"Overview" of Integration & Its Relationship to Differentiation

$$\begin{tikzcd} f(t) \arrow[r, bend left, "\text{Differentiate}"] & f'(t) \arrow[l, bend left, "\text{Integration}"] \end{tikzcd}$$

$f(t)$: Quantity Function

$f'(t)$: Rate Function (giving rate of change of $f(t)$ wrt t)

An Example to Keep in Mind:

$p(t)$ = position @ time t

$v(t)$ = velocity @ time t

$a(t)$ = acceleration @ time t

$$\begin{tikzcd} p(t) \arrow[r, bend left, "\text{Differentiate}"] & v(t) \arrow[l, bend left, "\text{Integrate}"] \arrow[r, bend left, "\text{Differentiate}"] & a(t) \arrow[l, bend left, "\text{Integrate}"] \end{tikzcd}$$

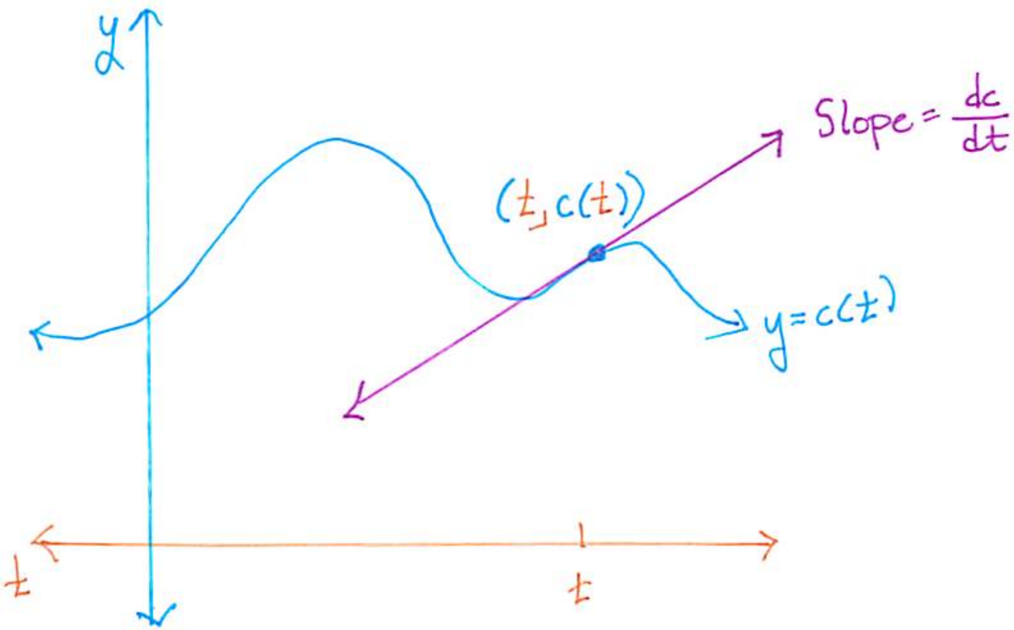
<u>Goal</u>: Answering questions like

If quantity changes @ a nonfixed rate, how much did the quantity increase over a particular period of time

Integration Purpose Preamble

Recall:

IF the **cost** $c(t)$ is a function of the time t,
THEN the rate of change of $c(t)$ (wrt t) is $\frac{dc}{dt}$



