# A STUDY OF $L$-FUNCTIONS: AT THE EDGE OF THE CRITICAL STRIP AND WITHIN.

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# Abstract

In analytic number theory, and increasingly in other surprising places, $L$-functions arise naturally when describing algebraic and geometric phenomena. For example, when attempting to prove the Prime Number Theorem the values of $L$-functions on the 1-line played a crucial role. In this thesis we discuss the theory of $L$-functions in two different settings.

In the classical context we provide results which give estimates for the size of a general $L$-function on the right edge of the critical strip $\mathrm{Re}(s) = 1$ and provide a bound for the number of zeros for the classical Riemann zeta function inside the critical strip commonly referred to as a zero density estimate.

In the second setting we study $L$-functions over the polynomial ring $\mathbb{A} = \mathbb{F}_q[T]$, where $\mathbb{F}_q$ is a finite field. As $\mathbb{A}$ and $\mathbb{Z}$ have similar structure, $\mathbb{A}$ is a natural candidate for analyzing classical number theoretic questions. Additionally, the truth of the Riemann Hypothesis (RH) in $\mathbb{A}$ yields deeper unconditional results currently unattainable over $\mathbb{Z}$. We will focus on the distribution of values of specific $L$-functions in two different places: On the right edge of the critical strip, that is $\mathrm{Re}(s) = 1$ and values where $\frac{1}{2} < \mathrm{Re}(s) < 1$.

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# Table of Contents

# List of Tables

# List of Figures

# Notations

We introduce some basic notations and definitions for reference.

**Classical Setting:**

For this thesis $p$ denotes a prime, $q, a, n$ are integers, $x$ with or without subscript is a real number and $s = \sigma + it$ is a complex number with $\sigma, t \in \mathbb{R}$. The notation $p \equiv a \pmod q$ means there exists $n$ such that $p = nq + a$.

We use the following notation to describe the growth of functions:

- $f$ is *asymptotic* to $g$, written $f(x) \sim g(x)$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$,
- $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there is a constant $C$ such that for all $x$ large enough, $|f(x)| \le Cg(x)$,
- Similarly, let $h$ be a parameter, then $f(x) \ll_h g(x)$, means the constant $C$ depends on $h$.
- $f(x) = \Omega(g(x))$ if and only if $\limsup_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| > 0$.
- $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

An *arithmetic function* is any function $f : \mathbb{N} \to \mathbb{C}$. Of particular interest are *multiplicative* arithmetic functions which satisfy the condition: $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. We say the function is *completely multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

We use the following arithmetic functions throughout.

- The *von Mangoldt function* $\Lambda(n)$ is given by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k, p \text{ prime}, k \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

- The *Euler phi function* $\varphi(q)$ is given by $\varphi(q) = \#\{n \mid 1 \le n \le q \text{ and } (n,q) = 1\}$, where $(a,q) = \gcd(a,q)$ is the *greatest common divisor* of $a$ and $q$. We may also define $\varphi(q) = \#(\mathbb{Z}/q\mathbb{Z})^\times$. Note that $\varphi$ is multiplicative.

- The *Möbius function* is given by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not squarefree,} \\ (-1)^k & \text{if } n = p_1 \cdots p_k, p_i \text{ distinct primes.} \end{cases}$$

- The *divisor function* $d(n)$ is given by $\sum_{d|n} 1$. Note that $d(n)$ multiplicative.

- For $z \in \mathbb{C}$, the *generalized divisor function* $d_z$ is the multiplicative function defined on prime powers as follows

$$d_z(p^a) = \frac{\Gamma(z+a)}{\Gamma(z)a!}$$

and extending multiplicatively to all of $\mathbb{Z}_{\ge 0}$.

The *summatory function* of an arithmetic function $f$ is given by

$$F(x) = \sum_{n \le x} f(n).$$

- The following prime counting functions are examples of summatory functions:

$$\pi(x) = \sum_{p \le x} 1, \ \theta(x) = \sum_{p \le x} \log(p), \quad \text{and} \quad \psi(x) = \sum_{n \le x} \Lambda(n).$$

- More generally for $(a,q)=1$ we have

$$\pi(x,q,a)=\sum_{\substack{p\le x\\ p\equiv a (\mathrm{mod}\ q)}} 1,\ \theta(x,q,a)=\sum_{\substack{p\le x\\ p\equiv a (\mathrm{mod}\ q)}} \log(p) \ \text{ and } \ \psi(x,q,a)=\sum_{\substack{n\le x\\ n\equiv a (\mathrm{mod}\ q)}} \Lambda(n).$$

A *Dirichlet series* is a function of a complex variable $s$ given by

$$g(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

for $\{a_n\}_n$ a complex valued sequence. For an arithmetic function $f$, the *Dirichlet series associated to* $f$ is

$$g(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}.$$

The following are examples of Dirichlet series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \qquad \text{and} \qquad L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

where $\chi$ denotes a Dirichlet character modulo $q$.

We note that Dirichlet characters are completely multiplicative arithmetic functions. We have the following additional notation regarding them:

- $\chi_0$ denotes the *principal character* modulo $q$.
- The *Legendre symbol* is a special completely multiplicative arithmetic function which induces a Dirichlet character given by: Let $p$ be an odd prime and $(a,p)=1$ then

$$\left(\frac{a}{p}\right)=\begin{cases}1 & \text{if } x^2\equiv a \pmod{p} \text{ has a solution,}\\ -1 & \text{if } x^2\equiv a \pmod{p} \text{ does not have a solution.}\end{cases}$$

- Summation written as $\sum_{\chi \pmod{q}}$ means to vary over the characters $\chi$ modulo $q$.

The following are related to the zeros of $\zeta(s)$:

- $\rho = \beta + i\gamma$ with $0 < \beta < 1, \gamma \in \mathbb{R}$ denotes a *non-trivial zero* of $\zeta(s)$,

- $Z = \{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, 0 < \beta < 1\}$, represents the set of non-trivial zeros for $\zeta(s)$,

- Let $T > 0$, $N(T) = \{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ 0 < \beta < 1 \text{ and } 0 < \gamma \leq T\}$,

- Let $0 < \sigma < 1$, $T > 0$, $N(\sigma, T) = \{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ \sigma < \beta < 1 \text{ and } 0 < \gamma \leq T\}$,

We make use the following special functions:

- Let $s \in \mathbb{C}$, the *Gamma function* is given by

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx.$$

- Let $f$ be a function. The *Mellin transform of $f$* is given by

$$M_f(s) = \int_0^\infty x^{s-1} f(x) dx,$$

  and the *inverse Mellin transform of $f$* is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_f(s) x^{-s} ds.$$

  Note that $\Gamma$ is the Mellin transform of $e^{-x}$.

**Function Fields setting:**

Throughout the terms below $q = p^e$ with $e \geq 1$. $P$ will denote a monic irreducible polynomial.

- $\mathbb{F}_q$ denotes the finite field with $q$ elements.

- $\mathbb{A} = \mathbb{F}_q[T]$ the polynomial ring over $\mathbb{F}_q$ with indeterminate $T$.

- For $f \in \mathbb{A}$ with $f \neq 0$, when we write $f(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \cdots + \alpha_1 T + \alpha_0$, where $\alpha_i \in \mathbb{F}_q$ and $\alpha_n \neq 0$, we have the following notation:

  - The *degree* of $f$, denoted $\deg(f)$, is $n$.

- The *sign* of $f$, denoted $\operatorname{sgn}(f)$, is $\alpha_n$.
- If $\operatorname{sgn}(f) = 1$, we say that $f$ is *monic*. The monic polynomials in $\mathbb{A}$ are analogous with the positive integers in $\mathbb{Z}$.

- Note that $\mathbb{A}$ is a Principal Ideal Domain (PID). By $f\mathbb{A}$ we mean the ideal generated by $f$ and

$$|f| = \begin{cases} 0 & \text{if } f = 0, \\ \#(\mathbb{A}/f\mathbb{A}) = q^{\deg(f)} & \text{otherwise.} \end{cases}$$

  Note that $|\cdot|$ is a non-archimedean norm as $|f+g| \leq \max\{|f|, |g|\}$ where equality holds if $|f| \neq |g|$.

- The arithmetic functions have analogous definitions in $\mathbb{A}$. For example, the *Euler phi function* over $\mathbb{A}$ is given by $\varphi(f) = \#(\mathbb{A}/f\mathbb{A})^\times = |f| \prod_{P|f}(1 - \frac{1}{|P|})$.

- For $z \in \mathbb{C}$, the *generalized divisor function* $d_z$ is the multiplicative function defined on prime powers as follows

$$d_z(P^a) = \frac{\Gamma(z+a)}{\Gamma(z)a!}$$

  and extending multiplicatively to all of $\mathbb{Z}_{\geq 0}$.

- The prime counting functions over $\mathbb{A}$ are defined as follows:

$$\pi_q(k) = \#\{P : \deg(P) = k\} \text{ and } \pi_q(k, a, m) = \#\{P : P \equiv a \pmod{m}, \deg(P) = k\}.$$

- The *zeta function* and *Dirichlet L-series* over $\mathbb{A}$ are defined as follows: For $s \in \mathbb{C}$,

$$\zeta_{\mathbb{A}}(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} \text{ and } L(s, \chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s},$$

  where $\chi$ is a Dirichlet character modulo $g$ for some $g \in \mathbb{A}$.

- We have an analogous Legendre symbol: For $(a, P) = 1$, we define $\left(\frac{a}{|P|}\right)$ as follows:

$$a^{\frac{|P|-1}{2}} \equiv \left(\frac{a}{|P|}\right) \pmod{P}.$$

- The Legendre symbol satisfies the Law of Quadratic Reciprocity as well: For irreducibles $Q, P$ with $\deg(P) = \delta, \deg(Q) = \nu$, we have

$$\left(\frac{Q}{P}\right) = (-1)^{\frac{q-1}{2}\delta\nu}\left(\frac{P}{Q}\right).$$

  Note: the size of the finite field affects the sign. As such, if $q \equiv 1 \pmod 4$, then $\left(\frac{Q}{P}\right) = \left(\frac{P}{Q}\right)$.

# 1 Introduction

## 1.1 History and general description of the problem

This thesis's main focus is the study of $L$-functions, which are used extensively in analytic number theory to study discrete objects. For example, we can use $L$-functions to study the analytic properties of prime numbers. Our focus is on the behaviour of these functions in the critical strip, a special region of the complex plane where knowledge of an $L$-function's behaviour translates into knowledge of the discrete objects, say the primes. Additionally, there are many open problems concerning the behaviour of $L$-functions in this region including the location of the zeros within this strip.

The simplest $L$-function is the Riemann zeta function $\zeta(s)$, which was critical in the proof of the Prime Number Theorem. Analytic techniques are not always required to understand questions about the primes. For example, Euclid proved there are infinitely many primes using only elementary properties of the integers. However, the study of $\zeta(s)$ led to deeper results about the primes: Hadamard and de la Vallée Poussin proved the Prime Number Theorem by studying the behaviour of $\zeta(1+it)$ where $t \in \mathbb{R}$. To describe their approach, consider the following prime counting functions:

$$\pi(x) = \#\{p \leq x \mid p \text{ is a prime}\} = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1,$$

$$\theta(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} \log(p), \quad \text{and} \quad \psi(x) = \sum_{n \leq x} \Lambda(n), \tag{1.1}$$

where $\Lambda(n)$ is the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k, p \text{ prime }, k \in \mathbb{N} \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

In 1792, Gauß' computations led him to conjecture that $\pi(x) \sim x/\log(x)$. Note that

$$\pi(x) \sim \frac{x}{\log x} \iff \theta(x) \sim x \iff \psi(x) \sim x.$$

Hadamard and de la Vallée Poussin focused on $\psi(x)$, the reason for which will become clear below.

In 1859, Riemann [92] considered the function $\zeta(s) = \sum_{n\geq 1} 1/n^s$, $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > 1$. In this note, among other facts, he showed that $\zeta(s)$ can be analytically extended to $\mathbb{C}$ with the exception of a pole at $s = 1$. Furthermore, he conjectured the following explicit formula for $\psi(x)$: if $x$ is not a prime power, then

$$\psi(x) = x - \sum_{\rho\in Z} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}), \tag{1.3}$$

where

$$Z = \{\rho = \beta + i\gamma \in \mathbb{C} : \zeta(\rho) = 0 \text{ and } 0 \leq \beta \leq 1\}. \tag{1.4}$$

In 1895, von Mangoldt [76] proved this explicit formula, which Hadamard [41] and de la Vallée Poussin [109] independently used to prove Gauß' conjecture the next year. They showed that if $\rho \in Z$ then $\beta \neq 1$, or equivalently $\zeta(1 + it) \neq 0$ for $t \in \mathbb{R} \setminus \{0\}$. This breakthrough is the first demonstration of the value in understanding the behaviour of $\zeta(s)$ in this region. Indeed, as we will see throughout, understanding $L$-functions on the 1-line is a central theme of current research.

The above technique is not limited to the prime numbers. Encoding sequences in an $L$-function and analyzing its behaviour has yielded many important results. In 1837, Dirichlet [25] proved that there are infinitely many primes in arithmetic progression, $p = nq + a$ for $a, q$ relatively prime, by studying Dirichlet $L$-functions (see Section 2.4 for a precise definition). Dirichlet proved this result by showing $L(1, \chi) \neq 0$ for all non-trivial $\chi$. Here $L(s, \chi)$ is defined as follows: for $\chi$ a nontrivial character associated to $(\mathbb{Z}/q\mathbb{Z})^\times$ and $\text{Re}(s) > 0$, we define $L(s, \chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$, where $\chi(n + q) = \chi(n)$ if $\gcd(n, q) = 1$ and $\chi(n) = 0$ otherwise. Following Riemann's lead, we can analytically extend $L(s, \chi)$ to $\mathbb{C}$ and obtain a proof of the prime number theorem in arithmetic progressions (cf. de la Vallée Poussin [108]).

Let us focus on another problem which is influenced by $L$-functions: consider the field $\mathbb{Q}(\sqrt{d})$ defined by

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\},$$

where for now think of $d \equiv 1 \pmod 4$. We want to understand the structure $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, the ring of integers of $\mathbb{Q}(\sqrt{d})$. What do the elements of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ look like? Do its elements uniquely factorize into irreducible elements (primes)? If yes, we might ask what the distribution of these irreducible elements look like. If no, we would like to investigate why not.

An important tool for studying $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is the class group, denoted $\mathcal{C}l(\mathbb{Q}(\sqrt{d}))$ (see Definition 2.10), and $h_d = |\mathcal{C}l(\mathbb{Q}(\sqrt{d}))|$, the class number. It is well known that $h_d = 1$ if and only if $\mathcal{C}l(\mathbb{Q}(\sqrt{d}))$ is a unique factorization domain (see Chapter 2). The class group can be quite arduous to compute for a specific field and nearly impossible for generic fields. Instead of calculating $h_d$ for specific $d$, analytic number theorists ask more qualitative questions about the growth of $h_d$ as $|d| \to \infty$, or about how many $d \in \mathbb{Z}$ satisfy $h_d = 1$.

Let us consider the second question: How many $d \in \mathbb{Z}$ satisfy $h_d = 1$? If $d < 0$, then it is known that only finitely many $h_d = 1$. Thus, the growth of $h_d$ is straightforward: In 1801, Gauß conjectured (see *Disquisitiones Arithmeticae* [36])

$$\frac{1}{N} \sum_{k \le N} h_{-4k} \sim \frac{4\pi}{21\zeta(3)} \sqrt{N}. \tag{1.5}$$

Siegel [101] settled (1.5) by utilizing Dirichlet's class number formula (proven in [24]), which connects $h_d$ to $L(1, \chi_d)$, where $\chi_d$ is the quadratic character modulo $|d|$ associated to $d$.

If $d > 0$, then it is still an open question whether there are infinitely many $h_d = 1$. This question is more complicated since the structure of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is harder to study. Indeed, Gauß conjectured

$$\frac{1}{N} \sum_{k \le N} h_{4k} R_{4k} \sim \frac{4\pi^2}{21\zeta(3)} \sqrt{N}, \tag{1.6}$$

where $R_d$ is the regulator of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ (see Definition 2.11). Siegel also settled (1.6) using the second case of Dirichlet's class number formula. Equations (1.5) and (1.6) provide the first moment of $h_d$ and $h_d R_d$, respectively. It is an active area of research to understand the distribution and extreme values of $h_d$ as $|d| \le x$ ranges over fundamental discriminants. In this setting, Granville and Soundararajan [40] and

Lamzouri (several works) have investigated these questions. Section 5.1 explains their results in more detail.

The examples provided are among the simplest $L$-functions. As the discrete objects we study increase in complexity so do the associated $L$-functions. There are invariants of $L$-functions that aid in classifying and understanding of these objects. These invariants measure complexity and group $L$-functions into classes with the goal of giving a general description for all members of the class. Two prominent invariants are the degree and the conductor of an $L$-function. Both $\zeta(s)$ and $L(s,\chi)$ have degree 1, while $\zeta(s)$ has conductor 1 and $L(s,\chi)$ has conductor $q$ where $\chi$ is a character of $(\mathbb{Z}/q\mathbb{Z})^{\times}$. See Section 2.5 for more details.

This thesis will be divided into results of two different types, quantitative and qualitative. The quantitative results give explicit information about the conditional growth of $L$-functions on the line $1+it$. They will also provide unconditional counts for other associated quantities, such as the number of zeros we expect $\zeta(s)$ to have away from the line $1/2+it$ in absence of RH. The qualitative results will adapt techniques of Granville, Lamzouri, Lester, Radziwiłł and Soundararajan to a new space which shares many properties in common with $\mathbb{Q}$ but enjoys the luxury of RH being true. We focus on providing results for $L$-functions over the new space taking $s=1$ first and then generalizing to $1/2<\sigma<1$ later. In these works we will uncover some surprising results which are not present in the original setting.

The rest of this chapter describes the results we prove.

## 1.2 Explicit Classical Problems: Counting Zeros

The Prime Number Theorem, independently proven by Hadamard and de la Vallée Poussin in 1896, implies the weighted prime counting function $\psi(x) \sim x$ as $x \to \infty$. The proof shows that the sum of the zeros in the explicit formula in (1.3) has a smaller order of magnitude than $x$, the desired main term. Obtaining sharp bounds for $E(x) := |\psi(x) - x|$ is a classic problem in explicit number theory. There has been a recent flurry of publications attempting to reduce the size of $E(x)$, for example [27], [28] and [17], as results of this form are often useful in surprising places. For example, Hough [47] used Faber and Kadiri [32] to provide a solution to the minimum modulus problem for covering systems.

There are many ingredients which are used to obtain explicit results for $E(x)$. In 1941 Rosser [95] initiated a program for determining explicit bounds for $E(x)$. His results address the fact that (1.3) cannot explicitly evaluate $E(x)$, since the sum of the zeros in (1.3) is not absolutely convergent. He and Schoenfeld [97] improved on these ideas. Recently, Faber and Kadiri [32] developed a smoothing technique which generalizes Rosser and Schoenfeld's results allowing more flexibility in evaluating the sum over the zeros.

