we have the decomposition of the above character into irreducible characters of S_n.

Proposition 1.2.16. *(see Proposition 6.5.8) Let $FK^{(2)}_{q_2^2q_1^{n-4}}(n)$ be Fomin-Kirillov algebra of general n and of usual degree 2 and set partition degree type $q_2^2q_1^{n-4}$. Then we have the following decomposition into irreducible characters of S_n*

$$char_{S_n}[FK^{(2)}_{q_2^2q_1^{n-4}}(n)] = \chi_{n-2,2}+\chi_{n-3,2,1}+\chi_{n-3,1,1,1}+\chi_{n-4,2,2}+\chi_{n-4,1,1,1,1},\ n \geq 6. \quad (1.10)$$

While the above decomposition into irreducible characters for $n = 3,\ 4,\ 5$, do not relate to each other, but eventually become stable for enough big numbers $n \geq 6$. Similar is true for usual degree ≥ 2 and set partition types other than $q_2^2q_1^{n-4}$ for some enough big number for n.

In Chapter 7 we introduce a quotient of the algebra $FK(n)$ over the ideal generated by the missing edges in an n-cycle compared to the complete graph on n-vertexes associated to $FK(n)$. For this quotient algebra, denoted by $\overline{FK}_{C_n}(n)$, after having a couple of cases analyzed for finite n, we get into the general case of $\overline{FK}_{C_n}(n)$. We calculate Gröbner basis and the basis of this algebra and come to the conclusion that the dimension of this algebra equals the number of matchings in an n-cycle, i.e., the Lucas number, for $n \geq 3$. By definition a matching in a graph is a subset of the edges of the graph that do not have a common vertex. we came to the following Theorem.

Theorem 1.2.17. *(see Theorem 7.3.8) Dimension of $\overline{FK}_{C_n}(n)$ equals the number of matchings in an n-cycle C_n.*

As well the following Corollary

Corollary 1.2.18. *(see Corollary 7.3.9) Dimension of $\overline{FK}_{C_n}(n)$ equals Lucas number L_n.*

We also find the character of the dihedral group action of $\overline{FK}_{C_n}(n)$ for general n, and its decomposition into irreducible characters. Also we study the decomposition of the character in different degrees.

In Chapter 8 we discuss the commutative Formi-Kirillov algebra denoted $FK^c(n)$ and defined as follows.

Definition 1.2.19. *(see Def. 8.0.1) Commutative Fomin-Kirillov Algebra $FK^c(n)$ is defined by letting the generators of Fomin-Kirillov algebra commute with each other. [18]*

In this chapter we find Gröbner basis of the ideal generated by the relations of $FK^c(n)$, via which we find the basis of $FK^c(n)$ and prove the following proposition for the dimension of commutative Fomin-Kirillov algebra $FK^c(n)$.

Proposition 1.2.20. *(see Proposition 8.4.10)*

1. *There is a bijection between the basis of $FK^c(n)$ and the symmetric group S_n.*
2. *dim $(FK^c(n)) = n!$.*

This result previously was attributed by Fomin and Kirillov [14] to Varchenko by another method.

Chapter 2

Gröbner Basis

2.1 Non-commutative Buchberger theory

Buchberger theory was introduced in 1965 by Bruno Buchberger in his PhD thesis [6], [7]. Originally this was developed for commutative polynomials and commutative ideals. But very soon after it was extended to non-commutative setting [19].

Here, since the Fomin-Kirillov algebra is non-commutative, we will cover only the non-commutative version of Buchberger theory and refer the commutative one to any classic text.

A Gröbner basis in this theory, is a special generating set of an ideal in a polynomial ring over a field $K\langle x_1, x_2, \cdots, x_n\rangle$, where the variables $x_1, x_2, \cdots, x_n$ do not commute. A Gröbner basis allows many important properties of the ideal and the associated algebraic variety such as the dimension be derived easily.

We adapt the standard theory to a case where the Gröbner basis is computed degree by degree.

In the following, we introduce some basic notions of the non-commutative Buchberger theory specially those that are necessary to express the results of our work.

Definition 2.1.1. *A "monomial ordering" is a well ordering "$\geq$" on the set of monomials (i.e., any nonempty subset has a least element), with the condition that if $m \geq m'$ then $m_1 m m_2 \geq m_1 m' m_2$ for any two monomials m_1, m_2.*

Remark 2.1.2. *It is worth noting that when sorting the monomials in a polynomial, we put the bigger degree first as we want the leading term to be the biggest degree, otherwise, the reduction algorithm (yet to be defined later) may not stop.*

Different monomial orderings result in different Gröbner bases. The monomial ordering that we use in this work is a "graded lexicographic ordering" (glex) defined below.

Definition 2.1.3. *For monomials M_1 and M_2 in the polynomial ring with non commutative variables $x_{i,j}$ where $1 \leq i < j \leq n$, we say $M_1 <_{glex} M_2$ if*

$$\begin{cases} degM_1 < degM_2, \text{ or} \\ degM_1 = degM_2 \text{ and } M_1 <_{lex} M_2, \end{cases}$$

where the lexicographic ordering (lex) of monomials that we use in this work is defined by first *introducing a variable ordering by*

$$x_{ij} > x_{kl} \text{ if } \begin{cases} j < l, or \\ j = l \text{ and } i > k. \end{cases}$$

Then with this variable ordering, the following rule completes the definition of our lexicographic monomial ordering. For monomials M_1 and M_2 of the same usual degree d, $M_1 <_{lex} M_2$ if the first variable of M_1 is less than the first variable of M_2. If the first k variables happen to be the same, then compare the $k+1$ st variables.

Example 2.1.4. $x_{23}x_{13}x_{23} >_{glex} x_{14}x_{23}x_{13}$, $x_{23}x_{13}x_{23} >_{glex} x_{13}x_{23}x_{13}$, $x_{13}x_{23} <_{glex} x_{23}x_{13}x_{23}$.

Now for any fixed monomial order (as in Definition 2.1.3), the monomial that comes first in a polynomial P, is called "leading monomial of P", denoted by $LM(P)$, and the coefficient of the leading monomial is called "leading coefficient", denoted by $LC(P)$. Then the leading term of P is $LT(P) = LC(P)LM(P)$.

Definition 2.1.5. *For monomials $M = x_{i_1}x_{i_2}\cdots x_{i_l}$ and $N = x_{j_1}x_{j_2}\cdots x_{j_k}$ in non-commuting variables $x_i,\ i = 1, 2, \cdots$, we say M is divisible by N if $k \leq l$ and for some positive integer m we have $x_{j_1}x_{j_2}\cdots x_{j_k} = x_{i_m}x_{i_{m+1}}\cdots x_{i_{m+k-1}}$.*

Definition 2.1.6. [13], [8]. *A Gröbner basis for an ideal I in the polynomial ring $K\langle x_1, \cdots, x_n\rangle$ is a set G of generators for I whose leading terms also generate the ideal generated by all leading terms in I, denoted by $\langle LT(I)\rangle$, i.e.,*

$$G \text{ is a Gröbner basis if } I = \langle g : g \in G\rangle \text{ and } \langle LT(I)\rangle = \langle LT(g) : g \in G\rangle. \tag{2.1}$$

A finite Gröbner basis is said to be a 'reduced Gröbner basis' if no monomial appearing in *an element is divisible by any leading monomial of another element, and if all the leading coefficients are* 1*. A 'reduced Gröbner basis' is unique given a fixed monomial ordering.*

It is worth mentioning that every element of I can be written in terms of elements of its Gröbner basis up to some degree, in the sense that if $f \in \langle g_i : g_i \in G^{(k)} \rangle$, where $G^{(k)}$ is the set of elements of degree $\leq k$ in Gröbner basis , then $f = \sum_{g_i \in G^{(k)}} h_i g_i h'_i$, where h_i and h'_i are polynomials in $K\langle x_1, \cdots, x_n \rangle$. This is what we mean by the terminology "basis" in the expression " Gröbner basis", and not in the sense of say a linear basis of a vector space.

2.1.1 S-polynomial

To define the S-polynomial of two polynomials, we need to introduce the notion of "overlap" of two monomials as follows.

Definition 2.1.7. *Overlap of two monomials. Let* $M_1 = x_{i_1} x_{i_2} \cdots x_{i_n}$ *and* $M_2 = x_{j_1} x_{j_2} \cdots x_{j_m}$ *be two monomials and let given positive integer* d *such that* $max(n, m) \leq d \leq n + m$*. If* $x_{i_{n-s+1}} \cdots x_{i_n} = x_{j_1} x_{j_2} \cdots x_{j_s}$*, then we call the monomial* $v = x_{j_1} x_{j_2} \cdots x_{j_s}$ *the "overlap" of* M_1 *and* M_2 *with respect to* d*, and* $l(v) = m + n - d$*, the length of the overlap, which clearly equals* s*.*

Definition 2.1.8. *S-polynomial between two polynomials*
For $g_i, g_j \in K\langle x_{ab} | 1 \leq a < b \leq n \rangle$ *(for characteristic zero field* K*), let the leading monomials of* g_i *and* g_j *be* $LM(g_i) = w_i$ *and* $LM(g_j) = w_j$ *respectively, and* $n_i = l(w_i)$*,* $n_j = l(w_j)$*, where* l *stands for "length of". Then for a given positive integer* d *such that*

max$(n_i, n_j) \leq d \leq n_i + n_j$*, we define the* S*-polynomial of* g_i *and* g_j *of degree* d *by*

$$S(g_i, g_j, d) = \begin{cases} g_i y - \frac{LC(g_i)}{LC(g_j)} u.g_j & \textit{if } \exists\, u, v, y, \textit{ s.t. } \omega_i = uv,\ \omega_j = vy \textit{ and } l(v) = n_i + n_j - d, \\ 0 \quad \textit{otherwise}. \end{cases}$$

$$\overbrace{\underbrace{---}_{u}\underbrace{----------}_{v}}^{w_i} \qquad LM(g_i) = w_i = u.v$$

$$\underbrace{\overbrace{----------}^{v}\overbrace{---}^{y}}_{w_j} \qquad LM(g_j) = w_j = v.y$$

$$\leftarrow ------ d -------- \rightarrow \qquad . \tag{2.2}$$

It is clear that in general $S(g_i, g_j, d) \neq S(g_j, g_i, d)$.

2.1.2 Reduction of a Polynomial

Definition 2.1.9. *(Reduction of a polynomial). For a polynomial* $P = c_1w_1 + c_2w_2 + \cdots$*, the reduction of* P *with respect to a list of polynomials* G *in some order, is defined explicitly by the following algorithm.*

```
INPUT: Polynomial $P$, List $G = [g_1, \cdots, g_t]$
OUTPUT: Reduction of P, w.r.t. list $G = [g_1, \cdots, g_t]$
REPEAT
$Q := P$
```

```
FOR $P = c_1w_1 + c_2w_2 + \cdots + c_lw_l$, DO
Find the first $i$, index of $w_i$, and for a fixed $i$
find the smallest $k$, index of $g_k$, for fixed $i,k$ find
the shortest $u$ such that
$w_i = uLM(g_k)v$ for $g_k \in G$.
$P := P - \frac{c_i}{LC(g_k)} ug_kv$
UNTIL $Q = P$
OUTPUT: $P$
```

Remark 2.1.10. *Since our ordering in this work is glex, in the above algorithm, the polynomial* $P = c_1w_1 + c_2w_2 + \cdots + c_lw_l$ *should be in glex order.*

Remark 2.1.11. *(why the algorithm terminates) The input polynomial* P *is sorted by 'the bigger term comes first' under glex monomial ordering (a well ordering), so there is a biggest summand as well as a smallest one. In each FOR-DO cycle of the algorithm, the first (the biggest) term divisible by* g_k *for some* k *is deleted (in a unique way). Since there is a finite number of* ω_i*'s, as the number of possible monomials smaller than any fixed monomial is finite, the algorithm ends up with zero or a polynomial containing no monomial divisible by* $LM(g_k)$ *for any* k*.*

The process of reduction is also called multivariate division or "normal form" computation. The result of the process of reduction of P with respect to a list G is denoted by $\overline{P}^G$.

2.2 Non-commutative Buchberger criterion

In this section we will prove a theorem called non-commutative Buchberger criterion, which results in an algorithm for finding the non-commutative reduced Gröbner basis w.r.t a monomial order for an ideal (we adapt the commutative case from Cox [8] to prove our following non-commutative Buchberger criterion). However before getting into that, a few lemmas/corollaries are in order.

Lemma 2.2.1. *For a glex monomial ordering, let $\{f_1, \cdots, f_s\}$ be a set of polynomials in $K\langle x_1, x_2, \cdots, x_n\rangle$ (for a characteristic zero field K), with $LT(f_i)$ equal to γ for all i. Also suppose we have a linear combination $\sum_{j=1}^{s} c_j f_j$, such that $LT(\sum_{j=1}^{s} c_j f_j) <_{glex} \gamma$, where $c_j \in K$. Then $\sum_{j=1}^{s} c_j f_j$ can be written as a linear combination, with coefficients in K, of S-polynomials $S(f_j, f_k, d)$ for $1 \leq j, k \leq s$, and positive integers d such that for each $S(f_j, f_k, d)$ we have $LT[S(f_j, f_k, d)] <_{glex} \gamma$.*

Proof. Let $d_j = LC(f_j)$, so that $c_j d_j$ is the leading coefficient of $c_j f_j$. Since by hypothesis we have $LT \sum_{j=1}^{s} c_j f_j <_{glex} \gamma$, some cancellation must have been happened among the leading terms in the sum $\sum_{j=1}^{s} c_j f_j$ (otherwise, no cancellation among the leading terms in the sum implies that $LT \sum_{j=1}^{s} c_j f_j = c\gamma$, for a constant c), so the coefficients sum to zero, i.e., we have $\sum_{j=1}^{s} c_j d_j = 0$.

We define polynomial $p_j = f_j / d_j$, so p_j has leading coefficient 1. Consider the telescoping sum;

$$\begin{aligned}\sum_{j=1}^{s} c_j f_j = \sum_{j=1}^{s} c_j d_j p_j = c_1 d_1 (p_1 - p_2) + (c_1 d_1 + c_2 d_2)(p_2 - p_3) + \cdots + (c_1 d_1 + \\ + \cdots + c_{s-1} d_{s-1})(p_{s-1} - p_s) + (c_1 d_1 + \cdots + c_s d_s) p_s.\end{aligned} \tag{2.3}$$

Now the fact that $LT(f_i) = LT(f_j)$ for all pairs (i, j) implies that $LM(f_i) = LM(f_j)$ for all pairs (i, j).

In the S-polynomial in (2.2), by substituting f_i and f_j for g_i and g_j respectively, we will have $S(f_i, f_j, d) = f_i y - \frac{LC(f_i)}{LC(f_j)} u f_j$, which for specific values $y = 1$ and $u = 1$ reduces to

$$S(f_i, f_j, d) = f_i - \frac{LC(f_i)}{LC(f_j)} f_j = LC(f_i)(\frac{f_i}{LC(f_i)} - \frac{f_j}{LC(f_j)}) = d_i(\frac{f_i}{d_i} - \frac{f_j}{d_j}) = d_i(p_i - p_j).$$

Therefore we have

$$S(f_i, f_j, d) = d_i(p_i - p_j) \tag{2.4}$$

Here the degree d in $S(f_i, f_j, d)$ is justified, as $p_i - p_j$ has degree d, the common degree of $LT(f_i)$ for all i. Substituting the above equation into Equation (2.3) and considering the fact that $\sum_{j=1}^{s} c_j d_j = 0$, which results in vanishing the last term in Equation 2.3, our telescoping sum of Equation 2.3 reduces to

$$\begin{aligned}\sum_{j=1}^{s} c_j f_j =& c_1 d_1 S(f_1, f_2, d)/d_1 + (c_1 d_1 + c_2 d_2) S(f_2, f_3, d)/d_2 + \cdots + \\ & + (c_1 d_1 + \cdots + c_{s-1} d_{s-1}) S(f_{s-1}, f_s, d)/d_{s-1} \\ =& c_1 S(f_1, f_2, d) + (c_1 \frac{d_1}{d_2} + c_2) S(f_2, f_3, d) + (c_1 \frac{d_1}{d_3} + c_2 \frac{d_2}{d_3} + c_3) S(f_3, f_4, d) + \cdots \\ & \cdots + (c_1 \frac{d_1}{d_{s-1}} + c_2 \frac{d_2}{d_{s-1}} + \cdots c_{s-2} \frac{d_{s-2}}{d_{s-1}} + c_{s-1}) S(f_{s-1}, f_s, d)\end{aligned} \tag{2.5}$$

which is a linear combination of S-polynomials $S(f_t, f_{t+1}, d)$ for $t = 1, \cdots, s-1$.

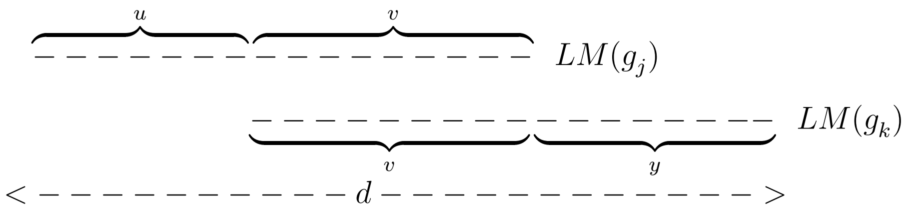
Since the leading monomials of f_i and f_j are equal, the leading monomials of p_i and p_j are equal. Therefore the leading monomials are canceled in $p_i - p_j$ and so from 2.4 we have $LT(S(f_i, f_j, d)) = LT(d_i(p_i - p_j)) <_{glex} LT(d_i p_i) = LT(f_i)$. Hence $LT(S(f_i, f_j, d)) <_{glex} LT(f_i)$, i.e., $LT[S(f_j, f_k, d)] <_{glex} \gamma$. This completes the proof. □

Lemma 2.2.2. *Let monomials $\alpha, \beta, \gamma, \eta$ and homogeneous polynomials $g_j, g_k \in K\langle x_1, x_2, \cdots, x_n\rangle$. Let $S(g_j, g_k, d) \neq 0$ for some positive integer d, and let $d' = d + l(\alpha) + l(\beta)$, where $l(\alpha) = deg(\alpha)$ and $l(\beta) = deg(\beta)$. Then*

$$\begin{aligned} S(\alpha g_j, g_k\beta, d') &= \alpha S(g_j, g_k, d)\beta, \\ S(g_j\gamma, \eta g_k, d) &= S(g_j, g_k, d), \ \textit{or} \ S(g_j\gamma, \eta g_k, d) = 0. \\ S(\alpha g_j\gamma, \eta g_k\beta, d') &= \alpha S(g_j, g_k, d)\beta, \ \textit{or} \ S(\alpha g_j\gamma, \eta g_k\beta, d') = 0 \end{aligned} \tag{2.6}$$

Proof. Without loss of generality consider $LC(g_j) = LC(g_k) = 1$. $S(g_j, g_k, d) \neq 0$ implies that there is a monomial v (an overlap), and monomials u and y as in the following diagram, such that

$$LM(g_j) = uv, \ LM(g_k) = vy, \ \text{and } S(g_j, g_k, d) = ug_k - g_jy. \tag{2.7}$$



In the diagram above the overlap between $LM(g_j)$ and $LM(g_k)$ is the monomial v. By cutting the overlap v from the l.h.s of $LM(g_k)$ we are left with a monomial that we call it the right margin of the overlap or just right margin. Here according to the diagram above we have $y =$ the right margin. Similarly by cutting the overlap v from the r.h.s of $LM(g_j)$

we are left with a monomial that we call it the left margin of the overlap or just the left margin. Here in the diagram above $u =$ the left margin. Then we can read the S-polynomial $S(g_j, g_k, d) = ug_k - g_j y$ in (2.7), as

$$S(g_j, g_k, d) = ug_k - g_j y = \text{(the left margin)} \times g_k - g_j \times \text{(the right margin)} \qquad (2.8)$$

Now for $S(\alpha g_j, g_k\beta, d'),\ d' = d + l(\alpha) + l(\beta)$, with the depicted overlap in the diagram below, we have the same overlap v as in the above, however now with right margin $y\beta$ and left margin αu, as well $g_j,\ g_k$ to be changed to $\alpha g_j,\ g_k\beta$ respectively . Therefore by (2.7) we have

$$S(\alpha g_j, g_k\beta, d') = (\alpha u)(g_k\beta) - (\alpha g_j)(y\beta) = \alpha(ug_k - g_j y)\beta = \alpha S(g_j, g_k, d)\beta.$$

Hence we have

$$S(\alpha g_j, g_k\beta, d') = \alpha S(g_j, g_k, d)\beta,\ d' = d + l(\alpha) + l(\beta).$$

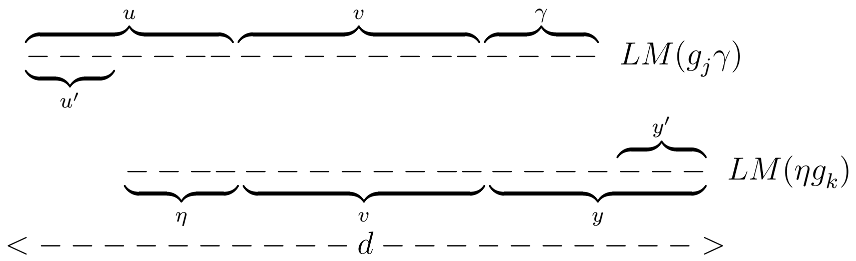
$$\begin{array}{l}
\qquad\quad < - - - - - - - - - - - - - d - - - - - - - - > \\
\overbrace{- - - -}^{\alpha}\overbrace{- - - - - - - - - -}^{u}\overbrace{- - - - - - - - - - -}^{v}\ LM(\alpha g_j) \\
\qquad\qquad\qquad\qquad\qquad\ \underbrace{- - - - - - - - - - -}_{v}\underbrace{- - - - - - -}_{y}\underbrace{- - - - -}_{\beta}\ LM(g_k\beta) \\
< - - - - - - - - - - - - - - - - - d' - - - - - - - - - - - >
\end{array}$$

For $S(g_j\gamma, \eta g_k, d)$ according to the diagram below, if $u = u'\eta$ and $y = \gamma y'$, then we have an overlap $\eta v\gamma$. Then with respect to the same d, we have

$$S(g_j\gamma, \eta g_k, d) = u'(\eta g_k) - (g_j\gamma)y' = (u'\eta)g_k - g_j(\gamma y') = ug_k - g_j y = S(g_j, g_k,\ d).$$

Otherwise we have no overlap and so zero S-polynomial. Hence

$$S(g_j\gamma, \eta g_k, d) = \begin{cases} S(g_j, g_k, d) & \text{if } u = u'\eta \text{ and } y = \gamma y', \\ 0 & \text{otherwise.} \end{cases}$$



Similar reasoning to the above shows that if $u = u'\eta$ and $y = \gamma y'$, then for $d' = d + l(\alpha) + l(\beta)$, we will have the following result;

$$S(\alpha g_j\gamma, \eta g_k\beta, d') = \alpha S(g_j\gamma, \eta g_k, d)\beta = \alpha S(g_j, g_k, d)\beta.$$

Otherwise we have no overlap, so we will have zero S-polynomial.

Hence we have

$$S(\alpha g_j\gamma, \eta g_k\beta, d') = \begin{cases} \alpha S(g_j, g_k, d)\beta & \text{if } u = u'\eta \text{ and } y = \gamma y', \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. □

Remark 2.2.3. *It is possible to set up Buchberger theory to be done degree by degree, for this, many statements/equations are regarded up to a fixed degree. In the sequel we emphasize the up to degree validity of expressions depending on cases either by words or by conditional* *equality sign ($=_{\leq}$) instead of ($=$) sign when necessary. The sign ($=_{\leq k}$) means that the equality is valid if the terms involved are of deg $\leq k$ or the equality is valid up to degree k, but could be invalid beyond.*

Lemma 2.2.4. *Let G be a Gröbner basis, with respect to a glex monomial order for ideal $I \subset K\langle x_1, x_2, \cdots, x_n\rangle$ and $G^{(k)}$ the set of elements of deg $\leq k$ in G. Then*

1. $\langle G^{(k)}\rangle =_{\leq k} I$.
2. $\langle LT(G^{(k)})\rangle =_{\leq k} \langle LT(I)\rangle$.

Proof. 1. For $f \in K\langle x_1, \cdots, x_n\rangle$ and $deg(f) \leq k$, we show that $f \in I \iff f \in \langle G^{(k)}\rangle$.

($\Leftarrow$) $f \in \langle G^{(k)}\rangle \to f = \sum_{g_i \in G^{(k)}} h_i g_i h_i' \xrightarrow{G^{(k)} \subset G} f = \sum_{g_i \in G} h_i g_i h_i' \in \langle G\rangle = I$, hence $f \in \langle G^{(k)}\rangle \implies f \in I$.

($\Rightarrow$) Let $f \in I$, and $deg(f) \leq k$. we need to show that $f \in \langle G^{(k)}\rangle$.

$f \in I = \langle G\rangle$, so

$$f = \sum_{g_i \in G} l_i g_i l_i', \tag{2.9}$$

where $l_i, l_i' \in K\langle x_1, x_2, \cdots, x_n\rangle$. Then $f \in I = \langle G\rangle \to LT(f) \in \langle LT(I)\rangle = \langle LT(G)\rangle$ (as G is Gröbner basis), so we have

$$LT(f) = \sum_{g_i \in G} h_i LT(g_i) h_i'. \tag{2.10}$$

where $deg(LT(f)) \leq k$ by hypothesis. For the summands $h_i LT(g_i) h_i'$ in (2.10), we consider the following cases.

- If $deg(h_i LT(g_i) h_i') \leq k$ for all i, then $degLT(g_i) \leq k$ for all i, therefore we have $g_i \in G^{(k)}$ for all i, which turns (2.9) into $f = \sum_{g_i \in G^{(k)}} l_i g_i l_i'$, i.e., $f \in \langle G^{(k)}\rangle$.
- Otherwise if $deg(h_i LT(g_i) h_i') > k$ for some i s, then there must be an integer

$k_1 > k$ where $deg(h_i LT(g_i)h_i') \leq k_1$, for all i. Then we rewrite (2.10) as

$$\underbrace{LT(f)}_{\leq k} = \sum_{g_i \in G} \underbrace{h_i LT(g_i) h_i'}_{deg \leq k_1}, \ k < k_1. \tag{2.11}$$

We then isolate the terms of the highest degrees in h_i and h_i' in (2.11) by defining

$$h_i = h_i^{(1)} + h_i^{(2)}, \text{ where } \begin{cases} h_i^{(1)} = \text{sum of the monomials in } h_i \text{ of highest degree,} \\ h_i^{(2)} = \text{the rest of } h_i. \end{cases}$$

and

$$h_i' = h_i^{'(1)} + h_i^{'(2)}, \text{ where } \begin{cases} h_i^{'(1)} = \text{sum of the monomials in } h_i' \text{ of highest degree,} \\ h_i^{'(2)} = \text{the rest of } h_i'. \end{cases}$$

We then substitute the above into (2.11). Therefore we have

$$\underbrace{LT(f)}_{\leq k} = \sum_{g_i \in G} \underbrace{h_i LT(g_i) h_i'}_{\leq k_1} = \sum_{g_i \in G} (h_i^1 + h_i^2) LT(g_i)(h_i^{'1} + h_i^{'2}),$$

that results in the following equation:

$$\begin{aligned} \underbrace{LT(f)}_{\leq k} &= \sum_{g_i \in G} \underbrace{h_i^1 LT(g_i) h_i^{'1}}_{deg = k_1} \\ &\quad + \sum_{g_i \in G} \underbrace{h_i^1 LT(g_i) h_i^{'2}}_{deg \leq k_1 - 1} + \sum_{g_i \in G} \underbrace{h_i^2 LT(g_i) h_i^{'1}}_{deg \leq k_1 - 1} + \sum_{g_i \in G} \underbrace{h_i^2 LT(g_i) h_i^{'2}}_{deg \leq k_1 - 2} \end{aligned}$$

In the above however, on the l.h.s., $deg(LT(f)) \leq k$, so on the r.h.s. the first sum has to be zero as $k < k_1$ (cancellation of the leading terms). So we have

$$\underbrace{LT(f)}_{\leq k} = \sum_{g_i \in G} \underbrace{h_i^1 LT(g_i) h_i^{'2}}_{deg \leq k_1 - 1} + \sum_{g_i \in G} \underbrace{h_i^2 LT(g_i) h_i^{'1}}_{deg \leq k_1 - 1} + \sum_{g_i \in G} \underbrace{h_i^2 LT(g_i) h_i^{'2}}_{deg \leq k_1 - 2}. \tag{2.12}$$

Now in this equation, if $k_1 - 1 = k$, we have 3 sums on the right hand side, the summands of each are of degree $\leq k$, which implies $deg(LT(g_i)) \leq k$ (as the left hand side is of degree $\leq k$). Otherwise if $k_1 - 1 > k$, we repeat this process of isolating the highest degree terms.

We see that each time we repeat the isolation of the highest degree monomials, it reduces the degrees of the summands at least by one unit. Since the number of monomials of degree less than a finite fixed degree is finite, by repeating the isolation of the leading terms, we eventually end up with $k_1 - j = k$ for some j.

Then we are left with a number of sums with summands each of degree $\leq k$, which implies $degLT(g_i) \leq k \ \forall i$ as before, i.e., $g_i \in G^{(k)}$. This results in rewriting Equation (2.9) as $f = \sum_{g_i \in G^{(k)}} l_i g_i l'_i$, which implies $f \in \langle G^{(k)} \rangle$. This completes the proof of part 1.

2. For $f \in K\langle x_1, \cdots, x_n \rangle$, $deg(f) \leq k$, we show $f \in \langle LT(I) \rangle \iff f \in \langle LT(G^{(k)}) \rangle$.

$$(\Leftarrow)\ f \in \langle LT(G^{(k)}) \rangle \to f = \sum_{g_i \in G^{(k)}} h_i LT(g_i) h'_i \xrightarrow{G^{(k)} \subset G} f = \sum_{g_i \in G} h_i LT(g_i) h'_i.$$

So $f \in \langle LT(G) \rangle = \langle LT(I) \rangle$ (as G is Gröbner basis for I). Hence we have shown that

$$f \in \langle LT(G^{(k)}) \rangle \implies f \in \langle LT(I) \rangle.$$

$(\Rightarrow)$ $f \in \langle LT(I) \rangle = \langle LT(G) \rangle$, therefore we have

$$f = \sum_{g_i \in G} h_i LT(g_i) h'_i. \tag{2.13}$$

Following the same steps we took for the r.h.s. of Equation (2.10) in the proof of part 1, second item, we come up with the same result $g_i \in G^{(k)}$, which turns (2.9) into

$f = \sum_{g_i \in G^{(k)}} h_i LT(g_i) h'_i$, which implies $f \in \langle LT(G^{(k)})\rangle$. Hence we have shown

$$f \in \langle LT(I)\rangle \Rightarrow f \in \langle LT(G^{(k)})\rangle.$$

This completes the proof. $\square$

Lemma 2.2.5. *Let $G^{(k)} = \{g_1, g_2 \cdots, g_t\}$ be the set of elements of deg $\leq k$ in the reduced Gröbner basis w.r.t. a glex monomial order, for an ideal $I \subset K\langle x_1, \cdots x_n\rangle$, and $G^{(k)}$ contains the generators of deg $\leq k$ of I. Then for f of deg $\leq k$ in $K\langle x_1, x_2, \cdots, x_n\rangle$ there exists a unique r of deg $\leq k$ in $K\langle x_1, x_2, \cdots, x_n\rangle$ with the following properties:*

1. *No term of r is divisible by any element of the set $\{LT(g_1), LT(g_2), \cdots, LT(g_t)\}$,*
2. *There exists an element g of deg $\leq k$ in I such that $f = g + r$.*

Proof. With our monomial order, let the monomials w_i of deg $\leq k$ and the polynomial $f = c_1w_1 + c_2w_2 + \cdots + c_sw_s$ be in $K\langle x_1, x_2, \cdots, x_n\rangle$ and let $g_i \in G^{(k)}$.
Here we recall the reduction algorithm of f, w.r.t. the list $[g_1, g_2, \cdots, g_t]$, introduced in Definition 2.1.9, according to which for $f = c_1w_1 + c_2w_2 + \cdots + c_sw_s$, we find first j, the index of w, and for fixed j we find the smallest l, the index of the elements in the list, and for our fixed j and l we find the shortest u such that $w_j = uLM(g_l)v$ for g_l in list $[g_1, g_2, \cdots, g_t]$. Then $f := f - \frac{c_j}{LC(g_l)} u g_l v$. Therefore to each term c_jw_j of f there are associated $\frac{c_j}{LC(g_l)}u,\ g_l$ and v that, in order to adapt to the notation of this section, by change of index, we denote them simply by u_i, g_i and v_i respectively, such that $u_i LT(g_i) v_i = c_i w_i$.

Therefore in each step, the algorithm drops from polynomial f a term c_iw_i, if it is divisible by leading term of an element in list $G^{(k)}$, and adds lower monomials w.r.t. the monomial order. Since the algorithm terminates (as explained in Remark 2.1.11), eventually we are left with an expression for polynomial f with none of its monomials divisible by the leading term of any element in list $G^{(k)} = [g_1, \cdots, g_t]$. Hence we are left with the reduction of f w.r.t. the list $G^{(k)} = [g_1, \cdots, g_t]$, i.e. we are left with $\overline{f}^{G^{(k)}} = f - \sum_{i, s.t.\ LT(g_i)|w_i \text{ for some i}} u_i g_i v_i$ which is clearly of deg $\leq k$. Therefore we have

$$f = \sum_{i, s.t.\ LT(g_i)|w_i \text{ for some i}} u_i g_i v_i + \overline{f}^{G^{(k)}} \tag{2.14}$$

where no term in $\overline{f}^{G^{(k)}}$ is divisible by $LT(g_i)$ for any $g_i \in G^{(k)}$. Hence part 1 of the lemma is satisfied if we let $r = \overline{f}^{G^{(k)}}$.

Part 2 of the lemma is satisfied by letting $g = \sum_{i, s.t.\ LT(g_i)|w_i \text{ for some i}} u_i g_i v_i$, of degree $\leq k$ from Equation (2.14). This proves the existence of g.

We prove the uniqueness of r by contradiction. If r is not unique, then there is another polynomial $r' \neq r$ of deg $\leq k$ in $K\langle x_1, x_2, \cdots, x_n\rangle$ with no monomial divisible by $LT(g_i)$ for any $g_i \in G^{(k)}$, such that there is a deg $\leq k$ element $g' \in I$ satisfying $f = g' + r'$. Then we have

$$g + r = g' + r' \tag{2.15}$$

Then

$$r - r' = g' - g \in I \text{ as } g, g' \in I. \tag{2.16}$$

Then $r - r' \in I \implies LT(r - r') \in \langle LT(I)\rangle =_{\leq k} \langle LT(G^{(k)})\rangle$ (by Lemma 2.2.4, as $G^{(k)} = \{g_1, \cdots, g_t\}$ is the set of elements of deg $\leq k$ in the reduced gröbner basis for ideal I). Therefore $LT(r - r')$ is divisible by $LT(g_i)$ for some $g_i \in G^{(k)}$, which is impossible as none of the terms in r and r' are to be divisible by $LT(g_i)$ for any $g_i \in G^{(k)}$ by the first part of the Lemma. Hence $r - r'$ must be zero, i.e. $r = r'$, contradiction!. Hence r is unique. This completes the proof. □

Corollary 2.2.6. *Let $G^{(k)} = \{g_1, \cdots, g_t\}$ be the set of elements of deg $\leq k$ in the reduced Gröbner basis with respect to a glex monomial order, for ideal $I \subset K\langle x_1, \cdots x_n\rangle$. Then for f of deg $\leq k$ in $K\langle x_1, \cdots, x_n\rangle$, we have $f \in I$, if and only if $\overline{f}^{G^{(k)}} = 0$.*

Proof. ($\Rightarrow$) If $\overline{f}^{G^{(k)}} = 0$, then from (2.14), $f = \sum_{i,\ s.t.\ LT(g_i)|w_i \text{ for some i}} u_i g_i v_i \in I$.
($\Leftarrow$) Conversely given $f \in I$, then $f = f + 0$ satisfies the two conditions of the lemma (2.2.5). It follows that 0 is $\overline{f}^{G^{(k)}}$. This completes the proof. □

Theorem 2.2.7. *Non-commutative Buchberger criterion.*
For an ideal $I \subset K\langle x_1, \cdots, x_n\rangle$, $G^{(k)}$ is the set of elements of deg $\leq k$ in the reduced Gröbner basis w.r.t. a glex monomial order, if and only if $\langle G^{(k)}\rangle =_{\leq k} I$; all elements $g \in G^{(k)}$ are of $deg \leq k$ and no monomial of them is divisible by any other leading term, and for any positive integer $d \leq k$ and for any pair (g_i, g_j) where $g_i, g_j \in G^{(k)}$ we have $\overline{S(g_i, g_j, d)}^{G^{(k)}} = 0$.

Proof. ($\Rightarrow$) If $G^{(k)}$ is the set of elements of deg $\leq k$ in the reduced Gröbner basis for I, then by Lemma 2.2.4 we have $\langle G^{(k)}\rangle =_{\leq k} I$. Also for any pair (g_i, g_j) where $g_i, g_j \in G^{(k)}$, and for any positive integer $d \leq k$, we have $\deg(S(g_i, g_j, d)) = d \leq k$ and it belongs to I (by construction of S-polynomial). Hence by corollary 2.2.6, we have $\overline{S(g_i, g_j, d)}^{G^{(k)}} = 0$.
($\Leftarrow$) Let $\langle G^{(k)}\rangle =_{\leq k} I$ and $\overline{S(g_i, g_j, d)}^{G^{(k)}} = 0$ for all pairs (g_i, g_j) where $g_i, g_j \in G^{(k)}$ and for any positive integer $d \leq k$. For polynomial $f \in \langle G^{(k)}\rangle$, we need to prove that $LT(f) \in \langle LT(G^{(k)})\rangle$.
Strategy of the proof:

We have $\langle G^{(k)} \rangle =_{\leq k} I$, i.e., for a polynomial f of deg $\leq k$ in I we have $f \in \langle G^{(k)} \rangle$, i.e., there are polynomials h_i and $h'_i \in K\langle x_1, x_2, \cdots, x_n \rangle$ such that

$$f = \sum_{g_i \in G^{(k)}} h_i g_i h'_i, \tag{2.17}$$

where

$$LT(f) \leq_{glex} LT(h_i g_i h'_i) \text{ for some } g_i \in G^{(k)}. \tag{2.18}$$

The reason for the inequality in the above equation is the idea that the sum in (2.17) may undergo cancellation of leading terms, otherwise if all the leading terms are different, the equality occurs. Equation (2.17) to (2.18) satisfies the hypothesis of Lemma 2.2.1 except for the equality in Equation (2.18). Therefore if equality does not occur in Equation (2.18), then some cancellation among the leading terms in the sum in Equation (2.17) must have occurred.

Since $LT(f) <_{glex} LT(h_i g_i h'_i)$ for some $g_i \in G^{(k)}$, by Lemma 2.2.1, we can rewrite the right hand side of Equation (2.17) as a linear combination $f = \sum_{j,l} c_{jl} S(h_j g_j h'_j, h_l g_l h'_l, d')$, with $LT[S(h_j g_j h'_j, h_l g_l h'_l, d')] \leq_{glex} LT(f) \leq_{glex} LT(h_i g_i h'_i)$ for some $g_i \in G^{(k)}$ by our graded lexicographical monomial order. However by Lemma 2.2.2, $S(h_j g_j h'_j, h_l g_l h'_l, d') = h_j S(g_j, g_l, d) h'_l$ if not zero.

Then we consider our hypothesis that for any positive integer $d \leq k$ and for any pair (g_i, g_j) where $g_i, g_j \in G^{(k)}$ we have $\overline{S(g_i, g_j, d)}^{G^{(k)}} = 0$; in other words, the remainder of multivariate division of $S(g_i, g_j, d)$ by $G^{(k)}$ is zero. Therefore, it allows us by division algorithm, to replace $S(g_j, g_l, d)$ by expressions that are lexicographically less than or equal to our S-polynomials, as it follows.

$$
\begin{aligned}
f = \sum_{g_i \in G^{(k)}} h_i g_i h'_i = &\sum_{j,l} c_{jl} S(h_j g_j h'_j, h_l g_l h'_l, d'),\ \text{(by Lemma 2.2.1)},\\
=&\sum_{j,l} c_{jl} h_j S(g_j, g_l, d) h'_l,\ \text{if not zero (by Lemma 2.2.2)},\\
=&\sum_{j,l} c_{j,l} h_j (\sum_r h''_r g_r h'''_r) h'_l,\ \text{if not zero},\\
&\text{(by Lemma 2.2.5 and our hypothesis } \overline{S(g_j, g_l, d)}^{G^{(k)}} = 0),\\
=&(\sum_{j,l,r} c_{j,l,r} h_j h''_r g_r h'''_r h'_l) = \sum_{j,l,r} h_{jr} g_r h_{rl},\ \text{if not zero.}
\end{aligned}
$$

In the above, $LT(h''_r g_r h'''_r) \leq_{glex} LT(S(g_j, g_l, d))$ by division algorithm, so the last sum in the above involves less cancellation of leading terms than the first sum. Hence while we have inequality in (2.18) we can rewrite (2.17) such that it involves less cancellation of leading terms. Continuing this way, we will eventually find an expression for f (i.e. specific h_i and h'_i for given g_i) with no cancellation of leading terms, so we have equality in Equation (2.18). Hence we have $f = \sum_{g_i \in G^{(k)}} h_i g_i h'_i$ such that $LT(f) = LT(h_i g_i h'_i) = LT(h_i) LT(g_i) LT(h'_i)$ for some $g_i \in G^{(k)}$. In other words $LT(f)$ is divisible by some $LT(g_i) \in LT(G^{(k)})$, i.e., $LT(f) \in \langle LT(G^{(k)}) \rangle$, which is what we want to prove.

Now the details of the proof. Consider all possible ways that f can be written in the form of Equation (2.17). Since monomial ordering is a well ordering (a total ordering of a set with a least element for every non-empty subset), we have a minimal way where the minimal glex degree for leading terms in the expression of f in Equation (2.17) will occur (in other words where the maximal cancellation of the leading terms on r.h.s. of Equation (2.17) happens).

Claim: We claim that at this minimal way, described in the above paragraph, we have equality in Equation 2.18, i.e., we have

$$LT(f) = LT(h_{i_m} g_{i_m} h'_{i_m}) = LT(h_{i_m}) LT(g_{i_m}) LT(h'_{i_m})$$

for some $g_{i_m} \in G^{(k)}$, i.e., $LT(f)$ is divisible by some $LT(g_{i_m}) \in LT(G^{(k)})$, so $LT(f) \in \langle LT(G^{(k)}) \rangle$. We prove this claim by contradiction.

If equality in Equation (2.18) does not occur at this minimal way, then we are going to have $LT(f) <_{glex} LT(h_{i_m} g_{i_m} h'_{i_m})$. This means that the leading terms of the summands in (2.17) are partly equal to $LT(h_{i_m} g_{i_m} h'_{i_m})$ and partly less. Then we rewrite (2.17) as

$$f = \sum_{g_i \in G^{(k)}} h_i g_i h'_i = \sum_{LT(h_i g_i h'_i) = LT(h_{i_m} g_{i_m} h'_{i_m})} h_i g_i h'_i + \sum_{LT(h_i g_i h'_i) <_{glex} LT(h_{i_m} g_{i_m} h'_{i_m})} h_i g_i h'_i. \tag{2.19}$$

By dividing the first sum on r.h.s. of the above equation we rewrite it as

$$\begin{aligned} f = & \sum_{LT(h_i g_i h'_i) = LT(h_{i_m} g_{i_m} h'_{i_m})} LT(h_i) g_i LT(h'_i) \\ & + \sum_{LT(h_i g_i h'_i) = LT(h_{i_m} g_{i_m} h'_{i_m})} (h_i - LT(h_i)) g_i LT(h'_i) \\ & + \sum_{LT(h_i g_i h'_i) = LT(h_{i_m} g_{i_m} h'_{i_m})} LT(h_i) g_i (h'_i - LT(h'_i)) \\ & + \sum_{LT(h_i g_i h'_i) = LT(h_{i_m} g_{i_m} h'_{i_m})} (h_i - LT(h_i)) g_i (h'_i - LT(h'_i)) \\ & + \sum_{LT(h_i g_i h'_i) <_{glex} LT(h_{i_m} g_{i_m} h'_{i_m})} h_i g_i h'_i. \end{aligned} \tag{2.20}$$

The leading terms of the polynomials on r.h.s. of Equation (2.20), except for the 1st one, are already graded lexicographically less than $LT(h_{i_m} g_{i_m} h'_{i_m})$. Thus our assumption that $LT(f) <_{glex} LT(h_{i_m} g_{i_m} h'_{i_m})$, implies that the 1st sum is also graded lexicographically less than $LT(h_{i_m} g_{i_m} h'_{i_m})$, i.e.

$$\sum_{LT(h_i g_i h'_i)=LT(h_{i_m} g_{i_m} h'_{i_m})} LT(h_i) g_i LT(h'_i) <_{glex} LT(h_{i_m} g_{i_m} h'_{i_m}) \tag{2.21}$$

Then by Lemma (2.2.1), the sum in Equation (2.21) can be written as a linear combination of S-polynomials, each of them graded lexicographically less than $LT(h_{i_m} g_{i_m} h'_{i_m})$, i.e.

$$\sum_{LT(h_i g_i h'_i)=LT(h_{i_m} g_{i_m} h'_{i_m})} LT(h_i) g_i LT(h'_i) = \sum_{j,r} c_{jr} S(LT(h_j) g_j LT(h'_j), LT(h_r) g_r LT(h'_r), d'). \tag{2.22}$$

Now by applying Lemma 2.2.2 with $\alpha = LT(h_j)$, and $\beta = LT(h'_r)$, we get the following expression for the summand on the r.h.s. of Equation (2.22).

$$S(LT(h_j) g_j LT(h'_j), LT(h_r) g_r LT(h'_r), d') = LT(h_j) S(g_j, g_r, d) LT(h'_r), \text{if not zero,} \tag{2.23}$$

where $d' = deg(LT(h_j)) + d + deg(LT(h'_r))$. Substituting Equation (2.23) into Equation (2.22) we have

$$\sum_{LT(h_i g_i h'_i)=LT(h_{i_m} g_{i_m} h'_{i_m})} LT(h_i) g_i LT(h'_i) = \sum_{j,r} c_{jr} LT(h_j) S(g_j, g_r, d) LT(h'_r), \text{if not zero} \tag{2.24}$$

On the other hand, since $deg(S(g_j, g_r, d)) = d \leq k$, then by Corollary 2.2.6 from equation $\overline{S(g_j, g_r, d)}^{G^{(k)}} = 0$ we come to the conclusion that $S(g_j, g_r, d) \in I$, so we have

$$S(g_j, g_r, d) = \sum_i c_{ijk} l_i g_i l'_i, \text{ where } LT(l_i g_i l'_i) \leq_{glex} LT(S(g_j, g_r, d)), \tag{2.25}$$

with polynomials l_i and l'_i in $K\langle x_1, \cdots, x_n \rangle$. By substituting $S(g_j, g_r, d)$ from Equation (2.25) into Equation (2.24) we get the following.

$$
\begin{aligned}
\sum_{LT(h_i g_i h'_i)=LT(h_{i_m} g_{i_m} h'_{i_m})} LT(h_i) g_i LT(h'_i) &= \sum_{i,j,k} c_{jk} c_{ijk} LT(h_j) l_i g_i l'_i LT(h'_k) \\
&= \sum_{i,j,k} c'_{ijk} q_{ji} g_i q_{ik}, \text{ if not zero,}
\end{aligned}
\tag{2.26}
$$

where $q_{ji} = LT(h_j) l_i,\ q_{ik} = l'_i LT(h'_k)$ and $c'_{ijk} = c_{jk} c_{ijk}$.

Then

$$
\begin{aligned}
LT(q_{ji} g_i q_{ik}) &= LT[LT(h_j) l_i g_i l'_i LT(h'_k)] = \\
&= LT(h_j) LT(l_i g_i l'_i) LT(h'_k) \\
&\leq_{glex} LT(h_j) LT[S(g_j, g_k, d)] LT(h'_k) \text{ (by (2.25))} \\
&= LT[LT(h_j) S(g_j, g_k, d) LT(h'_k)] \\
&= LT[S(LT(h_j) g_j LT(h'_j), LT(h_k) g_k LT(h'_k), d')] \text{ (by (2.23))} \\
&<_{glex} LT(h_{i_m} g_{i_m} h'_{i_m}) \text{ (by Lemma 2.2.1).}
\end{aligned}
$$

Therefore

$$
LT(q_{ji} g_i q_{ik}) <_{glex} LT(h_{i_m} g_{i_m} h'_{i_m}). \tag{2.27}
$$

Equation (2.27) together with Equation (2.26) yields

$$
\begin{aligned}
\sum_{LT(h_i g_i h'_i)=LT(h_{i_m} g_{i_m} h'_{i_m})} LT(h_i) g_i LT(h'_i) &= \sum_{i,j,k} c'_{ijk} q_{ji} g_i q_{ik}, \quad \text{if not zero,} \\
&\qquad \text{where } LT(q_{ji} g_i q_{ik}) <_{glex} LT(h_{i_m} g_{i_m} h'_{i_m}).
\end{aligned}
\tag{2.28}
$$

Therefore the first sum on the r.h.s. of Equation (2.20) either is zero or can be written as a sum of polynomials $q_{ji} g_i q_{ik}$, each graded lexicographically less than $LT(h_{i_m} g_{i_m} h'_{i_m})$. Since the other sums in Equation (2.20), are also graded lexicographically less than $LT(h_{i_m} g_{i_m} h'_{i_m})$, we come up with the conclusion that f in Equation (2.20) is a sum of polynomials, each graded lexicographically less than $LT(h_{i_m}g_{i_m}h'_{i_m})$. However this is contradictory to the assumption of minimality for $h_{i_m}g_{i_m}h'_{i_m}$. Therefore the assumption of inequality in Equation (2.18) is invalid, so equality occurs there when the minimal $h_{i_m}g_{i_m}h'_{i_m}$ is chosen.

Hence for polynomial f of degree $\leq k$, having $f \in \langle G^{(k)} \rangle$, there is a minimal way of writing $f = \sum_{g_{i_m} \in G^{(k)}} h_{i_m}g_{i_m}h'_{i_m}$, where in this minimal way we have equality $LT(f) = LT(h_{i_m}g_{i_m}h'_{i_m})$, which implies that $LT(f) = LT(h_{i_m})LT(g_{i_m})LT(h'_{i_m})$, i.e., $LT(f)$ is divisible by some $LT(g_{i_m}) \in LT(G^{(k)})$, so $LT(f) \in \langle LT(G^{(k)}) \rangle$.
Hence we have shown that for a polynomial f of deg $\leq k$ in $\langle G^{(k)} \rangle$, we have $LT(f) \in \langle LT(G^{(k)}) \rangle$, which is what we wanted to prove. This completes the proof. $\square$

Remark 2.2.8. *The above theorem is the essence of the following Buchberger Algorithm for calculating the reduced non-commutative Gröbner basis w.r.t. a monomial order, degree by degree.*
If we have already Gröbner basis up to degree k denoted by $G^{(k)}$, we start a new step of the algorithm by adding the set of generators of degree $k+1$ (if any), denoted $H^{(k+1)}$ to form the set $G^{(k)} \cup H^{(k+1)}$. Then for any pair of elements (p,q) in $G^{(k)} \cup H^{(k+1)}$ we calculate $\overline{S(p,q,k+1)}^{G^{(k)} \cup H^{(k+1)}}$ (in case of Fomin-Kirillov algebra for $k \geq 2$, we calculate $\overline{S(p,q,k+1)}^{G^{(k)}}$, as $H^{(k+1)}$ is empty for $k \geq 2$). If it is non-zero, we add it to $G^{(k)} \cup H^{(k+1)}$ and continue until it is zero.
Then we have the output of this step, i.e, the set of the reduced Gröbner basis w.r.t. our monomial order, of degree $\leq k+1$ denoted $G^{(k+1)}$, and use it as the input of the next step of algorithm.

```
INPUT: $k$ and $G^{(k)}$: set of elements of Grobner basis of
deg $\leq k$ and $H^{(k+1)}$ the set of generators of degree $k+1$.
OUTPUT: $G$, Here $G = G^{(k+1)}$, the set of elements of Grobner
basis of deg $\leq k+1$.
FOR $k$ (assume all elements of $G^{(k)}$ have been computed)
$G := G^{(k)} \cup H^{(k+1)}$
REPEAT
    $G' := G$
    FOR each pair $(p,q)$, such that $p,q \in G'$, DO
        $S := \overline{S(p,q,k+1)}^{G'}$
        IF $S \neq 0$ THEN $G := G \cup \{S\}$
UNTIL $G = G'$.
```

Remark 2.2.9. *The above algorithm works, as by non-commutative Buchberger criterion, Theorem 2.2.7, if for any pair of elements p, q in $G^{(k)}$ and for any positive integer $d \leq k$ we have $\overline{S(p,q,d)}^{G^{(k)}} = 0$, where $G^{(k)}$ contains the generators of deg $\leq k$ of ideal, then $G^{(k)}$ is the set of elements of deg $\leq k$ in Gröbner basis.*
The fact that the algorithm does end is due to restricted degree; the vector space of all polynomials of degree at most k is finite dimensional, so we have at most finitely many elements of deg $\leq k$ in a Gröbner basis.

The dimension of the quotient algebra is not finite if the Gröbner basis of the associated ideal is not finite. However it could be finite or infinite if the Gröbner basis is finite.
Since the basis of the free algebra generated by a set of generators consists of all the words on the generators, the dimension of the quotient of this free algebra on the ideal generated by the Gröbner basis is the cardinal number of the set of all the words on generators of algebra that

are not divisible by the leading terms of the elements of the Gröbner basis. Therefore one of the useful applications of Gröbner basis is to find the dimension of quotient algebras.

$$\begin{gathered}\dim\Big[\frac{F\langle x_1, \cdots, x_n\rangle}{\langle g_i : g_i \in \text{Gröbner basis}\rangle}\Big] = \\ |\text{words on } x_i \text{ not divisible by leading terms of Gröbner basis}|\end{gathered} \tag{2.29}$$

Chapter 3

Symmetric properties of $FK(n)$

In this section we study the symmetric properties of Fomin-Kirillov algebra.

3.1 Action of the symmetric group on $FK(n)$

We say a group G defines an action on a set S if there is a binary operation $* : G \times S \to S$ such that

1. Action of identity: $e * s = s$ for all s in S;
2. Preserves the structure, i.e, for $g_1, g_2 \in G$, we have $(g_1 g_2) * s = g_1 * (g_2 * s)$.

Here we define the action of symmetric group S_n on a monomial in variables $x_{ij},\ i<j$ by defining the action of the permutation $\sigma \in S_n$ on the variable by

$$\sigma x_{ij} = \begin{cases} x_{\sigma(i)\sigma(j)} & \text{if } \sigma(i) < \sigma(j) \\ -x_{\sigma(j)\sigma(i)} & \text{otherwise} \end{cases} \tag{3.1}$$

and extend it multiplicatively to monomials. We have the following lemmas.

Lemma 3.1.1. *The ideal associated to $FK(n)$ is S_n-invariant.*

Proof. Here we recall the set of relations of $FK(n)$ from (1.2).

$$\begin{aligned} R =&\{x_{ij}^2 = 0,\ 1 \le i < j \le n; \\ & x_{ij}x_{kl} - x_{kl}x_{ij} = 0;\ \text{for distinct } i,j,k,l \text{ such that } 1 \le i,j,k < l \le n; \\ & x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0;\ \text{if } 1 \le i < j < k \le n; \\ & x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk} = 0,\ \text{if } 1 \le i < j < k \le n\}. \end{aligned} \tag{3.2}$$

One can easily check that the above relations as a set is symmetric under any transposition of indexes $s_{ij} \in S_n$, i.e., under any transposition, a relation in the above set R, either stays unchanged or it turns into another relation in the set. So the set of relations is symmetric under any transposition, so is symmetric under any permutation in S_n. We show this for the relation

$$x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0,\ 1 \le i < j < k \le n,$$

in the following, the rest are similar.

$$s_{ij}(x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}) = x_{ji}x_{ik} - x_{ik}x_{jk} - x_{jk}x_{ji} = -x_{ij}x_{ik} - x_{ik}x_{jk} + x_{jk}x_{ij}$$

$$= -(x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk}).$$

So

$$s_{ij}(x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0) \to (x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk} = 0) \in R.$$

Also

$$s_{jk}(x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}) = -x_{ik}x_{jk} + x_{jk}x_{ij} - x_{ij}x_{ik} = -(x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk}).$$

So

$$s_{jk}(x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0) \to (x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk} = 0) \in R.$$

Also

$$s_{ik}(x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}) = (x_{kj}x_{ji} + x_{ji}x_{ik} + x_{ik}x_{kj}) = (x_{jk}x_{ij} - x_{ij}x_{ik} - x_{ik}x_{jk})$$

$$= -(x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk}).$$

So

$$s_{ik}(x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0) \to (x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk} = 0) \in R.$$

Therefore the set of the relations of $FK(n)$ is S_n-invariant. Hence the ideal generated by the r.h.s. of these relations is S_n-invariant. This completes the proof. $\square$

Definition 3.1.2. *We define the following map for any $\sigma \in S_n$,*

$$\begin{aligned} \sigma :& FK(n) \to FK(n) \text{ by} \\ & f(x_{i_1j_1}, \cdots, x_{i_kj_k}) + I \mapsto f(\sigma(x_{i_1j_1}), \cdots, \sigma(x_{i_kj_k})) + I, \end{aligned} \tag{3.3}$$

where $\sigma(x_{ij})$ is defined in (3.1).

The above map is well-defined on $FK(n)$ as

$$\begin{aligned} & f + I = g + I \\ & \qquad \to \ f - g \in I, \\ & \qquad \to \ \sigma(f - g) \in I \text{ (as } I \text{ is } S_n\text{-invariant by Lemma 3.1.1)} \\ & \qquad \to \ \sigma(f) - \sigma(g) \in I \\ & \qquad \to \ \sigma(f) + I = \sigma(g) + I. \end{aligned}$$

Proposition 3.1.3. *The symmetric group S_n, defines an action on $FK(n)$.*

Proof. The map $\sigma : FK(n) \to FK(n)$ defined by
$f(x_{i_1j_1}, \cdots, x_{i_kj_k}) + I \mapsto f(\sigma(x_{i_1j_1}), \cdots, \sigma(x_{i_kj_k}) + I$ defines an action on $FK(n)$ because

1. It is well defined (as shown in the above).
2. (identity axiom): $\epsilon(f(x_{i_1j_1}, \cdots, x_{i_kj_k}) + I) = f(x_{i_1j_1}, \cdots, x_{i_kj_k}) + I$.
3. (compatibility axiom):

$$\begin{aligned} (\sigma_1\sigma_2)(f(x_{i_1j_1}, \cdots, x_{i_kj_k}) + I) =& f((\sigma_1\sigma_2)x_{i_1j_1}, \cdots, (\sigma_1\sigma_2)x_{i_kj_k}) + I \\ =& \sigma_1 f(\sigma_2 x_{i_1j_1}, \cdots, \sigma_2 x_{i_kj_k}) + I \\ =& \sigma_1(\sigma_2 f(x_{i_1j_1}, \cdots, x_{i_kj_k})) + I. \end{aligned}$$

Therefore the symmetric group defines an action on $FK(n)$. □

3.1.1 An anti-automorphism on $FK(n)$

In plain language we may say that when there is an automorphism or an anti- automorphism on a mathematical object, then the structure of that object is basically preserved. Strictly speaking, an anti- homomorphism between two algebras is a map between the two algebras that preserves addition, but unlike homomorphism reverses the order of multiplication. Therefore $\phi : X \to Y$ is an algebra anti- homomorphism if and only if

1. $\phi(1) = 1$,
2. $\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y)$ for x and y in X and for α and β in the underlying field of the algebra,
3. $\phi(xy) = \phi(y)\phi(x)$ for x and y in X.

Moreover if ϕ is one-to-one and surjective as well, it is called an anti-isomorphism. An anti-isomorphism from X to itself is called anti-automorphism.

Definition 3.1.4. *Reverse word [1]. Let $w = w_1w_2 \cdots w_k$ be a word of degree k with $w_j = x_{i_j}$ for some $1 \leq i_j \leq n$. We call $\overleftarrow{w} = w_kw_{k-1} \cdots w_1$ the reverse word of w and extend it linearly to polynomials.*

By the following lemma we have an anti-automorphism on $FK(n)$.