$\frac{dc}{dt}$ gives the slope at the time t

New

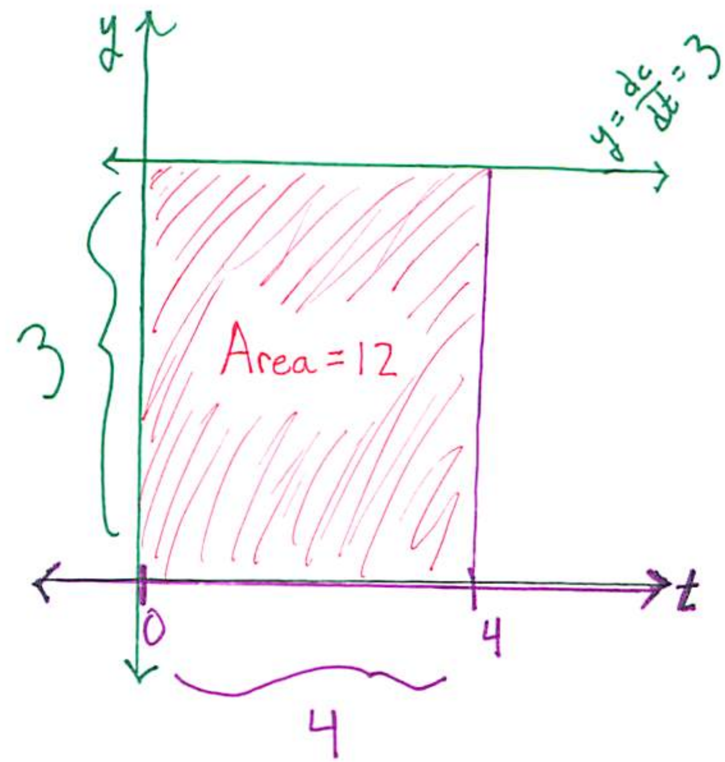
Q: What if we start with the rate of change $\frac{dc}{dt}$ of the cost $c(t)$? Can we retrieve the cost increase over a time interval?

A: Yes! This will be integration!

<u>Set-Up</u>:

Warm-Up Example

SUPPOSE $\begin{cases} \frac{dc}{dt} = 3 \text{ (cost is increasing at a rate of 3 \$/month)} \\ \text{Time interval is } [0,4] \text{, in moths} \end{cases}$



Cost Change = (Rate of Change $\frac{dc}{dt}$) × (Time)

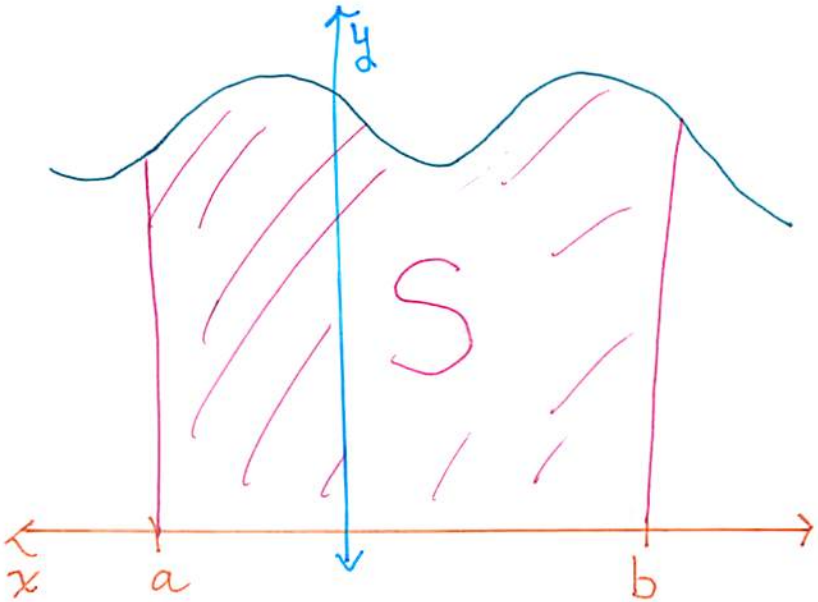
$$3 \times 4 = \boxed{12}$$

<u>BIG QUESTION</u>: What if rate of change isn't constant (isn't a constant function)!?!

Riemann sums & integration!

Goal of Riemann Sums

Find Area S



RIEMANN SUMS ARE NOT CRAZY THEORETICAL NONSENSE!!!