The Riemann Hypothesis, the most famous conjecture of Riemann's memoir [92], conjectures where the zeros of $\zeta(s)$ lie:

**Conjecture 1.1** (Riemann Hypothesis (RH), 1859)**.** *Let $Z$ be defined as in* (1.4)*. If $\rho = \beta + i\gamma \in Z$, then $\beta = \frac{1}{2}$.*

Note that RH implies that $E(x) \ll \sqrt{x} \log^2 x$, which is a far cry from what we can prove unconditionally.

Explicitly determining $E(x)$ revolves around a careful analysis of the zeros. This analysis requires splitting the critical strip into regions based on the following explicit information about the zeros:

- A partial verification of RH: for some fixed $H_0$, if $\rho \in Z$ and $|\mathfrak{Im}(\rho)| \le H_0$, then $\mathfrak{Re}(\rho) = 1/2$,
- The zero-free region: an explicit region inside the critical strip with no zeros,
- An explicit bound for the zero counting function

$$N(T) = \#\{\rho | \rho \in Z, 0 < \mathfrak{Re}(\rho) < 1 \text{ and } |\mathfrak{Im}(\rho)| \le T\} \sim \frac{T}{2\pi} \log T. \tag{1.7}$$

Faber and Kadiri [32] showed that more detailed information about the density of the zeros in the critical strip can further reduce the error term. More specifically, they needed an explicit zero density estimate for

$$N(\sigma, T) = \#\{\rho | \rho \in Z, \sigma \le \mathfrak{Re}(\rho) < 1 \text{ and } 0 \le \mathfrak{Im}(\rho) \le T\}.$$

Under RH, $N(\sigma, T) = 0$ for any $\sigma > 1/2$. Unconditionally, we know $N(\sigma, T) = o(N(T))$. The first such result is due to Kadiri [56, Theorem 1.1]:

**Theorem 1.1** (Kadiri)**.** *Let $\sigma \ge 0.55$ and $T \ge H_0$, where $H_0$ refers to the partial verification of RH. Let $\sigma_0$ and $H$ be such that $0.5208 < \sigma_0 < 0.9723$ and $10^3 \le H \le H_0$. Then there exists positive constants $b_1$, $b_2$ and $b_3$ depending on $\sigma, \sigma_0$ and $H$ such that*

$$N(\sigma, T) \le b_1(T - H) + b_2(\log TH)^2 + b_3.$$

*Numerical values of the $b_i'$s are recorded in [56, Table 1].*

Since (1.7) counts the total number of zeros in the strip, Theorem 1.1 implies that, for any $\epsilon > 0$, most zeros satisfy $|\beta - \frac{1}{2}| < \epsilon$. Kadiri, Ng and I [59] (see Chapter 3 of this thesis) significantly improved the explicit results for $N(\sigma, T)$. The main result of Chapter 3 is as follows:

**Theorem 1.2** (Kadiri, L, Ng)**.** *Let* $\frac{10^9}{H_0} \le k \le 1, d > 0, H \in [1002, H_0)$, $\alpha > 0$, $\delta \ge 1$, $\eta_0 = 0.23622\ldots$, $1 + \eta_0 \le \mu \le 1 + \eta$, *and* $\eta \in (\eta_0, \frac{1}{2})$ *be fixed. Let* $\sigma > \frac{1}{2} + \frac{d}{\log H_0}$. *Then there exist* $\mathcal{C}_1, \mathcal{C}_2 > 0$ *such that, for any* $T \ge H_0$,

$$N(\sigma, T) \le \frac{(T-H)(\log T)}{2\pi d} \log\Big(1 + \frac{\mathcal{C}_1(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)} T^{\frac{8}{3}(1-\sigma)}}{T-H}\Big) + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2, \tag{1.8}$$

*where* $\mathcal{C}_1 = \mathcal{C}_1(\alpha, d, \delta, k, H, \sigma)$ *and* $\mathcal{C}_2 = \mathcal{C}_2(d, \eta, k, H, \mu, \sigma)$ *are defined in* (3.117) *and* (3.118). *Since* $\log(1+x) \le x$ *for* $x \ge 0$, (1.8) *implies*

$$N(\sigma, T) \le \frac{\mathcal{C}_1}{2\pi d}(\log(kT))^{2\sigma}(\log T)^{5-4\sigma} T^{\frac{8}{3}(1-\sigma)} + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2. \tag{1.9}$$

*In addition, numerical results are displayed in tables in Section 3.5.*

We give both forms of the bound for $N(\sigma, T)$ because (1.9) is significantly easier to apply when attempting to provide results for $E(x)$. This work combines ideas due to Ingham, Montgomery, Ramaré, Vaughan and Gallagher.

This result has several applications. We use the above result for explicit bounds of $E(x)$ and other related problems. For example, Kadiri and I [58] use Theorem 1.1 to improve on explicit results related to Bertrand's postulate. More specifically, we find constants $\Delta$ such that, for $x$ large enough, there is always a prime in the interval $(x(1 - \Delta^{-1}), x)$, where $\Delta$ is an explicitly computed large constant. Theorem 1.2 could be used to increase $\Delta$, which would further shrink the interval $(x(1 - \Delta^{-1}), x)$.

## 1.3 Explicit Classical Problems: Conditional Results for a general $L$-function on the 1-line

Let $f$ be an arithmetic or geometric object, for example a Dirichlet character or an elliptic curve over $\mathbb{Q}$, and $L(s,f)$ be the associated $L$-function. This portion of the thesis is used to provide explicit upper and lower bounds for $|L(1,f)|$. As with Riemann zeta function, we have a conjecture regarding the location of the non-trivial zeros of these $L$-functions:

**Conjecture 1.2** (Grand Riemann Hypothesis (GRH))**.** *Let $L(s,f)$ be an $L$-function defined in Section 2.5. Then all zeros of $L(s,f)$ such that $0 < \mathfrak{Re}(s) < 1$ are on the critical line $\mathfrak{Re}(s) = \frac{1}{2}$.*

We first consider Dirichlet $L$-functions given their historical importance. Unconditionally, for any $\epsilon > 0$ and any $\chi$, a Dirichlet character modulo $q$, we have

$$\frac{1}{q^{\epsilon}} \ll |L(1,\chi)| \ll \log q.$$

Louboutin [72] has obtained explicit upper bounds of this shape. Upon GRH, Littlewood [71] showed

$$\frac{\zeta(2)(1+o(1))}{2e^{\gamma}\log\log q} \leq |L(1,\chi)| \leq (2e^{\gamma}+o(1))\log\log q. \tag{1.10}$$

In 2015, Lamzouri, Li and Soundararajan [65] gave an explicit expression for the $o(1)$ term:

$$\frac{\frac{\zeta(2)}{2e^{\gamma}}}{\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q} + \frac{14\log\log q}{\log q}} \leq |L(1,\chi)| \leq 2e^{\gamma}\left(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q}\right). \tag{1.11}$$

By adapting the ideas used to prove (1.11) I provide explicit bounds for degree $d$ $L$-functions on the 1-line in terms of the analytic conductor, $C(f)$ (see Theorem 1.3 below). The analytic conductor of an $L$-function is an invariant which combines information about the degree of the $L$-function, the conductor of $f$ and some other local parameters, $\kappa_i$ for $1 \leq i \leq d$. These local parameters are not well understood objects in all cases and in order to give our bounds we also assume the Ramanujan-Petersson conjecture (see Conjecture 2.1).

Assuming the Ramanujan-Petersson conjecture and GRH, Littlewood's method applied to degree $d$ $L$-

functions yields:

$$\left(\frac{\zeta(2)}{2e^{\gamma}\log\log C(f)}\right)^{d}(1+o(1)) \ \le\ |L(1,f)| \ \le\ (1+o(1))(2e^{\gamma}\log\log C(f))^{d}. \tag{1.12}$$

Chapter 4 of this thesis provides an explicit expression for the $o(1)$ term (see also [73]). More specifically, the upper bound is as follows:

**Theorem 1.3** (L.)**.** *Let $d \geq 1$ be a fixed positive integer and let $L(s,f)$ be an $L$-function of degree $d$ with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s,f)$. Then, for $C(f)$ chosen such that $\log C(f) \geq 23d$, we have*

$$|L(1,f)| \ \le\ 2^{d}e^{d\gamma}\left((\log\log C(f)-\log 2d)^{d}+\frac{d}{2}(\log\log C(f)-\log 2d)^{d-1}+\frac{dK(d)}{4}(\log\log C(f)-\log 2d)^{d-2}\right), \tag{1.13}$$

*where*

$$K(d) = 2.31 + \frac{22.59}{d}(e^{0.31d} - 1 - 0.31d). \tag{1.14}$$

There is a similar expression for the lower bound (see Theorem 4.3). Expanding (1.13) we see that the coefficient of the $(\log\log C(f))^{d-1}$ is negative. Thus we improve the quality of Littlewood's result.

## 1.4 Classical Problems over Function Fields: Distribution of values of $L(1,\chi_D)$

The second portion of the thesis considers the qualitative problems mentioned at the end of Section 1.1. We study $L$-functions over $\mathbb{A} = \mathbb{F}_q[T]$, where $T$ is an indeterminate and requiring that $\mathbb{F}_q$ is a finite field with $q$ elements. Note that $q = p^e$, where $p$ a prime and $e \in \mathbb{N}$, we have $\mathbb{F}_q$ has characteristic $p$. Recall the following notation for $f \in \mathbb{A}$ where $f(T) = \alpha_n T^n + \alpha_{n-1} + \cdots + \alpha_0$, for $\alpha_i \in \mathbb{F}_q$ with $\alpha_n \neq 0$:

- The *degree* of $f$ is $n$. We write $\deg(f) = n$
- We say $f$ is *monic* if $\alpha_n = 1$.
- The *norm* of $f$ in $\mathbb{A}$ is 0 if $f = 0$ and $q^{\deg(f)}$ otherwise. We write $|f|$ for the norm of $f$.

We note that $\mathbb{A}$ shares many properties with $\mathbb{Z}$. Since $\mathbb{F}_q$ is a field, $\mathbb{A}$ is a Principal Ideal Domain (PID), and as such $\mathbb{A}$ is a Unique Factorization Domain (UFD). Thus, every element of $\mathbb{A}$ can be factored uniquely (up to order and multiplication by units) into products of monic irreducible elements. Further, since we are in a PID, the primes of $\mathbb{A}$ correspond with the monic irreducible elements of $\mathbb{A}$. So, the monic irreducible polynomials in $\mathbb{A}$ are directly analogous to the prime numbers in $\mathbb{Z}$.

Given the above similarities between $\mathbb{A}$ and $\mathbb{Z}$ it is natural to ask which results from $\mathbb{Z}$ carry over to $\mathbb{A}$. In $\mathbb{Z}$ we asked about the distribution of primes, and in $\mathbb{A}$ we want the same information: Let $\pi_q(k)$ denote the number of monic irreducible polynomials in $\mathbb{A}$ with degree $k$. The following relation

$$\sum_{k|m} k\pi_q(k) = q^m, \tag{1.15}$$

implies the prime number theorem for $\mathbb{A}$ (cf. [94, Theorem 2.2]):

$$\pi_q(n) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right). \tag{1.16}$$

Let $x = q^n$ in (1.16). Then the main term is $\frac{x}{\log_q x}$ just as in the classic prime number theorem. Note that (1.16) gives an error term of $\frac{\sqrt{x}}{\log_q x}$, which even RH does not imply over $\mathbb{Z}$. Weil [112] proved RH in $\mathbb{A}$, see Theorem 2.9. However the proof of (1.15) does not require such a deep result, despite the involvement of the zeta function, $\zeta_{\mathbb{A}}(s)$. Although RH is not needed to obtain (1.16), we can obtain deeper unconditional results in $\mathbb{A}$ with it. We focus on these results in the last two chapters.

The first question we will focus on is the class number associated to quadratic extensions of $\mathbb{F}_q(T)$, the fraction field of $\mathbb{A}$: Let $D \in \mathbb{A}$ be square-free, and let $K = \mathbb{F}_q(T)$. Then as before, we study $K(\sqrt{D})$ and its integral elements, denoted $\mathcal{O}_{K(\sqrt{D})}$. Recall that class numbers are intricately linked with UFDs. More specifically, the class number, $h_d$, is 1 if and only if $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a UFD. In Chapter 5, we study the class number associated to $K(\sqrt{D})$. We note that there are some slight differences from the number field case. For example, the class number of $K(\sqrt{D})$ is the size of Picard group associated to $\mathcal{O}_{K(\sqrt{D})}$, whereas $h_d$ is the size of the class group. See Section 2.6 for more details about the differences.

Recall the difference between the positive and negative cases of the class number $h_d$. This discrepancy strengthens when considering the class number formula, which connects $h_d$ to a particular $L$-function. A similar discrepancy occurs in function fields for the degree of square-free polynomial $D$: $d < 0$ corresponds to $\deg(D)$ odd, and $d > 0$ corresponds to $\deg(D)$ even. Artin [5] also developed an analogous class number formula which relates $h_D$ with a special value of a Dirichlet $L$-function over $\mathbb{A}$. More precisely, he connected $h_D$ to $L(1, \chi_D)$, where $\chi_D$ is the Kronecker symbol associated to $D$.

When the degree of $D$ is odd, the class number formula ties the value of $L(1, \chi_D)$ directly to $h_D$. Hoffstein and Rosen [46] used the class number formula to obtain the following average result: For odd $M > 0$,

$$\frac{1}{q^M} \sum_{\substack{D \text{ monic} \\ \deg(D)=M}} h_D = \frac{\zeta_{\mathbb{F}_q[T]}(2)}{\zeta_{\mathbb{F}_q[T]}(3)} q^{(M-1)/2} - q^{-1}. \tag{1.17}$$

Note that $h_D$ and $h_d$ have similar average growth since $\sqrt{|D|} = q^{M/2}$.

When the degree of $D$ is even, the class number formula ties the value of $L(1, \chi_D)$ to $h_D R_D$, where $R_D$ is the regulator of $\mathcal{O}_{K(\sqrt{D})}$. Hoffstein and Rosen [46] provide an average result for this case: For even $M > 0$,

$$\frac{1}{q^M} \sum_{\substack{D \text{ monic} \\ \deg(D)=M \\ M\neq\square}} h_D R_D = \frac{1}{q-1} \left( \frac{\zeta_{\mathbb{F}_q[T]}(2)}{\zeta_{\mathbb{F}_q[T]}(3)} q^{M/2} - (2 + (1 - q^{-1})(M-1)) \right). \tag{1.18}$$

There are two natural limits to consider in $\mathbb{F}_q$. Hoffstein and Rosen [46] considered $\deg(D)$ fixed while $q \to \infty$. We consider $q$ fixed while $\deg(D) \to \infty$. As such, we introduce some new notation.

Let $q$ be fixed and $D \in \mathcal{H}_n$, where

$$\mathcal{H}_n = \{F \in \mathbb{A} : F \text{ monic}, \ \deg(F) = n, F \text{ square-free}\}.$$

Andrade [4] (2012) proved that the mean value of $h_D$ with $D$ taken over $\mathcal{H}_{2g+1}$ with $g \to \infty$, is asymptotic to $C_q\sqrt{|D|}$ for $C_q$ a constant depending on $q$. This result agrees with the expectation seen in both (1.5) and (1.17). Jung [53, 54] (2014) proved the mean value of $h_D R_D$ with $D$ taken over $\mathcal{H}_{2g+2}$ with $g \to \infty$ is asymptotic to $C'_q\sqrt{|D|}$ for $C'_q$ a constant depending on $q$. Again, this result agrees with expectation seen in (1.6) and (1.18). The approaches of [4, 53, 54] adapt Siegel's proof of the average growth for class numbers associated to number fields.

Note the above yields information about the first moments for $h_D$ and $h_D R_D$. In Chapter 5 we expand the knowledge of the distribution of $h_D$ and $h_D R_D$ over $\mathcal{H}_{2g+1}$ and $\mathcal{H}_{2g+2}$, respectively, as $g \to \infty$. We achieve this goal by studying the complex moments of the appropriate $L$-functions over $\mathcal{H}_n$. Additionally, we use the information about the distribution to give us insight into the extreme values that the $L$-functions can attain, and thus the extremal class numbers. Below we list some of the results proven in Chapter 5 along with how they compare the the corresponding number field results.

The first result we discuss is about extreme values of $L(1, \chi)$:

**Proposition 1.1** (L.)**.** *Let $F$ be a monic polynomial, and $\chi$ be a non-trivial character on $(\mathbb{A}/F\mathbb{A})^{\times}$. For any complex number $s$ with $\mathfrak{Re}(s) = 1$ we have*

$$\frac{\zeta_{\mathbb{A}}(2)}{2e^{\gamma}}(\log_q \log_q |F| + O(1))^{-1} \ \le\ |L(s, \chi)| \ \le\ 2e^{\gamma} \log_q \log_q |F| + O(1). \tag{1.19}$$

With the above discussion in mind, if $\chi$ is the Kronecker symbol $\chi_D$ for some $D \in \mathcal{H}_n$ in (1.19), then

$$\frac{\zeta_{\mathbb{A}}(2)}{2e^{\gamma}}(\log_q \log_q |D| + O(1))^{-1} \le |L(1, \chi_D)| \le 2e^{\gamma} \log_q \log_q |D| + O(1). \tag{1.20}$$

Proposition 1.1 is a direct analogue to Littlewood's result [70] (see (1.10)) which requires GRH for $L(1, \chi)$. Recall the refined result in [65] (see (1.11)). We see that the main term on either side aligns with exactly what is known assuming GRH in the classical situation.

Building on earlier work of Montgomery and Odlyzko [82], Montgomery and Vaughan probabilistically argue that the number of fundamental discriminants $d$ which satisfy both $|d| \le x$ and $L(1, \chi_d) > e^{\gamma}\tau$ must lie between $\exp(-Ce^{\tau}/\tau)$ and $\exp(-ce^{\tau}/\tau)$ for some $C > c > 0$ constants. In addition, they also conjecture:

**Conjecture 1.3** (Montgomery-Vaughan)**.** *For any $\epsilon > 0$ there are only finitely many $d$ with*

$$L(1, \chi_d) > e^{\gamma}(\log \log |d| + (1 + \epsilon) \log \log \log |d|)$$

*or with*

$$L(1, \chi_d) < \frac{\zeta(2)}{e^{\gamma}}(\log \log |d| + (1 + \epsilon) \log \log \log |d|)^{-1}.$$

In line with this, we make the following conjecture based on some distribution results in Chapter 5:

**Conjecture 1.4** (L.)**.** *Let $n$ be large.*

$$\max_{D \in \mathcal{H}_n} L(1, \chi_D) = e^{\gamma}(\log_q n + \log_q \log_q n) + O(1),$$

*and*

$$\min_{D\in\mathcal{H}_n} L(1,\chi_D) = \zeta_{\mathbb{A}}(2)e^{-\gamma}(\log_q n + \log_q\log_q n + O(1))^{-1}.$$

Note that (1.20) falls short of Conjecture 1.4 by a factor of 2 for both the upper and lower bounds for $L(1,\chi_D)$. A similar difference occurs in the classical case.

We use the method of moments (see Section 2.7) to obtain the desired distribution results. As such, we need as many moments of $L(1,\chi_D)$ as possible.