Lemma 3.1.5. *Let $S = \{x_{ij} : 1 \leq i < j \leq n\}$ be the set of generators of $FK(n)$, and let w be a word on the letters in S. Then we have an anti-automorphism $\phi : FK(n) \to FK(n)$*

defined by $\phi(w+I) = \overleftarrow{w} + I$ *and linear extension of it to* $FK(n)$*, where* $\overleftarrow{w}$ *is the reverse word of* ω*.*

Proof. The basis of the free algebra $K\langle x_{ij}\rangle$ consists of all the words on the letters $\{x_{ij} : 1 \leq i < j \leq n\}$, so is closed under reverse of words. On the other hand, we observe that the set of the generators of the ideal associated to $FK(n)$ in (1.3) is invariant under the reverse of word, because as we see in the following, the reverse of word of any element in the set of the generators results in another element of the set.

$$\begin{aligned}
\overleftarrow{x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}} &= \overleftarrow{x_{ij}x_{jk}} - \overleftarrow{x_{jk}x_{ik}} - \overleftarrow{x_{ik}x_{ij}} \\
&= x_{jk}x_{ij} - x_{ik}x_{jk} - x_{ij}x_{ik} \\
&= -(x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk}), \\
\overleftarrow{x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk}} &= \overleftarrow{x_{ij}x_{ik}} - \overleftarrow{x_{jk}x_{ij}} + \overleftarrow{x_{ik}x_{jk}} \\
&= x_{ik}x_{ij} - x_{ij}x_{jk} + x_{jk}x_{ik} \\
&= -(x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}), \\
\overleftarrow{x_{ij}x_{kl} - x_{kl}x_{ij}} &= \overleftarrow{x_{ij}x_{kl}} - \overleftarrow{x_{kl}x_{ij}} \\
&= x_{kl}x_{ij} - x_{ij}x_{kl} = -(x_{ij}x_{kl} - x_{kl}x_{ij}), \\
\overleftarrow{x_{ij}x_{ij}} &= x_{ij}x_{ij}.
\end{aligned}$$

Hence the set of the generators of the ideal associated to $FK(n)$ in (1.3) is invariant up to sign under the reverse of words. So the ideal generated by these generators is invariant under the reverse of words. Therefore the operator of the reverse of words is well defined on quotient, i.e., $\phi(w+I) = \overleftarrow{w} + I$ is well defined as

1. $a + I = b + I \to a - b \in I \to \overleftarrow{a-b} \in I \to \overleftarrow{a} - \overleftarrow{b} \in I \to \overleftarrow{a} + I = \overleftarrow{b} + I.$

Also as an algebra the addition and multiplication operations in $FK(n)$ are preserved under ϕ as

2. $\phi((a+I)+(b+I)) = \phi(a+b+I) = \overleftarrow{a+b}+I = \overleftarrow{a}+\overleftarrow{b}+I = (\overleftarrow{a}+I)+(\overleftarrow{b}+I) = \phi(a+I)+\phi(b+I)$.
3. $\phi((a+I).(b+I)) = \phi(ab+I) = \overleftarrow{ab}+I = \overleftarrow{b}\,\overleftarrow{a}+I = (\overleftarrow{b}+I).(\overleftarrow{a}+I) = \phi(b+I)\phi(a+I)$.

   Also ϕ is 1-to-1 and surjective as

4. $\phi(a+I) = \phi(b+I) \Rightarrow \overleftarrow{a}+I = \overleftarrow{b}+I \rightarrow \overleftarrow{a}-\overleftarrow{b} \in I \rightarrow \overleftarrow{a-b} \in I \rightarrow a-b \in I \rightarrow a+I = b+I$.
5. $\forall\, y = \overleftarrow{a}+I,\ \exists\, a+I,$ such that $\phi(a+I) = \overleftarrow{a}+I$.

Therefore we have an anti-isomorphism from $FK(n)$ to itself. $\square$

Chapter 4

Finite degree Gröbner basis associated with $FK(n)$

Definition 4.0.1. *We define* $GB(n)$*, as the reduced Gröbner basis associated for* $FK(n)$ *with respect to the special monomial order defined in Definition 1.2.1. We also define* $G^{(d)}(n)$*, as the subset of* $GB(n)$ *of degree* $\leq d$*, and* $G^{(2)}(n)$ *as the subset of* $GB(n)$ *of degree* 2*, or simply the degree* 2 *terms of* $GB(n)$*.*

In this chapter we calculate $GB(n)$ defined in Definition 4.0.1 for degrees 3 and 4. In Propositions 4.1.1 and 4.2.1 we prove that the sets of degrees 3 and 4 elements of $GB(n)$ consists of polynomials that follow a special geometric pattern defined as z-star in Definition 1.2.3.

4.1 Degree 3 for general n

Proposition 4.1.1. *Let $T^{(3)}(n)$ be the set of degree 3 elements of $GB(n)$, the reduced Gröbner basis for the monomial order defined in 1.2.1, associated with $FK(n)$. Then*

$$T^{(3)}(n) = \{x_{bz}x_{az}x_{bz} + x_{az}x_{bz}x_{az} : 1 \leq a < b < z \leq n\}, \tag{4.1}$$

i.e., $T^{(3)}(n)$ consists of z-star polynomials (defined in Definition 1.2.3) for values of z such that $3 \leq z \leq n$.

Proof. Here we use Buchberger algorithm introduced at the end of Chapter 2. We consider the polynomials in (1.3), as the degree 2 elements of $GB(n)$. We recall (1.3) here with their leading terms underlined and here after on we refer it as the set of degree 2 elements of $GB(n)$ denoted by $GB^{(2)}(n)$:

$$G^{(2)}(n) : \begin{cases} (i) : \underline{x_{ij}^2},\ 1 \leq i < j \leq n, \\ (ii) : \underline{x_{ij}x_{kl}} - x_{kl}x_{ij};\ \text{for distinct } i,j,k,l \text{ such that, } 1 \leq i,j,k < l \leq n, \\ (iii) : \underline{x_{ij}x_{jk}} - x_{jk}x_{ik} - x_{ik}x_{ij};\ \text{if } 1 \leq i < j < k \leq n, \\ (iii') : \underline{x_{ij}x_{ik}} - x_{jk}x_{ij} + x_{ik}x_{jk},\ \text{if } 1 \leq i < j < k \leq n. \end{cases} \tag{4.2}$$

Then the degree 3 terms of $GB(n)$ are the results of the reduction of S-polynomials w.r.t. $G^{(2)}(n)$ between all the pairs (p, q), where $p, q \in G^{(2)}(n)$.

Out of all the S-polynomials between all pairs of $p, q \in G^{(2)}$, those that result in non zero expressions are the following S-polynomials of (iii) and (i), and also for (iii') and (i);

$$S(\underline{x_{ab}x_{bz}} - x_{bz}x_{az} - x_{az}x_{ab},\ \underline{x_{bz}x_{bz}}, 3),\ \text{and}\ \ S(\underline{x_{ab}x_{az}} - x_{bz}x_{ab} + x_{az}x_{bz},\ \underline{x_{az}x_{az}}, 3),$$

where $1 \leq a < b < z \leq n$. They reduce w.r.t. $G^{(2)}(n)$ to

$$\underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az},\ \text{and}\ -(\underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az})$$

respectively which are the same up to sign.

This implies that $T^{(3)}(n)$ in the statement includes all the non zero reduced S-polynomials. It could also be the set of all the degree 3 elements of $GB(n)$ if it includes the degree 3 generators of I as well. However in this cases of $FK(n)$ the generators of I, i.e., the polynomials in (4.2) are only of degree 2, as we have no relation of degree 3, so the set of degree 3 generators is empty. Hence $T^{(3)}(n)$ is indeed the set of all degree 3 elements of $GB(n)$. This completes the proof. □

Example 4.1.1. *For* $n = 3,\ 1 \leq a < b < z \leq n$, *the only options for* a, b *and* z *are* $1, 2$ *and* 3 *respectively. Then we get* $\underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az} = \underline{x_{23}x_{13}x_{23}} + x_{13}x_{23}x_{13}$,

$$\overbrace{\underline{x_{23}x_{13}x_{23}} + x_{13}x_{23}x_{13}}^{3-star}$$

Example 4.1.2. *For* $n = 4$,

$$\underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az} = \begin{cases} z = 3, & \underline{x_{23}x_{13}x_{23}} + x_{13}x_{23}x_{13},\ for\ a = 1,\ b = 2 \\ z = 4, & \begin{cases} \underline{x_{34}x_{14}x_{34}} + x_{14}x_{34}x_{14},\ for\ a = 1,\ b = 3 \\ \underline{x_{24}x_{14}x_{24}} + x_{14}x_{24}x_{14},\ for\ a = 1,\ b = 2 \\ \underline{x_{34}x_{24}x_{34}} + x_{24}x_{34}x_{24},\ for\ a = 2,\ b = 3 \end{cases} \end{cases}$$

The above z-star polynomials for $n = 4$ is depicted in the following chart

$$\overbrace{\underline{x_{34}x_{14}x_{34}} + x_{14}x_{34}x_{14}, \quad \underline{x_{34}x_{24}x_{34}} + x_{24}x_{34}x_{24},}^{4-stars}$$

$$\underline{x_{24}x_{14}x_{24}} + x_{14}x_{24}x_{14}$$

$$\overbrace{\underline{x_{23}x_{13}x_{23}} + x_{13}x_{23}x_{13}}^{3-star}$$

4.1.1 The number of degree 3 elements of $GB(n)$

The number of degree 3 terms of the $GB(n)$ associated with $FK(n)$ is the number of polynomials of the form $\underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az}$ with the conditions $1 \leq a < b < z \leq n$. This number equals the number of ways you can choose a 3-set $\{a, b, z\}$ out of n objects, i.e., $\binom{n}{3}$. Therefore

$$N(n) = \binom{n}{3} = \frac{n(n-1)(n-2)}{6} \tag{4.3}$$

Example 4.1.3. *$GB(6)$ has 20 degree 3 elements.*

4.2 Degree 4 for general n

Similar to the previous proposition we prove the following proposition which again gives rise to z-star elements of $GB(n)$.

Proposition 4.2.1. *Let $T^{(4)}(n)$ be the set of degree 4 elements of $GB(n)$, the reduced Gröbner basis for the monomial order defined in 1.2.1, associated with $FK(n)$. Then*

$$\begin{aligned} T^{(4)}(n) = \{&\underline{x_{cz}x_{bz}x_{az}x_{cz}} + x_{bz}x_{az}x_{cz}x_{bz} + x_{az}x_{cz}x_{bz}x_{az}, \\ &\underline{x_{cz}x_{az}x_{bz}x_{cz}} + x_{bz}x_{cz}x_{az}x_{bz} + x_{az}x_{bz}x_{cz}x_{az}, \\ &\text{where } 1 \le a < b < c < z \le n\}, \end{aligned} \tag{4.4}$$

$i.e., T^{(4)}(n)$ consists of z-star polynomials for different values of z, $4 \le z \le n$.

Remark 4.2.1. *In the set $T^{(4)}(n)$ in the above proposition, the second element is 'reverse' of the first one.*

proof of Proposition 4.2.1. We denote an element of $T^{(3)}(n)$ in Equation (4.1) by g. Then out of all S-polynomials between 2-terms and g only

$$S(\underline{x_{ij}x_{kl}} - x_{kl}x_{ij}, g, 4) \text{ and } S(\underline{x_{ij}x_{ik}} - x_{jk}x_{ij} + x_{ik}x_{jk}, g, 4)$$

and out of all possible cases for 2-terms in here only the ones in

$$\begin{aligned} &S(\underline{x_{ac}x_{bz}} - x_{bz}x_{ac},\ \underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az}, 4), \text{ and} \\ &S(\underline{x_{bc}x_{bz}} - x_{cz}x_{ac} + x_{az}x_{bz},\ \underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az}, 4) \end{aligned}$$

result in nonzero S-polynomials, where these reduce w.r.t. $G^{(3)}(n)$, respectively to

$$\begin{aligned} g =& \underline{x_{cz}x_{az}x_{bz}x_{cz}} + x_{bz}x_{cz}x_{az}x_{bz} + x_{az}x_{bz}x_{cz}x_{az}, \\ \text{and its reverse: } \overleftarrow{g} =& \underline{x_{cz}x_{bz}x_{az}x_{cz}} + x_{bz}x_{az}x_{cz}x_{bz} + x_{az}x_{cz}x_{bz}x_{az}. \end{aligned}$$

Since the set of degree 4 ideal generators is empty, these are the only degree 4 element of

$GB(n)$ as before. This completes the proof. $\square$

4.2.1 The number of degree 4 elements of $GB(n)$

The number of degree 4 terms of the $GB(n)$ associated to $FK(n)$ is the number of polynomials of the forms

$$x_{cz}x_{az}x_{bz}x_{cz}+x_{bz}x_{cz}x_{az}x_{bz} + x_{az}x_{bz}x_{cz}x_{az}, \text{ and}$$
$$x_{cz}x_{bz}x_{az}x_{cz}+x_{bz}x_{az}x_{cz}x_{bz} + x_{az}x_{cz}x_{bz}x_{az}$$

with the conditions $1 \leq a < b < c < z \leq n$. Since each of the two equal the number of 4-sets $\{a, b, c, z\}$ we can choose in n object, i.e., $\binom{n}{4}$, we have

$$N(n) = 2\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{12} \tag{4.5}$$

Example 4.2.2. *$GB(6)$ has 30 degree 4 elements.*

Chapter 5

General degree Gröbner basis associated with $FK(n)$

In this chapter we prove Proposition 5.2.3 that would be the inductive step for a proof to Theorem 5.2.4 on condition that Conjecture 5.2.1 is true. Theorem 5.2.4 asserts that every element of $GB(n)$, defined in 4.0.1 as the reduced Gröbner basis associated for $FK(n)$ with respect to the special monomial order defined in Definition 1.2.1, is a z-star polynomial for $n \geq 3$.

Since elements of $GB(n)$ are the result of reduction of S-polynomials, to prove Theorem 5.2.4, we need to check S-polynomials. However for every z-star polynomials p and q we have $S(p,q)$ reduces to a z-star polynomial w.r.t. a list of z-stars (as we will show in the following lemma). Therefore we need to check only S-polynomials $\overline{S(p,q,k+1)}^{G^{(k)}(n)}$, where $p \in G^{(2)}(n)$ (the degree 2 subset of $G^{(k)}(n)$ in (4.2)) and $q \in G^{(k)}(n)$ (assumed to be z-star for $k > 2$), also where $q \in G^{(2)}(n)$ and $p \in G^{(k)}(n)$ (assumed to be z-star for $k > 2$).

Remark 5.0.1. *We use notation $G^{(d)}(n)$ to denote the set of elements of $GB(n)$ of degree $\leq d$, also the notation $G^{(2)}(n)$ to denote the 2-terms of $GB(n)$. A general degree d element of $GB(n)$ is denoted by $\underline{x_{a_1z}x_{a_2z}\cdots x_{a_dz}}+M_2+M_3+\cdots+M_l$, or just by $g=\underline{M_1}+M_2+\cdots+M_l$, where $M_1=x_{a_1z}x_{a_2z}\cdots x_{a_dz}$ is the leading term LT and x_{a_1z} is the leading variable of the leading term. We underline the leading terms when necessary. Another situation we use underline is when emphasizing, in the chain of calculations, where we are working or doing a reduction.*

As in Remark 2.2.8, calculation of Gröbner basis of $FK(n)$ is via $\overline{S(g_1,g_2,d+1)}^{G^{(d)}(n)}$ as there is no generators of deg ≥ 3 for the ideal associated to $FK(n)$. In the following lemma, we find this if g_1 and g_2 are both z-stars.

Lemma 5.0.2. *Let g_1 and g_2 be z-star elements of degrees $s\geq 3$ and $t\geq 3$ respectively in $G^{(d)}(n)$. Assume all the elements of degree ≥ 3 in $G^{(d)}(n)$ are z-stars. Then $\overline{S(g_1,g_2,d+1)}^{G^{(d)}(n)}$ is either a z-star or zero.*

Proof. Here we need to calculate $S(g_1,g_2,d+1)$. Referring to Definition 2.1.7, the length of the overlap v between the leading monomials of g_1 and g_2 with respect to integer $d+1$ is $l(v)=s+t-(d+1)$. We first look at case $l(v)=1$. For such an overlap to a available we need $1=s+t-(d+1)$, so we have $d+1=s+t-1$, otherwise there would be no overlap available, so results in zero.

Then to have an overlap of length 1 between the leading terms of g_1 and g_2 we need them to be of the following forms (otherwise there would be no overlap, so a zero result):

$g_1=\underline{x_{a_1z}x_{a_2z}\cdots x_{a_sz}}+M_2+\cdots+M_p,$ and $g_2=\underline{x_{a_sz}x_{a_{s+1}z}\cdots x_{a_{s+t-1}z}}+M'_2+\cdots+M'_{p'},$

where $M_2, \cdots, M_p$ and $M'_2, \cdots, M'_{p'}$ are z-star monomials. Then we will have

$$\begin{aligned}
S(g_1, g_2,&d+1) = S(g_1, g_2, s+t-1) = \\
&= x_{a_1z}x_{a_2z}\cdots x_{a_{s-1}z}M'_2 + x_{a_1z}x_{a_2z}\cdots x_{a_{s-1}z}M'_3 + \cdots + x_{a_1z}x_{a_2z}\cdots x_{a_{s-1}z}M'_p \\
&\quad - M_2x_{a_{s+1}z}\cdots x_{a_{s+t-1}z} - M_3x_{a_{s+1}z}\cdots x_{a_{s+t-1}z} - \cdots - M_px_{a_{s+1}z}\cdots x_{a_{s+t-1}z}.
\end{aligned}$$

However it is obvious that all the monomials in $S(g_1, g_2, s+t-1)$ in the above equation are z-stars, so are not divisible by elements of $G^{(2)}(n)$ (as they are not z-stars). So their reduction w.r.t. degree ≥ 3 terms in $G^{(d)}(n)$ (by assumption z-stars) would be a z-star polynomial or zero. Therefore $\overline{S(g_1, g_2, s+t-1)}^{G^{(d)}(n)}$ is either a z-star polynomial or zero. For case $l(v) > 1$, the proof is similar. □

The above theorem guarantees that S-polynomials between two z-star polynomials in $G^{(d)}(n)$ reduces to a z-star element or zero, under reduction w.r.t. $G^{(d)}(n)$. Therefore we need to look into the cases $S(p, q)$ where $p \in G^{(2)}(n)$ and q is a z-star also where $q \in G^{(2)}(n)$ and p is a z-star.

5.1 S-polynomial of a degree 2 term of Gröbner basis and a z-star

The following lemma is applied frequently in the sequel. In this lemma, for a z-star monomial of general form $M = x_{a_1z}x_{a_2z}\cdots x_{a_dz}$, we derive a formula for the reduction of $x_{\beta\alpha}M$ with respect to 2-terms in (4.2), denoted $G^{(2)}(n)$. To be clear we need to emphasis that this reduction could be incomplete, we denote it by the symbol $\xrightarrow{\text{mod } G^{(2)}(n)}$ to show the equivalence modulo (the ideal generated by) $G^{(2)}(n)$ or in general $\xrightarrow{\text{mod } G^{(l)}(n)}$ to show the equivalence modulo (the ideal generated by) $G^{(l)}(n)$, the symbols we use frequently in the sequel.

Lemma 5.1.1. *Let $\tau_{\beta\alpha}$ denote transposition of indexes α and β where $1 \leq \beta < \alpha < z \leq n$, and consider a z-star monomial $M = x_{a_1 z} x_{a_2 z} \cdots x_{a_d z}$. Then we have the following equivalence modulo $G^{(2)}(n)$ of $x_{\beta\alpha}M$ where $G^{(2)}(n)$ is the set of* 2*-terms of $GB(n)$ in* (4.2) *associated with $FK(n)$.*

$$\begin{aligned} x_{\beta\alpha}M &\xrightarrow{\textit{mod } G^{(2)}(n)} \tau_{\beta\alpha}(M)x_{\beta\alpha} \\ &+ \sum_{s=1}^{d} \tau_{\beta\alpha}(x_{a_1 z} \cdots x_{a_{s-1} z}) \left\{ \begin{array}{ll} x_{\alpha z}x_{\beta z}, & a_s = \alpha \\ -x_{\beta z}x_{\alpha z}, & a_s = \beta \\ 0, & \textit{otherwise} \end{array} \right\} x_{a_{s+1} z} \cdots x_{a_d z}. \end{aligned} \tag{5.1}$$

Proof. We first prove the following claim.

CLAIM I: Let $1 \leq \beta < \alpha < z \leq n$ and $a < z$, then we have the following equivalence.

$$x_{\beta\alpha}x_{az} \xrightarrow{\text{mod } G^{(2)}(n)} \tau_{\beta\alpha}(x_{az})x_{\beta\alpha} + \begin{cases} x_{\alpha z}x_{\beta z}, & \text{if } a = \alpha, \\ -x_{\beta z}x_{\alpha z}, & \text{if } a = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

The terms on the r.h.s. of Equation (5.2) are not divisible by any leading 2-term, so it is indeed a reduction but, as far as it comes to the process of the proof of the statement, for now, equivalence modulo $G^{(2)}(n)$ is enough. We prove this claim by cases.

Applying $FK(n)$ relations, $ii): \underline{x_{\beta\alpha}x_{\gamma z}} - x_{\gamma z}x_{\beta\alpha}$, $iii): \underline{x_{\beta\alpha}x_{\alpha z}} - x_{\alpha z}x_{\beta z} - x_{\beta z}x_{\beta\alpha}$, and $iii'): \underline{x_{\beta\alpha}x_{\beta z}} - x_{\alpha z}x_{\beta\alpha} + x_{\beta z}x_{\alpha z}$, with underlined leading terms, we have the following cases.

1. If $a = \alpha$, then $x_{\beta\alpha}x_{\alpha z} \to x_{\alpha z}x_{\beta z} + x_{\beta z}x_{\beta\alpha} = x_{\alpha z}x_{\beta z} + \tau_{\beta\alpha}(x_{az})x_{\beta\alpha}$,
2. If $a = \beta$, then $x_{\beta\alpha}x_{\beta z} \to x_{\alpha z}x_{\beta\alpha} - x_{\beta z}x_{\alpha z} = \tau_{\beta\alpha}(x_{\beta z})x_{\beta\alpha} - x_{\beta z}x_{\alpha z} = \tau_{\alpha\beta}(x_{az})x_{\beta\alpha} - x_{\beta z}x_{\alpha z}$,
3. If $a \neq \alpha, \beta$, then $x_{\beta\alpha}x_{az} \to x_{az}x_{\beta\alpha} = \tau_{\beta\alpha}(x_{az})x_{\beta\alpha}$.

We summarize the above items in

$$x_{\beta\alpha}x_{az} \to \begin{cases} \tau_{\beta\alpha}(x_{az})x_{\beta\alpha} + x_{\alpha z}x_{\beta z}, & \text{if } a = \alpha, \\ \tau_{\beta\alpha}(x_{az})x_{\beta\alpha} - x_{\beta z}x_{\alpha z}, & \text{if } a = \beta, \\ \tau_{\beta\alpha}(x_{az})x_{\beta\alpha}, & \text{otherwise,} \end{cases} \tag{5.3}$$

which is equivalent to Equation (5.2). However Equation (5.2) could be seen, in the context of the monomial $M = x_{a_1z}x_{a_2z}\cdots x_{a_dz}$, as the result of passing $x_{\beta\alpha}$ over x_{a_1z} to the right, as in

$$x_{\beta\alpha}(x_{a_1z}\cdots x_{a_dz}) \to \tau_{\alpha\beta}(x_{a_1z})x_{\beta\alpha}x_{a_2z}\cdots x_{a_dz} + \left\{ \begin{array}{ll} x_{\alpha z}x_{\beta z} & \text{if } a_1 = \alpha \\ -x_{\beta z}x_{\alpha z} & \text{if } a_1 = \beta \\ 0 & \text{otherwise} \end{array} \right\} x_{a_2z}\cdots x_{a_dz}, \tag{5.4}$$

and extension of this idea to the case of passing $x_{\beta\alpha}$ to the right over the whole monomial $M = x_{a_1z}x_{a_2z}\cdots x_{a_dz}$, motivates the following claim.

CLAIM II: With the same setting of indexes as in CLAIM I, and for $1 \leq t \leq d$, we have the following equivalence modulo $G^{(2)}(n)$.

$$x_{\beta\alpha}M = x_{\beta\alpha}(x_{a_1z}\cdots x_{a_dz}) \xrightarrow[x_{\beta\alpha}\text{ passes over }x_{a_1z}\cdots x_{a_tz}]{\text{mod. }G^{(2)}(n)} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_tz})\underline{x_{\beta\alpha}x_{a_{t+1}z}}\cdots x_{a_dz}$$
$$+\sum_{s=1}^{t} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})\left\{\begin{array}{ll} x_{\alpha z}x_{\beta z}, & \text{if } a_s=\alpha\\ -x_{\beta z}x_{\alpha z}, & \text{if } a_s=\beta\\ 0, & \text{otherwise}\end{array}\right\} x_{a_{s+1}z}\cdots x_{a_{t+1}z}\cdots x_{a_dz}. \tag{5.5}$$

To prove CLAIM II, we consider Equation (5.4) as the base of induction. For inductive step, we apply Equation (5.2) to the underlined part in (5.5) to get an expression for $t+1$. The underlined two variables in the above is the only divisible by 2-terms. As $x_{\beta\alpha}$ moves to the right, step by step, makes new divisible parts.

Therefore we have

$$x_{\beta\alpha}M = x_{\beta\alpha}(x_{a_1z}\cdots x_{a_dz}) \xrightarrow{\text{mod } G^{(2)}(n)} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_tz})\Big[\tau_{\beta\alpha}(x_{a_{t+1}z})x_{\beta\alpha}$$
$$+\left\{\begin{array}{ll} x_{\alpha z}x_{\beta z}, & \text{if } a_{t+1}=\alpha;\\ -x_{\beta z}x_{\alpha z}, & \text{if } a_{t+1}=\beta, \Big]\\ 0 & \text{otherwise}\end{array}\right. x_{a_{t+2}z}\cdots x_{a_dz}$$
$$+\sum_{s=1}^{t} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})\left\{\begin{array}{ll} x_{\alpha z}x_{\beta z}, & \text{if } a_s=\alpha\\ -x_{\beta z}x_{\alpha z}, & \text{if } a_s=\beta\\ 0, & \text{otherwise}\end{array}\right\} x_{a_{s+1}z}\cdots x_{a_dz} =$$

$$= \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_tz})\tau_{\beta\alpha}(x_{a_{t+1}z})x_{\beta\alpha}x_{a_{t+2}z}\cdots x_{a_dz}$$

$$+\tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_tz})\left\{\begin{array}{lr} x_{\alpha z}x_{\beta z}, & \text{if } a_{t+1}=\alpha; \\ -x_{\alpha z}x_{\alpha z}, & \text{if } a_{t+1}=\beta, \\ 0 & \text{otherwise} \end{array}\right\} x_{a_{t+2}z}\cdots x_{a_dz}$$

$$+\sum_{s=1}^{t}\tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})\left\{\begin{array}{lr} x_{\alpha z}x_{\beta z}, & \text{if } a_s=\alpha \\ -x_{\beta z}x_{\alpha z}, & \text{if } a_s=\beta \\ 0, & \text{otherwise} \end{array}\right\}\times x_{a_{s+1}z}\cdots x_{a_dz}$$

$$= \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_tz}x_{a_{t+1}z})x_{\beta\alpha}x_{a_{t+2}z}\cdots x_{a_dz}$$

$$+\tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_tz})\left\{\begin{array}{lr} x_{\alpha z}x_{\beta z}, & \text{if } a_{t+1}=\alpha; \\ -x_{\beta z}x_{\alpha z}, & \text{if } a_{t+1}=\beta, \\ 0 & \text{otherwise} \end{array}\right\} x_{a_{t+2}z}\cdots x_{a_dz}$$

$$+\sum_{s=1}^{t}\tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})\left\{\begin{array}{lr} x_{\alpha z}x_{\beta z}, & \text{if } a_s=\alpha \\ -x_{\beta z}x_{\alpha z}, & \text{if } a_s=\beta \\ 0, & \text{otherwise} \end{array}\right\} x_{a_{s+1}z}\cdots x_{a_dz} =$$

$$= \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_tz}x_{a_{t+1}z})x_{\beta\alpha}x_{a_{t+2}z}\cdots x_{a_dz}$$

$$+\sum_{s=1}^{t+1}\tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})\left\{\begin{array}{lr} x_{\alpha z}x_{\beta z}, & \text{if } a_s=\alpha \\ -x_{\beta z}x_{\alpha z}, & \text{if } a_s=\beta \\ 0, & \text{otherwise} \end{array}\right\} x_{a_{s+1}z}\cdots x_{a_dz},$$

which completes the proof of CLAIM II.

Now in the statement of CLAIM II, Equation (5.5), letting $t = d$, yields the following.

$$
\begin{aligned}
x_{\beta\alpha}M \xrightarrow{\text{mod } G^{(2)}(n)} & \tau_{\beta\alpha}(M)x_{\beta\alpha}+ \\
& + \sum_{s=1}^{d} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z}) \left\{ \begin{array}{ll} x_{\alpha z}x_{\beta z}, & a_s=\alpha \\ -x_{\beta z}x_{\alpha z}, & a_s=\beta \\ 0, & \text{otherwise} \end{array} \right\} x_{a_{s+1}z}\cdots x_{a_dz}.
\end{aligned}
\tag{5.6}
$$

This completes the proof. □

Definition 5.1.2. *For a z-star monomial $M = x_{a_1z}x_{a_2z}\cdots x_{a_dz}$ we define its associated polynomial w.r.t α and β, denoted by $\tilde{M}(\alpha,\beta)$ by*

$$\tilde{M}(\alpha,\beta) = \sum_{1\le s\le d:\ a_s=\alpha} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})(x_{\alpha z}x_{\beta z})x_{a_{s+1}z}\cdots x_{a_dz}. \tag{5.7}$$

Similarly for any z-star polynomial $g = M_1 + \cdots + M_t$, we call

$$
\begin{aligned}
\tilde{g}(\alpha,\beta) = \tilde{M}_1(\alpha,\beta) + \cdots + \tilde{M}_t(\alpha,\beta) & \text{ associated polynomial to } g \text{ if} \\
\tilde{M}_i(\alpha,\beta) & \text{ is associated to } M_i, \text{ for } i = 1,\cdots,t.
\end{aligned}
\tag{5.8}
$$

Remark 5.1.3. *From* (5.7) *we notice that $\tilde{M}$ is a z-star but could be a monomial or a homogeneous polynomial with deg$(\tilde{M})$ = deg$(M) + 1$. Also it is clear that in general $\tilde{M}(\alpha,\beta) \neq \tilde{M}(\beta,\alpha)$ and $\tilde{M}(\alpha,\beta) \neq \tau_{\alpha\beta}\tilde{M}(\beta,\alpha)$. Also $\tilde{M}(\alpha,\beta) = 0$ if $a_s \neq \alpha$ for $s = 1,2,\cdots,d$ or if $x_{\alpha z}$ and $x_{\beta z}$ appear in M only as consecutive variables $x_{\alpha z}x_{\beta z}$, as then the defining sum of $\tilde{M}(\alpha,\beta)$ consists of only one term which is equal to zero. Similarly $\tilde{M}(\beta,\alpha) = 0$ if $a_s \neq \beta$ for $s = 1,2,\cdots,d$ or if $x_{\alpha z}$ and $x_{\beta z}$ appear in M only as consecutive variables $x_{\beta z}x_{\alpha z}$. It is also clear from* (5.8) *that $\tilde{g}$ is z-star, as it is the sum of z-star summands $\tilde{M}$.*

Example 5.1.4. *Let $M = \underline{x_{a_1z}}x_{a_2z}\cdots x_{a_{d-1}z}\underline{x_{a_1z}}$ where a_i, for $i = 1,\cdots,d-1$ are distinct* *(notice that the variable x_{a_1z} appears at two ends of M). Then by* (5.7) *for* $\beta \neq a_i, i = 1, 2, \cdots, d-1$ *we have*

$$\tilde{M}(a_1, \beta) =$$

$$= \tau_{\beta a_1}(x_{a_1z} \cdots x_{a_0z})(x_{a_1z}x_{\beta z})x_{a_2z}x_{a_3z} \cdots x_{a_{d-1}z}x_{a_1z} + \tau_{\beta a_1}(x_{a_1z}x_{a_2z} \cdots x_{a_{d-1}z})(x_{a_1z}x_{\beta z})$$

$$= x_{a_1z}x_{\beta z}x_{a_2z}x_{a_3z} \cdots x_{a_{d-1}z}x_{a_1z} + x_{\beta z}x_{a_2z}x_{a_3z} \cdots x_{a_{d-1}z}x_{a_1z}x_{\beta z}.$$

Hence

$$\tilde{M}(a_1, \beta) = x_{a_1z}x_{\beta z}x_{a_2z}x_{a_3z} \cdots x_{a_{d-1}z}x_{a_1z} + x_{\beta z}x_{a_2z}x_{a_3z} \cdots x_{a_{d-1}z}x_{a_1z}x_{\beta z}. \qquad (5.9)$$

However $\tilde{M}(\beta, a_1) = 0$ *as there is no variable* x_{a_sz} *in* M *with* $a_s = \beta$ *as* $\beta \neq a_i$ *by assumption.*

Example 5.1.5. *Let* $M_j = x_{a_jz}x_{a_{j+1}z} \cdots x_{a_{d-1}z}\underline{x_{a_1z}}x_{a_2z} \cdots x_{a_jz}$ *where the indexes* $a_s, s = 1, \cdots d-1$ *are distinct. Then by* (5.7) *for* $\beta \neq a_i, i = 1, 2, \cdots, d-1$ *we have* $\tilde{M}(a_1, \beta) = \tau_{\beta a_1}(x_{a_jz}x_{a_{j+1}z} \cdots x_{a_{d-1}z})(x_{a_1z}x_{\beta z})x_{a_2z}x_{a_3z} \cdots x_{a_jz}$. *Hence*

$$\tilde{M}_j(a_1, \beta) = x_{a_jz}x_{a_{j+1}z} \cdots x_{a_{d-1}z}\underline{x_{a_1z}x_{\beta z}}x_{a_2z}x_{a_3z} \cdots x_{a_jz}. \qquad (5.10)$$

Proposition 5.1.6. *For a z-star monomial* $M = x_{a_1z}x_{a_2z} \cdots x_{a_dz}$ *and for* $\alpha, \beta < z$ *we have*

$$x_{\beta\alpha}M \xrightarrow{mod\ G^{(2)}(n)} \tau_{\beta\alpha}(M)x_{\beta\alpha} + \tilde{M}(\alpha, \beta) - \tilde{M}(\beta, \alpha). \qquad (5.11)$$

Example 5.1.7. *Let* $\beta \neq a, b$ *and let* $M = x_{az}x_{bz}x_{az}$ *then* $a_1 = a$, $a_2 = b$ *and* $a_3 = a$. *Then*
$\tilde{M}(a, \beta) = \sum_{\{m:\ a_m = a,\ i.e,\ m=1,3\}} \tau_{\beta a}(x_{a_1z} \cdots x_{a_{m-1}z})x_{az}x_{\beta z}(x_{a_{m+1}z} \cdots x_{a_dz})$
$= \tau_{\beta a}(1)x_{az}x_{\beta z}(x_{bz}x_{az}) + \tau_{\beta a}(x_{az}x_{bz})x_{az}x_{\beta z} = x_{az}x_{\beta z}x_{bz}x_{az} + x_{\beta z}x_{bz}x_{az}x_{\beta z}$.
Also$\tilde{M}(\beta, a) = 0$ *as by assumption* $\beta \neq a, b$ *or in other words*
$\tilde{M}(\beta, a) = \sum_{\{m: a_m = \beta\} \to \phi} \tau_{\beta a}(x_{a_1z} \cdots x_{a_{m-1}z})x_{\beta z}x_{az}(x_{a_{m+1}z} \cdots x_{a_dz}) \to 0$, *as the sum is*

over empty set.

Proof of Proposition 5.1.6. Equation (5.1) could be rewritten as

$$\begin{aligned} x_{\beta\alpha}M \xrightarrow{\text{mod } G^{(2)}(n)} & \tau_{\beta\alpha}(M)x_{\beta\alpha} \\ & + \sum_{1\le s\le d:\ a_s=\alpha} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})(x_{\alpha z}x_{\beta z})x_{a_{s+1}z}\cdots x_{a_dz} \\ & - \sum_{1\le s\le d:\ a_s=\beta} \tau_{\beta\alpha}(x_{a_1z}\cdots x_{a_{s-1}z})(x_{\beta z}x_{\alpha z})x_{a_{s+1}z}\cdots x_{a_dz}. \end{aligned} \tag{5.12}$$

Now an application of Definition 5.1.2 to the above equation yields

$$x_{\beta\alpha}M \xrightarrow{\text{mod } G^{(2)}(n)} \tau_{\beta\alpha}(M)x_{\beta\alpha} + \tilde{M}(\alpha,\beta) - \tilde{M}(\beta,\alpha). \tag{5.13}$$

This completes the proof. $\square$

Example 5.1.8. *Let $M = x_{a_1z}x_{a_2z}\cdots x_{a_dz}$, with distinct $a_i < z$ for $i = 1, 2, \cdots, d$. Then for indexes α and β, such that $a_i \neq \alpha, \beta$ we have $(\tau_{\beta\alpha}M)x_{\beta\alpha} = Mx_{\beta\alpha}$, and $\tilde{M}(\alpha,\beta) = 0 = \tilde{M}(\beta,\alpha)$. Therefore $x_{\beta\alpha}M = Mx_{\beta\alpha}$, as expected.*

Example 5.1.9. *For indexes $\alpha, \beta < z$, Let $M = x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z}$, with distinct indexes $a_i, i = 1, 2, \cdots d-1$. According to the notation of indexes in Proposition 5.1.6 we have $\alpha = a_1, a_i \neq \beta, i = 1, 2, \cdots, d-1$. Then*

$$\begin{aligned} (\tau_{\beta a_1}M)x_{\beta a_1} =& x_{\beta z}x_{a_2z}x_{a_3z}\cdots x_{a_{d-1}z}x_{\beta z}x_{\beta a_1}, \\ \tilde{M}(a_1,\beta) =& x_{a_1z}x_{\beta z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z} + \tau_{\beta a_1}(x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z})x_{a_1z}x_{\beta z}. \end{aligned}$$

Also $\tilde{M}(\beta, a_1) = 0$ as by assumption we have $\beta \neq a_i$.

Hence we have the following.

$$
\begin{aligned}
x_{\beta a_1}M = & x_{\beta z}x_{a_2 z}x_{a_3 z}\cdots x_{a_{d-1}z}x_{\beta z}x_{\beta a_1} + x_{a_1 z}x_{\beta z}x_{a_2 z}\cdots x_{a_{d-1}z}x_{a_1 z}\\
& + x_{\beta z}x_{a_2 z}x_{a_3 z}\cdots x_{a_{d-1}z}x_{a_1 z}x_{\beta z}.
\end{aligned}
$$

Example 5.1.10. *let $M = x_{az}x_{bz}x_{az}$ with a and b distinct. Let indexes $\alpha, \beta < z$ such that $\beta \neq a, b$, then according to the notation of Proposition 5.1.6, we have $a_1 = a$, $a_2 = b$ and $a_3 = a$. Then $x_{\beta a}M = x_{\beta a}(x_{az}x_{bz}x_{az}) \xrightarrow{\text{mod } G^{(2)}(n)} \tau_{\beta a}(x_{az}x_{bz}x_{az})x_{\beta a} + \tilde{M}(a,\beta) + \tilde{M}(\beta, a)$. However from Example 5.1.7 we have $\tilde{M}(a,\beta) = x_{az}x_{\beta z}x_{bz}x_{az} + x_{\beta z}x_{bz}x_{az}x_{\beta z}$, and $\tilde{M}(\beta, a) = 0$ as $\beta \neq a, b$, which upon substitution into the above we get*

$x_{\beta a}M = (x_{\beta z}x_{bz}x_{\beta z})\underline{x_{\beta a}} + x_{az}x_{\beta z}x_{bz}x_{az} + x_{\beta z}x_{bz}x_{az}x_{\beta z}$.

Proposition 5.1.6 is useful in proving the following Lemmas 5.1.11 to 5.1.15 where we prove that for $p \in G^{(2)}(n)$, $\overline{S(p, g, d+1)}^{G^{(d)}(n)}$ equals either a z-star or zero, where g is a degree d element of $GB(n)$.

Lemma 5.1.11. *Let $g = \sum_{i=1}^{l} M_i = \underline{x_{a_1 z}x_{a_2 z}\cdots x_{a_d z}} + M_2 + \cdots + M_l$ be a general degree d element of $GB(n)$ where $x_{a_1 z}x_{a_2 z}\cdots x_{a_d z}$ is the leading monomial. Assume all the elements of degree ≥ 3 in $G^{(d)}(n)$ are z-stars. Then $\overline{S(x_{ij}x_{ij}, g, d+1)}^{G^{(d)}(n)}$ and $\overline{S(g, x_{ij}x_{ij}, d+1)}^{G^{(d)}(n)}$, are z-star elements of $GB(n)$ or zero.*

Proof. From all the possible values for the indexes in $x_{ij}x_{ij}$ the only cases for the 2-term to result in non-zero S-polynomial with our z-star g, are z-star 2-terms $x_{a_1 z}x_{a_1 z}$ and $x_{a_d z}x_{a_d z}$. Then by Lemma 5.0.2, S-polynomial of two z-stars reduces to z-star or zero w.r.t. $G^{(d)}(n)$, as all the elements of degree ≥ 3 in $G^{(d)}(n)$ are z-stars by hypothesis. This completes the proof. □

Remark 5.1.12. *In the following Lammas, while g may not be a complete reduction, we arrive at an intermediate step where we have only z-star. Any further reduction (as seen in Lemma 5.0.2) preserve the z-star property.*

Lemma 5.1.13. *Let* $g = \sum_{i=1}^{l} M_i = \underline{x_{a_1z}x_{a_2z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$ *be a general degree* d*,* z*-star element of* $GB(n)$ *where* $x_{a_1z}x_{a_2z}\cdots x_{a_dz}$ *is the leading monomial. For* $\alpha, \beta < z$ *let the monomials in* g *contain variables* $x_{\alpha z}$ *and/or* $x_{\beta z}$ *with any multiplicity. Assume all the elements of degree* ≥ 3 *in* $G^{(d)}(n)$ *are* z*-stars. Then* $S(\underline{x_{ij}x_{kl}} - x_{kl}x_{ij}, g, d+1)$ *under reduction w.r.t.* $G^{(d)}(n)$ *reduces to the following* z*-star polynomial or zero.*

$$g' = \overline{\tilde{g}(\beta,\alpha) - \tilde{g}(\alpha,\beta)}^{G^{(d)}(n)}, \tag{5.14}$$

where $\tilde{g}(\beta,\alpha)$ *and* $\tilde{g}(\alpha,\beta)$ *are defined in Definition 5.1.2.*

Proof of Lemma 5.1.13. From all possibilities for $\underline{x_{ij}x_{kl}} - x_{kl}x_{ij}$, the one's that results in a possibly nonzero S-polynomial with our z-star g are

$$\underline{x_{\alpha\beta}x_{a_1z}} - x_{a_1z}x_{\alpha\beta},$$

where α, β, a_1, z are distinct. So we have:

$$x_{\alpha\beta}x_{a_1z} \xrightarrow{\text{mod } G^{(2)}(n)} x_{a_1z}x_{\alpha\beta}.$$

Therefore we are restricted to

$$S(\underline{x_{\alpha\beta}x_{a_1z}} - x_{a_1z}x_{\alpha\beta},\ \underline{x_{a_1z}x_{a_2z}\cdots x_{a_dz}} + M_2 + \cdots + M_l, d+1).$$

Then

$$\begin{aligned}
&S(x_{\alpha\beta}x_{a_1z} - x_{a_1z}x_{\alpha\beta},\ \underline{x_{a_1z}x_{a_2z}\cdots x_{a_dz}} + M_2 + \cdots + M_l, d+1) \\
&\quad = x_{\alpha\beta}M_2 + \cdots + x_{\alpha\beta}M_l + x_{a_1z}x_{\alpha\beta}x_{a_2z}\cdots x_{a_dz} \\
&\quad = x_{\alpha\beta}M_2 + \cdots + x_{\alpha\beta}M_l + x_{\alpha\beta}\underline{x_{a_1z}x_{a_2z}\cdots x_{a_dz}} \\
&\quad = x_{\alpha\beta}\sum_{i=1}^{l} M_i \\
&\quad \xrightarrow{\text{mod } G^{(2)}(n)} \sum_{i=1}^{l}[(\tau_{\alpha\beta}M_i)x_{\alpha\beta} + \tilde{M}_i(\beta,\alpha) - \tilde{M}_i(\alpha,\beta)] \\
&\quad = \tau_{\alpha\beta}[\sum_{i=1}^{l} M_i]x_{\alpha\beta} + \sum_{i=1}^{l}[\tilde{M}_i(\beta,\alpha) - \tilde{M}_i(\alpha,\beta)] \\
&\quad = \tau_{\alpha\beta}(g)x_{\alpha\beta} + \tilde{g}(\beta,\alpha) - \tilde{g}(\alpha,\beta).
\end{aligned}$$

However $\tau_{\alpha\beta}(g)$ is in the ideal as ideal is closed under S_n. Since $\tau_{\alpha\beta}(g)$ is in the ideal and it is of degree $\leq d$ it is divisible w.r.t. $G^{(d)}(n)$, and so is its multiple $\tau_{\alpha\beta}(g)x_{\alpha,\beta}$. Therefore $\overline{\tau_{\alpha\beta}(g)x_{\alpha,\beta}}^{G^{(d)}(n)} = 0$, so we are left with the z-star polynomial $\tilde{g}(\beta,\alpha) - \tilde{g}(\alpha,\beta)$. While this could be not a complete reduction and it could be subject to further reduction, however being a z-star polynomial, under any further reduction, it reduces to either a z-star polynomial $g' = \overline{\tilde{g}(\beta,\alpha) - \tilde{g}(\alpha,\beta)}^{G^{(d)}(n)}$ or zero by Lemma 5.0.2. This completes the proof. □

Lemma 5.1.14. *Let* $g = \sum_{i=1}^{l} M_i = \underline{x_{a_1z}x_{a_2z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$ *be a general degree* d*,* z*-star element of* $GB(n)$ *where* $x_{a_1z}x_{a_2z}\cdots x_{a_dz}$ *is the leading monomial. For* $\beta < z$ *let monomials in* g *contain variable* $x_{\beta z}$ *with any multiplicity. Assume all the elements of degree* ≥ 3 *in* $G^{(d)}(n)$ *are* z*-stars. Then* $S(\underline{x_{ij}x_{jk}} - x_{jk}x_{ik} - x_{ik}x_{ij}, g, d+1)$ *under reduction w.r.t.* $G^{(d)}(n)$ *reduces to the following* z*-star element of* $GB(n)$ *or zero.*

$$g' = \overline{\tilde{g}(a_1,\beta) - \tilde{g}(\beta,a_1)}^{G^{(d)}(n)}. \tag{5.15}$$

Proof. As before restricting our 2-term to those not result in obviously zero S-polynomials, we calculate $S(\underline{x_{\beta a_1}x_{a_1z}} - x_{a_1z}x_{\beta z} - x_{\beta z}x_{\beta a_1},\ \underline{x_{a_1z}x_{a_2z}\cdots x_{a_dz}} + M_2 + \cdots + M_l, d+1)$.

Calculations similar to those for Lemma 5.1.13 gives S-polynomial

$$\begin{gathered} S(x_{\beta a_1}x_{a_1z} - x_{a_1z}x_{\beta z} - x_{\beta z}x_{\beta a_1},\ \ x_{a_1z}\cdots x_{a_dz} + M_2 + \cdots + M_l, d+1) = \\ x_{\beta a_1}M_2 + \cdots + x_{\beta a_1}M_l + x_{a_1z}x_{\beta z}x_{a_2z}\cdots x_{a_dz} + x_{\beta z}x_{\beta a_1}x_{a_2z}\cdots x_{a_dz}. \end{gathered} \tag{5.16}$$

However in Equation (5.16), consider the last term and let $x_{\beta z}x_{\beta a_1}x_{a_2z}\cdots x_{a_dz} = x_{\beta z}x_{\beta a_1}M'$, where $M' = x_{a_2z}\cdots x_{a_dz}$. Then

$$x_{\beta z}x_{\beta a_1}x_{a_2z}\cdots x_{a_dz} = x_{\beta z}x_{\beta a_1}M' \xrightarrow{\text{mod } G^{(2)}(n)} x_{\beta z}[\tau_{\beta a_1}(M')x_{\beta a_1} + \tilde{M}'(a_1,\beta) - \tilde{M}'(\beta,a_1)].$$

Therefore

$$x_{\beta z}x_{\beta a_1}x_{a_2z}\cdots x_{a_dz} \xrightarrow{\text{mod } G^{(2)}(n)} x_{\beta z}\tau_{\beta a_1}(M')x_{\beta a_1} + x_{\beta z}\tilde{M}'(a_1,\beta) - x_{\beta z}\tilde{M}'(\beta,a_1) \tag{5.17}$$

where $M' = x_{a_2z}\cdots x_{a_dz}$.

We now calculate the three terms on r.h.s. of (5.17).

1. $x_{\beta z}\tau_{\beta a_1}(M')x_{\beta a_1} = x_{\beta z}\tau_{\beta a_1}(x_{a_2z}\cdots x_{a_dz})x_{\beta a_1} = \tau_{\beta a_1}(x_{a_1z}x_{a_2z}\cdots x_{a_dz})x_{\beta a_1}$
   $= \tau_{\beta a_1}(M_1)x_{\beta a_1}$.

2. $x_{\beta z}\tilde{M}'(\beta,a_1) \overset{def}{=} x_{\beta z}\sum_{2\le s\le d:\ a_s=a_1}\tau_{\beta a_1}(x_{a_2z}\cdots x_{a_{s-1}z})(x_{a_1z}x_{\beta z})x_{a_{s+1}z}\cdots x_{a_dz} =$
   $\sum_{1\le s\le d:\ a_s=a_1}\tau_{\beta a_1}(x_{a_1z}x_{a_2z}\cdots x_{a_{s-1}z})(x_{a_1z}x_{\beta z})x_{a_{s+1}z}\cdots x_{a_dz} \overset{def}{=} \tilde{M}_1(\beta,a_1)$.

3. similar calculations yields , $x_{\beta z}\tilde{M'}(a_1,\beta) \xrightarrow{\text{mod } G^{(2)}(n)} \tilde{M}_1(a_1,\beta) - x_{a_1z}x_{\beta z}x_{a_2z}\cdots x_{a_dz}$.

Substitution the above three items into (5.17) yields

$$\begin{aligned} &x_{\beta z}x_{\beta a_1}x_{a_2z}\cdots x_{a_dz} \xrightarrow{\text{mod } G^{(2)}(n)} \\ &\qquad (\tau_{\beta a_1}M_1)x_{\beta a_1} + \tilde{M}_1(a_1,\beta) - \tilde{M}_1(\beta,a_1) - x_{a_1z}x_{\beta z}x_{a_2z}\cdots x_{a_dz}. \end{aligned} \tag{5.18}$$

Also

$$x_{\beta a_1}M_2+\cdots+x_{\beta a_1}M_l = \sum_{i=2}^{l} x_{\beta a_1}M_i \xrightarrow{\text{mod } G^{(2)}(n)} \sum_{i=2}^{l} \tau_{\beta a_1}(M_i)x_{\beta a_1} + \tilde{M}_i(a_1,\beta) - \tilde{M}_i(\beta,a_1). \tag{5.19}$$

Now substitution of (5.18), and (5.19) into (5.16) yields:

$$\begin{aligned} &S(x_{\beta a_1}x_{a_1z} - x_{a_1z}x_{\beta z} - x_{\beta z}x_{\beta a_1},\ x_{a_1z}\cdots x_{a_dz} + M_2 + \cdots + M_l, d+1) \\ &\quad \xrightarrow{\text{mod } G^{(2)}(n)} \sum_{i=1}^{l}[\tau_{\beta a_1}M_i x_{\beta a_1} + \tilde{M}_i(a_1,\beta) - \tilde{M}_i(\beta,a_1)] \\ &\quad = \tau_{\beta a_1}(\sum_{i=1}^{l} M_i)x_{\beta a_1} + \sum_{i=1}^{l}[\tilde{M}_i(a_1,\beta) - \tilde{M}_i(\beta,a_1)] \\ &\quad = \tau_{\beta a_1}(g)x_{\beta a_1} + \tilde{g}(a_1,\beta) - \tilde{g}(\beta,a_1) \\ &\quad \xrightarrow{\text{mod } G^{(d)}(n)} \tilde{g}(a_1,\beta) - \tilde{g}(\beta,a_1), \text{as } \tau_{\beta a_1}(g)x_{\beta a_1} \xrightarrow{\text{red. } w.r.t.\ G^{(d)}(n)} 0, \text{ as before.} \end{aligned}$$

Hence

$$\begin{aligned} &S(x_{\beta a_1}x_{a_1z} - x_{a_1z}x_{\beta z} - x_{\beta z}x_{\beta a_1},\ x_{a_1z}\cdots x_{a_dz} + M_2 + \cdots + M_l, d+1) \\ &\qquad\qquad \xrightarrow{mod\ G^{(d)}(n)} \tilde{g}(a_1,\beta) - \tilde{g}(\beta,a_1), \end{aligned}$$

which is z-star and under any further reduction, it reduces to a z-star polynomial

$g' = \overline{\tilde{g}(a_1,\beta) - \tilde{g}(\beta,a_1)}^{G^{(d)}(n)}$ or zero by Lemma 5.0.2. This completes the proof. $\square$

Lemma 5.1.15. *Let* $g = \sum_{i=1}^{l} M_i = \underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$ *be a general degree* d*, z-star element of* $GB(n)$ *where* $x_{a_1z}\cdots x_{a_dz}$ *is the leading monomial. For* $\beta < z$ *let g involves variable* $x_{\beta z}$ *with any multiplicity. Assume all the elements of degree* ≥ 3 *in* $G^{(d)}(n)$ *are z-stars. Then,* $S(\underline{x_{ij}x_{ik}} - x_{jk}x_{ij} + x_{ik}x_{jk}, g, d+1)$ *under reduction w.r.t.* $G^{(d)}(n)$*, reduces to a z-star element of* $GB(n)$ *by*

$$g' = \overline{\tilde{g}(\beta,a_1) - \tilde{g}(a_1,\beta)}^{G^{(d)}(n)}. \tag{5.20}$$

Proof. Like before restricting our degree 2 elements of $GB(n)$ to the ones that do not result in obviously zero S-polynomials, we calculate $S(\underline{x_{a_1\beta}x_{a_1z}} - x_{\beta z}x_{a_1,\beta} + x_{a_1z}x_{\beta z}, g, d+1)$. Calculations give

$$\begin{aligned} &S(\underline{x_{a_1\beta}x_{a_1z}} - x_{\beta z}x_{a_1,\beta} + x_{a_1z}x_{\beta z},\ \underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l, d+1) = \\ &\qquad x_{a_1\beta}M_2 + \cdots + x_{a_1\beta}M_l + x_{\beta z}x_{a_1\beta}x_{a_2z}\cdots x_{a_dz} - x_{a_1}x_{\beta z}x_{a_2z}\cdots x_{a_dz}. \end{aligned} \tag{5.21}$$

Calculations similar to those for the proof of Lemma 5.1.14 shows that

$$\begin{aligned} &x_{\beta z}x_{a_1\beta}x_{a_2z}\cdots x_{a_dz} \xrightarrow{\text{mod } G^{(2)}(n)} \\ &\qquad (\tau_{a_1\beta}M_1)x_{a_1\beta} + \tilde{M}_1(\beta,a_1) - \tilde{M}_1(a_1,\beta) + x_{a_1z}x_{\beta z}x_{a_2}\cdots x_{a_dz}. \end{aligned} \tag{5.22}$$

and

$$x_{\beta a_1}M_2 + \cdots + x_{\beta a_1}M_l \xrightarrow{\text{mod } G^{(2)}(n)} \sum_{i=2}^{l} \tau_{a_1\beta}M_i x_{a_1\beta} + \tilde{M}_i(\beta,a_1) - \tilde{M}_i(a_1,\beta). \tag{5.23}$$

Now substitution of (5.22), and (5.23) into (5.21) yields the following.

$$
\begin{aligned}
&S(x_{a_1\beta}x_{a_1z} - x_{\beta z}x_{a_1,\beta} + x_{a_1z}x_{\beta z},\ \ x_{a_1z}\cdots x_{a_dz} + M_2 + \cdots + M_l, d+1) \\
&\qquad\qquad \xrightarrow{\text{mod } G^{(2)}(n)} \tau_{a_1\beta}(\sum_{i=1}^{l} M_i)x_{a_1\beta} + \sum_{i=1}^{l}\{\tilde{M}_i(\beta, a_1) - \tilde{M}_i(a_1,\beta)\} \\
&\qquad\qquad = \tau_{a_1\beta}(g)x_{a_1\beta} + \tilde{g}(\beta, a_1) - \tilde{g}(a_1,\beta) \\
&\qquad\qquad \xrightarrow{\text{red. } w.r.t.\ G^{(d)}(n)} \tilde{g}(\beta, a_1) - \tilde{g}(a_1,\beta),\ \text{as } \tau_{a_1\beta}(g)x_{a_1\beta} \xrightarrow{\text{red. } w.r.t.\ G^{(d)}(n)} 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&S(\underline{x_{a_1\beta}x_{a_1z}} - x_{\beta z}x_{a_1,\beta} + x_{a_1z}x_{\beta z},\ \ \underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l, d+1) \\
&\qquad\qquad\qquad\qquad \xrightarrow{mod\ G^{(d)}(n)} \tilde{g}(\beta, a_1) - \tilde{g}(a_1,\beta),
\end{aligned}
\tag{5.24}
$$

which is a z-star polynomial and under any further reduction, it reduces to a z-star polynomial $g' = \overline{\tilde{g}(\beta, a_1) - \tilde{g}(a_1,\beta)}^{G^{(d)}(n)}$ or zero by Lemma 5.0.2. This completes the proof. □

Remark 5.1.16. *Having a general z-star element g of degree d in $G^{(d)}(n)$, Equations* (5.14), (5.15) *and* (5.20) *provide concise formulas for finding the higher degree element g' of degree $d+1$. We summarize Lemmas 5.1.13, 5.1.14 and 5.1.15 in Table 5.1 with over line on the terms in the right column as emphasis on the point made in Remark 5.1.12. We see that the result for degree* 2 *elements of $GB(n)$*

$$
\underline{x_{\beta a_1}x_{a_1z}} - x_{a_1z}x_{\beta z} - x_{\beta z}x_{\beta a_1} \ \textit{and}\ \ \underline{x_{a_1\beta}x_{a_1z}} - x_{\beta z}x_{a_1\beta} + x_{a_1z}x_{\beta z}
$$

are the same up to sign.

We summarize Lemmas 5.1.11, 5.1.13, 5.1.14 and 5.1.15 in the following proposition.

Proposition 5.1.17. *Let $g = \sum_{i=1}^{l} M_i = \underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$ be a general degree*

Table 5.1: $\overline{S(p,g,d+1)}^{G^{(d)}(n)},\ p \in G^{(2)}$

$p \in G^{(2)}(n)$	g: element of $GB(n)$ of degree d	$\overline{S(p,g,d+1)}^{G^{(d)}(n)}$
$\underline{x_{\alpha\beta}x_{a_1z}} - x_{a_1z}x_{\alpha\beta}$	$\underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$	$\overline{\tilde{g}(\beta,\alpha) - \tilde{g}(\alpha,\beta)}^{G^{(d)}(n)}$
$\underline{x_{\beta a_1}x_{a_1z}} - x_{a_1z}x_{\beta z} - x_{\beta z}x_{\beta a_1}$	$\underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$	$\overline{\tilde{g}(a_1,\beta) - \tilde{g}(\beta,a_1)}^{G^{(d)}(n)}$
$\underline{x_{a_1\beta}x_{a_1z}} - x_{\beta z}x_{a_1\beta} + x_{a_1z}x_{\beta z}$	$\underline{x_{a_1z}\cdots x_{a_dz}} + M_2 + \cdots + M_l$	$\overline{-\tilde{g}(a_1,\beta) + \tilde{g}(\beta,a_1)}^{G^{(d)}(n)}$

d, z-star element of $G^{(d)}(n)$. Assume that all the elements of degree ≥ 3 in $G^{(d)}(n)$ are z-stars. Then $S(p,g,d+1)$, where $p \in G^{(2)}$, under reduction w.r.t. $G^{(d)}(n)$ reduces either to zero or a z-star element of $GB(n)$.

5.2 The conjecture on the vanishing of some S-polynomials

Our calculations in the next section yields some lemmas and propositions that strongly suggest the following conjecture.

