Tracial Simplex of Every Unital C*-Algebra Is a Choquet Simplex

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Abstract

C^*-algebras are norm-closed self-adjoint subalgebras of bounded linear operators on a complex Hilbert space. Choquet simplex is a special type of compact convex set with a unique representation property. The goal of this thesis is to find a self-contained and easily accessible proof of the classical fact that the set of tracial states of a unital C^*-algebra is a Choquet simplex.

Contents

Chapter 1

Introduction

In this thesis, we are trying to find a self-contained and easily accessible proof that the set of tracial states in a unital C*-algebra is a Choquet simplex. The tracial simplex $\mathbf{T}(A)$ is an important invariant for a C*-algebra. After showing that $\mathbf{T}(A)$ is a Choquet simplex, the tracial decomposition follows.

With the help of Choquet theory on real locally convex linear space (about the Choquet order), one can prove the Ergodic Decomposition Theorem for a general G-Abelian system [5]. However, in the most important case, one may just consider the system of the unitary group of a C*-algebra A acting on A by conjugation. In this thesis, we will show the tracial simplex $\mathbf{T}(A)$ is a Choquet simplex. Furthermore, we will explain why this result is significant, particularly in relation to Elliott's invariant for a unital C*-algebra. Elliott's Program encompasses several conjectures that are expected to play essential roles in the classification of specific types of C*-algebras [8].

Chapter 2

Background of the Theory of Operator Algebras and the Choquet Simplex

This section provides the basics of the theory of operator algebras and the Choquet theory related to the Choquet simplex. It aims to give the reader the fundamental background of the two theories, although the proof of almost every result in this section will be omitted.

2.1 C*-algebras and von Neumann algebras

2.1.1 C*-algebras

Definition 2.1.1 (Complex Hilbert Space)
A Hilbert space is a vector space H over complex number $\mathbb{C}$, together with a form: $(-|-) : H \times H \longrightarrow \mathbb{C}$*, such that for all v, ξ, η and ζ in H and λ, λ' in $\mathbb{C}$:*

- $(\lambda\xi + \lambda'\xi|\eta) = \lambda(\xi|\eta) + \lambda'(\xi|\eta)$
- $(\xi|\eta) = \overline{(\eta|\xi)}$
- $(\xi|\xi) \geq 0$
- $(\xi|\xi) = 0$ *if and only if* $\xi = \mathbf{0}$ *in* H.

Define the norm of ξ in H

$$\|\xi\|_2 := \sqrt{(\xi|\xi)}$$

The vector space H is required to be complete with respect to this norm.

Throughout this thesis, unless otherwise stated, all Hilbert spaces are assumed to be complex Hilbert spaces. To begin the introduction of operator algebras, we first need to define *Banach algebras*.

Definition 2.1.2 (Banach algebras)
Let A be a complete, complex normed space (i.e., a Banach space) equipped with a multiplication operation on A that satisfies the following properties for all $x, y, z \in A$ and $\lambda \in \mathbb{C}$:

- $x(yz) = (xy)z$
- $(x + y)z = xz + yz$
- $z(x + y) = zx + zy$
- $\lambda(xy) = (\lambda x)y = x(\lambda y)$

The norm on A is required to satisfy the inequality:

$$\|xy\| \leq \|x\| \ \|y\|$$

We call such a space a Banach algebra. If there is an element $\mathbf{1}_A$ (or just $\mathbf{1}$ when A is clear) in A, such that for all x:

$$\mathbf{1}x = x\mathbf{1} = x, \ \|\mathbf{1}\| = 1$$

then we call this element the identity of A, and A is called a unital Banach algebra.

When the norm structure of a Banach algebra is ignored, it is simply referred to as a *complex algebra* or an *algebra*.

An important class of Banach algebras is the algebra of bounded linear operators on a Hilbert space.

Definition 2.1.3 (Bounded Linear Operators)
Let H be a Hilbert space. A linear operator $t : H \longrightarrow H$ is bounded if

$$\|t\|_\infty := \sup_{\|\xi\|_2 = 1} \|t\xi\|_2 < \infty$$

The set of all bounded linear operators on H forms a Banach algebra with norm $\|\cdot\|_\infty$. The multiplication of operators is defined to be the composition of operators. This algebra is denoted as $\mathscr{B}(H)$.

There is an important operator $^* : \mathscr{B}(H) \longrightarrow \mathscr{B}(H)$, which is called involution, defined by the equation:

$$(t\xi|\eta) = (\xi|t^*\eta)$$

With this definition, there are 4 properties related to it:

1. $(\lambda t + \lambda' r)^* = \bar{\lambda} t^* + \bar{\lambda}' r^*$.
2. $(t^*)^* = t$.
3. $(tr)^* = r^* t^*$.
4. $\|t^* t\| = \|t\|^2$.

The 4th property is known as the C*-identity. The Banach algebra $\mathscr{B}(H)$, with the involution, forms a C*-algebra. To be more precise, an abstract C*-algebra can be defined in the following way:

Definition 2.1.4 (C*-algebra and its Dual)
A C-algebra is a complex Banach algebra A with a operator $^* : A \longrightarrow A$ satisfying:*

1. $(\lambda a + \lambda' b)^* = \bar{\lambda} a^* + \bar{\lambda}' b^*$.
2. $(a^*)^* = a$.
3. $(ab)^* = b^* a^*$.
4. $\|a^* a\| = \|a\|^2$.

For all $\lambda, \ \lambda' \in \mathbb{C}$ and $a, \ b \in A$. We say A is unital if it is also a unital Banach algebra.

The dual space of A is denoted as $A^\sharp$, containing all bounded linear functionals (a linear functional is a linear map $\rho : \ A \longrightarrow \mathbb{C}$) on A.

Again, when the norm structure of a C*-algebra is ignored, it is referred to as a *-algebra.

The proper maps between C*-algebras are defined as follows:

Definition 2.1.5 (*-homomorphisms and *-isomorphisms)
Let A and B be two unital C-algebras. A linear map $\varphi : \ A \longrightarrow B$ is a *-homomorphism if it is multiplicative, carries $\mathbf{1}_A$ to $\mathbf{1}_B$, and $\varphi(a^*) = \varphi(a)^*$ for all a in A. If a *-homomorphism is injective, then we call it a *-isomorphism.*

In many other branches of mathematics, an isomorphism is a bijective map that preserves the structure of the objects being studied. However, in the context of C*-algebras, a *-isomorphism is an injective map between C*-algebras. We will use the term **-isomorphic* to describe the relation of two C*-algebras, such that there exists a surjective *-isomorphism between them. Additionally, it can be shown that any *-homomorphism is a bounded linear map with the supremum norm at most 1. In particular, all *-isomorphisms are isometries. For more details on the terminology, please check [2, Page 237].

A key point for C*-algebra that every C*-algebra is *-isomorphic to a norm closed C*-subalgebra of $\mathscr{B}(H)$. The technique used here is called the *Gelfand–Naimark–Segal (GNS) Construction*. This tool helps us isometrically embed the C*-algebra into $\mathscr{B}(H)$. This construction uses linear functionals of a special kind, therefore we need to define the positivity in C*-algebras.

Recall the positive operators in $\mathscr{B}(H)$:

Definition 2.1.6 (Positive Operators in $\mathscr{B}(H)$)
An operator t in $\mathscr{B}(H)$ for some Hilbert space H is positive if for all ξ in H, we have

$$\omega_\xi(t) := (t\xi|\xi) \geq 0$$

Definition 2.1.7 (Spectrum)
In a unital C-algebra A, the spectrum of an element a is the set*

$$Sp_A(a) = \{\lambda \in \mathbb{C} | \lambda \mathbf{1} - a \text{ is not invertible}\}$$

One can prove the following in $\mathscr{B}(H)$ [2, Theorem 4.2.6]:

$$t \text{ is positive} \iff t = t^* \text{ and } Sp_A(t) \subseteq [0, +\infty)$$

This allows us to define positivity and other classes of elements in an abstract C*-algebra:

Definition 2.1.8 (Classes of Elements in an Abstract C*-algebra A)
In a unital C-algebra A,*

- *An element h is self-adjoint if $h = h^*$. A_{sa} denotes the set of all self-adjoint elements.*
- *A self-adjoint element h is positive if $Sp_A(h) \subseteq [0, +\infty)$. We say self-adjoint elements $a_1 \leq a_2$ if $a_2 - a_1$ is positive. A_+ denotes the set of all positive elements.*

- *A element u is called unitary if $uu^* = u^*u = \mathbf{1}$. All unitary elements in A form a multiplicative group denoted as $\mathbf{U}(A)$.*

Note that in any *-algebra A, the self-adjoint elements in A form a real vector space. It can be checked easily by the definition of the involution operation.

We come back to the definition of the functional $\omega_\xi(t) := (t\xi|\xi)$ on $\mathscr{B}(H)$. Particularly, it satisfies the following two properties:

- For all positive t, $\omega_\xi(t) \geq 0$
- $\omega_\xi(t) := (t\xi|\xi) = \overline{(\xi|t\xi)} = \overline{(t^*\xi|\xi)} = \overline{\omega_\xi(t^*)}$

We can define the positivity in the dual of a unital C*-algebra A as follows:

Definition 2.1.9 (Different Types of Linear Functionals)
Suppose ρ is a linear functional on a unital C*-algebra A, then:

- *ρ is called hermitian or self-adjoint if $\rho(a) = \rho^*(a) := \overline{\rho(a^*)}$.*
- *ρ is positive if for all $a \in A$ positive, $\rho(a) \geq 0$. We say $\rho_1 \leq \rho_2$ as self-adjoint bounded linear functionals if $\rho_2 - \rho_1$ is positive. Denote the set of positive linear functionals on A as $A^\sharp_+$ in $A^\sharp$.*
- *A positive linear functional ρ on a unital A is a state if $\rho(\mathbf{1}) = 1$. The set of all states, known as the state space, is denoted as $\mathbf{S}(A)$.*
- *A state ρ is called faithful on A, if for any a in A, $\rho(a^*a) = 0$ implies $a = 0$ in A.*
- *A tracial state is a state such that for all a, b in A, $\rho(ab) = \rho(ba)$. The set of all tracial states is denoted as $\mathbf{T}(A)$ in $\mathbf{S}(A)$.*

By function calculus on a unital C*-algebra, one can represent any element in A to a linear combination of at most four unitaries (see [2, Theorem 4.1.7]). Therefore, an equivalent definition for a state to be tracial is:

Let τ be a state, then it is tracial if for all a in A and u in $\mathbf{U}(A)$, $\tau(a) = \tau(uau^*)$.

For any a in A, the evaluation functional on the dual of A with respect to a is defined in the following way:

$$\hat{a}(\rho) := \rho(a)$$

which is an element in the second dual of A. Notice $\hat{a}$ is weak*-continuous on $A^\sharp$.

There is another property worth mentioning:

Proposition 2.1.10 (Equivalent Condition of Positive Linear Functionals, [2, Theorem 4.3.2])
ρ is a linear functional on A, then:

$$\rho \text{ is positive} \iff \rho \text{ is bounded, and } \|\rho\| = \rho(\mathbf{1})$$

Therefore, all the inclusive relations of sets in Definition 2.1.9 make sense.