**Theorem 1.4** (L.)**.** *Let $n$ a positive integer and $z\in\mathbb{C}$ be such that $|z|\le \frac{n}{260\log_q n\log\log_q n}$ and let $c_0>0$ be a constant. Then*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n} L(1,\chi_D)^z = \sum_{f\ monic}\frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\left(1+O\left(\frac{1}{n^{11}}\right)\right)+O\left(q^{-\frac{n}{c_0\log_q n}}\right).$$

Here $d_z(f)$ is the generalized divisor function defined in (5.6). I prove Theorem 1.4 by connecting $L(1,\chi_D)$ to a random model, $L(1,\mathbb{X})$, described at the end of Section 2.7, where $\mathbb{X}$ is a random variable chosen to mimic $\chi_D$.

I then use Theorem 1.4 to prove the tail of the distribution of values of $L(1,\chi_D)$ over $D\in\mathcal{H}_n$ decays doubly exponentially by combining Theorems 1.5 and 1.6.

**Theorem 1.5** (L.)**.** *Let $n$ be large. Uniformly in $1\le\tau\le\log_q n - 2\log_q\log_q n - \log_q\log_q\log_q n$ we have*

$$\frac{1}{|\mathcal{H}_n|}|\{D\in\mathcal{H}_n : L(1,\chi_D) > e^{\gamma}\tau\}| = \Phi_{\mathbb{X}}(\tau)\left(1+O\left(\frac{e^{\tau}(\log n)^2\log_2 n}{n}\right)\right),$$

*and*

$$\frac{1}{|\mathcal{H}_n|}|\{D\in\mathcal{H}_n : L(1,\chi_D) < \frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau}\}| = \Psi_{\mathbb{X}}(\tau)\left(1+O\left(\frac{e^{\tau}(\log n)^2\log_2 n}{n}\right)\right).$$

Here $\Phi_{\mathbb{X}}(\tau)$ and $\Psi_{\mathbb{X}}(\tau)$ are probability functions associated to the random model defined as follows:

$$\Phi_{\mathbb{X}}(\tau) := \mathbb{P}(L(1,\mathbb{X}) > e^{\gamma}\tau) \text{ and } \Psi_{\mathbb{X}}(\tau) := \mathbb{P}\left(L(1,\mathbb{X}) < \frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau}\right),$$

where $\mathbb{P}$ is the probability of an event occurring. Finally, $\Phi_{\mathbb{X}}(\tau)$ and $\Psi_{\mathbb{X}}(\tau)$ behave as follows:

**Theorem 1.6** (L.). *For any large $\tau$ we have*

$$\Phi_{\mathbb{X}}(\tau) = \exp\left(-C_1(q^{\{\log_q \kappa(\tau)\}})\frac{q^{\tau - C_0(q^{\{\log_q \kappa(\tau)\}})}}{\tau}\left(1 + O\left(\frac{\log \tau}{\tau}\right)\right)\right), \tag{1.21}$$

*where $\kappa(\tau)$ is defined by* (5.29)*, $C_0(t) = G_2(t)$, $C_1(t) = G_2(t) - G_1(t)$ and $G_i(t)$ are defined in* (5.34) *and* (5.36) *respectively. Furthermore we have*

$$-\frac{1}{\log q} + \log(\cosh(c))/c - \tanh(c) < -C_1(q^{\{\log_q \kappa(\tau)\}}) < \log(\cosh(q))/q - \tanh(q),$$

*where $c = 1.28377....$ In particular, $C_1(t) > 0$. The same results hold for $\Psi_{\mathbb{X}}$.*
*Additionally, if we let $0 < \lambda < e^{-\tau}$, then*

$$\Phi_{\mathbb{X}}(e^{-\lambda}\tau) = \Phi_{\mathbb{X}}(\tau)(1 + O(\lambda e^{\tau})) \text{ and } \Psi_{\mathbb{X}}(e^{-\lambda}\tau) = \Psi_{\mathbb{X}}(\tau)(1 + O(\lambda e^{\tau})). \tag{1.22}$$

To put this in context, in the classical case of Dirichlet $L$-functions $L(1, \chi_d)$, Granville and Soundararajan [40] (case $d < 0$) and separately Lamzouri and Dahl [20] ($d > 0$, a special shape) find an expression for $\Phi(\tau)$ with a similar structure to (1.21) except, in both [40, 20], $C_0$ and $C_1$ are constants, whereas in (1.21), they vary depending on $\tau$.

## 1.5 Classical Problems over Function Fields: Distribution of values of $L(\sigma, \chi_D)$ for $\frac{1}{2} < \sigma < 1$

The distribution of values of $L$-functions at points within (and very near to) the critical strip has received a lot of attention. In the early twentieth century Bohr showed the following result: Let $\epsilon > 0$ and $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then

$$|\{s \in \mathbb{C} : 1 < \mathfrak{Re}(s) < 1 + \epsilon, \zeta(s) = c\}| = \infty.$$

Later, in 1930, Bohr and Jessen [9] refined these ideas using probabilistic methods to describe the behaviour of

$$\Phi_T^{\zeta}(\sigma, \tau) = \frac{1}{T}\text{meas}\{t \in [T, 2T] : \log|\zeta(\sigma + it)| > \tau\} \text{ with } \mathfrak{Re}(s) > 1/2.$$

In their paper they prove that $\Phi_T^{\zeta}(\sigma,\tau)$ has a limiting distribution:

$$\lim_{T\to\infty} \Phi_T^{\zeta}(\sigma,\tau) = f(\sigma,\tau).$$

Nearly 60 years later Montgomery and Odlyzko prove that this limiting distribution is bounded between two functions. More specifically they prove there exists constants $b_1, b_2 > 0$ such that

$$\exp\left(-b_1\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\right) \le f(\sigma,\tau) \le \exp\left(-b_2\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\right).$$

One reason we are interested in understanding the distribution of values of $\zeta$ is that it sheds light on the extreme values of $\zeta$. For example, under the Riemann Hypothesis Titchmarsh proves that for $1/2 < \sigma < 1$ and $t \ge 3$

$$\log|\zeta(\sigma+it)| \ll_\sigma \frac{(\log t)^{2(1-\sigma)}}{\log\log t}.$$

On the other hand Montgomery proves that for $T$ large and $c > 0$ a constant depending on $\sigma$ we have

$$\max_{t\in[T,2T]} \log|\zeta(\sigma+it)| \ge c\frac{(\log T)^{1-\sigma}}{(\log\log T)^{\sigma}}.$$

In 2011, Lamzouri [62] proves results that, due to their uniformity, indicate that Montgomery's result is the correct estimate for extremal behaviour:

**Theorem 1.7** (Lamzouri, 2011)**.** *Let $1/2 < \sigma < 1$ and $T$ be large. Then there exists $c_1(\sigma) > 0$ such that uniformly in the range $1 \ll \tau \ll c_1(\sigma)(\log T)^{1-\sigma}/\log\log T$ we have*

$$\Phi_T^{\zeta}(\sigma,\tau) = \exp\left(-A_1(\sigma)\tau^{\frac{1}{1-\sigma}}(\log\tau)^{\frac{\sigma}{1-\sigma}}\left(1+O\left(R(\log T,\tau)\right)\right)\right),$$

*with*

$$R(y,\tau) = \frac{1}{\sqrt{\log\tau}} + \left(\frac{\tau}{y^{1-\sigma}(\log y)^{-1}}\right)^{\frac{\sigma-1/2}{1-\sigma}},$$

*and $A_1(\sigma) > 0$ a constant.*

A parallel line of research breakthroughs follow for our special $L$-functions $L(\sigma,\chi_d)$ with $\chi_d$ the Kronekcer symbol and $d$ is taken to be a fundamental discriminant. That is to say, in 1973, P.D.T.A. Elliot [29] established an analogue of Bohr and Jessen's results for $L(s,\chi_d)$ at a fixed point $s$ for $1/2 < \mathfrak{Re}(s) \le 1$. We have a similar discrepancy in the extreme values and, in 2011, Lamzouri [62] finds strong evidence that the $\Omega$-results are correct.

The final chapter of this thesis can be thought of as extension the work described in Section 1.4 to $L(\sigma, \chi_D)$ for $\frac{1}{2} < \sigma < 1$, or as an analogy to what is already known over number fields. We recall that the calculation of complex moments of $L(1, \chi_D)$ for a large range of complex values $z$ played a crucial role in obtaining the pertinent distribution results. In fact, the range of complex values which we provide an asymptotic formula for directly effects the uniformity of related distribution results and the range for which we can calculate complex moments depends on the maximal order of our $L$-function. Recall that Conjecture 1.4 and the unconditional (1.20) differ by a factor of 2. However, inside the critical strip, just as was described for number fields, this discrepancy is much larger. More specifically, under GRH, the truncated sum of $\log L(\sigma, \chi_D)$ satisifes

$$\sum_{\substack{f \text{ monic} \\ \deg f \leq Y}} \frac{\Lambda(f)\chi(f)}{\deg(f)|f|^{\sigma}} \ll \frac{(\log_q |D|)^{2(1-\sigma)}}{\log_q \log_q |D|} \qquad \text{for all } D \in \mathcal{H}_n, \tag{1.23}$$

(see our Proposition 6.1). Probabilistic arguments suggest

$$\sum_{\substack{f \text{ monic} \\ \deg f \leq Y}} \frac{\Lambda(f)\chi(f)}{\deg(f)|f|^{\sigma}} \ll \frac{(\log_q |D|)^{(1-\sigma)}}{\log_q \log_q |D|} \qquad \text{for all } D \in \mathcal{H}_n. \tag{1.24}$$

Since the discrepancy in the order of magnitude now appears in the exponent, we must be more careful with the analysis of the complex moments. In Chapter 6, we describe the asymptotic nature of $L(\sigma, \chi_D)^z$ for $D \in \widetilde{\mathcal{H}}_{n,g}$, where for $g(\sigma)$ a function satisfying $2\sigma - 1 \leq g(\sigma) \leq \sigma$ for $\frac{1}{2} < \sigma < 1$, we define

$$\widetilde{\mathcal{H}}_{n,g} = \left\{ D \in \mathcal{H}_n \Big| \sum_{\substack{f \text{ monic} \\ \deg f \leq Y}} \frac{\Lambda(f)\chi(f)}{\deg(f)|f|^{\sigma}} \leq \frac{(\log_q |D|)^{1-g(\sigma)}}{\log_q \log_q |D|} \right\}. \tag{1.25}$$

Then the first main result of Chapter 6 is as follows.

**Theorem 1.8** (L.)**.** *Let $n$ be large, $1/2 < \sigma < 1$, $B > 2$ a constant be fixed. There exists a positive constant*

$b_3 := b_3(\sigma, B, A)$ *such that for* $z \in \mathbb{C}$ *with* $|z| \leq b_3 n^{g(\sigma)}$ *we have*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \widetilde{\mathcal{H}}_{n,g}} L(\sigma, \chi_D)^z = \mathbb{E}(L(\sigma, \mathbb{X})^z) + O\left(\frac{\mathbb{E}(L(\sigma, \mathbb{X})^{\mathfrak{Re}(z)})}{n^{B-(g(\sigma)+1)}}\right), \tag{1.26}$$

*where* $2\sigma - 1 \leq g(\sigma) \leq \sigma$*, and we have for some constants* $C, C' > 0$ *that*

$$|\mathcal{H}_n \setminus \widetilde{\mathcal{H}}_{n,g}| \ll |\mathcal{H}_n| \exp\left(-Cn(\sigma - g(\sigma)) - C' \frac{n}{\log_q n}\right).$$

Note that, since $\widetilde{\mathcal{H}}_{n,g} \subseteq \mathcal{H}_n$ and the size of $L(\sigma, \chi_D)$ is restricted by $g(\sigma)$, if we let $g = g_0$ where $g_0(\sigma) = 2\sigma - 1$, then we require all $D \in \widetilde{\mathcal{H}}_{n,g_0}$ to be bounded as in (1.23) hence $\mathcal{H}_n = \widetilde{\mathcal{H}}_{n,g_0}$.

For other functions $g$ satisfying $2\sigma - 1 \leq g(\sigma) \leq \sigma$, $|\mathcal{H}_n \setminus \widetilde{\mathcal{H}}_{n,g}|$ decays exponentially. As such, Theorem 1.8 yields information about the tail of the distribution of values of $\log L(\sigma, \chi_D)$ as $D$ varies over $\mathcal{H}_n$ using an asymptotic formula for $L(\sigma, \chi_D)^z$ taken over $\widetilde{\mathcal{H}}_{n,g}$ with $g(\sigma) = \sigma$. That is, for $3 \leq \tau \ll n^{1-\sigma} (\log n)^{-1/\sigma}$, we obtain

$$\frac{1}{|\mathcal{H}_n|} |\{D \in \mathcal{H}_n : \log L(\sigma, \chi_D) > \tau\}| = \Phi_\sigma(\tau) \left(1 + O\left(\frac{(\tau \log \tau)^{\frac{\sigma}{1-\sigma}} \log n}{n^\sigma}\right)\right),$$

uniformly where

$$\Phi_\sigma(\tau) = \exp\left(-A_{\mathbb{X}}(q, \sigma, \tau) \tau^{\frac{1}{1-\sigma}} (\log_q \tau)^{\frac{\sigma}{1-\sigma}} \left(1 + O\left(\frac{\log \log \tau}{(\log \tau)^{2-\frac{1}{\sigma}}}\right)\right)\right).$$

Recall that $q$ and $\sigma$ are fixed. So, $A_{\mathbb{X}}(q, \sigma, \tau)$ varies with $\tau$. However, $A_{\mathbb{X}}(q, \sigma, \tau)$ is bounded from above and below by values depending on $q$ and $\sigma$. Analogous results by Lamzouri, Lester and Radziwiłł [64] hold for the classical case. In the classical case, however, $A_{\mathbb{X}}(\sigma)$ is a constant. Further research is needed to determine the nature of $A_{\mathbb{X}}(q, \sigma, \tau)$.

Based on the uniformity of our distribution results it seems that just as in the number field case the extreme behaviour is much closer to $\Omega$-result given by (1.24). We are able to construct characters which display this extremal behaviour:

**Theorem 1.9** (L.)**.** *Let* $N$ *be large. Let* $1/2 < \sigma < 1$ *be fixed. There exist a constant* $\beta_q(\sigma) > 0$ *and* *irreducible polynomials* $T_1$ *and* $T_2$ *of degree* $N$*, such that*

$$\log L(\sigma, \chi_{T_1}) \geq \beta_q(\sigma) \frac{(\log_q |T_1|)^{1-\sigma}}{(\log_q \log_q |T_1|)^{\sigma}}, \tag{1.27}$$

*and*

$$\log L(\sigma, \chi_{T_2}) \leq -\beta_q(\sigma) \frac{(\log_q |T_2|)^{1-\sigma}}{(\log_q \log_q |T_2|)^{\sigma}}. \tag{1.28}$$

# 2 Background

In what follows we will develop the necessary definitions and concepts for the results in the final four chapters to be understood.

## 2.1 Algebraic Structures and Motivation

We assume in this thesis that the reader is familiar with some basic definitions of abstract algebra, however, for the sake of completeness we will recall some important definitions and theorems of commutative algebra which are supporting the structures we study in the later chapters of the text. For the most part this section will provide definitions taken over the rationals, however any information which needs further explanation in terms of function fields will be done in Section 2.6.

We begin with a basic object of interest which makes an appearance frequently throughout.

**Definition 2.1.** *For a set $G$ and an operation $\cdot$, often shortened to $G = (G, \cdot : G \times G \to G)$, which is closed under the operation $\cdot$, is called a group if the following axioms hold:*

1. *Associativity: $g, h, f \in G$ then $g \cdot (f \cdot h) = (g \cdot f) \cdot h$.*
2. *Unity: there exists $e \in G$ such that $\forall g \in G$, $eg = ge = g$.*
3. *Inverse: for all $g \in G$ there exists $g^{-1}$ such that $gg^{-1} = g^{-1}g = e$.*

The main examples we will deal with: Consider $q > 1$ an integer. Then the following are groups:

$$(\mathbb{Z}/q\mathbb{Z}, +) = \{0, 1, 2, 3, \ldots, q-1\} \text{ and } \left((\mathbb{Z}/q\mathbb{Z})^\times, \cdot\right) = \{1 \leq a \leq q-1 : (a, q) = 1\}.$$

In both of these examples the element $n$ is a representative of the equivalence class of integers which are congruent to $n$ modulo $q$.

Adding a second operation to the mix creates many new objects called rings. Ascribing certain additional properties to these rings births new spaces which are interesting for different reasons. First:

**Definition 2.2.** *A* **ring** $(R, +, \cdot)$ *is a set* $R$ *such that*

0. $+ : R \times R \to R$ *and* $\cdot : R \times R \to R$.

1. $(R, +)$ *is an abelian group with* $0$ *as the unity.*

2. $(R, \cdot)$ *is associative with* $1$ *as unity.*

3. $+, \cdot$ *are compatible, that is, we can distribute the* $\cdot$ *over the* $+$.

*A* **subring** *is a set* $S$ *such that* $S \subseteq R$, $S$ *is a ring under the same operations as* $R$ *and has the same* $0$ *element.*

In general, the operation association to "multiplication" $\cdot$, need not be commutative, and if the ring $R$ is commutative, it is in reference to the multiplication. However, for the purposes of this thesis, all rings we will come across are commutative. We further suppose that our ring has the property that $ab = 0$ implies that one of $a$ or $b$ is $0$, these rings are called **integral domains**. The main example here is the integers $\mathbb{Z}$.

**Definition 2.3.** *Let* $R$ *be a ring, a* **unit** *of* $R$ *is an element* $u \in R$ *such that there exists* $x \in R$ *satisfying the equation* $ux = 1$. *The set of units in* $R$ *is denoted* $R^\times$. *If* $R^\times = R \setminus \{0\}$ *then we say that* $R$ *is a* **field**.

In our main example, $\mathbb{Z}$ we have $\mathbb{Z}^\times = \{\pm 1\}$. There are other rings where the group of units may be much larger. For example, if we consider $\mathbb{Z}[\sqrt{7}]$, the smallest subring of $\mathbb{R}$ which contains both $\mathbb{Z}$ and $\sqrt{7}$, then the element $8 + 3\sqrt{7}$ is a unit since $(8 + 3\sqrt{7})(8 - 3\sqrt{7}) = 1$. In fact, for any $n \in \mathbb{N}$ we have that $(8 \pm 3\sqrt{7})^n$ is a unit so we can see that the set $\mathbb{Z}[\sqrt{7}]^\times$ is very different from $\mathbb{Z}^\times$ and there may be some additional interesting structure associated to the units. It is quite clear that neither of these examples returns a field. Starting from the integers we can obtain fields in a lot of ways, but the most familiar one is called the field of fractions and in the case of $\mathbb{Z}$ is given by $\mathbb{Q}$.

Now, the integers also happen to satisfy the property that its elements can be uniquely factored (up to order and sign) into products of irreducible elements, called primes. This makes the ring a **unique factorization domain** (UFD). This is a special property of the integers which is not satisfied by all integral domains and even small manipulations done to $\mathbb{Z}$ can break it. For example, $\mathbb{Z}[\sqrt{-10}]$ is not a UFD since the element $14 = 2 \cdot 7 = (2+\sqrt{-10})(2+\sqrt{-10})$ and it is easy to check that each of these factorizations is made up of different irreducible elements.