Conjecture 5.2.1. *Let g be an element of degree $d \geq 3$ in $G^{(d)}(n)$, the set of elements of the reduced Gröbner basis with degree $\leq d$, for the ideal associated to $FK(n)$ with respect to the ordering defined in Definition 1.2.1. Assume all of the elements of degree ≥ 3 in $G^{(d)}(n)$ are z-stars. Then for $p \in G^{(2)}(n)$, $S(g,p,d+1)$, w.r.t. $G^{(d)}(n)$ reduces to zero.*

Remark 5.2.2. *If Conjecture 5.2.1 is true, then for $g = \sum_{i=1}^{l} M_i$, a z-star element of degree d in $G^{(d)}(n)$, there is a higher degree element*

$$g' = \overline{S(p,\ g = \sum_{i=1}^{l} M_i,\ d+1)}^{G^{(d)}(n)} = \overline{\tilde{g}(\beta,\alpha) - \tilde{g}(\alpha,\beta)}^{G^{(d)}(n)},$$

where $p \in G^{(2)}(n)$ and where the indexes $\alpha, \beta < z$ are in p. According to Remark 5.1.3, for

$\tilde{M}_i(\alpha, \beta)$ and/or $\tilde{M}_i(\beta, \alpha)$ in $\tilde{g}(\alpha, \beta)$ and/or $\tilde{g}(\beta, \alpha)$ respectively, not to be zero, we need the variables $x_{\alpha z}$ and/or $x_{\beta z}$ appear in monomials $M_i(\alpha, \beta)$ and/or $M_i(\beta, \alpha)$ respectively at least once but not as consecutive variables $x_{\alpha z}x_{\beta z}$ and/or $x_{\beta z}x_{\alpha z}$ respectively.
For a finite n (so finite number of indexes), consider $\overline{S(p,\ g = \sum_{i=1}^{l} M_i,\ d+1)}^{G^{(d)}(n)}$ for any $p \in G^{(2)}(n)$ as an operation of p on g from left. Then, for any fixed pair of indexes (α, β) where $\alpha, \beta < z$, after a finite number of successive applications of elements $p \in G^{(2)}$ from left on g (no matter successive applications are with the same element or with different elements having the same fixed indexes (α, β)), we are eventually left with monomials wherein the variables $x_{\alpha z}$ and $x_{\beta z}$ appear only as consecutive variables $x_{\alpha z}x_{\beta z}$ or $x_{\beta z}x_{\alpha z}$, so we come up (eventually) with zero, i.e., the generation of new terms terminates.
Application of the above considerations reflected from Table 5.1 provides not only a conjecturally faster computer program for calculating Gröbner basis but also could give some hint on how to attack the problem of dimension for $n = 6$.

On condition that Conjecture 5.2.1 holds Proposition 5.1.17 together with Lemma 5.0.2 yield the following proposition.

Proposition 5.2.3. *On condition that Conjecture 5.2.1 holds, let $G^{(d)}(n)$ be the set of elements of Gröbner basis of degree $\leq d$ for the ideal associated to $FK(n)$ with respect to the ordering defined in Definition 1.2.1. Then if all the elements of $G^{(d)}(n)$ of degree ≥ 3 are z-stars, same is true for $G^{(d+1)}(n)$.*

Proposition 4.1.1 as the base of induction, together with Proposition 5.2.3 as the inductive step prove the following Theorem which is one of the main goals of my project.

Theorem 5.2.4. *On condition that Conjecture 5.2.1 holds, let $G^{(d)}(n)$ be the set of elements of degree $\leq d$ of $GB(n)$, the reduced Gröbner basis w.r.t. ordering defined in 1.2.1 for the* *ideal associated to* $FK(n)$*. Then all the elements of the* $G^{(d)}(n)$ *with degree* ≥ 3 *are* z*-star polynomials for* $3 \leq z \leq n$*.*

5.3 Evidences supporting our conjecture

The material of this section are in support of conjecture 5.2.1. Here we recognize some important classes of Gröbner basis elements g for the ideal associated to $FK(n)$ for which if g is of degree d then $S(g, p, d+1)$ for $p \in G^{(2)}$, reduce to zero by reduction w.r.t lower degree terms. We start with the following definition.

Definition 5.3.1. $GB(n)$ *elements of type-one. For distinct indexes* a_i *and* z*, for* $1 \leq a_i < z \leq n$*, we call any homogeneous polynomial of the following form a type-one element of* $GB(n)$ *for ideal associated to* $FK(n)$ *if it is of the following form,*

$$g = \sum_{j=1}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}, \tag{5.25}$$

where d *is the degree of the polynomial,* $3 \leq d \leq n$*, and where the number of monomials in the polynomial is* $d-1$*.*

Remark 5.3.2. *In the sequel we denote a type- one degree* d *element of* $GB(n)$ *alternatively by*

$$\begin{aligned} g = \underline{x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z}} + \sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_jz},\ \textit{or} \\ g = \underline{M_1} + M_1 + \cdots + M_{d-1}, \end{aligned} \tag{5.26}$$

where $M_i = x_{a_iz}x_{a_{i+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_iz}$*, and, where* $M_1 = x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z}$

is the leading term of g shown underlined.

Example 5.3.3. *The set of elements of degree 3 in $GB(n)$ found in Proposition 4.1.1 and recalled in here*

$$T^{(3)}(n) = \left\{\underline{x_{bz}x_{az}x_{bz}} + x_{az}x_{bz}x_{az} : 1 \leq a < b < z \leq n\right\}, \tag{5.27}$$

as well as all the degree 4 elements of $GB(n)$ derived in Equation 4.4 are examples of type-one elements of $GB(n)$.

Remark 5.3.4. *Since the length of g in Definition 5.3.1 is d and the maximum value for d is n (as variables $x_{a_i z}$ are distinct except for the first and the last one in each monomial, and $a_i < z \leq n, i = 1, 2, \cdots d-1$), the highest degree of type-one elements of $GB(n)$ for $FK(n)$ is n.*

Remark 5.3.5. *It is worth to refer to a more general form of type-one elements of $GB(n)$ derived from 5.3.1 by relabeling variable indexes, like the following*

$$g = \sum_{j=m}^{d-1} x_{a_j z}x_{a_{j+1} z} \cdots x_{a_{d-1} z}x_{a_m z} \cdots x_{a_j z}, \tag{5.28}$$

which is a type-one but of degree $d-m+1$, or even more general forms

$$g = \sum_{j=m-l}^{d-1} x_{a_j z}x_{a_{j+1} z} \cdots x_{a_{d-1} z}x_{a_1 z}x_{a_2 z} \cdots x_{a_l z}x_{a_m z}x_{a_{m+1} z} \cdots x_{a_j z}, \tag{5.29}$$

which is of degree $d+l-m+1$.

Another form of type-one but of degree $(d-i)$ is the following.

$$g = \sum_{t=1}^{m-i} x_{a_t z} x_{a_{t+1} z} \cdots \underline{x_{a_{m-i} z} x_{a_{m+1} z}} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_t z} + \sum_{t=m+1}^{d-1} x_{a_t z} x_{a_{t+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_t z}. \tag{5.30}$$

We are going to use these general forms alternatively in the sequel. It is also worth noting that the number of monomials in a type-one element is always one unit less than the degree (it is clear from the structure of type-one).

Remark 5.3.6. *The family of elements of type-one is in the ideal of Fomin-Kirillov Algebra as it is the result of the reduction of S-polynomials. To prove this we show that the S-polynomial between at least a specific 2-term and a degree d type-one element of $G^{(d)}(n)$ reduces w.r.t. $G^{(d)}(n)$ to a type-one degree $d+1$ element of $GB(n)$, and conversely any type-one element is the result of the reduction of S-polynomial between a 2-term and a type-one. We cover this in the following lemma.*

Lemma 5.3.7. *(also shown in [14], Lemma 7.2)*

Let

$$g_1 = \underline{x_{a_1 z} x_{a_2 z} \cdots x_{a_{d-1} z} x_{a_1 z}} + \sum_{j=2}^{d-1} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j z}$$

be a type-one element in $G^{(d)}(n)$. Then for 2-term $\underline{x_{\beta a_1} x_{a_1 z}} - x_{a_1 z} x_{\beta z} - x_{\beta z} x_{\beta a_1}$, with the assumption that $\beta \neq a_i,\ i = 1, 2, \cdots, d-1$, and $\beta < a_1$, the S-polynomial $S(x_{\beta a_1} x_{a_1 z} - x_{a_1 z} x_{\beta z} - x_{\beta z} x_{\beta a_1}, g_1, d+1)$ reduces w.r.t. $G^{(d)}(n)$ to a degree $d+1$ type-one element of Gröbner basis. Conversely, given a type-one degree d polynomial g_1, there is a type-one degree $d-1$ polynomial g_2 and a degree 2 element R of $GB(n)$ such that

$$S(R, g_2, d) \xrightarrow{\textit{red. w.r.t. } G^{(d-1)}(n)} g_1.$$

Proof. 1. We rewrite g_1 as $g_1 = M_1 + \sum_{j=2}^{d-1} M_j$, where

$$M_1 = x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z},$$

and

$$M_j = x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}.$$

By Equation (5.24) the S-polynomial in the statement reduces w.r.t. $G^{(d)}(n)$ to

$$g_1' = \tilde{g_1}(a_1,\beta) - \tilde{g_1}(\beta,a_1).$$

However

$$\tilde{g_1}(\beta,a_1) = \tilde{M}_1(\beta,a_1) + \sum_{j=2}^{d-1}\tilde{M}_j(\beta,a_1) = 0,$$

as by definition of the associated monomial $\tilde{M}$, Equation (5.7), we have

$$\tilde{M}_1(\beta,a_1) = \tilde{M}_j(\beta,a_1) = 0, j = 2,3,\cdots,d-1,$$

as no a_i equals β (the sum in Equation (5.7) in this case is over an empty set) and

$$\tilde{g_1}(a_1,\beta) = \tilde{M}_1(a_1,\beta) + \sum_{j=2}^{d-1}\tilde{M}_j(a_1,\beta),$$

where by Examples 5.1.4 and 5.1.5, we have

$$\tilde{M}_1(a_1,\beta) = x_{a_1z}\underline{x_{\beta z}}x_{a_2z}x_{a_3z}\cdots x_{a_{d-1}z}x_{a_1z} + \underline{x_{\beta z}}x_{a_2z}x_{a_3z}\cdots x_{a_{d-1}z}x_{a_1z}\underline{x_{\beta z}},$$

and

$$\tilde{M}_j(a_1,\beta) = x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\underline{x_{\beta z}}x_{a_2z}x_{a_3z}\cdots x_{a_jz}.$$

Hence we have

$$\begin{aligned}
g_1' =&\tilde{g}_1(a_1,\beta)-\tilde{g}_1(\beta,a_1)\\
=&\tilde{g}_1(a_1,\beta)\\
=&\tilde{M}_1(a_1,\beta)+\sum_{j=2}^{d-1}\tilde{M}_j(a_1,\beta)\\
=&x_{a_1z}\underline{x_{\beta z}}x_{a_2z}x_{a_3z}\cdots x_{a_{d-1}z}x_{a_1z}+\underline{x_{\beta z}}x_{a_2z}x_{a_3z}\cdots x_{a_{d-1}z}x_{a_1z}\underline{x_{\beta z}}\\
+&\sum_{j=2}^{d-1}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\underline{x_{\beta z}}x_{a_2z}x_{a_3z}\cdots x_{a_jz}.
\end{aligned}$$

By relabeling the indexes as

$$a_1=b_1,\ \beta=b_2,\ a_i=b_{i+1},\ i\geq 2,$$

we get

$$\begin{aligned}
g_1'=\underline{x_{b_1z}x_{b_2z}\cdots x_{b_dz}x_{b_1z}}&+x_{b_2z}x_{b_3z}\cdots x_{b_dz}x_{b_1z}x_{b_2z}\\
&+\sum_{j=2}^{d-1}x_{b_{j+1}z}x_{b_{j+2}z}\cdots x_{b_dz}x_{b_1z}x_{b_2z}\cdots x_{b_{j+1}z}.
\end{aligned}$$

In the above, absorbing the 2nd monomial as $j=1$ term in the sum we get

$$g_1'=\underline{x_{b_1z}x_{b_2z}\cdots x_{b_dz}x_{b_1z}}+\sum_{j=1}^{d-1}x_{b_{j+1}z}x_{b_{j+2}z}\cdots x_{b_dz}x_{b_1z}x_{b_2z}\cdots x_{b_{j+1}z}$$

and readjusting the sum limits in the above yields

$$g_1'=\underline{x_{b_1z}x_{b_2z}\cdots x_{b_dz}x_{b_1z}}+\sum_{j=2}^{d}x_{b_jz}x_{b_{j+1}z}\cdots x_{b_dz}x_{b_1z}x_{b_2z}\cdots x_{b_jz}, \tag{5.31}$$

which is a type-one (see (5.26)) degree $d+1$ element of $GB(n)$.

2. Conversely, given a type-one degree d polynomial

$$g_1 = \underline{x_{b_1z}x_{b_2z}\cdots x_{b_{d-1}z}x_{b_1z}} + \sum_{j=2}^{d-1} x_{b_jz}x_{b_{j+1}z}\cdots x_{b_{d-1}z}x_{b_1z}x_{b_2z}\cdots x_{b_jz},$$

there is a 2-term

$$R = \underline{x_{b_2b_1}x_{b_1z}} - x_{b_1z}x_{b_2z} - x_{b_2z}x_{b_2b_1},$$

and there is degree $d-1$ type-one element in $G^{(d)}(n)$,

$$g_2 = \underline{x_{b_1z}x_{b_3z}}x_{b_4z}\cdots x_{b_{d-1}z}x_{b_1z} + \sum_{j=3}^{d-1} x_{b_jz}x_{b_{j+1}z}\cdots x_{b_{d-1}z}\underline{x_{b_1z}x_{b_3z}}x_{b_4x}\cdots x_{b_jz},$$

(made by deleting monomial $M_2 = x_{b_2z}x_{b_3z}\cdots x_{b_{d-1}z}x_{b_1z}x_{b_2z}$ from g_1 as well deleting x_{b_2z} from every other monomial in g_1), such that

$$S(R, g_2, d) \xrightarrow{mod\ G^{(d-1)}(n)} g_1,$$

where g_1 is of type-one and degree d, i.e., we need to show the following.

$$\begin{aligned} S(R, g_2,d) = S(&\underline{x_{b_2b_1}x_{b_1z}} - x_{b_1z}x_{b_2z} - x_{b_2z}x_{b_2b_1}, \\ &\underline{x_{b_1z}x_{b_3z}}x_{b_4z}\cdots x_{b_{d-1}z}x_{b_1z} + \sum_{j=3}^{d-1} x_{b_jz}x_{b_{j+1}z}\cdots x_{b_{d-1}z}\underline{x_{b_1z}x_{b_3z}}x_{b_4x}\cdots x_{b_jz}, d) \\ &\xrightarrow{mod\ G^{(d-1)}(n)} g_1. \end{aligned} \tag{5.32}$$

Our simplified S-polynomial is

$$\begin{aligned} S(R, g_2, d) =& x_{b_2b_1} \sum_{j=3}^{d-1} x_{b_jz}x_{b_{j+1}z} \cdots x_{b_{d-1}}x_{b_1z}x_{b_3z}x_{b_4z} \cdots x_{b_jz} \\ & + x_{b_1z}x_{b_2z}x_{b_3z} \cdots x_{b_{d-1}z}x_{b_1z} + x_{b_2z}x_{b_2b_1}x_{b_3z}x_{b_4z} \cdots x_{b_{d-1}z}x_{b_1z} \\ & \xrightarrow{mod\ G^{(2)}(n)} \sum_{j=3}^{d-1} x_{b_jz}x_{b_{j+1}z} \cdots x_{b_{d-1}z}\underline{x_{b_2b_1}x_{b_1z}}x_{b_3z}x_{b_4z} \cdots x_{b_jz} \\ & + x_{b_1z}x_{b_2z}x_{b_3z} \cdots x_{b_{d-1}z}x_{b_1z} + x_{b_2z}x_{b_3z}x_{b_4z} \cdots x_{b_{d-1}z}\underline{x_{b_2b_1}x_{b_1z}} \end{aligned}$$

Applying

$$x_{b_2b_1}x_{b_1z} \xrightarrow{mod\ G^{(2)}(n)} x_{b_1z}x_{b_2z} + x_{b_2z}x_{b_2b_1},$$

in the above and simplifying further, we will have the following.

$$S(R, g_2, d) = \xrightarrow{mod\ G^{(2)}(n)} \sum_{j=3}^{d-1} x_{b_j z} x_{b_{j+1} z} \cdots x_{b_{d-1}} x_{b_1 z} x_{b_2 z} \cdots x_{b_j z}$$

$$+ \sum_{j=3}^{d-1} x_{b_j z} x_{b_{j+1} z} \cdots x_{b_{d-1}} x_{b_2 z} x_{b_3 z} \cdots x_{b_j z} x_{b_2 b_1}$$

$$+ x_{b_1 z} x_{b_2 z} \cdots x_{b_{d-1} z} x_{b_1 z} + x_{b_2 z} x_{b_3 z} \cdots x_{b_{d-1} z} x_{b_1 z} x_{b_2 z}$$

$$+ x_{b_2 z} x_{b_3 z} \cdots x_{b_{d-1} z} x_{b_2 z} x_{b_2 b_1}$$

$$= \sum_{j=1}^{d-1} x_{b_j z} x_{b_{j+1} z} \cdots x_{b_{d-1}} x_{b_1 z} x_{b_2 z} \cdots x_{b_j z}$$

$$+ (\sum_{j=2}^{d-1} x_{b_j z} x_{b_{j+1} z} \cdots x_{b_{d-1}} x_{b_2 z} x_{b_3 z} \cdots x_{b_j z}) x_{b_2 b_1}$$

$$= \sum_{j=1}^{d-1} x_{b_j z} x_{b_{j+1} z} \cdots x_{b_{d-1}} x_{b_1 z} x_{b_2 z} \cdots x_{b_j z}$$

$$+ (g_3) x_{b_2 b_1}\ (g_3, \text{type-one } deg(d-1))$$

$$\xrightarrow{mod\ G^{(d-1)}(n)} \sum_{j=1}^{d-1} x_{b_j z} x_{b_{j+1} z} \cdots x_{b_{d-1}} x_{b_1 z} x_{b_2 z} \cdots x_{b_j z}$$

$$= g_1, \text{ type-one degree } d.$$

In the last step in the above, g_3 of degree $d-1$ is of type-one, so is in I. So a multiple of it, $g_3 x_{b_2 b_1}$ is in I and reduces to zero on reduction by $G^{(d-1)}$. Hence $\overline{S(R, g_2, d)}^{G^{(d-1)}} = g_1$.

This completes the proof. $\square$

In support of our conjecture we show that for a type one elements g of degree d in $GB(n)$ and $p \in G^{(2)}(n)$, $S(g, p, d+1) \to 0$ on reduction w.r.t lower degree terms. However before getting into that we need to prove the following lemma which is used in proofs in the sequel.

In the process of reduction when we are involved in a monomial $M = M_1 LT(g) M_2$, where g is of degree d and M_1 and M_2 are monomials and where M is not divisible by any $g' \in GB(n)$ of degree $< d$, then the following lemma is helpful.

Lemma 5.3.8. *For any type-one element* $g = \sum_{j=1}^{d-1} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z} \in GB(n)$*, we have*

$$x_{a_1 z} x_{a_2 z} \cdots x_{a_{d-1} z} x_{a_1 z} \xrightarrow{mod\ g} -\sum_{j=2}^{d-1} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j}. \tag{5.33}$$

Proof. Recall the algorithm of reduction $p \to p - \frac{c_i}{LC(g)} ugv$. Let $p = LT(g)$, then by taking $u = v = 1$ we have

$$LT(g) \xrightarrow{\text{mod } g} LT(g) - g = -\sum_{j=2}^{d-1} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j}, \tag{5.34}$$

i.e. the reduction of the leading term of an element in $GB(n)$ w.r.t. the element itself is the rest of the terms with negative sign. This is also a complete reduction, as non of the summands in (5.34) are divisible by any leading term of elements in $GB(n)$ as they are monomials of a reduced element. This completes the proof. □

Remark 5.3.9. *In proof of Lemma 5.3.8, while the reduction w.r.t. type-one g is used, however since the reduction is for a monomial of an element of $GB(n)$, and so is not divisible by any other lower term, we can also say reduction 'w.r.t. g and lower terms'. Also while we made the statement for type-one $g \in GB(n)$ but we can apply it to any z-star monomial of the form $x_{a_1 z} x_{a_2 z} \cdots x_{a_{d-1} z} x_{a_1 z}$ as one can always consider an associated type-one $g \in GB(n)$ for it.*

The following lemma facilitates our calculations in the future.

Lemma 5.3.10. *For distinct* a_i *and* $z < z'$*, for* $i = 1, 2, \cdots, d-1$*, we have the following*

$$x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{zz'} \xrightarrow{mod\ G^{(2)}}$$

$$x_{zz'}x_{a_1z'}\cdots x_{a_{d-1}z'} + \sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}\cdots x_{a_lz},$$

and

$$x_{a_jz}x_{a_{j+1}z}\cdots x_{a_lz}x_{zz'} \xrightarrow{mod\ G^{(2)}}$$

$$x_{zz'}x_{a_jz'}\cdots x_{a_lz'} + \sum_{t=j}^{l} x_{a_tz'}x_{a_{t+1}z'}\cdots x_{a_lz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_tz}. \tag{5.35}$$

Proof. 1st expression. We use relation $x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}$ to find the following equivalences.

$$x_{a_1z}x_{a_2z}\cdots x_{a_{d-2}z}\underline{x_{a_{d-1}z}x_{zz'}} \xrightarrow{mod\ G^{(2)}}$$

$$x_{a_1z}x_{a_2z}\cdots x_{a_{d-2}z}x_{zz'}x_{s_{d-1}z'} + x_{a_1z}x_{a_2z}\cdots x_{a_{d-2}z}x_{a_{d-1}z'}x_{a_{d-1}z}$$

$$\xrightarrow{mod\ G^{(2)}} x_{a_1z}x_{a_2z}\cdots \underline{x_{a_{d-2}z}x_{zz'}}x_{s_{d-1}z'} + x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z} \text{ (as } a_i \text{ are distinct)}$$

$$\xrightarrow{mod\ G^{(2)}} x_{a_1z}x_{a_2z}\cdots \underline{x_{a_{d-3}z}x_{zz'}}x_{a_{d-2}z'}x_{s_{d-1}z'}$$

$$+ x_{a_{d-2}z'}x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_{d-2}z} + x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}.$$

So after 2 iteration we come up with

$$x_{a_1z}x_{a_2z}\cdots \underline{x_{a_{d-1}z}x_{zz'}} \xrightarrow{\text{mod } G^{(2)}} x_{a_1z}x_{a_2z}\cdots \underline{x_{a_{d-3}z}x_{zz'}}x_{a_{d-2}z'}x_{s_{d-1}z'}$$

$$+ \sum_{l=d-2}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_lz}.$$

In the above, in each iteration, $x_{zz'}$ moves one step to the left (in first term on r.h.s. in the above). As well the lower limit of the sum decreases one unit (the second term on r.h.s. in the above).

It is easily shown that after $d-1$ iteration we have the following equivalence.

$$\begin{aligned} x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{zz'} \xrightarrow{\text{mod } \ G^{(2)}} & x_{zz'}x_{a_1z'}x_{a_2z'}\cdots x_{s_{d-1}z'} \\ & + \sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_lz}. \end{aligned}$$

Proof of the 2nd expression is similar to the 1st one. This completes the proof. □

Proposition 5.3.11. *Let g be a type-one degree d element of $GB(n)$ in $G^{(d)}(n)$, and let $p \in G^{(2)}$. Then*

$$S(g, p, d+1) \xrightarrow{\textit{red. w.r.t. } G^{(d)}(n)} 0. \tag{5.36}$$

Proof. We prove the proposition for different cases of 2-terms in (4.2) in the following Lemmas 5.3.12, 5.3.14, 5.3.15, 5.3.16 and 5.3.17. □

Lemma 5.3.12. *Let $g = \underline{x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z}} + \sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}$ be a type-one degree d element of $GB(n)$, Then $S(g, x_{ij}x_{ij}, d+1) \xrightarrow{\textit{red. w.r.t. } G^{(d)}(n)} 0$.*

Remark 5.3.13. *This Lemma for type-one elements of Gröbner basis is stronger than Lemma 5.1.11 for general case as here we prove that the associated S-polynomial reduces to zero.*

proof of Lemma 5.3.12. As before we restrict our 2-terms looking for a possibly nonzero S-polynomial and calculate

$$S(x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z} + \sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz},\ x_{a_1z}x_{a_1z}, d+1).$$

$$S(x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z}+\sum_{j=2}^{d-1}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz},\ x_{a_1z}x_{a_1z},d+1)$$

$$=-\sum_{j=2}^{d-1}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}\underline{x_{a_1z}x_{a_2z}\cdots x_{a_jz}x_{a_1z}}$$

$$\xrightarrow{mod\ g}\sum_{j=2}^{d-1}\sum_{l=2}^{j}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}\{x_{a_lz}x_{a_{l+1}z}\cdots x_{a_jz}x_{a_1z}x_{a_2z}\cdots x_{a_lz}\}(\text{by (5.33)})$$

$$=\sum_{j=2}^{d-1}\sum_{l=2}^{j}\underline{x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_lz}x_{a_{l+1}z}\cdots x_{a_jz}}x_{a_1z}x_{a_2z}\cdots x_{a_lz}$$

$$=(\sum_{j=2}^{l-1}+\sum_{j=l}^{d-1})\sum_{l=2}^{j}\underline{x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_lz}x_{a_{l+1}z}\cdots x_{a_jz}}x_{a_1z}x_{a_2z}\cdots x_{a_lz}$$

$$=\sum_{l=2}^{j}\sum_{j=l}^{d-1}\underline{x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_lz}x_{a_{l+1}z}\cdots x_{a_jz}}x_{a_1z}x_{a_2z}\cdots x_{a_lz}$$

$$+\sum_{j=2}^{l-1}\sum_{l=2}^{j}\underline{x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_lz}x_{a_{l+1}z}\cdots x_{a_jz}}x_{a_1z}x_{a_2z}\cdots x_{a_lz}$$

$$=\sum_{l=2}^{j}g_2x_{a_1z}x_{a_2z}\cdots x_{a_lz}\quad(g_2,\text{a type-one degree } d-l+1\leq d-1,\text{as } l=2,3,\cdots j)$$

$$+\sum_{j=2}^{l-1}\sum_{l=2}^{j}\underline{x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_lz}x_{a_{l+1}z}\cdots x_{a_jz}}x_{a_1z}x_{a_2z}\cdots x_{a_lz}$$

(double sum $\to 0$ due to limits)

$$=\sum_{l=2}^{j}g_2x_{a_1z}x_{a_2z}\cdots x_{a_lz}\xrightarrow{red.\ w.r.t.\ G^{(d-1)}(n)}0,$$

because

$$g_2=\sum_{j=l}^{d-1}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_lz}x_{a_{l+1}z}x_{a_jz},$$

is of degree $d+1-l,\ l=2,3,\cdots j$, so deg $(g_2)=d+1-l\leq d-1$, and it is type-one so is in I by Remark 5.3.6, so $g_2\in G^{(d-1)}$ and its multiples are in I, and reduce to zero on reduction w.r.t. $G^{(d-1)}$. This completes the proof. □

Lemma 5.3.14. *Let* $g=\underline{x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z}}+\sum_{j=2}^{d-1}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}$ *be a* *type-one degree* d *element of* $GB(n)$ *and let* $(ii) : x_{ij}x_{kl} - x_{kl}x_{ij}$*, where* $k \neq a_i$*. Then* $S(g,(ii),d+1) \xrightarrow{red.\ w.r.t.\ G^{(d)}(n)} 0$.

Proof. As before we restrict our 2-terms looking for a possibly nonzero S-polynomial and calculate instead for

$$x_{a_1z}x_{cz'} - x_{cz'}x_{a_1z},$$

for $c \neq a_j,\ j = 1, 2, \cdots, d-1$. Then we have

$$\begin{aligned}
&S(g, x_{a_1z}x_{cz'} - x_{cz'}x_{a_1z}, d+1) = \\
&\quad = [x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z} + \sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}]x_{cz'} \\
&\quad - x_{a_1z}\cdots x_{a_{d-1}z}(x_{a_1z}x_{cz'} - x_{cz'}x_{a_1z}) \\
&\quad = [\sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}]x_{cz'} + x_{a_1z}\cdots x_{a_{d-1}z}x_{cz'}x_{a_1z} \\
&\quad \xrightarrow{mod.\ G^{(2)}} x_{cz'}[\sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}] + x_{cz'}x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z} \\
&\quad = x_{cz'}[\sum_{j=1}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}] \\
&\quad = x_{cz'}g \xrightarrow{red.\ w.r.t.\ G^{(d)}(n)} 0,
\end{aligned}$$

as g of degree d is a type-one, and as before its multiple $x_{cz'}g$ is in I and reduces to zero on reduction by $G^{(d)}$. This completes the proof. $\square$

The following lemma covers the cases not covered by Lemma 5.3.14 by relaxing a condition in it.

Lemma 5.3.15. *Let* $g = \underline{M_1} + M_2 + \cdots + M_m + \cdots M_{d-1}$*, where*

$$M_i = x_{a_i z} x_{a_{i+1} z} \cdots x_{a_{m-1} z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_i z},$$

be a type-one degree d *element of* $GB(n)$*. Also let* 2*-terms* $\underline{x_{ij} x_{kl}} - x_{kl} x_{ij}$ *such that the first index of the second variable in its leading term could occur in* M_i*. Then we have*

$$S(g,\ \underline{x_{ij} x_{kl}} - x_{kl} x_{ij},\ d+1) \xrightarrow{red.\ w.r.t.\ G^{(d)}(n)} 0.$$

Proof. As before looking for a possibly nonzero S-polynomial restricts our 2-terms to $\underline{x_{a_1 z} x_{a_m z'}} - x_{a_m z'} x_{a_1 z}$ where the first index of the 2nd variable of its leading term, i.e. $a_m,\ m \geq 2$, occurs in

$$M_i = x_{a_i z} x_{a_{i+1} z} \cdots x_{a_{m-1} z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_i z}.$$

Since a_is are distinct, a_m could occur only once in each monomial of g, except for the one monomial in which a_m appears in both the first and last variables. Then

$$\begin{aligned} & S(g,\ x_{a_1 z} x_{a_m z'} - x_{a_m z'} x_{a_1 z}, d+1) = \\ = & x_{a_1 z} \cdots x_{a_{m-1} z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} (x_{a_1 z} x_{a_m z'} - x_{a_m z'} x_{a_1 z}) \\ & - (M_1 + M_2 + \cdots + M_m + \cdots M_{d-1}) x_{a_m z'} \\ = & - x_{a_1 z} \cdots x_{a_{m-1} z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_m z'} x_{a_1 z} \\ & \quad - M_2 x_{a_m z'} - \cdots - M_m x_{a_m z'} - \cdots - M_{d-1} x_{a_m z'}. \end{aligned}$$

Therefore we come up with the following equation.

$$\begin{aligned}
S(g,&x_{a_1 z}x_{a_m z'} - x_{a_m z'}x_{a_1 z}, d+1) = \\
&- x_{a_1 z}x_{a_2 z}\cdots x_{a_m z}x_{a_{m+1} z}\cdots x_{a_{d-2} z}x_{a_{d-1} z}x_{a_m z'}x_{a_1 z} \\
&- \sum_{j=2}^{m-1} x_{a_j z}x_{a_{j+1} z}\cdots x_{a_m z}x_{a_{m+1} z}\cdots x_{a_{d-2} z}x_{a_{d-1} z}x_{a_1 z}x_{a_2 z}\cdots x_{a_j z}x_{a_m z'} \\
&- x_{a_m z}x_{a_{m+1} z}\cdots x_{a_{d-1} z}x_{a_1 z}x_{a_2 z}\cdots x_{a_{m-1} z}x_{a_m z}x_{a_m z'} \\
&- \sum_{j=m+1}^{d-1} x_{a_j z}x_{a_{j+1} z}\cdots x_{a_{d-1} z}x_{a_1 z}x_{a_2 z}\cdots x_{a_{m-1} z}x_{a_m z}x_{a_{m+1} z}\cdots x_{a_j z}x_{a_m z'}.
\end{aligned} \tag{5.37}$$

We now find the equivalence modulo to each of the above terms such that the index z' on the r.h.s. of the terms shift to the l.h.s. (here we suffice to put the result of our calculations).

$$\begin{aligned}
-x_{a_m z}&x_{a_{m+1} z}\cdots x_{a_{d-1} z}x_{a_1 z}x_{a_2 z}\cdots x_{a_{m-1} z}x_{a_m z}x_{a_m z'} \xrightarrow{mod.\ G^{(2)}} \\
&- x_{zz'}x_{a_m z'}x_{a_{m+1} z'}\cdots x_{a_{d-1} z'}x_{a_1 z'}x_{a_2 z'}\cdots x_{a_{m-1} z'}x_{a_m z} \\
&- \sum_{l=m}^{d-1} x_{a_l z'}x_{a_{l+1} z'}\cdots x_{a_{d-1} z'}x_{a_1 z'}x_{a_2 z'}\cdots x_{a_{m-1} z'}x_{a_m z}x_{a_{m+1} z}\cdots x_{a_l z}x_{a_m z} \\
&- \sum_{l=1}^{m-1} x_{a_l z'}x_{a_{l+1} z'}\cdots x_{a_{m-1} z'}x_{a_m z}x_{a_{m+1} z}\cdots x_{a_{d-1} z}x_{a_1 z}x_{a_2 z}\cdots x_{a_l z}x_{a_m z} \\
&+ x_{zz'}\sum_{l=m}^{d-1} x_{a_l z'}x_{a_{l+1} z'}\cdots x_{a_{d-1} z'}x_{a_1 z'}x_{a_2 z'}\cdots x_{a_{m-1} z'}x_{a_m z}x_{a_{m+1} z}\cdots x_{a_l z} \\
&+ x_{zz'}\sum_{l=1}^{m-1} x_{a_l z'}x_{a_{l+1} z'}\cdots x_{a_{m-1} z'}x_{a_m z}\cdots x_{a_{d-1} z}x_{a_1 z}x_{a_2 z}\cdots x_{a_l z} \\
&- \sum_{l=m+1}^{d-1} x_{a_m z'}x_{zz'}x_{a_l z'}x_{a_{l+1} z'}\cdots x_{a_{d-1} z'}x_{a_1 z'}\cdots x_{a_{m-1} z'}x_{a_{m+1} z}\cdots x_{a_l z} \\
&- \sum_{l=1}^{m-1} x_{a_m z'}x_{zz'}x_{a_l z'}x_{a_{l+1} z'}\cdots x_{a_{m-1} z'}x_{a_{m+1} z}\cdots x_{a_{d-1} z}x_{a_1 z}x_{a_2 z}\cdots x_{a_l z}.
\end{aligned} \tag{5.38}$$

Also

$$
\begin{aligned}
&- \sum_{j=m+1}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_mz}x_{a_{m+1}z}\cdots x_{a_jz}x_{a_mz'} \xrightarrow{mod.\ G^{(2)}}\\
&- \sum_{j=m+1}^{d-1} x_{zz'}x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{m-1}z'}x_{a_mz}x_{a_{m+1}z}\cdots x_{a_jz}\\
&- \sum_{j=m+1}^{d-1}\sum_{l=j}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{m-1}z'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_lz}x_{a_mz}x_{a_{m+1}z}\cdots x_{a_jz}\\
&- \sum_{j=m+1}^{d-1}\sum_{l=1}^{m-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{m-1}z'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz}x_{a_mz}x_{a_{m+1}z}\cdots x_{a_jz}\\
&+ \sum_{j=m+1}^{d-1} x_{a_mz'}x_{zz'}x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{m-1}z'}x_{a_{m+1}z}x_{a_{m+2}z}\cdots x_{a_jz}\\
&+ \sum_{j=m+1}^{d-1}\sum_{l=j}^{d-1} x_{a_mz'}x_{a_lz'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{m-1}z'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_lz}x_{a_{m+1}z}x_{a_{m+2}z}\cdots x_{a_jz}\\
&+ \sum_{j=m+1}^{d-1}\sum_{l=1}^{m-1} x_{a_mz'}x_{a_lz'}\cdots x_{a_{m-1}z'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz}x_{a_{m+1}z}x_{a_{m+2}z}\cdots x_{a_jz},
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
&- \sum_{j=2}^{m-1} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z} x_{a_m z'} \\
&\xrightarrow{mod.\ G^{(2)}} - \sum_{j=2}^{m-1} x_{zz'} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{m-1} z'} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z} \\
&- \sum_{j=2}^{m-1} \sum_{l=j}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z} \\
&+ \sum_{j=2}^{m-1} x_{a_m z'} x_{zz'} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z} \\
&+ \sum_{j=2}^{m-1} \sum_{l=j}^{m-1} x_{a_m z'} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_{m+1} z} x_{a_{m+2} z} \cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z},
\end{aligned}
\tag{5.40}
$$

and

$$
\begin{aligned}
&-x_{a_1 z} x_{a_2 z} \cdots x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-2} z} x_{a_{d-1} z} x_{a_m z'} x_{a_1 z} \\
&\qquad \xrightarrow{mod\ G^{(2)}} -x_{zz'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \\
&\qquad - \sum_{l=1}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_1 z} x_{a_2 z} \cdots x_{a_l z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \\
&\qquad + x_{a_m z'} x_{zz'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} \\
&\qquad + \sum_{l=1}^{m-1} x_{a_m z'} x_{a_l z'} \cdots x_{a_{m-1} z'} x_{a_1 z} x_{a_2 z} \cdots x_{a_l z} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z}.
\end{aligned}
\tag{5.41}
$$

Adding (5.41), (5.40), (5.38) and (5.39), we come up with the following terms for S-polynomial:

$$
S(g, x_{a_1 z} x_{a_m z'} - x_{a_m z'} x_{a_1 z}, d+1) \xrightarrow{mod\ G^{(2)}} A + B + C, \tag{5.42}
$$

where

$$\begin{aligned}
A = {} & -x_{zz'}x_{a_1z'}\cdots x_{a_{m-1}z'}x_{a_mz}\cdots x_{a_{d-1}z}x_{a_1z} \\
& - x_{zz'}\sum_{j=2}^{m-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{m-1}z'}x_{a_mz}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz} \\
& - x_{zz'}x_{a_mz'}\cdots x_{a_{d-1}z'}x_{a_1z'}\cdots x_{a_{m-1}z'}x_{a_mz} \\
& + x_{zz'}\sum_{l=m}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}\cdots x_{a_{m-1}z'}x_{a_mz}x_{a_{m+1}z}\cdots x_{a_lz} \\
& + x_{zz'}\sum_{l=1}^{m-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{m-1}z'}x_{a_mz}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_lz} \\
& - x_{zz'}\sum_{l=m+1}^{d-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}\cdots x_{a_{m-1}z'}x_{a_mz}x_{a_{m+1}z}\cdots x_{a_jz}.
\end{aligned} \tag{5.43}$$

In the above, by absorbing the term 1 into the sum in term 2 and absorbing the term 3 into the sum in term 6 we have the following equation.

$$\begin{aligned}
A = {} & - x_{zz'}\sum_{j=1}^{m-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{m-1}z'}x_{a_mz}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz} \\
& + x_{zz'}\sum_{l=m}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}\cdots x_{a_{m-1}z'}x_{a_mz}x_{a_{m+1}z}\cdots x_{a_lz} \\
& + x_{zz'}\sum_{l=1}^{m-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{m-1}z'}x_{a_mz}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_lz} \\
& - x_{zz'}\sum_{l=m}^{d-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}\cdots x_{a_{m-1}z'}x_{a_mz}x_{a_{m+1}z}\cdots x_{a_jz} = 0,
\end{aligned} \tag{5.44}$$

$$\begin{aligned}
B =& x_{a_m z'} x_{zz'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} \\
&+ x_{a_m z'} z_{zz'} \sum_{j=2}^{m-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j z} \\
&- x_{a_m z'} x_{zz'} \sum_{l=m+1}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} \cdots x_{a_l z} \\
&- x_{a_m z'} z_{zz'} \sum_{l=1}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_l z} \\
&+ x_{a_m z'} x_{zz'} \sum_{j=m+1}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} \cdots x_{a_j z}.
\end{aligned} \tag{5.45}$$

In the above equation we absorb the first term into the second term to come up with:

$$\begin{aligned}
B =& x_{a_m z'} z_{zz'} \sum_{j=1}^{m-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j z} \\
&- x_{a_m z'} x_{zz'} \sum_{l=m+1}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} \cdots x_{a_l z} \\
&- x_{a_m z'} z_{zz'} \sum_{l=1}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} x_{a_{m+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_l z} \\
&+ x_{a_m z'} x_{zz'} \sum_{j=m+1}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_{m+1} z} \cdots x_{a_j z} = 0,
\end{aligned} \tag{5.46}$$

as in the above equation, the 1st term cancels the 3rd one and the 2nd term cancels the 4th one.

$$
\begin{aligned}
C = &-\sum_{l=1}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_1 z} \cdots x_{a_l z} x_{a_m z} \cdots x_{a_{d-1} z} x_{a_1 z} \\
&-\sum_{l=2}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_m z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_l z} x_{a_m z} \\
&-\sum_{j=2}^{m-1} \sum_{l=j}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_l z} x_{a_m z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j z} \\
&-\sum_{j=m+1}^{d-1} \sum_{l=1}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_l z} x_{a_m z} \cdots x_{a_j z} \\
&+\sum_{j=2}^{m-1} \sum_{l=j}^{m-1} x_{a_m z'} x_{a_l z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_l z} x_{a_{m+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j z} \\
&+\sum_{j=m+1}^{d-1} \sum_{l=1}^{m-1} x_{a_m z'} x_{a_l z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_l z} x_{a_{m+1} z} \cdots x_{a_j z} \\
&-\sum_{l=m}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_m z} \cdots x_{a_l z} x_{a_m z} \\
&-\sum_{j=m+1}^{d-1} \sum_{l=j}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_l z} x_{a_m z} \cdots x_{a_j z} \\
&+\sum_{j=m+1}^{d-1} \sum_{l=j}^{d-1} x_{a_m z'} x_{a_l z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_l z} x_{a_{m+1} z} \cdots x_{a_j z}.
\end{aligned}
\tag{5.47}
$$

However the first 4 terms in Equation (5.47) add up to make

$$
\begin{aligned}
C_1 = &-\sum_{j=m}^{d-1} \sum_{l=1}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_l z} x_{a_m z} \cdots x_{a_j z} \\
&-\sum_{j=1}^{m-1} \sum_{l=j}^{m-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_l z} x_{a_m z} \cdots x_{a_{d-1} z} x_{a_1 z} \cdots x_{a_j z}.
\end{aligned}
\tag{5.48}
$$

In C_1, l runs in $1 \leq l \leq m-1$ in both sums. To cover all the values of l in $1 \leq l \leq m-1$, we calculate C_1 at $l = m - i,\ i = 1, \cdots, m-1$. Then

$$
\begin{aligned}
C_1(l = m-i) &= x_{a_{m-i}z'} x_{a_{m-i+1}z'} \cdots x_{a_{m-1}z'} \times \\
&\Big[x_{a_1z} x_{a_2z} \cdots x_{a_{m-i}z} x_{a_mz} x_{a_{m+1}z} \cdots x_{a_{d-1}z} x_{a_1z} \\
&\quad + x_{a_2z} \cdots x_{a_{m-i}z} x_{a_mz} x_{a_{m+1}z} \cdots x_{a_{d-1}z} \quad x_{a_1z} x_{a_2z} \\
&\qquad \vdots \\
&\qquad \vdots \\
&\quad + x_{a_{m-i}z} x_{a_mz} x_{a_{m+1}z} \cdots x_{a_{d-1}z} x_{a_1z} x_{a_2z} \cdots x_{a_{m-i}z} \\
&\quad + x_{a_mz} x_{a_{m+1}z} \cdots x_{a_{d-1}z} x_{a_1z} x_{a_2z} \cdots x_{a_{m-i}z} \cdots x_{a_mz} \\
&\quad + x_{a_{m+1}z} \cdots x_{a_{d-1}z} x_{a_1z} x_{a_2z} \cdots x_{a_{m-i}z} \cdots x_{a_{m+1}z} \\
&\qquad \vdots \\
&\qquad \vdots \\
&\quad + x_{a_{d-1}z} x_{a_1z} x_{a_2z} \cdots x_{a_{m-i}z} \cdots x_{a_{d-1}z} \Big] \\
&= -x_{a_{m-i}z'} x_{a_{m-i+1}z'} \cdots x_{a_{m-1}z'} \times \\
&\quad \times \Big[\sum_{j=m}^{d-1} x_{a_jz} x_{a_{j+1}z} \cdots x_{a_{d-1}z} x_{a_1z} \cdots \underline{x_{a_{m-i}z} x_{a_mz}} x_{a_{m+1}z} \cdots x_{a_jz} \\
&\qquad + \sum_{j=1}^{m-i} x_{a_jz} x_{a_{j+1}z} \cdots \underline{x_{a_{m-i}z} x_{a_mz}} x_{a_{m+1}z} \cdots x_{a_{d-1}z} x_{a_1z} \cdots x_{a_jz} \Big] \\
&= -x_{a_{m-i}z'} x_{a_{m-i+1}z'} \cdots x_{a_{m-1}z'} g_1,
\end{aligned}
\tag{5.49}
$$

where by Equation (5.30), g_1 is type-one of degree $d - i + 1 \leq d$, for $i = 1, 2, \cdots, m-1$ (we find degree of g_1 say by calculating the degree of the summand in one of the sums in (5.49)). Since g_1 is type-one, it is in I by Remark 5.3.6, so its multiples are in I and reduce to zero on reduction w.r.t. $G^{(d)}(n)$ (as $\deg(g_1) \leq d$).

Hence we have

$$C_1 = -\sum_{i=1}^{m-1} x_{a_{m-i}z'} x_{a_{m-i+1}z'} \cdots x_{a_{m-1}z'} g_1 \xrightarrow{\text{red. w.r.t } G^{(d)}(n)} 0. \tag{5.50}$$

The next 2 terms in Equation (5.47) add up to make

$$C_2 = \sum_{j=1}^{m-1}\sum_{l=j}^{m-1} x_{a_m z'} x_{a_l z'} x_{a_{l+1}z'} \cdots x_{a_{m-1}z'} x_{a_j z} x_{a_{j+1}z} \cdots x_{a_l z} x_{a_{m+1}z} \cdots x_{a_{d-1}z} x_{a_1 z} \cdots x_{a_j z} + \sum_{j=m+1}^{d-1}\sum_{l=1}^{m-1} x_{a_m z'} x_{a_l z'} x_{a_{l+1}z'} \cdots x_{a_{m-1}z'} x_{a_j z} x_{a_{j+1}z} \cdots x_{a_{d-1}z} x_{a_1 z} \cdots x_{a_l z} x_{a_{m+1}z} x_{a_{m+2}z} \cdots x_{a_j z}. \tag{5.51}$$

However l run in $1 \leq l \leq m-1$ in both sums. For $l = m-i,\ i = 1, \cdots m-1$ we have

$$\begin{aligned} C_2(l = m-i) =& x_{a_m z'} x_{a_{m-i}z'} x_{a_{m-i+1}z'} \cdots x_{a_{m-1}z'} \times \\ & \times \Big[\sum_{j=1}^{m-i} x_{a_j z} x_{a_{j+1}z} \cdots x_{a_{m-i}z} x_{a_{m+1}z} x_{a_{m+2}z} \cdots x_{a_{d-1}z} x_{a_1 z} \cdots x_{a_j z} \\ & + \sum_{j=m+1}^{d-1} x_{a_j z} x_{a_{j+1}z} \cdots x_{a_{d-1}z} x_{a_1 z} \cdots x_{a_{m-i}z} x_{a_{m+1}z} \cdots x_{a_j z} \Big] \\ & = x_{a_m z'} x_{a_{m-i}z'} x_{a_{m-i+1}z'} \cdots x_{a_{m-1}z'} g_2. \end{aligned}$$

Hence similar calculations done for C_1, yields for C_2 as in the following.

$$C_2 = -x_{a_m z'} \sum_{i=1}^{m-1} x_{a_{m-i}z'} x_{a_{m-i+1}z'} \cdots x_{a_{m-1}z'} g_2 \xrightarrow{\text{red. w.r.t } G^{(d-1)}(n)} 0, \tag{5.52}$$

as by Equation (5.30), g_2 is a type-one degree $d-i \leq d-1$ for $i = 1, 2, \cdots, m-1$. So

$g_2 \in G^{(d-1)}(n)$, and as before its multiples are in I and reduce to zero on reduction with respect to $G^{(d-1)}(n)$.

We consider term 7 in Equation (5.47),

$$-\sum_{l=m}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_m z} \cdots x_{a_l z} x_{a_m z}.$$

It can be seen as the case $j = m$ of the term 8:

$$-\sum_{j=m+1}^{d-1} \sum_{l=j}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} x_{a_j z} \cdots x_{a_l z} x_{a_m z} \cdots x_{a_j z}.$$

Therefore the term 7 is absorbed, as $j = m$ term, in term 8 to make the following.

$$\begin{aligned}
C_3 &= \\
&= -\sum_{j=m}^{d-1} \sum_{l=j}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_j z} \\
&= -\sum_{l=m}^{d-1} \sum_{j=m}^{l} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_j z} \\
&= -\sum_{l=m}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} \sum_{j=m}^{l} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_m z} x_{a_{m+1} z} \cdots x_{a_j z} \\
&= -\sum_{l=m}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{m-1} z'} g_3.
\end{aligned} \tag{5.53}$$

Here deg$(g_3) = l - m + 2 \leq d - 1$ (as $l \leq d - 1$ and $m \geq 2$), and g_3 is a type-one, thus as before its multiples are in I and reduce to zero on reduction by $G^{(d-1)}(n)$.

Hence we have

$$C_3 = -\sum_{l=m}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_{l+1} z'} \cdots x_{a_{m-1} z'} g_3 \xrightarrow{\text{red. w.r.t } G^{(d-1)}(n)} 0. \tag{5.54}$$

Finally the last term in Equation (5.47) is

$$\begin{aligned} C_4 &= \sum_{l=m+1}^{d-1} \sum_{j=m+1}^{l} x_{a_m z'} x_{a_l z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_{m+1} z} \cdots x_{a_j z} \\ &= \sum_{l=m+1}^{d-1} x_{a_m z'} x_{a_l z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} \sum_{j=m+1}^{l} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_{m+1} z} \cdots x_{a_j z} \\ &= x_{a_m z'} \sum_{l=m+1}^{d-1} x_{a_l z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} g_4, \end{aligned} \tag{5.55}$$

where

$$g_4 = \sum_{j=m+1}^{l} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_{m+1} z} \cdots x_{a_j z},$$

is of degree $l - m + 1 \leq d - 2$ (as $l \leq d - 1$ and $m \geq 2$) and is a type-one by Equation (5.30), so as before its multiples are in I and reduce to zero on reduction by $G^{(d-2)}$.

Hence we have the following result.

$$C_4 = x_{a_m z'} \sum_{l=m+1}^{d-1} x_{a_l z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_{m-1} z'} g_4 \xrightarrow{\text{red. w.r.t } G^{(d-2)}(n)} 0. \tag{5.56}$$

Now adding up Equations (5.64), (5.65), (5.50), (5.52), (5.54) and (5.56), we have

$$\begin{aligned} S(g, x_{a_1 z} x_{a_m z'} - x_{a_m z'} x_{a_1 z}, d+1) &\xrightarrow{mod\ G^{(2)}} \underbrace{A + B}_{0} + C \\ &= C = C_1 + C_2 + C_3 + C_4 \xrightarrow{\text{red. w.r.t } G^{(d)}(n)} 0. \end{aligned} \tag{5.57}$$

Hence we have

$$\overline{S(g,\ x_{a_1z}x_{a_mz'} - x_{a_mz'}x_{a_1z},\ d+1)}^{G^{(d)}(n)} = 0.$$

This completes the proof. $\square$

Lemma 5.3.16. . *Let* $g = \underline{M_1} + M_2 + \cdots + M_{d-1}$*, where*

$$M_i = x_{a_iz}x_{a_{i+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_iz},$$

be a type-one degree d *element of* $GB(n)$*. Then for* 2*-term* $\underline{x_{ij}x_{ik}} - x_{jk}x_{ij} + x_{ik}x_{jk}$*, we have*

$$S(g,\ \underline{x_{ij}x_{ik}} - x_{jk}x_{ij} + x_{ik}x_{jk}, d+1) \xrightarrow{red.\ w.r.t.\ G^{(d)}(n)} 0. \tag{5.58}$$

Proof. As before we restrict our 2-term to

$$x_{a_1z}x_{a_1z'} - x_{zz'}x_{a_1z} + x_{a_1z'}x_{zz'}$$

and calculate the following S-polynomial.

$$\begin{aligned} &S(g,\ x_{a_1z}x_{a_1z'} - x_{zz'}x_{a_1z} + x_{a_1z'}x_{zz'},\ d+1) = \\ &\quad =(M_1 + \cdots + M_{d-1})x_{a_1z'} \\ &\qquad - x_{a_1z}\cdots x_{a_{d-1}z}(x_{a_1z}x_{a_1z'} - x_{zz'}x_{a_1z} + x_{a_1z'}x_{zz'}) \end{aligned}$$

Therefore we have:

$$\begin{aligned} S(g,\ x_{a_1z}x_{a_1z'} - x_{zz'}x_{a_1z} + x_{a_1z'}x_{zz'},\ d+1) =& M_2x_{a_1z'} + \cdots + M_{d-1}x_{a_1z'} \\ &+ x_{a_1z}\cdots x_{a_{d-1}z}x_{zz'}x_{a_1z} \\ &- x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z'}x_{zz}. \end{aligned}$$

Substituting M_i into the above we will have the following equation.

$$\begin{aligned} S(g,\ x_{a_1z}x_{a_1z'} - x_{zz'}x_{a_1z} + x_{a_1z'}x_{zz'},\ d+1) = & \sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz}x_{a_1z'} \\ & + x_{a_1z}\cdots x_{a_{d-1}z}x_{zz'}x_{a_1z} \\ & - x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z'}x_{zz}. \end{aligned} \tag{5.59}$$

Here we apply 'mod $G^{(2)}$' on the terms in (5.59).

By Lemma (5.3.10) we come up with

$$\begin{aligned} x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{zz'}x_{a_1z} \xrightarrow{mod\ G^{(2)}} & \ x_{zz'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{d-1}z'}x_{a_1z} \\ & + \sum_{l=2}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_lz}x_{a_1z}. \end{aligned} \tag{5.60}$$

Also we have the following reduction,

$$\begin{aligned} \sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_jz}x_{a_1z'} \xrightarrow{mod\ G^{(2)}(n)} & \\ & \sum_{j=2}^{d-1}\sum_{j=l}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_lz}x_{a_1z}x_{a_2z}\cdots x_{a_jz} \\ & + \sum_{j=2}^{d-1} x_{zz'}x_{a_jz'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_jz} \\ & - \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} x_{a_1z'}x_{a_lz'}\cdots x_{a_{d-1}z'}x_{a_jz}\cdots x_{a_lz}x_{a_2z}x_{a_3z}\cdots x_{a_jz} \\ & - \sum_{j=2}^{d-1} x_{a_1z'}x_{zz'}x_{a_jz'}\cdots x_{a_{d-1}z'}x_{a_2z}x_{a_3z}\cdots x_{a_jz}. \end{aligned} \tag{5.61}$$

as well as

$$\begin{aligned}-x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z'}x_{zz} \xrightarrow{mod\ G^{(2)}(n)} &-x_{zz'}\sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_lz}\\ &+ x_{a_1z'}x_{zz'}\sum_{l=2}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_2z}x_{a_3z}\cdots x_{a_lz}.\end{aligned} \tag{5.62}$$

Substitution of (5.60), (5.62) and (5.61) into (5.59) yields the following.

$$S(g,\ x_{a_1z}x_{a_1z'} - x_{zz'}x_{a_1z} + x_{a_1z'}x_{zz'},\ d+1) \xrightarrow{mod\ G^{(2)}(n)} A+B+C, \tag{5.63}$$

where A, B and C are defined and calculated below.

$$\begin{aligned}A = x_{zz'}\Big[&x_{a_1z'}\cdots x_{a_{d-1}z'}x_{a_1z} - \sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_lz}\\ &+\sum_{j=2}^{d-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_jz}\Big]\\ = x_{zz'}\Big[&-\sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_lz}\\ &+\sum_{j=1}^{d-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_jz}\Big] = 0,\end{aligned} \tag{5.64}$$

as the last two sums cancel each other.

$$\begin{aligned}B = x_{a_1z'}x_{zz'}\Big[&\sum_{l=2}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_2z}x_{a_3z}\cdots x_{a_lz}\\ &-\sum_{j=2}^{d-1} x_{a_jz'}\cdots x_{a_{d-1}z'}x_{a_2z}x_{a_3z}\cdots x_{a_jz}\Big] = 0,\end{aligned} \tag{5.65}$$

as the two sums cancel each other.

$$
\begin{aligned}
C = \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} & x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_1 z} \cdots x_{a_j z} \\
& - x_{a_1 z'} \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_j z} \cdots x_{a_l z} x_{a_2 z} \cdots x_{a_j z} \\
& + \sum_{l=2}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z} \cdots x_{a_l z} x_{a_1 z} \qquad (5.66) \\
= \sum_{j=1}^{d-1}\sum_{l=j}^{d-1} & x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_1 z} \cdots x_{a_j z} \\
- x_{a_1 z'} \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} & x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_2 z} \cdots x_{a_j z}.
\end{aligned}
$$

In the above equation, the sums are over l and j with $l \geq j$. The application of the following interchange

$$
\sum_{j=1}^{d-1}\sum_{l=j}^{d-1} \rightarrow \sum_{l=1}^{d-1}\sum_{j=1}^{l},
$$

in the above equation results in the following result.

$$
\begin{aligned}
C &= \sum_{l=1}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} \sum_{j=1}^{l} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z} \\
&- x_{a_1 z} \sum_{l=2}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} \sum_{j=2}^{l} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_2 z} x_{a_3 z} \cdots x_{a_j z}.
\end{aligned} \tag{5.67}
$$

Hence

$$
C = \sum_{l=1}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} g_1 - x_{a_1 z'} \sum_{l=2}^{d-1} x_{a_l z'} \cdots x_{a_{d-1} z'} g_2, \tag{5.68}
$$

where

$$
g_1 = \sum_{j=1}^{l} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z}
$$

is type-one degree $l+1 \leq d$, and

$$g_2 = \sum_{j=2}^{l} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z} x_{a_2 z} \cdots x_{a_j z}$$

is type-one degree $l \leq d-1$, by Equation (5.3.5).