Moreover, $\mathbf{S}(A)$ and $\mathbf{T}(A)$ (if it is not empty) are two convex, weak*-compact subsets of $A^\sharp$ if A is unital. The weak*-compactness of them can be checked in the following way, note:

$$\mathbf{S}(A) = \bigcap_{a \in A_+} \{\rho \in A^\sharp |\ \hat{\mathbf{1}}(\rho) = 1,\ \hat{a}(\rho) \geq 0\}$$

and

$$\mathbf{T}(A) = \bigcap_{a,b \in A} \{\rho \in \mathbf{S}(A) | (\hat{ab} - \hat{ba})(\rho) = 0\}$$

Since every set in the intersection is a weak*-closed set, and $\mathbf{S}(A)$ and $\mathbf{T}(A)$ are subset of a weak*-compact set, the closed unit ball of $A^\sharp$ (following from the Banach–Alaoglu Theorem), therefore the compactness for them follows.

Note that by the Krein–Milman Theorem, we know that the set $\mathbf{P}(A) = \partial\mathbf{S}(A)$ of extreme points in $\mathbf{S}(A)$ contains "many" states. To be more precise, the weak*-closure of the convex hull of $\mathbf{P}(A)$ is $\mathbf{S}(A)$, that is $\overline{\text{co}}(\mathbf{P}(A)) = \mathbf{S}(A)$. Similarly, $\overline{\text{co}}(\partial\mathbf{T}(A)) = \mathbf{T}(A)$ if $\mathbf{T}(A)$ is not empty.

One more thing to be mentioned is the subset of all bounded, tracial, hermitian functionals on a unital C*-algebra A forms a real Banach space, which will be closely related to the proof in the last section.

Now we introduce the *Gelfand–Naimark–Segal (GNS) Construction*. There is an important proposition worth mentioning for a state:

Proposition 2.1.11 (Cauchy-Schwartz Inequality and the Left Kernel, [2, Proposition 4.3.1 and Proposition 4.5.1])
Let ρ be a state for a unital C*-*algebra A, then:*

- $|\rho(b^*a)| \le \rho(a^*a)\rho(b^*b)$, *for all* a, b *in* A.
- *Let*

$$\mathscr{L}_\rho := \{a \in A | \rho(a^*a) = 0\}$$

  then $\mathscr{L}_\rho$ *is a closed left ideal.*

Let

$$a_\rho := a + \mathscr{L}_\rho$$

then we can have that $A/\mathscr{L}_\rho$ with $(a_\rho|b_\rho) := \rho(b^*a)$ (It can be checked that is well-defined!) forms a *pre-Hilbert space*, which is a linear space with an inner product. The completion of the space $A/\mathscr{L}_\rho$ is the Hilbert Space H_ρ we want. For all a in A, we can difeine:

$$\pi_\rho(a)(b + \mathscr{L}_\rho) := ab + \mathscr{L}_\rho$$

This operator has a unique linear extension on H_ρ. Therefore, we have a GNS construction triplets $(\pi_\rho, H_\rho, \mathbf{1}_\rho)$ associated with a tracial state ρ. The *-homomorphism π_ρ maps A to $\mathscr{B}(H_\rho)$. More than that, if we have

$$H := \sum_{\mathbf{P}(A)} \bigoplus H_\rho$$

and take $\pi := \sum_{\mathbf{P}(A)} \bigoplus \pi_\rho$, then π will be a *-isomorphism from A to $\mathscr{B}(H)$, therefore an isometry (The full argument can be found in [2, Section 4.5]).

The last thing about the C*-algebra is the following representation theorem:

Theorem 2.1.12 (The Function Representation of Commutative C*-algrbas, [2, Theorem 4.4.3])
Any unital commutative C**-algebra is* **-isomorphic to the space of complex continuous functions on a compact Hausdorff space* X *equiped with the supremum norm and the involution being the conjugation of a function.*

This representation theorem says that for any unital commutative C*-algebra, it is just a complex continuous function space on some compact Hausdorff space X. In this representation, positive elements are mapped to positive functions, and self-adjoint elements are mapped to real-valued functions.

2.1.2 von Neumann Algebras

First, we introduce two important topologies on $\mathscr{B}(H)$.

Definition 2.1.13
There are two locally convex topologies defined on $\mathscr{B}(H)$:

- *The **Strong Operator Topology (SOT)** is the topology generated by the following family of separating seminorms: $\{\|\cdot\xi\|_2\}_{\xi\in H}$, where $\|\cdot\xi\|_2(t) := \|t\xi\|_2$ for a fixed ξ in H.*
- *The **Weak Operator Topology (WOT)** is the topology generated by the following family of separating seminorms: $\{|(\cdot\xi|\eta)|\}_{\xi,\eta\in H}$, where $|(\cdot\xi|\eta)|(t) := |(t\xi|\eta)|$ for fixed ξ and η in H.*

The comparison of the three topologies on $\mathscr{B}(H)$ is:

$$\text{Operator norm topology} \geq \text{SOT} \geq \text{WOT}$$

where "$\geq$" above means "stronger than".

Definition 2.1.14 (von Neumann Algebras)
Consider a complex Hilbert space H, and the bounded operator algebra $\mathscr{B}(H)$. A *-subalgebra M including $\mathbf{1}_H$ of $\mathscr{B}(H)$ is a von Neumann algebra if it is WOT closed.

Note that the WOT and the SOT closure of a convex set can be proved equal in $\mathscr{B}(H)$ (see [2, Theorem 5.1.2]).

There are some fundamental results we will use in this thesis about von Neumann algebra:

Theorem 2.1.15 (The Double Commutant Theorem, [2, Theorem 5.3.1])
Let H be a complex Hilbert space. If M is a *-subalgebra of $\mathscr{B}(H)$ containing $\mathbf{1}_H$, then the WOT closure of M is equal to the SOT closure of M and equal to the bicommutant of M. The commutant of M is defined to be

$$M' := \{r \in \mathscr{B}(H) \mid rt = tr,\ t \in M\}$$

and the bicommutant of M is $M'' := (M')'$.

One remark should be made in the proof of the Double Commutant Theorem: for any self-adjoint set S in $\mathscr{B}(H)$ (for every x in S, x^* is in S), S' is a von Neumann algebra.

We also need concepts related to projections in $\mathscr{B}(H)$:

Proposition 2.1.16 (Orthogonal Decomposition of a Hilbert space)
Let $K \subset H$ be a closed linear subspace, then the set $K^{\perp} := \{\xi \in H | (\xi|\eta) = 0,\ \eta \in K\}$ is a closed linear subspace of H, where $H = K \bigoplus K^{\perp}$.

By $H = K \bigoplus K^{\perp}$, any vector in H can be uniquely orthogonally decomposed to the sum of two components (vectors) in K and $K^{\perp}$. Then we have the definition of a projection with respect to a subspace K:

Definition 2.1.17 (Orthogonal Projections)
Given a closed linear subspace K of H, the orthogonal projection, or just projection, p from H onto K is the map assigning every vector to the K-component in the orthogonal decomposition corresponding to K and $K^{\perp}$.

Notice every projection p is linear, bounded, positive (therefore self-adjoint), and $p^2 = p$. The image of any vector ξ under the projection p is the vector in the subspace K with the minimum distance to ξ.

The following proposition gives some equivalent conditions of the order on the set of projections:

Proposition 2.1.18 (Equivalent Conditions of Comparing Projections, [2, Proposition 2.5.2])
Let p and p' be projections from a Hilbert space H into closed subspaces K and K'. Then the following conditions are equivalent:

1. $p \leq p'$ ($p' - p$ *is positive*).
2. $K \subseteq K'$.
3. $pp' = p$.
4. $p'p = p$.

From the definition of positive operators in $\mathscr{B}(H)$, it follows that ordering of projections (as self-adjoint operators) corresponds to the ordering of closed subspaces by the inclusion relation $\subseteq$.

Therefore, considering a family of projections $\{p_\alpha\}_{\mathbb{A}}$ on H, it is corresponding to a family of closed subspaces $\{K_\alpha\}_{\mathbb{A}}$ of H (p_α takes K_α as its range). Both the supremum and the infimum of $\{p_\alpha\}_{\mathbb{A}}$ exist (as projections). To be precise, the range of $\bigvee_{\mathbb{A}} p_\alpha$ is:

$$\overline{\text{span}}\{\bigcup_{\mathbb{A}} K_\alpha\}$$

and the range of the $\bigwedge_{\mathbb{A}} p_\alpha$ is:

$$\bigcap_{\mathbb{A}} K_\alpha$$

The above result can be found in [2, Section 2.5].

Now we can define the central support of a projection in some von Neumann algebra M

Definition 2.1.19 (Central Support of a Projection)
Let p be a projection in a von Neumann algebra M on a Hilbert space H. The central support of p in M is the projection $\bigwedge_{\mathbb{A}} z_\alpha$, where $\{z_\alpha\}_{\mathbb{A}}$ is the family of projections in the center of M (denoted as $\mathbf{Z}(M) := M \bigcap M'$) such that each $z_\alpha \geq p$.

The last concept related to the orthogonal decomposition of H is the concept of reducing subspaces of a subset S of $\mathscr{B}(H)$.

Definition 2.1.20 (Reducing Subspaces of a Subset of $\mathscr{B}(H)$)
Let S be a subset of $\mathscr{B}(H)$, then a closed subspace K of H is reducing for S if for all s in S, $sK := \{s\xi | \xi \in K\} \subset K$ and $sK^\perp \subset K^\perp$. Denote $SK := \{s\xi | \xi \in K,\ s \in S\}$.

There is a useful lemma about reducing subspaces for a *-algebra, we will recall that lemma when we are going to use it.

2.2 Choquet Theory Related to Choquet Simplex

We are not going to recall the entire Choquet's theory here. Instead, we are going to give the definition of the Choquet simplex and state his theorem about equivalent the condition for being a Choquet simplex in metrizable cases. We start with defining some related types of sets in a real vector space.

Definition 2.2.1 (Generating Cones and Bases)
In a real vector space V, a generating cone is a set P in V satisfying the following properties:

- *If x and $-x$ are both in P, then $x = \mathbf{0}$.*
- *If $\lambda \geq 0$ and x in P, then λx is in P.*

- *If x and y are in P, then $x + y$ is in P.*
- $V = P - P$

A base of a generating cone is a subset C of P such that:

- $\mathbf{0}$ *is not in C,*
- *C is convex, and*
- *For any $y \neq \mathbf{0}$ in C, there is a unique pair (λ, x) in $(0, +\infty) \times C$ where $y = \lambda x$.*

It can be checked that A_+ and $A^\sharp_+$ are generating cones for the real subspaces consisting of self-adjoint elements in A and self-adjoint functionals in $A^\sharp$, and $\mathbf{S}(A)$ is a base of $A^\sharp_+$.

Given the existence of a generating cone P on V, one may define a partial order on V by the following:

Definition 2.2.2 (Vector Order induced by a Cone)
Let x, y are in V, $x \leq y$ if $y - x$ is in P.

One can check this definition gives a partial order on V. There are three obvious observations:

- $\mathbf{0} \leq x$ if and only if x is in P.
- For all $x \leq y$ in V and z in V,
$$x + z \leq y + z$$
- For any $x \leq y$ in V and λ a positive number, $\lambda x \leq \lambda y$.