**Definition 2.4.** *Let $R$ be a ring and $I \subseteq R$. Then, we say that $I$ is an* **ideal** *of $R$ if the following hold:*

1. *$(I, +)$ is a subgroup of $(R, +)$.*

2. *Let $r \in R$, then $rI$ and $Ir$ are both contained in $I$.*

*Let $a_1, a_2, \ldots, a_n \in R$. Then the set $(a_1, a_2, \ldots, a_n) = a_1R + a_2R + \cdots + a_nR = \{\sum_{i=1}^{n} a_i x_i : x_i \in R\}$ is an ideal of $R$. Ideals written as $(a) = aR$ are called* **principal ideals***. An ideal $I$ is called* **prime** *if it satisfies the following property: Let $a, b \in R$ such that $ab \in I$ then we have $a \in I$ or $b \in I$. An ideal is called* **maximal** *if it satisfies the following property: suppose $J$ is an ideal such that $I \subseteq J \subseteq R$ then $J = I$ or $J = R$.*

**Remark 2.1.** *Given two ideals $I$ and $J$ of a ring $R$ we may define addition and multiplication of them:*

$$I + J = \{a + b : a \in I, b \in J\}$$

*and*

$$IJ = \left\{ \sum_{\textit{finite}} ab : a \in I, b \in J \right\}.$$

*In both of these cases, the collection of ideals does not form a group structure.*

**Remark 2.2.** *Let $a, b \in R$. Note that $a \mid b$ if and only if $bR \subseteq aR$. Using this we can also define a notion of division between ideals. Let $I, J \subseteq R$ be ideals. Then $I$ divides $J$, written $I \mid J$, if and only if $J \subseteq I$.*

In our running example of the integers, $(n) = n\mathbb{Z} = \{zn : z \in \mathbb{Z}\}$ is the ideal generated by $n$. In fact this is what a typical ideal in $\mathbb{Z}$ looks like since any ideal $(a, b) \subseteq \mathbb{Z}$ can be described using a single generator $d$ where $d$ is the greatest common divisor of $a$ and $b$. This property makes the integers a **principal ideal domain** (PID). An integral domain is a PID if every ideal is principal. In PIDs the notion of divisibility of the elements and divisibility of the ideals are equivalent. Even very simple adjustments to the integers can remove this property. In fact, $\mathbb{Z}[\sqrt{-10}]$ also fails to be a PID. Naturally, one might begin to suspect that UFD and PID are really synonymous, however these examples have been chosen with a particular goal in mind. It is true that all PIDs are UFDs but there are UFDs which are not PIDs. An easy example is demonstrating this fact is $\mathbb{Z}[x]$, which generates all polynomials with integer coefficients. For all the examples we will encounter in this thesis PID and UFD can be thought of interchangeably but the reason it holds comes from the following theorem.

**Theorem 2.1.** *If $R$ is a UFD such that every non-zero prime ideal is maximal then $R$ is also a PID.*

Now, we list a characterization of prime ideals and maximal ideals that allows us to use the previous theorem easily. We first introduce the notion of a quotient ring:

**Definition 2.5.** *Let $R$ be a ring and $I \subseteq R$ be an ideal. Then the quotient ring of $R$ by $I$ is the set of cosets $x + I$ for $x \in R$ with the operations*

$$(x + I) + (y + I) = (x + y + I) \text{ and } (x + I)(y + I) = (xy + I).$$

*If $R$ is a commutative ring with identity, then $R/I$ is too with additive identity $0 + I = I$ and multiplicative identity $1 + I$.*

**Theorem 2.2.** *Let $I$ be an ideal of $R$. Then*

*1) $I$ is a prime ideal if and only if $R/I$ is an integral domain and*

*2) $I$ is a maximal ideal if and only if $R/I$ is a field.*

For example, consider the ideal $7\mathbb{Z}$. It is easy to show that this is a prime ideal and indeed we have $\mathbb{Z}/7\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6\}$ is an integral domain since if we multiple any pair of nonzero elements we return back another non-zero one. Additionally, we can see that $\mathbb{Z}/7\mathbb{Z}$ is also a field meaning that $7\mathbb{Z}$ was also maximal. In general, for any prime $p$ we obtain $\mathbb{Z}/p\mathbb{Z}$ is a field. We see that if $m$ is composite then the ideal $m\mathbb{Z}$ fails to satisfy both of these conditions.

**Definition 2.6.** *Let $R$ be an integral domain. We say that $R$ has* **Krull dimension 1** *if every non-zero prime ideal is maximal.*

**Proposition 2.1.** *Every PID has Krull dimension 1.*

Combining Theorem 2.2 and Proposition 2.1 gives us an easy way to check that $\mathbb{Z}[x]$ is not a PID! It is clear that the ideal generated by $x$, written $x\mathbb{Z}[x]$ is prime, so if it were to also be maximal it would mean that $\mathbb{Z}[x]/x\mathbb{Z}[x]$ would have to be a field, however, it is easy to see that this space is isomorphic to $\mathbb{Z}$. Where isormorphic means there is a map between the two spaces which preserves all operations and is a bijection.

We have one final special type of ring we will encounter. Afterward we will describe precisely the question associated to these problems which is tackled in this thesis.

**Definition 2.7.** *Let $R$ be an integral domain. A* **Euclidean function** *$f$ is a function from $R \setminus \{0\}$ to $\mathbb{N} \cup \{0\}$ satisfying the following division algorithm property: Let $a, b \in R$ be non-zero. Then there exists unique $q, r \in R$ such that $a = bq + r$ with $0 \leq f(r) < f(b)$. $R$ is said to be a* **Euclidean domain** *if it can be endowed with at least one Euclidean function.*

Of course $\mathbb{Z}$ is a Euclidean domain with the absolute value acting as the Euclidean function. Being a Euclidean domain implies one is a PID and thus also a UFD. One other very important example of a Euclidean domain can be constructed as follows. Let $k$ be a field and $T$ be an indeterminate. Then the polynomial ring $k[T]$ has the euclidean function given by the degree of the elements. In particular, as will be discussed in Section 2.6, since $\mathbb{F}_q$ is a field we have $\mathbb{F}_q[T]$ is a Euclidean domain and so shares many properties in common with $\mathbb{Z}$.

Now that we have notions about factorization and the potential lack of uniqueness of this factorization, one might wonder if it is possible to easily describe if such a property is satisfied by our rings. For this we explain with more rigour the idea of a class group, a field and its "ring of integers" and the ultimate tool we use to measure these things.

**Definition 2.8.** *Suppose that $E$ and $F$ are two fields such that $F \subseteq E$ and the operations done in $F$ are the same as those in $E$. Then we say that $F$ is a* **subfield** *of $E$ or that $E$ is a* **field extension** *of $F$. The field $E$ is a $F$-vector space and the dimension of this vector space is called the* **degree** *of the extension, denoted $[E : F]$. Let $E$ be a field extension of $F$ and $\alpha \in E$. The minimal polynomial of $\alpha$ over $F$ is the monic polynomial $f(x) \in F[x]$ with the smallest degree such that $f(\alpha) = 0$.*

As an example consider $F = \mathbb{Q}$ and $E = \mathbb{Q}(\sqrt{n})$ with $n$ a square-free integer. Then $[E : F] = 2$ and is called a **quadratic extension** of $\mathbb{Q}$. The minimal polynomial for $\sqrt{n}$ is $f(x) = x^2 - n$. The **ring of integers** inside of $\mathbb{Q}(\sqrt{n})$, denoted $\mathcal{O}_{\mathbb{Q}(\sqrt{n})}$, are all elements $\alpha \in \mathbb{Q}(\sqrt{n})$ such that the minimal polynomial $g$ has coefficients in the ring of integers for $\mathbb{Q}$, that is $g(x) \in \mathbb{Z}[x]$. Thus we see trivially that $\mathbb{Z} \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{n})}$.

In general, just as we can think of field extensions as vector spaces over the base field, for any finite extension $K$ of $\mathbb{Q}$ (also called a number field) we can consider $\mathcal{O}_K$ to be a free module over $\mathbb{Z}$. The proof of this is non-trivial but can be found in a standard introduction to algebraic number theory. Considering $\mathcal{O}_K$ as a free module over $\mathbb{Z}$ means that we have a basis that can describe the integral elements in relation to $\mathbb{Z}$. In the above examples where we found unique factorization to break down we had always chosen our $n$ in such a way that we could write $\mathcal{O}_{\mathbb{Q}(\sqrt{n})} = \mathbb{Z}[\sqrt{n}]$. For quadratic extensions of $\mathbb{Q}$ the elements of the ring of integers have the following shape:

$$\mathcal{O}_{\mathbb{Q}(\sqrt{n})} = \begin{cases} \mathbb{Z}[\sqrt{n}] & \text{if } n \equiv 2, 3 \pmod 4 \\ \mathbb{Z}[\frac{1+\sqrt{n}}{2}] & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

**Remark 2.3.** *From this we can see that the minimal polynomial which is associated to the ring of integers may differ from the one associated to the field extension. An important invariant which will appear frequently in later discussion, called the* **discriminant** *of the field, is calculated from the minimal polynomial associated to the ring of integers. In the case of quadratic extensions above if $n \equiv 2, 3 \pmod 4$ the discriminant is $d = 4n$ and if $n \equiv 1 \pmod 4$ then the discriminant $d = n$. In the literature, these $d$ are referred to as* **fundamental discriminants**.

Even in this case, where the extension of $\mathbb{Q}$ is fairly straight forward, having the smallest non-trivial degree, it is quite difficult to predict whether or not unique factorization will fail. In order to study this property we introduce a few more definitions.

**Lemma 2.1.** *Let $K$ be a finite extension of $\mathbb{Q}$. Then for every $\alpha \in K$ there exists $d > 0 \in \mathbb{Z}$ such that $d\alpha \in \mathcal{O}_K$.*

The proof of this is straightforward, simply use the minimal polynomial of $\alpha$.

**Definition 2.9.** *Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. A* **fractional ideal** *of $\mathcal{O}_K$ is an $\mathcal{O}_K$-submodule $M \subseteq K$ such that there exists a nonzero $d \in \mathcal{O}_K$ satisfying $dM \subseteq \mathcal{O}_K$. A fractional ideal is* **principal** *if it has the form $\alpha\mathcal{O}_K$ for some $\alpha \in K$.*

The following gives us some structure to the collection of nonzero fractional ideals and also lets us know that, even if our ring $\mathcal{O}_K$ is not a UFD, we still may obtain unique factorization into ideals.

**Theorem 2.3.** *Let $K$ be a number field. Then*

*i. the collection of nonzero fractional ideals of $\mathcal{O}_K$ forms a commutative group under ideal multiplication where $\mathcal{O}_K = (1)$ behaves as identity element 1.*

*ii. For each fractional ideal $I$ of $\mathcal{O}_K$ we may write*

$$I = \prod_{P \subseteq \mathcal{O}_K prime\ ideals} P^{n_P},$$

*where $n_P$ is an integer which is nonzero for only finitely many primes. Furthermore, this expression is unique (up to order) and the integral ideals of $\mathcal{O}_K$ have all $n_P \geq 0$.*

This result requires some work to prove, but any standard introductory course in algebraic number theory would have it. Now, with this in mind, we may finally define properly the class group which is associated to our field extension.

**Definition 2.10.** *Let $K$ be a number field. The* **class group** *of $K$ is given by*

$$\mathcal{C}l(K) = \{nonzero\ fractional\ ideals\ of\ \mathcal{O}_K\}/\{nonzero\ principal\ fractional\ ideals\ of\ \mathcal{O}_K\}.$$

*The* **class number** *is the size of $\mathcal{C}l(K)$. In the quadratic extension case, where $K = \mathbb{Q}(\sqrt{n})$, the class number is denoted $h_d$, where $d$ is the associated discriminant.*

**Remark 2.4.** *From this definition we can see that if we can prove the class group is trivial, that is that it has class number 1 then we know for sure that $\mathcal{O}_K$ is a PID and thus we have it is a UFD. Additionally, $\mathcal{O}_K$ satisfies the condition that all the nonzero prime ideals are in fact maximal, so that if $\mathcal{O}_K$ has class number $> 1$ then $\mathcal{O}_K$ cannot be a UFD.*

Finally, we recall from the overview that when studying quadratic extensions there is a dichotomy which occurs depending if the square free integer $d$ is positive or negative. The main difference stems from the fact that the group of units $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{d})}$ is extremely different. If $d < 0$, the group of units is finite and if $d < -6$ then $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{d})} = \mathbb{Z}^{\times}$. If $d > 0$ then the group of units need not be finite and has another invariant which is associated to it.

**Definition 2.11.** *The* **fundamental unit**, *$\epsilon$ say, is a generator of $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{d})}$ modulo the roots of unity when the group has rank 1, ie that it has an infinite cyclic component. The* **regulator** *of $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{d})}$, denoted $R_d$, is* $\ln|\epsilon|$, *and it measures the density of units in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*

**Remark 2.5.** *The theory of $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{d})}$ for $d > 0$ essentially comes down to understanding solutions to Pell's equation $x^2 - dy^2 = 1$.*

Dirichlet's unit theorem [30] proves that $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{d})}$ has rank 1 precisely when $d > 0$. In the overview we mentioned that the study of $h_d$ associated to number fields has been thoroughly explored by analytic number theorists. The plan of attack is to calculate moments of these class numbers in order to describe the distribution of values we can take. The average values were given by Siegel in equation (1.5) and (1.6). One does not study directly the class group for this type of analysis. Instead we use the link Dirichlet discovered to analyze this information from the perspective of $L$-functions. The background for this can be found in the following sections.

## 2.2 Analytic Tools

We begin with some simple definitions to set the stage of discourse.

**Definition 2.12.** *Let $u_n(z)_{n\geq 1}$ be a sequence of complex functions for $z \in \mathbb{C}$. We say that the sequence $u_n(z)$ is* **uniformly convergent** *to $u(z)$ in a domain $D \subseteq \mathbb{C}$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|u_n(z) - u(z)| < \epsilon$ for every $n \geq N$ and $z \in D$.*

So suppose that $u_n(z) \to u(z)$ uniformly as in the definition. If $u_n(z)$ is continuous/differentiable/integrable for all $n$ then so is $u(z)$. Not only this, but uniform convergence allows for the swapping of order of operations. For example, the limits of the sum of uniform convergent series, is the sum of the limits of the individual functions. Similar results hold for the integrals of these functions.

We also consider a uniformly convergent infinite product, as this will play a key role in the later chapters.

**Definition 2.13.** *Let $u_n(z)_{n\geq 1}$ be a sequence of complex functions of for $z \in \mathbb{C}$. We say that the infinite product $\prod_{n\geq 1}(1 + u_n(z))$ is* **uniformly convergent** *in any region where the series $\sum_{n\geq 1} |u_n(z)|$ converges to a bounded sum. It is important to note that the infinite product can be convergent only if the limit of its partial products is never 0.*

A fundamental object of our study is an analytic function. We provide the definition here.

**Definition 2.14.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We say $f$ is analytic in the domain $D \subseteq \mathbb{C}$ if and only if for every point $s \in D$ we have $f$ is represented by a power series in some neighbourhood of $s$.*

The proof of the prime number theorem was first given using complex analysis, and as described in Chapter 1, we may use similar ideas to solve other problems about discrete objects.

First, we recall the definition of an arithmetic function and it's associated summatory function.

**Definition 2.15.** *We say $f$ is an* **arithmetic function** *if $f : \mathbb{N} \to \mathbb{C}$, that is it is a function whose domain is $\mathbb{N}$.*

- *We say that $f$ is* **multiplicative** *if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$ and* **completely multiplicative** *if it holds for any $m, n \in \mathbb{N}$.*

- *We say that $f$ is* **additive** *if $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$ and* **completely additive** *if it holds for any $m, n \in \mathbb{N}$.*

Arithmetic functions may be multiplied, divided (provided they do not vanish), added and subtracted to create new arithmetic functions. In addition we have another method for combining these functions defined as follows:

**Definition 2.16.** *Let $f, g$ be two arithmetic functions. We define their* **Dirichlet convolution** *by*

$$(f \star g) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

For example, consider the function $1(n) = 1$ for all $n \in \mathbb{N}$. Then

$$1 \star 1 = \sum_{d|n} 1(d)1(n/d) = \sum_{d|n} 1,$$

which is the divisor function $d(n)$.

**Definition 2.17.** *The* ***summatory function*** *of an arithmetic function $f$ is given by*

$$F(x) = \sum_{n \leq x} f(n).$$

**Definition 2.18.** *A Dirichlet series is a function of a complex variable $s$ given by*

$$g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

*with $\{a_n\}$ a complex valued sequence.*

Many Dirichlet series of interest to us have $a_n = f(n)$ where $f$ is an arithmetic function. Each Dirichlet series has an abscissa of convergence $\sigma_c$ which defines the region for which it converges. These regions of convergence are half-planes in $\mathbb{C}$, that is the series will converge for all $s \in \mathbb{C}$ such that $\mathfrak{Re}(s) > \sigma_c$.

One can multiply two Dirichlet series together and obtain a new Dirichlet series in the following way. Let $f(s) = \sum_{n\geq 1} \frac{a(n)}{n^s}$ and $g(s) = \sum_{n\geq 1} \frac{b(n)}{n^s}$ be two Dirichlet series. Then

$$f(s)g(s) = \sum_{n\geq 1} \frac{(a \star b)(n)}{n^s}.$$

A Dirichlet series can be represented using Mellin transforms.

**Definition 2.19.** *Let $f$ be a function. The Mellin transform of $f$ is given by*

$$M_f(s) = \int_0^\infty x^{s-1} f(x) dx,$$

*and the inverse Mellin transform is*

$$f(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} M_f(s) x^{-s} ds,$$

*where $C \in \mathbb{R}$ such that $C > \sigma_c$.*

The relationship follows from [79, Theorem 1.3], which says that given the summatory function $F(x)$ of $f$, we can write the Dirichlet series associated to $f$ as

$$g(s) = \sum_{n=1}^\infty \frac{f(n)}{n^s} = s \int_1^\infty F(x) x^{-s-1} dx, \tag{2.1}$$

if $\mathfrak{Re}(s) > \sigma_c$.

Mellin transforms have an inverse relationship, which allows one to reverse the operation above. To demonstrate this fact, we state Perron's formula, which asserts that

$$F(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} g(s) \frac{x^s}{s} ds, \tag{2.2}$$

for $\sigma_0 > \sigma_c$. If $f(n) = 1$ for all $n \in \mathbb{N}$, then the Dirichlet series associated to $f$ gives the Riemann zeta-function,

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad \mathfrak{Re}(s) > 1.$$

As an example for understanding the way powers of $\zeta$ work consider:

$$\zeta(s)^2 = \sum_{n\geq 1} \frac{(1\star 1)(n)}{n^s} = \sum_{n\geq 1} \frac{d(n)}{n^s}.$$

We can also express $\zeta(s)$ as an Euler product which is well defined in the same region,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The Euler product can be used to prove that there exists infinitely many primes. It is also easy to see from this definition that $\zeta(s) \neq 0$ for all $\mathfrak{Re}(s) > 1$. Dirichlet utilized a similar argument in order to prove that there exists infinitely many primes $p \equiv a \pmod{q}$. His technique used special arithmetic functions that characterize the congruence condition which will be introduced in section 2.4.