Therefore g_1 and g_2 are type-one, so they are in I by Remark 5.3.6, thus their multiples are in I and reduce to zero on reduction by $G^{(d)}(n)$ and $G^{(d-1)}(n)$ respectively. Hence we have the following reduction for C.

$$C \xrightarrow{red\ w.r.t.\ G^{(d)}(n)} 0. \tag{5.69}$$

Now adding up A, B and C from (5.64), (5.65), and (5.69), we have

$$S(g, x_{a_1 z} x_{a_1 z'} - x_{zz'} x_{a_1 z} + x_{a_1 z'} x_{zz'}, d+1) \xrightarrow{red.\ w.r.t. G^{(d)}(n)} 0. \tag{5.70}$$

This completes the proof. □

Lemma 5.3.17. *Let* $g = \underline{M_1} + M_2 + \cdots M_{d-1}$, *where*

$$M_i = x_{a_i z} x_{a_{i+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_i z},$$

be a type-one degree d *element of* $GB(n)$*. Then*

$$S(g, \underline{x_{ij} x_{jk}} - x_{jk} x_{ik} - x_{ik} x_{ij}, d+1) \xrightarrow{red.\ w.r.t.\ G^{(d)}(n)} 0.$$

Proof. As before we restrict our general 2-term $\underline{x_{ij} x_{jk}} - x_{jk} x_{ik} - x_{ik} x_{ij}$ to

$$\underline{x_{a_1 z} x_{zz'}} - x_{zz'} x_{a_1 z'} - x_{a_1 z'} x_{a_1 z},$$

and calculate the S-polynomial.

$$\begin{aligned}S(g,\ &\underline{x_{a_1z}x_{zz'}} - x_{zz'}x_{a_1z'} - x_{a_1z'}x_{a_1z}, d+1)\\ &=(\underline{M_1} + \cdots + M_{d-1})x_{zz'}\\ &\quad - x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}(x_{a_1z}x_{zz'} - x_{zz'}x_{a_1z'} - x_{a_1z'}x_{a_1z})\end{aligned}$$

Cancellation of the leading terms in the above parenthesis results in:

$$\begin{aligned}S(g,\ \underline{x_{a_1z}x_{zz'}} - x_{zz'}x_{a_1z'} - x_{a_1z'}x_{a_1z}, d+1) =&M_2x_{zz'} + M_3x_{zz'}\cdots + M_{d-1}x_{zz'}\\ &+ x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{zz'}x_{a_1z'}\\ &+ x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z'}x_{a_1z}.\end{aligned}$$

Substituting M_i into the above equation we will have the following expression for the above S-polynomial.

$$\begin{aligned}S(g,\ \underline{x_{a_1z}x_{zz'}} - x_{zz'}x_{a_1z'} - x_{a_1z'}x_{a_1z}, d+1) =&(\sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_jz})x_{zz'}\\ &+ x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{zz'}x_{a_1z'}\\ &+ x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z'}x_{a_1z}.\end{aligned} \tag{5.71}$$

We get into evaluating the terms on r.h.s. of the above equation.

$$\begin{aligned}&x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{zz'}x_{a_1z'} \xrightarrow{mod\ G^{(2)}(n)}\\ &\quad[x_{zz'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{d-1}z'} + \sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z}x_{a_2z}\cdots x_{a_lz}]x_{a_1z'}\\ &\quad= x_{zz'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{d-1}z'}x_{a_1z'} + \sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}\underline{x_{a_1z}x_{a_1z'}}x_{a_2z}x_{a_3z}\cdots x_{a_lz}.\end{aligned}$$

Thus

$$\begin{aligned} x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{zz'}x_{a_1z'} \xrightarrow{mod\ G^{(2)}(n)} & \ x_{zz'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{d-1}z'}x_{a_1z'} \\ & + \sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{zz'}x_{a_1z}x_{a_2z}\cdots x_{a_lz} \\ & - \sum_{l=1}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{zz'}x_{a_2z}x_{a_3z}\cdots x_{a_lz}. \end{aligned} \tag{5.72}$$

Also

$$\begin{aligned} x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z'}x_{a_1z} \xrightarrow{mod\ G^{(2)}(n)} & \ x_{zz'}x_{a_1z}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z} \\ & - x_{a_1z'}x_{zz'}x_{a_2z}x_{a_3z}\cdots x_{a_{d-1}z}x_{a_1z}. \end{aligned} \tag{5.73}$$

And

$$\begin{aligned} & \sum_{j=2}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_jz}x_{zz'} \xrightarrow{mod\ G^{(2)}(n)} \\ & \quad \sum_{j=2}^{d-1} x_{zz'}x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_jz'} \\ & \quad + \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{j-1}z'}x_{zz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_lz} \\ & \quad - \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{j-1}z'}x_{a_jz'}x_{zz'}x_{a_{j+1}z}x_{a_{j+2}z}\cdots x_{a_lz} \\ & \quad + \sum_{j=2}^{d-1}\sum_{l=1}^{j} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{zz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz} \\ & \quad - \sum_{j=2}^{d-1}\sum_{l=1}^{j} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{a_jz'}x_{zz'}x_{a_{j+1}z}x_{a_{j+2}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz}. \end{aligned} \tag{5.74}$$

Adding up Equations (5.73), (5.72), and (5.74) together, we come up with the following equation.

$$
\begin{aligned}
&S(g,\ \underline{x_{a_1z}x_{zz'}} - x_{zz'}x_{a_1z'} - x_{a_1z'}x_{a_1z}, d+1) \xrightarrow{mod\ G^{(2)}(n)} \\
&\quad + x_{zz'}x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z} \\
&\quad - x_{a_1z'}x_{zz'}x_{a_2z}\cdots x_{a_{d-1}z}x_{a_1z} \\
&\quad + x_{zz'}x_{a_1z'}\cdots x_{a_{d-1}z'}x_{a_1z'} \\
&\quad + \sum_{l=1}^{d-1} x_{a_lz'}\cdots x_{a_{d-1}z'}x_{zz'}x_{a_1z}x_{a_2z}\cdots x_{a_lz} \\
&\quad - \sum_{l=1}^{d-1} x_{a_lz'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{zz'}x_{a_2z}\cdots x_{a_lz} \\
&\quad + x_{zz'}\sum_{j=2}^{d-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}\cdots x_{a_jz'} \\
&\quad + \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{j-1}z'}x_{zz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_lz} \\
&\quad - \sum_{j=2}^{d-1}\sum_{l=j}^{d-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}x_{a_2z'}\cdots x_{a_{j-1}z'}x_{a_jz'}x_{zz'}x_{a_{j+1}z}x_{a_{j+2}z}\cdots x_{a_lz} \\
&\quad + \sum_{j=2}^{d-1}\sum_{l=1}^{j} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{zz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz} \\
&\quad - \sum_{j=2}^{d-1}\sum_{l=1}^{j} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{a_jz'}x_{zz'}x_{a_{j+1}z}x_{a_{j+2}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_lz}.
\end{aligned}
\tag{5.75}
$$

From the above 10 terms,

$$
\begin{aligned}
&\text{term } 3 + \text{term } 6 = \\
&= - x_{zz'}\sum_{j=1}^{d-1} x_{a_jz'}x_{a_{j+1}z'}\cdots x_{a_{d-1}z'}x_{a_1z'}\cdots x_{a_jz'} = -x_{zz'}g_1 \xrightarrow{\text{red. w.r.t. } G^{(d)}(n)} 0,
\end{aligned}
\tag{5.76}
$$

as before and the fact that g_1 is type-one of degree d in $G^{(d)}(n)$.

We can see the term 1, $x_{zz'}x_{a_1z}\cdots x_{a_{d-1}z}x_{a_1z}$ as the $j=1$ case of the term 9. Absorbing term 1 in term 9, as the $j=1$ case we will have:

term 1 + term 9 =

$$\begin{aligned}
&=\sum_{j=1}^{d-1}\sum_{l=1}^{j} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{zz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz}\\
&=\sum_{j=1}^{d-1} x_{zz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_jz}\ \text{(terms with } l=j)\\
&+\sum_{j=1}^{d-1}\sum_{l=1}^{j-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{zz'}x_{a_jz}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz}\ \text{(terms with } l<j).
\end{aligned} \tag{5.77}$$

Therefore

term 1 + term 9 =

$$\begin{aligned}
=&x_{zz'}\sum_{j=1}^{d-1} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_jz}\\
&+\sum_{j=1}^{d-1}\sum_{l=1}^{j-1} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{zz'}x_{a_jz}x_{a_{j+1}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz}.
\end{aligned} \tag{5.78}$$

Also rewrite term 10 as

$$\begin{aligned}
-&\sum_{j=2}^{d-1}\sum_{l=1}^{j} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_{j-1}z'}x_{a_jz'}x_{zz'}x_{a_{j+1}z}x_{a_{j+2}z}\cdots x_{a_{d-1}z}x_{a_1z}\cdots x_{a_lz}\\
&=-\sum_{j=1}^{d-1}\sum_{l=1}^{j} x_{a_lz'}x_{a_{l+1}z'}\cdots x_{a_jz'}x_{zz'}x_{a_{j+1}z}x_{a_{j+2}z}\cdots x_{a_{d-1}z}x_{a_1z}x_{a_2z}\cdots x_{a_lz}\\
&\quad+x_{a_1z'}x_{zz'}x_{a_2z}x_{a_3z}\cdots x_{a_{d-1}z}x_{a_1z}.
\end{aligned} \tag{5.79}$$

Then adding up the above results we come up with the following equation.

$$\text{term1} + \text{term9} + \text{term10} + \text{term2} = x_{zz'} \sum_{j=1}^{d-1} x_{a_j z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_j z}$$
$$+ \sum_{j=1}^{d-1} \sum_{l=1}^{j-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{j-1} z'} x_{zz'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_l z} \tag{5.80}$$
$$- \sum_{j=1}^{d-1} \sum_{l=1}^{j} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_j z'} x_{zz'} x_{a_{j+1} z} x_{a_{j+2} z} \cdots x_{a_{d-1} z} x_{a_1 z} x_{a_2 z} \cdots x_{a_l z}.$$

However in the above equation the first term is nothing but $x_{zz'} g_2$ for g_2 of degree d and so $x_{zz'} g_2$ reduces to zero by reduction w.r.t. $G^{(d)}(n)$ as before. also the two double sums cancel telescopically and leave us with the first term of one and the last term of the other one, thus we have;

$$\text{term1} + \text{term9} + \text{term10} + \text{term2} = -\sum_{l=1}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{zz'} x_{a_1 z} x_{a_2 z} \cdots x_{a_l z}. \tag{5.81}$$

However the sum in the above equation is nothing but the negative of the sum in term 4. Therefore

$$\text{term1} + \text{term9} + \text{term10} + \text{term2} + \text{term4} \xrightarrow{\text{red. w.r.t. } G^{(d)}(n)} 0. \tag{5.82}$$

We now consider the term 8:

$$-\sum_{j=2}^{d-1} \sum_{l=j}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{j-1} z'} x_{a_j z'} x_{zz'} x_{a_{j+1} z} x_{a_{j+2} z} \cdots x_{a_l z}$$

We rewrite term 8 as it follows.

$$\begin{aligned}
&\text{term } 8 = \\
&= -\sum_{j=2}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'} x_{zz'} \text{ (terms with } l = j) \\
&\quad - \sum_{j=2}^{d-1} \sum_{l=j+1}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} \cdots x_{a_j z'} x_{zz'} x_{a_{j+1} z} x_{a_{j+2} z} \cdots x_{a_l z} \text{ (terms with } l > j) \\
&\quad - \sum_{l=1}^{d-1} x_{a_l z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{zz'} x_{a_2 z} x_{a_3 z} \cdots x_{a_l z}.
\end{aligned} \tag{5.83}$$

Now

$$\begin{aligned}
\text{term } 8 + \text{term } 7 = &-\sum_{j=2}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'} x_{zz'} \\
&- \sum_{j=2}^{d-1} \sum_{l=j+1}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'} x_{zz'} x_{a_{j+1} z} x_{a_{j+2} z} \cdots x_{a_l z} \\
&+ \sum_{j=2}^{d-1} \sum_{l=j}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{j-1} z'} x_{zz'} x_{a_j z} x_{a_{j+1} z} \cdots x_{a_l z},
\end{aligned} \tag{5.84}$$

where the two double sums in the above telescopically cancel intermediate terms, Therefore we are left with:

$$\begin{aligned}
\text{term } 8 + \text{term } 7 = &-\sum_{j=2}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'} x_{zz'} \\
&- \sum_{l=2}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{zz'} x_{a_2 z} x_{a_3 z} \cdots x_{a_l z} \\
&- \Big(\sum_{l=d}^{d-1} \ldots = 0 \text{ because of the sum limits}\Big).
\end{aligned} \tag{5.85}$$

Then

$$\begin{aligned}
\text{term } 8 + \text{term } 7 + \text{term } 5 = &-\sum_{j=2}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'} x_{zz'} \\
&-\sum_{l=2}^{d-1} x_{a_l z'} x_{a_{l+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{zz'} x_{a_2 z} x_{a_3 z} \cdots x_{a_l z} \\
&-\sum_{l=1}^{d-1} x_{a_l z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{zz'} x_{a_2 z} x_{a_3 z} \cdots x_{a_l z} = \\
&-\sum_{j=2}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'} x_{zz'} \\
&- x_{a_1 z'} x_{a_2 z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{zz'} \\
&= -[\sum_{j=1}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'} x_{zz'}] \\
&= -g_3 x_{zz'} \xrightarrow{\text{red. w.r.t. } G^{(d)}(n)} 0, \text{ as before}
\end{aligned} \tag{5.86}$$

where $g_3 = -\sum_{j=1}^{d-1} x_{a_j z'} x_{a_{j+1} z'} \cdots x_{a_{d-1} z'} x_{a_1 z'} x_{a_2 z'} \cdots x_{a_j z'}$ is type-one of degree d. Hence

$$\text{term } 8 + \text{term } 7 + \text{term } 5 = \xrightarrow{\text{red. w.r.t. } G^{(d)}(n)} 0. \tag{5.87}$$

We now substitute Equations (5.76), (5.82), and (5.87) into Equation (5.75).

$$\begin{aligned}
S(g,\ \underline{x_{a_1 z} x_{zz'}} - x_{zz'} x_{a_1 z'} - x_{a_1 z'} x_{a_1 z}, d+1) &\xrightarrow{mod\ G^{(2)}(n)} \\
-x_{zz'} g_1 + x_{zz'} g_2 - g_3 x_{zz'} &\xrightarrow{\text{red. w.r.t. } G^{(d)}(n)} 0
\end{aligned} \tag{5.88}$$

as g_1, g_2 and g_3 are of degree d and the multiples of them each reduce is zero on reduction by $G^{(d)}(n)$ as before. This completes the proof. □

Remark 5.3.18. *Lemmas 5.3.12, 5.3.14, 5.3.15, 5.3.16 and 5.3.17 prove Proposition 5.3.11.*

5.3.1 Type-two and type-three

It is important to remark that from type-one elements we can produce new types of elements in the Gröbner basis. For example for $p \in G^{(2)}$ and a type one q, while $S(p,q)$, when p introduces a new index not appearing in type-one q, reduces w.r.t. lower terms into another type-one, but if p does not introduce a new index then the S-polynomial reduces to a new type, called type-two. Also S-polynomial between two type-one, when the one with lower degree introduces no new index to the one with higher degree, could results in a new type called type-three.

Example 5.3.19. *For a degree* 5 *type-one element of Gröbner basis* $g = \underline{x_{a_1z}x_{a_2z}x_{a_3z}x_{a_4z}x_{a_1z}} + \sum_{j=2}^{4} x_{a_jz}x_{a_{j+1}z}\cdots x_{a_4z}x_{a_1z}x_{a_2z}\cdots x_{a_jz}$ *and for* 2*-term* $R = \underline{x_{a_3a_1}x_{a_1z}} - x_{a_1z}x_{a_3z} - x_{a_3z}x_{a_3a_1}$*, with the assumption that* $a_3 > a_4$*, the reduction of* $S(R,g,6)$ *w.r.t. lower degree terms, gives the following element of degree* 6 *called type-two.*

$$\begin{aligned} g = {} & \big[\underline{x_{a_1z}x_{a_3z}x_{a_2z}x_{a_3z}x_{a_4z}x_{a_1z}} - x_{a_1z}x_{a_4z}x_{a_3z}x_{a_2z}x_{a_3z}x_{a_1z}\big] \\ & + \big[x_{a_3z}x_{a_4z}x_{a_1z}x_{a_4z}x_{a_2z}x_{a_3z} - x_{a_3z}x_{a_2z}x_{a_4z}x_{a_1z}x_{a_4z}x_{a_3z}\big] \\ & + \big[x_{a_3z}x_{a_2z}x_{a_3z}x_{a_4z}x_{a_1z}x_{a_4z} - x_{a_4z}x_{a_1z}x_{a_4z}x_{a_3z}x_{a_2z}x_{a_3z}\big] \\ & + \big[x_{a_2z}x_{a_3z}x_{a_4z}x_{a_1z}x_{a_4z}x_{a_2z} - x_{a_2z}x_{a_4z}x_{a_1z}x_{a_4z}x_{a_3z}x_{a_2z}\big] \\ & + \big[x_{a_4z}x_{a_1z}x_{a_3z}x_{a_2z}x_{a_3z}x_{a_4z} - x_{a_4z}x_{a_3z}x_{a_2z}x_{a_3z}x_{a_1z}x_{a_4z}\big]. \end{aligned} \tag{5.89}$$

When the only common variable between two type-one elements is their leading variables, the S-polynomial between them reduces to a new type called type-three.

Example 5.3.20. *Let* $g_1 = \underline{x_{a_1z}x_{a_2z}x_{a_3z}x_{a_1z}} + x_{a_2z}x_{a_3z}x_{a_1z}x_{a_2z} + x_{a_3z}x_{a_1z}x_{a_2z}x_{a_3z}$ *and let* $g_2 = \underline{x_{a_1z}x_{a_2z}x_{a_1z}} + x_{a_2z}x_{a_1z}x_{a_2z}$ *be type-one elements of degrees* 4 *and* 3*, respectively. Then*

$S(g_1, g_2, 6)$ *reduces w.r.t. lower degree terms to*

$$g = \underline{x_{a_1z}x_{a_2z}x_{a_3z}x_{a_2z}x_{a_1z}x_{a_2z}} - x_{a_3z}x_{a_1z}x_{a_2z}x_{a_3z}x_{a_2z}x_{a_1z}. \tag{5.90}$$

When two type-one have more common variables, then new and more complicated types emerge by reduction of S-polynomial between them, as well when taking S-polynomial between different types they become more and more complicated and difficult to introduce them and we must restrict ourselves in this work to the above new types.

Remark 5.3.21. *We have checked the conjecture 5.2.1 on some other types, but the computations involved increase so much that there is no room here to add any of those details. It is a case by case study but this may not end. Thus, we will limit our evidence to type-one only.*

Chapter 6

Decomposition of $FK(n)$

In this chapter we review the usual degree, S_n-degree and set partition degree and find Hilbert series, character map and Frobenius image of some finite examples of $FK(n)$ along with different decompositions. We find character and Frobenius image of degree 1 component of $FK(n)$. Under S_n for Fomin-Kirillov algebra of general n, with usual degree 2 and set partition degree types $q_2^2 q_1^{n-4}$ and $q_3 q_1^{n-3}$, we find the character decomposition and explain why for any other degrees of Fomin-Kirillov algebra we also have representation stability.

6.1 Homogeneous decomposition w.r.t. different degrees

Definition 6.1.1. *Let N be a monoid with operation $\odot$. An algebra A is called N-graded if $A = \bigoplus_{\alpha \in N} A^\alpha$ such that if $x \in A^\alpha$ and $y \in A^\beta$, then $xy \in A^{\alpha \odot \beta}$.*

6.1.1 Usual degree homogeneous decomposition

Usual degree of a monomial is defined by the number of variables in the monomial. A polynomial is said to be homogeneous if all its monomial have the same degree. For example $x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}$ for $1 \leq i < j < k \leq n$, that is one of the relations of $FK(n)$, is homogeneous as every monomial appearing in it is of the same degree 2. It turns out that all the relations of $FK(n)$, i.e., the generators of the ideal of $FK(n)$ are homogeneous w.r.t. usual degree, so the ideal is homogeneous w.r.t usual degree, i.e., $I = \bigoplus_d I^{(d)}$, as an ideal generated by homogeneous generators is homogeneous (see Appendix B.2).

Definition 6.1.2. *We define the usual degree on $FK(n)$ by the map $deg : FK(n) \to N$ defined by*

$$x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} + I \mapsto \begin{cases} 0 & \textit{if } x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} \in I, \\ k & \textit{otherwise.} \end{cases}$$

One can check that the above definition is well-defined (see Appendix C.1).

Since the free algebra generated by the generators of $FK(n)$ and the ideal of $FK(n)$ are both homogeneous w.r.t. usual degree d, and we have a well defined degree for $FK(n)$, then the quotient of them is a graded algebra w.r.t. this degree, i.e., we have

$$FK(n) = \bigoplus_{d \geq 0} FK^d(n), \tag{6.1}$$

where $FK^d(n)$ is the usual degree d component of $FK(n)$. The above decomposition is both a homogeneous decomposition and representation decomposition into S_n-invariant parts, as the action S_n given in Proposition 3.1.3 preserves the degree, hence the above decomposition is an S_n decomposition into invariant parts.

6.1.2 S_n-degree homogeneous decomposition

For symmetric group S_n, we define S_n-degree for a monomial in variables x_{ij} by map $x_{ij} \longmapsto (ij)$ and multiplicative extension to monomials, where (ij) is the transposition of i and j.

Example 6.1.3. *$x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}$ for $1 \leq i < j < k \leq n$ is one of the relations of $FK(n)$. The S_n-degree of the 1st, 2nd and the 3-rd monomial in this relation are $(ij)(jk) = (ijk), (jk)(ik) = (ijk)$ and $(ik)(ij) = (ijk)$ respectively that are the same, so the relation is homogeneous.*

We denoted S_n-degree by σ. One can check that all of the relations of $FK(n)$ are homogeneous w.r.t. S_n- degree, so we have homogeneous ideal as before, i.e., $I = \bigoplus_\sigma I^\sigma$, where I^σ is the set of all homogeneous elements of I with the same S_n-degree σ.

Definition 6.1.4. *We define the S_n-degree on $FK(n)$ by the map*

$$S_n\text{-}degree : FK(n) \to S_n$$

defined by

$$x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} + I \mapsto \begin{cases} \epsilon & \text{if } x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} \in I, \\ \sigma = (i_1j_1)(i_2j_2)\cdots(i_kj_k) & \text{otherwise}, \end{cases}$$

where $(i_l j_l) \in S_n$ *is the transposition of indexes* i_l *and* j_l*.*

One can check that the above definition is well-defined (see Appendix C.2).

Since the free algebra generated by the generators of $FK(n)$ and the defining ideal of $FK(n)$ are both homogeneous w.r.t. S_n-degree, and we have a well-defined S_n-degree, the quotient of them is a graded algebra. Thus we have the following homogeneous decomposition w.r.t. S_n-degree;

$$FK(n) = \bigoplus_{\sigma} FK^{\sigma}(n), \tag{6.2}$$

where FK^{σ} is the set of elements of $FK(n)$ of the same S_n degree $\sigma \in S_n$. However this is not a representation decomposition (for which we need a coarser decomposition by conjugacy class. See Section 6.2).

6.1.3 Set-partition degree homogeneous decomposition

Definition 6.1.5. 1. *We call* $\alpha = \{A_1, A_2, \cdots, A_t\}$ *a set partition on set* A*,*

$$\text{if } \begin{cases} A_i \subset A, \\ A_i \cap A_j = \phi, \text{ for } i \neq j, \\ A_i \cup A_2 \cup \cdots \cup A_t = A. \end{cases}$$

2. *For a monomial* M *in variables* x_{ij}*, the set partition degree of* M *denoted by sp-deg*(M) *is defined as the finest set partition of* $[n]$ *for which* i *and* j *lie in the same block if and only if* x_{ij} *appears in* M*. If we denote the blocks by* $m_k,\ k = 1, 2, \cdots, t$*, then we have sp-deg* $(M) = \{m_1, m_2, \cdots, m_t\}$*, where* m_k *are disjoint subsets of* $[n]$ *such that*

$\cup_{k=1}^{t} m_k = [n]$ *and such that* i *and* j *lie in the same* m_k *if and only if* x_{ij} *appears in* M. *Two monomials* M *and* M' *with different set partition degrees, i.e.,* $m_k \neq m'_k$ *for some* k*, could have the same cardinal number* $|m_k| = |m'_k|$ *for* $k = 1, 2, \cdots, t$*, i.e., they could belong to the same set partition type (sp-type) denoted by* $(|m_1|, |m_2|, \cdots, |m_t|)$ *with* $|m_k|$ *here sorted in decreasing order.*

3. *We define the product of two set partitions as the finest set partition that coarsens both.*

Example 6.1.6. *Let* $M = x_{13}x_{24}x_{35}x_{15}$. *Then the set partition degree of* M *is sp-deg*$(M) = \{\{135\}, \{24\}\}$*, and Then the set partition type of* M *is sp-type*$(M) = (3, 2)$.

Example 6.1.7. $x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij}$ *for* $1 \leq i < j < k \leq n$ *is one of the relations of* $FK(n)$. *Every monomial in this polynomial is of set-partition degree* $\{\{ijk\}\}$ *(set partition type* (3)*). So this relation is homogeneous w.r.t. set-partition degree.*

One can check that all of the relations of $FK(n)$ are homogeneous w.r.t. the set partition degree, so we have homogeneous ideal w.r.t. this degree as before, i.e., $I = \bigoplus_{\lambda} I^{\lambda}$, where I^{λ} is the set of all homogeneous elements of I with the same set partition degree λ .

Definition 6.1.8. *We define set-partition degree on* $FK(n)$ *by the map*
sp-degree: $FK(n) \to P(n)$*, where* $P(n)$ *is the set of all partitions of* $[n]$*, defined by*

$$M + I \mapsto \begin{cases} \emptyset & \text{if } M \in I, \\ \text{sp-deg}(M) & \text{otherwise.} \end{cases}$$

One can check that the above definition is well-defined (see Appendix C.3).

Since the free algebra generated by the generators of $FK(n)$ and the ideal of $FK(n)$ are both homogeneous w.r.t. set partition degree, and we have a well-defined set partition degree for $FK(n)$, then the quotient of them is a graded algebra w.r.t. this degree, i.e., we have

$$FK(n) = \bigoplus_{\lambda} FK^{\lambda}(n), \tag{6.3}$$

where FK^{λ} is the sum of homogeneous components with the same set partition degree λ. However the above decomposition is not a representation decomposition for which we need a coarser decomposition by set partition type. See Section 6.2.

6.2 Different representation decompositions

6.2.1 Representation decomposition w.r.t. usual degree

Here we recall Equation (6.1),

$$FK(n) = \bigoplus_{d \geq 0} FK^{d}(n), \tag{6.4}$$

which is both a homogeneous decomposition and representation decomposition into S_n-invariant parts $FK^d(n)$ (as S_n defines action on $FK(n)$ but does not change the number of variables, i.e., usual degree d is invariant under S_n).

6.2.2 Representation decomposition w.r.t. conjugacy class

$$FK(n) = \bigoplus_{\mu} FK^{\mu}(n), \tag{6.5}$$

where $FK^{\mu}(n)$ is the sum of all degree σ elements, where σ belongs to the conjugacy class indexed by μ. It is both a homogeneous decomposition and representation decomposition into S_n-invariant parts $FK^{\mu}(n)$ (as S_n defines action on $FK(n)$ and while the action of S_n changes the S_n-degree but does not change the conjugacy class, as a monomial of degree σ, when acted by a permutation ν is sent to a monomial of degree $\nu^{-1}\sigma\nu$ which is a conjugate of σ by definition of conjugacy, i.e., conjugacy class is invariant under S_n and so $FK^{\mu}(n)$ is S_n-invariant).

6.2.3 Representation decomposition by set partition type

$$FK(n) = \bigoplus_{\phi} FK^{\phi}(n), \tag{6.6}$$

where $FK^{\phi}(n)$ is the sum of all elements with set partition degree λ with the same set partition type indexed by ϕ. It is both a homogeneous decomposition and representation decomposition into S_n-invariant parts $FK^{\phi}(n)$ (as S_n defines action on $FK(n)$ and while it changes the set partition degree of a monomial by changing the indexes in each part of the set partition degree of the monomial, but does not change the number of indexes in each part, i.e., does not change the set partition type. For example when a monomial of set partition degree $\{\{i,j,k\},\{l,m\}\}$ with set partition type $(3,2)$ is acted upon by permutation σ, the set partition degree changes to $\{\{\sigma(i),\sigma(j),\sigma(k)\},\{\sigma(l),\sigma(m)\}\}$, where the indexes in each part change but the number of indexes in each part stays the same 3 and 2, i.e., the set partition type $(3,2)$ is preserved, i.e., set partition type is S_n-invariant).

Notations

We use notation q^i (superscript i) for usual degree i. We use t_i to denote a cycle of length i when denoting a conjugacy class. For example the conjugacy class indexed by $\mu = (123)(45)(67)(8)$ is denoted by $t_3t_2^2t_1$. We use notation q_i (subscript) to denote a part of size i when showing a set partition type. For example a monomial with set partition degree $\{\{134\},\{25\},\{67\}\}$ belongs to set partition type $(3,2,2)$ denoted by $q_3q_2^2$. As another example $FK^{(2)}_{q_3q_1^2}(5)$ denotes elements of $FK(5)$ of usual degree 2 and set partition type $(3,1,1)$ denoted by $q_3q_1^2$.
Here we investigate some cases of representation decomposition by usual degree, conjugacy class and set partition type.

6.2.4 Representation decomposition of $FK(3)$ by different degrees

Relations

The set of relations of $FK(3)$ from Equation (1.2) are

$$R(3) = \{x_{12}^2 = 0, x_{23}^2 = 0, x_{13}^2 = 0, x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12} = 0, x_{12}x_{13} - x_{23}x_{12} + x_{13}x_{23} = 0\} \tag{6.7}$$

Gröbner basis

$$GB(3) = \{x_{12}^2, x_{23}^2, x_{13}^2, x_{12}x_{23}-x_{23}x_{13}-x_{13}x_{12}, x_{12}x_{13}-x_{23}x_{12}+x_{13}x_{23}, x_{23}x_{13}x_{23}+x_{13}x_{23}x_{13}\}. \tag{6.8}$$

Basis

$$\begin{aligned} B[FK(3)] = \{&1, x_{12}, x_{23}, x_{13}, x_{23}x_{12}, x_{23}x_{13}, x_{13}x_{12}, x_{13}x_{23},\\ &x_{23}x_{13}x_{12},\ x_{13}x_{23}x_{13},\ x_{13}x_{23}x_{12}, x_{13}x_{23}x_{13}x_{12}\}. \end{aligned} \tag{6.9}$$

Hilbert series

From the above basis we can read the Hilbert series.

$$H_3 = q^4 + 3q^3 + 4q^2 + 3q + 1 \tag{6.10}$$

Remark 6.2.1. *In our character tables, conjugacy classes and irreducible characters are indexed by partition. So for example in Table 6.1, conjugacy class* c_{21} *stands for the one with partition* $3 = 2+1$*, with conjugacy class representative say*$(12)(3)$ *and with its corresponding irreducible character* χ_{21}.

Remark 6.2.2. *Under the action of* S_n*, the component of the character of* $FK(n)$ *on conjugacy class indexed by* $\sigma \in S_n$*, is the sum of the coefficients of* $z \in B[FK(n)]$ *in* $\sigma(z)$ *expanded in elements of* $B[FK(n)]$*, when* z *runs in* $B[FK(n)]$.

Table 6.1: Character table of S_3. Conjugacy classes and irred. characters indexed by partitions.

	$\lvert c_{111}\rvert = 1$	$\lvert c_{21}\rvert = 3$	$\lvert c_3\rvert = 2$	Frobenius image
χ_3	1	1	1	s_3
χ_{21}	2	0	-1	s_{21}
χ_{111}	1	-1	1	s_{111}

Character

$$char_{S_3}[FK(3)] = (12,0,0) = 2\chi_3 + 4\chi_{21} + 2\chi_{111}, \tag{6.11}$$

where χ_3, χ_{21} and χ_{111} are irreducible characters of S_3 in Table 6.1.

Remark 6.2.3. *In decomposition of a character χ in irreducible characters $\chi^{(i)}$ of a group G, the coefficients of $\chi^{(i)}$ are derived as follows*

$$\chi = \sum_i m_i \chi^{(i)}, \;\; m_i = \langle \chi, \chi^{(i)} \rangle = \frac{1}{|G|} \sum_K |K| \chi_K \overline{\chi}_K^{(i)}, \tag{6.12}$$

where sum is over conjugacy classes. We apply this formula frequently here after on.

Frobenius image

$$F_{S_3}(char_{S_3}[FK(3)]) = 2s_3 + 4s_{21} + 2s_{111}, \tag{6.13}$$

where s_3, s_{21} and s_{111} are Schur functions.

Representation decomposition by usual degree

$B[FK(3)]$ in Equation (6.9) can be divided into following sets by usual degree.

$$\begin{gathered} B_0 = \{1\},\ B_1 = \{x_{12}, x_{23}, x_{13}\},\ B_2 = \{x_{23}x_{12}, x_{23}x_{13}, x_{13}x_{12}, x_{13}x_{23}\}, \\ B_3 = \{x_{23}x_{13}x_{12}, x_{13}x_{23}x_{13}, x_{13}x_{23}x_{12}\},\ B_4 = \{x_{13}x_{23}x_{13}x_{12}\}. \end{gathered} \tag{6.14}$$

Characters are

$$\begin{aligned} char_{S_3}[FK^{(0)}(3)] &= (1,1,1) = \chi_3, \\ char_{S_3}[FK^{(1)}(3)] &= (3,-1,0) = \chi_{21} + \chi_{111}, \\ char_{S_3}[FK^{(2)}(3)] &= (4,0,-2) = 2\chi_{21}, \\ char_{S_3}[FK^{(3)}(3)] &= (3,-1,0) = \chi_{21} + \chi_{111}, \\ char_{S_3}[FK^{(4)}(3)] &= (1,1,1) = \chi_3, \end{aligned} \tag{6.15}$$

where $\chi_3, \chi_{21}, \chi_{111}$ are irreducible characters of S_3, in Table 7.1. Also with Frobenius images

$$\begin{aligned} F_{S_3}(char_{S_3}[FK^{(0)}(3)]) &= s_3, \\ F_{S_3}(char_{S_3}[FK^{(1)}(3)]) &= s_{21} + s_{111}, \\ F_{S_3}(char_{S_3}[FK^{(2)}(3)]) &= 2s_{21}, \\ F_{S_3}(char_{S_3}[FK^{(3)}(3)]) &= s_{21} + s_{111}, \\ F_{S_3}(char_{S_3}[FK^{(4)}(3)]) &= s_3. \end{aligned} \tag{6.16}$$

Therefore

$$char_{S_3}[FK(3)](q) = \chi_3 + (\chi_{21} + \chi_{111})q + 2\chi_{21}q^2 + (\chi_{21} + \chi_{111})q^3 + \chi_3 q^4, \tag{6.17}$$

with Frobenius image

$$F_{S_3}(char_{S_3}[FK(3)](q) = (1+q^4)s_3 + (q+2q^2+q^3)s_{21} + (q+q^3)s_{111}. \tag{6.18}$$

Remark 6.2.4. *Equations* (6.17) *and* (6.18) *provide finer decompositions than Equation* (6.11) *and* (6.13) *and reduce to them upon putting* $q = 1$ *in them.*

Representation decomposition by conjugacy class

The basis of $FK(3)$ is partitioned into

$$\begin{aligned} B_{t_1^3} =& \{1, x_{13}x_{23}x_{13}x_{12}\},\ B_{t_2t_1} = \{x_{12}, x_{13}, x_{23}, , x_{23}x_{13}x_{12}, x_{13}x_{23}x_{13}, x_{13}x_{23}x_{12}\}, \\ &\text{and } B_{t_3} = \{x_{23}x_{12}, x_{23}x_{13}, x_{13}x_{12}, x_{13}x_{23}\} \end{aligned}$$

of conjugacy classes denoted by t_1^3, t_2t_1 and t_3 respectively. One can easily check that each of them are closed under S_3 and that S_3 defines an actions on each of $FK^{t_1^3}(3)$, $FK^{t_2t_1}(3)$ and $FK^{t_3}(3)$, with the following characters.

$$\begin{aligned} char_{S_3}[FK^{t_1^3}(3)] &= (2,2,2) = 2\chi_3, \\ char_{S_3}[FK^{t_2t_1}(3)] &= (6,-2,0) = 2\chi_{111} + 2\chi_{21}, \\ char_{S_3}[FK^{t_3}(3)] &= (4,0,-2) = 2\chi_{21}, \end{aligned} \tag{6.19}$$

with Frobenius images

$$\begin{aligned} F_{S_3}(char_{S_3}[FK^{t_1^3}(3)]) &= 2s_3, \\ F_{S_3}(char_{S_3}[FK^{t_2t_1}(3)]) &= 2s_{21} + 2s_{111}, \\ F_{S_3}(char_{S_3}[FK^{t_3}(3)]) &= 2s_{21}. \end{aligned} \tag{6.20}$$

Therefore we have

$$char_{S_3}[FK(3)](t_1, t_2, t_3) = 2\chi_3 t_1^3 + (2\chi_{111} + 2\chi_{21})t_2t_1 + 2\chi_{21}t_3, \tag{6.21}$$

with Frobenius image

$$F_{S_3}(char_{S_3}[FK(3)](t_1, t_2, t_3)) = (2t_1^3)s_3 + 2(t_2t_1 + t_3)s_{21} + (2t_2t_1)s_{111}. \tag{6.22}$$

Remark 6.2.5. *Equations* (6.21) *and* (6.22) *are finer than Equations* (6.11) *and* (6.13) *respectively and reduce to them upon letting* $t_1 = t_2 = t_3 = 1$*. We also notice that letting* $t_1 = t_2 = t_3 = q$ *in* (6.21) *and* (6.22) *does not relate them to* (6.17) *and* (6.18)*, the reason is that monomials with the same permutation degree (*S_n *degree) could have different usual degrees. For example while in the basis of* $FK(3)$ *both* x_{12} *and* $x_{13}x_{23}x_{13}$ *have the same permutation degree* $(12)(3)$*, but they are of different usual degrees* 1 *and* 3*.*

Representation decomposition by set partition type

The basis of $FK(3)$ can be divided into the following 3 parts by set partition degrees type,
$B_{q_1^3} = \{1\}$, $B_{q_2q_1} = \{x_{12}, x_{23}, x_{13}\}$ and
$B_{q_3} = \{x_{23}x_{12}, x_{23}x_{13}, x_{13}x_{12}, x_{13}x_{23}, x_{23}x_{13}x_{12}, x_{13}x_{23}x_{13}, x_{13}x_{23}x_{12}, x_{13}x_{23}x_{13}x_{12}\}$ of set

partition types denoted by q_1^3, q_2q_1 and q_3 respectively. Then

$$\begin{aligned} char_{S_3}[FK_{q_1^3}(3)] &= (1,1,1) = \chi_3, \\ char_{S_3}[FK_{q_2q_1}(3)] &= (3,-1,0) = \chi_{21} + \chi_{111}, \\ char_{S_3}[FK_{q_3}(3)] &= (8,0,-1) = \chi_3 + 3\chi_{21} + \chi_{111}. \end{aligned} \tag{6.23}$$

with Frobenius images

$$\begin{aligned} F_{S_3}(char_{S_3}[FK_{q_1^3}(3)]) &= s_3, \\ F_{S_3}(char_{S_3}[FK_{q_2q_1}(3)]) &= s_{21} + s_{111}, \\ F_{S_3}(char_{S_3}[FK_{q_3}(3)]) &= s_3 + 3s_{21} + s_{111}. \end{aligned} \tag{6.24}$$

Therefore

$$char_{S_3}[FK(3)](q_1,q_2,q_3) = (\chi_3)q_1^3 + (\chi_{21} + \chi_{111})q_2q_1 + (\chi_3 + 3\chi_{21} + \chi_{111})q_3, \tag{6.25}$$

with Frobenius image

$$F_{S_3}(char_{S_3}[FK(3)](q_1,q_2,q_3)) = (q_1^3 + q_3)s_3 + (q_2q_1 + 3q_3)s_{21} + (q_2q_1 + q_3)s_{111}. \tag{6.26}$$

Remark 6.2.6. *Equations* (6.25) *and* (6.26) *represent finer decompositions of* $FK(3)$ *than Equations* (6.11) *and* (6.13) *respectively and reduce to them upon letting* $q_1 = q_2 = q_3 = 1$ *in Equations* (6.25) *and* (6.26)*. We also notice that letting* $q_1 = q_2 = q_3 = q$ *in Equations* (6.25) *and* (6.26) *does not relate them to Equations* (6.17) *and* (6.18)*, the reason is that monomials of the same set partition degree could have different usual degrees. For example in the basis* $FK(3)$*, while* $x_{23}x_{12}$ *and* $x_{13}x_{23}x_{13}x_{12}$ *both have the same set partition degree* $\{\{1,2,3\}\}$ *but are of usual degrees* 2 *and* 4*.*

Representation decomposition by set partition type and usual degree

Using the basis of $FK(3)$ partitioned by usual degree in Equation (6.14) we derive easily the following character map and Frobenius image in terms of set partition type. This together with the ones for higher $n = 4$, $n = 5$,...will be helpful later on to find a pattern for general n.

$$\begin{aligned} char_{S_3}[FK^{(0)}(3)] = char_{S_3}[FK^{(0)}_{q_1^3}(3)] = (1,1,1) = \chi_3. \\ F_{S_3}(char_{S_3}[FK^{(0)}_{q_1^3}(3)]) = s_3. \end{aligned} \tag{6.27}$$

$$\begin{aligned} char_{S_3}[FK^{(1)}_{q_2q_1}(3)] = char_{S_3}[FK^{(1)}(3)] = (3,-1,0) = \chi_{21} + \chi_{111}. \\ F_{S_3}(char_{S_3}[FK^{(1)}_{q_2q_1}(3)]) = F(char_{S_3}[FK^{(1)}(3)]) = s_{21} + s_{111}. \end{aligned} \tag{6.28}$$

$$\begin{aligned} char_{S_4}[FK^{(1)}_{q_2q_1^2}(4)] = (6,0,-2,0,0) = \chi_{31} + \chi_{211}. \\ F_{S_4}(char_{S_4}[FK^{(1)}_{q_2q_1^2}(4)]) = s_{31} + s_{211}. \end{aligned} \tag{6.29}$$

$$\begin{aligned} char_{S_5}[FK^{(1)}_{q_2q_1^3}(5)] = (10,2,-2,1,-1,0,0) = \chi_{41} + \chi_{311}. \\ F_{S_5}(char_{S_5}[FK^{(1)}_{q_2q_1^3}(5)]) = s_{41} + s_{311}. \end{aligned} \tag{6.30}$$

$$\begin{aligned} char_{S_3}[FK^{(2)}_{q_3}(3)] = (4,0,-2) = 2\chi_{21}. \\ F_{S_3}(char_{S_3}[FK^{(2)}_{q_3}(3)]) = 2s_{21}. \end{aligned} \tag{6.31}$$

$$\begin{aligned} char_{S_3}[FK^{(3)}_{q_3}(3)] = (3,-1,0) = \chi_{21} + \chi_{111}. \\ F_{S_3}(char_{S_3}[FK^{(3)}_{q_3}(3)]) = s_{21} + s_{111}. \end{aligned} \tag{6.32}$$

$$\begin{aligned} char_{S_3}[FK^{(4)}_{q_3}(3)] = (1,1,1) = \chi_3. \\ F_{S_3}(char_{S_3}[FK^{(4)}_{q_3}(3)]) = s_3. \end{aligned} \tag{6.33}$$

6.3 Decomposition of $FK^d(4),\ d = 2, 3, 4$, conjugacy class, set partition type, representation decompositions.

Relations of $FK(4)$

$$\begin{aligned}R(4) =\{&x_{12}^2 = 0,\ x_{13}^2 = 0,\ x_{14}^2 = 0,\ x_{23}^2 = 0,\ x_{24}^2 = 0,\ x_{34}^2 = 0,\\ &x_{12}x_{34} - x_{34}x_{12} = 0,\ x_{13}x_{24} - x_{24}x_{13} = 0,\ x_{23}x_{14} - x_{14}x_{23} = 0,\\ &x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12} = 0,\ x_{12}x_{13} - x_{23}x_{12} + x_{13}x_{23} = 0,\\ &x_{12}x_{24} - x_{24}x_{14} - x_{14}x_{12} = 0,\ x_{12}x_{14} - x_{24}x_{12} + x_{14}x_{24} = 0,\\ &x_{13}x_{34} - x_{34}x_{14} - x_{14}x_{13} = 0,\ x_{13}x_{14} - x_{34}x_{13} + x_{14}x_{34} = 0,\\ &x_{23}x_{34} - x_{34}x_{24} - x_{24}x_{23} = 0,\ x_{23}x_{24} - x_{34}x_{23} + x_{24}x_{34} = 0\}.\end{aligned} \tag{6.34}$$

Gröbner basis

$$\begin{aligned}GB(4) =\{&x_{12}^2, x_{13}^2,\ x_{14}^2,\ x_{23}^2,\ x_{24}^2, x_{34}^2,\\ &x_{12}x_{34} - x_{34}x_{12},\ x_{13}x_{24} - x_{24}x_{13},\ x_{23}x_{14} - x_{14}x_{23},\\ &x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12},\ x_{12}x_{13} - x_{23}x_{12} + x_{13}x_{23},\ x_{12}x_{24} - x_{24}x_{14} - x_{14}x_{12},\\ &x_{12}x_{14} - x_{24}x_{12} + x_{14}x_{24},\ x_{13}x_{34} - x_{34}x_{14} - x_{14}x_{13},\ x_{13}x_{14} - x_{34}x_{13} + x_{14}x_{34},\\ &x_{23}x_{34} - x_{34}x_{24} - x_{24}x_{23},\ x_{23}x_{24} - x_{34}x_{23} + x_{24}x_{34},\\ &x_{23}x_{13}x_{23} + x_{13}x_{23}x_{13},\ x_{24}x_{14}x_{24} + x_{14}x_{24}x_{14},\\ &x_{34}x_{14}x_{34} + x_{14}x_{34}x_{14},\ x_{34}x_{24}x_{34} + x_{24}x_{34}x_{24}\}.\end{aligned} \tag{6.35}$$

Table 6.2: character table of S_4

c.c.	$\|c_{1111}\| = 1$	$\|c_{211}\| = 6$	$\|c_{22}\| = 3$	$\|c_{31}\| = 8$	$\|c_4\| = 6$	
c.c. rep	(1)(2)(3)(4)	(1,2)(3)(4)	(1,2)(3,4)	(1,2,3)(4)	(1,2,3,4)	F.I
χ_4	1	1	1	1	1	s_4
χ_{31}	3	1	-1	0	-1	s_{31}
χ_{22}	2	0	2	-1	0	s_{22}
χ_{211}	3	-1	-1	0	1	s_{211}
χ_{1111}	1	-1	1	1	-1	s_{1111}

6.3.1 Degree 2 component of $FK(4)$

Basis of $FK^{(2)}(4)$

$$\begin{aligned} B[FK^{(2)}(4)] =\{ & x_{13}x_{23}, x_{23}x_{13}, x_{14}x_{24}, x_{24}x_{14}, x_{14}x_{34}, x_{34}x_{14}, x_{24}x_{34}, x_{34}x_{24}, x_{23}x_{12}, x_{24}x_{12}, \\ & x_{34}x_{13}, x_{34}x_{23}, x_{34}x_{12}, x_{24}x_{13}, x_{14}x_{23}, x_{24}x_{23}, x_{14}x_{13}, x_{14}x_{12}, x_{13}x_{12}\}. \end{aligned} \tag{6.36}$$

Character map

$$char_{S_4}FK^{(2)}(4) = (19, -1, 3, -2, -1). \tag{6.37}$$

However this character is reducible and is written in terms of irreducible characters of S_4 in Table 6.2.

$$char_{S_4}FK^{(2)}(4) = 2\chi_{31} + 3\chi_{22} + 2\chi_{211} + \chi_{1111}. \tag{6.38}$$

Table 6.3: $char_{S_4} FK^{(i)}(4)$ for $i = 1, 2, 3, 4$.

	$\lvert c_{1111}\rvert = 1$	$\lvert c_{211}\rvert = 6$	$\lvert c_{22}\rvert = 3$	$\lvert c_{31}\rvert = 8$	$\lvert c_4\rvert = 6$
	$(1)(2)(3)(4)$	$(1,2)(3)(4)$	$(1,2)(3,4)$	$(1,2,3)(4)$	$(1,2,3,4)$
$char_{S_4}[FK^{(0)}(4)]$	1	1	1	1	1
$char_{S_4}[FK^{(1)}(4)]$	6	0	-2	0	0
$char_{S_4}[FK^{(2)}(4)]$	19	-1	3	-2	-1
$char_{S_4}[FK^{(3)}(4)]$	42	-2	-6	0	-2
$char_{S_4}[FK^{(4)}(4)]$	71	-1	7	-1	-1

We have also the Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(2)}(4)]) = 2s_{31} + 3s_{22} + 2s_{211} + s_{1111}. \tag{6.39}$$

Conjugacy class decomposition of $FK^{(2)}(4)$

In terms of conjugacy class we divide the basis of $FK^{(2)}(4)$, Equation (6.36), into 2 sets:

$$\begin{aligned} B_{t_2^2} =& \{x_{34}x_{12}, x_{24}x_{13}, x_{14}x_{23}\}, \text{ and} \\ B_{t_3t_1} =& \{x_{13}x_{23}, x_{23}x_{13}, x_{14}x_{24}, x_{24}x_{14}, x_{14}x_{34}, x_{34}x_{14}, x_{24}x_{34}, x_{34}x_{24}, x_{23}x_{12}, x_{24}x_{12}, \\ & x_{34}x_{13}, x_{34}x_{23}, x_{24}x_{23}, x_{14}x_{13}, x_{14}x_{12}, x_{13}x_{12}\}. \end{aligned} \tag{6.40}$$

with the following character and Frobenius maps:

$$\begin{aligned} char_{S_4}[FK^{(2)}_{t_2^2}(4)] = (3, -1, 3, 0, -1) = \chi_{22} + \chi_{1111}. \\ F(char_{S_4}[FK^{(2)}_{t_2^2}(4)]) = s_{22} + s_{1111}. \end{aligned} \tag{6.41}$$

And

$$\begin{aligned} char_{S_4}[FK^{(2)}_{t_3t_1}(4)] = (16,0,0,-2,0) = 2\chi_{31} + 2\chi_{22} + 2\chi_{211}. \\ F(char_{S_4}[FK^{(2)}_{t_3t_1}(4)]) = 2s_{31} + 2s_{22} + 2s_{211}. \end{aligned} \tag{6.42}$$

Therefore

$$char_{S_4}[FK^{(2)}(4)](t_1,t_2,t_3) = (\chi_{22} + \chi_{1111})t_2^2 + (2\chi_{31} + 2\chi_{22} + 2\chi_{211})t_3t_1, \tag{6.43}$$

with Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(2)}(4)](t_1,t_2,t_3)) = 2t_3t_1s_{31} + (2t_3t_1 + t_2^2)s_{22} + 2t_3t_1s_{211} + t_2^2s_{1111}. \tag{6.44}$$

Set partition type decomposition of $FK^{(2)}(4)$

The basis of $FK^{(2)}(4)$, Equation (6.36), is divided into 2 sets:

$$B_{q_2^2} = \{x_{34}x_{12}, x_{24}x_{13}, x_{14}x_{23}\},$$

of set partition type q_2^2 and

$$\begin{aligned} B_{q_3q_1} = \{&x_{13}x_{23}, x_{23}x_{13}, x_{14}x_{24}, x_{24}x_{14}, x_{14}x_{34}, x_{34}x_{14}, x_{24}x_{34}, x_{34}x_{24}, x_{23}x_{12}, x_{24}x_{12}, \\ &x_{34}x_{13}, x_{34}x_{23}, x_{24}x_{23}, x_{14}x_{13}, x_{14}x_{12}, x_{13}x_{12}\}, \end{aligned}$$

of set partition type denoted by q_3q_1.

However this same partition is invariant under conjugacy class as well as set partition type, so we are going to have essentially the same representation decomposition.

$$\begin{aligned} char_{S_4}[FK^{(2)}_{q_2^2}(4)] &= (3,-1,3,0,-1) \\ &= \chi_{22} + \chi_{1111}, \end{aligned} \tag{6.45}$$

with Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(2)}_{q_2^2}(4)]) = s_{22} + s_{1111}. \tag{6.46}$$

and

$$\begin{aligned} char_{S_4}[FK^{(2)}_{q_3q_1}(4)] &= (16,0,0,-2,0) \\ &= 2\chi_{31} + 2\chi_{22} + 2\chi_{211}, \end{aligned} \tag{6.47}$$

with Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(2)}_{q_3q_1}(4)]) = 2s_{31} + 2s_{22} + 2s_{211}, \tag{6.48}$$

or just

$$\begin{aligned} F_{S_4}(char_{S_4}[FK^{(2)}(4)])(q_1,q_2,q_3) =& (2q_3q_1)s_{31} + (2q_3q_1 + q_2^2)s_{22} \\ &+ (2q_3q_1)s_{211} + q_2^2 s_{1111}. \end{aligned} \tag{6.49}$$

Remark 6.3.1. *Equation* (6.49) *is a finer decomposition than Equation* (6.39) *and reduces to that upon putting* $q_1 = q_2 = q_3 = 1$.

Table 6.4: $F_{S_4}[FK^{(i)}(4)]$ and $F_{S_3}[FK^{(i)}(3)]$, $i = 1, 2, 3, 4$.

	$F.I$		$F.I$
$F_{S_4}(char_{S_4}[FK^{(0)}(4)])$	s_4	$F_{S_3}(char_{S_3}[FK^{(0)}(3)]$	s_3
$F_{S_4}(char_{S_4}[FK^{(1)}(4)])$	$s_{31} + s_{211}$	$F_{S_3}(char_{S_3}[FK^{(1)}(3)])$	$s_{21} + s_{111}$
$F_{S_4}(char_{S_4}[FK^{(2)}(4)])$	$2s_{31} + 3s_{22} + 2s_{211} + s_{1111}$	$F_{S_3}(char_{S_3}[FK^{(2)}(3)])$	$2s_{21}$
$F_{S_4}(char_{S_4}[FK^{(3)}(4)])$	$6s_{31} + 3s_{22} + 6s_{211} + 2s_{1111}$	$F_{S_3}(char_{S_3}[FK^{(3)}(3)])$	$s_{21} + s_{111}$
$F_{S_4}(char_{S_4}[FK^{(4)}(4)])$	$3s_4 + 8s_{31} + 8s_{22} + 8s_{211} + 4s_{1111}$		

6.3.2 Degree 3 component of $FK(4)$

$$\begin{aligned} B[FK^{(3)}(4)] = \{ & x_{14}x_{13}x_{12}, x_{14}x_{13}x_{23}, x_{14}x_{23}x_{12}, x_{14}x_{23}x_{13}, x_{14}x_{34}x_{12}, x_{14}x_{34}x_{13}, \\ & x_{14}x_{34}x_{23}, x_{14}x_{34}x_{24}, x_{14}x_{34}x_{14}, x_{14}x_{24}x_{12}, x_{14}x_{24}x_{13}, x_{14}x_{24}x_{23}, \\ & x_{14}x_{24}x_{34}, x_{14}x_{24}x_{14}, x_{23}x_{13}x_{12}, x_{24}x_{14}x_{12}, x_{24}x_{14}x_{13}, x_{24}x_{14}x_{23}, \\ & x_{24}x_{14}x_{34}, x_{24}x_{13}x_{12}, x_{24}x_{13}x_{23}, x_{24}x_{23}x_{12}, x_{24}x_{23}x_{13}, x_{34}x_{24}x_{12}, \\ & x_{34}x_{24}x_{13}, x_{34}x_{24}x_{23}, x_{34}x_{24}x_{14}, x_{34}x_{14}x_{12}, x_{34}x_{14}x_{13}, x_{34}x_{14}x_{23}, \\ & x_{34}x_{23}x_{12}, x_{34}x_{23}x_{13}, x_{34}x_{13}x_{12}, x_{34}x_{13}x_{23}, x_{13}x_{23}x_{12}, x_{13}x_{23}x_{13}, \\ & x_{24}x_{34}x_{12}, x_{24}x_{34}x_{13}, x_{24}x_{34}x_{23}, x_{24}x_{34}x_{24}, x_{24}x_{34}x_{14}, x_{34}x_{14}x_{24} \}. \end{aligned} \tag{6.50}$$

Character map

$$\begin{aligned} char_{S_4}[FK^{(3)}(4)] &= (42, -2, -6, 0, -2) \\ &= 6\chi_{31} + 2\chi_{22} + 6\chi_{211} + 2\chi_{1111}, \end{aligned} \tag{6.51}$$

where χ_i are irreducible characters of S_4 in Table 6.2. Also with Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(3)}(4)]) = 6s_{31} + 2s_{22} + 6s_{211} + 2s_{1111}. \tag{6.52}$$

Representation decomposition of $FK^{(3)}(4)$ by conjugacy class

Here we divide degree 3 part of $FK(4)$ basis into 2 sets according to their conjugacy classes denoted by t_4 and $t_2t_1^2$.

$$\begin{aligned}
B_{t_4} = \{&x_{14}x_{13}x_{12}, x_{14}x_{23}x_{13}, x_{24}x_{14}x_{23}, x_{24}x_{13}x_{12}, x_{34}x_{24}x_{14}, x_{34}x_{14}x_{12}, x_{14}x_{13}x_{23},\\
&x_{14}x_{23}x_{12}, x_{24}x_{14}x_{13}, x_{34}x_{14}x_{23}, x_{24}x_{34}x_{14}, x_{14}x_{34}x_{12}, x_{24}x_{14}x_{34}, x_{34}x_{23}x_{13},\\
&x_{34}x_{13}x_{12}, x_{24}x_{34}x_{13}, x_{14}x_{34}x_{23}, x_{14}x_{24}x_{34}, x_{34}x_{23}x_{12}, x_{34}x_{13}x_{23},\\
&x_{24}x_{34}x_{12}, x_{14}x_{34}x_{24}, x_{14}x_{24}x_{23}, x_{24}x_{23}x_{13}, x_{34}x_{24}x_{13}, x_{14}x_{24}x_{13},\\
&x_{24}x_{13}x_{23}, x_{24}x_{23}x_{12}, x_{34}x_{24}x_{12}, x_{34}x_{14}x_{24}\}.\\
B_{t_2t_1^2} = \{&x_{14}x_{34}x_{13}, x_{24}x_{34}x_{23}, x_{14}x_{34}x_{14}, x_{23}x_{13}x_{12}, x_{14}x_{24}x_{12}, x_{34}x_{24}x_{23},\\
&x_{14}x_{24}x_{14}, x_{13}x_{23}x_{13}, x_{24}x_{14}x_{12}, x_{34}x_{14}x_{13}, x_{13}x_{23}x_{12}, x_{24}x_{34}x_{24}\}.
\end{aligned} \tag{6.53}$$

One can check that B_{t_4} and $B_{t_2t_1^2}$ are closed under the action of S_4.

Decomposition of character in irreducible characters/Frobenius image

$$\begin{aligned}
char_{S_4}[FK^{(3)}_{t_4}(4)] &= (30, 0, -6, 0, -2)\\
&= 5\chi_{31} + \chi_{22} + 4\chi_{211} + \chi_{1111}.\\
F_{S_4}(char_{S_4}[FK^{(3)}_{t_4}(4)]) &= 5s_{31} + s_{22} + 4s_{211} + s_{1111}.\\
char_{S_4}[FK^{(3)}_{t_2t_1^2}(4)] &= (12, -2, 0, 0, 0)\\
&= \chi_{31} + \chi_{22} + 2\chi_{211} + \chi_{1111}.\\
F_{S_4}(char_{S_4}[FK^{(3)}_{t_2t_1^2}(4)]) &= s_{31} + s_{22} + 2s_{211} + s_{1111}.
\end{aligned} \tag{6.54}$$

Adding them up we have

$$\begin{aligned} char_{S_4}[FK^{(3)}(4)](t_1,t_2,t_4) =&(5\chi_{31}+\chi_{22}+4\chi_{211}+\chi_{1111})t_4 \\ &+(\chi_{31}+\chi_{22}+2\chi_{211}+\chi_{1111})t_2t_1^2, \end{aligned} \tag{6.55}$$

with Frobenius image

$$\begin{aligned} F_{S_4}(char_{S_4}[FK^{(3)}(4)](t_1,t_2,t_4)) =&(5t_4+t_2t_1^2)s_{31}+(t_4+t_2t_1^2)s_{22} \\ &+(4t_4+2t_2t_1^2)s_{211}+(t_4+t_2t_1^2)s_{1111}. \end{aligned} \tag{6.56}$$

Remark 6.3.2. *Decompositions in Equations* (6.56) *is a finer decomposition than the ones in Equation* (6.52) *and reduces to that at* $t_1=t_2=t_4=1$.

Representation decomposition of $FK^3(4)$ by set partition type

As before, by set partition type we will have essentially the same decomposition as we had in conjugacy class. The reason is that referring to Equation (6.53) we see that the set of set partition type q_3q_1 is nothing but $B_{t_2t_1^2}$ and the set of set partition type q_4 is nothing but B_{t_4}, i.e., we have the same partition of basis invariant under conjugacy class and set partition type. Hence set partition decomposition is the same as conjugacy class and here we put them briefly just for reference.

$$\begin{aligned} char_{S_4}[FK^{(3)}_{q_4}(4)] &= (30,0,-6,0,-2) \\ &= 5\chi_{31}+\chi_{22}+4\chi_{211}+\chi_{1111}, \end{aligned} \tag{6.57}$$

with Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(3)}_{q_4}(4)] = 5s_{31} + s_{22} + 4s_{211} + s_{1111}. \tag{6.58}$$

We have also

$$\begin{aligned} char_{S_4}[FK^{(3)}_{q_3q_1}(4)] =&(12, -2, 0, 0, 0) \\ =&\chi_{31} + \chi_{22} + 2\chi_{211} + \chi_{1111}, \end{aligned} \tag{6.59}$$

with Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(3)}_{q_3q_1}(4)]) = s_{31} + s_{22} + 2s_{211} + s_{1111}. \tag{6.60}$$

Hence we have

$$\begin{aligned} char_{S_4}[FK^{(3)}(4)](q_1, q_3, q_4) =&(\chi_{31} + \chi_{22} + 2\chi_{211} + \chi_{1111})q_3q_1 \\ &+ (5s_{31} + s_{22} + 4s_{211} + s_{1111})q_4, \end{aligned} \tag{6.61}$$

with Frobenius image

$$\begin{aligned} F_{S_4}(char_{S_4}[FK^{(3)}(4)])(q_1, q_3, q_4) =&(q_3q_1 + 5q_4)s_{31} + (q_3q_1 + q_4)s_{22} \\ &+ (2q_3q_1 + 4q_4)s_{211} + (q_3q_1 + q_4)s_{1111}. \end{aligned} \tag{6.62}$$

Remark 6.3.3. *Equation* (6.62) *is a finer decomposition than Equation* (6.52) *and reduces to that upon putting* $q_1 = q_3 = q_4 = 1$.