Sometimes we call the generating cone in the order defined above the positive cone.

On the self-adjoint portions of a unital C*-algebra A and the dual $A^\sharp$, the usual orders are exactly induced by A_+ and $A^\sharp_+$.

Definition 2.2.3 (Vector Lattice)
Suppose we have a real vector space with a vector order induced by a positive cone P, such that both $x \wedge y := \sup\{z \in V \mid z \leq x,\ z \leq y\}$ and $x \vee y := \inf\{z \in V \mid z \geq x,\ z \geq y\}$ exists in V for all x and y. Then we say V with the order $\leq$ forms a vector lattice.

Now we are ready to define the Choquet simplex.

Definition 2.2.4 (Choquet simplex)
In a real locally convex vector space V, a convex and compact set C is a Choquet simplex

if there this a vector order induced by a positive cone P which takes C as a base, and the vector space V with this order forms a vector lattice.

To state Choquet's theorem, the concept of barycenter needs to be defined:

Definition 2.2.5 (Probability Radon Measures on Compact Convex Set)
Let X be a compact convex set in a real locally convex space V. Let

$$M_{+,1}(X) := \{\mu \in M(X) \mid \mu \geq 0, \|\mu\| = 1\}$$

where $M(X)$ consists of all signed Radon measures on the Borel σ-algebra of X. $\|\mu\|$ is the total variation of μ.

Definition 2.2.6 (Barycenter for a Measure)
Let μ in $M_{+,1}(X)$ as above. The barycenter of μ is a point x in X such that for every continuous linear functionals f, $\int_X f d\mu = f(x)$.

An essential result of Choquet simplex is the following:

Theorem 2.2.7 (Choquet's Theorem, [6, Theorem 11.12])
For a metrizable, compact and convex set C in a real locally convex vector space V, the following are equivalent:

- *C is a Choquet simplex*
- *For any x in C, there is a unique probability Radon measure μ on C, such that x is the barycenter of μ, and $\mu(\partial C) = 1$.*

For metrizable Choquet simplices, they form a fairly complicated category, and all of them are realized as $\mathbf{T}(A)$ for a separable unital C*-algebra A [7, Theorem 3.10].

Now we give an example of Choquet simplex in the finite-dimensional case. Choquet simplices in such cases are just the n-simplices defined in the common way:

$$\Delta_n := \{x \in \mathbb{R}^n \mid 0 \leq x_j \leq 1,\ \forall 0 \leq j \leq n,\ \sum_{j=1}^{n} x_j = 1\}$$

It can be shown Δ_n's are the only Choquet simplices in the finite dimensional case (see [6, Theorem 11.1]). We can check they are Choquet simplices from the two equivalent conditions in Theorem 2.2.7.

On one hand, the positive cone defined by Δ_n is simply the set $\{x \in \mathbb{R}^n \mid 0 \leq x_j,\ 1 \leq j \leq n\}$. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the positive cones are just 1^{st} quadrant and 1^{st} octant. For any x in $\mathbb{R}^n$, the set $\{z \in \mathbb{R}^n \mid x \leq z\}$ is just obtained by shifting the positive cone from $\mathbf{0}$ to x. For any x and y in $\mathbb{R}^n$, the intersection of $\{z \in \mathbb{R}^n \mid x \leq z\}$ and $\{z \in \mathbb{R}^n \mid y \leq z\}$ is also a cone shifted by the positive cone, with the vertex of this intersection being the element $x \vee y$. Similarly, one can find $x \wedge y$ in $\mathbb{R}^n$.

On the other hand, proven by induction on n, we know that every point x in Δ_n is the barycenter of a unique μ on $M_{+,1}(\partial\Delta_n)$, where $\partial\Delta_n = \{e_j \mid 0 \leq j \leq n\}$ (the standard base for $\mathbb{R}^n$). This is because any μ in $M_{+,1}(\partial\Delta_n)$ has the form $\sum_{j=1}^n \lambda_j \delta_{e_j}$, where $\sum_{j=1}^n \lambda_j = 1$ and $\lambda_j \geq 0$. We can perform the easy calculation below, let x be the barycenter for $\mu \in M_{+,1}(\partial\Delta_n)$:

$$\sigma(x) = \int_{\{e_j|0\leq j\leq n\}} \sigma(y)\ d\mu(y) = \sum_{j=1}^{n} \mu(\{e_j\})\sigma(e_j) = \sigma(\sum_{j=1}^{n} \mu(\{e_j\})e_j)$$

for all σ in $(\mathbb{R}^n)^\sharp$, so $x = \sum_{j=1}^n \mu(\{e_j\})e_j$.

Chapter 3

Notations Will Be Used in Following Sections

Definition 3.0.1 (Notations)
In this thesis, the following notations are frequently used.

- *In this thesis, H will always denote a complex Hilbert space, A will always denote a unital* C*-algebra, *while τ denotes a tracial state on A. The element $\mathbf{1}$ denotes the multiplicative unit in A.*

- *Letters like let a, b and c denote elements in A. The notation $\|\cdot\|$ denotes the norm on A and the norm on $A^\sharp$. Recall $A^\sharp$ is the Banach space dual of A, and the norm is defined to be the operator norm.*

- *The map $\pi_\tau(\cdot)$ is the representation in GNS construction associated with the tracial state τ, and H_τ is the Hilbert space under this construction. Recall $\pi_\tau(\cdot)$ is a *-homomorphism from A to $\mathscr{B}(H_\tau)$. $\|\cdot\|_\infty$ denotes the operator norm in $\mathscr{B}(H_\tau)$. Note that the range of π_τ is denoted by $\pi_\tau[A]$.*

- *$\mathscr{L}_\tau := \{a \in A| \ \tau(a^*a) = 0\}$ is the left kernel of τ. Since τ is tracial, $\mathscr{L}_\tau$ is also an closed ideal of A, which means $A/\mathscr{L}_\tau$ is a* C*-algebra. *Recall $a_\tau := a + \mathscr{L}_\tau$ denotes a vector in H_τ. The set of all vectors formed in this way is denoted by A_τ. Notice the vector $\mathbf{1}_\tau$ is cyclic in the GNS representation by τ. By being cyclic, we mean the norm closure of $\pi_\tau[A]\mathbf{1}_\tau$ is H_τ.*

- *Let u be in $\mathbf{U}(A)$. Let $c_\tau(u)$ be the unique extension of the linear map*

$$a_\tau \mapsto (uau^*)_\tau \tag{3.1}$$

in $\mathscr{B}(H_\tau)$*. If we consider* c_τ *as a map, the range of it is denoted by* $c_\tau[\mathbf{U}(A)]$.

- *Denote the closed invariant subspace of* H_τ *under the action of* $c_\tau[\mathbf{U}(A)]$ *as* E_τ*. That is:*

$$E_\tau = \{\xi \in H_\tau \mid c_\tau(u)\xi = \xi,\ u \in \mathbf{U}(A)\} \tag{3.2}$$

$E_\tau = \{\xi \in H_\tau \mid c_\tau(u)\xi = \xi,\ u \in \mathbf{U}(A)\}$*. The projection from* H_τ *into this space is denoted as* p_τ.

- *The lowercase letters like* x, y, r *and* t *denote the operators on some Hilbert space.*

- *Define:*

$$M_\tau = \pi_\tau[A]'' \tag{3.3}$$

Note that M_τ *is a von Neumann algebra in* $\mathscr{B}(H_\tau)$ *by the Double Commutant Theorem, and it is the SOT closure of the* C^**- subalgebra* $\pi_\tau[A]$.

- *The notations* $\mathbf{U}(M_\tau)$ *and* $\mathbf{U}(A)$ *denote the unitary groups of* M_τ *and* A.

- $\mathbf{Z}(M_\tau)$ *denotes the center of* M_τ.

- *If* K *is a subset of a normed space,* K_1 *or* $(K)_1$ *denotes the closed unit ball in* K *corresponding to the norm. The number 1 can be changed to any other positive number as the radius of the closed ball.*

- *Let* S *be a subset of a locally convex space* V*, then the convex hull of* S*,* $\operatorname{co} S$*, is defined to be:*

$$\operatorname{co} S := \{\sum_{j=1}^{n} \lambda_j s_j \mid \forall 0 \le j \le n,\ s_j \in K,\ \lambda_j \ge 0 \text{ and } \sum_{j=1}^{n} \lambda_j = 1,\ n \in \mathbb{N}\}$$

The close convex hull of S *will be denoted as* $\overline{\operatorname{co}}\ S$*, which is defined to be the closure of* $\operatorname{co} S$*. Notice both* $\operatorname{co} S$ *and* $\overline{\operatorname{co}}\ S$ *are convex.*

Chapter 4

The System $\{A, \mathbf{U}(A)\}$ Is Abelian

In this section, we show one of the fundamental results of the action of $\mathbf{U}(A)$ on A. There is a family of operators in the GNS construction associated with τ which is commutative. This result will be essential to help us prove the set of tracial states on A forming a Choquet simplex. The argument will be established through several steps in the following sections.

4.1 The Actions of $\mathbf{U}(A)$ on A and on $\pi_\tau[A]$

To start our proof, let us clarify the definition of the action of $\mathbf{U}(A)$ on A:

Definition 4.1.1 (The Action of $\mathbf{U}(A)$ on A and its Invariant States)
The action of $\mathbf{U}(A)$ *on* A *is given by* $(a, u) \mapsto \operatorname{Ad} u(a) = uau^*$. *The states which are invariant under this action are precisely the tracial states.*

Note that for all u in $\mathbf{U}(A)$, $a \mapsto \operatorname{Ad} u(a)$ is a *-automorphism on A. Any *-automorphism is an isometry from A onto A, since it is a *-isomorphism. For continuous functionals on A, we may define the action of $\mathbf{U}(A)$ on $A^\sharp$ by $\operatorname{Ad} u(f)(a) \coloneqq f(uau^*)$. $\operatorname{Ad} u(f)$ is positive if and only if f is positive, and $\operatorname{Ad} u(f)$ has the same norm as f.

There are two essential properties of this action, the first one is:

Proposition 4.1.2
For a unital C^*- *algebra, we have the following properties for the map* c_τ (3.1)*:*

1. *The map:*

$$c_\tau :\ \mathbf{U}(A) \longrightarrow \mathbf{U}(\mathscr{B}(H_\tau))$$

is a unitary representation with additional condition $c_\tau(u^*) = c_\tau(u)^*$ for all u in $\mathbf{U}(A)$.

2. *For any u in $\mathbf{U}(A)$ and a in A, we have:*

$$c_\tau(u)\pi_\tau(a)c_\tau(u^*)b_\tau = \pi_\tau(\mathrm{Ad}\, u(a))b_\tau$$

Proof. For any u in $\mathbf{U}(A)$, we have

- For all a, b in A

$$(c_\tau(u)a_\tau|b_\tau) = \tau(b^*uau^*) = \tau(u^*b^*uau^*u) = \tau((u^*bu)^*a) = (a_\tau|c_\tau(u^*)b_\tau)$$

  This equation shows $c_\tau(u^*) = c_\tau(u)^*$.