1. Approximation is extremely important
2. Allow us to "see" how rates change into quantities
3. Help us to see appropriate integral to compute

This is the same process as

rate $\rightsquigarrow$ quantity

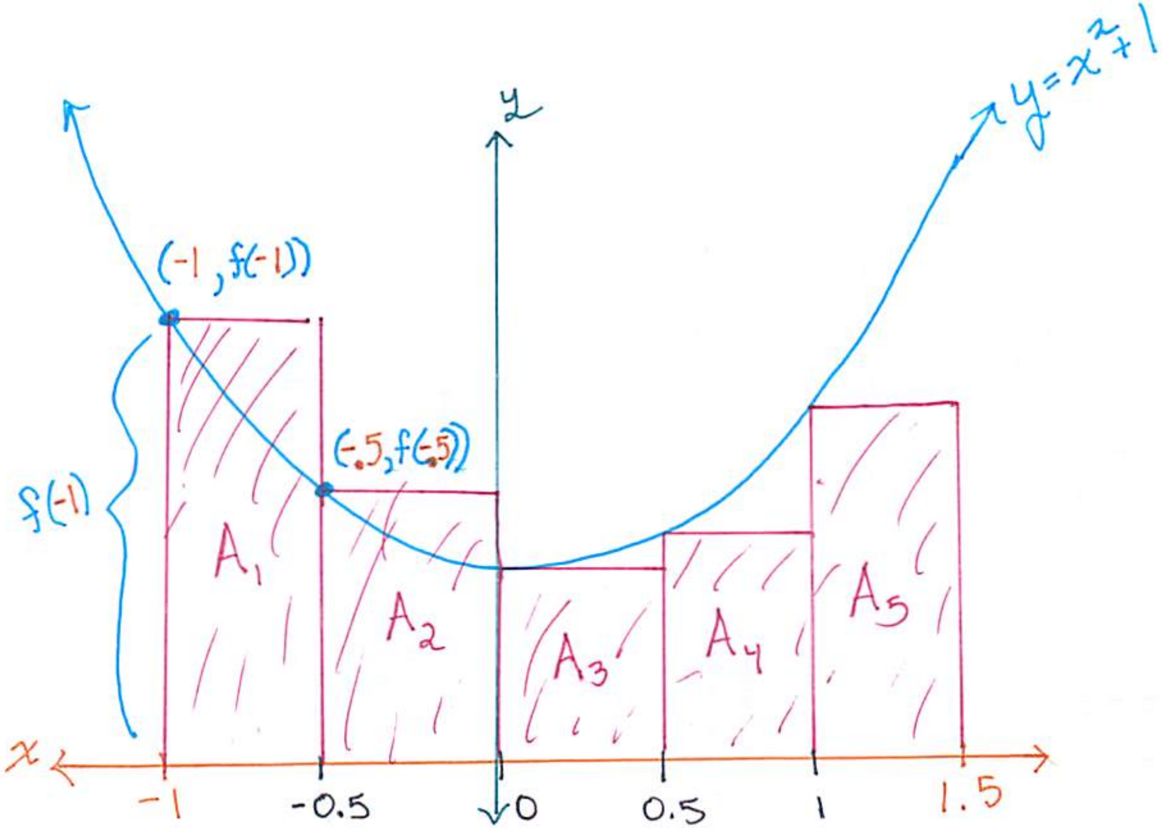
f $\rightsquigarrow$ area under f

f $\rightsquigarrow$ antiderivative / integral of f

rate f $\rightsquigarrow$ quantity F

Approximating the Area Under a Curve Using Left Endpoints

Task: Area under the curve $f(x) = x^2 + 1$ on the interval $[-1, 1.5]$ with 5 subintervals



Width of the Rectangle:

$$\Delta x = \frac{\text{Total width}}{\#\text{ intervals}} = \frac{(1.5 - (-1))}{5} = \frac{2.5}{5} = 0.5$$

Approximate Area

$$= A_1 + A_2 + A_3 + A_4 + A_5$$

$$= \Delta x\, f(-1) + \Delta x\, f(-.5) + \Delta x\, f(0) + \Delta x\, f(.5) + \Delta x\, f(1)$$

$$= .5\left(2 + 5/4 + 1 + 5/4 + 2\right) = \boxed{15/4}$$

left endpoints

This is $\boxed{L_5}$ 5 subintervals

i.e. $\boxed{L_5 = 15/4}$

Approximating the Area Under a Curve Using Left Endpoints & Fancy Useful Notation