In keeping with the theme of using $\zeta(s)$ as a model for how to study discrete objects we make note that choosing $f(n) = \Lambda(n)$ in Definition 2.18 we obtain

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{2.3}$$

This Dirichlet series also has abscissa $\sigma_c = 1$, and we recall that the summatory function of $\Lambda(n)$ is $\psi(x)$, so that by (2.2) and (2.3) we have

$$\psi(x) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds, \text{ for some } \sigma_0 > 1. \tag{2.4}$$

The explicit formula (1.3) is derived from here using the following theorem.

**Theorem 2.4** (Residue Theorem). *Let $U \subseteq \mathbb{C}$ be open, simply connected and containing a finite list of points $u_1, u_2, \ldots, u_n$ and let $f : U \setminus \{u_1, u_2, \ldots, u_n\} \to \mathbb{C}$ be a holomorphic function, and let $\gamma$ be a rectifiable path in $U$ which does not cross any of the $u_i$. Then*

$$\frac{1}{2\pi i}\oint_\gamma f(z)dz = \sum_{i=1}^{n} I(\gamma, u_i)Res(f, u_i),$$

*where $I(\gamma, u_i)$ counts the number of times $\gamma$ encircles $u_i$ and $Res(f, u_i)$ is the residue of $f$ at the point $u_i$. Note that if $\gamma$ is a positively oriented simply connected closed curve then $I(\gamma, u_i) = 1$ or $0$ depending on* *whether $u_i$ falls in the interior of $\gamma$ or not.*

To prove (1.3) one computes the integral in (2.4) around a rectangle $R$ with sides $\gamma_1$:$\sigma_0 - iT$ to $\sigma_0 + iT$, $\gamma_2$:$\sigma_0 + iT$ to $k + iT$, $\gamma_3$:$k + iT$ to $-k - iT$ and $\gamma_4$: $-k - iT$ to $\sigma_0 - iT$, where $T$ is a large positive real number which does not coincide with a zero of $\zeta(s)$ and $k$ is an odd positive integer. The choice of $k$ becomes clear in the following section. By the residue theorem we have

$$\oint_R -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds = x - \sum_{\substack{\rho \in Z \\ |\mathfrak{Im}(\rho)| \leq T}} \frac{x^\rho}{\rho} - \log(2\pi) - \sum_{i=1}^{(k-1)/2} \frac{x^{-2i}}{2i}.$$

Taking the limit as $T, k \to \infty$ provides the explicit formula. On the other hand

$$\frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds = \lim_{T\to\infty}\int_{\gamma_1} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds,$$

so the proof is completed after showing

$$\lim_{T,k\to\infty}\int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds = 0.$$

This technique will be used frequently in Chapter 4. This argument is often described simply as "shifting the contour to the left".

Finally, we record Cauchy's argument principle, which is useful for Chapter 3.

**Theorem 2.5** (Cauchy's Argument Principle)**.** *Suppose that $f$ is a meromorphic function inside of a simple closed contour $C$, and $f$ has no poles or zeros on the contour, then*

$$\frac{1}{2\pi i}\oint_C \frac{f'}{f}(z)dz = Z_0 - P,$$

*where $Z_0$ represents the number of zeros of $f$ (with multiplicity) inside of $C$ and $P$ the number of poles inside $C$.*

## 2.3 Zeros of the Riemann Zeta Function.

Chapter 3 focuses on understanding in a more precise way, in absence of RH, the location of the non-trivial zeros of $\zeta(s)$. Here we provide some classical information related to this.

In 1859 in his famous memoir Riemann showed that $\zeta(s)$ admits a meromorphic continuation to the complex plane, with a simple pole at $s = 1$. He also proves that $\zeta(s)$ satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.5}$$

As a consequence $\zeta(s)$ vanishes at $s = -2, -4, -6, \ldots$ (these are called the *trivial zeros*). The remaining zeros are called *non-trivial* and we denote $Z$ as the set of zeros in the strip $0 < \mathfrak{Re}(s) < 1$, we recall definition (1.4):

$$Z = \{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, 0 < \beta < 1\}.$$

We have the following facts about $Z$. $|Z|$ is countably infinite and there are finitely many $\rho \in Z$ such that $|\gamma| \leq T$, with $T > 0$. In fact, for $N(T)$ the number of zeros $\rho = \beta + i\gamma$ such that $0 < \beta < 1$ and $0 < \gamma \leq T$, Riemann conjectured (proven by Von Mangoldt in 1905, for a more recent treatment see [21, Chapter 15]) that

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + O(\log(T)). \tag{2.6}$$

From the functional equation (2.5) we see the zeros in the critical strip are symmetric with respect to both the real axis and the line $\mathfrak{Re}(s) = \frac{1}{2}$. Riemann conjectured that if $\rho \in Z$ then $\rho$ lies on the vertical line, $\mathfrak{Re}(s) = \frac{1}{2}$, which we already stated as Conjecture 1.1.

To date there has not been a proof or disproof of RH but there are a number of results which seem to hint at its truth.

In 1914 Hardy [43] showed that infinitely many of the zeros lie on the $\frac{1}{2}$-line and, in 1942, Selberg [100] showed that, of the total number of zeros, a positive proportion of them must lie on the $\frac{1}{2}$-line. In 1974, Levison [69] showed that at least 34.2% of the zeros must lie on the critical line and, in 1989, Brian Conrey [19] improved this to 40.7%. Progress has been slower recently as it is becoming more and more difficult to handle the mollifiers which are used to study these proportions. In 2010, Bui, Conrey and Young [16] managed to obtain 41.05% on the critical line and then in 2012, S. Feng improved this to 41.28% [33].

We also have a number of numerical results that can be considered as indicators of its truth. For example, a number of authors have worked on the partial verification of the Riemann Hypothesis. So far all the zeros that have been found lie on the the line $\mathfrak{Re}(s) = \frac{1}{2}$ and are simple. Let $H > 0$ be a constant, Table 2.1 lists authors who have shown for $\rho \in Z$ if $\gamma \leq H$ then $\beta = \frac{1}{2}$ for larger and larger $H$ values. There is a list of the different numerical verifications below in Table 2.1.

Table 2.1: History of Partial Verification of RH

| Year | Authors | Verification Height |
|---|---|---:|
| 1859 | Riemann | 25.010 |
| 1903 | Gram [38] | 65.112 |
| 1914 | Backlund [6] | 198.015 |
| 1925 | Hutchinson [48] | 299.840 |
| 1935 | Titchmarsh [103] | 388.846 |
| 1936 | Titchmarsh [104] | $1,467.477$ |
| 1953 | Turing [107] | $1,540.030$ |
| 1956 | Lehmer [68] | $9,878.910$ |
| 1956 | Lehmer [67] | $21,942.593$ |
| 1958 | Meller [78] | $29,750.745$ |
| 1966 | Lehman [66] | $170,570.745$ |
| 1968 | Rosser, Schoenfeld & Yohe [98] | $1,893,194.452$ |
| 1977 | Brent [13] | $18,114,537.803$ |
| 1979 | Brent [12] | $35,018,261.243$ |
| 1982 | Brent, van de Lune, te Riele, Winter [14] | $81,702,130.190$ |
| 1983 | van de Lune,te Riele [74] | $119,590,809.282$ |
| 1986 | van de Lune, te Riele &Winter [75] | $545,439,823.215$ |
| 2003 | Wedeniwski* [111] | $57,292,877,670.307$ |
| 2004 | Gourdon* [37] | $2,445,999,556,030.000$ |
| 2011 | Platt [84] | $30,610,046,000.000$ |

(*unpublished)

On the other hand we know that these zeros do not lie "too close" to the 1-line either. This result is known as the zero free region and its existence was the final step in the proof of the PNT. The classical zero free region is given by the following theorem.

**Theorem 2.6.** *[21, §13, pg. 86] There exists a positive numerical constant $R_0$ such that $\zeta(s)$ has no zero in the region*

$$\mathfrak{Re}(s) \geq 1 - \frac{1}{R_0 \log(|\mathfrak{Im}(s)|)}, \ \mathfrak{Im}(s) \geq 2.$$

We give a history of the explicit constant $R_0$ in Table 2.2.

Table 2.2: History of Increasing Zero Free Region

| Year | Authors | Value of $R_0$ |
|---|---|---|
| 1899 | de la Vallée Poussin [108] | 34.82 |
| 1939 | Rosser [95] | 19 |
| 1962 | Rosser & Schoenfeld [96] | 17.516 |
| 1975 | Rosser & Schoenfeld [97] | 9.645908801 |
| 2002 | Ford [34] or [35] | 8.463 |
| 2005 | Kadiri [57] | 5.69693 |
| 2015 | Mossinghoff & Trudgian [83] | 5.573412 |

Lastly, we also have some information about the density of zeros which could potentially have real part $\sigma > \frac{1}{2}$. Define the function $N(\sigma, T)$ to be the number of elements in $Z$ such that $\sigma \le \beta < 1$ and $0 < \gamma \le T$. Then it can be shown for any $\sigma > \frac{1}{2}$,

$$N(\sigma, T) = O(T).$$

In some specific regions for $\sigma$ we may prove instead that for some $\theta < 1$,

$$N(\sigma, T) = O(T^{\theta}).$$

In chapter 3 we obtain something stronger, which has a simplified expression as

$$N(\sigma, T) \le c_1(\sigma) T^{\frac{8}{3}(1-\sigma)} (\log T)^{5-2\sigma} + c_2(\sigma)(\log(T))^2.$$

As part of Chapter 3 we describe more precisely the various zero density estimates and their subsequent explicit improvements. For completeness we will include the very short history associated to explicit zero density estimates in Table 2.3.

Table 2.3: History of Explicit Bounds for $N(\sigma, T)$

| Year | Authors | $N(\sigma, T) \le f(\sigma, T)$ | Region of validity |
|---|---|---|---|
| 2013 | Kadiri [56] | $f(\sigma, T) = c_1(\sigma)T + c_2(\sigma)\log T - c_3(\sigma)$ | $\sigma \ge 0.55$, $T \ge H_0^*$ |
| 2015 | Ramaré [90] | $f(\sigma, T) = 965(3T)^{\frac{8(1-\sigma)}{3}}(\log T)^{5-2\sigma} + 51.5(\log T)^2$ | $\sigma \ge 0.52$, $T \ge 2000$ |

* $H_0$ is any value taken from Table 2.1.

The content of Chapter 3 discusses an improvement on the work done in [90].

## 2.4 Dirichlet $L$-functions and Characters

We describe in some detail the definition of a standard Dirichlet $L$-function and in Section 2.6 we will alter the definitions to match up with the setting over function fields.

**Definition.** *Let $G$ be a finite group. A character of $(G, \cdot)$ is a homomorphism*

$$\chi : G \longrightarrow \mathbb{C}^\times$$

*such that for any $g_1, g_2 \in G$ we have*

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2).$$

*Let $q \in \mathbb{N}$. A Dirichlet character is a group character defined on $((\mathbb{Z}/q\mathbb{Z})^\times, \cdot)$. We call $q$ the modulus of $\chi$.*

The first example of such a character is called the trivial one and it is denoted $\chi_0$. This takes the value 1 for all $n \in (\mathbb{Z}/q\mathbb{Z})^\times$.

We have that $|(\mathbb{Z}/q\mathbb{Z})^\times| = \varphi(q)$ and the set of characters for this group, $G_q$ say, is isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$, that is $|G_q| = \varphi(q)$. We denote the principal character as $\chi_0$, whose value is 1 for all $n \in (\mathbb{Z}/q\mathbb{Z})^\times$. For $\chi$ a Dirichlet character we have $\chi(1) = 1$ and multiplication of two characters is given by $\chi_1\chi_2(n) = \chi_1(n)\chi_2(n)$. Let $n \in \mathbb{N}$. Since $n^{\varphi(q)} \equiv 1 \pmod q$, $\chi(n^{\varphi(q)}) = 1$ and consequently $\chi(n)^{\varphi(q)} = 1$. From this we see that the values taken by characters are precisely the $\varphi(q)$-th roots of unity. Let $\overline{\chi} = \overline{\chi(n)}$, where $\overline{\chi(n)}$ means the complex conjugate of $\chi(n)$, hence we have $\chi\overline{\chi}(n) = \chi_0(n) = 1$.

The following is a corollary of [79, Lemma 4.2] and is known as an **orthogonality relation** for Dirichlet characters.

**Lemma 2.2.** *[79, Corollary 4.5] For fixed $a, n \in (\mathbb{Z}/q\mathbb{Z})^\times$ we have*

$$\frac{1}{\varphi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a)\chi(n) = \begin{cases} 1 & \text{if } n \equiv a (\mathrm{mod}\ q) \\ 0 & \text{otherwise.} \end{cases}$$

We now extend the definition of Dirichlet characters from $(\mathbb{Z}/q\mathbb{Z})^\times$ to $\mathbb{Z}$:

$$\chi(n) = \begin{cases} \chi(n (\text{mod } q)) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $d \mid q$ and that $\chi^*$ is a character modulo $d$. Then set

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if } (n,q) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\chi$ is completely multiplicative and has period $q$. Then by [79, Theorem 4.7] we have that $\chi$ is a Dirichlet character modulo $q$. In this situation we say that $\chi^*$ induces $\chi$.

**Lemma 2.3.** *Let $\chi$ be a character modulo $q$. We say that $d$ is a quasiperiod of $\chi$ if $\chi(m) = \chi(n)$ whenever $(mn, q) = 1$ and $m \equiv n \pmod{d}$. The least quasiperiod of $\chi$ is called the conductor of $\chi$ and is a divisor of $q$.*

**Definition 2.20.** *We say that a character $\chi$ modulo $q$ is primitive if its conductor is $q$.*

The associated Dirichlet series as defined in Definition 2.18 are called Dirichlet $L$-functions and are given by:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

If $\chi = \chi_0$ then $L(s, \chi)$ has an abscissa of convergence $\sigma_c = 1$ and if $\chi \neq \chi_0$ then $\sigma_c = 0$.

In this thesis, we will be concerned with taking powers of Dirichlet $L$-functions, so we give a brief example,

$$L(s, \chi)^2 = \sum_{n \geq 1} \frac{(\chi \star \chi)(n)}{n^s} = \sum_{n \geq 1} \frac{d(n)\chi(n)}{n^s},$$

where the last equality follows from the fact that Dirichlet characters are completely multiplicative.

The logarithmic derivative of $L(s, \chi)$ is

$$\frac{L'}{L}(s, \chi) = -\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}.$$

In the overview we mentioned that in order to prove the Prime Number Theorem in Arithmetic Progressions that de la Vallée Poussin adapted his argument of the PNT. He proved that $\psi(x;q,a) \sim \frac{x}{\varphi(q)}$, where

$$\psi(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} \overline{\chi}(a)\psi(x,\chi),$$

and

$$\psi(x,\chi) = \sum_{n \leq x} \Lambda(n)\chi(n).$$

The key to such a proof relies on finding an explicit formula for $\psi(x,\chi)$. There are certainly more technicalities involved when calculating this but it follows the same general pattern as was described for $\psi(x)$ in Section 2.2. This is mentioned to highlight the general theme, or approach when using $L$-functions to analyze our discrete sequences. As we will see in Chapter 4 the ideas are flexible and can be used on even more general $L$-functions.

As was the case for $\zeta(s)$ we have an Euler product:

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

This is a key feature of the $L$-functions we are interested in formalizing as will become clear in the following section.

Showing that $L(1,\chi) \neq 0$ for each character is a key step in the proof of Dirichlet's theorem (there are infinitely many primes in each valid arithmetic progression). We refer the reader to [21, Chapter 1] for reference.

Finally, the Euler product of $L(s,\chi)$ where $\chi$ is not primitive, may be written in terms of the primitive character which induces it, $\chi^*$,

$$L(s,\chi) = L(s,\chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right).$$

Similarily for $L(s,\chi_0)$ we have

$$L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

and hence, in some sense we see that $\zeta(s)$ is the primitive character which induces $\chi_0$ for any $q$.

We will end this section by defining a special Dirichlet character which has been mentioned in the overview, the Kronecker symbol. First, the set up,

**Definition 2.21.** *Let $p$ be a prime and $a$ an integer such that $p \nmid a$. Then $a$ is said to be a* **quadratic residue** *modulo $p$ if the congruence*

$$x^2 \equiv a(\mathrm{mod} p)$$

*has a solution $x \in \mathbb{Z}$; otherwise it is called a* **quadratic nonresidue** *modulo $p$.*

Given the following we can induce a real primitive character modulo an odd prime $p$:

**Definition 2.22.** *Let $p$ be an odd prime and $a \in \mathbb{Z}$ such that $p \nmid a$. The Legendre symbol, written as $\left(\frac{a}{p}\right)$, is defined by*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

We can extend this to all integers by defining $\left(\frac{a}{p}\right) = 0$ if $p \mid a$. The Legendre symbol is completely multiplicative so that we need only understand how it behaves on primes and for $-1$. For information on how $\left(\frac{2}{p}\right)$ and $\left(\frac{-1}{p}\right)$ behave one can consult any introductory text on elementary number theory. A key tool for studying the behavior of $\left(\frac{q}{p}\right)$ where $q \neq p$ are both odd primes is called the Law of Quadratic Reciprocity.

**Theorem 2.7** (The Law of Quadratic Reciprocity)**.** *Let $p$ and $q$ be odd primes such that $p \neq q$. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Then, for any $d$ we may define

**Definition 2.23.** *Let $n \neq 0$ be an integer with prime factorization $n = up_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, with $u$ a unit, then the* **Kronecker symbol** *associated to $d$ is*

$$\left(\frac{d}{n}\right) = \left(\frac{d}{u}\right)\prod_{i=1}^{k}\left(\frac{d}{p_i}\right)^{a_i}.$$

Note that $\left(\frac{d}{1}\right) = 1$ for all $d$, $\left(\frac{d}{-1}\right) = \mathrm{sgn}(d)$ and $\left(\frac{d}{2}\right) = 0$ if $d$ is even, 1 if $d \equiv \pm 1 \pmod 8$ and $-1$ if $d \equiv \pm 3 \pmod 8$.

The Legendre symbol induced a real primitive character, $\chi_p$, for $p$ a prime. It is not true that the Kronecker symbol will induce a primitive character for every composite $d$. In order to obtain a primitive character, $d$ should only have relatively prime factors of the form $-4$, $\pm 8$ and $(-1)^{(p-1)/2}p$ for odd primes $p$, cf. [21, Chapter 5, p. 40]. Put another way, we need to take $d$ to be the discriminant of minimal polynomial associated to the ring of integers of quadratic extension of $\mathbb{Q}(\sqrt{N})$. If $N \equiv 2, 3 \pmod 4$ this gives $4N$ and if $N \equiv 1 \pmod 4$ we obtain $N$.