- Easy to check $c_\tau(u_1u_2) = c_\tau(u_1)c_\tau(u_2)$.
- $c_\tau(u^*)c_\tau(u)b_\tau = (u^*ubu^*u)_\tau = (uu^*buu^*)_\tau = c_\tau(u)c_\tau(u^*)b_\tau = b_\tau$. Therefore $c_\tau(u)$ is a unitary operator in $\mathscr{B}(H_\tau)$.

Therefore $u \mapsto c_\tau(u)$ is a unitary representation for $\mathbf{U}(A)$.

For the second property, we have

$$c_\tau(u)\pi_\tau(a)c_\tau(u^*)b_\tau = (uau^*buu^*)_\tau = \pi_\tau(uau^*)b_\tau = \pi_\tau(\mathrm{Ad}\, u(a))b_\tau$$

□

4.2 The 2–norm on M

The goal for this section is to show on M_τ (3.3), there is another norm that can be defined with respect to the vector $\mathbf{1}_\tau$. The first thing to be checked is that $\omega_{\mathbf{1}_\tau}(x) := (x\mathbf{1}_\tau|\mathbf{1}_\tau)$ is induced by the state τ on A. By "induced", we mean:

$$\omega_{\mathbf{1}_\tau} \circ \pi_\tau = \tau$$

This functional also can be written as $\tilde{\tau}$ because of this relation. The proof is trivial by checking the definition of the GNS construction. Additionally, the SOT-continuous for this state is also essential.

Proposition 4.2.1

On $\mathscr{B}(H_\tau)$, ω_ξ is SOT continuous, for all ξ in H.

Proof. Let $t_\lambda \longrightarrow t$ with SOT topology in M_τ. We have $|((t_\lambda - t)\xi|\xi)| \le \|(t_\lambda - t)\xi\|_2\|\xi\|_2$. So as $t_\lambda \longrightarrow t$ in SOT topology, $\|(t_\lambda - t)\xi\|_2 \longrightarrow 0$. Hence $(t_\lambda\xi|\xi) \longrightarrow (t\xi|\xi)$. □

Recall that we aim to check the seminorm $\|\cdot\mathbf{1}_\tau\|_2$ on M_τ(3.3) to be a norm. It is sufficient to show the state $\tilde{\tau}$ is faithful on M_τ.

Before proving this result, we need to mention three results related to some type of elements in A. The references for them are noted after them respectively.

Proposition 4.2.2
Let A be a unital C*-*algebra.*

- *If a is self-adjoint, then $a \le \|a\|\mathbf{1}$ (see [2, Proposition 4.2.3]).*
- *For any a in A, aa^* is positive (see [2, Theorem 4.2.6]).*
- *If a is positive, for any b in A, b^*ab is positive (see [2, Corollary 4.2.7]).*

Recall that we have the definition of a faithful state from Definition 2.1.9.

Proposition 4.2.3
The state $\tilde{\tau}$ is faithful on M_τ.

Proof. Let t in M_τ, and assume that $(t^*t\mathbf{1}_\tau|\mathbf{1}_\tau) = (t\mathbf{1}_\tau|t\mathbf{1}_\tau) = \|t\mathbf{1}_\tau\|_2^2 = 0$. Since M_τ is SOT closure of $\pi_\tau[A]$, suppose there is a net $\{\pi_\tau(a_\lambda)\}_\Lambda$, such that $\pi_\tau(a_\lambda)\xrightarrow[\lambda]{} t$ in SOT. Then, we have for all b in A, $\|\pi_\tau(a_\lambda)b_\tau\|_2 \xrightarrow[\lambda]{} \|tb_\tau\|_2$. But for $\|\pi_\tau(a_\lambda)b_\tau\|_2^2 = \|(a_\lambda b)_\tau\|_2^2 = \tau(b^*a_\lambda^*a_\lambda b) = \tau(a_\lambda bb^*a_\lambda^*)$. As $bb^* \le \|b\|^2\mathbf{1}$ (combine the facts stated before this proposition and the C*-identity), $a_\lambda(\|b\|^2\mathbf{1} - bb^*)a_\lambda^* \ge \mathbf{0}$. Therefore $\tau(a_\lambda bb^*a_\lambda^*) \le \|b\|^2\tau(a_\lambda a_\lambda^*) = \|b\|^2\|(a_\lambda)_\tau\|_2^2$. And by definition, $\|(a_\lambda)_\tau\|_2 = \|\pi_\tau(a_\lambda)\mathbf{1}_\tau\|_2 \longrightarrow \|t\mathbf{1}_\tau\|_2 = 0$. Hence we have $tb_\tau = 0$ for all b in A. Since $\{b_\tau|b \in A\}$ is dense in H_τ, t is the zero operator. □

In this proof, it is easy to see for any two operators in M_τ, $\|(t-t')\mathbf{1}_\tau\|_2 = 0$ implies $t = t'$. It actually shows $\|t\|_2 := \|t\mathbf{1}_\tau\|_2$ defining a norm on M_τ. In other words, we say the vector $\mathbf{1}_\tau$ is *separating* for the von Neumann algebra M_τ.

4.3 The Closed Convex Hull of Orbits under Some Group Actions

In this section, we will study some properties of the closed convex hulls of some sets of vectors and operators.

First consider the orbit of a_τ under the action of $c_\tau[\mathbf{U}(A)]$(3.1). Let ξ be in H_τ, and denote:

$$k_\xi := \overline{\text{co}}^{\|\cdot\|_2}\{c_\tau(u)\xi|\ u \in \mathbf{U}(A)\} \tag{4.1}$$

Therefore $k_{a_\tau} = \overline{\text{co}}^{\|\cdot\|_2}\{(uau^*)_\tau \mid u \in \mathbf{U}(A)\}$. The set k_{a_τ} has an important property with respect to the invariant space E_τ.

Recall that p_τ is the projection from H_τ into the space E_τ(3.2), the invariant space under the action of $c_\tau[\mathbf{U}(A)]$.

Lemma 4.3.1
For any a in A, $k_{a_\tau} \cap E_\tau = \{p_\tau a_\tau\}$.

Proof. Notice that k_{a_τ} is a closed convex set in H_τ. There is a unique vector ξ in k_{a_τ}, such that $\|\xi\|_2 = \inf_{\zeta\in k_{a_\tau}} \|\zeta\|_2$. For all u in $\mathbf{U}(A)$ and for any convex combination $\sum_{j=0}^n \lambda_j c_\tau(u_j)a_\tau$, $c_\tau(u)(\sum_{j=0}^n \lambda_j c_\tau(u_j)a_\tau) = \sum_{j=0}^n \lambda_j c_\tau(uu_j)a_\tau$ remains in the set $\text{co}\{c_\tau(u)a_\tau \mid u \in \mathbf{U}(A)\}$. Therefore $c_\tau(u)\xi$ is in k_{a_τ}, because ξ is a norm limit of a sequence in $\text{co}\{c_\tau(u)a_\tau \mid u \in \mathbf{U}(A)\}$. By uniqueness of ξ, and $c_\tau(u)$ is an isometry for all u, $c_\tau(u)\xi = \xi$. Therefore ξ is in E_τ.

On another hand, for all u in $\mathbf{U}(A)$, ζ in k_{a_τ} and η in E_τ, we have $\|\zeta - \eta\|_2 = \|c_\tau(u)\zeta - \eta\|_2$, because $c_\tau(u)$ is an isometry and η is invariant under $c_\tau(u)$. This demonstrates that the closest vector to both ζ and $c_\tau(u)\zeta$ in E_τ is identical. Hence the projection is invariant with respect to the composition with $c_\tau(u)$ (i.e., $p_\tau c_\tau(u) = p_\tau$). Therefore, we have at most one element in $p_\tau(\text{co}\{c_\tau(u)a_\tau \mid u \in \mathbf{U}(A)\})$ and $p_\tau k_{a_\tau}$. Hence we have $p_\tau(\text{co}\{c_\tau(u)a_\tau \mid u \in \mathbf{U}(A)\}) = p_\tau k_{a_\tau} = \{p_\tau a_\tau\} = \{\xi\}$. □

Now, let's consider a bounded subset, denoted as S, of M_τ(3.3). The closure of S under SOT remains bounded.

Recall the definition of *lower semicontinuity*:

Definition 4.3.2 (Lower Semicontinuity)

Let f be a function from a topological space X to $\mathbb{R}$. We say f is lower semicontinuous if for any ζ_0 in X, for any $\alpha < f(\zeta_0)$, there is an open set U containing ζ_0, such that $f(\zeta) > \alpha$ for all ζ in U.

Lemma 4.3.3

The norm function $\|\cdot\|_\infty$ on M_τ is lower semi-continuous with respect to SOT.

Proof. Let ξ in $(H_\tau)_1$, and assume there is a net $\{t_\lambda\}_\Lambda$ in $(M_\tau)_L$ $(L > 0)$ converging to t. Then $\|t_\lambda\xi\|_2 \longrightarrow \|t\xi\|_2$ and $\|t_\lambda\xi\|_2 \leq \|t_\lambda\|_\infty\|\xi\|_2 \leq L$ for all λ. That is $\|t\|_\infty \leq L$. Therefore $(M_\tau)_L$ is SOT-closed. Hence for any x with $\|x\|_\infty > L$, there is a SOT-open set U containing x with $U \bigcap (M_\tau)_L = \varnothing$, which implies for all y in U, $\|y\|_\infty > L$. Since L and x are arbitrary, the lower semi-continuity follows.

□

By the proof Lemma 4.3.3, we have for any S as a subset of $(M_\tau)_L$, $\overline{S}^{SOT}$ remains as subset of $(M_\tau)_L$. Moreover, if S is convex, $\overline{S}^{SOT} = \overline{S}^{WOT}$ as a bounded set in $(M_\tau)_L$ (see [2, Theorem 5.1.2]).

Recall that fixing any unitary g in M_τ, the map $t \mapsto gtg^*$ is an isometry on M_τ (see the paragraph below Definition 4.1.1). Building on that, we have the following result for all convex combination:

$$\|\sum_{j=1}^{n} \lambda_j g_j t g_j^*\|_\infty \leq \sum_{j=1}^{n} \lambda_j \|t\|_\infty = \|t\|_\infty$$

Therefore the convex hull $\mathrm{co}\{gtg^* \mid U \in \mathbf{U}(M_\tau)\}$ is bounded by $\|t\|_\infty$. Combining with Lemma 4.3.3, $\overline{\mathrm{co}}^{SOT}\{gtg^* \mid g \in \mathbf{U}(M_\tau)\} = \overline{\mathrm{co}}^{WOT}\{gtg^* \mid g \in \mathbf{U}(M_\tau)\}$ is bounded by $\|t\|_\infty$ as well. Hence we have proven that the set $\overline{\mathrm{co}}^{SOT}\{gtg^* \mid g \in \mathbf{U}(M_\tau)\}$ is bounded:

Proposition 4.3.4

For any t in M_τ, the set $\overline{co}^{SOT}\{gtg^* \mid g \in \mathbf{U}(M_\tau)\}$ is bounded by $\|t\|_\infty$.

Following Proposition 4.3.4, we can prove there is an operator in $\mathbf{Z}(M_\tau)$ represent $p_\tau a_\tau$ in E_τ, for all a in A.