Dividing interval $[a,b]$ into n subintervals

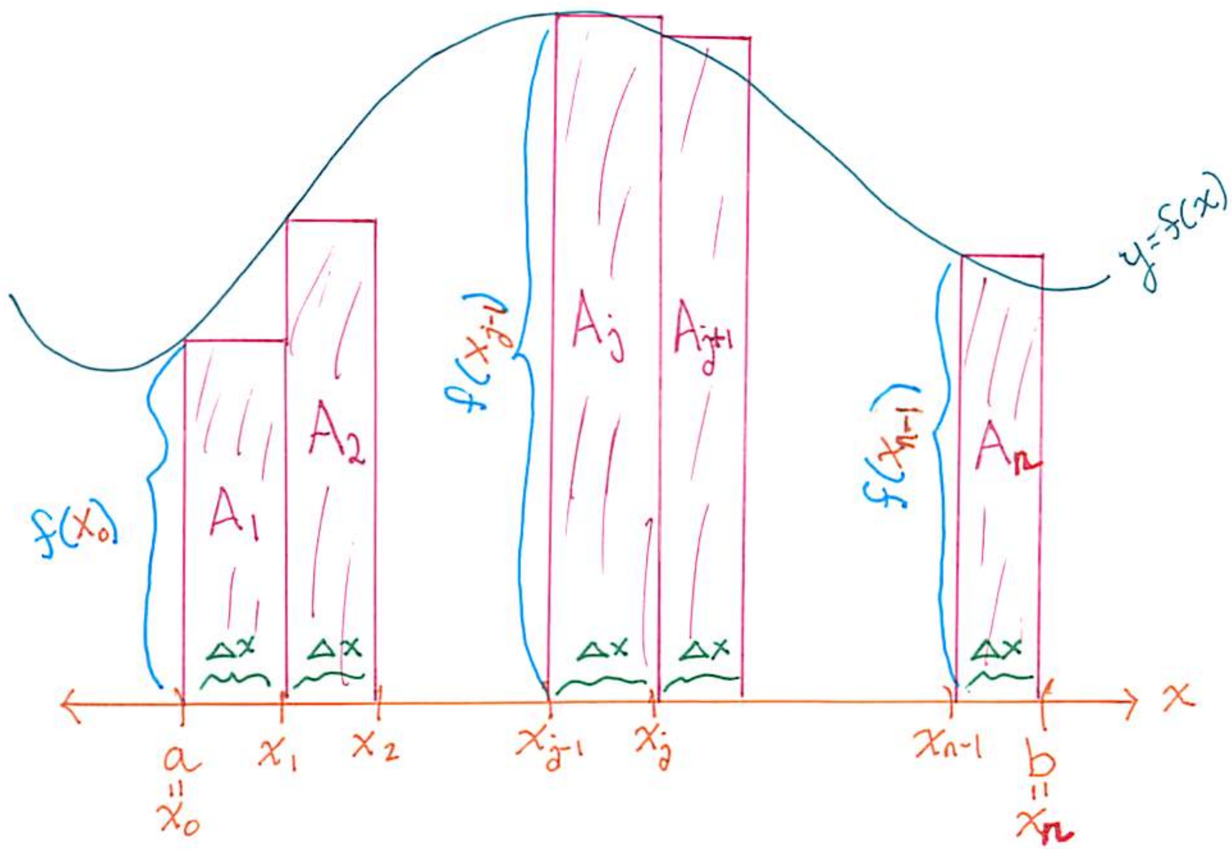
$$\Delta x = \frac{b-a}{n}$$

$$x_j = a + j\Delta x$$

$[x_{j-1}, x_j]$ is the j^{th} subinterval

Test: $[x_0, x_1]$ is the 1^{st} subinterval

$[x_1, x_2]$ is the 2^{nd} subinterval



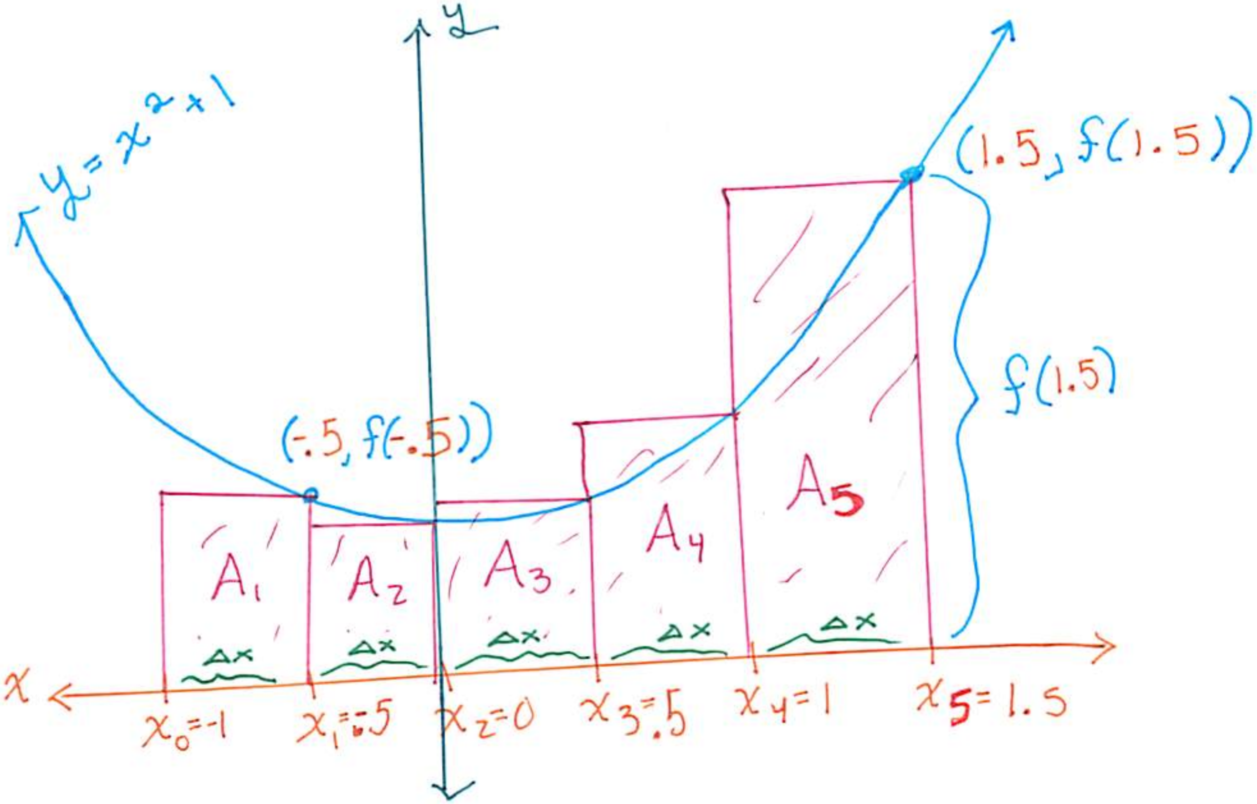
$$L_n = A_1 + A_2 + \cdots + A_j + \cdots + A_n$$

$$= \Delta x f(x_0) + \Delta x f(x_1) + \cdots + \Delta x f(x_j) + \cdots + \Delta x f(x_{n-1})$$

$$= \underbrace{\sum_{j=1}^{n} f(x_{j-1})\Delta x = \Delta x \sum_{j=1}^{n} f(x_{j-1})}_{\text{Sigma Notation}}$$

Approximating the Area Under the Curve Using Right Endpoints

<u>Task</u>: Area under the curve $f(x) = x^2 + 1$ on the interval $[-1, 1.5]$ with **5 subintervals**