Finally, Dirichlet proved that the class number $h_d$ defined in Definition 2.10 associated to such a quadratic extension can be calculated by studying the Dirichlet $L$-function generated by $\chi_d$, cf. [21, Chapter 6].

$$h_d = \begin{cases} \frac{w\sqrt{|d|}}{2\pi} L(1, \chi_d) & \text{if } d < 0 \\ \frac{\sqrt{d}}{R_d} L(1, \chi_d) & \text{if } d > 0, \end{cases} \tag{2.7}$$

where $R_d$ is the regulator of Definition 2.11 and

$$w = \begin{cases} 2 & \text{if } d < -6 \\ 4 & \text{if } d = -4, \\ 6 & \text{if } d = -6. \end{cases}$$

## 2.5 General $L$-functions

In what follows we describe the generic definition of an $L$-function given in [52, Chapter 5]. To begin, let $d \geq 1$ be a fixed positive integer, which will be called the **degree** of the $L$-function. Let $L(s, f)$ be given by the Dirichlet series and Euler product

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j,f}(p)}{p^s}\right)^{-1},$$

where $\lambda_f(n) \in \mathbb{C}$ is a function which is associated to $f$. We normalize $\lambda_f(1) = 1$, and insist that both the series and product are absolutely convergent in $\mathrm{Re}(s) > 1$. In general, the $|\alpha_{j,f}(p)| < p$, however in the context of this thesis we will assume a stronger bound.

**Conjecture 2.1** (Ramanujan-Petersson)**.** *Let $L(s, f)$ be a degree $d$ $L$-function with the following Euler* *product expansion:*

$$L(s, f) = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j,f}(p)}{p^s}\right)^{-1}.$$

*Then we have that* $|\alpha_{j,f}(p)| \leq 1$ *for all primes* $p$ *and* $1 \leq j \leq d$.

Further, a key property of $L$-functions is that they can be analytically extended to all of $\mathbb{C}$ with a finite number of poles, cf. (2.8). The formula for this extension involves a **gamma factor** described as follows

$$\gamma(s, f) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \kappa_j}{2}\right),$$

where $\kappa_j$ are complex numbers that depend on $f$. These $\kappa_j$ are called the **local parameters at infinity** and may be referred to as such throughout. If $\kappa$ is a local parameter at infinity then so is $\overline{\kappa}$. In general, it is assumed that $\mathrm{Re}(\kappa_j) > -1$, in our case the Ramanujan-Petersson conjecture guarantees that $\mathrm{Re}(\kappa_j) \geq 0$. This last condition ensures that $\gamma(s, f)$ has no pole in $\mathrm{Re}(s) > 0$.

Associated to each $L$-function is an integer $q(f) \geq 1$, called the **conductor** of $L(s, f)$. The conductor satisfies the property that $\alpha_{j,f}(p) \neq 0$ whenever $p \nmid q(f)$. We say that primes $p$ such that $p \nmid q(f)$ are **unramified**. With this in mind, we can define the so-called **completed $L$-function**,

$$\xi(s, f) = q(f)^{s/2} \gamma(s, f) L(s, f). \tag{2.8}$$

$\xi(s, f)$ has finite order. This completion satisfies a functional equation

$$\xi(s, f) = \epsilon(f) \xi(1 - s, \overline{f}),$$

where $\epsilon(f)$ is a complex number of absolute value 1, and $\xi(s, \overline{f}) = \overline{\xi(\overline{s}, f)}$. In this notation, $\overline{f}$ is called the dual of $f$, in relation to the other invariants this dual must satisfy $\lambda_{\overline{f}}(n) = \lambda_f(n)$, $\gamma(s, \overline{f}) = \gamma(s, f)$ and $q(\overline{f}) = q(f)$.

Finally, we come to the most important invariant associated to our $L$-functions. When providing estimates for a single $L$-function the only parameter is $s$, however, if we want to consider the $L$-functions as a class we need to consider how a number of invariants, like conductor and degree, might effect the uniformity of our estimates.

Thus, in order to facilitate the quest for uniform estimates, it is convenient to state the results in terms of the **analytic conductor** which combines together all the invariants we must keep track of.

**Definition 2.24.** *For $s \in \mathbb{C}$, let $L(s, f)$ be a degree $d$ $L$-function as described above. Then, the associated analytic conductor of $L$ is defined by*

$$C(f, s) := \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{s + \kappa_j}{2} \right|.$$

*where $q(f)$ is the conductor and the $\kappa_j$ are the local parameters at infinity.*

In this thesis we are interested mainly in studying the value of $L(1, f)$ and so we introduce the shorthand notation

$$C(f) := C(f, 1) = \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{1 + \kappa_j}{2} \right|.$$

We note that in [18] the author uses $C(f) = C(f, 1/2)$. This definition is very similar to the one given in Iwaniec and Kowalski [52] and only differs by a constant factor to the power of the degree of the $L$-function. To help orient the reader, we give an example in the form of the analytic conductor of a Dirichlet $L$-function. Let $\chi$ be a Dirichlet character modulo $q$ then the associated $L$-function has analytic conductor:

$$C(\chi) = q\frac{1 + \mathfrak{a}}{2\pi}, \text{ where } \mathfrak{a} = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1. \end{cases}$$

## 2.6 Function Fields and the analogies

For completeness sake we will do a bit of exposition on how to construct a finite field and repeat some definitions of Section 2.1 in order to be sure the reader is comfortable with the setting. This section is important for reading the last two chapters comfortably.

We begin by noting that any finite field must have a prime power order. The first example of a finite field is $\mathbb{Z}/p\mathbb{Z}$. This is a field with characteristic $p$, meaning that summing any element with itself $p$ times returns 0.

**Theorem 2.8.** *Let $p$ be any prime in $\mathbb{Z}$. If $\pi(x)$ is an irreducible polynomial of degree $n$ in $\mathbb{Z}/p\mathbb{Z}$ then $(\mathbb{Z}/p\mathbb{Z})[x]/\pi(x)(\mathbb{Z}/p\mathbb{Z})$ is a field of order $p^n$.*

**Remark 2.6.** *Any such field will have an embedding of $\mathbb{Z}/p\mathbb{Z}$ inside of it and so must also have characteristic*

$p$.

*Proof.* The elements in $(\mathbb{Z}/p\mathbb{Z})[x]/\pi(x)(\mathbb{Z}/p\mathbb{Z})$ are the possible remainders of $\pi(x)$ with coefficients taken from $\mathbb{Z}/p\mathbb{Z}$. That is, if $f \in \mathbb{Z}/p\mathbb{Z}$ then

$$f(x) = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n.$$

Since we have $p$ choices for each of the $a_i$ coefficients, there are exactly $p^n$ different elements. Since $\pi(x)$ is irreducible and $\mathbb{Z}/p\mathbb{Z}[x]$ is a PID we have that $\pi(x)(\mathbb{Z}/p\mathbb{Z})$ is in fact maximal, hence $(\mathbb{Z}/p\mathbb{Z})[x]/\pi(x)(\mathbb{Z}/p\mathbb{Z})$ is a field. □

Let's construct a quick example: Let $p = 2$. Take $\pi(x) = x^3 + x + 1$, this is irreducible over $(\mathbb{Z}/2\mathbb{Z})[x]$, then

$$(\mathbb{Z}/p\mathbb{Z})[x]/\pi(x)(\mathbb{Z}/p\mathbb{Z}) = \{0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1\},$$

is a field of order $2^3$. Clearly, I could have generated another field with $2^3$ elements by choosing a different irreducible polynomial of degree 3. As it turns out, one can show that the choice of irreducible polynomial is irrelevant since all fields of $2^3$ elements are isomorphic. This is true in general, all finite fields of size $q$, say, are isomorphic to one another.

We shall now fix some notation, as the above was rather cumbersome. For this section and in Chapters 5 and 6 we will take $q = p^e$, for $p$ a prime number and $e \geq 1$ an integer. We denote the finite field with $q$ elements as $\mathbb{F}_q$.

Now, let $T$ be an indeterminate and fix $\mathbb{A} = \mathbb{F}_q[T]$, the polynomial ring over $\mathbb{F}_q$. In Section 2.1 we already pointed out that this generates a Euclidean domain and so it shares many properties in common with the integers including, being a PID and UFD. However, there are also some differences, which will be highlighted momentarily.

**Definition 2.25.** *Let $f(T) \in \mathbb{A}$. Then,*

$$f(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \cdots + \alpha_0, \text{ for } \alpha_i \in \mathbb{F}_q.$$

*The* **degree** *of $f$, written as* $\deg(f)$*, is $n$ if $\alpha_n \neq 0$. The* **sign** *of $f$, written as sgn$(f)$ is $\alpha_n$. We say that $f$*

*is* **monic** *if* $sgn(f) = 1$.

Let $f, g \in \mathbb{A}$ then

$$\deg(fg) = \deg(f) + \deg(g),\ \deg(f+g) \leq \max\{\deg(f), \deg(g)\}$$

and

$$\operatorname{sgn}(fg) = \operatorname{sgn}(f)\operatorname{sgn}(g).$$

**Proposition 2.2** (Division Algorithm for Polynomials)**.** *Let $f, g \in \mathbb{A}$ such that $g \neq 0$. Then there exists elements $q, r \in \mathbb{A}$ such that $f = gq + r$, where $\deg(r) < \deg(g)$. The $q$ and $r$ are unique.*

**Definition 2.26.** *If $f \in \mathbb{A}$ and $f \neq 0$, then $|f| = \#(\mathbb{A}/f\mathbb{A}) = q^{\deg(f)}$, if $f = 0$ then $|f| = 0$.*

Note that we must have $\mathbb{A}^\times = \mathbb{F}_q^\times$, since if $f, g \in \mathbb{A}$ and $fg = 1$, then $\deg(fg) = \deg(1) = 0$, hence $\deg(f) + \deg(g) = 0$. The degree of a polynomial is nonnegative, hence we must have that $\deg(f) = \deg(g) = 0$, so that $f, g \in \mathbb{F}_q^\times$. In fact,

**Proposition 2.3.** *The multiplicative group of units of a finite field is cyclic. In particular, in our case, we have that $|\mathbb{F}_q^\times| = |\mathbb{A}^\times| = q - 1$ and is cyclic.*

We recall that $\mathbb{Z}^\times = \{\pm 1\}$ is also a cyclic group. With this in mind, we point out that the notion of "positive integer" in $\mathbb{A}$ is the monic polynomials. Additionally, the role of the primes in $\mathbb{Z}$ are played by the monic irreducible polynomials in $\mathbb{A}$. Since $\mathbb{A}$ has so many things in common with $\mathbb{Z}$ it is natural to ask the same number theoretic questions about it. For example, we might wonder how the primes in $\mathbb{A}$ are distributed. To answer this we consider the Riemann zeta function, as defined in [94, Chapter 2]:

$$\zeta_{\mathbb{A}}(s) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{1}{|P|^s}\right)^{-1} = \sum_{f \text{ monic}} \frac{1}{|f|^s},\ \mathfrak{Re}(s) > 1.$$

Since there are $q^k$ different monic polynomials of degree $k$, we can also rewrite $\zeta_{\mathbb{A}}(s)$ by collecting all the terms with respect to their degree:

$$\zeta_{\mathbb{A}}(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \sum_{k \geq 0} \frac{1}{q^{k(s-1)}} = \frac{1}{1 - q^{1-s}}.$$

Which has an analytic extension to a meromorphic function with a simple pole at $s = 1$. Let

$$\pi_q(n) = \#\{a \mid a \text{ monic}, \deg(a) = n \text{ and } a \text{ is irreducible}\}.$$

The prime number theorem for polynomials gives the following about $\pi_q(n)$ (cf. [94, Theorem 2.2]):

$$\sum_{k|m} k\pi_q(k) = q^m, \tag{2.9}$$

and

$$\pi_q(n) = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right). \tag{2.10}$$

The next natural question is to consider generalizing this idea to primes in arithmetic progressions. As we saw in Sections 2.2 and 2.4 we should understand Dirichlet characters associated to $(\mathbb{A}/f\mathbb{A})^\times$.

**Definition 2.27** (Chapter 4 [94])**.** *Let $F \in \mathbb{A}$ such that $\deg(F) > 0$. A Dirichlet character modulo $F$, $\chi : \mathbb{A} \to \mathbb{C}$, satisfies*

1. $\chi(a + bF) = \chi(a)$ *for all* $a, b \in \mathbb{A}$,
2. $\chi(a)\chi(b) = \chi(ab)$ *for all* $a, b \in \mathbb{A}$,
3. $\chi(a) \neq 0 \Leftrightarrow (a, F) = 1$.

Just as in the classical case these characters obey an orthogonality relation like in Lemma 2.2.

**Proposition 2.4.** *Let $\chi, \psi$ be two Dirichlet characters modulo $m$ and $a, b \in \mathbb{A}/m\mathbb{A}$. Then*

$$\sum_{a \in (\mathbb{A}/m\mathbb{A})} \chi(a)\overline{\psi}(a) = \Phi(m)\delta(\chi, \psi),$$

*and*

$$\sum_{\chi \pmod m} \chi(a)\overline{\chi}(b) = \Phi(m)\delta(a, b),$$

*where $\delta(m, n) = 1$ if $m = n$ and $0$ otherwise.*

In the above, $\Phi(m)$ is defined in the usual way, that is $\Phi(m) = \#(\mathbb{A}/m\mathbb{A})^\times$.

Before defining our associated Dirichlet $L$-functions, let us first consider a non-trivial example of a Dirichlet character here. In particular, we will make an analogue of the Legendre symbol defined in Definition 2.22.

**Definition 2.28.** *If $f \in A$ such that $\deg(f) > 0$ and $(a, f) = 1$, then we say that $a$ is a $d$-***th power residue** *modulo $f$ if $X^d \equiv a (\mod f)$ is solvable.*

Let $P \in \mathbb{A}$ be irreducible and $d|q-1$. If $a \in \mathbb{A}$ and $(a, P) = 1$ then $X^d \equiv a \pmod P$ has solutions if and only if $a^{\frac{|P|-1}{d}} \equiv 1 \pmod P$. If $a^{\frac{|P|-1}{d}} \not\equiv 1 \pmod P$, it is still true that it is an element of order $d$ in $(\mathbb{A}/P\mathbb{A})^{\times}$. And, since $\mathbb{F}_q^{\times} \hookrightarrow (\mathbb{A}/P\mathbb{A})^{\times}$, there is a unique $\alpha \in \mathbb{F}_q^{\times}$ such that $a^{\frac{|P|-1}{d}} \equiv \alpha \pmod P$.

**Definition 2.29.** *If $P$ is an irreducible and $P \nmid a$ then let $\left(\frac{a}{P}\right)_d$ be the unique element in $\mathbb{F}_q^*$ such that*

$$a^{\frac{|P|-1}{d}} \equiv \left(\frac{a}{P}\right)_d \pmod P.$$

*If $P|a$, then $\left(\frac{a}{P}\right)_d = 0$. This symbol is called the $d$-***th power residue symbol**.

**Remark 2.7.** *The case $d = 2$ is exactly defining the Legendre symbol. For the remaining $d > 2$ we should point out that this returns an element of $\mathbb{F}_q^{\times}$ and not $\mathbb{C}^{\times}$, which implies that $\left(\frac{a}{P}\right)_d$ does not automatically generate an order $d$ Dirichlet character modulo $P$, cf. Definition 2.27.*

We do have a reciprocity law for all $d$:

**Proposition 2.5** (Reciprocity Law)**.** *Let $P$ and $Q$ be monic irreducible polynomials of degree $\delta$ and $\nu$ respectively. Then*

$$\left(\frac{Q}{P}\right)_d = (-1)^{\frac{q-1}{d}\delta\nu} \left(\frac{P}{Q}\right)_d.$$

We extend the definition of $\left(\frac{a}{P}\right)_d$ to hold when $P$ is not irreducible and we state the general reciprocity rule.

**Definition 2.30.** *Let $b \in \mathbb{A}$ have prime decomposition $b = \beta Q_1^{f_1} Q_2^{f_2} \cdots Q_s^{f_s}$. If $a \in \mathbb{A}$, then*

$$\left(\frac{a}{b}\right)_d = \prod_{i=1}^{s} \left(\frac{a}{Q_i}\right)_d^{f_i}.$$

**Remark 2.8.** *Taking $d = 2$ this returns the analogous Kronecker symbol defined in Definition 2.23.*

**Proposition 2.6.** *[94, Theorem 3.5][General Reciprocity Law] Let $a, b \in \mathbb{A}$ such that $(a, b) = 1$. Then,*

$$\left(\frac{a}{b}\right)_d \left(\frac{b}{a}\right)_d^{-1} = (-1)^{\frac{q-1}{d}\deg(a)\deg(b)} \left(\mathrm{sgn}(a)^{\deg(b)}\mathrm{sgn}(b)^{\deg(a)}\right)^{\frac{q-1}{d}}.$$

**Remark 2.9.** *If both $a, b$ are monic then we can see clearly that if $d$=2 and $q \equiv 1 \pmod 4$ that this product will be 1 for any $a, b \in \mathbb{A}$. This informs our later restriction on $q$.*

Now that we have seen some examples of Dirichlet characters in this space, we can introduce the notion of our Dirichlet $L$-function over $\mathbb{A}$:

$$L(s,\chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s} = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1} \text{ for } s \in \mathbb{C}.$$

As with $\zeta_{\mathbb{A}}(s)$ we may collect terms with respect to the degree of the polynomial and write $L(s,\chi)$ as follows:

$$L(s,\chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s} = \sum_{k \geq 0} \frac{1}{q^{ks}} \sum_{\substack{f \text{ monic} \\ \deg(f)=k}} \chi(f).$$

By [94, Proposition 4.3], if $\chi$ is a nontrivial Dirichlet character and $k \geq \deg F$ then

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=k}} \chi(f) = 0. \tag{2.11}$$

That is to say that $L(s,\chi)$ is actually a polynomial in $q^{-s}$, whose degree is at most $\deg(F) - 1$. Hence, we may also express it as a finite product of linear terms : $(1 - \alpha_j(\chi) q^{-s})$, for $j = 1, 2, \ldots, n \leq \deg(F) - 1$.

Consider $P$ an irreducible polynomial and let $\Lambda(f) = \deg P$ if $f = P^k$ and 0 otherwise, the function field analogue of the Von Mangoldt function, then from the proof of [94, Theorem 4.8] we see

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=k}} \Lambda(f)\chi(f) = -\sum_{j=1}^{\deg F - 1} \alpha_j(\chi)^k. \tag{2.12}$$

We mentioned in Section 1.4 that A. Weil [112] proved the analogue of the Riemann Hypothesis. In this setting RH implies that $|\alpha_j(\chi)| = 1$ or $|\alpha_j(\chi)| = \sqrt{q}$. From this we deduce

$$\sum_{\deg P = k} \chi(P) \ll \frac{q^{k/2}}{k} \deg F, \tag{2.13}$$

and that the Euler product representation of $L(s,\chi)$

$$L(s,\chi) = \prod_{P} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1},$$

is actually valid for $\mathrm{Re}(s) > 1/2$.