Proposition 4.3.5

If a is in A, then there is an unique operator x_a in $\mathbf{Z}(M_\tau)$ such that $x_a\mathbf{1}_\tau = p_\tau a_\tau$.

Proof. Given a in A and assuming $\|a\| = 1$, and denote

$$B_a := \text{co}\{uau^* | u \in \mathbf{U}(A)\}$$

Note that B_a is in A_1. Recall that $c_\tau(u)a_\tau = (uau^*)_\tau = \pi_\tau(uau^*)\mathbf{1}_\tau$. For any convex combination $\sum_{j=0}^n \lambda_j c_\tau(u_j)a_\tau$, we have:

$$\sum_{j=0}^n \lambda_j c_\tau(u_j)a_\tau = \sum_{j=0}^n \lambda_j \pi_\tau(u_j a u_j^*)\mathbf{1}_\tau = \pi_\tau(\sum_{j=0}^n \lambda_j u_j a u_j^*)\mathbf{1}_\tau$$

Since, by Lemma 4.3.1, $p_\tau a_\tau$ is contained in $k_{a_\tau} = \overline{co}^{\|\cdot\|_2}\{(uau^*)_\tau | u \in \mathbf{U}(A)\}$ (4.1), there is a sequence $\{\pi_\tau(b_n)\mathbf{1}_\tau\}_{\mathbb{N}}$ for some b_n in B_a, such that $\pi_\tau(b_n)\mathbf{1}_\tau \xrightarrow[n]{} p_\tau a_\tau$ in H_τ with the norm topology. Since the norm topology is finer than the weak topology on the Hilbert space, $\pi_\tau(b_n)\mathbf{1}_\tau \xrightarrow[n]{} p_\tau a_\tau$ weakly.

Consider the sequence $\{\pi_\tau(b_n)\}_{\mathbb{N}}$ in $(M_\tau)_1$ (assuming $\|a\| = 1$ and notice $\|\pi_\tau(b_\lambda)\|_\infty \le \|b_\lambda\| \le \|a\|$). Since $(M_\tau)_1$ is WOT compact (see [2, Theorem 5.3.3]), there is a subnet $\{\pi_\tau(b_\lambda)\}_\Lambda$ of $\{\pi_\tau(b_n)\}_{\mathbb{N}}$ that converges to an operator x_a in $(M_\tau)_1$ with WOT topology. In other words, we have $(\pi_\tau(b_\lambda)\mathbf{1}_\tau|\xi) \xrightarrow[\lambda]{} (x_a\mathbf{1}_\tau|\xi)$ for all ξ in H_τ. Therefore $\pi_\tau(b_\lambda)\mathbf{1}_\tau \xrightarrow[\lambda]{} x_a\mathbf{1}_\tau$ in weak topology of H_τ. As a subnet of $\{\pi_\tau(b_n)\mathbf{1}_\tau\}_{\mathbb{N}}$, $\pi_\tau(b_\lambda)\mathbf{1}_\tau \xrightarrow[k]{} p_\tau a_\tau$. Since the weak topology of H_τ is Hausdorff, $x_a\mathbf{1}_\tau = p_\tau a_\tau$. Notice that the vector $\mathbf{1}_\tau$ is separating for M_τ, therefore x_a is the only operator in M_τ satisfies $x_a\mathbf{1}_\tau = p_\tau a_\tau$.

Now we prove x_a is in $\mathbf{Z}(M_\tau)$. Let t in M_τ, consider a net $\{\pi_\tau(a_\lambda)\}_\Lambda$, such that $\pi_\tau(a_\lambda) \xrightarrow[\lambda]{} t$ in SOT. Then for u in $\mathbf{U}(A)$, we have

$$\pi_\tau(u)\pi_\tau(a_\lambda)\pi_\tau(u^*) = \pi_\tau(ua_\lambda u^*) \xrightarrow[\lambda]{} \pi_\tau(u)t\pi_\tau(u^*)$$

in SOT.

Recall that if we fix y in $\mathscr{B}(H)$ for some Hilbert space H, the two maps $t \mapsto ty$ and $t \mapsto yt$ are both SOT continuous (see [2, Remark 2.5.10]). Therefore taking the map $t \mapsto \pi_\tau(u)t\pi_\tau(u^*)$ as the composition of maps $t \mapsto \pi_\tau(u)t$ and $t \mapsto t\pi_\tau(u^*)$, it is still SOT-continuous.

Since for all u in $\mathbf{U}(A)$, $\pi_\tau(ua_\lambda u^*)\mathbf{1}_\tau = c_\tau(u)\pi_\tau(a_\lambda)\mathbf{1}_\tau$ and $c_\tau(u)\pi_\tau(a_\lambda) \xrightarrow[\lambda]{} c_\tau(u)t$ in SOT in $\mathscr{B}(H_\tau)$, then we have:

$$\pi_\tau(u)t\pi_\tau(u^*)\mathbf{1}_\tau = c_\tau(u)t\mathbf{1}_\tau$$

Therefore, we have $c_\tau(u)x_a\mathbf{1}_\tau = \pi_\tau(u)x_a\pi_\tau(u^*)\mathbf{1}_\tau$, since x_a is in M_τ. Notice $x_a\mathbf{1}_\tau = p_\tau a_\tau$ is in E_τ, then we have $c_\tau(u)x_a\mathbf{1}_\tau = x_a\mathbf{1}_\tau = \pi_\tau(u)x_a\pi_\tau(u^*)\mathbf{1}_\tau$. Since $\mathbf{1}_\tau$ is a separating vector on M_τ, we have $x_a = \pi_\tau(u)x_a\pi_\tau(u^*)$ for all u in $\mathbf{U}(A)$. All elements in A can be expended as a linear combination of at most four unitary elements in A (see [2, Theorem 4.1.7]), which means x_a commutes with all elements in $\pi_\tau[A]$. Therefore x_a commutes with all elements in $\pi_\tau[A]''$ as the SOT closure of $\pi_\tau[A]$ by the Double Commutant Theorem. □

With the above proposition, we have the most important lemma in this section:

Lemma 4.3.6
For τ as a tracial state on A, we have

$$\overline{\mathbf{Z}(M_\tau)\mathbf{1}_\tau} = E_\tau$$

Proof. Since $E_\tau = p_\tau H_\tau$, and $A_\tau := A/\mathscr{L}_\tau$ is dense in H_τ, we have $p_\tau A_\tau$ is dense in E_τ. By Proposition 4.3.4, we have $E_\tau \subseteq \overline{\mathbf{Z}(M_\tau)\mathbf{1}_\tau}$.

Conversely, for all y in $\mathbf{Z}(M_\tau)$, we have a net $\{\pi_\tau(a_\lambda)\}_\Lambda$ such that $\pi_\tau(a_\lambda)\xrightarrow[\lambda]{SOT} y$. Because $c_\tau(u)y\mathbf{1}_\tau = \lim_\lambda c_\tau(u)\pi_\tau(a_\lambda)\mathbf{1}_\tau = \lim_\lambda \pi_\tau(u)\pi_\tau(a_\lambda)\pi_\tau(u^*)\mathbf{1}_\tau = \pi_\tau(u)y\pi_\tau(u^*)\mathbf{1}_\tau = y\mathbf{1}_\tau$, we have $y\mathbf{1}_\tau$ is in E_τ. Therefore $\overline{\mathbf{Z}(M_\tau)\mathbf{1}_\tau} = E_\tau$. □

4.4 $p_\tau\pi_\tau[A]p_\tau$ Is a Family of Commutative Operators

In this section, we prove an essential property of a GNS construction associated with a tracial state τ, namely, the family of operators $p_\tau\pi_\tau[A]p_\tau$ consists of commutative operators.

The first result is that E_τ(3.2) is reducing for $\mathbf{Z}(M_\tau)$ (the definition can be checked at Definition 2.1.20). Recall the definition of M_τ(3.3). There is a useful lemma for reducing subspaces for *-algebras:

Lemma 4.4.1 (Reducing Subspaces for *-Algebras, [9, Lemma 2.2.3])
Let M be a *-algebra in $\mathscr{B}(H)$, let $K \subset H$ be a closed subspace, and let p_K be the projection into K. Then the following are equivalent:

- *K is reducing for M.*
- *$MK \subset K$.*
- *p_k is in M'.*

Combining this lemma with the Double Commutant Theorem, one can show the following:

Theorem 4.4.2 (Compression of von Neumann algebras, [9, Theorem 2.5.7])
Let M be a von Neumann algebra on H, and p is a projection in M. Then pMp and $M'p$ are von Neumann algebras on $\mathscr{B}(pH)$.

Moreover,

$$(pMp)' = M'p,\ (M'p)' = pMp$$

in $\mathscr{B}(pH)$.

Proposition 4.4.3
The projection p_τ (mapping H_τ into E_τ) is in $(\mathbf{Z}(M_\tau))'$.

Proof. Let ξ be in E_τ(3.2). By Lemma 4.3.6, there is a sequence $\{x_n\}_{\mathbb{N}}$ in $\mathbf{Z}(M_\tau)$ such that $\{x_n\mathbf{1}_\tau\}_{\mathbb{N}}$ converging to ξ. Let x be in $\mathbf{Z}(M_\tau)$, there is a net $\{\pi_\tau(a_\lambda)\}_\Lambda$ converging to x in SOT topology.

By Lemma 4.3.6, $x\mathbf{1}_\tau$ is in E_τ. Notice for all n, xx_n is in $\mathbf{Z}(M_\tau)$. Since we have $x_n\mathbf{1}_\tau \xrightarrow[n]{\|\cdot\|_2} \xi$ in E_τ, with x_n in $\mathbf{Z}(M_\tau)$, $c_\tau(u)x\xi = \lim_n c_\tau(u)xx_n\mathbf{1}_\tau = \lim_n xx_n\mathbf{1}_\tau = x\xi$. Therefore we have that E_τ is reducing for $\mathbf{Z}(M_\tau)$. By Proposition 4.4.1, p_τ in $(\mathbf{Z}(M_\tau))'$. □

There is one more theorem we need about abelian von Neumann algebras:

Theorem 4.4.4 (A Sufficient Condition for a Maximal Commutative Algebra, [3, Theorem 7.1.16])
If M is a commutative von Neumann algebra with a cyclic vector in $\mathscr{B}(H)$, then it is maximal commutative, which means $M' = M$.

Then we are ready to prove the main result in this chapter:

Proposition 4.4.5
Let τ be a tracial state on A. The set of operators $p_\tau\pi_\tau[A]p_\tau$ is commutative.

Proof. Since p_τ is in $(\mathbf{Z}(M_\tau))'$, by Theorem 4.4.2, we have in $\mathscr{B}(E_\tau)$:

$$(p_\tau(\mathbf{Z}(M_\tau))'p_\tau)' = \mathbf{Z}(M_\tau)p_\tau$$

Moreover, $\mathbf{Z}(M_\tau) = M_\tau \cap M_\tau' = (M_\tau \cup M_\tau')'$, which implies $(\mathbf{Z}(M_\tau))' = (M_\tau \cup M_\tau')''$. Notice that $p_\tau(M_\tau \cup M_\tau')''p_\tau$ contains $p_\tau\pi_\tau[A]p_\tau$.