Width $\Delta x = \dfrac{1.5-(-1)}{5} = 0.5$

Approximate Area

$= A_1 + A_2 + A_3 + A_4 + A_5$

$= \Delta x f(-.5) + \Delta x f(0) + \Delta x f(.5) + \Delta x f(1) + \Delta x f(1.5)$

$= 0.5\left(5/4 + 1 + 5/4 + 2 + 13/4\right) = \boxed{35/8}$

More Generally $R_n = \Delta x f(x_1) + \cdots + \Delta x f(x_n)$

<u>Sigma Notation</u>: $R_n = \sum_{j=1}^{n} \Delta x f(x_j)$

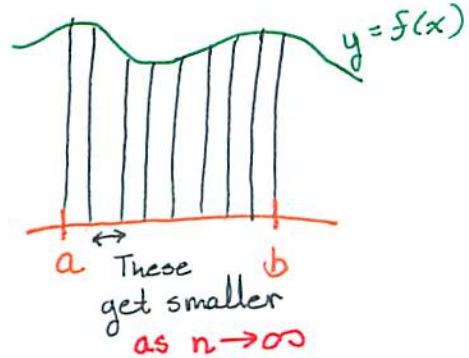
Limit of Riemann Sums is Area Under Curve

Area under curve:

$$\lim_{n\to\infty} L_n = \text{Area} = \lim_{n\to\infty} R_n$$

when these limits equal

Picture:



Example where equal:

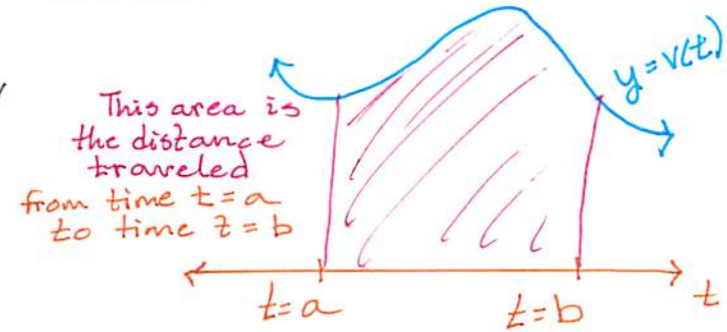
When $\left\{ \begin{array}{l} f \geq 0 \text{ AND} \\ f \text{ continuous} \end{array} \right\}$

on $[a, b]$

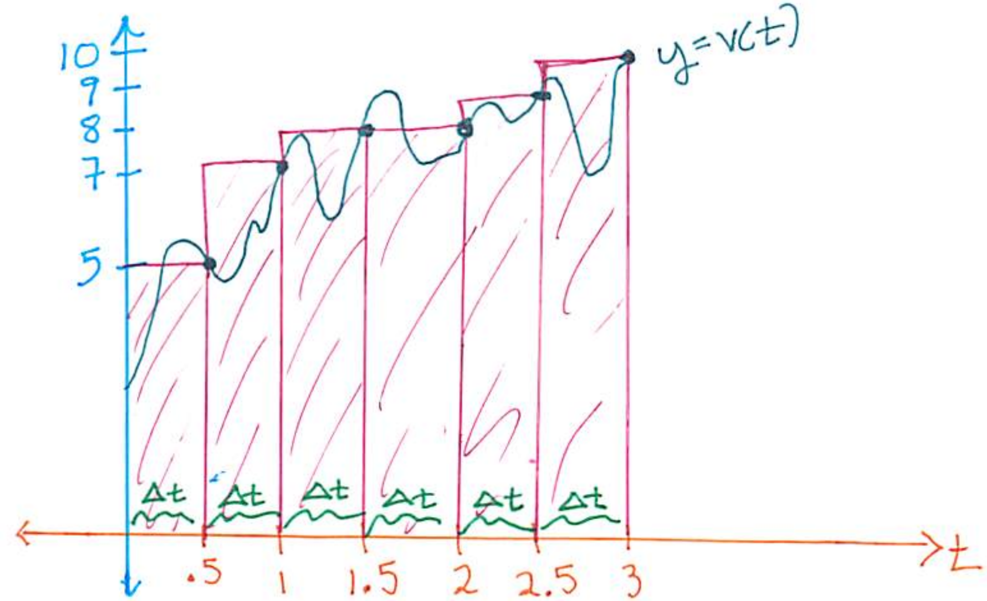
General / Specific Application of Approximating the Area Under the Curve