Finally, as was discussed in Section 1.4 we are interested in obtaining statistical information for $L$-functions taken over the following set:

$$\mathcal{H}_n = \{D \in \mathbb{A} : D \text{ monic, squarefree and } \deg D = n\}.$$

When $n > 1$ then from [94, Proposition 2.3]:

$$|\mathcal{H}_n| = q^{n-1}(q-1).$$

The reason for this choice is that these polynomials are exactly those which will be of interest when considering quadratic extensions of $K = \mathbb{F}_q(T)$, the field of fractions associated to $\mathbb{A}$. We are interested in studying the class number, $h_D$ associated to $K(\sqrt{D})$ where $D \in \mathcal{H}_n$. Over function fields the class number is the size of $\mathrm{Pic}(\mathcal{O}_K)$, the Picard group of $\mathcal{O}_K$ which is given by

$$\mathrm{Pic}(\mathcal{O}_K) = \{\text{non-zero invertible fractional ideals}\}/\{\text{non-zero principal ideals}\}. \tag{2.14}$$

If one takes $D$ to be square free then we have $\mathcal{O}_K$ is a Dedekind domain, which tells us that every non-zero fractional ideal is invertible and hence $\mathrm{Pic}(\mathcal{O}_K) = \mathcal{C}\ell(K)$. So that in the context of square free polynomials we are indeed studying a class number in the traditional sense.

To give a little bit of a geometric flavour, and to more easily introduce the analogue of RH in this space, we consider the following. Let $n$ be an integer, then it can be written as $2g+1$ when $n$ is odd and $2g+2$ when it is even. Let $n > 4$, $D \in \mathcal{H}_n$ and consider

$$C_D : y^2 = D(x). \tag{2.15}$$

This defines a hyperelliptic curve over $\mathbb{F}_q$ with genus $g$. In this instance $h_D$ is associated to the number of $\mathbb{F}_q$-rational points on the Jacobian of $C_D$.

Let $u = q^{-s}$ then the zeta function associated to $C_D$ is defined by

$$Z_{C_D}(s) = \frac{P_{C_D}(u)}{(1-u)(1-qu)}.$$

Weil [112] proved $P_{C_D}(u)$ is a polynomial of degree $2g$. In fact, from [94, Propositions 14.6 and 17.7] we have

$$P_{C_D}(u) = L(s, \chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s},$$

where $\chi_D(f)$ is given by the Kronecker symbol from Definition 2.30:

$$\chi_D(f) = \left(\frac{D}{f}\right)_2 = \left(\frac{D}{f}\right).$$

So from the point of view of (5.3) and (5.16) it's natural that $h_D$ should be associated to this hyperelliptic curve.

We now have enough to state a watered down version of the RH which is sufficient for our use.

**Theorem 2.9** (Riemann Hypothesis over Function Fields)**.** *Let $C_D$ be as in* (2.15) *and let $Z_{C_D}$ be the associated zeta function. Then all the roots of $Z_{C_D}(s)$ lie on the line $\mathfrak{Re}(s) = 1/2$.*

For the complete statement one refers to Rosen's Number Theory in Function Fields, [94, Theorem 5.10]. The appendix of [94] also contains a different proof than Weil's, originally given by Bombieri [10].

The truth of this statement allows us to obtain some immediate consequences about $h_D$. For example, we have the following bounds: $(\sqrt{q}-1)^{2g} \leq h_D \leq (\sqrt{q}+1)^{2g}$, cf [94, Proposition 5.11]. In particular, we already know that if $q > 4$ then $h_D \neq 1$.

## 2.7 Probabilistic Tools

In this section we briefly introduce the reader to some concepts from probability which will be useful in the last two chapters of the thesis. Much of this information is taken from Rick Durrett's *Probability: Theory and Examples* [26].

**Definition 2.31.** *A* **probability space** *is a triple* $(\Omega, \mathcal{F}, P)$ *where* $\Omega$ *is a* $\sigma$*-algebra of outcomes,* $\mathcal{F}$ *is a set* *of events and $P$ is a measure which assigns probability to events, $P : \mathcal{F} \to [0, 1]$.*

We remind the reader that $(\Omega, \mathcal{F})$ is called a **measure space**. An associated **measure** is a nonnegative function which satisfies the following properties

(i) $\mu(A) \geq \mu(\varnothing) = 0$ for every $A \in \mathcal{F}$,

(ii) if $A_i$ is a countable sequence of pairwise disjoint sets then $\mu(\bigcup A_i) = \sum_i \mu(A_i)$,

(iii) if $A \subseteq B$ then $\mu(A) \leq \mu(B)$,

(iv) if $A \subseteq \bigcup U A_i$ then $\mu(A) \leq \sum_i \mu(A_i)$,

(v) if $A_i \uparrow A$ then $\mu(A_i) \uparrow \mu(A)$.

Finally, if $\mu(\Omega) = 1$ then we call the measure a **probability measure**.

On its own a probability space is not that interesting, however, we can assign to these the real object of interest: a **random variable**.

**Definition 2.32.** *A (real valued)* **random variable** *is a function $\mathbb{X} : \Omega \to \mathbb{R}$ such that for every Borel set, $B \subseteq \mathbb{R}$ we have $\mathbb{X}^{-1}(B) = \{\omega | \mathbb{X}(\omega) \in B\} \in \mathcal{F}$. Another name for this is a* **measurable function**.

As a simple example consider the following. The indicator function: Let $A \in \mathcal{F}$ then

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that if we have two measurable maps which we can compose, the result will also be measurable. Since random variables are measurable maps this allows us to apply such maps to our random variable and return another random variable. In particular, we may add or multiply random variables together to create a new one.

Each random variable induces a probability measure on $\mathbb{R}$ which is called its **distribution** by setting $\mu(A) = P(\mathbb{X}^{-1}(A))$. These distributions are normally described by giving its **distribution function**: $F(x) = P(\mathbb{X} \leq x)$. Associated to these distribution functions are **probability density functions**, which in the below examples are denoted as $f$. For continuous random variables they have the following relationship:

$$F(x) = \int_{-\infty}^{x} f(u)du.$$

We consider the following examples to orient ourselves. First, the uniform distribution on the interval $(0,1)$: Let $f(x) = 1$ if $x \in (0,1)$ and 0 otherwise. Then it has distribution function

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1. \end{cases}$$

Next, the exponential distribution with a specified rate $\lambda$: Let $f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$ and 0 otherwise. Then it has distribution function

$$F(x) = \begin{cases} 0 & 0 \leq x \\ 1 - e^{-x} & x \geq 0. \end{cases}$$

Finally, the most famous and one which is often featured in the study of the distribution of values of $L$-functions: The standard normal distribution: Let $f(x) = (2\pi)^{-1/2}e^{-x^2/2}$. There is no closed form for this distribution function, however we have for $x > 0$

$$(x^{-1} - x^{-3})\exp(-x^2/2) \leq \int_{x}^{\infty} \exp(-y^2/2)dy \leq x^{-1}\exp(-x^2/2).$$

Other means for understanding how a random variable behaves come in the form of **moments** of this variable. The first moment, provides information about the average, the second about variance, the third about the skewness of its distribution. As the moments go higher the more information we have about the extreme values that could be obtained by a function of a random variable.

**Definition 2.33.** *Let $\mathbb{X}$ be a random variable with outcomes $x_1, x_2, x_3, \ldots$ occurring with probabilities $p_1, p_2, p_3, \ldots$ respectively, such that $\sum_{i\geq 1}|x_i|p_i$ converges. Then the* **expected value**, *also referred to as the* **first moment** *of $\mathbb{X}$, $E[\mathbb{X}]$ is defined as*

$$E[\mathbb{X}] = \sum_{i\geq 1} x_i p_i.$$

*If $\mathbb{X}$ is instead a continuous random variable then we integrate against its probability measure. Suppose $\mathbb{X} \geq 0$*

*is a random variable on* $(\Omega, \mathcal{F}, P)$ *then*

$$E[\mathbb{X}] = \int \mathbb{X} dP.$$

*This integral always makes sense, but may be* $\infty$. *For a general random variable we consider* $x^+ = \max\{x, 0\}$ *as the positive part of* $x$ *and* $x^- = \max\{-x, 0\}$ *as the negative part, then* $E[\mathbb{X}] = E[\mathbb{X}^+] - E[\mathbb{X}^-]$ *whenever the subtraction makes sense.*

Suppose we do not already have a probability distribution function for our random variable, in some cases we can use information about "all" moments associated to discover the probability distribution function. The remainder of this section will be used to give a history about the use of probabilistic ideas and techniques in number theory.

Using tools from probability has made appearances in number theory beginning in the early 20th century with investigations of Hardy and Ramanujan surrounding the function

$$\omega(n) = \#\{p : p \mid n\}.$$

In 1917 [89], they proved that $\omega(n) \sim \log\log x$ for almost all $n \leq x$. The next natural question to ask is about how $\omega(n) - \log\log(n)$ behaves. In 1934, Turán [106] proved

$$\frac{1}{x}\sum_{n \leq x}(\omega(n) - \log\log n)^2 = \log\log x(1 + o(1)).$$

So far, the results have given us information about the average value of $\omega(n)$ and the average difference between its value and its mean. From the perspective of probability, one should immediately consider whether there is a distribution function for $\omega(n)$. In the 1930's Kac noted the resemblance these results had to developments in probability and he conjectured that $(\omega(n) - \log\log n)/\sqrt{\log\log n}$ is normally distributed. Indeed, in 1940 Erdös and Kac announced [31] the following: let $\tau \in \mathbb{R}$ then

$$\frac{1}{x}|\{n \leq x : \frac{\omega(n) - \log\log(n)}{\sqrt{\log\log(n)}} \leq \tau\}| \to \int_{-\infty}^{\tau} e^{-t^2/2} dt,$$

as $x \to \infty$.

Along the same lines, in 1946 Selberg [99] published his central limit theorem for $\log|\zeta(\frac{1}{2} + it)|$.

**Theorem 2.10** (Selberg's Central Limit Theorem)**.** *Let $V \in \mathbb{R}$ be fixed. Then for all large $T$,*

$$\frac{1}{T}\text{meas}\{t \in [T, 2T] : \log|\zeta(\tfrac{1}{2}+it)| \geq \tfrac{1}{2}V\sqrt{\log\log T}\} \sim \frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-u^2/2}du.$$

Recently, Radziwiłł and Soundararajan [88] found a simpler proof of Selberg's theorem. Proofs of this form apply a technique in probability theory known as the **Method of moments**.

**Definition 2.34.** *let $F_n$ be a sequence of distribution functions. Then we say this sequence* **converges weakly** *to a limit $F$ if $F_n(y) \to F(y)$ for all $y$ that are continuity points for $F$. Let $\mathbb{X}_n$ be a sequence of random variables. Then $\mathbb{X}_n$ is said to* **converge in distribution***, to a limit $\mathbb{X}$ if the distribution functions $F_n(x) = P(\mathbb{X}_n \leq x)$ converge weakly.*

The method of moments is a way to prove convergence in distribution:

**Theorem 2.11** (Method of Moments)**.** *Suppose that $\mathbb{X}$ is a random variable such that $\mathbb{E}(\mathbb{X}^k)$ exist for all $k$ and that $\mathbb{X}_n$ is a sequence of random variables. If in addition, the probability distribution of $\mathbb{X}$ is completely determined by its moments then if*

$$\lim_{n\to\infty}\mathbb{E}(\mathbb{X}_n^k) = \mathbb{E}(\mathbb{X}^k),$$

*for all values of $k$ we have that $\mathbb{X}_n$ converges to $\mathbb{X}$ in distribution.*

Since the advent of Erdös-Kac theorem and Selberg Central Limit Theorem there have been many other attempts to provide central limit theorem type results for different $L$-functions with some success in special cases and under some strong assumptions regarding the central value (that is the value at the point $s = \frac{1}{2}$).

As we saw in Sections 1.4 and 1.5 the use of probability to understand $L$-functions goes beyond proving these types of results on the critical line. Indeed there is a rich history of research considering the distribution of values of $L$-functions occurring at other points within the critical strip. We have already mentioned the conjectures of Montgomery and Vaughan and the work of Granville and Soundararajan and separately Lamzouri associated to values $L(1, \chi_d)$, with $\chi_d$ the Kronecker symbol. In these works, there is some limitation on the range of moments which can be computed, thus they do not return a description of a full probability distribution function but they are able to describe the behaviour of the tails of these distributions. We prove similar theorems over function fields in Chapter 5. In this chapter we will explain more clearly the likenesses and the surprising behaviour we find which diverges from the number field case. Chapter 6 investigates what can be said about $L(\sigma, \chi_D)$, with $\sigma \in (1/2, 1)$ fixed taken over function fields.

The key idea of Chapters 5 and 6 is to study how the $L$-function behaves by comparing it and as many moments as possible to a random model. This random model is built out of random variables which mimic the behaviour of the characters associated to the $L$-function of interest. In the case of $\zeta(\sigma+it)$ one considers a sequence of random variables $\{\mathbb{X}(p)\}_{p \text{ prime}}$ which is uniformly distributed on the unit circle and build the following random model:

$$\zeta(\sigma,\mathbb{X}) = \prod_p \left(1 - \frac{\mathbb{X}(p)}{p^\sigma}\right)^{-1}.$$

For the case of the $L(\sigma,\chi_d)$ we again build a random model by considering a sequence of random variables $\{\mathbb{X}(p)\}_{p \text{ prime}}$ which have some appropriate discrete probabilities assigned to them (depending on which character you want to model) and consider

$$L(\sigma,\mathbb{X}) = \prod_p \left(1 - \frac{\mathbb{X}(p)}{p^\sigma}\right)^{-1}.$$

In the last chapters the we choose the probabilities as follows:

$$\mathbb{X}(P) = \begin{cases} 0 & \text{with probability } \frac{1}{|P|+1} \\ \pm 1 & \text{with probability } \frac{|P|}{2(|P|+1)}. \end{cases}$$

The reason we make this choice is based on the fact that our discriminants $D$ are in the set $\mathcal{H}_n$. This means that for any irreducible $P$ the residue classes $D$ can occupy are restricted to $|P|^2 - 1$ of the $|P|^2$ classes modulo $P^2$. Since there are $|P| - 1$ of these which will not be co-prime to $P$ we see that the Kronecker symbol on those classes would have value 0. Therefore we choose $(|P|-1)/(|P|^2-1) = 1/(|P|+1)$ as the probability we obtain 0 for our random variables. In the remaining residue classes the Kronecker symbol takes the values 1 exactly half of the time and $-1$ otherwise.

# 3 Zero Density Results for the Riemann Zeta Function

This chapter is joint work with Dr. Habiba Kadiri and Dr. Nathan Ng from the University of Lethbridge.

## 3.1 Introduction

Throughout this chapter $\zeta(s)$ denotes the Riemann zeta function and $\varrho$ denotes a non-trivial zero of $\zeta(s)$ lying in the critical strip, $0 < \mathfrak{Re}(s) < 1$. Let $\frac{1}{2} < \sigma < 1, T > 0$, and define

$$N(\sigma, T) = \#\{\varrho = \beta + i\gamma : \ \zeta(\varrho) = 0, 0 < \gamma < T \text{ and } \sigma < \beta < 1\}. \tag{3.1}$$

We shall prove a non-trivial, explicit upper bound for $N(\sigma, T)$. Such a bound is commonly referred to as a zero-density estimate. We denote RH the Riemann Hypothesis and RH($H_0$) the statement:

$$\mathrm{RH}(H_0): \text{ all non-trivial zeros } \varrho \text{ of } \zeta(s) \text{ with } |\mathfrak{Im}(\varrho)| \le H_0 \text{ satisfy } \mathfrak{Re}(\varrho) = \frac{1}{2}. \tag{3.2}$$

Currently, the best published value of $H_0$ for which (3.2) is true is due to David Platt [84]:

$$H_0 = 3.0610046 \cdot 10^{10}$$

with $N(H_0) = 103\,800\,788\,359$. Other strong evidence towards the RH is the large body of zero-density estimates for $\zeta(s)$. Namely, very good bounds for $N(\sigma, T)$ in various ranges of $\sigma$.
Let $\sigma > \frac{1}{2}$. In 1913 Bohr and Landau [8] showed that

$$N(\sigma, T) = \mathcal{O}_\sigma\left(\frac{T}{\sigma - \frac{1}{2}}\right) \tag{3.3}$$

for $T$ asymptotically large. This result implies that for any fixed $\varepsilon > 0$, almost all zeros of $\zeta(s)$ lie in the band $|\frac{1}{2} - \mathfrak{Re}(s)| < \varepsilon$. This was improved in 1937 by Ingham [51], who showed uniformly in $1/2 \le \sigma \le 1$ as $T \to \infty$ that

$$N(\sigma, T) = \mathcal{O}\left(T^{(2+4c)(1-\sigma)}(\log T)^5\right) \tag{3.4}$$

under the assumption that $\zeta(\frac{1}{2} + it) = \mathcal{O}(t^{c+\epsilon})$. In particular, the Lindelöf Hypothesis $\zeta(\frac{1}{2} + it) = \mathcal{O}(t^{\epsilon})$ implies that $N(\sigma, T) = \mathcal{O}\left(T^{2(1-\sigma)+\epsilon}\right)$, also known as the Density Hypothesis. There is a prolific literature on the bounds for $\zeta(s)$, starting with the convexity bound of $c = \frac{1}{4} = 0.25$ (Lindelöf), the first subconvexity bound of Hardy & Littlewood [42] $c = \frac{1}{6} = 0.1666\ldots$, to some more recent results of Huxley [49] (2005) $c = \frac{32}{205} = 0.1560\ldots$ and of Bourgain [11] (2017) $c = \frac{13}{84} = 0.1547\ldots$. In addition, there are also many articles on estimates for $N(\sigma, T)$. A selection of some notable results may be found in [49], [50], [55], and [11]. On the other hand, there are few explicit bounds for $N(\sigma, T)$. We refer the reader to a result of the first author [57] for an explicit version of Bohr and Landau's bound. The method provides two kind of results: for $T$ asymptotically large, as in $N(0.90, T) \le 0.4421T + 0.6443 \log T - 363\,301$, and for $T$ taking a specific value, as in $N(0.90, H_0) \le 96$. These bounds are useful to improve estimates of prime counting functions, as in [32], [28], [85], [105] and in [58] to find primes in short intervals. Ramaré had earlier proven a version of (3.4) in his D.E.A. memoire, which remained unpublished until recently. Let $\sigma \ge 0.52$ be fixed. In [90] he proves [1] that for any $T \ge 2000$

$$N(\sigma, T) \le 965(3T)^{\frac{8(1-\sigma)}{3}}(\log T)^{5-2\sigma} + 51.5(\log T)^2, \tag{3.5}$$

which gives $N(0.90, T) < 1293.48(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 51.50(\log T)^2$, which gives the bound for $T = H_0$: $N(0.90, H_0) < 2.1529 \cdot 10^{10}$. The purpose of this article is to bound $N(\sigma, T)$ by applying Ingham's argument with a general weight and to improve both [57] and [90].