Considering $\mathbf{Z}(M_\tau)p_\tau$ on E_τ with a cyclic vector $\mathbf{1}_\tau$, we can apply Theorem 4.4.4 for $\mathbf{Z}(M_\tau)p_\tau$, which shows this von Neumann algebra is maximal commutative.

Therefore, $\mathbf{Z}(M_\tau)p_\tau$ containing $p_\tau(M_\tau \cup M'_\tau)''p_\tau$ and containing $p_\tau\pi_\tau[A]p_\tau$. Therefore we proved that $p_\tau\pi_\tau[A]p_\tau$ consists of commutative operators. □

In general, the pair of $\{A, G\}$, where G is a subgroup of the *-automorphism on A, with such a property is called an *abelian system*. In this case, we need τ to be one of the G-invariant states. Let the invariant subspace $E_\tau := \{\xi \in H_\tau \mid c_\tau(g)\xi = \xi,\ g \in G\}$ and the projection corresponding to this space is denoted by p_τ. Notice $c_\tau(g)$ is the unique linear extension on H_τ of the map $a_\tau \mapsto (ga)_\tau$.

Definition 4.4.6 (Abelian Systems)
Let G be a subgroup of the *-automorphism group on A, the system $\{A, G\}$ is abelian if for any G-invariant state τ, (i.e., for all a in A, g in G, we have $\tau(a) = \tau(ga)$, where ga is the image of a under the action of g) we have $p_\tau\pi_\tau[A]p_\tau$ consisits of commutative operators.

It can be checked that the proof in the following chapter works for all abelian systems.

Chapter 5

$\mathbf{T}(A)$ is a Choquet Simplex

The result $\mathbf{T}(A)$ (Recall $\mathbf{T}(A)$ is the set of tracial state on A) is a Choquet simplex for any unital C*-algebra will be proved in this chapter. Moreover the tracial decomposition theorem for $\mathbf{T}(A)$ will follow from the Choquet's Theorem. Recall $c_\tau[\mathbf{U}(A)]$(3.1) is the set of all linear extension of $a_\tau \longrightarrow (uau^*)_\tau$, for u in $\mathbf{U}(A)$. Denote

$$U_\tau := \pi_\tau[A] \bigcup c_\tau[\mathbf{U}(A)] \tag{5.1}$$

5.1 U'_τ is a commutative von Neumann algebra

Since that U_τ(5.1) is a self-adjoint set, by the Double Commutant Theorem, we have U'_τ as a von Neumann algebra.

The following lemma related to a projection and its central support (see Definition 2.1.19) will be useful to us.

Lemma 5.1.1 ([9, Lemma 5.3.11])
Let M be a von Neumann algebra on H, and p in M is a projection, with central support e. Then the map $ye \mapsto yp$ from $M'e \longrightarrow M'p$ is a *- isomorphism.

The theorem that U'_τ is a commutative von Neumann algebra will essentially help us to prove $\mathbf{T}(A)$ is a Choquet simplex. This result is directly regarding the commutativity of $p_\tau \pi_\tau[A] p_\tau$.

Theorem 5.1.2

Consider a tracial state τ on a unital C*-*algebra A, then $U'_\tau(5.1)$ is commutative.*

Proof. For any h in $U'_\tau \subseteq c_\tau[\mathbf{U}(A)]'$, we have $hc_\tau(u)\xi = h\xi = c_\tau(u)h\xi$ for all ξ in E_τ and u in $\mathbf{U}(A)$. This shows E_τ is reducing for U'_τ. Therefore, by Lemma 4.4.1, p_τ is in U''_τ. Hence $p_\tau U''_\tau p_\tau$, $U'_\tau p_\tau$ are von Neumann algebras on E_τ. We have the following calculations:

i. $p_\tau \pi_\tau(a) c_\tau(u) p_\tau = p_\tau \pi_\tau(a) p_\tau$.

ii. $p_\tau c_\tau(u) \pi_\tau(a) p_\tau = p_\tau \pi_\tau(a) p_\tau$, since we have shown $p_\tau c_\tau(u) = p_\tau$ in the proof of 4.3.1.

iii $p_\tau \pi_\tau(b) c_\tau(u) \pi_\tau(a) p_\tau = p_\tau c_\tau(u^*) \pi_\tau(b) c_\tau(u) \pi_\tau(a) p_\tau = p_\tau \pi_\tau(u^* bua) p_\tau$.

Notice $p_\tau U''_\tau p_\tau = p_\tau \overline{\mathcal{A}^*(U_\tau)}^{SOT} p_\tau$, where $\mathcal{A}^*(U_\tau)$ is the *-algebra generates by U_τ.

To be precise, $\mathcal{A}^*(U_\tau) = \{q(\bar{s}) \mid q$ is a multi-variable complex polynomial, $\bar{s} \in (U_\tau)^n,\ n \in \mathbb{N}\}$. Recall U_τ is self-adjoint. Notice for every $q(\bar{s})$ in $\mathcal{A}^*(U_\tau)$, $p_\tau q(\bar{s}) p_\tau$ has the form:

$$\sum_{j=0}^{n} \lambda_j p_\tau q_j p_\tau$$

where all q_j has the form $c_\tau(u_{j_1})\pi_\tau(a_{j_1})c_\tau(u_{j_2})\pi_\tau(a_{j_2})......\pi_\tau(a_{j_m})$ (because both π_τ and c_τ are multiplicative homomorphisms). Using i , ii and iii, $p_\tau c_\tau(u_{j_1})\pi_\tau(a_{j_1})......\pi_\tau(a_{j_m})p_\tau$ can be simplified to $p_\tau \pi_\tau(a_j) p_\tau$ for some a_j in A.

Because $p_\tau \pi_\tau[A] p_\tau$ is commutative, we have $p_\tau \mathcal{A}^*(U_\tau) p_\tau$ is a commutative *-algebra. By taking limits with respect to SOT, it can be shown that $p_\tau U''_\tau p_\tau$ is a commutative von Neumann algebra on E_τ. (Let x and y be in U''_τ, first show $p_\tau x p_\tau$ commutes with all $p_\tau q(\bar{s}) p_\tau$ in $p_\tau \mathcal{A}^*(U_\tau) p_\tau$, by representing $p_\tau x p_\tau$ as a limit of some net $\{p_\tau q_\lambda(\bar{s}) p_\tau\}_\Lambda$. Then show $p_\tau x p_\tau$ commutes with $p_\tau y p_\tau$ by representing $p_\tau y p_\tau$ as a limit of some net $\{p_\tau q_{\lambda'}(\bar{s}) p_\tau\}_{\Lambda'}$.)

We claim p_τ has the central support $\mathbf{1}_{H_\tau}$ in U''_τ. Let e be a projection in $U''_\tau \cap U'_\tau$, and $ep_\tau = p_\tau e = p_\tau$. Then for all a in A:

$$e\pi_\tau(a)\mathbf{1}_\tau = \pi_\tau(a)e\mathbf{1}_\tau = \pi_\tau(a)ep_\tau\mathbf{1}_\tau = \pi_\tau(a)\mathbf{1}_\tau = a_\tau$$

Therefore $\overline{e\pi_\tau[A]\mathbf{1}_\tau} = H_\tau$. Hence the range of e is H_τ, so e is $\mathbf{1}_{H_\tau}$. Therefore the central support of p_τ is $\bigwedge\{e\} = \mathbf{1}_{H_\tau}$.

By Lemma 5.1.1, the map $y \mapsto yp_\tau$, from $U'_\tau \longrightarrow U'_\tau p_\tau$, is a *-isomorphism. Moreover, we have $(p_\tau U''_\tau p_\tau)' = p_\tau U'_\tau p_\tau$ on E_τ by Theorem 4.4.2. Note that $\overline{p_\tau U''_\tau p_\tau \mathbf{1}_\tau} = E_\tau$, since $\pi_\tau[A] \subseteq U_\tau$.

Therefore $p_\tau U_\tau'' p_\tau$ is maximal commutative in $\mathscr{B}(E_\tau)$, therefore $p_\tau U_\tau' p_\tau = U_\tau' p_\tau$ (contained in $p_\tau U_\tau'' p_\tau$) is commutative, which implies U_τ' is commutative. □

5.2 The Radon–Nikodym Theorem for Tracial States

In this section, we prove the second core tool for proving the final result. Denote $\mathbf{0}^\sharp$ as the zero functional on A. That comes from the Radon–Nikodym Theorem for the states:

Theorem 5.2.1 (The Radon–Nikodym Theorem for States, [1, Theorem 3.6.1])
Let ρ be a state on A. Then there is an order isomorphism between $\mathcal{K}_0 := \{h \in \pi_\rho[A]' \mid \mathbf{0}_{H_\rho} \leq h \leq \mathbf{1}_{H_\rho}\}$ and $\mathcal{Y}_0 := \{\psi \in A^\sharp \mid \mathbf{0}^\sharp \leq \psi \leq \rho\}$. The isomorphism is given by

$$\psi_h(a) := (\pi_\rho(a)h\mathbf{1}_\rho|\mathbf{1}_\rho)$$

To show $\mathbf{T}(A)$ is a Choquet simplex, it is essential to identify the real locally convex space in which the tracial simplex is.

Definition 5.2.2 (The Space of Tracial Self-adjoint Functionals)
Let E denote the real vector space consisting of all tracial and self-adjoint states on A.

Recall that $\mathbf{T}(A)$ is a compact convex set in E with the weak*-topology restricted on E. E is a subspace of a locally convex space $A^\sharp$ with the weak*-topology, therefore E is also locally convex.

Recall the unique orthogonal decompositions for self-adjoint elements in A and self-adjoint functionals on A (see [2, Proposition 4.2.3] and [2, Theorem 4.3.6]).

- Let h be a self-adjoint element in A, then there are two unique positive elements h_+ and h_+ in A satisying the condition $h_+h_- = h_-h_+ = \mathbf{0}_A$ and $h = h_+ - h_-$. Such decomposition also satisfies $\|h\| = \max\{\|h_+\|, \ \|h_-\|\}$.
- Let f be a self-adjoint functional on A, then there are two unique positive linear functionals f^+ and f^-, such that $f = f^+ - f^-$ and $\|f\| = \|f^+\| + \|f^-\|$.

Let f be in E and $f = f^+ - f^-$ be the unique orthogonal decomposition of f in the dual of A. Then we can check that $\operatorname{Ad} u(f) = f = \operatorname{Ad} u(f^+) - \operatorname{Ad} u(f^-)$ is also an orthogonal decomposition. $\operatorname{Ad} u$ is a automorphism on A, hence $\|f\| = \|\operatorname{Ad} u(f^+)\| + \|\operatorname{Ad} u(f^-)\|$ still holds. By uniqueness of orthogonal decomposition, $f^+ = \operatorname{Ad} u(f^+)$ and $f^- = \operatorname{Ad} u(f^-)$ for all

u in $\mathbf{U}(A)$, therefore f^+ and f^- are also tracial.

Therefore, we have proven the following:

Proposition 5.2.3
The space E takes the set of positive tracial functionals as the generating cone, and $\mathbf{T}(A)$ is a base for the generating cone.