Recall: Riemann sums approximating a rate function ⇝ Value function

Set-Up: SUPPOSE $\begin{cases} v(t) = \text{velocity at time } t \\ v(t) \geq 0 \text{ whenever } a \leq t \leq b \end{cases}$ THEN



	time (seconds)	velocity (ft/sec)
	.5	5
$\Delta t = 1.0 - .5$	1.0	7
$\Delta t = .5$	1.5	8
$\Delta t = .5$	2.0	8
$\Delta t = .5$	2.5	9
$\Delta t = .5$	3.0	10



Task: Approximate

Area Approximates distance traveled in 3 secs $= \Delta t\, v(.5) + \Delta t\, v(1) + \Delta t\, v(1.5) + \Delta t\, v(2) + \Delta t\, v(2.5) + \Delta t\, v(3)$

$= .5(5 + 7 + 8 + 8 + 9 + 10) = \boxed{47/2 \text{ ft}}$

Riemann Sum ⇝ Definite Integral

Recall: To approximate the area under the curve $f(x)$ on $[a,b]$ with n subintervals

Have

$$\Delta x = \frac{b-a}{n}$$

$$x_j = a + j\Delta x$$

- Left Riemann Sum $\boxed{L_n} = \underbrace{f(x_0)}_{\text{height}}\overbrace{\Delta x}^{\text{width}} + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \Delta x \sum_{j=1}^{n} f(x_{j-1})$

- Right Riemann Sum $\boxed{R_n} = \underbrace{f(x_1)}_{\text{height}}\overbrace{\Delta x}^{\text{width}} + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \Delta x \sum_{j=1}^{n} f(x_j)$

$$\lim_{n\to\infty} L_n = \text{Area} = \lim_{n\to\infty} R_n$$

New: $\text{Area} = \underbrace{\lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{f(x_j^*)}_{\text{height}} \overbrace{\Delta x}^{\text{width}}}_{\text{Riemann Sum}}$ for any $x_{j-1} \leq x_j^* \leq x_j$

THEN

$$\int_{a}^{b} \overbrace{f(x)}^{\text{Integrand}} \underbrace{dx}_{\text{Tells what integrating "wrt"}} = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$

b ← upper limit

a ← lower limit

called $\boxed{\text{definite integral}}$ of f from a to b

When this limit exists & same for all choices x_j^* say f is $\boxed{\text{integrable}}$ on $[a,b]$

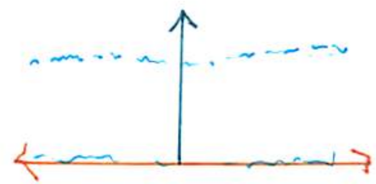
Integrability

NOT EVERYTHING'S INTEGRABLE EVERYWHERE!

Not Integrable

① $f(x) = \frac{1}{x}$ not integrable on $[0,1]$ since not defined at 0!

② $f(x) = \begin{cases} 0 & \text{if } x \text{ rational } (\text{integer}/\text{integer}) \\ 1 & \text{if } x \text{ irrational} \end{cases}$



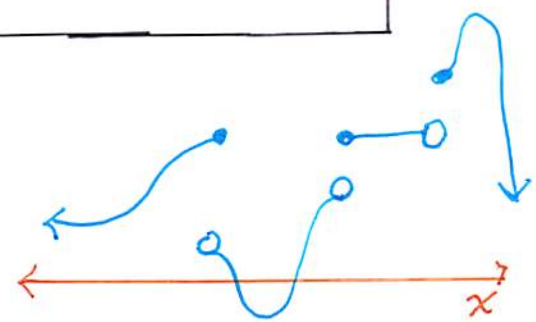
integrable nowhere!