6.3.3 Degree 4 component of $FK(4)$

Basis

$$\begin{aligned}
B[FK^{(4)}(4)] = \{&x_{13}x_{23}x_{13}x_{12},\ x_{14}x_{34}x_{14}x_{13},\ x_{14}x_{24}x_{14}x_{12},\ x_{24}x_{34}x_{24}x_{23},\\
&x_{14}x_{13}x_{23}x_{12},\ x_{14}x_{34}x_{13}x_{12},\ x_{14}x_{34}x_{23}x_{13},\ x_{14}x_{34}x_{24}x_{12},\\
&x_{14}x_{34}x_{14}x_{24},\ x_{14}x_{24}x_{13}x_{23},\ x_{14}x_{24}x_{23}x_{12},\ x_{14}x_{24}x_{34}x_{13},\\
&x_{14}x_{24}x_{34}x_{24},\ x_{14}x_{24}x_{14}x_{34},\ x_{24}x_{14}x_{13}x_{12},\ x_{24}x_{14}x_{23}x_{13},\\
&x_{24}x_{14}x_{34}x_{14},\ x_{24}x_{14}x_{34}x_{23},\ x_{24}x_{23}x_{13}x_{12},\ x_{24}x_{34}x_{13}x_{23},\\
&x_{24}x_{34}x_{23}x_{12},\ x_{24}x_{34}x_{24}x_{14},\ x_{24}x_{34}x_{14}x_{12}, x_{34}x_{24}x_{13}x_{12},\\
&x_{34}x_{24}x_{23}x_{13},\ x_{34}x_{24}x_{14}x_{13},\ x_{34}x_{14}x_{13}x_{23},\ x_{34}x_{14}x_{23}x_{12},\\
&x_{34}x_{13}x_{23}x_{13},\ x_{34}x_{14}x_{24}x_{14},\ x_{34}x_{14}x_{24}x_{23},\ x_{14}x_{13}x_{23}x_{13},\\
&x_{14}x_{23}x_{13}x_{12},\ x_{14}x_{34}x_{13}x_{23},\ x_{14}x_{34}x_{23}x_{12},\ x_{14}x_{34}x_{24}x_{13},\\
&x_{14}x_{34}x_{24}x_{14},\ x_{14}x_{34}x_{24}x_{23},\ x_{14}x_{34}x_{14}x_{12},\ x_{14}x_{34}x_{14}x_{23},\\
&x_{14}x_{24}x_{13}x_{12},\ x_{14}x_{24}x_{23}x_{13},\ x_{14}x_{24}x_{34}x_{12},\ x_{14}x_{24}x_{34}x_{14},\\
&x_{14}x_{24}x_{34}x_{23},\ x_{14}x_{24}x_{14}x_{13},\ x_{14}x_{24}x_{14}x_{23},\ x_{24}x_{14}x_{13}x_{23},\\
&x_{24}x_{14}x_{23}x_{12},\ x_{24}x_{14}x_{34}x_{12},\ x_{24}x_{14}x_{34}x_{13},\ x_{24}x_{14}x_{34}x_{24},\\
&x_{24}x_{13}x_{23}x_{12},\ x_{24}x_{13}x_{23}x_{13},\ x_{24}x_{34}x_{13}x_{12},\ x_{24}x_{34}x_{23}x_{13},\\
&x_{24}x_{34}x_{24}x_{12},\ x_{24}x_{34}x_{24}x_{13},\ x_{24}x_{34}x_{14}x_{13},\ x_{24}x_{34}x_{14}x_{23},\\
&x_{24}x_{34}x_{14}x_{24},\ x_{34}x_{24}x_{13}x_{23},\ x_{34}x_{24}x_{23}x_{12},\ x_{34}x_{24}x_{14}x_{12},\\
&x_{34}x_{24}x_{14}x_{23},\ x_{34}x_{14}x_{13}x_{12},\ x_{34}x_{14}x_{23}x_{13},\ x_{34}x_{23}x_{13}x_{12},\\
&x_{34}x_{13}x_{23}x_{12},\ x_{34}x_{14}x_{24}x_{12},\ x_{34}x_{14}x_{24}x_{13}\}.
\end{aligned} \tag{6.63}$$

Character map and Frobenius images

$$\begin{aligned} char_{S_4}[FK^{(4)}(4)] =&(71, -1, 7, -1, -1) \\ =&3\chi_4 + 8\chi_{31} + 8\chi_{22} + 8\chi_{211} + 4\chi_{1111}, \end{aligned} \tag{6.64}$$

where χ_i are irreducible characters of S_4 in Table 6.2. Also with Frobenius image

$$F_{S_4}(char_{S_4}[FK^{(4)}(4)]) = 3s_4 + 8s_{31} + 8s_{22} + 8s_{211} + 4s_{1111}. \tag{6.65}$$

Representation decomposition of $FK^{(4)}(4)$ by conjugacy class

Basis

Basis of $FK^4(4)$ in Equation (6.63) is decomposed in conjugacy class invariant parts

$$B(FK^{(4)}(4)) = B(FK^{(4)}_{t_1^4}(4)) \cup B(FK^{(4)}_{t_2^2}(4)) \cup B(FK^{(4)}_{t_3 t_1}(4)), \tag{6.66}$$

where

$$B(FK^{(4)}_{t_1^4}(4)) = \{x_{13}x_{23}x_{13}x_{12},\ x_{14}x_{34}x_{14}x_{13},\ x_{14}x_{24}x_{14}x_{12},\ x_{24}x_{34}x_{24}x_{23}\}. \tag{6.67}$$

$$
\begin{aligned}
B(FK^{(4)}_{t_2^2}(4)) = \{&x_{14}x_{13}x_{23}x_{12},\ x_{14}x_{34}x_{13}x_{12},\ +x_{14}x_{34}x_{23}x_{13},\ +x_{14}x_{34}x_{24}x_{12},\\
&x_{14}x_{34}x_{14}x_{24},\ x_{14}x_{24}x_{13}x_{23},\ x_{14}x_{24}x_{23}x_{12},\ x_{14}x_{24}x_{34}x_{13},\\
&x_{14}x_{24}x_{34}x_{24},\ x_{14}x_{24}x_{14}x_{34},\ x_{24}x_{14}x_{13}x_{12},\ x_{24}x_{14}x_{23}x_{13},\\
&x_{24}x_{14}x_{34}x_{14},\ x_{24}x_{14}x_{34}x_{23},\ x_{24}x_{23}x_{13}x_{12},\ x_{24}x_{34}x_{13}x_{23},\\
&x_{24}x_{34}x_{23}x_{12},\ x_{24}x_{34}x_{24}x_{14},\ x_{24}x_{34}x_{14}x_{12},\ x_{34}x_{24}x_{13}x_{12},\\
&x_{34}x_{24}x_{23}x_{13},\ x_{34}x_{24}x_{14}x_{13},\ x_{34}x_{14}x_{13}x_{23},\ x_{34}x_{14}x_{23}x_{12},\\
&x_{34}x_{13}x_{23}x_{13},\ x_{34}x_{14}x_{24}x_{14},\ x_{34}x_{14}x_{24}x_{23}\},
\end{aligned}
\tag{6.68}
$$

and

$$
\begin{aligned}
B(FK^{(4)}_{t_3t_1}(4)) = \{&x_{14}x_{13}x_{23}x_{13},\ x_{14}x_{23}x_{13}x_{12},\ x_{14}x_{34}x_{13}x_{23},\ x_{14}x_{34}x_{23}x_{12},\\
&x_{14}x_{34}x_{24}x_{13},\ x_{14}x_{34}x_{24}x_{14},\ x_{14}x_{34}x_{24}x_{23},\ x_{14}x_{34}x_{14}x_{12},\\
&x_{14}x_{34}x_{14}x_{23},\ x_{14}x_{24}x_{13}x_{12},\ x_{14}x_{24}x_{23}x_{13},\ x_{14}x_{24}x_{34}x_{12},\\
&x_{14}x_{24}x_{34}x_{14},\ x_{14}x_{24}x_{34}x_{23},\ x_{14}x_{24}x_{14}x_{13},\ x_{14}x_{24}x_{14}x_{23},\\
&x_{24}x_{14}x_{13}x_{23},\ x_{24}x_{14}x_{23}x_{12},\ x_{24}x_{14}x_{34}x_{12},\ x_{24}x_{14}x_{34}x_{13},\\
&x_{24}x_{14}x_{34}x_{24},\ x_{24}x_{13}x_{23}x_{12},\ x_{24}x_{13}x_{23}x_{13},\ x_{24}x_{34}x_{13}x_{12},\\
&x_{24}x_{34}x_{23}x_{13},\ x_{24}x_{34}x_{24}x_{12},\ x_{24}x_{34}x_{24}x_{13},\ x_{24}x_{34}x_{14}x_{13},\\
&x_{24}x_{34}x_{14}x_{23},\ x_{24}x_{34}x_{14}x_{24},\ x_{34}x_{24}x_{13}x_{23},\ x_{34}x_{24}x_{23}x_{12},\\
&x_{34}x_{24}x_{14}x_{12},\ x_{34}x_{24}x_{14}x_{23},\ x_{34}x_{14}x_{13}x_{12},\ x_{34}x_{14}x_{23}x_{13},\\
&x_{34}x_{23}x_{13}x_{12},\ x_{34}x_{13}x_{23}x_{12},\ x_{34}x_{14}x_{24}x_{12},\ x_{34}x_{14}x_{24}x_{13}\}.
\end{aligned}
\tag{6.69}
$$

Characters and Frobenius images:

$$\begin{aligned} char_{S_4}[FK^{(4)}_{t_1^4}(4)] &= (4,2,0,1,0) \\ &= \chi_4 + \chi_{31}. \end{aligned} \tag{6.70}$$

$$F_{S_4}(char_{S_4}[FK^{(4)}_{t_1^4}(4)]) = s_4 + s_{31}. \tag{6.71}$$

$$\begin{aligned} char_{S_4}[FK^{(4)}_{t_2^2}(4)] &= (27,-3,7,0,-1) \\ &= \chi_4 + 2\chi_{31} + 4\chi_{22} + 3\chi_{211} + 3\chi_{1111}. \end{aligned} \tag{6.72}$$

$$F_{S_4}(char_{S_4}[FK^{(4)}_{t_2^2}(4)]) = s_4 + 2s_{31} + 4s_{22} + 3s_{211} + 3s_{1111}. \tag{6.73}$$

$$\begin{aligned} char_{S_4}[FK^{(4)}_{t_3t_1}(4)] &= (40,0,0,-2,0) \\ &= \chi_4 + 5\chi_{31} + 4\chi_{22} + 5\chi_{211} + \chi_{1111}. \end{aligned} \tag{6.74}$$

$$F_{S_4}(char_{S_4}[FK^{(4)}_{t_3t_1}(4)]) = s_4 + 5s_{31} + 4s_{22} + 5s_{211} + s_{1111}. \tag{6.75}$$

Hence we have

$$\begin{aligned} F_{S_4}(char_{S_4}[FK^{(4)}(4)]) =& (t_3t_1 + t_2^2 + t_1^4)s_4 + (5t_3t_1 + 2t_2^2 + t_1^4)s_{31} \\ & + 4(t_3t_1 + t_2^2)s_{22} + (5t_3t_1 + 3t_2^2)s_{211} + (t_3t_1 + 3t_2^2)s_{1111}. \end{aligned} \tag{6.76}$$

Representation decomposition of $FK^{(4)}(4)$ by set partition type

By set partition type, unlike conjugacy class, the basis of $FK^4(4)$ is decomposed into only two invariant parts as it follows.

$$B(FK^{(4)}(4)) = B(FK^{(4)}_{q_3q_1}(4)) \cup B(FK^{(4)}_{q_4}(4)), \tag{6.77}$$

where

$$B(FK^{(4)}_{q_3q_1}(4)) = \{x_{13}x_{23}x_{13}x_{12},\ x_{14}x_{34}x_{14}x_{13},\ x_{14}x_{24}x_{14}x_{12},\ x_{24}x_{34}x_{24}x_{23}\}, \tag{6.78}$$

and

$$B(FK^{(4)}_{q_4}(4)) = B(FK^{(4)}(4)) \setminus B(FK^{(4)}_{q_3q_1}(4)). \tag{6.79}$$

Characters and Frobenius images

$$\begin{aligned} char_{S_4}[FK^{(4)}_{q_3q_1}(4)] &= (4, 2, 0, 1, 0) \\ &= \chi_4 + \chi_{31}. \end{aligned} \tag{6.80}$$

$$F_{S_4}(char_{S_4}[FK^{(4)}_{q_3q_1}(4)]) = s_4 + s_{31}. \tag{6.81}$$

$$\begin{aligned} char_{S_4}[FK^{(4)}_{q_4}(4)] &= (67, -3, 7, -2, -1) \\ &= 2\chi_4 + 7\chi_{31} + 8\chi_{22} + 8\chi_{211} + 4\chi_{1111}. \end{aligned} \tag{6.82}$$

$$F_{S_4}(char_{S_4}[FK^{(4)}_{q_4}(4)]) = 2s_4 + 7s_{31} + 8s_{22} + 8s_{211} + 4s_{1111}. \tag{6.83}$$

Hence we have

$$\begin{aligned} &F_{S_4}(char_{S_4}[FK^{(4)}(4)])(q_1, q_3, q_4) = \\ &\qquad (2q_4 + q_3q_1)s_4 + (7q_4 + q_3q_1)s_{31} + 8q_4s_{22} + 8q_4s_{211} + 4q_4s_{1111}. \end{aligned} \tag{6.84}$$

Remark 6.3.4. *Equations* (6.84) *and* (6.76) *are finer decompositions than Equation* (6.65) *and reduce to that upon putting* $q_1 = q_3 = q_4 = 1$ *and* $t_1 = t_2 = t_3 = 1$ *respectively.*

6.3.4 Degree one component $FK^{(1)}(5)$

$$\begin{aligned} char_{S_5}[FK^{(1)}_{q_2q_1^3}(5)] = char_{S_5}[FK^{(1)}(5)] &= (10, 2, -2, 1, -1, 0, 0) \\ &= \chi_{41} + \chi_{311}, \end{aligned} \tag{6.85}$$

where χ_i are irreducible characters of S_5 in Table 6.5, and with Frobenius image

$$F_{S_5}(char_{S_5}[FK^{(1)}_{q_2q_1^3}(5)]) = s_{41} + s_{311}. \tag{6.86}$$

6.3.5 Degree 2 component of $FK(5)$

$$\begin{aligned} char_{S_5}[FK^{(2)}(5)] =&(55, 1, 3, -2, -2, -1, 0) \\ =&2\chi_{41} + 3\chi_{32} + 2\chi_{311} + 3\chi_{221} + \chi_{2111} + \chi_{11111}, \end{aligned} \tag{6.87}$$

where χ_i are irreducible characters of S_5 in Table 6.5, and with Frobenius image

$$F_{S_5}(char_{S_5}[FK^{(2)}(5)]) = 2s_{41} + 3s_{32} + 2s_{311} + 3s_{221} + s_{2111} + s_{11111}. \tag{6.88}$$

6.3.6 Representation decomposition of $FK^2(5)$ by set partition type

$$B(FK^{(2)}(5)) = B(FK^{(2)}_{q_3q_1^2}(5)) \cup B(FK^{(2)}_{q_2^2q_1}(5)) \tag{6.89}$$

Table 6.5: character table of S_5.

	$\lvert c_{11111}\rvert = 1$	$\lvert c_{2111}\rvert = 10$	$\lvert c_{221}\rvert = 15$	$\lvert c_{311}\rvert = 20$	$\lvert c_{32}\rvert = 20$	$\lvert c_{41}\rvert = 30$	$\lvert c_{5}\rvert = 24$	F.I.
χ_5	1	1	1	1	1	1	1	s_5
χ_{41}	4	2	0	1	-1	0	-1	s_{41}
χ_{32}	5	1	1	-1	1	-1	0	s_{32}
χ_{311}	6	0	-2	0	0	0	1	s_{311}
χ_{221}	5	-1	1	-1	-1	1	0	s_{221}
χ_{2111}	4	-2	0	1	1	0	-1	s_{2111}
χ_{11111}	1	-1	1	1	-1	-1	1	s_{11111}

where

$$\begin{aligned} B(FK^{(2)}_{q_3q_1^2}(5)) =\{ & x_{15}x_{12}, x_{15}x_{13}, x_{15}x_{14}, x_{15}x_{25}, x_{15}x_{35}, x_{15}x_{45}, x_{14}x_{12}, x_{14}x_{13}, x_{14}x_{24}, \\ & x_{14}x_{34}, x_{25}x_{12}, x_{25}x_{15}, x_{25}x_{24}, x_{25}x_{13}, x_{25}x_{45}, x_{25}x_{35}, x_{35}x_{13}, x_{35}x_{15}, \\ & x_{35}x_{23}, x_{35}x_{25}, x_{35}x_{34}, x_{35}x_{45}, x_{45}x_{14}, x_{45}x_{15}, x_{45}x_{24}, x_{45}x_{25}, x_{45}x_{34}, \\ & x_{45}x_{35}, x_{24}x_{12}, x_{24}x_{14}, x_{24}x_{23}, x_{24}x_{34}, x_{34}x_{13}, x_{34}x_{14}, x_{34}x_{23}, x_{34}x_{24}, \\ & x_{13}x_{12}, x_{13}x_{23}, x_{23}x_{12}, x_{23}x_{13}\} \end{aligned} \tag{6.90}$$

and

$$\begin{aligned} B(FK^{(2)}_{q_2^2q_1}(5)) =\{ & x_{15}x_{23}, x_{15}x_{24}, x_{15}x_{34}, x_{14}x_{23}, x_{25}x_{13}, x_{25}x_{14}, x_{25}x_{34}, \\ & x_{35}x_{12}, x_{35}x_{14}, x_{35}x_{24}, x_{45}x_{12}, x_{45}x_{13}, x_{45}x_{23}, x_{24}x_{13}, x_{34}x_{12}\}. \end{aligned} \tag{6.91}$$

with associated characters and Frobenius images

$$\begin{aligned} char_{S_5}[FK^{(2)}_{q_3q_1^2}(5)] &= (40, 4, 0, -2, -2, 0, 0) \\ &= 2\chi_{41} + 2\chi_{32} + 2\chi_{311} + 2\chi_{221}, \end{aligned} \tag{6.92}$$

$$F_{S_5}(char_{S_5}[FK^{(2)}_{q_3q_1^2}(5)]) = 2s_{41} + 2s_{32} + 2s_{311} + 2s_{221}, \tag{6.93}$$

Table 6.6: Character/Frobenius image of $[FK^{(2)}_{q_2^2}(4)]$, $[FK^{(2)}_{q_2^2q_1}(5)]$, and $[FK^{(2)}_{q_2^2q_1^2}(6)]$.

$FK^{(2)}_{q_2^2}(4)$	$FK^{(2)}_{q_2^2q_1}(5)$	$FK^{(2)}_{q_2^2q_1^2}(6)$
$(3,-1,3,0,-1)$	$(15,-3,3,0,0,-1,0)$	$(45,-3,1,9,0,0,0,-1,-1,0,0)$
$s_{22}+s_{1111}$	$s_{32}+s_{221}+s_{2111}+s_{11111}$	$s_{42}+s_{321}+s_{3111}+s_{222}+s_{21111}$

and

$$\begin{aligned} char_{S_5}[FK^{(2)}_{q_2^2q_1}(5)] &= (15,-3,+3,0,0,-1,0) \\ &= \chi_{32}+\chi_{221}+\chi_{2111}+\chi_{11111}, \\ F_{S_5}(char_{S_5}[FK^{(2)}_{q_2^2q_1}(5)]) &= s_{32}+s_{221}+s_{2111}+s_{11111}, \end{aligned} \tag{6.94}$$

where χ_{ab} are irreducible characters of S_5 in Table 6.5. Combining the above we have:

$$\begin{aligned} F_{S_5}[FK^{(2)}(5)](q_1,q_2,q_3) =&(2q_3q_1^2)s_{41}+(q_2^2q_1+2q_3q_1^2)s_{32}+(2q_3q_1^2)s_{311} \\ &+(q_2^2q_1+2q_3q_1^2)s_{221}+(q_2^2q_1)s_{2111}+(q_2^2q_1)s_{11111}, \end{aligned} \tag{6.95}$$

which reduces to Equation (6.88) upon letting $q_1 = q_2 = q_3 = 1$, as expected.

Remark 6.3.5. *Here we have the same decomposition of the basis invariant under conjugacy classes indexed by $t_2^2t_1$ and $t_3t_1^2$ as under set partition types $q_2^2q_1$ and $q_3^2q_1^2$ respectively. So the conjugacy class decomposition of $FK^{(2)}(5)$ is essentially the same as set partition type decompositions.*

6.3.7 Representation decomposition of $FK^3(5)$ by set partition type

$$B(FK^{(3)}(5)) = B(FK^{(3)}_{q_3q_1^2}(5)) \cup B(FK^{(3)}_{q_3q_2}(5)) \cup B(FK^{(3)}_{q_4q_1}(5)), \tag{6.96}$$

where

$$\begin{aligned} B(FK^{(3)}_{q_3q_1^2}(5)) =\{&x_{15}x_{35}x_{15}, x_{15}x_{25}x_{12}, x_{15}x_{45}x_{14}, x_{15}x_{35}x_{13}, x_{15}x_{45}x_{15}, x_{15}x_{25}x_{15},\\ &x_{14}x_{24}x_{12}, x_{14}x_{24}x_{14}, x_{14}x_{34}x_{13}, x_{14}x_{34}x_{14}, x_{25}x_{15}x_{12}, x_{25}x_{35}x_{23},\\ &x_{25}x_{35}x_{25}, x_{25}x_{45}x_{24}, x_{25}x_{45}x_{25}, x_{35}x_{15}x_{13}, x_{35}x_{25}x_{23}, x_{35}x_{45}x_{34},\\ &x_{35}x_{45}x_{35}, x_{45}x_{15}x_{14}, x_{45}x_{25}x_{24}, x_{45}x_{35}x_{34}, x_{24}x_{14}x_{12}, x_{24}x_{34}x_{23},\\ &x_{24}x_{34}x_{24}, x_{34}x_{14}x_{13}, x_{24}x_{24}x_{23}, x_{13}x_{23}x_{12}, x_{13}x_{23}x_{13}, x_{23}x_{13}x_{12}\} \end{aligned} \tag{6.97}$$

with character

$$\begin{aligned} char_{S_5}[FK^{3}_{q_3q_1^2}(5)] &= (30, 0, -2, 0, 0, 0, 0)\\ &= \chi_{41} + \chi_{32} + 2\chi_{311} + \chi_{221} + \chi_{2111}, \end{aligned} \tag{6.98}$$

where χ_{ab} are irreducible characters of S_5 in Table 6.5, and with Frobenius image

$$F_{S_5}[FK^{3}_{q_3q_1^2}(5)] = s_{41} + s_{32} + 2s_{311} + s_{221} + s_{2111}. \tag{6.99}$$

Also

$$\begin{aligned} B(FK^{(3)}_{q_3q_2}(5)) =\{&x_{15}x_{14}x_{23}, x_{15}x_{24}x_{13}, x_{15}x_{24}x_{23}, x_{15}x_{24}x_{34}, x_{15}x_{25}x_{34}, x_{15}x_{34}x_{12},\\ &x_{15}x_{34}x_{23}, x_{15}x_{34}x_{24}, x_{15}x_{35}x_{24}, x_{15}x_{45}x_{23}, x_{25}x_{14}x_{13}, x_{25}x_{14}x_{23},\\ &x_{25}x_{14}x_{34}, x_{25}x_{15}x_{34}, x_{25}x_{24}x_{13}, x_{25}x_{34}x_{12}, x_{25}x_{34}x_{13}, x_{25}x_{34}x_{14},\\ &x_{25}x_{35}x_{14}, x_{25}x_{45}x_{13}, x_{35}x_{14}x_{12}, x_{35}x_{14}x_{23}, x_{35}x_{14}x_{24}, x_{35}x_{15}x_{24},\\ &x_{35}x_{24}x_{12}, x_{35}x_{24}x_{13}, x_{35}x_{24}x_{14}, x_{35}x_{25}x_{14}, x_{35}x_{34}x_{12}, x_{35}x_{45}x_{12},\\ &x_{45}x_{13}x_{23}, x_{45}x_{13}x_{12}, x_{45}x_{14}x_{23}, x_{45}x_{15}x_{23}, x_{45}x_{23}x_{12}, x_{45}x_{23}x_{13},\\ &x_{45}x_{24}x_{13}, x_{45}x_{25}x_{13}, x_{45}x_{34}x_{12}, x_{45}x_{35}x_{12}\} \end{aligned} \tag{6.100}$$

Table 6.7: Character/Frobenius image of $FK_{q_4}^{(3)}(4)$ and $FK_{q_4q_1}^{(3)}(5)$

$FK_{q_4}^{(3)}(4)$	$FK_{q_4q_1}^{(3)}(5)$
$(30, 0, -6, 0, -2)$	$(150, 0, -6, 0, 0, -2, 0)$
$5s_{31} + s_{22} + 4s_{211} + s_{1111}$	$5s_{41} + 6s_{32} + 9s_{311} + 5s_{221} + 5s_{2111} + s_{11111}$

with character

$$\begin{aligned} char_{S_5}[FK^3_{q_3q_2}(5)] &= (40, -4, 0, -2, +2, 0, 0) \\ &= 2\chi_{32} + 2\chi_{311} + 2\chi_{221} + 2\chi_{2111}, \end{aligned} \tag{6.101}$$

with Frobenius image

$$F_{S_5}[FK^3_{q_3q_2}(5)] = 2s_{32} + 2s_{311} + 2s_{221} + 2s_{2111}. \tag{6.102}$$

Also $B[FK_{q_4q_1}^{(3)}(5)]$ consists of 150 elements and result to the following character.

$$\begin{aligned} char_{S_5}[FK_{q_4q_1}^{(3)}(5)] =&(150, 0, -6, 0, 0, -2, 0) \\ =&5\chi_{41} + 6\chi_{32} + 9\chi_{311} + 5\chi_{221} + 5\chi_{2111} + \chi_{11111}. \end{aligned} \tag{6.103}$$

Frobenius image:

$$F_{S_5}[FK_{q_4q_1}^{(3)}(5)] = 5s_{41} + 6s_{32} + 9s_{311} + 5s_{221} + 5s_{2111} + s_{11111}. \tag{6.104}$$

Here we summarize the characters for usual degree up to 3 and for different set partition types for $n = 2, 3, 4, 5$ along with their associated Frobenius images in Tables 6.8, 6.6, and 6.7.

Table 6.8: $char_{S_n}[FK^{d}_{q_k q_1^{n-k}}(n)]$ for $d = 2, 3$, $k = 1, 2, 3$, and $n = 3, 4, 5$.

$char_{S_2}[FK^{(0)}(2)]$	$char_{S_3}[FK^{(0)}(3)]$	$char_{S_4}[FK^{(0)}(4)]$	$char_{S_5}[FK^{(0)}(5)]$
$(1,1)$	$(1,1,1)$	$(1,1,1,1,1)$	$(1,1,1,1,1,1,1)$
s_2	s_3	s_4	s_5
$char_{S_2}[FK^{(1)}_{q_2}(2)]$	$char_{S_3}[FK^{(1)}_{q_2q_1}(3)]$	$char_{S_4}[FK^{(1)}_{q_2q_1^2}(4)]$	$char_{S_5}[FK^{(1)}_{q_2q_1^3}(5)]$
$(1,-1)$	$(3,-1,0)$	$(6,0,-2,0,0)$	$(10,2,-2,1,-1,0,0)$
s_{11}	$s_{21}+s_{111}$	$s_{31}+s_{211}$	$s_{41}+s_{311}$
	$char_{S_3}[FK^{(2)}_{q_3}(3)]$	$char_{S_4}[FK^{(2)}_{q_3q_1}(4)]$	$char_{S_5}[FK^{(2)}_{q_3q_1^2}(5)]$
	$(4,0,-2)$	$(16,0,0,-2,0)$	$(40,4,0,-2,-2,0,0)$
	$2s_{21}$	$2s_{31}+2s_{22}+2s_{211}$	$2s_{41}+2s_{32}+2s_{311}+2s_{221}$
	$char_{S_3}[FK^{(3)}_{q_3}(3)]$	$char_{S_4}[FK^{(3)}_{q_3q_1}(4)]$	$char_{S_5}[FK^{(3)}_{q_3q_1^2}(5)]$
	$(3,-1,0)$	$(12,-2,0,0,0)$	$(30,0,-2,0,0,0,0)$
	$s_{21}+s_{111}$	$s_{31}+s_{22}+2s_{211}+s_{1111}$	$s_{41}+s_{32}+2s_{311}+s_{221}+s_{2111}$
		$char_{S_4}[FK^{(4)}_{q_3q_1}(4)]$	
		$(4,2,0,1,0)$	
		s_4+s_{31}	
		$char_{S_4}[FK^{(4)}_{q_4}(4)]$	
		$(67,-3,\ 7,-2,-1)$	
		$2s_4+7s_{31}+8s_{22}$ $+8s_{211}+4s_{1111}$	

6.4 Character of $FK(n)$ of usual degrees 0 and 1.

Proposition 6.4.1. *$F_{S_n}[FK^{(0)}(n)] = s_n$, $n \geq 2$, where s_n is Schur function.*

Proof. $FK^{(0)}(n) = \{1\}$, then for any $\sigma \in S_n$, we have $\sigma 1 = 1$. So we have

$$char_{S_n}[FK^{(0)}(n)] = (\underbrace{1, 1, \cdots, 1}_{p(n)}) = \chi_n, \text{ so } F_{S_n}(char_{S_n}[FK^{(0)}(n)]) = s_n.$$

This completes the proof. □

Remark 6.4.2. *In the sequel f^{λ} stands for the number of standard tableaux of shape $\lambda \vdash n$, where according to Hook Formula $f^{\lambda} = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$ where $h_{i,j}$ stands for Hook length of the associated node (i,j).*

Proposition 6.4.3. *Let* $\sigma \in S_n$ *be of type* $(1^{a_1}, 2^{a_2}, \cdots, n^{a_n})$*. Then the value of* $char_{S_n}[FK^{(1)}(n)]$ *on conjugacy class indexed by* σ *is*

$$char_{S_n}[FK^{(1)}(n)](\sigma) = -a_2 + \binom{a_1}{2}. \tag{6.105}$$

Proof. $B[FK^{(1)}(n)] = \{x_{ab},\ 1 \leq a < b \leq n\}$. Then $|B[FK^{(1)}(n)]|$ = number of 2-sets in n object = $\binom{n}{2}$. We consider $x_{ab} \in B[FK^{(1)}(n)]$. a and b could also appear in $\sigma \in S_n$. We consider the following cases of appearance of a, b in σ.

1. If a and b appear in σ as fixed points. $\sigma z = z$, contribution to the character at σ equals the number of 2-sets of fixed points in σ, which is nothing but $\binom{a_1}{2}$.
2. If a and b are not both fixed but are in different cycles, we have no contribution to the character.
3. If a and b both appear in σ in a 2-cycle, then $\sigma x_{ab} = (a, b)x_{ab} = -x_{ab}$, so a factor (-1) for each 2-cycle appear in σ, so a contribution of $-a_2$ to the character.
4. If a and b appear in k-cycles for $k > 2$, then it results in no contribution. The reason is that the action of $\sigma \in S_n$ on $z \in$ Basis, develops new indexes in z that did not exist before, so causes no eigenvector.

Adding up the contribution due to the above cases we have $char_{S_n}[FK^{(1)}(n)](\sigma) = -a_2 + \binom{a_1}{2}$. This completes the proof. □

Proposition 6.4.4. *Let permutation* $\sigma \in S_n$ *be of cycle type* $(1^{a_1}, 2^{a_2}, \cdots n^{a_n})$*, and let* $\chi_{n-1,1}$

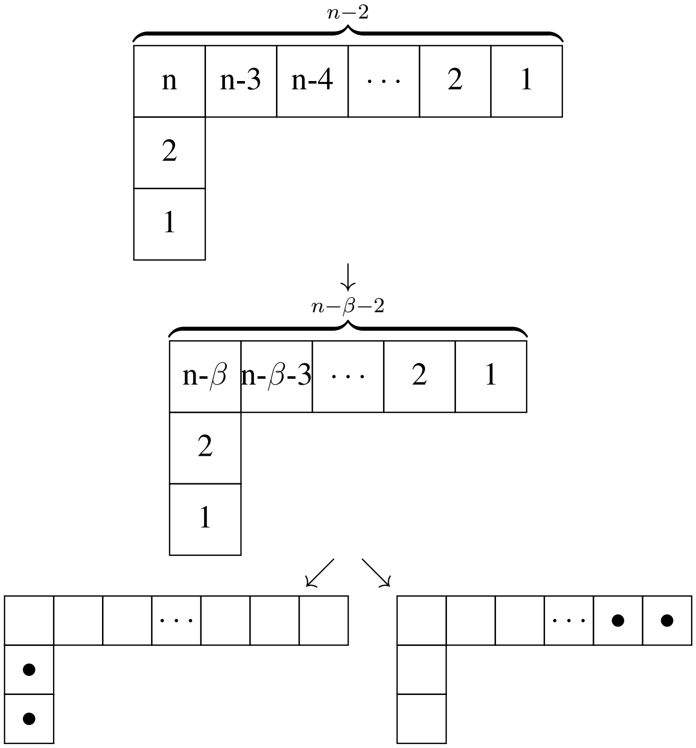
and $\chi_{n-2,1,1}$ *be irreducible characters of* S_n*. Then*

$$\chi_{n-1,1}(\sigma) + \chi_{n-2,1,1}(\sigma) = -a_2 + \binom{a_1}{2}. \tag{6.106}$$

Proof. Application of Murnaghan-Nakayama rule to evaluate $\chi_{n-2,1,1}$ at σ, at each step reduces the arm of the diagram of shape $\lambda = (n-2,1,1)$ by length of a cycle of length $i > 2$, until we have cut a total length of $\beta = \sum_{i>2} ia_i$ off the arm and are left instead of σ with σ' with only 1-cycles and/or 2-cycles, i.e., we are left with an involution $\sigma' = (\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})$.

$$\chi_{n-2,1,1}(\sigma) = \cdots = \chi_{n-2-\beta,1,1}(\sigma') = \chi_{n-2-\beta,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}), \tag{6.107}$$

where $\beta = \sum_{i>2} ia_i$. So $n - \beta = a_1 + 2a_2$. The following diagram illustrates the below process of calculation.



We apply Murnaghan-Nakayama Rule to calculate the following.

$$\begin{aligned}
\chi_{n-2,1,1}(\sigma) =&\chi_{n-2-\beta,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) = \chi_{a_1+2a_2-2,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
=&-\chi_{a_1+2a_2-2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1}) + \chi_{a_1+2a_2-2-2,1,1}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})\\
=&-1 + \chi_{a_1+2a_2-2-2,1,1}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})\\
=&-1 - \chi_{a_1+2a_2-2-2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1}) + \chi_{a_1+2a_2-2-2-2,1,1}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})\\
=&-1-1 + \chi_{a_1+2a_2-2-2-2,1,1}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})\\
&\vdots\\
=&\underbrace{-1-1-1\cdots-1}_{a_2}+\chi_{a_1+2a_2-2-2a_2,1,1}(1,1\cdots,1)\\
=&-a_2 + \chi_{a_1-2,1,1}(\underbrace{1,1\cdots,1}_{a_1})
\end{aligned}$$

Hence we have

$$\chi_{n-2,1,1}(\sigma) = -a_2 + \chi_{a_1-2,1,1}(\underbrace{1,1\cdots,1}_{a_1}). \tag{6.108}$$

However by Hook formula we have

$$\chi_{a_1-2,1,1}(\underbrace{1,1\cdots,1}_{a_1}) = \frac{a_1!}{(a_1-3)!a_1.2.1} = \frac{(a_1-1)(a_1-2)}{2}. \tag{6.109}$$

substituting (6.109) into (6.108) we have

$$\chi_{n-2,1,1}(\sigma) = -a_2 + \frac{(a_1-1)(a_1-2)}{2}. \tag{6.110}$$

On the other hand we have [21]

$$\chi_{n-1,1}(\sigma) = (\text{number of fixed points in } \sigma) - 1 = a_1 - 1. \tag{6.111}$$

Adding up (6.110) and (6.111) we have

$$\begin{aligned}\chi_{n-1,1}(\sigma) + \chi_{n-2,1,1}(\sigma) &= -a_2 + \frac{(a_1-1)(a_1-2)}{2} + a_1 - 1 \\ &= -a_2 + \frac{(a_1-1)a_1}{2} = -a_2 + \binom{a_1}{2}.\end{aligned}$$

Hence we have

$$\chi_{n-1,1}(\sigma) + \chi_{n-2,1,1}(\sigma) = -a_2 + \binom{a_1}{2}. \tag{6.112}$$

where a_1 and a_2 are the number of 1-cycles and 2-cycles in σ. This completes the proof. □

Proposition 6.4.5. *Let $FK^{(1)}(n)$ be degree one components of $FK(n)$. Then for $n \geq 3$, we have:*

$$\begin{gathered} char_{S_n}[FK^{(1)}(n)] = \chi_{n-1,1} + \chi_{n-2,1,1}. \\ F_{S_n}(char_{S_n}[FK^{(1)}(n)]) = s_{n-1,1} + s_{n-2,1,1}. \end{gathered} \tag{6.113}$$

Proof. We start from the right hand side of (6.113) for the character and come up with the left hand side.

$$\begin{aligned}\chi_{n-1,1} + \chi_{n-2,1,1} =& - a_2 + \binom{a_1}{2} \text{ (by (6.112))} \\ =& char_{S_n}[FK^{(1)}(n)]. \text{ (by Proposition 6.4.3)}\end{aligned}$$

The corresponding Frobenius image is straight forward. □

Corollary 6.4.6. *Let $FK^{(1)}(n)$ be degree one part of $FK(n)$ and let f^λ be the number of* *standard tableaux of shape* $\lambda \vdash n$*. Then we have*

$$dim[FK^{(1)}(n)] = \binom{n}{2} = f^{(n-1,1)} + f^{(n-2,1,1)}. \tag{6.114}$$

Proof. Evaluation of (6.112), $\chi_{n-1,1}(\sigma) + \chi_{n-2,1,1}(\sigma) = -a_2 + \binom{a_1}{2}$ at identity $\sigma = 1^n$ gives $f^{n-1,1} + f^{n-2,1,1} = \binom{n}{2}$. This completes the proof. $\square$

6.5 Character of $FK^{(2)}_{q_2^2 q_1^{n-4}}(n)$

6.5.1 Basis

The part of the basis of Fomin-Kirillov algebra, of usual degree 2 with set partition degree type $q_2^2 q_1^{n-4}$, consists of products of two variables with distinct indexes, such that the second index of the first variable is the biggest.

$$B[FK^{(2)}_{q_2^2 q_1^{n-4}}(n)] = \{x_{cd}x_{ab}, x_{ad}x_{bc}, x_{bd}x_{ac},\ 1 \leq a < b < c < d \leq n\}. \tag{6.115}$$

Example 6.5.1. $B[FK^{(2)}_{q_2^2 q_1}(5)] = \{x_{34}x_{12}, x_{24}x_{13}, x_{14}x_{23},\ x_{45}x_{12}, x_{45}x_{13}, x_{45}x_{23},$ $x_{35}x_{12}, x_{35}x_{14}, x_{35}x_{24},\ x_{25}x_{13}, x_{25}x_{14}, x_{25}x_{34},\ x_{15}x_{23}, x_{15}x_{24}, x_{15}x_{34}\}.$

6.5.2 Character

Proposition 6.5.2. *Let $\sigma \in S_n$ be of cycle type $(1^{a_1}, 2^{a_2}, \cdots, n^{a_n})$. Then the value of the character of $FK^{(2)}_{q_2^2 q_1^{n-4}}(n), n \geq 4$ at σ is*

$$char_{S_n}[FK^{(2)}_{q_2^2 q_1^{n-4}}(n)](\sigma) = 3\binom{a_1}{4} + 3\binom{a_2}{2} - \binom{a_1}{2}a_2 - a_4. \tag{6.116}$$

Proof. Referring to the basis of $FK^{(2)}_{q_2^2 q_1^{n-4}}(n)$ in (6.115), the indexes a, b, c, d appearing in the basis, as well could appear in σ (written in disjoint cycles) in different ways. In the following, we discuss the cases that could possibly make a contribution to the character upon the action of S_n.

1. For $m = 1, 2, 3, 4$, if only m out of a, b, c, d appear in a k-cycle for $k > m$, then no matter how the rest appear we have no contribution. The reason is that the action of $\sigma \in S_n$ on the basis elements, develops new indexes that was not in them previously, so it causes not to have an eigenvector. In specific if all of the indexes a, b, c, d appear in a k-cycle for $k > 4$, there would be no contribution to the character. Therefore character is a function of a_i (the number of i-cycles in $\sigma \in S_n$), for $1 \leq i \leq$ (the number of indexes in the basis element) $\leq 2d$, where d is the degree of the elements. In our specific case character is a function of a_i for $1 \leq i \leq 4$.
2. If a, b, c, d appear in σ as fixed points, then we have a contribution of $3\binom{a_1}{4}$ to character. Here $\binom{a_1}{4}$ is the number of 4-sets one can make with the number of fixed points a_1, and the coefficient 3 takes care of the three available type of degree 2 elements in the basis in (6.115).

3. If two out of a, b, c, d appear in σ in a 2-cycle and the other two in another 2-cycle, then we have a contribution $3\binom{a_2}{2}$ to character. Here $\binom{a_2}{2}$ is the number of 2-sets of 2-cycles in all the 2-cycles a_2 in σ.

4. If two out of a, b, c, d appear in σ in a 2-cycle and the other two appear as fixed points of σ, then we have a contribution $-\binom{a_1}{2}a_2$.

5. If three out of a, b, c, d appear in σ in a 3-cycle and the forth one appear as a fixed point, then no contribution, as is seen in the following

$$\sigma(x_{cd}x_{ab}) = (a,b,c)(d)x_{cd}x_{ab} = x_{ad}x_{bc}.$$

$$\sigma(x_{ad}x_{bc}) = (a,b,c)(d)x_{ad}x_{bc} = x_{bd}x_{ca} = -x_{bd}x_{ac}.$$

$$\sigma(x_{bd}x_{ac}) = (a,b,c)(d)x_{bd}x_{ac} = x_{cd}x_{ba} = -x_{cd}x_{ab}.$$

6. If a, b, c, d appear in σ in a 4-cycle, then we have a contribution $-a_4$ to the character, as

$$\sigma(x_{cd}x_{ab}) = (a,b,c,d)x_{cd}x_{ab} = x_{da}x_{bc} = -x_{ad}x_{bc}.$$

$$\sigma(x_{ad}x_{bc}) = (a,b,c,d)x_{ad}x_{bc} = \cdots = -x_{ab}x_{cd}.$$

$$\sigma(x_{bd}x_{ac}) = (a,b,c,d)x_{bd}x_{ac} = \cdots = -x_{bd}x_{ac} \to -1.$$

   I.e., we have a contribution of -1 per each 4-cycle in σ, that amounts to $-a_4$.

Summarizing the above items we come up with the conclusion that
$char_{S_n}[FK^{(2)}_{q_2^2 q_1^{n-4}}(n)](\sigma) = 3\binom{a_1}{4} + 3\binom{a_2}{2} - \binom{a_1}{2}a_2 - a_4$. This completes the proof. $\square$

The following examples of the above proposition with their associated decomposition into irreducible characters of S_n show a constant pattern for $n \geq 6$.

Example 6.5.3. $char_{S_4}[FK^{(2)}_{q_2^2}(4)] = (3,-1,3,0,-1) = \chi_{22} + \chi_{1111}$.

Example 6.5.4. $char_{S_5}[FK^{(2)}_{q_2^2q_1}(5)] = (15,-3,3,0,0,-1,0) = \chi_{32} + \chi_{221} + \chi_{2111} + \chi_{1111}$.

Example 6.5.5. $char_{S_6}[FK^{(2)}_{q_2^2q_1^2}(6)]$
$= (45,-3,1,9,0,0,0,-1,-1,0,0) = \chi_{42} + \chi_{321} + \chi_{3111} + \chi_{222} + \chi_{21111}$.

Example 6.5.6. $char_{S_7}[FK^{(2)}_{q_2^2q_1^3}(7)]$
$= (105,5,-3,9,3,-1,3,0,-1,-1,-1,0,0,0,0) = \chi_{52} + \chi_{421} + \chi_{4111} + \chi_{322} + \chi_{31111}$.

Example 6.5.7. $char_{S_8}[FK^{(2)}_{q_2^2q_1^4}(8)]$
$= (210,30,-6,6,18,15,-3,3,0,0,2,-2,2,-1,-2,0,0,0,0,0,0,0)$
$= \chi_{62} + \chi_{521} + \chi_{5111} + \chi_{422} + \chi_{41111}$.

The above examples suggest what we prove in the following proposition.

Proposition 6.5.8. *Let $FK^{(2)}_{q_2^2q_1^{n-4}}(n), n \geq 4$ be Fomin-Kirillov algebra of general n and of usual degree 2 and set partition degree type $q_2^2q_1^{n-4}$. Then we have the following decomposition into irreducible characters of S_n.*

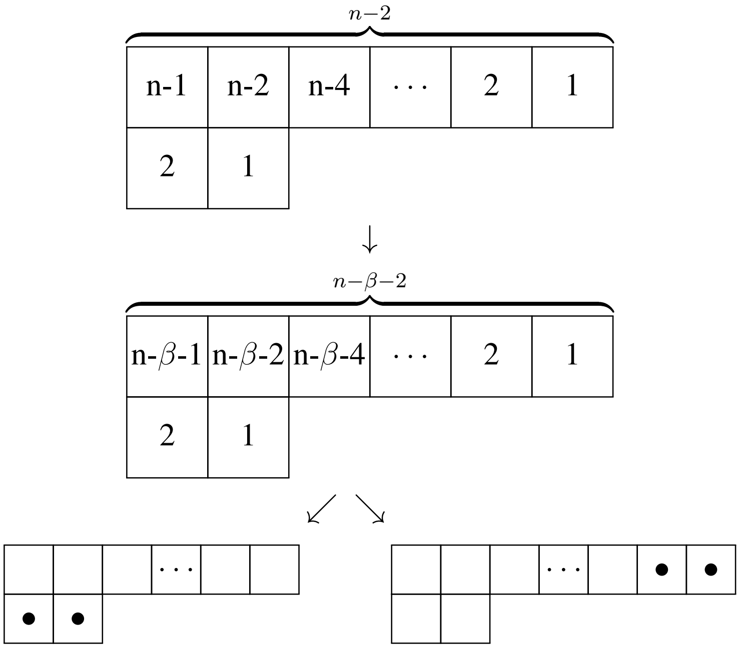
$$char_{S_n}[FK^{(2)}_{q_2^2q_1^{n-4}}(n)] = \begin{cases} \chi_{22} + \chi_{1111}, & n = 4 \\ \chi_{32} + \chi_{221} + \chi_{2111} + \chi_{1111}, & n = 5 \\ \chi_{n-2,2} + \chi_{n-3,2,1} + \chi_{n-3,1,1,1} + \chi_{n-4,2,2} + \chi_{n-4,1,1,1,1}. & n \geq 6 \end{cases} \tag{6.117}$$

Proof. We calculate the irreducible characters on the right hand side of (6.117) by Murnaghan-Nakayama rule, and show that their sum equals the left hand side. The first two cases, $n = 4$ and $n = 5$ are trivial, we prove the third cases $n \geq 6$.

Let $\sigma \in S_n$ be of type $(1^{a_1}, 2^{a_2}, \cdots, n^{a_n})$. Then

1. We show that $\chi_{n-2,2}(\sigma) = a_2 + \frac{1}{2}a_1(a_1 - 3)$.

   We start with applying Murnaghan-Nakayama rule to $\chi_{n-2,2}(\sigma)$ where at each step of the rule, a k-cycle, $k > 2$ is cut from σ, equivalently a block of more than 2 cells cut from the arm of the tableau (as only from the arm is a more than 2 cell cut available), so eventually instead of σ we are left σ' containing only 1-cycles and 2-cycles, equivalently left with a tableau with shorter arm $n - \beta - 2$ where $\beta = \sum_{i>2} ia_i$, i.e., $\chi_{n-2,2}(\sigma) \to \chi_{n-\beta-2}(\sigma')$, where $\beta = \sum_{i>2} ia_i$, and $\sigma' = (\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})$.
   So $n - \beta = a_1 + 2a_2$. Now we are left with $\chi_{n-\beta-2}(\sigma')$ and its associated tableau shown below where at each step of the rule 2-cycles could be cut from σ', equivalently 2 cells blocks cut from arm or leg of the tableau, until we are left with a permutation consisting of 1-cycles, i.e., with identity, where the value of character at identity is nothing but f^λ which is calculated by Hook's formula.



Then

$$\begin{aligned}
\chi_{n-2,2}(\sigma) =&\chi_{n-\beta-2,2}(\sigma') = \chi_{a_1+2a_2-2,2}(\sigma') = \chi_{a_1+2a_2-2,2}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
=&\chi_{a_1+2a_2-2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1}) + \chi_{a_1+2a_2-2-2,2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})\\
=&1 + \chi_{a_1+2a_2-2-2,2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})\\
=&1 + \chi_{a_1+2a_2-2-2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1}) + \chi_{a_1+2a_2-2-2-2,2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})\\
=&1 + 1 + \chi_{a_1+2a_2-2-2-2,2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})\\
&\vdots\\
=&\underbrace{1+1+\cdots+1}_{a_2} + \chi_{a_1+2a_2-2-2a_2,2}(\underbrace{11\cdots1}_{a_1})\\
=&a_2 + \chi_{a_1-2,2}(\underbrace{11\cdots1}_{a_1})
\end{aligned}$$

So

$$\chi_{n-2,2}(\sigma) = a_2 + \chi_{a_1-2,2}(\underbrace{11\cdots1}_{a_1}). \tag{6.118}$$

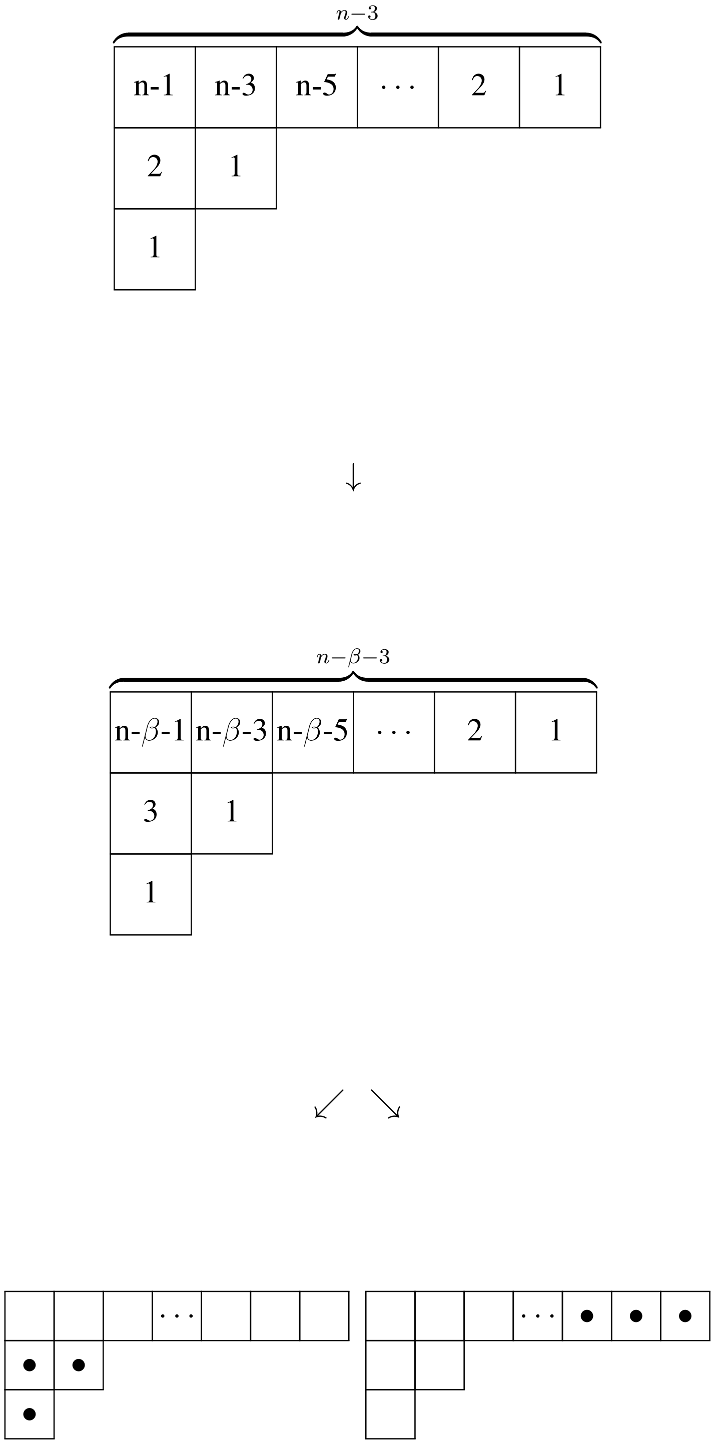
However, $\chi_{a_1-2,2}(\underbrace{11\cdots1}_{a_1}) = f^{a_1-2,2} = \frac{a_1(a_1-3)}{2}$ by Hook formula. Substituting this into (6.118), we have

$$\chi_{n-2,2}(\sigma) = a_2 + \frac{a_1(a_1-3)}{2}. \tag{6.119}$$

2. We now show $\chi_{n-3,2,1}(\sigma) = -a_3 + \frac{(a_1-4)(a_1-2)a_1}{3}$.

   $\chi_{n-3,2,1}(\sigma) \to \chi_{n-\beta-3,2,1}(\sigma')$, where $\beta = \sum_{i>3} ia_i$, and where

   $\sigma' = (\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})$. So $n-\beta = a_1 + 2a_2 + 3a_3$.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Then

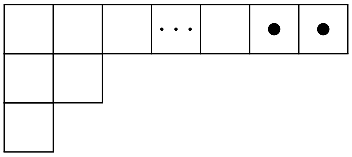
$$\begin{aligned}\chi_{n-3,2,1}(\sigma) =&\chi_{n-\beta-3,2,1}(\sigma') = \chi_{a_1+2a_2+3a_3-3,2,1}(\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) = \\ =&\chi_{a_1+2a_2+3a_3-3}(\underbrace{3\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\ &+ \chi_{a_1+2a_2+3a_3-3-3,2,1}(\underbrace{33\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\ =&-1 + \chi_{a_1+2a_2+3a_3-3-3,2,1}(\underbrace{33\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\ =&-1 - \chi_{a_1+2a_2+3a_3-3-3}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\ &+ \chi_{a_1+2a_2+3a_3-3-3-3,2,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\ =&-1-1 + \chi_{a_1+2a_2+3a_3-3-3-3,2,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\ &\vdots \\ =&\underbrace{-1-1-\cdots-1}_{a_3}+\chi_{a_1+2a_2+3a_3-3-3a_3,2,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\ =&-a_3 + \chi_{a_1+2a_2-3,2,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}).\end{aligned}$$

So we have

$$\chi_{n-3,2,1}(\sigma) = -a_3 + \chi_{a_1+2a_2-3,2,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}). \tag{6.120}$$

However

$$\begin{aligned}&\chi_{a_1+2a_2-3,2,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) = \\ &\chi_{a_1+2a_2-3-2,2,1}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1}) = \chi_{a_1+2a_2-3-2-2,2,1}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1}) \\ &= \cdots = \chi_{a_1+2a_2-3-2a_2,2,1}(\underbrace{11\cdots1}_{a_1}) = \chi_{a_1-3,2,1}(\underbrace{11\cdots1}_{a_1}).\end{aligned}$$



So

$$\chi_{a_1+2a_2-3,2,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) = \chi_{a_1-3,2,1}(\underbrace{11\cdots1}_{a_1}), \tag{6.121}$$

where

$$\chi_{a_1-3,2,1}(\underbrace{11\cdots1}_{a_1}) = \frac{a_1!}{(a_1-5)!(a_1-3)(a_1-1).3} = \frac{(a_1-4)(a_1-2)a_1}{3}, \tag{6.122}$$

by Hooks formula.

Now (6.122), (6.121) and (6.120) yield

$$\chi_{n-3,2,1}(\sigma) = -a_3 + \frac{(a_1-4)(a_1-2)a_1}{3}. \tag{6.123}$$

3. We now show

$$\chi_{n-3,1,1,1}(\sigma) = a_3 - a_2(a_1-1) + \frac{(a_1-1)(a_1-2)(a_1-3)}{6}.$$

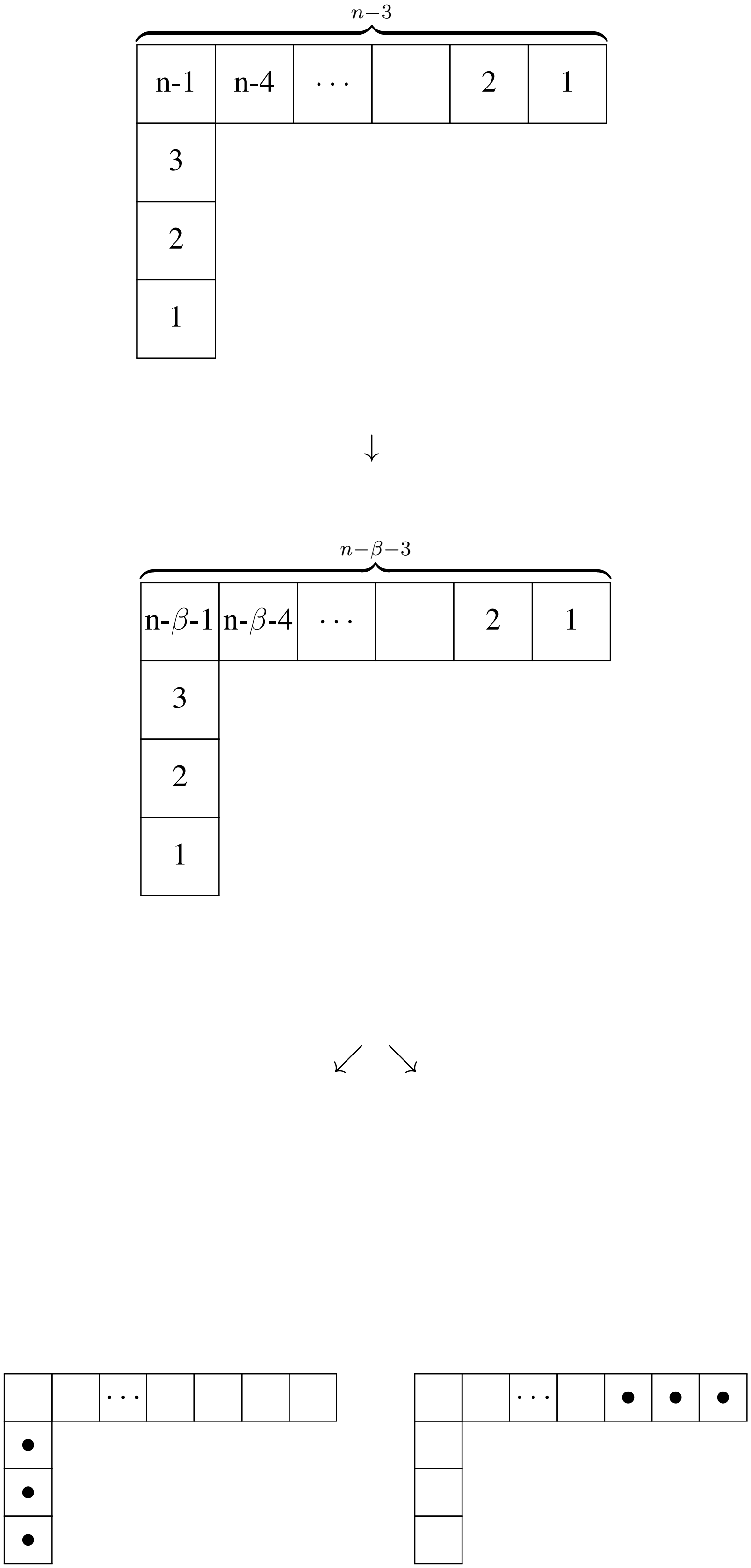
$\chi_{n-3,1,1,1}(\sigma) \to \chi_{n-\beta-3,1,1,1}(\sigma')$, where

$$\beta = \sum_{i>3} i a_i,$$

and where

$$\sigma' = (\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}).$$

So $n-\beta = a_1 + 2a_2 + 3a_3$.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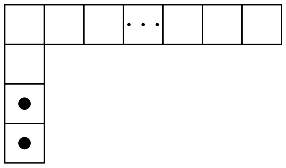
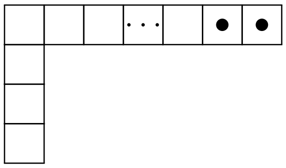
Then

$$
\begin{aligned}
&\chi_{n-3,1,1,1}(\sigma)\\
&= \chi_{n-\beta-3,1,1,1}(\sigma') = \chi_{a_1+2a_2+3a_3-3,1,1,1}(\underbrace{33\cdots3}_{a_3}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= \chi_{a_1+2a_2+3a_3-3}(\underbrace{3\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&\quad + \chi_{a_1+2a_2+3a_3-3-3,1,1,1}(\underbrace{33\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= 1 + \chi_{a_1+2a_2+3a_3-3-3,1,1,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&+ \chi_{a_1+2a_2+3a_3-3-3-3,1,1,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= 1 + 1 + \chi_{a_1+2a_2+3a_3-3-3-3,1,1,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&\vdots\\
&= \underbrace{1+1+\cdots+1}_{a_3} + \chi_{a_1+2a_2+3a_3-3-3a_3,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= a_3 + \chi_{a_1+2a_2-3,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}).
\end{aligned}
$$

So

$$\chi_{n-3,1,1,1}(\sigma) = a_3 + \chi_{a_1+2a_2-3,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}). \tag{6.124}$$





However

$$\chi_{a_1+2a_2-3,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) =$$

$$= - \chi_{a_1+2a_2-3,1}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1}) + \chi_{a_1+2a_2-3-2,1,1,1}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})$$

$$= - (a-1) + \chi_{a_1+2a_2-3-2,1,1,1}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1}) \text{ (by (6.111))}$$

$$= - (a-1) - \chi_{a_1+2a_2-3-2,1}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})$$

$$+ \chi_{a_1+2a_2-3-2-2,1,1,1}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})$$

$$= - (a_1-1) - (a_1-1) + \chi_{a_1+2a_2-3-2-2-2,1,1,1}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})$$

$$\vdots$$

$$= \underbrace{-(a_1-1) - (a_1-1) - \cdots - (a_1-1)}_{a_2} + \chi_{a_1+2a_2-3-2a_2,1,1,1}(\underbrace{11\cdots1}_{a_1})$$

$$= - a_2(a_1-1) + \chi_{a_1-3,1,1,1}(\underbrace{11\cdots1}_{a_1}).$$

So

$$\chi_{a_1+2a_2-3,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) = -a_2(a_1-1) + \chi_{a_1-3,1,1,1}(\underbrace{11\cdots1}_{a_1}). \tag{6.125}$$

However

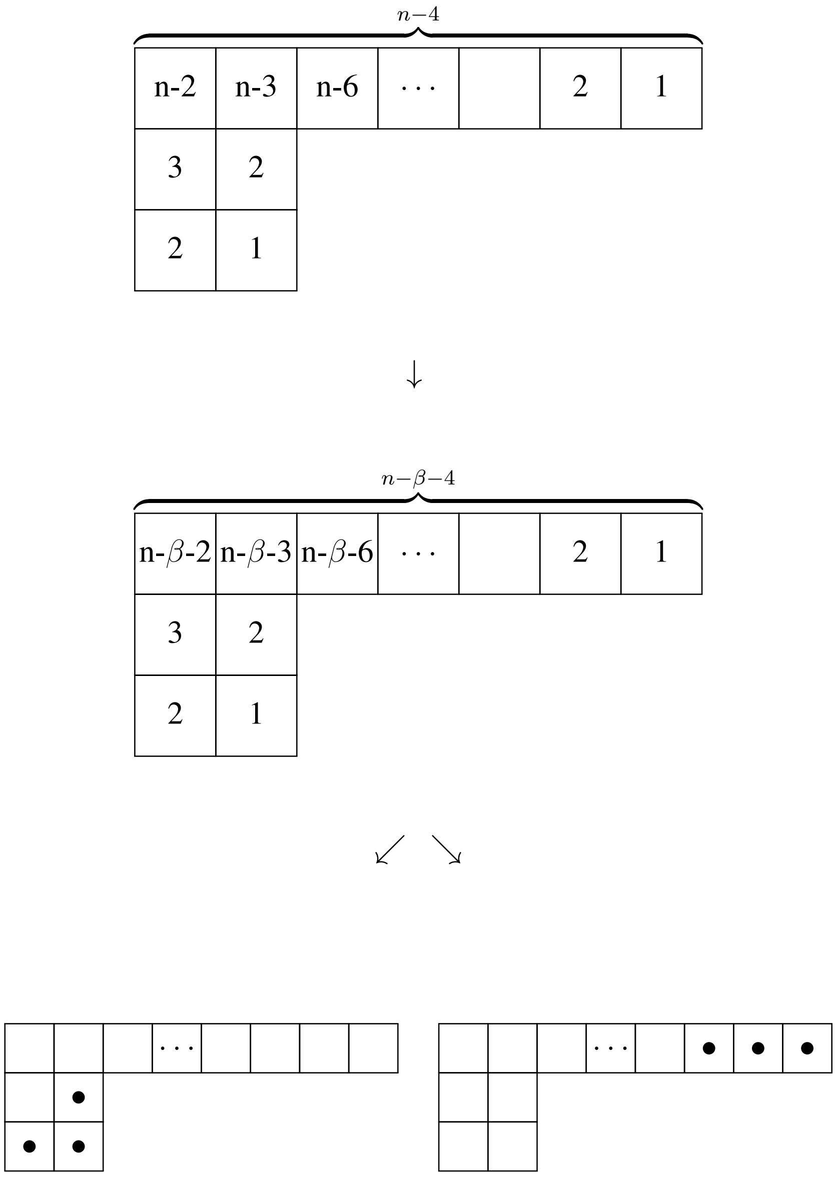
$$\chi_{a_1-3,1,1,1}(\underbrace{11\cdots1}_{a_1}) = \frac{(a_1-1)(a_1-2)(a_1-3)}{6}. \tag{6.126}$$

Hooks formula, and Equations (6.126), (6.125) and (6.124) yield

$$\chi_{n-3,1,1,1}(\sigma) = a_3 - a_2(a_1-1) + \frac{(a_1-1)(a_1-2)(a_1-3)}{6}. \tag{6.127}$$

4. We show that $\chi_{n-4,2,2} = -a_3(a_1-1) + a_2^2 - 2a_2 + \frac{a_1(a_1-1)(a_1-4)(a_{-}5)}{12}$.

   $\chi_{n-4,2,2}(\sigma) \to \chi_{n-\beta-4,2,2}(\sigma')$, where $\beta = \sum_{i>3} ia_i$, and where $\sigma' = (\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})$. So $n-\beta = a_1 + 2a_2 + 3a_3$.