Recall the definition of U_τ(5.1). We have an essential tool here:

Theorem 5.2.4 (The Radon–Nikodym Theorem for Tracial States)
Let τ be a tracial state on A. Then there is an order isomorphism between $\mathcal{K} := \{h \in U_\tau' \mid \mathbf{0}_{H_\tau} \leq h \leq \mathbf{1}_{H_\tau}\}$ and $\mathcal{Y} := \{\psi \in E \mid \mathbf{0}^\sharp \leq \psi \leq \tau\}$. The isomorphism is given by

$$\psi_h(a) := (\pi_\tau(a)h\mathbf{1}_\tau|\mathbf{1}_\tau)$$

Notice that the set $\mathcal{K}$ and $\mathcal{Y}$ can be scaled by any positive number.

Proof. It is sufficient to show that the isomorphism maps between $\mathcal{K}$ and $\mathcal{Y}$, other properties are guaranteed by Theorem 5.2.1.

Let h be in $\mathcal{K}$, therefore $\mathbf{0}_{H_\tau} \leq h \leq \mathbf{1}_{H_\tau}$ and h is in U_τ'. Consider $\psi_h(a) := (\pi_\tau(a)h\mathbf{1}_\tau|\mathbf{1}_\tau)$. By Theorem 5.2.1, $\mathbf{0}^\sharp \leq \psi_h \leq \tau$. Then consider $\operatorname{Ad} u(\psi_h)$, for all a in A and all u in $\mathbf{U}(A)$:

$$\begin{aligned}\operatorname{Ad} u(\psi_h)(a) &= (\pi_\tau(uau^*)h\mathbf{1}_\tau|\mathbf{1}_\tau)\\ &= (c_\tau(u)\pi_\tau(a)c_\tau(u^*)h\mathbf{1}_\tau|\mathbf{1}_\tau)\\ &= (\pi_\tau(a)hc_\tau(u^*)\mathbf{1}_\tau|c_\tau(u^*)\mathbf{1}_\tau)\end{aligned}$$

Since $\mathbf{1}_\tau$ is in E_τ, we have

$$(\pi_\tau(a)hc_\tau(u^*)\mathbf{1}_\tau|c_\tau(u^*)\mathbf{1}_\tau) = (\pi_\tau(a)h\mathbf{1}_\tau|\mathbf{1}_\tau) = \psi_h(a)$$

Hence ψ_h is in $\mathcal{Y}$.

Let ψ in $\mathcal{Y}$, therefore ψ is tracial and $\mathbf{0}^\sharp \leq \psi \leq \tau$. Let h in $\mathcal{K}_0$ be the preimage of ψ under the order isomorphism by Theorem 5.2.1.

From the proof of Theorem 5.2.1 (see [1, Theorem 3.6.1]), h is the unique operator in $\mathscr{B}(H_\tau)$

such that:

$$\psi(d^*b) = (hb_\tau|d_\tau)$$

Since $(c_\tau(u)hc_\tau(u^*)b_\tau|d_\tau) = (hc_\tau(u^*)b_\tau|c_\tau(u^*)d_\tau) = \psi(ud^*u^*ubu^*) = \psi(ud^*bu^*) = \psi(d^*b)$ for all b and d in A, the operator h and $c_\tau(u)hc_\tau(u^*)$ are corresponding to a same conjugate bilinear form on $H_\tau \times H_\tau$, which imples they are equal. Therefore h commutes with $c_\tau[\mathbf{U}(A)]$, and h is in $\mathcal{K}$. □

To extend Theorem 5.2.4 to self-adjoint cases, we define the following concept.

Definition 5.2.5 (Positive Decomposition under a Tracial State)
For any f in E, we say f has a N-positive decomposition under a tracial state τ, if there are two positive functionals f_+ and f_- in E, such that $f = f_+ - f_-$ and $f_+ + f_- \le N\tau$ for some positive number N.

Notice that the orthogonal decomposition for f is a N-positive decomposition if $|f| \le N\tau$. Recall the notation for the orthogonal decompositions here:

Let $f = f^+ - f^-$ in E be the orthogonal decomposition, denote $|f| := f^+ + f^-$. For any h' as a self-adjoint operator in U'_τ, $h' = h'_+ - h'_-$ are the unique decomposition such that $h'_+h'_- = 0$. Denote $|h'| := h'_+ + h'_-$.

The $'$ on h' shows that this element is in U'_τ. Now, we have four useful corollaries:

Corollary 5.2.6
Let τ be a tracial state on A.

1. *Suppose there exists a N-positive decomposition $f = f_+ - f_-$ for f under a tracial state τ. Then there is a unique self-adjoint h' in $U'_\tau(5.1)$, such that $f(a) := (\pi_\tau(a)h'\mathbf{1}_\tau|\mathbf{1}_\tau)$. Moreover, $\|h'\|_\infty \le 2N$.*
2. *For all self-adjoint h' in U'_τ, $f(a) := (\pi_\tau(a)h'\mathbf{1}_\tau|\mathbf{1}_\tau)$ is in E. Moreover, f is given by a $\|h'\|_\infty$-positive decomposition $f = f_+ - f_-$ under τ, corresponding to the decomposition $h' = h'_+ - h'_-$.*
3. *If we have self-adjoint $h'_1 \le h'_2$ in U'_τ, then $f_1 \le f_2$ in E, where $f_1(a) := (\pi_\tau(a)h'_1\mathbf{1}_\tau|\mathbf{1}_\tau)$ and $f_2(a) := (\pi_\tau(a)h'_2\mathbf{1}_\tau|\mathbf{1}_\tau)$.*
4. *If there exist N-positive decomposition for f and N'-positive decomposition for g under τ, and $f \le g$ in E, then $h'_f \le h'_g$ in U'_τ, where h'_f is the unique self-adjoint operator in*

U'_τ such that $f(a) := (\pi_\tau(a)h'_f\mathbf{1}_\tau|\mathbf{1}_\tau)$ and h'_g is the unique self-adjoint operator in U'_τ such that $g(a) := (\pi_\tau(a)h'_g\mathbf{1}_\tau|\mathbf{1}_\tau)$.

Proof. Remember for any tracial state τ, U'_τ is a commutative von Neumann algebra. Recall the Theorem 2.1.12, $U'_\tau \cong C_{\mathbb{C}}(X)$ for some Hausdorff compact space X. ζ in this proof will represent the point in space X.

For the first statement, let h'_1 and h'_2 be the positive operators in U'_τ corresponding to f_+ and f_- by Theorem 5.2.4, and notice $f_+, f_- \le N\tau$. Let $h' := h'_1 - h'_2$. Notice h' is self-adjoint. Since $f_+(a) := (\pi_\tau(a)h'_1\mathbf{1}_\tau|\mathbf{1}_\tau)$ and $f_-(a) := (\pi_\tau(a)h'_2\mathbf{1}_\tau|\mathbf{1}_\tau)$, $f(a) = f_+(a) - f_-(a) = (\pi_\tau(a)(h'_1 - h'_2)\mathbf{1}_\tau|\mathbf{1}_\tau)$. By Theorem 5.2.4, we have $h'_1 \le N\mathbf{1}_{H_\tau}$ and $h'_2 \le N\mathbf{1}_{H_\tau}$. Since they are all positive operators in the commutative algebra U'_τ, therefore by the function representation, $\|h'_1\|_\infty \le N$ and $\|h'_2\|_\infty \le N$. That gives $\|h'\|_\infty \le \|h'_1\|_\infty + \|h'_2\|_\infty \le 2N$.

Moreover, suppose we have self-adjoint h'_f and h'_F satisfying

$$f(a) = (\pi_\tau(a)h'_f\mathbf{1}_\tau|\mathbf{1}_\tau) = (\pi_\tau(a)h'_F\mathbf{1}_\tau|\mathbf{1}_\tau)$$

Hence $0 = (\pi_\tau(a)(h'_f - h'_F)\mathbf{1}_\tau|\mathbf{1}_\tau)$ for all a in A, which implies $0 = (\pi_\tau(d^*b)((h'_f - h'_F)\mathbf{1}_\tau|\mathbf{1}_\tau) = ((h'_f - h'_F)b_\tau|d_\tau)$ for all b and d in A. By 1-to-1 correspondence between the conjugate bilinear form and bounded operators, $h'_f - h'_F = \mathbf{0}_{H_\tau}$. Therefore, the uniqueness follows.

For the second statement, given a self-adjoint h' in U'_τ, consider the unique orthogonal decomposition $h' = h'_+ - h'_-$. In this decomposition:

$$\|h'\|_\infty = \max\{\|h'_+\|_\infty,\ \|h'_-\|_\infty\} \text{ and } h'_+h'_- = \mathbf{0}_{H_\tau}$$

Now, let $f_1(a) := (\pi_\tau(a)h'_+\mathbf{1}_\tau|\mathbf{1}_\tau)$ and $f_2(a) := (\pi_\tau(a)h'_-\mathbf{1}_\tau|\mathbf{1}_\tau)$, which are two positive functionals in E. Therefore $f(a) = (\pi_\tau(a)h'\mathbf{1}_\tau|\mathbf{1}_\tau) = (\pi_\tau(a)(h'_+ - h'_-)\mathbf{1}_\tau|\mathbf{1}_\tau)$ is in E. Considering the function representation for U'_τ, the two functions $h'_+(\zeta) = \max\{0, h'(\zeta)\}$ and $h'_-(\zeta) = -\min\{0, h'(\zeta)\}$ (following from $h'_+h'_- = \mathbf{0}$ pointwisely), hence $\||h'|\|_\infty = \|h'\|_\infty$. By the definition: $(f_1 + f_2)(a) = (\pi_\tau(a)|h'|\mathbf{1}_\tau \mid \mathbf{1}_\tau)$, $f_1 + f_2 \le \|h'\|_\infty\tau$, because $|h'| \le \|h'\|_\infty\mathbf{1}_{H_\tau}$.

For the third statement, $\mathbf{0}_{H_\tau} \le h'_g - h'_f \le \|h'_g - h'_f\|_\infty\mathbf{1}_{H_\tau}$. Therefore, $\mathbf{0}^\sharp \le g - f \le \|h'_g - h'_f\|_\infty\tau$.

For the fourth statement, the conditions imply $-N\tau \le f \le N\tau$ and $-N'\tau \le g \le N'\tau$. That is because, $f \le f_+ + f_-$ which impies $\mathbf{0}^\sharp \le N\tau - (f_+ + f_-) \le N\tau - f$; and $\mathbf{0}^\sharp \le 2f_+ = f_+ - f_- + f_+ + f_- \le f_+ - f_- + N\tau = f + N\tau$.

Therefore $\mathbf{0}^\sharp \leq g - f \leq (N + N')\tau$. That implies $\mathbf{0}_{H_\tau} \leq h'_g - h'_f \leq (N + N')\mathbf{1}_{H_\tau}$. □

5.3 The Final Result

Now we are completely ready for proving the Theorem:

Theorem 5.3.1
Let A be a unital C*-*algebra. Then the set of tracial states* $\mathbf{T}(A)$ *on A is a Choquet simplex if it is not empty.*

If A is commutative, we have $A \cong C_{\mathbb{C}}(X)$ for some compact Hausdorff space X. Furthermore, all state on A is tracial. The set $\mathbf{T}(A)$ is affinely homeomorphic to $M_{+,1}(X)$ (with the weak*-topology), which can be shown as a Choquet simplex by elementary measure theory (see [6, Example 11.4]). The proof for the non-commutative case is trying to reduce the situation to the commutative case.