But many functions are integrable:

Theorem:

IF $\begin{cases} f \text{ is continuous on } [a,b] \text{ OR} \\ f \text{ has only finitely many jump discontinuities} \end{cases}$

THEN f is integrable on $[a,b]$, i.e. $\int_a^b f(x)\,dx$ exists.



Integrals and "Net Area"

Have learned:

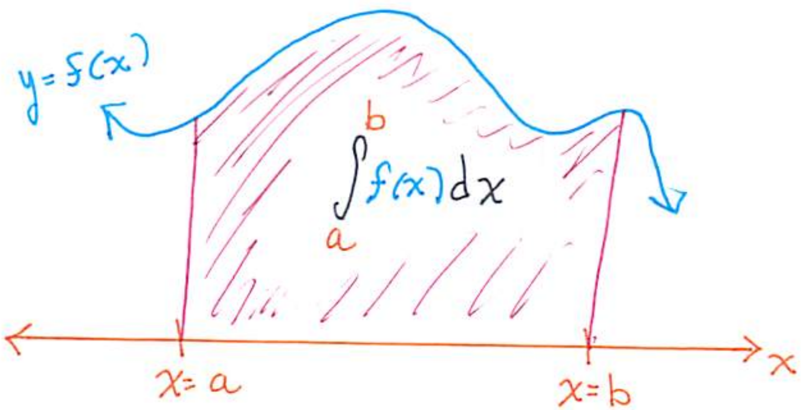
For a function f so that

$\begin{cases} f(x) \geq 0 \text{ for each } x \text{ in } [a,b] \text{ AND} \\ f \text{ is continuous on } [a,b] \end{cases}$

$\int_a^b f(x)\,dx$ is the area under the curve

$y = f(x)$

between $x = a$ AND $x = b$

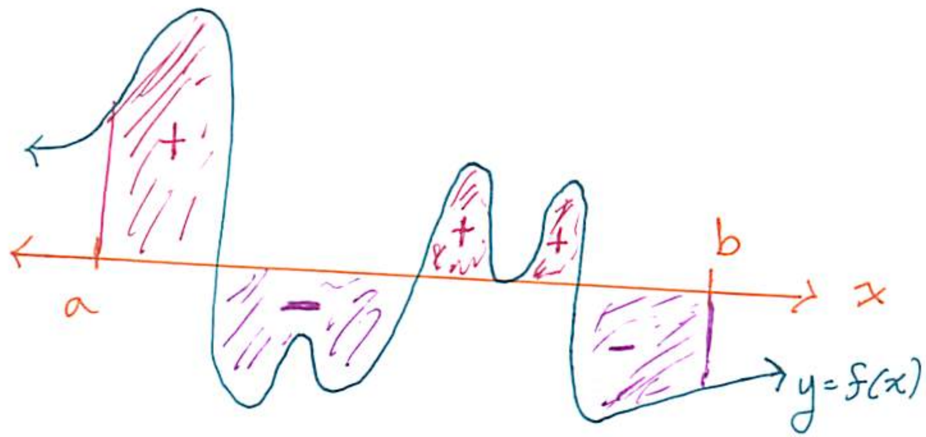


New

If don't have $f(x) \geq 0$

for all x values

$$\int_a^b f(x)\,dx = \text{Net Area} = \overbrace{A_1}^{\text{Area of region above x-axis \& below curve}} - \underbrace{A_2}_{\text{Area of region below x-axis \& above curve}}$$



Net Change Theorem

Net Change Theorem: Integral of Rate of Change = Net Change

Applications:

① $\frac{dn}{dt}$ = Rate of population growth (wrt)

$$\int_{t_1}^{t_2} \frac{dn}{dt} = n(t_2) - n(t_1)$$

= Net population change from time $= t_1$ to time $= t_2$

② $C(x)$ = Cost of producing x units of a commodity

$C'(x)$ = Marginal Cost

$$\int_{x_1}^{x_2} C'(x) = C(x_2) - C(x_1)$$

= Increase in cost when production increased from x_1 units to x_2 units