Then

$$\begin{aligned}
\chi_{n-4,2,2}(\sigma) &= \chi_{n-\beta-4,2,2}(\sigma') = \chi_{a_1+2a_2+3a_3-4,2,2}(\underbrace{33\cdots3}_{a_3}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -\chi_{a_1+2a_2+3a_3-4,1}(\underbrace{3\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&\quad + \chi_{a_1+2a_2+3a_3-4-3,2,2}(\underbrace{33\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -(a_1-1) - \chi_{a_1+2a_2+3a_3-4-3,1}(\underbrace{3\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&\quad + \chi_{a_1+2a_2+3a_3-4-3-3,2,2}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \text{ (by (6.111))} \\
&= -(a_1-1) - (a_1-1) \\
&\quad + \chi_{a_1+2a_2+3a_3-4-3-3,2,2}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&\vdots \\
&= \underbrace{-(a_1-1) - (a_1-1) + \cdots - (a_1-1)}_{a_3} \\
&\quad + \chi_{a_1+2a_2+3a_3-4-3a_3,2,2}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -a_3(a_1-1) + \chi_{a_1+2a_2-4,2,2}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})
\end{aligned}$$

So

$$\chi_{n-4,2,2}(\sigma) = -a_3(a_1-1) + \chi_{a_1+2a_2-4,2,2}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}). \tag{6.128}$$

However

$$\chi_{a_1+2a_2-4,2,2}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})$$

$$= \chi_{a_1+2a_2-4,2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1}) - \chi_{a_1+2a_2-4,1,1}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})$$

$$+ \chi_{a_1+2a_2-4-2,2,2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})$$

$$= [a_2 - 1 + \frac{a_1(a_1-3)}{2}] - [-(a_2-1) + \frac{(a_1-1)(a_1-2)}{2}]$$

$$+ \chi_{a_1+2a_2-4-2,2,2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1}) \text{ (by (6.119) and (6.110))}$$

$$= [a_2 - 1 + \frac{a_1(a_1-3)}{2}] - [-(a_2-1) + \frac{(a_1-1)(a_1-2)}{2}]$$

$$+ \chi_{a_1+2a_2-4-2,2,2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})$$

$$= [2(a_2-1)-1] + \chi_{a_1+2a_2-4-2,2,2}(\underbrace{22\cdots2}_{a_2-1}\underbrace{11\cdots1}_{a_1})$$

$$= [2(a_2-1)-1] + \chi_{a_1+2a_2-4-2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})$$

$$- \chi_{a_1+2a_2-4-2,1,1}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1}) + \chi_{a_1+2a_2-4-2-2,2,2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})$$

$$= [2(a_2-1)-1] + [a_2 - 2 + \frac{a_1(a_1-3)}{2}] - [-(a_2-2) + \frac{(a_1-1)(a_1-2)}{2}]$$

$$+ \chi_{a_1+2a_2-4-2-2,2,2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})$$

$$= [2(a_2-1)-1] + [2(a_2-2)-1] + \chi_{a_1+2a_2-4-2-2,2,2}(\underbrace{22\cdots2}_{a_2-2}\underbrace{11\cdots1}_{a_1})$$

$$\vdots$$

$$= \underbrace{[2(a_2-1)-1] + [2(a_2-2)-1] + \cdots + [-1]}_{a_2} + \chi_{a_1+2a_2-4-2a_2,2,2}(\underbrace{11\cdots1}_{a_1})$$

$$= a^2 - 2a_2 + \chi_{a_1-4,2,2}(\underbrace{11\cdots1}_{a_1}).$$

So

$$\chi_{a_1+2a_2-4,2,2}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) = a^2 - 2a_2 + \chi_{a_1-4,2,2}(\underbrace{11\cdots1}_{a_1}), \tag{6.129}$$

where by Hook formula we have

$$\begin{aligned}\chi_{a_1-4,2,2}(\underbrace{11\cdots1}_{a_1}) &= \frac{a_1!}{(a_1-6)!(a_1-3)(a_2-2).2.2.3} \\ &= \frac{a_1(a_1-1)(a_1-4)(a_1-5)}{12}.\end{aligned} \tag{6.130}$$

Equations (6.130), (6.129) and (6.128) yield

$$\begin{aligned}\chi_{n-4,2,2}(\sigma) =& - a_3(a_1-1) + a_2^2 - 2a_2 \\ &+ \frac{a_1(a_1-1)(a_1-4)(a_1-5)}{12}.\end{aligned} \tag{6.131}$$

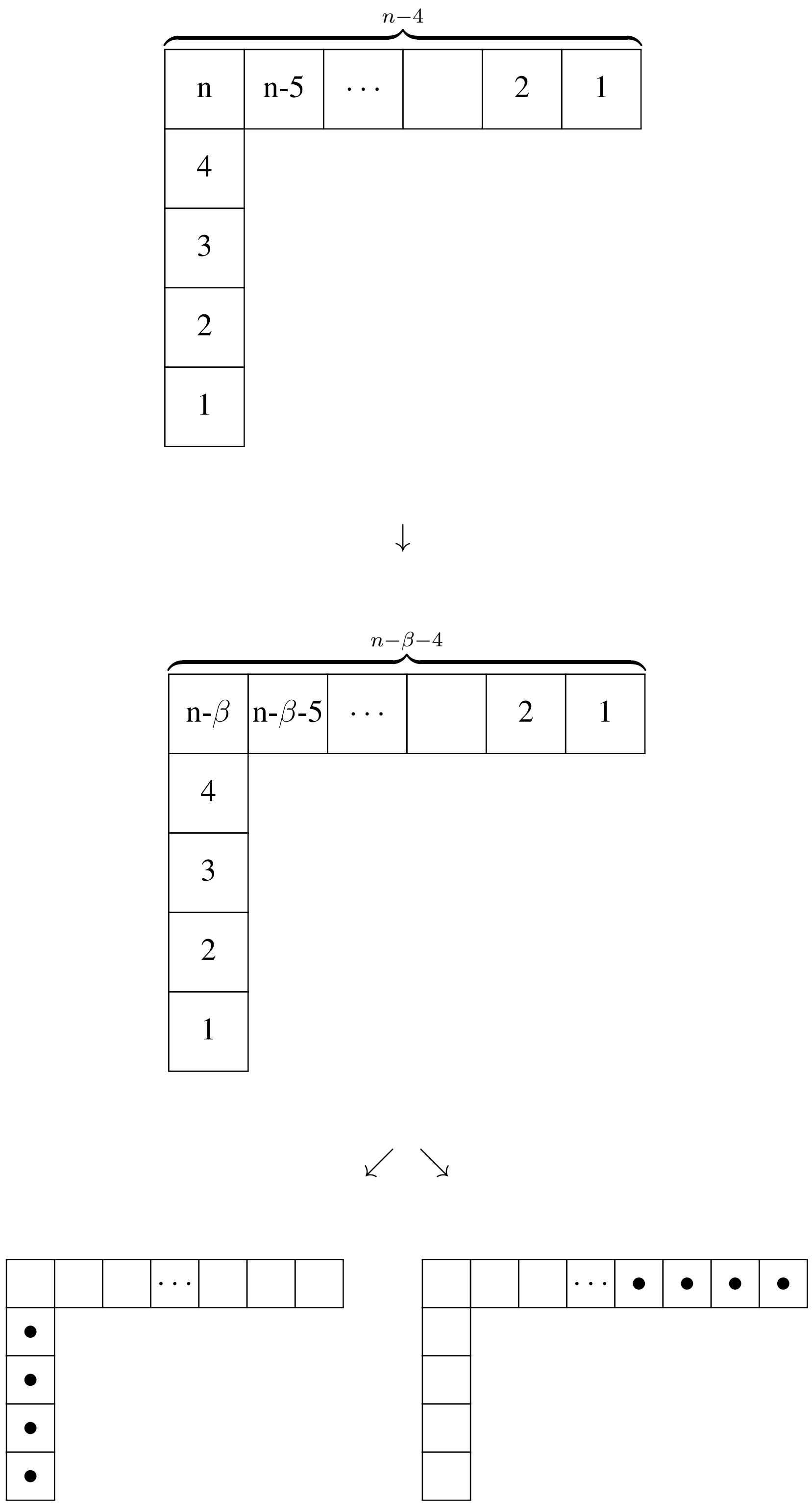
5. We show that

$$\begin{aligned}\chi_{n-4,1,1,1,1}(\sigma) =& - a_4 + (a_1-1)a_3 + \frac{a_2(a_2-1)}{2} - \frac{(a_1-1)(a_1-2)}{2}a_2 \\ &+ \frac{(a_1-4)(a_1-3)(a_1-2)(a_1-1)}{24}.\end{aligned}$$

$\chi_{n-4,1,1,1,1}(\sigma) = \chi_{n-\beta-4,1,1,1,1}(\sigma')$, where $\beta = \sum_{i>4} ia_i$, and where

$$\sigma' = (\underbrace{44\cdots4}_{a_4}\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}).$$

So $n-\beta = a_1 + 2a_2 + 3a_3 + 4a_4$.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n−β−4
n-β
n-β-5
· · ·
2
1
4
3
2
1
· · ·
· · ·

Then

$$
\begin{aligned}
\chi_{n-4,1,1,1,1}(\sigma) &= \chi_{n-\beta-4,1,1,1,1}(\sigma') \\
&= \chi_{a_1+2a_2+3a_3+4a_4-4,1,1,1,1}(\underbrace{44\cdots4}_{a_4}\underbrace{33\cdots3}_{a_3}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -\chi_{a_1+2a_2+3a_3+4a_4-4}(\underbrace{44\cdots4}_{a_4-1}\underbrace{3\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})+ \\
&\quad \chi_{a_1+2a_2+3a_3+4a_4-4-4,1,1,1,1}(\underbrace{44\cdots4}_{a_4-1}\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -1 + \chi_{a_1+2a_2+3a_3+4a_4-4-4,1,1,1,1}(\underbrace{44\cdots4}_{a_4-1}\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -1 - \chi_{a_1+2a_2+3a_3+4a_4-4-4}(\underbrace{44\cdots4}_{a_4-2}\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&\qquad + \chi_{a_1+2a_2+3a_3+4a_4-4-4-4,1,1,1,1}(\underbrace{44\cdots4}_{a_4-2}\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -1 - 1 \\
&\quad + \chi_{a_1+2a_2+3a_3+4a_4-4-4-4,1,1,1,1}(\underbrace{44\cdots4}_{a_4-2}\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&\vdots \\
&= \underbrace{-1-1-\cdots-1}_{a_4} \\
&\quad + \chi_{a_1+2a_2+3a_3+4a_4-4-4a_4,1,1,1,1}(\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \\
&= -a_4 + \chi_{a_1+2a_2+3a_3-4,1,1,1,1}(\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})
\end{aligned}
$$

So

$$
\chi_{n-4,1,1,1,1}(\sigma) = -a_4 + \chi_{a_1+2a_2+3a_3-4,1,1,1,1}(\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \tag{6.132}
$$

However

$$\begin{aligned}
&\chi_{a_1+2a_2+3a_3-4,1,1,1,1}(\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= \chi_{a_1+2a_2+3a_3-4,1}(\underbrace{33\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&\quad + \chi_{a_1+2a_2+3a_3-4-3,1,1,1,1}(\underbrace{33\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= (a_1-1) + \chi_{a_1+2a_2+3a_3-4-3,1,1,1,1}(\underbrace{33\cdots3}_{a_3-1}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) \text{ (by (6.111))}\\
&= (a_1-1) + \chi_{a_1+2a_2+3a_3-4-3,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&\quad + \chi_{a_1+2a_2+3a_3-4-3-3,1,1,1,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= (a_1-1) + (a_1-1) + \chi_{a_1+2a_2+3a_3-4-3-3,1,1,1,1}(\underbrace{33\cdots3}_{a_3-2}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&\vdots\\
&= \underbrace{(a_1-1)+(a_1-1)+\cdots+(a_1-1)}_{a_3}\\
&\quad + \chi_{a_1+2a_2+3a_3-4-3a_3,1,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1})\\
&= a_3(a_1-1) + \chi_{a_1+2a_2-4,1,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}).
\end{aligned}$$

So

$$\begin{aligned}
\chi_{a_1+2a_2+3a_3-4,1,1,1,1}(\underbrace{33\cdots3}_{a_3}\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}) &= a_3(a_1-1)\\
&\quad + \chi_{a_1+2a_2-4,1,1,1,1}(\underbrace{22\cdots2}_{a_2}\underbrace{11\cdots1}_{a_1}).
\end{aligned} \tag{6.133}$$

However

$$\chi_{a_1+2a_2-4,1,1,1,1}(\underbrace{22\cdots 2}_{a_2}\underbrace{11\cdots 1}_{a_1})$$

$$= -\chi_{a_1+2a_2-4,1,1}(\underbrace{22\cdots 2}_{a_2-1}\underbrace{11\cdots 1}_{a_1}) + \chi_{a_1+2a_2-4-2,1,1,1,1}(\underbrace{22\cdots 2}_{a_2-1}\underbrace{11\cdots 1}_{a_1})$$

$$= -[-(a_2-1) + \frac{(a_1-1)(a_1-2)}{2}] + \chi_{a_1+2a_2-4-2,1,1,1,1}(\underbrace{22\cdots 2}_{a_2-1}\underbrace{11\cdots 1}_{a_1}) \text{ (by (6.110))}$$

$$= -[-(a_2-1) + \frac{(a_1-1)(a_1-2)}{2}] + \chi_{a_1+2a_2-4-2,1,1}(\underbrace{22\cdots 2}_{a_2-2}\underbrace{11\cdots 1}_{a_1})$$

$$+ \chi_{a_1+2a_2-4-2-2,1,1,1,1}(\underbrace{22\cdots 2}_{a_2-2}\underbrace{11\cdots 1}_{a_1})$$

$$= -[-(a_2-1) + \frac{(a_1-1)(a_1-2)}{2}] - [-(a_2-2) + \frac{(a_1-1)(a_1-2)}{2}]$$

$$+ \chi_{a_1+2a_2-4-2-2,1,1,1,1}(\underbrace{22\cdots 2}_{a_2-2}\underbrace{11\cdots 1}_{a_1})$$

$$\vdots$$

$$= -[-(a_2-1) + \frac{(a_1-1)(a_1-2)}{2}] - [-(a_2-2) + \frac{(a_1-1)(a_1-2)}{2}] - \cdots$$

$$\cdots - [\frac{(a_1-1)(a_1-2)}{2}] + \chi_{a_1+2a_2-4-2a_2,1,1,1,1}(\underbrace{11\cdots 1}_{a_1})$$

$$= \frac{a_2(a_2-1)}{2} - \frac{(a_1-1)(a_1-2)}{2}a_2 + \chi_{a_1-4,1,1,1,1}(\underbrace{11\cdots 1}_{a_1}).$$

So we have

$$\begin{aligned}\chi_{a_1+2a_2-4,1,1,1,1}(\underbrace{22\cdots 2}_{a_2}\underbrace{11\cdots 1}_{a_1}) &= \frac{a_2(a_2-1)}{2} \\ &- \frac{(a_1-1)(a_1-2)}{2}a_2 \\ &+ \chi_{a_1-4,1,1,1,1}(\underbrace{11\cdots 1}_{a_1}).\end{aligned} \tag{6.134}$$

By Hook formula we have

$$\chi_{a_1-4,1,1,1,1}(\underbrace{11\cdots 1}_{a_1}) = \frac{(a_1-4)(a_1-3)(a_1-2)(a_1-1)}{24}. \tag{6.135}$$

Equations (6.135), (6.134), (6.133) and (6.132) yield

$$\begin{aligned}\chi_{n-4,1,1,1,1}(\sigma) = -a_4 + a_3(a_1-1) + \frac{a_2(a_2-1)}{2} - \frac{(a_1-1)(a_1-2)}{2}a_2 \\ + \frac{(a_1-4)(a_1-3)(a_1-2)(a_1-1)}{24}.\end{aligned} \tag{6.136}$$

We add up the Equations (6.119), (6.123), (6.127), (6.131) and (6.136) to find the right hand side of Equation (6.117).

$$\begin{aligned}&\chi_{n-2,2}(\sigma) + \chi_{n-3,2,1}(\sigma) + \chi_{n-3,1,1,1}(\sigma) + \chi_{n-4,2,2}(\sigma) + \chi_{n-4,1,1,1,1}(\sigma)\\ &= \left[a_2 + \frac{a_1(a_1-3)}{2}\right] + \left[-a_3 + \frac{(a_1-4)(a_1-2)a_1}{3}\right]\\ &+ \left[a_3 - a_2(a_1-1) + \frac{(a_1-1)(a_1-2)(a_1-3)}{6}\right]\\ &+ \left[-a_3(a_1-1) + a_2^2 - 2a_2 + \frac{a_1(a_1-1)(a_1-4)(a_1-5)}{12}\right]\\ &+ \left[-a_4 + a_3(a_1-1) + \frac{a_2(a_2-1)}{2} - \frac{(a_1-1)(a_1-2)}{2}a_2\right.\\ &\left.+ \frac{(a_1-4)(a_1-3)(a_1-2)(a_1-1)}{24}\right] = A + B,\end{aligned}$$

where

$$\begin{aligned}A &= a_2 - a_3 + a_3 - a_2(a_1-1) - a_3(a_1-1) + a_2^2 - 2a_2 - a_4 + a_3(a_1-1)\\ &+ \frac{a_2(a_2-1)}{2} - \frac{(a_1-1)(a_1-2)}{2}a_2 = a_2^2 - a_2 + \frac{a_2(a_2-1)}{2} - a_4 - \frac{a_2a_1^2}{2} + \frac{a_2a_1}{2}\\ &= 3\frac{a_2(a_2-1)}{2} - a_4 - \frac{a_1(a_1-1)}{2}a_2 = 3\binom{a_2}{2} - a_4 - \binom{a_1}{2}a_2,\end{aligned}$$

and

$$\begin{aligned}B =&\frac{a_1(a_1-3)}{2}+\frac{(a_1-4)(a_1-2)a_1}{3}+\frac{(a_1-1)(a_1-2)(a_1-3)}{6}\\&+\frac{a_1(a_1-1)(a_1-4)(a_1-5)}{12}+\frac{(a_1-4)(a_1-3)(a_1-2)(a_1-1)}{24}=3\binom{a_1}{4}.\end{aligned}$$

Hence

$\chi_{n-2,2}(\sigma)+\chi_{n-3,2,1}(\sigma)+\chi_{n-3,1,1,1}(\sigma)+\chi_{n-4,2,2}(\sigma)+\chi_{n-4,1,1,1,1}(\sigma)=A+B=$
$=3\binom{a_1}{4}+3\binom{a_2}{2}-a_4-\binom{a_1}{2}a_2=char_{S_n}[FK^{(2)}_{q_2^2q_1^{n-4}}(n)](\sigma)$ (by Proposition 6.5.2). This completes the proof. ☐

Corollary 6.5.9. *Let f^λ be the number of standard tableaux of shape λ. Then we have the following identity.*

$$f^{n-2,2}+f^{n-3,2,1}+f^{n-3,1,1,1}+f^{n-4,2,2}+f^{n-4,1,1,1,1}=3\binom{n}{4},\ n\geq 6. \tag{6.137}$$

Proof. Let permutation $\sigma\in S_n$ be of cycle type $\sigma=(1^{a_1},2^{a_2},\cdots,n^{a_n})$. From Propositions 6.5.2 and 6.5.8, for $n\geq 6$ we have

$\chi_{n-2,2}+\chi_{n-3,2,1}+\chi_{n-3,1,1,1}+\chi_{n-4,2,2}+\chi_{n-4,1,1,1,1}=3\binom{a_1}{4}+3\binom{a_2}{2}-\binom{a_1}{2}a_2-a_4.$
Since f^λs are the values of irreducible characters of S_n at identity, a straight forward evaluation of the above equation at $\sigma=1^n$ results in the identity of the statement. ☐

Remark 6.5.10. *We consider the character of $FK^{(2)}_{q_2^2q_1^{n-4}}(n)$ in* (6.117)*. It is a class function of S_n, so it can be written as a linear combination of irreducible characters χ^λ of S_n. As well it is a polynomial in n, a_1, a_2 and a_4 of degree 4. While the decomposition of $char_{S_n}[FK^{(2)}_{q_2^2q_1^{n-4}}(n)]$ into irreducible characters is not obviously related to each other as n grows, however eventually at some enough large n it stabilizes such that it becomes a tractable finite sum. The above stability is in general valid for $FK(n)$ with any usual degree and any set partition degree type. The reason is that the number of non-repeated indexes in an element of the basis of $FK(n)$,* *with usual degree d and any set partition degree type, is at most $2d$; if all the $2d$ indexes appear in $\sigma \in S_n$ in a k-cycle, for $k > 2d$, then we have no contribution to the character. Thus our character, a conjugacy class function, is a polynomial of degree at most $2d$, so by Theorem 1.1 of [15], decomposition of the character eventually stabilizes at some enough large n.*

In the next section we provide another example of representation stability.

6.6 Character of $FK^{(2)}_{q_3 q_1^{n-3}}(n)$

Basis

The basis of $FK(n)$ of usual degree 2 and set partition degree type $q_3 q_1^{n-3}$ is the set of all degree 2 combinations of generators $\{x_{ab}, 1 \leq a < b \leq n\}$ with set partition degree type $q_3 q_1^{n-3}$ that are not divisible by leading terms of the degree 2 elements of the associated Gröbner basis, i.e., by

$$LT(GB^{(2)}) = \{x_{ab}x_{ab}, x_{ab}x_{bc}, x_{ab}x_{ac}, x_{ab}x_{cd}, \text{ for distinct } a, b, c, d, 1 \leq a < b < c < d \leq n\}.$$

However

- Every degree 2 monomial with 3 indexes (associated to the number 3 in $q_3 q_1^{n-3}$) made by our generators is of set partition type $q_3 q_1^{n-3}$.

- The 1st variable of this degree 2 monomial should not be x_{ab} otherwise no matter what is the 2nd variable, this monomial is divisible by one of the elements in $LT(GB^{(2)})$.

The above conditions forces the basis of $FK(n)$ of usual degree 2 and set partition degree type $q_3q_1^{n-3}$, be the following.

$$B[FK^{(2)}_{q_3q_1^{n-3}}(n)] = \{x_{ac}x_{ab},\ x_{bc}x_{ab},\ x_{bc}x_{ac},\ x_{ac}x_{bc},\ 1 \leq a < b < c \leq n\}. \tag{6.138}$$

From (6.138), the cardinal number of this basis is $|B[FK^{(2)}_{q_3q_1^{n-3}}(n)]| = 4 \times$ (number of $3-$ sets in n object) $= 4\binom{n}{3}$, therefore $|B^{(2)}_{q_3q_1^{n-3}}(n)| = 4\binom{n}{3}$ (the number 4 in front of the binomial symbol is due to the 4 types of elements in the basis in Equation (6.138)).

Proposition 6.6.1. *Let $\sigma \in S_n$ be of type $(1^{a_1}, 2^{a_2}, 3^{a_3}, \cdots, n^{a_n})$. Then the value of character of $FK^{(2)}_{q_3q_1^{n-3}}(n)$ on conjugacy class indexed by permutation σ is*

$$char_{S_n}[FK^{(2)}_{q_3q_1^{n-3}}(n)] = -2a_3 + 4\binom{a_1}{3}. \tag{6.139}$$

Example 6.6.2. *$char_{S_4}[FK^{(2)}_{q_3q_1}(4)]$ on $\sigma = (123)(4)$ is $\chi(\sigma) = -2(1) + 4\binom{1}{3} = -2$.*

proof of Proposition 6.6.1. Here we discuss the following cases where the representative of an S_n conjugacy class σ, written as product of disjoint cycles, applies to $z \in B[FK^{(2)}_{q_3q_1^{n-3}}(n)]$ of Equation (6.138). Since in the following calculations σ is a representative of a conjugacy class, the order of indexes appearing in its cycles doesn't matter (as only cycle type matters). For $z \in B[FK^{(2)}_{q_3q_1^{n-3}}(n)]$, let $a, b, c,$ be the three indexes of z. However $a, b, c,$ could also appear in $\sigma \in S_n$ in different ways discussed in the following cases.

1. For $m = 1, 2, 3$, if only m out of a, b, c appear in a k-cycle for $k > m$, then no matter how the rest appear we have no contribution to the character, as before.
2. We have a contribution of $4\binom{a_1}{3}$ to $\chi(\sigma)$ if a, b, c, appear in σ as fixed points. This is because each 3-set of fixed points of σ contribute a 1 to $\chi(\sigma)$ by $\sigma z = z$, and the number of 3-sets of fixed points in σ is $\binom{a_1}{3}$ (the factor 4 in front of the binomial symbol is to cover the 4 elements in the basis in Equation (6.138)).
3. We have no contribution to $\chi(\sigma)$ if out of a, b, c, two of them say a, b appear in a 2-cycle and the third one c, appears as a fixed point in σ. The reason is that since in this case the only cycle of σ that acts on the elements of the basis is (ab), we have the following cases.
   - $\sigma(x_{ac}x_{ab}) = (ab)(x_{ac}x_{ab}) = -x_{bc}x_{ab}$
   - $\sigma(x_{bc}x_{ab}) = (ab)(x_{bc}x_{ab}) = -x_{ac}x_{ab}$
   - $\sigma(x_{bc}x_{ac}) = (ab)(x_{bc}x_{ac}) = x_{ac}x_{bc}$
   - $\sigma(x_{ac}x_{bc}) = (ab)(x_{ac}x_{bc}) = x_{bc}x_{ac}$

   So in none of the above cases we have $\sigma(z) = z$.
4. We have a contribution of $-2a_3$ to $\chi(\sigma)$ if a, b, c appear in a 3-cycle in σ. The reason is that since in this case σ acts on the elements only via cycle (abc), we have the following cases.
   - $\sigma(x_{ac}x_{ab}) = (abc)(x_{ac}x_{ab})$ [as the only cycle working on a, b, b is (abc)] $= -x_{ab}x_{bc} = -(x_{bc}x_{ac} + x_{ac}x_{ab})$, results in a coefficient -1 of $x_{ac}x_{ab}$ in $\sigma(x_{ac}x_{ab})$.
   - $\sigma(x_{bc}x_{ab}) = (abc)(x_{bc}x_{ab}) = -x_{ac}x_{bc}$.
   - $\sigma(x_{bc}x_{ac}) = (abc)(x_{bc}x_{ac}) = x_{ac}x_{ab}$.

- $\sigma(x_{ac}x_{bc}) = (abc)(x_{ac}x_{bc})$[as before]$= x_{ab}x_{ac} = x_{bc}x_{ab} - x_{ac}x_{bc}$, results in a coefficient -1 of $x_{ac}x_{bc}$ in $\sigma(x_{ac}x_{bc})$.

So a contribution of -2 for each 3-cycle appearing in σ, amount to $-2a_3$ contribution to $\chi(\sigma)$.

Adding up the non-zero contributions to the character due to the items 2 and 4, we get $\chi(\sigma) = -2a_3 + 4\binom{a_1}{3}$. This completes the proof. □

Decomposition of character in (6.139) into irreducible characters of S_n for $n = 4, 5, 6, 7, 8$ are listed below.

$$char_{S_n}[FK^{(2)}_{q_3q_1^{n-3}}(n)] = \begin{cases} 2\chi_{21}, & n = 3 \\ 2\chi_{31} + 2\chi_{2,2} + 2\chi_{211}, & n = 4 \\ 2\chi_{41} + 2\chi_{3,2} + 2\chi_{311} + 2\chi_{221}, & n = 5 \\ 2\chi_{51} + 2\chi_{4,2} + 2\chi_{411} + 2\chi_{321}, & n = 6 \\ 2\chi_{61} + 2\chi_{5,2} + 2\chi_{511} + 2\chi_{421}, & n = 7 \\ 2\chi_{71} + 2\chi_{6,2} + 2\chi_{611} + 2\chi_{521}. & n = 8 \end{cases}$$

We see that for $n \geq 5$ the decomposition into irreducible characters becomes stable and is of the form expressed in the next proposition.

Proposition 6.6.3. *Let $FK^{(2)}_{q_3q_1^{n-3}}(n), n \geq 3$ be Fomin-Kirillov algebra of general n and of usual degree 2 and set partition degree type $q_3q_1^{n-3}$. Then we have the following decomposition* *into irreducible characters of S_n.*

$$char_{S_n}[FK^{(2)}_{q_3q_1^{n-3}}(n)] = \begin{cases} 2\chi_{21}, & n=3 \\ 2\chi_{31}+2\chi_{2,2}+2\chi_{211}, & n=4 \\ 2\chi_{n-1,1}+2\chi_{n-2,2}+2\chi_{n-2,1,1}+2\chi_{n-3,2,1}. & n\geq 5 \end{cases} \tag{6.140}$$

Proof. Proof is similar to the proof of Proposition 6.5.8. □

Corollary 6.6.4. *We have the following identity for f^λs, the number of standard tableaux of shape λ.*

$$f^{n-1,1}+f^{n-2,2}+f^{n-2,1,1}+f^{n-3,2,1} = 2\binom{n}{3},\ n\geq 5. \tag{6.141}$$

Proof. By evaluating (6.140) at identity ($\sigma = 1^n$) by using (6.139), it is straight forward to come up with (6.141) like before. This completes the proof. □

Chapter 7

A quotient algebra of $FK(n)$

In this chapter we introduce a graph dependent algebra denoted by $\overline{FK}_G(n)$, where G is a subgraph of the complete graph on n vertexes associated to Fomin-Kirillov algebra.

$\overline{FK}_G(n)$ is a quotient of $FK(n)$ by the ideal generated by the missing edges in G compared to the associated complete graph.

$$\overline{FK}_G(n) = \frac{FK(n)}{I\langle \text{missing edges in subgraph } G\rangle} = \frac{F\langle x_{ij}\rangle}{I\langle \text{union of generators of the defining ideal of } FK(n) \text{ and missing edges in subgraph } G\rangle}, \tag{7.1}$$

where $x_{ij},\ 1 \leq i < j \leq n$ stands for the generators of $FK(n)$, in other words, the edges of a complete graph on n vertexes, a subgraph of which is G, and where $F\langle x_{ij}\rangle$ is the free algebra generated by the variables x_{ij}.

Remark 7.0.1. *The focus of this chapter however, is the special case when G is an n-cycle graph. An n-cycle graph denoted by C_n, is a regular polygon on $n \geq 3$ vertexes, with set of sides $\{x_{12}, x_{23}, \cdots x_{n-1,n}, x_{1n}\}$. The quotient of algebra $FK(n)$ associated with the graph C_n is denoted by $\overline{FK}_{C_n}(n)$.*

In sections 1 and 2, we analyze finite examples of $\overline{FK}_{C_n}(n)$ for $n = 5$ and 6. In each case we find relations, Gröbner basis, basis of $\overline{FK}_{C_n}(n)$, its dimension and its character refined by various gradings. In section 3 we get into the general case of n, where we find the basis, dimension and the character with its representation decomposition. We prove that the dimension of $\overline{FK}_{C_n}(n)$ equals Lucas number L_n.

According to the above description, to find the relations for $\overline{FK}_{C_n}(n)$ we can consider the relations of $FK(n)$ in (1.2) and set the edges of the complete graph that are missing in the subgraph C_n equal to zero. This way we will have the list of relations of $\overline{FK}_{C_n}(n)$ coming in the following definition of algebra $\overline{FK}_{C_n}(n)$.

Definition 7.0.2. *For $n > 3$, $\overline{FK}_{C_n}(n)$, the quotient algebra of $FK(n)$, is the algebra on generators $x_{1,n} = -x_{n,1}$ and $x_{m,m+1} = -x_{m+1,m}, 1 \leq m \leq n-1$, that satisfy the following relations.*

$$
\begin{aligned}
R_{C_n} = \{ & x^2_{m,m+1} = 0 : \ 1 \leq m \leq n-1; x^2_{1,n} = 0 \ , \\
& , \begin{cases} x_{m,m+1}x_{m+1,m+2} = 0, 1 \leq m \leq n-2 \\ x_{m+1,m+2}x_{m,m+1} = 0, 1 \leq m \leq n-2 \end{cases} , \begin{cases} x_{n-1,n}x_{1n} = 0 \\ x_{1n}x_{n-1,n} = 0 \end{cases} , \begin{cases} x_{12}x_{1n} = 0 \\ x_{1n}x_{12} = 0 \end{cases} \\
& , \begin{cases} \underline{x_{m,m+1}x_{l,l+1}} - x_{l,l+1}x_{m,m+1} = 0, \ 1 \leq m \leq l-2, \ 3 \leq l \leq n-1 \\ \underline{x_{m,m+1}x_{1,n}} - x_{1n}x_{m,m+1} = 0, \ 2 \leq m \leq n-2 \end{cases} \}.
\end{aligned}
\tag{7.2}
$$

From the relations in (7.2) we have the following set of generators for the defining ideal for $\overline{FK}_{C_n}(n)$.

$$\begin{gathered}\text{Generators of defining ideal of } \overline{FK}_{C_n}(n) = \\ = \{x^2_{m,m+1},\ 1 \le m \le n-1,\ x^2_{1,n}, \\ \begin{cases} x_{m,m+1}x_{m+1,m+2},\ 1 \le m \le n-2, \\ x_{m+1,m+2}x_{m,m+1},\ 1 \le m \le n-2, \end{cases} \quad \begin{cases} x_{n-1,n}x_{1n}, \\ x_{1n}x_{n-1,n}, \end{cases} \quad \begin{cases} x_{12}x_{1n}, \\ x_{1n}x_{12}, \end{cases} \\ \begin{cases} \underline{x_{m,m+1}x_{l,l+1}} - x_{l,l+1}x_{m,m+1},\ 1 \le m \le l-2,\ 3 \le l \le n-1, \\ \underline{x_{m,m+1}x_{1,n}} - x_{1n}x_{m,m+1},\ 2 \le m \le n-2 \end{cases} \}.\end{gathered} \tag{7.3}$$

Remark 7.0.3. *As mentioned above while the generators of $FK(n)$ are the edges of a complete graph, the generators of $\overline{FK}_{C_n}(n)$ are the sides of C_n, i.e., $\{x_{12}, x_{23}, \cdots, x_{n-1,n},\ x_{1n}\}$. Therefore while S_n defines an action on $FK(n)$, as the defining ideal of $FK(n)$ is stable under S_n, however S_n does not define an action on $\overline{FK}_{C_n}(n)$, as the defining ideal of $\overline{FK}_{C_n}(n)$ generated by the terms in* (7.3) *is not stable under S_n (for example x^2_{1n} in the set of generators of the defining ideal of $\overline{FK}_{C_n}(n)$, under transposition $(12) \in S_n$ goes to x^2_{2n} which is not in the set). However, as we will show later, the ideal is stable under dihedral group D_n, so D_n defines an action on $\overline{FK}_{C_n}(n)$ (an action is defined on a quotient if and only if the defining ideal is stable under the action).*

Dihedral group

Dihedral group is defined by

$$D_n = \langle r, s | s^2 = 1, r^n = 1, (rs)^2 = 1 \rangle.$$

We can realize group D_n in the permutation group S_n, where

$$s = \begin{cases} (1,n)(2,n-1)\cdots(\frac{n}{2},\frac{n}{2}+1), & \text{even } n, \\ (1,n)(2,n-1)\cdots(\frac{n-1}{2},\frac{n+3}{2})(\frac{n+1}{2}), & \text{odd } n, \end{cases} \tag{7.4}$$

$$\text{and } r = (12\cdots n),$$

are permutations on n vertexes (reflection and rotation in n-cycle C_n).

The stability of the defining ideal of $\overline{FK}_{C_n}(n)$ under D_n

Here we show that the defining ideal of $\overline{FK}_{C_n}(n)$ is stable under D_n, i.e., the generators of the ideal in (7.3) as a set is invariant under the action of generators of D_n, rotation and reflection.

1. In the first line of Equation (7.3) we have the set of all couples of same sides of C_n, so is invariant under rotation as well as reflection.
2. The three cases in the second line of (7.3) together, make the set of all couples of the successive sides, so is invariant under rotation as well as reflection, as rotation and reflection does not separate the successive sides.

3. In the third line of (7.3), the two items together, form the set of <u>all</u> couples of the sides which are separated by $l-2$ sides for $l = 3, 4, \cdots n-2$. So for each l we have an invariant set under the operations of rotation and reflection, as these operations do not affect the spacing between the sides.

The above three items show that the set of generators (7.3) of the defining ideal of $\overline{FK}_{C_n}(n)$ is invariant under generators of D_n, i.e., the defining ideal of $\overline{FK}_{C_n}(n)$ is stable under D_n. Hence D_n defines an action on $\overline{FK}_{C_n}(n)$ as follows.

7.0.1 The action of D_n on $\overline{FK}_{C_n}(n)$

We define the action of D_n realized in permutation group on $\overline{FK}_{C_n}(n)$ as

$$D_n : \overline{FK}_{C_n}(n) \to \overline{FK}_{C_n}(n) \text{ by } \sigma(M+I) \mapsto \sigma M + I, \tag{7.5}$$

where the action of $\sigma \in D_n$ on a monomial M is defined via action on a variable $x_{ij},\ j = i+1$ (as x_{ij} represents a side of an n-cycle), defined by

$$\sigma x_{ij} = \begin{cases} x_{\sigma(i)\sigma(j)} & \text{if } \sigma(i) < \sigma(j), \\ -x_{\sigma(j)\sigma(i)} & \text{otherwise,} \end{cases} \tag{7.6}$$

and multiplicative extension to monomial M. The algebra is closed under D_n as the defining ideal is stable under D_n as $\sigma \in D_n$ takes one side of C_n to another one, in other words, one monomial in generators $x_{ij},\ j = i+1$, to another one in them. We need to check the following items.

1. Well defined,

$$M + I = M' + I \to M - M' \in I \to \sigma(M - M') \in I (\text{as } I \text{ is stable under } D_n).$$

$$\text{Then } \sigma M - \sigma M' \in I \to \sigma M + I = \sigma M' + I \to \sigma(M + I) = \sigma(M' + I).$$

2. (identity axiom) $\epsilon(f(x_{i_1 j_1}, \cdots, x_{i_k j_k}) + I) = f(x_{i_1 j_1}, \cdots, x_{i_k j_k}) + I$,

3. (compatibility axiom)

$$\begin{aligned}(\sigma_1\sigma_2)(f(x_{i_1 j_1}, \cdots, x_{i_k j_k}) + I) =& f((\sigma_1\sigma_2)x_{i_1 j_1}, \cdots, (\sigma_1\sigma_2)x_{i_k j_k}) + I \\ =& \sigma_1 f(\sigma_2 x_{i_1 j_1}, \cdots, \sigma_2 x_{i_k j_k}) + I \\ =& \sigma_1(\sigma_2 f(x_{i_1 j_1}, \cdots, x_{i_k j_k})) + I.\end{aligned}$$

Hence the map in (7.5), defines an action on $\overline{FK}_{C_n}(n)$.

7.0.2 Representation decompositions of $\overline{FK}_{C_n}(n)$

- (Usual degree decomposition) The action of D_n on $\overline{FK}_{C_n}(n)$ defined in (7.5) permutes the indexes among themselves but does not change the number of variables in a monomial. Therefore usual degree remains invariant under D_n, so usual degree representation decomposition makes sense for $\overline{FK}_{C_n}(n)$ (i.e., D_n respects decomposition in usual degree).

$$\overline{FK}_{C_n}(n) = \bigoplus_{d \geq 0} \overline{FK}^{(d)}_{C_n}(n). \tag{7.7}$$

- (Conjugacy class decomposition) S_n-degree is well defined on $\overline{FK}_{C_n}(n)$, since the generators of the defining ideal in (7.3) are homogeneous with respect to S_n. The action of D_n on $\overline{FK}_{C_n}(n)$ defined in (7.5) does not change the cycle type of the S_n-degrees assigned to elements of $\overline{FK}_{C_n}(n)$, because a monomial of degree σ, when acted upon by a permutation ν is sent to a monomial of degree $\nu^{-1}\sigma\nu$ which is a conjugate of σ by definition of conjugacy. So permutation ν sends σ to a conjugate of σ, i.e., conjugacy class is invariant under D_n realized in permutation group. Hence the conjugacy class decomposition is a representation decomposition.

$$\overline{FK}_{C_n}(n) = \bigoplus_{\mu} \overline{FK}^{\mu}_{C_n}(n), \tag{7.8}$$

where $\overline{FK}^{\mu}_{C_n}(n)$ stands for all the elements of S_n-degree σ in $\overline{FK}_{C_n}(n)$ that belong to conjugacy class indexed by μ.

Remark 7.0.4. *Every permutation σ can be obtained using only transpositions $(1,2)$, $(2,3), \cdots (n-1,n)$, so $\overline{FK}^{\sigma}_{C_n}(n)$ is potentially not zero for any permutation σ. However for our special cases we will see in section 7.1.1 that for many of permutations it is zero. Having an S_n-conjugacy class, it is not true in general that it is a D_n-conjugacy class as well. For example transpositions $(2,3)$ and $(2,5)$ in the same S_n-conjugacy class for $n > 5$, are not conjugated via an element in D_n. However we will see in section 7.1.1 that for our very special basis, the S_n-conjugacy class elements form a D_n-conjugacy class (any two permutation in this S_n-conjugacy class are conjugated via an element of D_n). It is why* (7.8) *can not be refined.*

- (Set partition type decomposition) Set-partition degree is well defined on $\overline{FK}_{C_n}(n)$, since the generators of the defining ideal in (7.3) are homogeneous with respect to set-partition degree. For any monomial M in $\overline{FK}_{C_n}(n)$, the appearance of x_{ij} in M implies that i, j are in the same part of the set-partition degree. Any x_{ij} appearing in M, under the action of $\sigma \in D_n$ (by definition in (7.6)) goes to $x_{\sigma(i)\sigma(j)}$ if $\sigma(i) < \sigma(j)$

and to $-x_{\sigma(j)\sigma(i)}$ otherwise, so as a result of the action, now we have the indexes $\sigma(i)$ and $\sigma(j)$ in the same part. This means that while the action of D_n changes the indexes in a part but does not change the number of them in that part, i.e., set partition type is invariant under the action of D_n realized in permutation group. Hence partition in terms of set partition type is a D_n representation decomposition.

$$\overline{FK}_{C_n}(n) = \bigoplus_{\phi} \overline{FK}^{\phi}_{C_n}(n), \tag{7.9}$$

where $\overline{FK}^{\phi}_{C_n}(n)$ stands for all set partition degree λ elements that belong to the same set partition type indexed by ϕ.

Remark 7.0.5. *Unlike permutations, discussed in Remark 7.0.4, not every set-partition can be obtained by joining only consecutive entries modulo n. For example it is not possible to obtain the set-partition $\{\{1,3\},\{2\},\{4\}\}$. Therefore for some set-partition A, the space $\overline{FK}^{A}_{C_n}(n)$ is zero. We will see later in section 7.1.1, that for our very special basis, for any non-zero $\overline{FK}^{A}_{C_n}(n)$ and $\overline{FK}^{B}_{C_n}(n)$ such that A and B have same partition type, we can find an element of D_n that maps A into B. Hence* (7.9) *can not be refined.*

In the following two sections we get into the specific cases of $\overline{FK}_{C_5}(5)$ and $\overline{FK}_{C_6}(6)$ before getting into the general case of $\overline{FK}_{C_n}(n)$ in section 7.3.

7.1 Quotient algebra $\overline{FK}_{C_5}(5)$

Relations

$$\begin{aligned} R_{C_5} =&\{x_{12}^2 = 0,\ x_{23}^2 = 0,\ x_{34}^2 = 0,\ x_{45}^2 = 0,\ x_{15}^2 = 0 \\ &x_{12}x_{23} = 0,\ x_{23}x_{12} = 0,\ x_{23}x_{34} = 0,\ x_{34}x_{23} = 0, \\ &x_{34}x_{45} = 0,\ x_{45}x_{34} = 0,\ x_{15}x_{12} = 0,\ \ x_{15}x_{45} = 0,\ x_{12}x_{15} = 0, \\ &x_{12}x_{34} - x_{34}x_{12} = 0,\ x_{12}x_{45} - x_{45}x_{12} = 0,\ x_{23}x_{45} - x_{45}x_{23} = 0, \\ &x_{23}x_{15} - x_{15}x_{23} = 0,\ x_{34}x_{15} - x_{15}x_{34} = 0\} \end{aligned} \tag{7.10}$$

Gröbner basis

$$\begin{aligned} GB_{C_5} =&\{x_{12}^2,\ x_{23}^2,\ x_{34}^2,\ x_{45}^2,\ x_{15}^2 \\ &x_{12}x_{23},\ x_{23}x_{12},\ x_{23}x_{34},\ x_{34}x_{23},\ x_{34}x_{45},\ x_{45}x_{34},\ x_{15}x_{12},\ x_{12}x_{15}, \\ &x_{12}x_{34} - x_{34}x_{12},\ x_{12}x_{45} - x_{45}x_{12},\ x_{23}x_{45} - x_{45}x_{23},\ x_{23}x_{15} - x_{15}x_{23}, \\ &x_{34}x_{15} - x_{15}x_{34}, x_{45}x_{15}x_{34},\ x_{15}x_{34}x_{12},\ x_{15}x_{45}x_{12},\ x_{15}x_{45}x_{23}\} \end{aligned} \tag{7.11}$$

Basis

$$B(\overline{FK}_{C_5}(5)) = \{1,\ x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{15},\ x_{15}x_{34},\ x_{15}x_{23},\ x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23}\}. \tag{7.12}$$

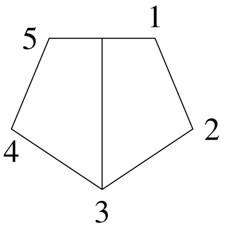


Figure 7.1: A 5-cycle with reflection $s = (15)(24)(3)$.

Hilbert series

$$H(\overline{FK}_{C_5}(5)) = 1 + 5q + 5q^2 \tag{7.13}$$

Remark 7.1.1. *dim*$[\overline{FK}_{C_5}(5)] = 11$.

Remark 7.1.2. *As mentioned before, D_n defines an action on Algebra $\overline{FK}_{C_n}(n)$. Under the action of D_n, the component of the character of $\overline{FK}_{C_n}(n)$ associated with the conjugacy class indexed by $\sigma \in D_n$, is the sum of the coefficients of $z \in B[\overline{FK}_{C_n}(n)]$ in $\sigma(z)$ expanded in elements of $B[\overline{FK}_{C_n}(n)]$, when z runs in $B[\overline{FK}_{C_n}(n)]$.*

7.1.1 Character and decompositions

Dihedral group D_5

Group $D_5 = \{r, s | r^5 = 1 = s^2, rsr = s\}$ has 4 conjugacy classes ϵ, $\{r, r^4\}$, $\{r^2, r^3\}$ and $\{s, sr, sr^2, sr^3, sr^4\}$ with generators r and s realized in permutation group as $r = (12345)$, and $s = (15)(24)(3)$. Figure 7.1 depicts a 5-cycle C_5 with reflection $s = (15)(24)(3)$.

Table 7.1: Character table of D_5 with $char_{D_5}\overline{FK}_{C_5}(5)$

	ϵ	$[r] = \{r, r^4\}$	$[r^2] = \{r^2, r^3\}$	$[s] = \{s, sr, sr^2, sr^3, sr^4\}$
	$\lvert c_{11111}\rvert = 1$	$\lvert c_5\rvert = 2$	$\lvert c_{5'}\rvert = 2$	$\lvert c_{221}\rvert = 5$
χ_{11111}	1	1	1	1
χ_{221}	1	1	1	-1
χ_5	2	$\frac{\sqrt{5}-1}{2}$	$-\frac{\sqrt{5}+1}{2}$	0
$\chi_{5'}$	2	$-\frac{\sqrt{5}+1}{2}$	$\frac{\sqrt{5}-1}{2}$	0
$char_{D_5}\overline{FK}_{C_5}(5)$	11	1	1	1

Remark 7.1.3. *In the character table of D_5, Table 7.1, c_5 stands for the D_5-conjugacy class represented by r realized in S_5 by permutation (12345) of cycle type 5, and $c_{5'}$ for D_5-conjugacy class represented by r^2, where by $(12345)(12345) = (13524)$ again is of cycle type (5) that we denote it by $c_{5'}$. Also c_{221} stand for S_5-conjugacy class represented by s, as $s = (15)(24)(3)$.*

As discussed before D_n defines an action on $\overline{FK}(n)$, so we have the following character (calculation explained in Remark 7.1.2).

Decomposition by irreducible representations

$$\begin{aligned} char_{D_5}\overline{FK}_{C_5}(5) &= (11, 1, 1, 1) \\ &= 2\chi_{11111} + \chi_{221} + 2\chi_{5'} + 2\chi_5, \end{aligned} \tag{7.14}$$

where $\chi_{11111}, \chi_{221}, \chi_5, \chi_{5'}$ are irreducible characters of D_5 in Table 7.1.

Representation decomposition by usual degree

The basis of $\overline{FK}_{C_5}(5)$, w.r.t. usual degree is divided into three sets:

$$\{1\},\ \{x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{15}\}\ and\ \{x_{15}x_{34},\ x_{15}x_{23},\ x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23}\}.$$

They consist of elements of degrees 0, 1, and 2 respectively. Since permutation of indexes does not change the number of variables, each of these sets are invariant under the action of D_5 whose conjugation class representatives, ϵ, r, r^2, and s have character vectors $(1, 1, 1, 1)$, $(5, 0, 0, -1)$ and $(5, 0, 0, 1)$ for degrees equal to 0, 1 and 2 respectively. Putting these characters together we have

$$\begin{aligned} char_{D_5}[\overline{FK}_{C_5}(5)] =&(1,1,1,1) + (5,0,0,-1)q + (5,0,0,1)q^2 \\ =&\chi_{11111} + (\chi_{221} + \chi_5 + \chi_{5'})q + (\chi_{11111} + \chi_5 + \chi_{5'})q^2, \end{aligned} \tag{7.15}$$

where $\chi_{11111}, \chi_{221}, \chi_5, \chi_{5'}$ are irreducible characters of D_5 in Table 7.1.

Representation decomposition by S_n-conjugacy class

Here the basis of $\overline{FK}_{C_5}(5)$ is divided by S_5-conjugacy class into sets

$$\{1\},\ \{x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{15}\},\ \text{and}\ \{x_{15}x_{34},\ x_{15}x_{23},\ x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23}\},$$

with associated S_5-conjugacy classes denoted by $t_1^5,\ t_2t_1^3$ and $t_2^2t_1$ respectively. We see that the S_5-conjugacy class elements here form a D_5-conjugacy class (any two permutation in this class are conjugated via element of D_5). For example the elements in $\{x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{15}\}$ with S_5-degrees $\{(12),\ (23),\ (34),\ (45),\ (15)\}$ are all in S_5-conjugacy class denoted by $t_2t_1^3$. Each pair of the permutations in $\{(12),\ (23),\ (34),\ (45),\ (15)\}$ however are conjugated by an element of D_5. Same is true for the other two sets in the above, i.e.,

$$\{1\},\ \text{and}\ \{x_{15}x_{34},\ x_{15}x_{23},\ x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23}\}.$$

It is why the S_5-decomposition is not refined. We will see that related to S_5-conjugacy classes $t_1^5,\ t_2t_1^3,$ and $t_2^2t_1$, we have the following characters.

$$\begin{aligned} char_{D_5}[\overline{FK}_{C_5}^{(t_1^5)}(5)] &= (1,1,1,1,1) = \chi_{11111}. \\ char_{D_5}[\overline{FK}_{C_5}^{(t_2t_1^3)}(5)] &= (5,0,0,-1) = \chi_{221} + \chi_5 + \chi_{5'}. \\ char_{D_5}[\overline{FK}_{C_5}^{(t_2^2t_1)}(5)] &= (5,0,0,1) = \chi_{11111} + \chi_5 + \chi_{5'}. \end{aligned} \tag{7.16}$$

Therefore we have

$$char_{D_5}[\overline{FK}_{C_5}(5)] = \chi_{11111}t_1^5 + (\chi_{221} + \chi_5 + \chi_{5'})t_2t_1^3 + (\chi_{11111} + \chi_5 + \chi_{5'})t_2^2t_1, \tag{7.17}$$

where $\chi_{11111}, \chi_{221}, \chi_5, \chi_{5'}$ are irreducible characters of D_5 in Table 7.1.

Representation decomposition by set partition type

We have the same decomposition of the basis of $\overline{FK}_{C_5}(5)$ invariant under usual degree as under set partition type and S_5-conjugacy classes as in the following.

$$\{1\},\ \{x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{15}\},\ \text{and}\ \{x_{15}x_{34},\ x_{15}x_{23},\ x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23}\}$$

with associated set-partition types denoted by q_1^5, $q_2q_1^3$ and $q_2^2q_1$ respectively. We see that the set-partition type conjugacy classes here form a D_5-conjugacy class. (any two set partition degree in each type are conjugated via an element of D_5). For example the elements of the set-partition type $q_2^2q_1$ in

$$\{x_{15}x_{34},\ x_{15}x_{23},\ x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23}\},$$

have set-partition degrees

$$\begin{aligned}&\{\{15\},\{34\},\{2\}\},\ \{\{15\},\{23\},\{4\}\},\ \{\{34\},\{12\},\{5\}\},\\&\{\{45\},\{12\},\{3\}\},\ \{\{45\},\{23\},\{1\}\},\end{aligned}$$

respectively, with each two of them conjugated by an element of D_5. Same is true for the other two sets in the above, i.e., $\{1\}$, and $\{x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{15}\}$. It is why the set-partition decomposition is not refined, as we will see in the next paragraph, we will come to the following decomposition:

$$\begin{aligned}char_{D_5}[\overline{FK}_{C_5}^{(q_1^5)}(5)] &= (1,1,1,1,1) = \chi_{11111}.\\ char_{D_5}[\overline{FK}_{C_5}^{(q_2q_1^3)}(5)] &= (5,0,0,-1) = \chi_{221}+\chi_5+\chi_{5'}.\\ char_{D_5}[\overline{FK}_{C_5}^{(q_2^2q_1)}(5)] &= (5,0,0,1) = \chi_{11111}+\chi_5+\chi_{5'}.\end{aligned} \tag{7.18}$$

Hence we have

$$char_{D_5}[\overline{FK}_{C_5}(5)] = \chi_{11111}q_1^5 + (\chi_{221}+\chi_5+\chi_{5'})q_2q_1^3 + (\chi_{11111}+\chi_5+\chi_{5'})q_2^2q_1. \tag{7.19}$$

Remark 7.1.4. *In case of $char_{D_5}[\overline{FK}_{C_5}(5)]$ decomposition w.r.t., usual degree, S_5-conjugacy class and set partition type, give essentially the same result.*

7.2 Quotient algebra $\overline{FK}_{C_6}(6)$

Relations

$$
\begin{aligned}
R_{C_6} = \{ & x_{12}^2 = 0,\ x_{23}^2 = 0,\ x_{34}^2 = 0, x_{45}^2 = 0, x_{56}^2 = 0, x_{16}^2 = 0, \\
& \begin{cases} x_{12}x_{23} = 0 \\ x_{23}x_{12} = 0 \end{cases}, \begin{cases} x_{23}x_{34} = 0 \\ x_{34}x_{23} = 0 \end{cases}, \begin{cases} x_{34}x_{45} = 0 \\ x_{45}x_{34} = 0 \end{cases}, \\
& \begin{cases} x_{45}x_{56} = 0 \\ x_{56}x_{45} = 0 \end{cases}, \begin{cases} x_{56}x_{16} = 0 \\ x_{16}x_{56} = 0 \end{cases}, \begin{cases} x_{16}x_{12} = 0 \\ x_{12}x_{16} = 0 \end{cases} \\
& , \begin{cases} x_{12}x_{34} - x_{34}x_{12} = 0, \\ x_{12}x_{45} - x_{45}x_{12} = 0 \\ x_{12}x_{56} - x_{56}x_{12} = 0 \end{cases}, \begin{cases} x_{23}x_{45} - x_{45}x_{23} = 0 \\ x_{23}x_{56} - x_{56}x_{23} = 0 \\ x_{23}x_{16} - x_{16}x_{23} = 0 \end{cases}, \\
& \begin{cases} x_{34}x_{56} - x_{56}x_{34} = 0 \\ x_{34}x_{16} - x_{16}x_{34} = 0 \end{cases}, \\
& \begin{cases} x_{45}x_{16} - x_{16}x_{45} = 0 \end{cases} \}.
\end{aligned}
\tag{7.20}
$$

Gröbner

$$
\begin{aligned}
GB_{C_6} = &\{x_{12}^2,\ x_{23}^2,\ x_{34}^2,\ x_{45}^2,\ x_{56}^2,\ x_{16}^2, \\
&, \begin{cases} x_{12}x_{23} \\ x_{23}x_{12} \end{cases}, \begin{cases} x_{23}x_{34} \\ x_{34}x_{23} \end{cases}, \begin{cases} x_{34}x_{45} \\ x_{45}x_{34} \end{cases}, \begin{cases} x_{45}x_{56} \\ x_{56}x_{45} \end{cases} \\
&, \begin{cases} x_{56}x_{16} \\ x_{16}x_{56} \end{cases}, \begin{cases} x_{12}x_{16} \\ x_{16}x_{12} \end{cases}, \begin{cases} x_{12}x_{34} - x_{34}x_{12} \end{cases} \\
&, \begin{cases} x_{12}x_{45} - x_{45}x_{12} \\ x_{23}x_{45} - x_{45}x_{23} \end{cases}, \begin{cases} x_{12}x_{56} - x_{56}x_{12} \\ x_{23}x_{56} - x_{56}x_{23} \\ x_{34}x_{56} - x_{56}x_{34} \end{cases} \\
&, \begin{cases} x_{23}x_{16} - x_{16}x_{23} \\ x_{34}x_{16} - x_{16}x_{34} \\ x_{45}x_{16} - x_{16}x_{45} \end{cases}, \begin{cases} x_{16}x_{34}x_{12} \\ x_{16}x_{45}x_{12} \end{cases} \\
&, \text{no deg-4 or higher terms}\}.
\end{aligned} \tag{7.21}
$$

Basis of $\overline{FK}_{C_6}(6)$

$$
\begin{aligned}
B[\overline{FK}_{C_6}(6)] = \{&1,\ x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{56},\ x_{16}, x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23},\ x_{56}x_{12},\ x_{56}x_{23}, \\
&x_{56}x_{34},\ x_{16}x_{23},\ x_{16}x_{34},\ x_{16}x_{45}, x_{16}x_{45}x_{23},\ x_{56}x_{34}x_{12}\}.
\end{aligned} \tag{7.22}
$$

Hilbert series

$$H(\overline{FK}_{C_6}(6)) = 1 + 6q + 9q^2 + 2q^3. \tag{7.23}$$

Remark 7.2.1. *dim*$[\overline{FK}_{C_6}(6)] = 18$.