Proof. We have shown $\mathbf{T}(A)$ is weak*-compact, convex and is a base for a generating cone in E. Also, E is locally convex. From the definition of a Choquet simplex (2.2.4), it is sufficient to show E (Definition 5.2.2) with the tracial positive state cone forms a vector lattice.

Let f_1 and f_2 in E, then $|f_j| = f_j^+ + f_j^-$ $(j = 1, 2)$. Set $\psi := \frac{|f_1|+|f_2|}{\|f_1\|+\|f_2\|}$. Therefore $|f_1|,\ |f_2| \leq (\|f_1\| + \|f_2\|)\psi$, and ψ is a tracial state. Consider the GNS construction associated with the tracial state ψ. By Corollary 5.2.6, there are unique self-adjoint h'_1 and h'_2 in U'_ψ, such that $f_j(a) = (\pi_\psi(a)h'_j\mathbf{1}_\psi|\mathbf{1}_\psi)$ $(j = 1, 2)$. Moreover, we have $\|h'_1\|_\infty \leq 2(\|f_1\| + \|f_2\|)$ and $\|h'_2\|_\infty \leq 2(\|f_1\| + \|f_2\|)$. Since $U'_\psi(5.1)$ is commutative, $U'_\psi \cong C_{\mathbb{C}}(X)$ for a Hausdorff and compact space X. The self-adjoint portion in U'_ψ forms a vector lattice (where the positive cone is the set of positive continuous functions) by the function representation of U'_ψ, then $h'_1 \vee h'_2$ exists and is self-adjoint in U'_ψ. Denote $f_3(a) := (\pi_\psi(a)(h'_1 \vee h'_2)\mathbf{1}_\psi|\mathbf{1}_\psi)$, hence $f_1,\ f_2 \leq f_3$ by the order isomorphism. Again by the function representation, we also have $|h'_1 \vee h'_2| \leq 2(\|f_1\| + \|f_2\|)\mathbf{1}_{H_\psi}$, which implies $\||h'_1 \vee h'_2|\|_\infty \leq 2(\|f_1\| + \|f_2\|)$. Therefore we have a $2(\|f_1\|+\|f_2\|)$-positive decomposition for f_3 under ψ, where $(f_3)_+(a) = (\pi_\psi(a)(h'_1 \vee h'_2)_+\mathbf{1}_\psi|\mathbf{1}_\psi)$, and $(f_3)_-(a) = (\pi_\psi(a)(h'_1 \vee h'_2)_-\mathbf{1}_\psi|\mathbf{1}_\psi)$. Hence $(f_3)_+ + (f_3)_- \leq 2(\|f_1\| + \|f_2\|)\psi$.

Now we show $f_3 = f_1 \bigvee f_2$. Suppose we have $f_j \leq f_4 (j = 1, 2)$ in E. Set $\psi_0 := \frac{\psi+|f_4|}{1+\|f_4\|}$ as a tracial state in E. In the GNS construction associated with the tracial state ψ_0, Corollary 5.2.6 give similar results: there are self-adjoint elements k', k'_j $(j = 1, 2, 3, 4)$ in U'_{ψ_0} such that $(\|f_1\| + \|f_2\|)\psi(a) = (\pi_{\psi_0}(a)k'\mathbf{1}_{\psi_0}|\mathbf{1}_{\psi_0})$, $f_j(a) = (\pi_{\psi_0}(a)k'_j\mathbf{1}_{\psi_0}|\mathbf{1}_{\psi_0})$. Notice $|f_1|$, $|f_2|$,

$(f_3)_+ + (f_3)_-$, $|f_4|$ and $(\|f_1\| + \|f_2\|)\psi$ are all less or equal to $N\psi_0$ for a positive scalar $N = 2(1 + \|f_1\| + \|f_2\|)(1 + \|f_4\|)$. By Corollary 5.2.6, f_1, $f_2 \le f_3$ gives k_1', $k_2' \le k_3'$. Similarly, we have k_1', $k_2' \le k_4'$. Since the self-adjoint portion of U'_{ψ_0} is also a vector lattice, $k_1' \vee k_2'$ exists.

Therefore $k_1' \vee k_2' \le k_3'$ as $k_1', k_2' \le k_3'$ in U'_{ψ_0}. Also $k_1' \vee k_2' \le k_4'$ as $k_1', k_2' \le k_4'$. Let $f_5(a) := (\pi_{\psi_0}(a)(k_1' \vee k_2')\mathbf{1}_{\psi_0}|\mathbf{1}_{\psi_0})$, we have $f_1, f_2 \le f_5$ and $f_5 \le f_3$ and $f_5 \le f_4$. As $|f_1|$, $|f_2| \le (\|f_1\| + \|f_2\|)\psi$, $-(\|f_1\| + \|f_2\|)\psi \le f_1, f_2 \le (\|f_1\| + \|f_2\|)\psi$. By the order isomorphism, we have $-k' \le k_1', k_2' \le k'$. Therefore we use the function representation to have $|k_1' \vee k_2'| \le k'$. From the proof of Corollary 5.2.6, we have a $(\|f_1\| + \|f_2\|)$-positive decomposition for f_5 under ψ, because

$$(f_5)_+ + (f_5)_- = (\pi_{\psi_0}(a)|k_1' \vee k_2'|\mathbf{1}_{\psi_0} \mid \mathbf{1}_{\psi_0}) \le (\pi_{\psi_0}(a)k'\mathbf{1}_{\psi_0}|\mathbf{1}_{\psi_0}) = (\|f_1\| + \|f_2\|)\psi$$

where $(f_5)_+(a) = (\pi_\psi(a)(k_1' \vee k_2')_+\mathbf{1}_{\psi_0}|\mathbf{1}_{\psi_0})$ and $(f_5)_-(a) = (\pi_\psi(a)(k_1' \vee k_2')_-\mathbf{1}_{\psi_0}|\mathbf{1}_{\psi_0})$. That is to say, we have h_5' in U'_ψ, $f_5(a) = (\pi_{\psi_0}(a)h_5'\mathbf{1}_{\psi_0}|\mathbf{1}_{\psi_0})$. Then $h_1', h_2' \le h_5'$ by $f_1, f_2 \le f_5$. Therefore gives $h_3' \le h_5'$ implying $f_3 = f_5$. Also, we have $f_5 \le f_4$, hence $f_1 \vee f_2 = f_3$.

One can make a similar argument for the existence of $f_1 \wedge f_2$. □

Now with the definition of a barycenter and Choquet's theorem, we have the final result of our thesis. Note that for all a in A,

$$\hat{a}(\tau) := \tau(a)$$

is a weak*-continuous linear functional on $A^\sharp$. We also have the fact: every a in A has the form:

$$a = a_r + ia_i$$

where i is the imaginary unit, while a_r and a_i are self-adjoint elements. Therefore $\hat{a} = \hat{a_r} + i\hat{a}_i$, and is a linear combination of two real weak*-continuous functional on the dual of A.

Theorem 5.3.2 (The Tracial Decomposition)
In an unital separable C*-*algebra* A *(implying* $\mathbf{T}(A)$ *is metrizable), for all* τ *in* $\mathbf{T}(A)$*, there exists a unique probability Radon measure* μ *on* $\mathbf{T}(A)$*, such that for all* a *in* A

$$\tau(a) = \hat{a}(\tau) = \int_{\mathbf{T}(A)} \hat{a}(\varphi)\ d\mu(\varphi)$$

and $\mu(\partial\mathbf{T}(A)) = 1$*, where* $\partial\mathbf{T}(A)$ *denotes the set of extreme points of* $\mathbf{T}(A)$*.*

Proof. Let A be a unital separable C*-algebra and τ be a tracial state on A. Therefore $\mathbf{T}(A)$ is metrizable. By Theorem 2.2.7, if $a = a_r + ia_i$ is the self-adjoint decomposition for an element a in A, there is a unique probability Radon measure μ on $\mathbf{T}(A)$ and $\mu(\partial\mathbf{T}(A)) = 1$ such that

$$\tau(a_r) = \hat{a_r}(\tau) = \int_{\mathbf{T}(A)} \hat{a_r}(\varphi)\ d\mu(\varphi)$$

$$\tau(a_i) = \hat{a_i}(\tau) = \int_{\mathbf{T}(A)} \hat{a_i}(\varphi)\ d\mu(\varphi)$$

Therefore we have

$$\tau(a) = \hat{a_r} + i\hat{a_i}(\tau) = \int_{\mathbf{T}(A)} \hat{a_r}(\varphi)\ d\mu(\varphi) + i\int_{\mathbf{T}(A)} \hat{a_i}(\varphi)\ d\mu(\varphi) = \int_{\mathbf{T}(A)} \hat{a}(\varphi)\ d\mu(\varphi)$$

□

Chapter 6

Further Applications

6.1 Examples For Nonempty $\mathbf{T}(A)$

The result we proved in the above sections demonstrates that for all unital C*-algebras $\mathbf{T}(A)$ is a Choquet simplex if it is nonempty. We list some examples of C*-algebra with nonempty $\mathbf{T}(A)$ [1].

- For every commutative unital C*-algebra, which is just a function space on a compact Hausdorff space X, every state on this algebra is a tracial state. Therefore its state space $\mathbf{S}(A)$ is not empty therefore is a Choquet simplex.
- For every complex matrix algebra $M_n(\mathbb{C})$, the map $\frac{1}{n}tr$ is a tracial state (actually, it is the unique tracial state on $M_n(\mathbb{C})$).
- A more interesting example is the so-called *UHF algebras* which is a tensor product of infinitely many matrix algebras. The UHF algebras also have a unique tracial state.

6.2 A Converse Result about AF-algebras and Choquet Simplex

As a specific kind of C*-algebra, AF-algebras, which AF stands for *approximately finite*, defined in the following way:

Definition 6.2.1 (AF-algebra)

Let A be a C*-*algebra, it is a AF-algebra if there are increasing sequence of finite-dimensional* C*-*subalgebra $\{B_n\}_{n\in\mathbb{N}}$ such that $\bigcup_{n=1}^{\infty} B_n$ is dense in A.*

And moreover, we say a C*-algebra is *simple* if it contains no none trivial closed 2-side ideal.

The following is [7, Theorem 3.10]:

Theorem 6.2.2
Let X be any metrizable Choquet simplex. Then there is a simple unital AF-algebra A with $\mathbf{T}(A)$ affinely homeomorphic to X.

6.3 Elliott's Invariant

For a unital C*-algebra A, one can define the *Elliott's invariant* "Ell(A)" for it. It consists of several parts, some of which are related to the K-theory of C*-algebra, which is far away from this thesis. The $\mathbf{T}(A)$ is a part of this invariant.

There are several conjectures related to the classification of C*-algebras based on the Elliott invariant. These conjectures can be found in [8]. The objective of the Elliott Program is to completely classify some specific types of C*-algebras just by classifying their Elliott's invariants.

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