7.2.1 Character and decompositions

Dihedral group D_6

Group $D_6 = \{r, s | r^6 = 1 = s^2, rsr = s\}$ has 6 conjugacy classes

$$\epsilon,\ \{r, r^5\},\ \{r^2, r^4\},\ \{r^3\},\ \{s, sr^2, sr^4\} \text{ and } \{sr, sr^3, sr^5\},$$

with generators r and s realized in permutation group as rotation $r = (123456)$, and reflection about $s = (16)(25)(34)$. As before D_6 defines an action on $\overline{FK}_{C_6}(6)$.

The below diagrams depict 6-cycle C_6 diagrams with reflections $s = (16)(25)(34)$ and $sr = (15)(24)(3)(6)$ in the diagrams on the left and right respectively.

Figure 7.2: 6-cycle diagrams with reflections $s = (16)(25)(34)$ on left hand side and $sr = (15)(24)(3)(6)$ on right hand side.

Table 7.2: Character table of D_6 with $char_{D_6}[\overline{FK}_{C_6}(6)]$. Irred. characters: The trivial, 3 reflections, and 2 rotations.

	ϵ	$\{r, r^5\}$	$\{r^2, r^4\}$	$\{r^3\}$	$\{s, sr^2, sr^4\}$	$\{sr, sr^3, sr^5\}$
	$\lvert c_{111111}\rvert = 1$	$\lvert c_6\rvert = 2$	$\lvert c_{33}\rvert = 2$	$\lvert c_{222}\rvert = 1$	$\lvert c_{222'}\rvert = 3$	$\lvert c_{2211}\rvert = 3$
χ_{111111}	1	1	1	1	1	1
χ_{222}	1	1	1	1	-1	-1
$\chi_{222'}$	1	-1	1	-1	+1	-1
χ_{2211}	1	-1	1	-1	-1	1
χ_6	2	1	-1	-2	0	0
χ_{33}	2	-1	-1	2	0	0
$char_{D_6}\overline{FK}_{C_6}(6)$	18	1	3	4	0	2

The character is calculated as before and is written in irreducible representations of D_6 in Table 7.2.

Remark 7.2.2. *In the character table of D_6, Table 7.2, c_{2211} stands for the D_6-conjugacy class with cycle type 2211, as $sr = (16)(25)(34)(123456) = (15)(24)(3)(6)$ has cycle type 2211. Similarly c_{33} stands for the D_6-conjugacy class with cycle type 33, as $r^2 = (123456)(123456) = (135)(246)$ has cycle type 33. Similarly for others.*

Decomposition in irreducible characters

$$\begin{aligned} char_{D_6}\overline{FK}_{C_6}(6) &= (18, 1, 3, 4, 0, 2) \\ &= 3\chi_{111111} + 2\chi_{222} + \chi_{222'} + 2\chi_{2211} + 2\chi_6 + 3\chi_{33}, \end{aligned} \tag{7.24}$$

where $\chi_{111111},\ \chi_{222},\ \chi_{222'},\ \chi_{2211},\ \chi_{6},\ \chi_{33}$ are the irreducible characters of Dihedral group D_6 in Table 7.2.

Representation decomposition by usual degree

Usual degree make the following partition on the basis.

$$\begin{aligned} B[\overline{FK}_{C_6}(6)] =&\{1\} \cup \{x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{56},\ x_{16}\} \\ &\cup \{x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23},\ x_{56}x_{12},\ x_{56}x_{23}, x_{56}x_{34},\ x_{16}x_{23},\ x_{16}x_{34},\ x_{16}x_{45}\} \\ &\cup \{x_{16}x_{45}x_{23},\ x_{56}x_{34}x_{12}\}. \end{aligned} \tag{7.25}$$

The D_6-conjugacy classes indexed by $c_{111111},\ c_{2211},\ c_{222},\ c_{222'},\ c_{33},\ c_{6}$ define actions on each of the components of the above partition and each component is invariant under D_6 (as before) so representation decomposition makes sense. Calculation of the character on each of the components as before yields.

$$\begin{aligned} char_{D_6}[\overline{FK}^{(0)}(6)] &= (1,1,1,1,1,1) = \chi_{111111}. \\ char_{D_6}[\overline{FK}^{(1)}(6)] &= (6,0,0,0,-2,0) = \chi_{222} + \chi_{2211} + \chi_{6} + \chi_{33}. \\ char_{D_6}[\overline{FK}^{(2)}(6)] &= (9,0,0,3,3,1) = 2\chi_{111111} + \chi_{222'} + \chi_{6} + 2\chi_{33}. \\ char_{D_6}[\overline{FK}^{(3)}(6)] &= (2,0,2,0,-2,0) = \chi_{222} + \chi_{2211}. \end{aligned} \tag{7.26}$$

Therefore we have

$$\begin{aligned} char_{D_6}\overline{FK}_{C_6}(6)(q) =&\chi_{111111} + (\chi_{222} + \chi_{2211} + \chi_{6} + \chi_{33})q \\ &+ (2\chi_{111111} + \chi_{222'} + \chi_{6} + 2\chi_{33})q^2 + (\chi_{222} + \chi_{2211})q^3, \end{aligned} \tag{7.27}$$

where $\chi_{111111}, \chi_{2211}, \chi_{222'}, \chi_{222}, \chi_{33}, \chi_6$ are irreducible characters of D_6 in Table 7.2.

Representation decomposition by S_n-conjugacy class

We have the same partition of the basis of $\overline{FK}_{C_6}(6)$ invariant under S_6-conjugacy class (that again correspond to a D_6-conjugacy class) as by usual degree, so we will have essentially the same decomposition as for usual degree (as before).

$$\begin{aligned} char_{D_6}\overline{FK}_{C_6}(6)(t_1,t_2) =& \chi_{111111}t_1^6 + (\chi_{222} + \chi_{2211} + \chi_6 + \chi_{33})t_2t_1^4 \\ &+ (2\chi_{111111} + \chi_{222'} + \chi_6 + 2\chi_{33})t_2^2t_1^2 \\ &+ (\chi_{222} + \chi_{2211})t_2^3, \end{aligned} \tag{7.28}$$

where $\chi_{111111},\ \chi_{2211},\ \chi_{222'},\ \chi_{222},\ \chi_{33},\ \chi_6$, are irreducible characters of D_6 in Table 7.2.

Representation decomposition by set partition type

Here we have the same partition of the basis invariant under set partition as under usual degree.

$$\begin{aligned} B[\overline{FK}_{C_6}(6)] =& \{1\} \cup \{x_{12},\ x_{23},\ x_{34},\ x_{45},\ x_{56},\ x_{16}\} \\ &\cup \{x_{34}x_{12},\ x_{45}x_{12},\ x_{45}x_{23},\ x_{56}x_{12},\ x_{56}x_{23},\ x_{56}x_{34},\ x_{16}x_{23},\ x_{16}x_{34},\ x_{16}x_{45}\} \\ &\cup \{x_{16}x_{45}x_{23},\ x_{56}x_{34}x_{12}\}. \end{aligned}$$

Therefore the calculation of the character as before results in the following decomposition (as before).

$$\begin{aligned} char_{D_6}\overline{FK}_{C_6}(6)(q_1,q_2) &= (1,1,1,1,1,1)q_1^6 + ((6,0,0,0,-2,0))q_2q_1^4 + \\ &+ (9,0,0,3,3,1)q_2^2q_1^2 + (2,0,2,0,-2,0)q_2^3 \\ &= \chi_{111111}q_1^6 + (\chi_{222} + \chi_{2211} + \chi_6 + \chi_{33})q_2q_1^4 \\ &+ (2\chi_{111111} + \chi_{222'} + \chi_6 + 2\chi_{33})q_2^2q_1^2 + (\chi_{222} + \chi_{2211})q_2^3, \end{aligned} \tag{7.29}$$

where $\chi_{111111},\ \chi_{2211},\ \chi_{222'},\ \chi_{222},\ \chi_{33},\ \chi_6$, are irreducible characters of D_6 in Table 7.2.

Remark 7.2.3. *Representation decomposition of $char_{D_6}[\overline{FK}_{C_6}(6)]$ by usual degree, S_n-conjugacy class and set partition type are essentially the same.*

In the next section we get into the general case of $\overline{FK}_{C_n}(n)$.

7.3 Quotient algebra $\overline{FK}_{C_n}(n)$

Relations of $\overline{FK}_{C_n}(n)$

We have already the relations of $\overline{FK}_{C_n}(n)$ in (7.2).

Defining ideal of $\overline{FK}_{C_n}(n)$

From (7.2), the defining ideal of $\overline{FK}_{C_n}(n)$ is generated by the generators in the following set.

$$\begin{aligned}
&\text{Generators of defining ideal of } \overline{FK}_{C_n}(n) = \\
&= \{x_{m,m+1}^2,\ 1 \leq m \leq n-1,\ x_{1,n}^2, \\
&\begin{cases} x_{m,m+1}x_{m+1,m+2},\ 1 \leq m \leq n-2, \\ x_{m+1,m+2}x_{m,m+1},\ 1 \leq m \leq n-2 \end{cases} \quad \begin{cases} x_{n-1,n}x_{1n} \\ x_{1n}x_{n-1,n} \end{cases} \quad \begin{cases} x_{12}x_{1n} \\ x_{1n}x_{12} \end{cases} \\
&\begin{cases} \underline{x_{m,m+1}x_{l,l+1}} - x_{l,l+1}x_{m,m+1},\ 1 \leq m \leq l-2,\ 3 \leq l \leq n-1 \\ \underline{x_{m,m+1}x_{1,n}} - x_{1n}x_{m,m+1},\ 2 \leq m \leq n-2 \end{cases} \}.
\end{aligned} \tag{7.30}$$

7.3.1 Gröbner basis

The set of generators of the defining ideal of $\overline{FK}_{C_n}(n)$ in (7.30) forms the degree 2 elements (2-terms) of the Gröbner basis. To find the 3-terms, we need to calculate S-polynomials between any two pair of 2-terms. We notice that the S-polynomial between two monomials is always zero. Moreover, the only S-polynomials that reduce to non-zero elements for Gröbner basis are

$$S(\underline{x_{m,m+1}x_{1n}} - x_{1n}x_{m,m+1}, x_{1n}x_{12}, 3), \text{ and } S(x_{1,n}x_{12}, x_{m,m+1}x_{l,l+1} - x_{l,l+1}x_{m,m+1}, 3),$$

that reduce respectively to $A = \{x_{1n}x_{m,m+1}x_{12},\ 3 \leq m \leq n-2\}$ (as here m has to be more than 2 otherwise it is divisible by the relation $x_{m+1,m+2}x_{m,m+1} = 0,\ 1 \leq m \leq n-2$ for $m = 1$) and $B = \{x_{1n}x_{l,l+1}x_{12},\ 3 \leq l \leq n-2\}$ (as l has to be at most $n-2$ for $x_{1n}x_{l,l+1}x_{12}$

not to be divisible by relation $x_{1n}x_{n-1,n} = 0$). However the sets A and B are the same. Hence the set of all 3-terms of Gröbner basis for the ideal associated to $\overline{FK}_{C_n}(n)$ is

$$\text{3-terms} : \{x_{1n}x_{m,m+1}x_{12} : \ 3 \leq m \leq n-2\} \tag{7.31}$$

Example 7.3.1. *For $n = 4$, we have no 3-terms. For $n = 5$, the only 3-term of the Gröbner basis is $x_{15}x_{34}x_{12}$. For $n = 6$, we have two 3-terms: $x_{16}x_{34}x_{12}$ and $x_{16}x_{45}x_{12}$.*

To find the degree 4 terms, the only S-polynomial is (as the rest of them goes zero) $S(\underline{x_{m_2,m_2+1}x_{1n}} - x_{1n}x_{m_2,m_2+1}, x_{1n}x_{m_1,m_1+1}x_{12}, 4)$ which reduce to $x_{1n}x_{m_2,m_2+1}x_{m_1,m_1+1}x_{12}$ with respect to terms of degree less than 4. However by the relations in Equation (7.2) these terms goes zero unless $m_2 - m_1 \geq 2$, $m_1 \geq 3$ and $m_2 \leq n-2$. Therefore

$$\text{4-terms} : \{x_{1n}x_{m_2,m_2+1}x_{m_1,m_1+1}x_{12} : \ m_2 - m_1 \geq 2, \ m_1 \geq 3 \text{ and } m_2 \leq n-2\}. \tag{7.32}$$

The forms of 3-terms and 4-terms in Equations (7.31) and (7.32) suggest a general form for the elements of the basis of $\overline{FK}_{C_n}(n)$ of degree k, in the following proposition.

Proposition 7.3.2. *For $k \geq 4$ any degree k element of Gröbner basis for the ideal associated to $\overline{FK}_{C_n}(n)$ is of the form*

$$\begin{aligned} & x_{1n}x_{m_{k-2}m_{k-2}+1}x_{m_{k-3}m_{k-3}+1}\cdots x_{m_2m_2+1}x_{m_1m_1+1}x_{12}, \ \textit{where}, \\ & m_i - m_{i-1} \geq 2, \ i = 2, 3, \cdots, k-2, \ \textit{and} \ \ m_1 \geq 3 \ \textit{and} \ m_{k-2} \leq n-2. \end{aligned} \tag{7.33}$$

Proof. Base of induction: Equation (7.32) serves as the base of induction.

Inductive step: Let the statement be valid for degree k elements of the Gröbner basis. We need to show that it is valid for degree $k+1$, i.e., we need to show that for degree $k+1$ we will have

$$x_{1n}x_{m_{k-1}m_{k-1}+1}x_{m_{k-2}m_{k-2}+1}x_{m_{k-3}m_{k-3}+1}\cdots x_{m_2m_2+1}x_{m_1m_1+1}x_{12},\ \text{where} \\ m_i - m_{i-1} \geq 2,\ i = 2,3,\cdots,k-1,\ \text{and}\ m_1 \geq 3 \text{ and } m_{k-1} \leq n-2. \tag{7.34}$$

The only S-polynomials available between the degree k element in the statement and a lower degree term that reduce to degree $k+1$ elements of the Gröbner basis is (as the rest of them go zero)

$$S(x_{m_{k-1}m_{k-1}+1}x_{1n} - x_{1n}x_{m,m+1},\ x_{1n}x_{m_{k-2}m_{k-2}+1}x_{m_{k-3}m_{k-3}+1}\cdots x_{m_2m_2+1}x_{m_1m_1+1}x_{12},\ k+1) \\ = x_{1n}\underline{x_{m_{k-1}m_{k-1}+1}x_{m_{k-2}m_{k-2}+1}}\cdots x_{m_1m_1+1}x_{12}. \tag{7.35}$$

We now focus on the underlined product in Equation (7.35), $C = \underline{x_{m_{k-1},m_{k-1}+1}x_{m_{k-2},m_{k-2}+1}}$. For C not to vanish or reduce to something else on reduction w.r.t. any of the 2-terms in Equation (7.2), we need to avoid the following cases.

1. $m_{k-1} = m_{k-2}$ where $C \to 0$ by relation $x^2_{m,m+1} = 0$.
2. $m_{k-1} = m_{k-2} - 1$ where $C \to 0$ by relation $x_{m,m+1}x_{m+1,m+2} = 0$.
3. $m_{k-1} \leq m_{k-2} - 2$ where $C \to x_{m_{k-2}m_{k-2}+1}x_{m_{k-1}m_{k-1}+1}$. (by reduction w.r.t. the leading term of the relation $\underline{x_{m,m+1}x_{l,l+1}} - x_{l,l+1}x_{m,m+1} = 0$)
4. $m_{k-1} \leq m_{k-2} + 1$ where $C \to 0$ by relation $x_{m+1,m+2}x_{m,m+1} = 0$.

To avoid the above cases we need that inequality $n-2 \geq m_{k-1} > m_{k-2}+1$ holds, which is nothing but $m_{k-1} - m_{k-2} \geq 2$ and $n-2 \geq m_{k-1}$, as well we need $m_1 \geq 3$, for $x_{m_1m_1+1}x_{12}$ not go zero by $x_{m+1,m+2}x_{m,m+1} = 0, 1 \leq m \leq n-2$. However these three inequalities

$m_{k-1} - m_{k-2} \geq 2,\ n-2 \geq m_{k-1}$, and $m_1 \geq 3$, meet the requirements of Equation (7.34) for validity for degree $k+1$. This completes the proof □

Putting together the k-terms for $k \geq 4$ in Equation (7.33), 3-term of Equation (7.31) and the 2-terms in Equation (7.2) we get the set of Gröbner basis for the ideal associated to $\overline{FK}_{C_n}(n)$.

$$\begin{aligned} GB_{C_n} =& \text{2-terms} : x^2_{m,m+1} :\ 1 \leq m \leq n-1;\ x^2_{1,n}, \\ & , \begin{cases} x_{m,m+1}x_{m+1,m+2}, 1 \leq m \leq n-2 \\ x_{m+1,m+2}x_{m,m+1}, 1 \leq m \leq n-2 \end{cases} , \begin{cases} x_{n-1,n}x_{1n} \\ x_{1n}x_{n-1,n} \end{cases} , \begin{cases} x_{12}x_{1n} \\ x_{1n}x_{12} \end{cases} \\ & , \begin{cases} x_{m,m+1}x_{l,l+1} - x_{l,l+1}x_{m,m+1},\ 1 \leq m \leq l-2,\ 3 \leq l \leq n-1 \\ x_{m,m+1}x_{1,n} - x_{1n}x_{m,m+1},\ 2 \leq m \leq n-2, \end{cases} \quad \text{and} \\ & \text{3-terms} : \{x_{1n}x_{m,m+1}x_{12} :\ 3 \leq m \leq n-2\} \\ & (k \geq 4)\text{-terms} : \{x_{1n}x_{m_{k-2}m_{k-2}+1}x_{m_{k-3}m_{k-3}+1} \cdots x_{m_2m_2+1}x_{m_1m_1+1}x_{12},\ \text{where} \\ & m_i - m_{i-1} \geq 2,\ i = 2, 3, \cdots, k-1,\ \text{and}\ m_1 \geq 3 \text{ and } m_{k-1} \leq n-2\}. \end{aligned} \tag{7.36}$$

Example 7.3.3. *For $n = 3$ we have*

$$GB = \{x^2_{12}, x^2_{23}, x^2_{13}, x_{12}x_{23}, x_{23}x_{13}, x_{12}x_{13}, x_{23}x_{12}, x_{13}x_{23}, x_{13}x_{12}\}.$$

Example 7.3.4. *For $n = 7$, with $x_{17}x_{m,m+1}x_{12}$, we have $m = 3, 4, 5$, so we have three 3-terms $\{x_{17}x_{34}x_{12}$, $x_{17}x_{45}x_{12}$, $x_{17}x_{56}x_{12}\}$, also with $x_{1n}x_{m_2m_2+1}x_{m_1m_1+1}x_{12}$ we have $m_1 = 3$, $m_2 = 5$, so we have $x_{17}x_{56}x_{34}x_{12}$.*

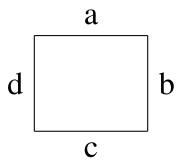
Remark 7.3.5. *We also notice that the set of elements of the Gröbner basis in Equation* (7.36), *for specific case of* $n = 6$ *reduce that for* $\overline{FK}_{C_6}(6)$ *in Equations* (7.21).

7.3.2 Basis and dimension

Definition 7.3.6. *1. A matching in a graph is a subset of the set of edges in the graph wherein no two edges have a common vertex.*

2. A matching on a set of line segments $V_n = \{\overline{12}, \overline{23}, \cdots \overline{n-1, n}\}$ *is a subset of* V_n *wherein no two line segments have a common vertex.*

Example 7.3.7. *In an square with successive sides/edges* a, b, c, d *(below diagram), we have* 7 *sets of edges wherein no two edges have a common vertex:* $\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, d\}$*. So the number of matchings in a square in* 7*.*



Theorem 7.3.8. *Dimensions of* $\overline{FK}_{C_n}(n)$ *equals the number of matchings in* n*-cycle* C_n*.*

Proof. Here the basis of the algebra $\overline{FK}_{C_n}(n)$ is the set of all the words on generators $\{x_{12}, x_{23}, x_{34}, \cdots, x_{n-1,n}; x_{1,n}\}$ that are not divisible by leading terms of the elements of the Gröbner basis in Equation (7.36). Therefore

1. Degrees 0 and 1 elements of $\overline{FK}_{C_n}(n)$ are just 1 and the generators $x_{m,m+1},\ m = 1, \cdots, n-1; x_{1n}$.

However for higher degrees, the structure of 2-terms and higher terms in Gröbner basis in Equation (7.36) is such that

2. Any monomial in our generators vanish by 2-terms in Gröbner basis if it has two adjacent 1st index differing by ±1 or 0.
3. Any monomial in our generators vanish if two adjacent 1st indexes differ by ≥ 2 but are increasing in 1st indexes.
4. If any two adjacent 2nd indexes are equal, then so are the 1st indexes, so vanishes by relations $x_{ij}x_{ij} = 0$.

Therefore the basis elements are: monomials with decreasing 1st indexes of variables (with an exception of x_{1n} when appears as the leading variable) such that any two successive 1st index differ by at least 2. Following the above points we come up with the following basis for $\overline{FK}_{C_n}(n)$.

$$\begin{aligned} B(\overline{FK}_{C_n}(n)) =\{1, & \\ & x_{m,m+1},\ \text{where } 1 \leq m \leq n-1,\ x_{1n}, \\ & x_{1n}x_{m_1m_1+1},\ \ \text{where } 2 \leq m_1 \leq n-2, \\ & x_{m_1m_1+1}x_{m_2m_2+1},\ \text{where } m_1 - m_2 \geq 2,\ m_2 \geq 1,\ m_1 \leq n-1. \\ & x_{m_1m_1+1}x_{m_2m_2+1}\cdots x_{m_km_k+1},\ \text{where } m_i - m_{i+1} \geq 2,\ m_k \geq 1,\ m_1 \leq n-1. \\ & x_{1n}x_{m_1m_1+1}x_{m_2m_2+1}\cdots x_{m_{k-1}m_{k-1}+1},\ \text{where } m_i - m_{i+1} \geq 2,\ m_{k-1} \geq 2, \\ & m_1 \leq n-2\}. \end{aligned} \tag{7.37}$$

The following argument shows why $B[\overline{FK}_{C_n}(n)]$ consists of all the matchings in C_n.

In the elements of the basis of $\overline{FK}_{C_n}(n)$ in Equation (7.37), if we view variables x_{mm+1} for

$m = 1, 2, \cdots, n-1$ and x_{1n}, as edges $\overline{12}, \overline{23}, \cdots \overline{n-1, n}$ and $\overline{1n}$ in an n-cycle C_n, then we can view an element of the basis as a subset of the set of the edges in C_n.

An element in the basis of $\overline{FK}_{C_n}(n)$ in (7.37), is a matching in an n-cycle C_n. The reason is that From $B[\overline{FK}_{C_n}(n)]$ in (7.37), no monomial in the basis includes two variables with a common index, so we can view an element of the basis as a subset of the set of the edges of the graph C_n such that no two edges have a common vertex i.e., a matching in C_n. Vice versa, we can view a matching in C_n as an element in the basis. Also from the domain of the indexes in Equation (7.37), we see that all the monomials in generators of $\overline{FK}_{C_n}(n)$ that have no repeated indexes have been covered in the basis.

The above considerations imply that there is a one-to-one correspondence between the set of all matchings in an n-cycle C_n and the basis of $\overline{FK}_{C_n}(n)$.

Therefore the number of elements in the basis of algebra $\overline{FK}_{C_n}(n)$ equals the number of matchings in an n-cycle graph C_n. Hence the dimensions of $\overline{FK}_{C_n}(n)$ equals the number of matchings in the n-cycle C_n. This completes the proof. $\square$

Corollary 7.3.9. *Dimension of $\overline{FK}_{C_n}(n)$ equals Lucas number L_n.*

Proof. From Theorem 7.3.8, dimension of $\overline{FK}_{C_n}(n)$ equals the number of matchings in C_n which equals Lucas number L_n [22]. Hence dimension of $\overline{FK}_{C_n}(n)$ equals Lucas number L_n. This completes the proof. $\square$

The highest degree elements in the basis

From $B(\overline{FK}_{C_n}(n))$ in Equation (7.37), the maximum degree of a monomial in the basis is attained when any two adjacent 1st indexes differ by 2 (equivalently if any two sides in the matching is separated by exactly one side in C_n). Therefore one can calculate easily that

$$\text{For } n \geq 4,\ \text{max (deg /size)} \ = \lfloor \frac{n}{2} \rfloor,\ \text{and}$$
$$\text{multiplisity of max (deg /size)} \ = \begin{cases} 2, & \text{for even } n \\ \\ n, & \text{for odd } n \end{cases} \tag{7.38}$$

Example 7.3.10. *The highest degree for elements in* $B[\overline{FK}(5)]$ *is* $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{5}{2} \rfloor = 2$*, and its multiplicity* 5.

7.3.3 ‘Matchings’ and Fibonacci number

Lemma 7.3.11. *Let* $M(n)$ *denote the number of matchings in the set of line segments*
$V_n = \{\overline{12}, \overline{23}, \cdots \overline{n-1, n}\}$*,* $n \geq 2$*. Then*

$$M(n) = F_n,\ n \geq 2, \tag{7.39}$$

where F_n *is* n*-th Fibonacci number with initial values* $F_0 = 1,\ F_1 = 1$.

Proof. We need to show

1. $M(2) = F_2$, and
2. $M(n+1) = M(n) + M(n-1)$, i.e., the recurrence for Fibonacci numbers[9], [10].

The 1st item is obvious as $M(2)$ is the number of matching in $\{\overline{12}\}$ which equals 2, including the empty matching, which equals $F(2)$ (as our starting values for Fibonacci sequence are $F_0 = 1, F_1 = 1$).

To show the 2nd item consider that $V_n = \{\overline{12}, \overline{23}, \cdots \overline{n-1, n}\} = V_{n-1} \cup \overline{n-1, n}$, where $V_{n-1} = \{\overline{12}, \overline{23}, \cdots \overline{n-2, n-1}\}$. Then addition of $\overline{n, n+1}$ to V_n make V_{n+1} and increases $M(n)$ by adding new matchings to it. However in order to avoid formation of common index/vertex, these new matchings have to be made by adding $\overline{n, n+1}$ to each matching only in V_{n-1}, as $\overline{n, n+1}$ does not have common index/vertex with elements in V_{n-1} but does have common index/vertex n with $\overline{n-1, n}$. Therefore the number of new matchings to be added to $M(n)$ is exactly the number of matchings in V_{n-1}, i.e., $M(n-1)$. Hence $M(n+1) = M(n) + M(n-1)$. This completes the proof. □

Lemma 7.3.12. *[9], [10]. Let $M(n,k)$ denote the number of matchings of size k in set of line segments $V_n = \{\overline{12}, \overline{23}, \cdots \overline{n-1, n}\}$. Then*

1. $M(n+1, k) = M(n, k) + M(n-1, k-1)$, *for* $n \geq 3$,
2. $M(n, k) = \binom{n-k}{k}$.

Proof. 1. Consider V_n as $V_n = V_{n-1} \cup \{\overline{n-1, n}\}$ where $V_{n-1} = \{\overline{12}, \overline{23}, \cdots \overline{n-2, n-1}\}$. Addition of segment $\overline{n, n+1}$ to V_n makes $V_{n+1} = \{\overline{n, n+1}\} \cup V_{n-1} \cup \{\overline{n-1, n}\}$. This addition increases $M(n,k)$ by adding $\overline{n,n+1}$ to each element of size $k-1$ in only V_{n-1} without developing common index/vertex, as the elements of V_{n-1} do not have indexes n or $n+1$. So the number of these new matchings to be added to $M(n,k)$ is exactly the number of elements in V_{n-1} of size $k-1$, i.e., $M(n-1,k-1)$. Hence we have $M(n+1,k) = M(n,k) + M(n-1,k-1)$.

2. To prove $M(n,k) = \binom{n-k}{k}$ we use double induction. We need to show that

   (a) base of induction: For $n=2$ and $k=1$ the statement $M(2,1) = \binom{2-1}{1}$ is true, as $M(2,1) = 1$ and $\binom{2-1}{1} = \binom{1}{1} = 1$.

   (b) Inductive step: We need to show that

   $$\text{If } M(n,k) = \binom{n-k}{k} \forall n \geq 2,\ k > 1, \text{ then } M(n+1,k) = \binom{n+1-k}{k},$$

   and

   $$\text{If } M(n,k) = \binom{n-k}{k} \forall n \geq 2,\ k > 1, \text{ then } M(n,k+1) = \binom{n-k-1}{k+1},$$

   as in it follows.

$$\begin{aligned} M(n+1,k) &= M(n,k) + M(n-1,k-1) \\ &= \binom{n-k}{k} + \binom{n-1-(k-1)}{k-1} = \binom{n-k}{k} + \binom{n-k}{k-1} \\ &= \frac{(n-k)!}{k!(n-2k)!} + \frac{(n-k)!}{(k-1)!(n-2k+1)!} = \cdots = \frac{(n-k+1)!}{k!(n-2k+1)!} \\ &= \binom{n+1-k}{k}. \end{aligned}$$

Also

$$\begin{aligned} M(n,k+1) &= M(n-1,k+1) + M(n-2,k) \\ &= \binom{n-1-(k+1)}{k+1} + \binom{n-2-k}{k} \end{aligned}$$

$$\begin{aligned}&= \binom{n-k-2}{k+1} + \binom{n-k-2}{k} \\ &= \frac{(n-k-2)!}{(k+1)!(n-2k-3)!} + \frac{(n-k-2)!}{k!(n-2k-2)!} \\ &\vdots \\ &= \frac{(n-k-1)!}{(k+1)!(n-2k-21)!} \\ &= \binom{n-k-1}{k+1}.\end{aligned}$$

This completes the proof. □

Proposition 7.3.13. *[9] [10]. Let $M_{C_n}(n,k)$ denote the number of matchings of size k in n-cycle C_n. Then we have*

$$\begin{aligned}M_{C_n}(n,k) =& M(n,k) + M(n-2,k-1),\ n \geq 4,\ k = 2,3,\cdots,\lfloor\frac{n}{2}\rfloor,\ \textit{and} \\ M_{C_n}(n,k) =& \frac{n}{n-k}\binom{n-k}{k}.\end{aligned} \tag{7.40}$$

where $M(n,0) = 1$, and $M(n,k) = \binom{n-k}{k}$ is the number of matchings of size k in $C_n \setminus \overline{\{1n\}}$.

Proof. $M(n,k)$ can be viewed as the number of matchings of size k in $C_n \setminus \{\overline{1n}\}$. We rewrite $C_n \setminus \{\overline{1n}\}$ as $C_n \setminus \{\overline{1n}\} = \{\overline{12}\} \cup V_{n-2} \cup \{\overline{n-1,n}\}$, where $V_{n-2} = \{\overline{23}, \overline{34}, \cdots, \overline{n-2,n-1}\}$. Now adding $\overline{1n}$ to $C_n \setminus \{\overline{1n}\}$ completes the n-cycle, as well increases $M(n,k)$ to $M_{C_n}(n,k)$ by making new matchings of size k. However to avoid making a common vertex we need to add $\overline{1n}$ to the elements of size $k-1$ only in V_{n-2}, as they do not involve in indexes 1 or n but $\overline{12}$ and $\overline{n-1,n}$ do involve in common indexes 1 or n. So the number of these new matchings is equal to the number of matchings of size $k-1$ in V_{n-2} i.e., $M(n-2,k-1)$. Hence $M_{C_n}(n,k) = M(n,k) + M(n-2,k-1)$. This completes the proof of the first part of the Lemma.

Substituting $M(n,k) = \binom{n-k}{k}$ into the first part we have

$$\begin{aligned} M_{C_n}(n,k) &= M(n,k) + M(n-2,k-1) = \binom{n-k}{k} + \binom{n-1-k}{k-1} \\ &= \binom{n-k}{k} + \frac{(n-1-k)!}{(k-1)!(n-2k)!} = \binom{n-k}{k} + \frac{\frac{(n-k)!}{(n-k)}}{\frac{k!}{k}(n-2k)!} \\ &= \binom{n-k}{k} + \frac{k}{n-k}\frac{(n-k)!}{k!(n-2k)!} = \binom{n-k}{k} + \frac{k}{n-k}\binom{n-k}{k} \\ &= \frac{n}{n-k}\binom{n-k}{k}. \end{aligned} \tag{7.41}$$

This completes the proof. □

7.3.4 Hilbert series and q-Fibonacci/q-Lucas numbers

From Proposition 7.3.13 and Theorem 7.3.8 and Corollary 7.3.9, we have

$$L_n = M_{C_n}(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_{C_n}(n,k) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k}.$$

The upper limit in the sum is due to the fact that maximum size of a matching in C_n or the maximum degree of an element in the basis of $\overline{FK}_{C_n}(n)$ is $\lfloor \frac{n}{2} \rfloor$. Hence taking sum over all the matchings of different sizes k in C_n (elements of different degrees k in the basis of $\overline{FK}_{C_n}(n)$), we have q-Lucas number [9], [10]:

$$L_n(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k} q^k, \ L_0 = 2. \tag{7.42}$$

Also from Lemmas 7.3.11 and 7.3.12 we have the total number of matchings in a set of line segments $V_n = \{\overline{12}, \overline{23}, \cdots \overline{n-1,n}\}$ to be $M_n = F_n$ and the number of matchings of size k

in V_n to be $M(n,k) = \binom{n-k}{k}$. This results in

$$F_n = M(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} M(n,k) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

Hence taking sum over all matchings of different sizes we have q-Fibonacci number [9][10]

$$F_n(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} q^k. \tag{7.43}$$

Proposition 7.3.14. *Hilbert series of* $\overline{FK}_{C_n}(n)$ *is* q*-Lucas number*

$$H_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} q^k = L_n(q). \tag{7.44}$$

Proof. We recall here (7.42)

$$L_n(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} q^k, \ L_0 = 2.$$

In the above expression, degree k, as the size of matching, is summed over to give us the Lucas number which is by Corollary 7.3.9 equal to the dimension of $\overline{FK}_{C_n}(n)$. Hence the above expression indeed demonstrates Hilbert series

$$H_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} q^k.$$

This completes the proof. □

Example 7.3.15. $H_5 = \sum_{k=0}^{2} \frac{5}{5-k} \binom{5-k}{k} q^k = \binom{5}{0} + \frac{5}{4}\binom{4}{1} q + \frac{5}{3}\binom{3}{2} q^2 = 1 + 5q + 5q^2$. *Similarly* $H_6 = 1 + 6q + 9q^2 + 2q^3$.

7.3.5 The action of D_n and decomposition

In this section we derive the character of $\overline{FK}_{C_n}(n)$ over D_n for even and odd n.

Lemma 7.3.16. *Let $r \in D_n$ realized in permutation group S_n as $r = (1, 2, \cdots, n)$. Then for integers $2 \leq p \leq n$ the trace of representation of $r^p \in D_n$ on $\overline{FK}_{C_n}(n)$ is*

$$\chi(r^p) = 1 + p\delta_{n,pm}, \tag{7.45}$$

for some $m = \frac{n}{p}$ if it is an integer. In q-counting according to total degree decomposition form we have

$$[\chi(r^p)](q) = 1 + pq^{\frac{n}{p}}, \text{ if } \frac{n}{p} \text{ is an integer}. \tag{7.46}$$

Proof. One can view a monomial $M \in B[\overline{FK}_{C_n}(n)]$ as a matching in an n-cycle (i.e., a set of sides in n-cycle with no common vertex). Also we can consider r as a clockwise rotation of angle $\frac{2\pi}{n}$ in C_n. Then $r^p M$ is the result of clockwise rotation of the matching M in n-cycle by angle $\frac{2\pi p}{n}$. So $r^p M = M$ means that after rotation of a matching M over p sides of the n-cycle, the matching M coincides with itself. We discuss the following cases.

1. The possibility that $r^p M = -M$ is absurd, i.e., the operator r^p does not have an eigen vector with eigen value -1. The reason is that in order for $r^p M$ to develop a minus sign we need M contains an x_{1n} to get rotated into $r^p x_{1n} = x_{1+p,n+p} = x_{1+p,p} = -x_{p,p+1}$, and thereby create a minus sign. Then since r^p rotates M as a matching onto itself there must be another variable in M like $x_{n-p,n+1-p}$ to be rotated onto x_{1n} to preserve M as a matching in C_n. However then $r^p x_{n-p,n+1-p} = x_{n,n+1} = x_{n,1} = -x_{1n}$; a second minus sign to cancel the first one. So $r^p M = -M$ can not hold.

2. $rM = M$ does not hold for non-empty M. The reason is that since r is a clockwise rotation of $\frac{2\pi}{n}$ in C_n, it takes a side in the matching M in C_n, into the side next to it (the neighbor side) in C_n. However this next side is not in M otherwise there would be a common vertex in M that is forbidden by the definition of a matching. Thus $rM = M$ can not hold for non-empty M.

3. For $2 \leq p \leq n$, and non-empty matching M for $r^pM = M$ to hold, we need $p|n$ and matching M be symmetric in C_n in the sense that any two successive sides in M be separated by the same number $p-1$ of sides in C_n. Therefore the number of non-empty matchings M such that $r^pM = M$, is equal to $p\delta_{n,pm}$ for some $m = \frac{n}{p}$ if it is an integer.

4. For any p there is always an empty matching (associated for identity element 1 in the basis) such that $r^pM = M(as\ r^p1 = 1)$.

Considering items $1-4$, we come up with the conclusion that for given positive integer p the number of matchings (monomials) M such that $r^pM = M$, i.e., the trace of representation of r^p is $1 + p\delta_{n,pm}$. Therefore for $2 \leq p \leq n$ we have $\chi(r^p) = 1 + p\delta_{n,pm}$, for some $m = \frac{n}{p}$ if it is an integer. The number 1 in $\chi(r^p) = 1 + p\delta_{n,pm}$ refers to the empty matching (degree zero element of the basis) and $p\delta_{n,pm}$ to the nonempty ones of size $m = \frac{n}{p}$ (degree $m = \frac{n}{p}$) if it is an integer. Therefore we can write the q-counting form of $\chi(r^p) = 1 + p\delta_{n,pm}$ as $[\chi(r^p)](q) = 1 + pq^{\frac{n}{p}}$ if $\frac{n}{p}$ is an integer (then $\frac{n}{p}$ =size of the non-empty matching that is eigen vector of r^p). This completes the proof. $\square$

Lemma 7.3.17. *For even $n \geq 4$, let $s \in D_n$, realized in permutation group as*
$s = (1n)(2, n-1)\cdots(\frac{n}{2}, \frac{n}{2}+1)$ in n-cycle graph C_n. Let $\chi(s)$ be the trace of representation

of s over $\overline{FK}_{C_n}(n)$. Then

$$[\chi(s)](q) = \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2}-k}{k} q^{2k} - 2\sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1} + \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2}, \tag{7.47}$$

and in q-Fibonacci form according to total degree decomposition we have

$$[\chi(s)](q) = F_{\frac{n}{2}}(q^2) - 2qF_{\frac{n}{2}-1}(q^2) + q^2F_{\frac{n}{2}-2}(q^2). \tag{7.48}$$

Proof. By Theorem 7.3.8 to any element of the basis of $\overline{FK}_{C_n}(n)$ there corresponds a matching in C_n. Let $T_1 = \{\overline{i_1, i_1+1}, \overline{i_2, i_2+1} \cdots \overline{i_m, i_m+1}\}$ be a matching of the set of line segments $S_1 = \{\overline{12}, \overline{23}, \cdots, \overline{\frac{n}{2}-1, \frac{n}{2}}\} \subset C_n$ defined in Definition 7.3.6, with its corresponding monomial

$$u = x_{i_1,i_1+1}x_{i_2,i_2+1}\cdots x_{i_m,i_m+1} \in B[\overline{FK}_{C_n}(n)].$$

Let its reflection in s, itself a matching (as n =even), be

$$T_1' = \{\overline{j_1, j_1+1}, \overline{j_2, j_2+1} \cdots \overline{j_m, j_m+1}\},$$

and its corresponding monomial be $u' = x_{j_1j_1+1}x_{j_2j_2+1}\cdots \cdots x_{j_mj_m+1}$. While matching T_1 is not symmetric about s, however matching $T_1 \cup T_1'$ is symmetric and its corresponding monomial $uu' \in B[\overline{FK}_{C_n}(n)]$ forms an eigen vector of s with eigen value $+1$ as $s(uu') = uu'$ (because $s(uu') = s(usu) = (su)s^2u = (su)u = u'u \to uu'$ on reduction w.r.t. 2-terms, because u and u' commute, as there is no common index among variables in uu'). The number of such eigen vectors in $\overline{FK}_{C_n}(n)$ equals the number of matchings $T_1 \cup T_1'$ in C_n which by construction equals the number of matchings $T_1 \subset \{\overline{12}, \overline{23}, \cdots, \overline{\frac{n}{2}-1, \frac{n}{2}}\}$ which is by

Equations (7.39) and (7.43) equal to

$$[M(\frac{n}{2})](q) = F_{\frac{n}{2}}(q) = \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2} - k}{k} q^{2k}, \tag{7.49}$$

where the factor 2 in the power of q^{2k} is to take care of the double number of sides in $T_1 \cup T_1'$ compared to T_1 and it reflects the even number of matchings in $T_1 \cup T_1'$. Hence our first contribution to $\chi(s)$ denoted $\chi_1(s)$ is

$$[\chi_1(s)](q) = \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2} - k}{k} q^{2k}. \tag{7.50}$$

To find the number of eigen vectors with eigen value -1 instead of considering the matching $T_1 \subset \{\overline{12}, \overline{23}, \cdots, \overline{\frac{n}{2} - 1, \frac{n}{2}}\}$ and its reflection T_1' in s, we need to consider the following cases

1. The matching $\{\overline{1n}\} \cup T_2$, with matching $T_2 \subset \{\overline{23}, \cdots, \overline{\frac{n}{2} - 1, \frac{n}{2}}\}$ (to prevent developing common vertex with $\{\overline{1n}\}$), and its reflection T_2' in s, with the corresponding monomials $v, v' \in B[\overline{FK}_{C_n}(n)]$, to T_2 and T_2' respectively.

2. The matching $\{\overline{\frac{n}{2}, \frac{n}{2} + 1}\} \cup T_3$ with matching $T_3 \subset \{\overline{12}, \cdots, \overline{\frac{n}{2} - 2, \frac{n}{2} - 1}\}$ (to prevent developing common vertex with $\{\overline{\frac{n}{2}, \frac{n}{2} + 1}\}$) and its reflection T_3' in s.

In case 1 while matching $\{\overline{1n}\} \cup T_2$ is not symmetric about s, however $\{\overline{1n}\} \cup T_2 \cup T_2'$ is a matching in C_n symmetric about s and its corresponding monomial $x_{1n}vv'$ forms an eigen vector of s with eigen value -1 as $s(x_{1n}vv') = -x_{1n}s(vv') = -x_{1n}vv'$ as before. By construction the number of matchings $\{\overline{1n}\} \cup T_2 \cup T_2'$ in C_n equals the number of matchings

T_2 in $\{\overline{23}, \cdots, \overline{\frac{n}{2}-1, \frac{n}{2}}\}$ which is by Equations (7.39) and (7.43) equal to

$$[M(\frac{n}{2}-1)](q) = F_{\frac{n}{2}-1}(q) = \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1}. \tag{7.51}$$

Here the odd power of q^{2k+1} reflects the odd size of the matching $\{\overline{1n}\} \cup T_2 \cup T_2'$ in C_n as well the negative eigen value of s. Considering a factor 2 to cover the 2nd case (as it exactly results to the same as case 1), as well a negative sine for the negative eigen value, we come up with our second contribution to $\chi(s)$

$$[\chi_2(s)](q) = -2 \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1}. \tag{7.52}$$

To cover the last contribution to $\chi(s)$, we consider the matching $\{\overline{1n}\} \cup \{\overline{\frac{n}{2}, \frac{n}{2}+1}\} \cup T_4$ and T_4', the reflection of T_4 in s, with matching $T_4 \subset \{\overline{23}, \overline{34}, \cdots, \overline{\frac{n}{2}-2, \frac{n}{2}-1}\}$ (to prevent developing common vertex with $\overline{1n}$ or $\overline{\frac{n}{2}, \frac{n}{2}+1}$) with w and w' the corresponding monomials to T_4 and T_4' respectively. By similar arguments we have eigen vectors with eigen values $+1$, with the number of such eigen vectors equal to the number of matchings $T_4 \subset \{\overline{23}, \overline{34}, \cdots, \overline{\frac{n}{2}-2, \frac{n}{2}-1}\}$ as before, which is by Equations (7.39) and (7.43) equal to

$$[M(\frac{n}{2}-2)](q) = F_{\frac{n}{2}-2}(q) = \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2}. \tag{7.53}$$

Hence our last contribution to $\chi(s)$ is

$$[\chi_3(s)](q) = \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2}. \tag{7.54}$$

Now adding the contributions in Equations (7.50), (7.52) and (7.54) we come up with

$$[\chi(s)](q) = \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2}-k}{k} q^{2k} - 2\sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1} + \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2},$$

which is written in the following q-Fibonacci form using Equation (7.43):

$$[\chi(s)](q) = F_{\frac{n}{2}}(q^2) - 2qF_{\frac{n}{2}-1}(q^2) + q^2 F_{\frac{n}{2}-2}(q^2).$$

This completes the proof. $\square$

Example 7.3.18. *For $n = 6$ we have*
$\chi(s) = \sum_{k=0}^{1} \binom{3-k}{k} q^{2k} - 2\sum_{k=0}^{1} \binom{2-k}{k} q^{2k+1} + \sum_{k=0}^{0} \binom{1-k}{k} q^{2k+2} = \binom{3}{0} + \binom{2}{1} q^2 - 2[\binom{2}{0} q + \binom{1}{1} q^3] + \binom{1}{0} q^2 = 1 + 2q^2 - 2[q + q^3] + q^2 = 1 - 2q + 3q^2 - 2q^3 \to 0$ *at* $q = 1$.

Example 7.3.19. *For $n = 6$ we have $\chi(s) = F_3(q^2) - 2qF_2(q^2) + q^2F_1(q^2)$. Let $q = 1$, then $\chi(s) \to F_3 - 2F_2 + F_1 = 3 - 2(2) + 1 = 0$.*

Lemma 7.3.20. *For even n, Let $s, r \in D_n$, realized in permutation group S_n as $s = (1, n)(2, n-1) \cdots (\frac{n}{2}, \frac{n}{2}+1)$ and $r = (1, 2, \cdots, n)$ in n-cycle graph C_n . Then*

$$[\chi(sr)](q) = \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k},\ n \geq 4, \tag{7.55}$$

and in q-Fibonacci form according to total degree decomposition

$$[\chi(sr)](q) = F_{\frac{n}{2}-1}(q^2), \tag{7.56}$$

where $\chi(sr)$ is the trace of the representation of sr over $\overline{FK}_{C_n}(n)$.

Proof. Here we have $sr = (n)(1, n-1)(2, n-2) \cdots (\frac{n}{2}-1, \frac{n}{2}+1)(\frac{n}{2})$ and so the only matchings in C_n symmetric about sr are $T \cup T'$ with matching $T \subset \{\overline{12}, \overline{23}, \cdots, \overline{\frac{n}{2}-2, \frac{n}{2}-1}\}$ and T', the reflection of T in sr. The corresponding monomials to T and T' are respectively w and w'. we have eigen vectors of sr with eigen value $+1$, as $sr(ww') = ww'$ as before. The number of such eigen vectors equals the number of matchings $T \cup T'$ in C_n which equals the number of matchings $T \subset \{\overline{12}, \overline{23}, \cdots, \overline{\frac{n}{2}-2, \frac{n}{2}-1}\}$ which is by Equations (7.39) and (7.43) equal to

$$[\chi(sr)](q) = M(\frac{n}{2}-1)(q) = F_{\frac{n}{2}-1}(q) = \sum_{k=0}^{\lfloor\frac{\frac{n}{2}-1}{2}\rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k}.$$

This completes the proof. □

Example 7.3.21. *For $n = 6$, we have $\chi(sr) = \sum_{k=0}^{1} \binom{2-k}{1} q^{2k} = \binom{2}{1} + \binom{1}{1} q^2 = 1 + q^2$.*

Lemma 7.3.22. *For odd n, Let $s \in D_n$ realized in permutation group S_n as $s = (1, n)(2, n-1) \cdots (\frac{n-1}{2}, \frac{n+3}{2})(\frac{n+1}{2})$ in an n-cycle graph C_n. Then*

$$[\chi(s)](q) = \sum_{k=0}^{\lfloor\frac{n-1}{4}\rfloor} \binom{\frac{n-1}{2}-k}{k} q^{2k} - \sum_{k=0}^{\lfloor\frac{n-3}{4}\rfloor} \binom{\frac{n-3}{2}-k}{k} q^{2k+1}, \tag{7.57}$$

with q-Fibonacci version

$$[\chi(s)](q) = F_{\frac{n-1}{2}}(q^2) - qF_{\frac{n-3}{2}}(q^2),$$

where $\chi(s)$ is the trace of the representation of s over $\overline{FK}_{C_n}(n)$.

Proof. Let

$$T_1 = \{\overline{i_1, i_1+1}, \overline{i_2, i_2+1} \cdots \overline{i_m, i_m+1}\}$$

be a matching of the set of line segments $S = \{\overline{12}, \overline{23}, \cdots, \overline{\frac{n-1}{2}-1, \frac{n-1}{2}}\}$, with its corresponding monomial

$$u = x_{i_1,i_1+1}x_{i_2,i_2+1}\cdots x_{i_m,i_m+1}.$$

Let its reflection in s, itself a matching, be

$$T_1' = \{\overline{j_1, j_1+1}, \overline{j_2, j_2+1} \cdots \overline{j_m, j_m+1}\},$$

and its corresponding monomial be $u' = x_{j_1j_1+1}x_{j_2j_2+1}\cdots\ \cdots x_{j_mj_m+1}$.

While the matching T_1 is not symmetric about s, however matching $T_1 \cup T_1'$ is so, and its corresponding monomial uu' forms an eigen vector of s with eigen value $+1$, as $s(uu') = uu'$ as before. The number of such eigen vectors equals the number of matching $T_1 \cup T_1'$ in C_n which equals the number of matchings in $T_1 \in \{\overline{12}, \overline{23}, \cdots, \overline{\frac{n-1}{2}-1, \frac{n-1}{2}}\}$ which is by (7.39) and (7.43) equal to

$$[M(\frac{n-1}{2})](q) = F_{\frac{n-1}{2}}(q) = \sum_{k=0}^{\lfloor\frac{n-1}{4}\rfloor} \binom{\frac{n-1}{2}-k}{k} q^{2k}, \tag{7.58}$$

where the factor 2 in the power of q^{2k} as before counts the double number of sides (double size=double degree). Hence our first contribution to $\chi(s)$ is

$$[\chi_1(s)](q) = \sum_{k=0}^{\lfloor\frac{n-1}{4}\rfloor} \binom{\frac{n-1}{2}-k}{k} q^{2k}. \tag{7.59}$$

We have another set of matchings symmetric about s. These matchings are made by adding the side $\overline{1n}$ (its corresponding variable x_{1n}), to avoid common vertex with $\overline{1n}$, only to each element of the matching $T_2 \subset \{\overline{23}, \overline{34}, \cdots, \overline{\frac{n-1}{2}-1, \frac{n-1}{2}}\}$. Let the reflection of T_2 in s be T_2', then $\{\overline{1n}\} \cup T_2 \cup T_2'$ is symmetric about s and its correspondent monomial $x_{12}ww'$ is an eigen vector of s with eigen value -1 as $s(x_{1n}ww') = -x_{12}s(ww') = -x_{12}ww'$ as before. The number of such eigen vectors is equal to the number of matchings $\{\overline{1n}\} \cup T_2 \cup T_2'$ in C_n which is equal to the number of matchings $T_2 \subset \{\overline{23}, \overline{34}, \cdots, \overline{\frac{n-1}{2}-1, \frac{n-1}{2}}\}$ which by Equations (7.39) and (7.43) is equal to $M(\frac{n-1}{2}-1) = M(\frac{n-3}{2}) = F_{\frac{n-3}{2}} = \sum_{k=0}^{\lfloor \frac{n-3}{4} \rfloor} \binom{\frac{n-3}{2}-k}{k} q^{2k+1}$.

Hence our second contribution to $\chi(s)$ is

$$[\chi_2(s)](q) = -\sum_{k=0}^{\lfloor \frac{n-3}{4} \rfloor} \binom{\frac{n-3}{2}-k}{k} q^{2k+1}. \tag{7.60}$$

Adding Equations (7.59) and (7.60) we come up with

$$[\chi(s)](q) = \sum_{k=0}^{\lfloor \frac{n-1}{4} \rfloor} \binom{\frac{n-1}{2}-k}{k} q^{2k} - \sum_{k=0}^{\lfloor \frac{n-3}{4} \rfloor} \binom{\frac{n-3}{2}-k}{k} q^{2k+1},$$

with its q-Fibonacci form according to total degree decomposition

$$[\chi(s)](q) = F_{\frac{n-1}{2}}(q^2) - qF_{\frac{n-3}{2}}(q^2).$$

This completes the proof. $\square$

Example 7.3.23. *For $n = 5$ we have $\chi(s) = \sum_{k=0}^{1} \binom{2-k}{k} q^{2k} - \sum_{k=0}^{0} \binom{1-k}{k} q^{2k+1} = \binom{2}{0} + \binom{1}{1} q^2 - \binom{1}{0} q = 1 - q + q^2 \to 1$ as $q \to 1$.*

Remark 7.3.24. *For odd n, there is no symmetry about sr in C_n, so we do not discuss the case of $\chi(sr)$.*

Table 7.3: D_n irred. characters for even n: The trivial and 3 reflections are 1-dimensional, the rest are 2-dimensional rotations. $[r^p]$ stands for $\{r^p, r^{n-p}\}$, $2 \leq p \leq \frac{n}{2} - 1$, $p|n$, its D_n-conjugacy class is denoted by $c_{(p)}$. $[s]$ stands for $\{s, sr^2, \cdots, sr^{n-2}\}$, $[sr]$ for $\{sr, sr^3, \cdots sr^{n-1}\}$, and $\chi_{(h)}$ for $\chi_{\frac{n}{h}h}$ when $h|n$.

	ϵ	$[r]$	$[r^p]$, $2 \leq p \leq \frac{n}{2} - 1$, such that $p\|n$	$\{r^{\frac{n}{2}}\}$	$[s]$	$[sr]$
	$\|c_{1^n}\| = 1$	$\|c_n\| = 2$	$\|c_{(p)}\| = 2$	$\|c_{2^{\frac{n}{2}}}\| = 1$	$\|c_{2^{\frac{n}{2}}{}'}\| = \frac{n}{2}$	$\|c_{2^{(\frac{n}{2}-1)}11}\| = \frac{n}{2}$
χ_{1^n}	1	1	1	1	1	1
$\chi_{2^{\frac{n}{2}}}$	1	1	1	1	-1	-1
$\chi_{2^{\frac{n}{2}}{}'}$	1	-1	$(-1)^p$	$(-1)^{\frac{n}{2}}$	1	-1
$\chi_{2^{(\frac{n}{2}-1)}11}$	1	-1	$(-1)^p$	$(-1)^{\frac{n}{2}}$	-1	1
$\chi_{(h)}$, $1 \leq h \leq \frac{n}{2} - 1$, such that $h\|n$	2	$2cos\frac{2h\pi}{n}$	$2cos\frac{2hp\pi}{n}$	$2(-1)^h$	0	0

Table 7.4: Character of $\overline{FK}_{C_n}(n)$ over D_n for even n

ϵ	$[r]$	$[r^p]$, $2 \leq p \leq \frac{n}{2}$ -1 such that $p\|n$	$\{r^{\frac{n}{2}}\}$	$[s]$	$[sr]$
$\|c_{1^n}\| = 1$	$\|c_n\| = 2$	$\|c_{(p)}\| = 2$	$\|c_{2^{\frac{n}{2}}}\| = 1$	$\|c_{2^{\frac{n}{2}}{}'}\| = \frac{n}{2}$	$\|c_{2^{(\frac{n}{2}-1)}11}\| = \frac{n}{2}$
$\sum_{k=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k}q^k$	1	$1 + pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}}$	$1 + \frac{n}{2}q^2$	$\sum_{k=0}^{\lfloor\frac{n}{4}\rfloor}\binom{\frac{n}{2}-k}{k}q^{2k}$ $-2\sum_{k=0}^{\lfloor\frac{\frac{n}{2}-1}{2}\rfloor}\binom{\frac{n}{2}-1-k}{k}q^{2k+1}$ $+\sum_{k=0}^{\lfloor\frac{\frac{n}{2}-2}{2}\rfloor}\binom{\frac{n}{2}-2-k}{k}q^{2k+2}$	$\sum_{k=0}^{\lfloor\frac{n-2}{4}\rfloor}\binom{\frac{n-2}{2}-k}{k}q^{2k}$

Table 7.5: Character of $\overline{FK}_{C_n}(n)$ over D_n for even n in q-Fibonacci/q-Lucas, with initial values $F_0 = 1$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$.

ϵ	$[r]$	$[r^p]$, $2 \leq p \leq \frac{n}{2}$ -1 such that $p\|n$	$\{r^{\frac{n}{2}}\}$	$[s]$	$[sr]$
$\|c_{1^n}\|$	$\|c_n\| = 2$	$\|c_{(p)}\| = 2$	$\|c_{2^{\frac{n}{2}}}\| = 1$	$\|c_{2^{\frac{n}{2}}{}'}\| = \frac{n}{2}$	$\|c_{2^{(\frac{n}{2}-1)}11}\| = \frac{n}{2}$
$L_n(q)$	1	$1 + pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}}$	$1 + \frac{n}{2}q^2$	$F_{\frac{n}{2}}(q^2) - 2qF_{\frac{n}{2}-1}(q^2) + q^2F_{\frac{n}{2}-2}(q^2)$	$F_{\frac{n-2}{2}}(q^2)$

7.3.6 Character of $\overline{FK}_{C_n}(n)$ over D_n for even n and decomposition

We summarize the results of Lemmas 7.3.16, 7.3.17, and 7.3.20 in Tables 7.4 and 7.5 as the character of $\overline{FK}_{C_n}(n)$ over D_n for even n.

Remark 7.3.25. *Tables 7.3, 7.4 and 7.5, are for irreducible characters of D_n, character of $\overline{FK}_{C_n}(n)$, with its q-Fibonacci form in Table 7.5. The cycle type of reflection s is $2^{\frac{n}{2}\prime}$, its D_n-conjugacy class is denoted by $c_{2^{\frac{n}{2}\prime}}$, and its associated irreducible character denoted by $\chi_{2^{\frac{n}{2}\prime}}$. The cycle type of reflection sr is $2^{\frac{n}{2}-1}11$, and its associated D_n-conjugacy class and irreducible character are denoted by $c_{2^{\frac{n}{2}-1}11}$ and $\chi_{2^{\frac{n}{2}-1}11}$ respectively. Similarly for other cycle types.*

Decomposition in irreducible characters

$$char_{D_n}[\overline{FK}_{C_n}(n=\text{even})] =$$

$$m_{1^n}\chi_{1^n} + m_{2^{\frac{n}{2}\prime}}\chi_{2^{\frac{n}{2}\prime}} + m_{2^{(\frac{n}{2}-1)}11}\chi_{2^{(\frac{n}{2}-1)}11} + m_{2^{\frac{n}{2}}}\chi_{2^{\frac{n}{2}}} + \sum_{1\le h\le \frac{n}{2}-1,\ h|n} m_{\frac{n}{h}^h}\chi_{\frac{n}{h}^h}, \tag{7.61}$$

where the coefficients m of the irreducible characters are calculated using character table 7.3 and either the character in q-Fibonacci form of Table 7.5 or the character in Table 7.4.

By Table 7.5 we have

$$
\begin{aligned}
m_{1^n} &= \frac{1}{2n}\Bigg\{L_n(q)+2+2\sum_{p=2}^{\frac{n}{2}-1}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})+(1+\frac{n}{2}q^2) \\
&\quad + \frac{n}{2}\Big[F_{\frac{n}{2}}(q^2)-2qF_{\frac{n}{2}-1}(q^2)+q^2F_{\frac{n}{2}-2}(q^2)\Big]+\frac{n}{2}F_{\frac{n-2}{2}}(q^2)\Bigg\}. \\
m_{2^{\frac{n}{2}}} &= \frac{1}{2n}\Bigg\{L_n(q)+2+2\sum_{p=2}^{\frac{n}{2}-1}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})+(1+\frac{n}{2}q^2) \\
&\quad - \frac{n}{2}\Big[F_{\frac{n}{2}}(q^2)-2qF_{\frac{n}{2}-1}(q^2)+q^2F_{\frac{n}{2}-2}(q^2)\Big]-\frac{n}{2}F_{\frac{n-2}{2}}(q^2)\Bigg\}. \\
m_{2^{\frac{n}{2}}{}'} &= \frac{1}{2n}\Bigg\{L_n(q)-2+2\sum_{p=2}^{\frac{n}{2}-1}(-1)^p(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})+(-1)^{\frac{n}{2}}(1+\frac{n}{2}q^2) \\
&\quad + \frac{n}{2}\Bigg[F_{\frac{n}{2}}(q^2)-2qF_{\frac{n}{2}-1}(q^2)+q^2F_{\frac{n}{2}-2}(q^2)\Bigg]-\frac{n}{2}F_{\frac{n-2}{2}}(q^2)\Bigg\}. \\
m_{2^{(\frac{n}{2}-1)}11} &= \frac{1}{2n}\Bigg\{L_n(q)-2+2\sum_{p=2}^{\frac{n}{2}-1}(-1)^p(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})+(-1)^{\frac{n}{2}}(1+\frac{n}{2}q^2) \\
&\quad - \frac{n}{2}\Bigg[F_{\frac{n}{2}}(q^2)-2qF_{\frac{n}{2}-1}(q^2)+q^2F_{\frac{n}{2}-2}(q^2)\Bigg]+\frac{n}{2}F_{\frac{n-2}{2}}(q^2)\Bigg\}. \\
m_{(h)} &= \frac{1}{2n}\Bigg\{2L_n+4cos\frac{2h\pi}{n}+4\sum_{p=2}^{\frac{n}{2}-1}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})cos\frac{2hp\pi}{n} \\
&\quad + 2(-1)^h(1+\frac{n}{2}q^2)\Bigg\},\ 1\le h\le\frac{n}{2}-1,\ \text{such that } h|n.
\end{aligned}
\tag{7.62}
$$

By Table 7.4 we get the following version of coefficients

$$
\begin{aligned}
m_{1^n} &= \frac{1}{2n}\Bigg\{ \sum_{k=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k} q^k + 2 + 2\sum_{p=2}^{\frac{n}{2}-1}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}}) + (1+\frac{n}{2}q^2) \\
&\quad + \frac{n}{2}\Bigg[\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2}-k}{k} q^{2k} - 2\sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1} + \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2} \Bigg] \\
&\quad + \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n-2}{4} \rfloor} \binom{\frac{n-2}{2}-k}{k} q^{2k} \Bigg\}. \\
m_{2^{\frac{n}{2}}} &= \frac{1}{2n}\Bigg\{ \sum_{k=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k} q^k + 2 + 2\sum_{p=2}^{\frac{n}{2}-1}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}}) + (1+\frac{n}{2}q^2) \\
&\quad - \frac{n}{2}\Bigg[\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2}-k}{k} q^{2k} - 2\sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1} + \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2} \Bigg] \\
&\quad - \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n-2}{4} \rfloor} \binom{\frac{n-2}{2}-k}{k} q^{2k} \Bigg\}. \\
m_{2^{\frac{n}{2}}{}'} &= \frac{1}{2n}\Bigg\{ \sum_{k=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k} q^k - 2 + 2\sum_{p=2}^{\frac{n}{2}-1}(-1)^p(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}}) + (-1)^{\frac{n}{2}}(1+\frac{n}{2}q^2) \\
&\quad + \frac{n}{2}\Bigg[\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2}-k}{k} q^{2k} - 2\sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1} + \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2} \Bigg] \\
&\quad - \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n-2}{4} \rfloor} \binom{\frac{n-2}{2}-k}{k} q^{2k} \Bigg\}. \\
m_{2^{(\frac{n}{2}-1)}11} &= \frac{1}{2n}\Bigg\{ \sum_{k=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k} q^k - 2 + 2\sum_{p=2}^{\frac{n}{2}-1}(-1)^p(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}}) + (-1)^{\frac{n}{2}}(1+\frac{n}{2}q^2) \\
&\quad - \frac{n}{2}\Bigg[\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{\frac{n}{2}-k}{k} q^{2k} - 2\sum_{k=0}^{\lfloor \frac{\frac{n}{2}-1}{2} \rfloor} \binom{\frac{n}{2}-1-k}{k} q^{2k+1} + \sum_{k=0}^{\lfloor \frac{\frac{n}{2}-2}{2} \rfloor} \binom{\frac{n}{2}-2-k}{k} q^{2k+2} \Bigg] \\
&\quad + \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n-2}{4} \rfloor} \binom{\frac{n-2}{2}-k}{k} q^{2k} \Bigg\}. \\
m_{(h)} &= \frac{1}{2n}\Bigg\{ 2\sum_{k=0}^{\frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k} q^k + 4cos\frac{2h\pi}{n} + 4\sum_{p=2}^{\frac{n}{2}-1}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})cos\frac{2hp\pi}{n} \\
&\quad + 2(-1)^h(1+\frac{n}{2}q^2) \Bigg\},\ 1 \le h \le \frac{n}{2}-1,\ \text{such that } h|n.
\end{aligned}
\tag{7.63}
$$

Example 7.3.26. *Let*

$char_{C_6}[\overline{FK}_{C_6}(6,q)] = m_{1^6}\chi_{1^6} + m_{222}\chi_{222} + m_{222'}\chi_{222'} + m_{2211}\chi_{2211} + m_6\chi_6 + m_{33}\chi_{33}$. *Then from Equation* (7.63) *we have*

$$m_{1^6} = 1 + 2q^2, m_{222} = q + q^3, m_{222'} = q^2,\ m_{2211} = q + q^3,\ m_6 = q + q^2,\ m_{33} = q + 2q^2.$$

Then substitution of the coefficients m *into the character* (7.61) *we have*

$$\begin{aligned} char_{D_6}[\overline{FK}_{C_6}(6,\ q)] &= (1 + 2q^2)\chi_{1^6} + (q + q^3)\chi_{222} + (q^2)\chi_{222'} \\ &+ (q + q^3)\chi_{2211} + (q + q^2)\chi_6 + (q + 2q^2)\chi_{33}. \end{aligned} \tag{7.64}$$

Sorting in q *we have the decomposition of* $char_{D_6}[\overline{FK}_{C_6}(6)]$ *in usual degrees.*

$$\begin{aligned} char_{D_6}[\overline{FK}_{C_6}(6)] &= \chi_{1^6} + (\chi_{222} + \chi_{2211} + \chi_6 + \chi_{33})q \\ &+ (2\chi_{1^6} + \chi_{222'} + \chi_6 + 2\chi_{33})q^2 + (\chi_{222} + \chi_{2211})q^3, \end{aligned} \tag{7.65}$$

which upon putting $q = 1$ *reduces to*

$$char_{D_6}\overline{FK}_{C_6}(6) = 3\chi_{1^6} + 2\chi_{222} + \chi_{222'} + 2\chi_{2211} + 2\chi_6 + 3\chi_{33},$$

in agreement with Equation (7.24).

7.3.7 Character of $\overline{FK}_{C_n}(n)$ over D_n for odd n and decomposition

We summarize the results of Lemmas 7.3.16 and 7.3.20 in Tables 7.7 and 7.8 as the character of $\overline{FK}_{C_n}(n)$ over D_n for odd n.

Table 7.6: Irreducible characters of D_n for odd n. The trivial and one reflection, the rest 2-dimensional rotations. $c_{(p)}$ stands for $c_{\frac{n}{p}^p}$ where $p \mid n$.

	ϵ	$\{r, r^{n-1}\}$	$\{r^p, r^{n-p}\},\ 2 \leq p \leq \frac{n-1}{2}$ such that $p \mid n$	$\{s, sr, \cdots, sr^{n-1}\}$
	$\lvert c_{1^n} \rvert = 1$	$\lvert c_n \rvert = 2$	$\lvert c_{(p)} \rvert = 2$	$\lvert c_{2^{\frac{n-1}{2}}1} \rvert = n$
χ_{1^n}	1	1	1	1
$\chi_{2^{\frac{n-1}{2}}1}$	1	1	1	-1
$\chi_{n^h}, 1 \leq h \leq \frac{n-1}{2}$ such that $h \mid n$	2	$2cos\frac{2h\pi}{n}$	$2cos\frac{2hp\pi}{n}$	0

Table 7.7: Character of $\overline{FK}_{C_n}(n)$ over D_n for odd n

ϵ	$[r]$	$[r^p],\ 2 \leq p \leq \frac{n-1}{2}$ such that $p \mid n$	$[s]$
$\lvert c_{1^n} \rvert = 1$	$\lvert c_n \rvert = 2$	$\lvert c_{(p)} \rvert = 2$	$\lvert c_{2^{\frac{n-1}{2}}} \rvert = n$
$\sum_{k=0}^{\frac{n-1}{2}} \frac{n}{n-k} \binom{n-k}{k} q^k$	1	$1 + pq^{\frac{n}{p}} \delta_{\frac{n}{p},\text{integer}}$	$\sum_{k=0}^{\lfloor \frac{n-1}{4} \rfloor} \binom{\frac{n-1}{2}-k}{k} q^{2k} - \sum_{k=0}^{\lfloor \frac{n-3}{4} \rfloor} \binom{\frac{n-3}{2}-k}{k} q^{2k+1}$

Table 7.8: Character of $\overline{FK}_{C_n}(n)$ over D_n for odd n and q-Fibonacci form

ϵ	$[r]$	$[r^p],\ 2 \leq p \leq \frac{n-1}{2}$ such that $p \mid n$	$[s]$
$\lvert c_{1^n} \rvert = 1$	$\lvert c_n \rvert = 2$	$\lvert c_{(p)} \rvert = 2$	$\lvert c_{2^{\frac{n-1}{2}}} \rvert = n$
$L_n(q)$	1	$1 + pq^{\frac{n}{p}} \delta_{\frac{n}{p},\text{integer}}$	$F_{\frac{n-1}{2}}(q^2) - qF_{\frac{n-3}{2}}(q^2)$

Remark 7.3.27. *In tables 7.6 and 7.7 the cycle type of reflection s is $2^{\frac{n-1}{2}}1$ and its associated D_n-conjugacy class and character are $c_{2^{\frac{n-1}{2}}1}$ and $\chi_{2^{\frac{n-1}{2}}1}$ respectively.*

Decomposition in irreducible characters

$$char_{D_n}[\overline{FK}_{C_n}(n = \text{odd})] = m_{1^n}\chi_{1^n} + m_{2^{\frac{n-1}{2}}}\chi_{2^{\frac{n-1}{2}}} + \sum_{0<h\leq\frac{n-1}{2},\ h \mid n} m_{n^h}\chi_{n^h}, \tag{7.66}$$

with the following coefficients calculated as before by using irreducible characters of D_n for odd n in Table 7.6 and character of $\overline{FK}_{C_n}(n = \text{odd})$ either Table 7.7 or the compact form of

q-Fibonacci in Table 7.8 it follows.

From Table 7.8 we have

$$\begin{aligned}
m_{1^n} &= \frac{1}{2n}\left[L_n(q)+2+2\sum_{p=2}^{\frac{n-1}{2}}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},integer})+n\Big(F_{\frac{n-1}{2}}(q^2)-qF_{\frac{n-3}{2}}(q^2)\Big)\right].\\
m_{2^{\frac{n-1}{2}}1} &= \frac{1}{2n}\left[L_n(q)+2+2\sum_{p=2}^{\frac{n-1}{2}}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},integer})-n\Big(F_{\frac{n-1}{2}}(q^2)-qF_{\frac{n-3}{2}}(q^2)\Big)\right].\\
m_{n^h} &= \frac{1}{2n}\left[2L_n(q)+4cos(\frac{2h\pi}{n})+4\sum_{p=2}^{\frac{n-1}{2}}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},integer})cos(\frac{2hp\pi}{n})\right],\\
&\quad 1\le h\le\frac{n-1}{2},\ h|n.
\end{aligned}\tag{7.67}$$

From Table 7.7 we have

$$\begin{aligned}
m_{1^n} =&\frac{1}{2n}\Bigg\{\sum_{k=0}^{\frac{n-1}{2}}\frac{n}{n-k}\binom{n-k}{k}q^k+2+2\sum_{p=2}^{\frac{n-1}{2}}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})\\
&+n\Big[\sum_{k=0}^{\lfloor\frac{n-1}{4}\rfloor}\binom{\frac{n-1}{2}-k}{k}q^{2k}-\sum_{k=0}^{\lfloor\frac{n-3}{4}\rfloor}\binom{\frac{n-3}{2}-k}{k}q^{2k+1}\Big]\Bigg\}.\\
m_{2^{\frac{n-1}{2}}1} =&\frac{1}{2n}\Bigg\{\sum_{k=0}^{\frac{n-1}{2}}\frac{n}{n-k}\binom{n-k}{k}q^k+2+2\sum_{p=2}^{\frac{n-1}{2}}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})\\
&-n\Big[\sum_{k=0}^{\lfloor\frac{n-1}{4}\rfloor}\binom{\frac{n-1}{2}-k}{k}q^{2k}-\sum_{k=0}^{\lfloor\frac{n-3}{4}\rfloor}\binom{\frac{n-3}{2}-k}{k}q^{2k+1}\Big]\Bigg\}.\\
m_{n^h} =&\frac{1}{2n}\Bigg\{2\sum_{k=0}^{\frac{n-1}{2}}\frac{n}{n-k}\binom{n-k}{k}q^k+4cos\frac{2h\pi}{n}+4\sum_{p=2}^{\frac{n-1}{2}}(1+pq^{\frac{n}{p}}\delta_{\frac{n}{p},\text{integer}})cos(\frac{2hp\pi}{n})\Bigg\},\\
&\text{where } 1\le h\le\frac{n-1}{2},\ h|n.
\end{aligned}\tag{7.68}$$

Example 7.3.28. *Let $n = 5$, then*

$$char_{D_5}[\overline{FK}_{C_5}(5)] = m_{1^5}\chi_{1^5} + m_{221}\chi_{221} + m_5\chi_5 + m_{5'}\chi_{5'},$$

where the m-coefficients calculated from Equation (7.68) *are:* $m_{1^5} = 1 + q,\ m_{221} = q,\ m_5 = q + q^2,\ m_{5'} = q + q^2$. *Substituting the coefficients into the character, Equation* (7.66)*, we have*

$$char_{D_5}[\overline{FK}_{C_5}(5)](q) = (1 + q^2)\chi_{1^5} + q\chi_{221} + (q + q^2)\chi_5 + (q + q^2)\chi_{5'}.$$

Sorting the above in terms of q yields the decomposition in usual degree:

$$char_{D_5}[\overline{FK}_{C_5}(5)](q) = \chi_{1^5} + (\chi_{221} + \chi_5 + \chi_{5'})q + (\chi_{1^5} + \chi_5 + \chi_{5'})q^2,$$

which upon putting $q = 1$ reduces to

$$char_{D_5}\overline{FK}_{C_5}(5) = 2\chi_{1^5} + \chi_{221} + 2\chi_{5'} + 2\chi_5,$$

in agreement with Equation (7.14).

Representation decomposition by conjugation class

Since the basis of $\overline{FK}_{C_n}(n)$ consists of monomials in variables with no common index (alternatively, matchings in n-cycle), the S_n-degree of an element of the basis is product of disjoint 2-cycles and 1-cycles. So it belongs to the S_n-conjugacy class denoted by $t_2^k t_1^{n-2k}$ where t_2 and t_1 stand for 2-cycle and 1-cycle respectively and where k is the degree of monomial. Since we have the same partition of the basis invariant under S_n-conjugacy class as under usual degree, we have essentially the same decomposition as by usual degree.

Representation decomposition by set partition type

Since no two variable appearing in a monomial $M \in B[\overline{FK}_{C_n}(n)]$ have common indexes, in each part of the set partition there is at most 2 indexes. So the set partition type of degree k monomial M is of the form $\alpha = (\underbrace{2, 2, \cdots, 2}_{k}, \underbrace{1, 1, \cdots, 1}_{n-2k})$ denoted by $2^k 1^{n-2k}$, where each 2 in α represent the two indexes of a variable if the variable appears in M, while 1s represent the indexes appearing in no variable in M. Since we have the same partition of the basis invariant under set partition type as under usual degree, we have essentially the same decomposition as by usual degree.

Chapter 8

Commutative Fomin-Kirillov Algebra $FK^c(n)$

In 1999 Sergey Fomin and Anatol N. Kirillov stated a proposition [14] according to which the commutative quotient of $FK(n)$ has dimension $n!$. In that paper they gave sketch of the proof using Gelfand-Varhenko algebra [16]. In this chapter we give a direct proof of this proposition.

Definition 8.0.1. *Commutative Fomin-Kirillov Algebra $FK^c(n)$ is defined by letting the generators of Fomin-Kirillov algebra commute with each other. [18]*

We define a slightly different monomial order than we did previously in Definition 1.2.1.

Definition 8.0.2. *The graded lexicographic ordering of monomials M_1 and M_2 in a polynomial is defined by first comparing their total degrees if they are of different degrees. If M_1 and M_2*

are of same degrees, we introduce a variable ordering by

$$x_{ij} > x_{kl} \textit{ if } \begin{cases} i < k, & \textit{or} \\ i = k, & \textit{and } j < l. \end{cases}$$

Then with this variable ordering, the following rule completes the definition of graded lexicographic monomial ordering. For monomials M_1 and M_2 of same degree d, $M_1 < M_2$ if the first variable of M_1 is less than the first variable of M_2. if the first m variables for $m = 1, \cdots, d-1$ happen to be the same, then compare the $m+1$ th variables.

Example 8.0.3. $x_{24} < x_{23}x_{24}$, $x_{13} > x_{23}$, $x_{23} > x_{24}$.

8.1 Relations

Out of the relations of $FK(n)$ in (1.2), relation (ii) : $x_{ij}x_{kl} - x_{kl}x_{ij} = 0$ for distinct i, j, k, l, is nothing but now a part of the definition of commutative algebra so this relation is redundant. Also now that the variables commute, we need to take care of our lexicographic ordering, i.e., in Equation (1.2), relation (iii) : $x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0,\ 1 \le i < j < k \le n$, now should be written as $x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk} = 0$ and similarly the relation (iii') : $x_{ij}x_{ik} - x_{jk}x_{ij} + x_{ik}x_{jk} = 0,\ 1 \le i < j < k \le n$ should be written as $x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk} = 0$ which is the same as our new (iii), i.e., relations (iii) and (iii') are equivalent when considering $FK^c(n)$. Therefore the set of relations for $FK^c(n)$ is

$$R^c = \{x_{ij}^2 = 0,\ \underline{x_{ij}x_{ik}} - x_{ij}x_{jk} + x_{ik}x_{jk} = 0,\ 1 \le i < j < k \le n\}. \tag{8.1}$$

Example 8.1.1. *For* $FK^c(3)$ $R^c = \{x_{12}^2 = x_{13}^2 = x_{23}^2 = 0,\ \underline{x_{12}x_{13}} - x_{12}x_{23} + x_{13}x_{23} = 0\}$

Example 8.1.2. *For* $FK^c(4)$ *we have following relations.*

$$\begin{aligned} R^c = &\{x_{12}^2 = x_{13}^2 = x_{14}^2 = x_{23}^2 = x_{24}^2 = x_{34}^2 = 0, \\ &\underline{x_{12}x_{13}} - x_{12}x_{23} + x_{13}x_{23} = 0,\ \underline{x_{12}x_{14}} - x_{12}x_{24} + x_{14}x_{24} = 0, \\ &\underline{x_{13}x_{14}} - x_{13}x_{34} + x_{14}x_{34} = 0,\ \underline{x_{23}x_{24}} - x_{23}x_{34} - x_{24}x_{34} = 0\}. \end{aligned}$$

The relations of $FK^c(n)$ are not homogeneous with respect to S_n-degree (1st term of the relation $x_{12}x_{13}$ is of S_n-degree $(12)(13) = (132)$ while the 2nd term $x_{12}x_{23}$ is of degree $(12)(23) = (123)$). So the ideal is not homogeneous w.r.t. S_n- degree, so this degree is not well defined in $FK^c(n)$. However the set-partition degree is well defined for this algebra as the relations are homogeneous w.r.t this degree.

From (8.1) we have the following set of generators for the defining ideal of $FK^c(n)$.

$$\begin{aligned} &\text{generators for defining ideal of } FK^c(n) \\ &= \{x_{ij}^2,\ \underline{x_{ij}x_{ik}} - x_{ij}x_{jk} + x_{ik}x_{jk},\ 1 \le i < j < k \le n\} \end{aligned} \tag{8.2}$$

8.2 Gröbner basis for $FK^c(n)$

We consider the polynomials in (8.2), as the degree 2 elements (2-terms) of Gröbner basis for the ideal associated with $FK^c(n)$. So the 2-terms of the Gröbner basis for the ideal associated to $FK^c(n)$ are

$$g_1 = \underline{x_{ij}x_{ij}} \text{ and } g_2 = \underline{x_{ij}x_{ik}} - x_{ij}x_{jk} + x_{ik}x_{jk}, \;\; 1 \leq i < j < k \leq n \tag{8.3}$$

where the leading terms are underlined. Then calculations of S-polynomials for commutative case by $S(g_1, g_2) = \frac{x^\gamma}{LT(g_1)}g_1 - \frac{x^\gamma}{LT(g_2)}g_2$ where $x^\gamma = LCM(LM(g_1), LM(g_2))$ show that $S(g_1, g_1, 3)$, $S(g_1, g_2, 3)$ and $S(g_2, g_2, 3) \to 0$ under reduction by lower degree terms, i.e., we get no elements of degree 3 and more for our Gröbner basis. Hence we have the following Gröbner basis for the ideal of $FK^c(n)$ which consists only of 2-terms.

$$GB^c = \{x_{ij}^2, \;\; \underline{x_{ij}x_{ik}} - x_{ij}x_{jk} + x_{ik}x_{jk}, \;\; 1 \leq i < j < k \leq n\}. \tag{8.4}$$

Example 8.2.1. *For* $FK^c(3)$, $GB^c = \{x_{12}^2, \; x_{13}^2, \; x_{23}^2, \; \underline{x_{12}x_{13}} - x_{12}x_{23} + x_{13}x_{23}\}$.

Example 8.2.2. *For* $FK^c(4)$ $GB^c = \{x_{12}^2, \; x_{13}^2, \; x_{14}^2, \; x_{23}^2, \; x_{24}^2, \; x_{34}^2,$
$\underline{x_{12}x_{13}} - x_{12}x_{23} + x_{13}x_{23}, \; \underline{x_{12}x_{14}} - x_{12}x_{24} + x_{14}x_{24},$
$\underline{x_{13}x_{14}} - x_{13}x_{34} + x_{14}x_{34}, \; \underline{x_{23}x_{24}} - x_{23}x_{34} + x_{24}x_{34}\}$

8.3 Basis of $FK^c(n)$

Theorem 8.3.1.

$$B[FK^c(n)] = \Bigg[x_{m_1l_1}x_{m_2l_2}\cdots x_{m_kl_k} : k \geq 0 \; (\textit{case } k = 0, \; \textit{empty product, so is } 1), \\ , \; m_i \textit{ are distict}, m_i < l_i, \; \begin{cases} l_i < l_{i+1}; \textit{or} \\ l_i = l_{i+1} \textit{ and } m_i < m_{i+1}. \end{cases} \Bigg] \tag{8.5}$$

Alternatively we can define $B[FK^c(n)]$ recursively as it follows.

1. Degrees 0 and 1 elements of $B[FK^c(n)]$ are respectively 1 and $x_{ml},\ 1 \le m < l \le n$;
2. For deg $k \ge 2$, let $x_{m_1l_1}x_{m_2l_2}\cdots x_{m_kl_k}$ be degree k element. Then degree $k+1$ element is $x_{m_1l_1}x_{m_2l_2}\cdots x_{m_kl_k}x_{m_{k+1},l_{k+1}}$ such that

$$m_{k+1} \ne m_i \text{ for } i = 1, \cdots, k \text{ and } \begin{cases} l_k < l_{k+1}, & \text{or} \\ l_k = l_{k+1}, & \text{and } m_k < m_{k+1}. \end{cases}$$

Proof. In general the basis of a quotient algebra consists of all the words on generators of the algebra that are not divisible by the leading monomials of the elements of Gröbner basis, i.e., from Equation (8.4) by x_{ij}^2 and $x_{ij}x_{ik}$. This restriction forces our basis to be consisting of terms in the form stated in (8.5) of the statement. This completes the proof. $\square$

Example 8.3.2. $B[FK^c(3)] = \{1, x_{12}, x_{13}, x_{23}, x_{12}x_{23}, x_{13}x_{23}\}$

8.4 Dimension of $FK^c(n)$

In this section we first define a bijection between cycles $C \in S_n$ and some elements of $B[FK^c(n)]$ and then extend it to define a bijection between arbitrary permutation $\sigma \in S_n$ and all the elements in $B[FK^c(n)]$.

Definition 8.4.1. *Let the cycle permutation $C = (c_1, c_2, \cdots, c_k)$ arranged such that the biggest entry c_k in C is the one on the far right of C, however the sequence $c_j, j = 1, 2, \cdots, k$ is not monotonic. Then*

1. *We define a "usual drop" at c_{j_i} of length l_{j_i} by sequence*

$$\begin{aligned} \bar{c}_{j_i l_{j_1}} &= (c_{j_i+1}, c_{j_i+2}, \cdots, c_{j_i+l_{j_i}}), \textit{such that} \\ c_{j_i+1} < c_{j_i+2} < \cdots &< c_{j_i+l_i} < c_{j_i} < c_{j_i+l_{j_i}+1} \end{aligned} \tag{8.6}$$

   (so the "usual drop" at c_{j_i} is a monotonic increasing sequence not including c_{j_i}). To this "usual drop" $\bar{c}_{j_i l_{j_1}}$, we associate monomial

$$T_{\bar{c}_{j_i l_{j_i}}} = x_{c_{j_i+1}c_{j_i}} x_{c_{j_i+2}c_{j_i}} \cdots x_{c_{j_i+l_{j_i}}c_{j_i}},\ i = 1, 2, \cdots, m, \tag{8.7}$$

   where m is the number of "usual drops" in cycle C. (so the associated monomial to a "usual drop" at c_{j_i}, is a degree l_{j_i} monomial in 2-index variables with increasing 1st indexes and common 2nd index c_{j_i}).

2. *We also define the "special drop" $\bar{c}_k$, where c_k is the biggest entry in the cycle C, by*

$$\bar{c}_k = C \backslash \bar{c}_{j_1 l_{j_1}} \backslash \bar{c}_{j_2 l_{j_2}} \backslash \cdots \backslash \bar{c}_{j_m, l_{j_m}}. \tag{8.8}$$

   By construction, the special drop $\bar{c}_k$ is an increasing sequence ending up with c_k. To the "special drop" we associate the monomial

$$T_{\bar{c}_k} = x_{(\bar{c}_k)_1 c_k} x_{(\bar{c}_k)_2 c_k} \cdots x_{(\bar{c}_k)_{k-(l_{j_1}+l_{j_2}+\cdots+l_{j_m})-1} c_k}, \tag{8.9}$$

   where $(\bar{c}_k)_i$ are the entries of $\bar{c}_k$ and where c_k, the biggest entry in C, now is the biggest entry in $\bar{c}_k$. By construction $T_{\bar{c}_k}$ is a monomial in 2-index variables with increasing 1st index and common 2nd index c_k, whose degree is $k - (l_{j_1} + l_{j_2} + \cdots + l_{j_m}) - 1$, where m is the number of usual drops.

3. *We define the associated monomial to the cycle C as*

$$T_C = T_{\bar{c}_{j_1} l_{j_1}} T_{\bar{c}_{j_2} l_{j_2}} \cdots T_{\bar{c}_{j_m} l_{j_m}} T_{\bar{c}_k}. \tag{8.10}$$

Remark 8.4.2. *1. by construction T_C is a monomial in variables piecewise increasing in 1st index and weakly increasing in the 2nd indexes $c_{j_1}, c_{j_2}, \cdots, c_{j_m}, c_k$ with the biggest value of them c_k, the biggest entry in C.*

2. *T_C is a monomial of degree $(l_{j_1} + l_{j_2} + \cdots + l_{j_m}) + k - (l_{j_1} + l_{j_2} + \cdots + l_{j_m}) - 1 = k - 1$.*

Example 8.4.3. *For cycle $C = (1, \mathbf{3}, 2, 7, \mathbf{8}, 4, 5, 6, \mathbf{9})$ (drop happens at components $3, 8$ and 9 written in bold). "usual drops" and their "associated monomials" are:*

- *1st usual drop $\overline{c}_{j_1}$, happens at $c_{j_1} = c_2 = 3$, with $\overline{c}_{j_1} = \overline{c}_2 = \overline{3} = (2)$, and with the associated monomial $T_{j_1 l_{j_1}} = x_{23}$.*

- *2nd usual drop $\overline{c}_{j_2}$, happens at $c_{j_2} = c_5 = 8$, and with the associated monomial $\overline{c}_{j_2} = \overline{c}_5 = \overline{8} = (4, 5, 6)$, with $T_{j_2 l_{j_2}} = x_{48} x_{58} x_{68}$.*

- *The"special drop" $\overline{c}_k$, happens at $c_k = c_9 = 9$, with $\overline{c}_k = \overline{c}_9 = C \backslash \overline{c}_{j_1} \backslash \overline{c}_{j_2} =$*
  $= (1, 3, 2, 7, 8, 4, 5, 6, 9) \backslash (2) \backslash (4, 5, 6) = (13789)$, and $T_k = x_{19} x_{39} x_{79} x_{89}$.

Now concatenating $T_{j_1 l_{j_1}} = x_{23}$, $T_{j_2 l_{j_2}} = x_{48} x_{58} x_{68}$ and $T_k = x_{19} x_{39} x_{79} x_{89}$ sorted in increasing 2nd index gives the monomial $T_C = x_{23} x_{48} x_{58} x_{68} x_{19} x_{39} x_{79} x_{89}$ associated to the cycle $C = (1, 3, 2, 7, 8, 4, 5, 6, 9)$.

Definition 8.4.4. *In S_n, let permutation $\sigma = C^{(1)} C^{(2)} \cdots C^{(l)}$, where $C^{(i)}$s are disjoint cycles (every permutation can be written in disjoint cycles), where $C^{(i)} = (c_1^{(i)}, c_2^{(i)}, \cdots, c_{k_i}^{(i)})$ such* *that for any i, $c_{k_i}^{(i)}$, the biggest entry in $C^{(i)}$, is on the far right of it, and disjoint $C^{(i)}$s in σ are sorted such that $c_{k_1}^{(1)} < c_{k_2}^{(2)} < \cdots < c_{k_l}^{(l)}$ (i.e., $C^{(i)}$s sorted in terms of their biggest entries). We define $T_\sigma = T_{C^{(1)}} T_{C^{(2)}} \cdots T_{C^{(l)}}$ where $T_{C^{(i)}}$ is the monomial associated to the cycle $C^{(i)}$.*

Remark 8.4.5. *From the above definition, and the definition of $T_{C^{(i)}}$, and the fact that $C^{(i)}$s are disjoint, the monomial T_σ can be sorted to be weakly increasing in 2nd indexes $c_{k_i}^{(i)}$ and increasing in 1st indexes for equal 2nd index. Also the 1st indexes are distinct (as firstly each cycle has distinct components, and secondly our cycles are disjoint), so satisfy the conditions of the basis of $FK^{(c)}(n)$ and so $T_\sigma \in B[FK^c(n)]$.*

Remark 8.4.6. *Each cycle has a "special drop", so the number of "special drops" in $\sigma \in S_n$ is equal to the number of cycles in the decomposition of σ into disjoint cycles.*

Example 8.4.7. *Let permutation $\sigma = (1,3,2,7,8,4,5,6,9)(10,11,14)$ (decomposed in disjoint cycles). Then from Example 8.4.3, the associated monomial to cycle $C^{(1)} = (1,3,2,7,8,4,5,6,9)$ is $T_{C^{(1)}} = x_{23}x_{48}x_{58}x_{68}x_{19}x_{39}x_{79}x_{89}$. Also since the cycle $C^{(2)} = (10,11,14)$ does not have "usual drop", its "special drop" is itself $C^{(2)} = (10,11,14)$ with the associated monomial $T_{C^{(2)}} = x_{10,14}x_{11,14}$. Therefore the associated monomial to $\sigma = (1,3,2,7,8,4,5,6,9)(10,11,14)$ is $T_\sigma = T_{C^{(1)}}T_{C^{(2)}} = x_{23}x_{48}x_{58}x_{68}x_{19}x_{39}x_{79}x_{89}x_{10,14}x_{11,14}$.*

Definition 8.4.8. *Let $\sigma = C^{(1)}C^{(2)} \cdots C^{(l)}$ be decomposition of permutation $\sigma \in S_n$ in disjoint cycles. Then we define the maps*

$$\begin{aligned} &B : B[FK^c(n)] \to S_n \text{ by } x_{m_1 l_1} \cdots x_{m_k l_k} \mapsto (m_k l_k) \cdots (m_1 l_1) \text{ and} \\ &B^{-1} : S_n \to B[FK^c(n)], \text{ by } \sigma = C^{(1)}C^{(2)} \cdots C^{(l)} \mapsto T_\sigma = T_{C^{(1)}}T_{C^{(2)}} \cdots T_{C^{(l)}}, \end{aligned} \tag{8.11}$$

where $T_{C^{(i)}}$ are the associated monomials to the cycles $C^{(i)}$ defined in Definition 8.4.4.

Example 8.4.9. *The associated monomial to permutation $\sigma = (1,3,2,7,8,4,5,6,9)(10,11,14)$ derived in Example 8.4.7 to be $T_\sigma = x_{23}x_{48}x_{58}x_{68}x_{19}x_{39}x_{79}x_{89}x_{10,14}x_{11,14}$.*

The map B defined in Definition 8.4.8 takes T_σ to

$(11,14)(10,14)(89)(79)(39)(19)(68)(58)(48)(23) = (2,7,8,4,5,6,9,1,3)(10,11,14) = \sigma$

In the following proposition we give a proof to the idea that the dimension of commutative Fomin-Kirillov algebra is $n!$. This proposition was stated in 1999 by Fomin and Kirillov [14], where in their paper they gave sketch of the proof using Gelfand-Varhenko algebra [16]. Here we give a direct proof to that proposition.

Proposition 8.4.10. *1. There is a bijection between $B[FK^c(n)]$ and symmetric group S_n.*

2. dim $(FK^c(n)) = n!$

Proof. We show that the map B defined in Definition 8.4.8, indeed is a bijection between $B[FK^c(n)]$ and symmetric group S_n.

Consider the above definition for the special case where our permutation is a cycle, i.e., $\sigma = C$. Also assume that C has m usual drops at $c_{j_1}, c_{j_2}, \cdots, c_{j_m}$ of lengths $l_{j_1}, l_{j_2}, \cdots, l_{j_m}$ respectively. The special drop occurs at c_k, the biggest component of C, arranged to be on the far right of C. Therefore our cycle would be of the form

$$C = (c_1, \cdots, c_{j_1}, c_{j_1+1}, \ldots c_{j_2}, c_{j_2+1}, \cdots, \cdots, c_{j_m}, c_{j_m+1}, \cdots, c_k).$$

Also we have $T_\sigma = T_C$. Then by Definitions 8.4.1, 8.4.4 and 8.4.8 we have usual drops

$$\begin{aligned}\bar{c}_{j_1,l_{j_1}} &=(c_{j_1+1}, c_{j_1+2}\cdots, c_{j_1+l_{j_1}}),\\ \bar{c}_{j_2,l_{j_2}} &=(c_{j_2+1}, c_{j_2+2}\cdots, c_{j_2+l_{j_2}}),\\ &\vdots\\ \bar{c}_{j_m,l_{j_m}} &=(c_{j_m+1}, c_{j_m+2}\cdots, c_{j_m+l_{j_m}})\end{aligned}\tag{8.12}$$

with their associated monomials

$$\begin{aligned}T_{\bar{c}_{j_1,l_{j_1}}} &=x_{c_{j_1+1},c_{j_1}}x_{c_{j_1+2},c_{j_1}}\cdots x_{c_{j_1+l_{j_1}},c_{j_1}},\\ T_{\bar{c}_{j_2,l_{j_2}}} &=x_{c_{j_2+1},c_{j_2}}x_{c_{j_2+2},c_{j_2}}\cdots x_{c_{j_2+l_{j_2}},c_{j_2}},\\ &\vdots\\ T_{\bar{c}_{j_m,l_{j_m}}} &=x_{c_{j_m+1},c_{j_m}}x_{c_{j_m+2},c_{j_m}}\cdots x_{c_{j_m+l_{j_m}},c_{j_m}}\end{aligned}\tag{8.13}$$

Also we have special drop

$$\begin{aligned}\bar{c}_k =&C \setminus \bar{c}_{j_1} \setminus \bar{c}_{j_2} \setminus \cdots \setminus \bar{c}_{j_m}\\ =&(c_1, c_2, \cdots, c_{j_1}, c_{j_1+l_{j_1}+1}, c_{j_1+l_{j_1}+2}, \ldots c_{j_2}, c_{j_2+l_{j_2}+1}, c_{j_2+l_{j_2}+2}, \cdots\\ &\cdots c_{j_m}, c_{j_m+l_{j_m}+1}, c_{j_m+l_{j_m}+2}, \cdots, c_k)\end{aligned}\tag{8.14}$$

with its associated monomial

$$\begin{aligned}T_{\bar{c}_k} = x_{c_1c_k}x_{c_2c_k}\cdots x_{c_{j_1}c_k}x_{c_{j_1+l_1+1}c_k}x_{c_{j_1+l_1+2}c_k}\cdots x_{c_{j_2}c_k}x_{c_{j_2+l_2+1}c_k}x_{c_{j_2+l_2+2}c_k}\cdots\\ \cdots x_{c_{j_m}c_k}x_{c_{j_m+l_m+1}c_k}x_{c_{j_m+l_m+2}c_k}\cdots x_{c_{k-1}c_k}.\end{aligned}\tag{8.15}$$

Now substituting Equations 8.13 and 8.15 into the equation

$$B^{-1}C =T_C = \underline{T_{\bar{c}_{j_1l_{j_1}}}T_{\bar{c}_{j_2l_{j_2}}}\cdots T_{\bar{c}_{j_ml_{j_m}}}}T_{\bar{c}_k}\tag{8.16}$$

yields:

$$\begin{aligned} B^{-1}C =& (x_{c_{j_1+1}c_{j_1}} x_{c_{j_1+2}c_{j_1}} \cdots x_{c_{j_1+l_{j_1}}c_{j_1}})(x_{c_{j_2+1}c_{j_2}} x_{c_{j_2+2}c_{j_2}} \cdots x_{c_{j_2+l_{j_2}}c_{j_2}}) \cdots \\ & \cdots (x_{c_{j_m+1}c_{j_m}} x_{c_{j_m+2}c_{j_m}} \cdots x_{c_{j_m+l_{j_m}}c_{j_m}}) \times [x_{c_1 c_k} x_{c_2 c_k} \cdots x_{c_{j_1} c_k} x_{c_{j_1+l_1+1} c_k} x_{c_{j_1+l_1+2} c_k} \cdots \\ & \cdots x_{c_{j_2} c_k} x_{c_{j_2+l_2+1} c_k} x_{c_{j_2+l_2+2} c_k} \cdots x_{c_{j_m} c_k} x_{c_{j_m+l_m+1} c_k} x_{c_{j_m+l_m+2} c_k} \cdots x_{c_{k-1} c_k}]. \end{aligned} \tag{8.17}$$

Then by Definition 8.4.8 we have:

$$\begin{aligned} B \circ B^{-1}C = BT_C =& (c_{k-1}, c_k) \cdots (c_{j_m+l_m+2}, c_k)(c_{j_m+l_m+1}, c_k)(c_{j_m}, c_k) \cdots \\ & \cdots (c_{j_2+l_2+2}, c_k)(c_{j_2+l_2+1}, c_k)(c_{j_2}, c_k) \\ & \cdots (c_{j_1+l_1+2}, c_k)(c_{j_1+l_1+1}, c_k)(c_{j_1}, c_k) \cdots \\ & \vdots \\ & \cdots (c_2, c_k)(c_1, c_k) \\ & (c_{j_m+l_{jm}}, c_{j_m}) \cdots (c_{j_m+2}, c_{j_m})(c_{j_m+1}, c_{j_m}) \cdots \\ & \vdots \\ & (c_{j_2+l_{j_2}}, c_{j_2}) \cdots (c_{j_2+2}, c_{j_2})(c_{j_2+1}, c_{j_2}) \cdots \\ & \cdots (c_{j_1+l_{j_1}}, c_{j_1}) \cdots (c_{j_1+2}, c_{j_1})(c_{j_1+1}, c_{j_1}). \end{aligned} \tag{8.18}$$

From Equation (8.18), we see that $(B \circ B^{-1}C)c_i = c_{i+1}$ for $i = 1, 2, \cdots, k-1$, and $(B \circ B^{-1}C)c_k = c_1$, as one can see from the structure of the equation that $c_{j_1}, c_{j_2}, \cdots c_{j_m}$ are taken to $c_{j_1+1}, c_{j_2+1}, \cdots c_{j_m+1}$ respectively, i.e., we have a cycle rotation. So we have

$$BT_C = B \circ B^{-1}C = C.$$

Hence B defines a 1-to-1 correspondence between any cycle $C^{(i)} \in S_n$ and its associated monomial $T_{C^{(i)}}$ defined above, and so between any permutation $\sigma = C^{(1)}C^{(2)} \cdots C^{(l)}$ and

$T_\sigma = T_{C^{(1)}} T_{C^{(2)}} \cdots T_{C^{(l)}}$ sorted in the orders defined above. Therefore B defines a bijection between $B[FK^c(n)]$ and S_n. Hence we have

$$dim[FK^c(n)] = |B[FK^c(n)]| = |S_n| = n!.$$

This completes th proof. $\square$

Appendix A

Schubert polynomials

A.1 Schubert polynomial

Here we calculate Schubert polynomial $\mathfrak{S}_{(2431)}$. From (1.1) we have

$$\mathfrak{S}_w = \sum_{a \in R(w)} \sum_{(i_1, \cdots, i_p) \in K(a)} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

$w = \binom{1234}{2431} = (124)(3)$ has 4 inversions, so $p = l(w) = 4$, $a = (a_1, a_2, a_3, a_4)$, where $1 \leq a_i \leq n - 1 = 4 - 1 = 3$. $a = (a_1, a_2, a_3, a_4)$ such that $s_{a_1} s_{a_2} s_{a_3} s_{a_4} = w = (124)(3)$, where $s_{a_i} = (a_i, a_i + 1)$. By $1 \leq a_i \leq n - 1$, we could have

$$\begin{aligned} a =&(a_1, a_2, a_3, a_4) \\ =&(1,2,3,2),(1,2,3,1),(1,3,2,3),(1,3,2,1),(2,1,3,1),(2,1,3,2), \\ &(2,3,1,2),(2,3,1,3),(3,1,2,3),(3,1,2,1),(3,2,1,2),(3,2,1,3) \end{aligned}$$

However by the condition $s_{a_1}s_{a_2}s_{a_3}s_{a_4} = w = (124)(3)$, out of the above list, we are left only with the following.

$$\begin{aligned} (1,2,3,2) &\rightarrow (12)(23)(34)(23) = (124)(3) = w, \\ (1,3,2,3) &\rightarrow (12)(34)(23)(34) = (124)(3) = w \\ (3,1,2,3) &\rightarrow (34)(12)(23)(34) = (124)(3) = w \end{aligned}$$

as the rest of them do not end up with w. Therefore we have

$$R(w) = \{(1,2,3,2),(1,3,2,3),(3,1,2,3)\}$$

However for

$$\begin{aligned} &a = (1,2,3,2), \text{ there is no } (i_1,i_2,i_3,i_4) \text{ available, so } K(1,2,3,2) = \{\} \\ &a = (1,3,2,3) \rightarrow (i_1,i_2,i_3,i_4) = (1,2,2,3), \text{ so } K(1,3,2,3) = \{(1,2,2,3)\} \\ &a = (3,1,2,3) \rightarrow (i_1,i_2,i_3,i_4) = (1,1,2,3), \text{ so } K(3,1,2,3) = \{(1,1,2,3)\} \end{aligned}$$

So we have

$$(i_1,i_2,i_3,i_4) = (1,2,2,3) \text{ and } (1,1,2,3).$$

Hence we have

$$\mathfrak{S}_{(2431)} = \sum_{a\in R(2431)} \sum_{(i_1,i_2,i_3 i_4)\in K(a)} x_{i_1}x_{i_2}x_{i_3}x_{i_4} = x_1x_2^2x_3 + x_1^2x_2x_3$$

A.2 Schubert polynomial

To calculate $\mathfrak{S}_{(n,n+1)}$, as in the above $w = (n, n+1) = \binom{1,2,\cdots,n,n+1}{1,2,\cdots,n+1,n}$, has one inversion. So $p = l(w) = 1$, and $a = (a_1)$, where $1 \leq a_1 \leq n$, and $s_{a_1} = w = (n, n+1)$, so $a_1 = n$, i.e., $R(w) = \{(n)\}$ and $(i_1) = (1), (2), \cdots, (n)$, i.e., $K(a) = \{(1), (2), \cdots, (n)\}$. Hence from (1.1) we have

$$\mathfrak{S}_{(n,n+1)} = \sum_{a\in R((n,n+1))} \sum_{(i_1)\in K(a)} x_{i_1} = x_1 + x_2 + \cdots + x_n.$$

A.3 Non-negativity conjecture

The 'non-negativity conjecture' says that for any $w \in S_n$, the Schubert polynomial evaluated at Dunkl elements, i.e., $\mathfrak{S}_w(\theta_1, \cdots, \theta_n)$ can be written as a linear combination of monomials in generators of $FK(n)$, x_{ij}, for $i < j$, with non-negative integers coefficients.

For $FK(3)$ we drive the Dunkl elements, by the formula $\theta_j = -\sum_{1\leq i<j} x_{ij} + \sum_{j<k\leq n} x_{jk}$. The result is

$$\theta_1 = x_{12} + x_{13},\ \theta_2 = -x_{12} + x_{23},\ \theta_3 = -x_{13} - x_{23}.$$

Like in A.1 we find

$$\mathfrak{S}_{id} = 1,\ \mathfrak{S}_{(12)(3)} = x_1,\ \mathfrak{S}_{(23)(1)} = x_1 + x_2,\ \mathfrak{S}_{(132)} = x_1^2,\ \mathfrak{S}_{(13)(2)} = x_1^2 x_2,\ \mathfrak{S}_{123} = x_1 x_2.$$

with their evaluation at Dunkl elements

$$\begin{aligned}
&\mathfrak{S}_{id}(\theta) = 1,\ \mathfrak{S}_{(12)(3)}(\theta) = \theta_1 = x_{12} + x_{13},\ \mathfrak{S}_{(23)(1)}(\theta) = \theta_1 + \theta_2 = x_{13} + x_{23},\\
&\mathfrak{S}_{(132)}(\theta) = \theta_1^2 = 2x_{12}x_{13},\ \mathfrak{S}_{(13)(2)}(\theta) = \theta_1^2\theta_2 = 2x_{23}x_{13}x_{12} + 2x_{13}x_{23}x_{12},\\
&\mathfrak{S}_{123}(\theta) = \theta_1\theta_2 = x_{23}x_{13} + x_{13}x_{23}.
\end{aligned}$$

where each of the above Schubert polynomials is a linear combination of monomials in generators of $FK(3)$, with non-negative integer coefficients, consistent with non-negativity conjecture.

If the non-negativity conjecture is true, then the following formula provides the structure constants of Schubert polynomials

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w,\ \ c_{uv}^w = \langle \text{coefficients of } w \text{ in } \mathfrak{S}_u(\theta)v \rangle.$$

where $\mathfrak{S}_u(\theta)v$ is defined by

$$x_{ij}v = \begin{cases} vs_{ij}, & \text{if } l(vs_{ij}) = l(v) + 1 \\ 0, & \text{otherwise} \end{cases}$$

Example A.3.1. *Let $u = (12)(3)$, $v = (23)(1)$, then $\mathfrak{S}_{(12)}\mathfrak{S}_{(23)} = \sum_w c_{(12)(23)}^w \mathfrak{S}_w$, where $c_{(12)(23)}^w$ is the coefficients of w in $\mathfrak{S}_u(\theta)v$. Then*

$$\mathfrak{S}_u(\theta)v = \mathfrak{S}_{(12)}(\theta)(23) = \theta_1(23) = (x_{12} + x_{13})(23) = x_{12}(23) + x_{13}(23) \tag{A.1}$$

However

$$x_{12}(23) = (23)s_{12} = (23)(12) = (132), \tag{A.2}$$

as $l((132)) = l\binom{123}{312} = 2 = l((23)) + 1$*, and*

$$x_{13}(23) = (23)s_{13} = (23)(13) = (123), \tag{A.3}$$

as $l((123)) = l\binom{123}{231} = 2 = l((23)) + 1$. *Substituting* (A.3) *and* (A.2) *into* (A.1) *we have*

$$\mathfrak{S}_{(12)}(\theta)(23) = (132) + (123)$$

So the coefficients of $w = (132)$ *and* $w = (123)$ *in* $\mathfrak{S}_{(12)}(\theta)(23)$ *are* 1 *and* 1*, so we have structure constants* $c^{(132)}_{(12)(23)} = c^{(123)}_{(12)(23)} = 1$ *(the rest are zero). Hence we have*

$$\mathfrak{S}_{(12)}\mathfrak{S}_{(23)} = \sum_{w} c^{w}_{(12)(23)}\mathfrak{S}_w = \mathfrak{S}_{(132)} + \mathfrak{S}_{(123)},$$

which is easily verified as the l.h.s. of the above equation is $\mathfrak{S}_{(12)}\mathfrak{S}_{(23)} = x_1(x_1 + x_2)$*, which equals to the r.h.s.* $\mathfrak{S}_{(132)} + \mathfrak{S}_{(123)} = x_1^2 + x_1x_2$.

Appendix B

Different degrees

B.1 Homogeneous ideal

Definition B.1.1. *Let polynomial* $p = \sum_{i=1}^{l} M_i$*, where* M_i*s are monomials of* P*. We define the projection of* p *on the degree* k *component of* I *by*

$$\prod_k (p) = \sum_{degM_i=k} M_i.$$

In particular if g *is a homogeneous polynomial of degree* k*, then we have*

$$\prod_k g = \begin{cases} g & \textit{if } deg(g) = k \\ 0 & \textit{otherwise} \end{cases}$$

Remark B.1.2. *The above definition applies to usual degree as well as to* S_n*-degree and set-partition degree.*

Definition B.1.3. *We call an ideal I homogeneous denoted $I = \oplus_k I^k$, if for any polynomial p, we have*

$$p \in I \Rightarrow \prod_k (p) \in I,$$

where $\prod_k(p)$ is the project of p on the degree k (usual, S_n, set-partition) component of I.

B.2 Ideal generated by homogeneous generators is homogeneous

B.2.1 Usual degree

To show that ideal generated by homogeneous generators is homogeneous we need to show that for polynomial p we have $p \in I \rightarrow \prod_k(p) \in I$, where $\prod_k(p)$ is the projection of p in component k of I. Let $p = \sum_i q_i g_i \in I$, where g_i are generators of I, then since

$$\prod_l (g_i) = \begin{cases} g_i & \text{if } l = deg(g_i) \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\prod_k (p) = \prod_k \sum_i (q_i g_i) = \sum_i \prod_k (q_i g_i) = \sum_i \prod_{k-d_i} (q_i) g_i \in I,$$

where $d_i = deg(g_i)$. In general we have

$$\prod_k (q_i g_i) = \begin{cases} \prod_{k-d_i}(q_i) g_i & \text{if } deg(g_i) = d_i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

B.2.2 S_n-degree

We need to show that $p \in I \to \prod_\sigma(p) \in I$, where p is a polynomial and $\prod_\sigma(p)$ is the projection of p in component I^σ of I. Let $p = \sum_i q_i g_i \in I$, then since

$$\prod_\sigma (g_i) = \begin{cases} g_i & \text{if } \sigma = S_n\text{-deg}(g_i) \\ 0 & \text{otherwise,} \end{cases}$$

and since S_n-degree$(MM') = (S_n$-degree$(M))(S_n$-degree$(M'))$ we have

$$\prod_\eta (p) = \prod_\eta \sum_i (q_i g_i) = \sum_i \prod_\eta (q_i g_i) = \sum_i \prod_{\eta \sigma_i^{-1}} (q_i) g_i \in I,$$

where $\sigma_i = S_n$-degree(g_i).

B.2.3 Set-partition degree

We need to show that $p \in I \to \prod_{\psi}(p) \in I$, where p is a polynomial and $\prod_{\psi}(p)$ is the projection of p in component I^{ψ} of I, where I^{ψ} be the set of elements of I with set partition degree ψ. Let $p = \sum_i q_i g_i \in I$, then since

$$\prod_{\phi}(g_i) = \begin{cases} g_i & \text{if } \phi = sp\text{-deg } (g_i) \\ 0 & \text{otherwise,} \end{cases}$$

and since the set partition degree of product of two monomials is the common coarsening of them, we have

$$\prod_{\psi}(p) = \prod_{\psi}\sum_{i}(q_i g_i) = \sum_{i}\prod_{\psi}(q_i g_i) = \sum_{i}\prod_{\xi}(q_i) g_i \in I,$$

where $\phi_i = sp\text{-deg}(g_i)$ and where ψ is the common coarsening of ϕ and ξ.

Appendix C

Different degrees on $FK(n)$

C.1 Usual degree is well defined on $FK(n)$

Definition C.1.1. *Usual degree is define on $FK(n)$ by the map $deg : FK(n) \to N$ defined by*

$$x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} + I \mapsto \begin{cases} 0 & \textit{if } x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} \in I \\ k & \textit{otherwise} \end{cases}$$

We need to check if the above definition is well-defined, i.e., for $A = x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} + I$ and $B = x_{t_1s_1}x_{t_2s_2}\cdots x_{t_ls_l} + I$, we need to show that

$$A = B \Rightarrow deg(A) = deg(B).$$

$$A = B \rightarrow x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} + I = x_{t_1s_1}x_{t_2s_2}\cdots x_{t_ls_l} + I$$
$$\rightarrow\ P = x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} - x_{t_1s_1}x_{t_2s_2}\cdots x_{t_ls_l} \in I = \bigoplus_{d\geq 0} I^d.$$

The last step in the above equation is valid because the ideal is homogeneous w.r.t. usual degree d (as the defining ideal of $FK(n)$ is generated by homogeneous polynomials). We then consider the following cases.

1. If $l \neq k$, then $\prod_k P = x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} \in I$ and $\prod_l P = x_{t_1s_1}x_{t_2s_2}\cdots x_{t_ls_l} \in I$, as I is homogeneous. So $degA = 0 = degB$, by definition.
2. If $l = k$, then considering $x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k} - x_{t_1s_1}x_{t_2s_2}\cdots x_{t_ls_l} \in I$ either both the monomials $x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k}$ and $x_{t_1s_1}x_{t_2s_2}\cdots x_{t_ls_l}$ lie in I or neither. If both are in I, again we have $degA = 0 = degB$ by definition. If neither are in I, then $degA = k = l = degB$, by definition.

Hence $A = B \implies degA = degB$. Therefore the above definition is well-defined.

Since the free algebra generated by the generators of FK and the ideal of $FK(n)$ are both homogeneous w.r.t. usual degree d, and we have a well defined degree for $FK(n)$, then the quotient of them is graded algebra w.r.t. this degree, i.e., we have

$$FK(n) = \bigoplus_{d\geq 0} FK^d(n), \tag{C.1}$$

where $FK^d(n)$ is the usual degree d component of $FK(n)$. The above decomposition is both a homogeneous decomposition and representation decomposition into S_n-invariant parts.

C.2 S_n-degree is well defined on $FK(n)$

Definition C.2.1. *We define S_n- degree on $FK(n)$ by the map $deg : FK(n) \to S_n$ defined by*

$$M + I \mapsto \begin{cases} \epsilon & \text{if } M \in I \\ \\ \sigma_M & \text{otherwise} \end{cases},$$

where σ_M for a monomial $M = x_{i_1j_1}x_{i_2j_2}\cdots x_{i_kj_k}$ is defined by

$$\sigma_M = (i_1j_1)(i_2j_2)\cdots(i_kj_k),$$

where (ij) is a transposition in S_n.

We need to check if the above definition is well-defined, i.e., if

$$\text{For } A = M + I \text{ and } B = M' + I,\ A = B \overset{?}{\to} S_n\text{-deg}(A) = S_n\text{-deg}(B).$$

It is well defined as:

$$A = B \ \to\ M + I = M' + I \ \to\ P = M - M' \in I = \bigoplus_{\sigma \in S_n} I^\sigma,$$

as ideal is homogeneous w.r.t. S_n-degree σ, because the relations generating the ideal are homogeneous w.r.t. S_n-degree. We then consider the following cases.

1. If $\sigma_M \neq \sigma_{M'}$, then considering $P = M - M' \in I = \bigoplus_{\sigma\in S_n} I^\sigma$, and using projection into homogeneous component we have $\prod_{\sigma_M} P = M \in I^{\sigma_M} \subset I$. Then by definition,

S_n-deg$(A) = \epsilon$. Also $\prod_{\sigma'_M} P = M' \in I^{\sigma_{M'}} \subset I$. Then by definition S_n-deg$(B) = \epsilon$. Then S_n-deg$(A) = \epsilon = S_n$-deg(B).

2. If $\sigma_M = \sigma_{M'}$, then considering $P = M - M' \in I = \bigoplus_{\sigma \in S_n} I^\sigma$ either M and M' are both in I or neither. If Both are in I then S_n-deg$(A) = \epsilon = S_n$-deg(B). If neither are in I, then S_n-deg$(A) = \sigma_M = \sigma_{M'} = S_n$-deg$(B)$.

Therefore the above definition is well-defined.

C.3 Set-partition deg is well defined on $FK(n)$

Definition C.3.1. *Set partition degree on $FK(n)$ is defined by map sp-deg:$FK(n) \to P(n)$ defined by*

$$M + I \mapsto \begin{cases} \emptyset & \text{if } M \in I \\ \text{sp-deg}(M) & \text{otherwise} \end{cases}$$

We need to show that the above definition is well defined. Let $A = M + I$ and $B = M' + I$. Then

$$A = B \to M + I = M' + I \to P = M - M' \in I = \bigoplus_{\phi} I^\phi,$$

as the ideal is homogeneous w.r.t. set partition degree, because the relations generating the ideal are homogeneous w.r.t. set partition degree. Here I^ϕ is the set of all elements of I with set partition degree ϕ. We then consider the following cases.

1. If sp-deg$(M) \neq$ sp-deg(M'), then considering $P = M - M' \in I = \bigoplus_\phi I^\phi$, and using projection into homogeneous component we will have

   $$\prod_\phi P = M \in I^\phi \subset I \to \ sp\text{-deg}(A) = 0.$$

   Also $\prod_{\phi'} P = M' \in I^{\phi'} \subset I \to$ sp-deg$(B) = \emptyset$, Hence sp-deg$(A) = \emptyset =$ sp-deg(B).

2. If sp-deg$(M) =$ sp-deg(M'), then considering $P = M - M' \in I = \bigoplus_\phi I^\phi$, either M and M' are both in I or neither. If both are in I, then sp-deg$(A) = \emptyset =$ sp-deg(B). If neither are in I, then sp-deg$(A) =$ sp-deg$(M) =$ sp-deg$(M') =$ sp-deg(B).

Therefore the above definition is well-defined.

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