**Theorem 3.1.** *Let* $\frac{10^9}{H_0} \le k \le 1, d > 0, H \in [1002, H_0)$*,* $\alpha > 0$*,* $\delta \ge 1$*,* $\eta_0 = 0.23622\ldots$*,* $1 + \eta_0 \le \mu \le 1 + \eta$*, and* $\eta \in (\eta_0, \frac{1}{2})$ *be fixed. Let* $\sigma > \frac{1}{2} + \frac{d}{\log H_0}$*.*

*Then there exist* $\mathcal{C}_1, \mathcal{C}_2 > 0$ *such that, for any* $T \ge H_0$*,*

$$N(\sigma, T) \le \frac{(T-H)(\log T)}{2\pi d} \log\left(1 + \frac{\mathcal{C}_1(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)}}{T-H}\right) + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2, \tag{3.6}$$

[1] Equation (1.1) [90, p. 326 ] gives the bound $N(\sigma, T) \le 4.9(3T)^{\frac{8(1-\sigma)}{3}}(\log T)^{5-2\sigma} + 51.5(\log T)^2$. However, there is a mistake in [90]. The authors have been in communication with Professor Ramaré and he has sent us a proof of the revised inequality (3.5).

*where* $\mathcal{C}_1 = \mathcal{C}_1(\alpha, d, \delta, k, H, \sigma)$ *and* $\mathcal{C}_2 = \mathcal{C}_2(d, \eta, k, H, \mu, \sigma)$ *are defined in* (3.117) *and* (3.118). *Since* $\log(1+x) \le x$ *for* $x \ge 0$, (3.6) *implies*

$$N(\sigma,T) \le \frac{\mathcal{C}_1}{2\pi d}(\log(kT))^{2\sigma}(\log T)^{5-4\sigma}T^{\frac{8}{3}(1-\sigma)} + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2. \tag{3.7}$$

*In addition, numerical results are displayed in tables in Section 3.5.*

For instance (3.7) gives $N(0.90,T) < 11.499(\log T)^{\frac{16}{5}}T^{\frac{4}{15}} + 3.186(\log T)^2$, and (3.6) gives $N(0.90,H_0) < 130.07$. This improves previous results both numerically and methodologically (one of the key ingredients is the choice of a more efficient weight function in Ingham's method). Note that choosing $k < 1$ and optimizing in $H$ can provide extra improvements to (3.5). In addition, we prove a stronger bound for the argument of a holomorphic function. We now explain the main ideas to prove Theorem 3.1.

## 3.2 Setting up the proof

### 3.2.1 Littlewood's classical method to count the zeros

Let $h(s) = \zeta(s)M(s)$ where $M(s)$ is entire and

$$N_h(\sigma,T) = \#\Big\{\varrho' = \beta' + i\gamma' \in \mathbb{C} \ : \ h(\varrho') = 0, \sigma < \beta' < 1, \text{ and } 0 < \gamma' < T\Big\}. \tag{3.8}$$

Then for a parameter $H \in (0, H_0)$, we have by (3.2) that

$$N(\sigma,T) = N(\sigma,T) - N(\sigma,H) \le N_h(\sigma,T) - N_h(\sigma,H)$$

for $T \ge H_0$. We compare the above number of zeros for $h$ to its average:

$$N_h(\sigma,T) - N_h(\sigma,H) \le \frac{1}{\sigma-\sigma'}\int_{\sigma'}^{\mu}(N_h(\tau,T) - N_h(\tau,H))\,d\tau$$

where $\mu > 1$ and $\sigma'$ is a parameter satisfying $\frac{1}{2} < \sigma' < \sigma$. Let $\mathcal{R}$ be the rectangle with vertices $\sigma' + iH$, $\mu + iH$, $\mu + iT$, and $\sigma' + iT$. We apply the classical lemma of Littlewood as stated in [102, (9.9.1)]:

$$\int_{\sigma'}^{\mu} \Big( N_h(\tau, T) - N_h(\tau, H) \Big) d\tau = -\frac{1}{2\pi i} \int_{\mathcal{R}} \log h(s) ds. \tag{3.9}$$

Thus

$$\begin{aligned} N(\sigma, T) \le \frac{1}{2\pi(\sigma - \sigma')} \Big( \int_H^T \log |h(\sigma' + it)| dt \\ + \int_{\sigma'}^{\mu} \arg h(\tau + iT) d\tau - \int_{\sigma'}^{\mu} \arg h(\tau + iH) d\tau - \int_H^T \log |h(\mu + it)| dt \Big). \end{aligned} \tag{3.10}$$

As $T$ grows larger, the main contribution arises from the first integral. The second and third integrals can be treated by using a general result for bounding $\arg f(s)$ for $f$ a holomorphic function. To do this we give an improvement of a lemma of Titchmarsh [102, p. 213] (see Proposition 3.1 and Corollary 3.1 below). The fourth integral can be estimated with a standard mean value theorem for Dirichlet polynomials (see Lemma 3.6). A key goal is to minimize the above expression over admissible functions $h$. We now give an idea of how to estimate the first integral in (3.10).

### 3.2.2 How the second mollified moment of $\zeta(s)$ occurs

Let $X \ge 1$ be a parameter and define the mollifier to be

$$M_X(s) = \sum_{n \le X} \frac{\mu(n)}{n^s} \tag{3.11}$$

where $\mu(n)$ is the Möbius function. Note that this is a truncation of the Dirichlet series for $\zeta(s)^{-1}$. These mollifiers were invented by Bohr and Landau [8] to help control the size of $\zeta(s)$ in the critical strip. Futhermore, let

$$f_X(s) = \zeta(s) M_X(s) - 1. \tag{3.12}$$

Note that the series expansion for $f_X$ is given by

$$f_X(s) = \sum_{n>X} \Big( \sum_{\substack{d|n \\ d\le X}} \mu(d) \Big) n^{-s} = \sum_{n\ge 1} \frac{\lambda_X(n)}{n^s}, \tag{3.13}$$

$$\text{with} \quad \lambda_X(n) = 0 \quad \text{if } n \le X, \quad \lambda_X(n) = \sum_{\substack{d|n \\ d\le X}} \mu(d) \quad \text{if } n > X. \tag{3.14}$$

We shall choose $h = h_X$ with

$$h_X(s) = 1 - f_X(s)^2 = \zeta(s)M_X(s)(2 - \zeta(s)M_X(s)). \tag{3.15}$$

Since we have

$$\frac{1}{b-a}\int_a^b \log f(t)dt \le \log\left(\frac{1}{b-a}\int_a^b f(t)dt\right),$$

for any $f$ non-negative and continuous, and $|h_X(s)| \le 1 + |f_X(s)|^2$, we deduce that

$$\int_H^T \log\left(|h_X(\sigma' + it)|\right) dt \le (T-H)\log\left(1 + \frac{1}{T-H}\int_H^T |f_X(\sigma' + it)|^2 dt\right). \tag{3.16}$$

We denote

$$F_X(\sigma, T) = \int_0^T |f_X(\sigma + it)|^2 dt \text{ where } \sigma \ge \frac{1}{2}. \tag{3.17}$$

To resume, the key point for getting a good bound on $N(\sigma, T) - N(\sigma, H)$ is to obtain a good bound for $F_X(\sigma, T)$. Following a classical method due to Ingham we compare it to a smoothed version of itself.

### 3.2.3 Ingham's smoothing method

Let $\sigma_1$ and $\sigma_2$ be such that $\sigma_1 < \sigma < \sigma_2$. Let $T > 0$ and $g = g_T$ be a non-negative, real valued function, depending on the parameter $T$, and holomorphic in $\sigma_1 \le \mathfrak{Re}(s) \le \sigma_2$. We define

$$\mathcal{M}_{g,T}(X, \sigma) = \int_{-\infty}^{+\infty} |g(\sigma + it)|^2 |f_X(\sigma + it)|^2 dt. \tag{3.18}$$

We shall consider $g$ of a special shape. For $\alpha, \beta > 0$, assume that there exist positive functions $\omega_1, \omega_2$ such that $g$ satisfies, for all $\sigma \in [\sigma_1, \sigma_2]$,

$$|g(\sigma+it)| \le \omega_1(\sigma, T, \alpha) e^{-\alpha\left(\frac{|t|}{T}\right)^\beta} \quad \text{for all } t \in \mathbb{R}, \tag{3.19}$$

$$\omega_2(\sigma, T, \alpha) \le |g(\sigma+it)| \quad \text{for all } t \in [H, T]. \tag{3.20}$$

In addition, we assume that $|g|$ is even in $t$:

$$|g(\sigma - it)| = |g(\sigma+it)| \text{ for } \sigma \in (\sigma_1, \sigma_2) \text{ and } t \in \mathbb{R}. \tag{3.21}$$

Thus $F_X(\sigma, T) \ll_g \mathcal{M}_{g,T}(X, \sigma)$, and more precisely

$$F_X(\sigma, T) \le \frac{\mathcal{M}_{g,T}(X, \sigma)}{2(\omega_2(\sigma, T, \alpha))^2}. \tag{3.22}$$

In this article, we shall choose a family of weights of the form

$$g(s) = g_T(s) = \frac{s-1}{s} e^{\alpha\left(\frac{s}{T}\right)^2}, \text{ where } \alpha > 0. \tag{3.23}$$

These weights will satisfy the above conditions with $\beta = 2$. We remark that Ingham [51] made use of the weight $g(s) = \frac{s-1}{s\cos(\frac{1}{2T})}$ and Ramaré [90] used $g(s) = \frac{s-1}{s(\cos s)^{\frac{1}{2T}}}$. These weights satisfy (3.19) with $\beta = 1$. We also studied the weights $g(s) = \frac{s-1}{s(\cos s)^{\frac{\alpha}{T}}}$ and $g(s) = \frac{s-1}{s(\cos\frac{\alpha}{T})}$. However, we obtained the best results with $g$ given by (3.23). The functions $g$ are chosen so that for fixed $\sigma$, $g(\sigma+it)$ behave likes the indicator function, $\mathbb{1}_{[0,T]}(t)$, and for $t$ large, $g(\sigma+it)$ has rapid decay. Nevertheless, it is an open problem to determine the best weights $g$ to use in this problem.

### 3.2.4 Final bound

Finally, to bound the integral $\mathcal{M}_{g,T}$, we appeal to a convexity estimate for integrals (see [44]). For $\sigma_2 > 1$ (and $\sigma_2$ close to 1), if $\frac{1}{2} \le \sigma \le \sigma_2$, then

$$\mathcal{M}_{g,T}(X, \sigma) \le \mathcal{M}_{g,T}(X, \tfrac{1}{2})^{\frac{\sigma_2 - \sigma}{\sigma_2 - \frac{1}{2}}} \mathcal{M}_{g,T}(X, \sigma_2)^{\frac{\sigma - \frac{1}{2}}{\sigma_2 - \frac{1}{2}}}. \tag{3.24}$$

The largest contribution arises from $\mathcal{M}_{g,T}(X, \frac{1}{2})$. To bound this we make use of:

- bounds (3.19), (3.20) for $g$ (see Lemma 3.7),
- a version of Montgomery and Vaughan's Mean Value Theorem for Dirichlet polynomials (see Lemma 3.6),
- bounds for arithmetic sums to bound the second moment of the mollifier $M_X$ (we use Ramaré's bounds, see Lemma 3.3 and 3.4),
- the most recent explicit subconvexity bound for the Riemann zeta function (due to Hiary [45], see Lemma 3.2).

## 3.3 Preliminary lemmas

### 3.3.1 Bounds for the Riemann zeta function

In this section we record a number of bounds for the zeta function. Rademacher [87, Theorem 4] established the following explicit convexity bound.

**Lemma 3.1.** *For $-\frac{1}{2} \le -\eta \le \sigma \le 1+\eta \le \frac{3}{2}$, we have*

$$|\zeta(s)| \le 3\frac{|1+s|}{|1-s|}\left(\frac{|1+s|}{2\pi}\right)^{\frac{1}{2}(1-\sigma+\eta)}\zeta(1+\eta). \tag{3.25}$$

The next lemma is an explicit version of van der Corput's subconvexity bound for $\zeta$ on the critical line, recently proven by Hiary. [45].

**Lemma 3.2.** *We have*

$$|\zeta(\tfrac{1}{2}+it)| \le a_1 t^{\frac{1}{6}} \log t \qquad \text{for all } \; t \ge 3, \tag{3.26}$$

$$\max_{|t|\le T} |\zeta(\tfrac{1}{2}+it)| \le a_1 T^{\frac{1}{6}} \log T + a_2 \qquad \text{for all } \; T > 0, \tag{3.27}$$

*with*

$$a_1 = 0.63 \text{ and } a_2 = 2.851. \tag{3.28}$$

*Proof of Lemma 3.2.* Statement (3.26) is [45, Theorem 1.1]. For $t \in [0,3]$, [45, Theorem 1.1] provides that $|\zeta(\frac{1}{2}+it)| \leq 1.461$. We find that the minimum of the function $t^{\frac{1}{6}}\log(t)$ occurs when $t = e^{-6}$. We require the polynomial $a_1 t^{\frac{1}{6}}\log(t) + a_2 \geq 1.461$, choosing $a_2$ as in the statement of the lemma achieves this. $\square$

### 3.3.2 Bounds for arithmetic sums

We list here some preliminary lemmas from [90] providing estimates for finite arithmetic sums. Let

$$b_1 = 0.62,\ b_2 = 1.048,\ b_3 = 0.605,\ \text{and } b_4 = 0.529. \tag{3.29}$$

**Lemma 3.3.** *We have*

$$\sum_{n\leq X}\mu^2(n) \leq b_1 X \quad \textit{for all} \quad X \geq 1700, \tag{3.30}$$

$$\sum_{n\leq X}\frac{\mu^2(n)}{n} - \frac{6}{\pi^2}\log X \leq b_2 \quad \textit{for all} \quad X \geq 1002. \tag{3.31}$$

(3.30) is [90, Lemma 3.1] and (3.31) is [90, Lemma 3.4].

**Lemma 3.4.** *Let $\tau > 1, \delta > 0$, $X \geq 10^9$, and $\gamma$ denotes Euler's constant. Then*

$$\sum_{X<n<5X}\frac{\lambda_X(n)^2}{n^2} \leq \frac{b_3}{X}, \tag{3.32}$$

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^\tau} \leq \frac{b_4\tau^2}{\tau-1}e^{\gamma(\tau-1)}\log X, \tag{3.33}$$

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{1+\frac{\delta}{\log X}}} \leq \frac{b_4}{\delta}\left(1+\frac{\delta}{\log X}\right)^2 e^{\frac{\delta\gamma}{\log X}}(\log X)^2, \tag{3.34}$$

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^{2+\frac{2\delta}{\log X}}} \leq \frac{b_4}{5\delta e^\delta}\left(1+\frac{\delta}{\log X}\right)^2 e^{\frac{\delta(\gamma-\log 5)}{\log X}}\frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta}}{X}. \tag{3.35}$$

*Proof.* (3.32) is [90, Lemma 5.6] and (3.33) is [90, Lemma 5.5]. (3.34) is a direct consequence of (3.33), taking $\tau = 1 + \frac{\delta}{\log X}$.

For (3.35) we set $\tau = 2 + \frac{2\delta}{\log X}$. Since $\lambda_X(n)^2 = 0$ when $1 \leq n \leq X$, then

$$\sum_{n\geq 1}\frac{\lambda_X(n)^2}{n^\tau} = \sum_{X<n<5X}\frac{\lambda_X(n)^2}{n^\tau} + \sum_{n\geq 5X}\frac{\lambda_X(n)^2}{n^\tau}.$$

Since $\tau \geq 2$, we use (3.32) and find that the first sum is

$$\leq \frac{1}{X^{\tau-2}} \sum_{X<n<5X} \frac{\lambda_X(n)^2}{n^2} \leq \frac{1}{X^{\tau-2}} \frac{b_3}{X} = \frac{b_3 e^{-2\delta}}{X}.$$

We bound the second sum using $n^\tau \geq (5X)^{1+\frac{\delta}{\log X}} n^{1+\frac{\delta}{\log X}}$ and (3.34). We find that it is

$$\leq \frac{1}{(5X)^{1+\frac{\delta}{\log X}}} \frac{b_4}{\delta} \left(1 + \frac{\delta}{\log X}\right)^2 e^{\frac{\delta\gamma}{\log X}} (\log X)^2.$$

Combining bounds

$$\sum_{n\geq 1} \frac{\lambda_X(n)^2}{n^\tau} \leq \frac{b_4}{5\delta e^\delta} \left(1 + \frac{\delta}{\log X}\right)^2 e^{\frac{\delta(\gamma - \log 5)}{\log X}} \frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta}}{X}.$$

□

**Lemma 3.5.** *Let $\tau > 1$ and $\gamma$ is Euler's constant. Then for $X \geq 1$,*

$$\sum_{n\geq X} \frac{d(n)}{n^\tau} \leq \frac{\tau}{X^{\tau-1}} \left(\frac{\log X}{\tau - 1} + \frac{1}{(\tau-1)^2} + \frac{\gamma}{\tau - 1} + \frac{7}{12\tau X}\right) \tag{3.36}$$

*and for $X \geq 47$,*

$$\sum_{n\geq X} \frac{d(n)^2}{n^\tau} \leq \frac{2\tau}{X^{\tau-1}} \Big(\frac{(\log X)^3}{\tau-1} + \frac{3\log^2 X}{(\tau-1)^2} + \frac{6\log X}{(\tau-1)^3} + \frac{6}{(\tau-1)^4}\Big). \tag{3.37}$$

*Proof.* By partial summation, we have

$$\sum_{n\geq X} \frac{d(n)}{n^\tau} \leq \tau \int_X^\infty \frac{\sum_{n\leq t} d(n)}{t^{\tau+1}} dt.$$

Using $\sum_{n\leq t} d(n) \leq t(\log t + \gamma + \frac{7}{12t})$, for $t \geq 1$, which follows from [91, Equation 3.1], we have

$$\sum_{n\geq X} \frac{d(n)}{n^\tau} \leq \tau \left(\int_X^\infty \frac{\log t}{t^\tau} dt + \gamma \int_X^\infty \frac{dt}{t^\tau} + \frac{7}{12} \int_X^\infty \frac{dt}{t^{\tau+1}}\right).$$

By applying the integrals

$$\int_X^\infty \frac{\log t}{t^c} dt = \frac{\log X}{(c-1)X^{c-1}} + \frac{1}{(c-1)^2 X^{c-1}} \text{ and } \int_X^\infty \frac{dt}{t^c} = \frac{1}{(c-1)X^{c-1}}, \text{ where } c > 1,$$

we obtain (3.36). The second estimate is similar. We have

$$\sum_{n\geq X}\frac{d(n)^2}{n^\tau}\leq \tau\int_X^\infty \frac{\sum_{n\leq t}d(n)^2}{t^{\tau+1}}\,dt.$$

It suffices to use the elementary bound $\sum_{n\leq t}d(n)^2\leq t(\log t+1)^3\leq 2t\log^3 t$ for $t\geq 47$, derived by Gowers [110]. Thus

$$\sum_{n\geq X}\frac{d(n)^2}{n^\tau}\leq 2\tau\int_X^\infty\frac{\log^3 t}{t^\tau}\,dt=2\tau\left(\frac{\frac{(\log X)^3}{\tau-1}+\frac{3\log^2 X}{(\tau-1)^2}+\frac{6\log X}{(\tau-1)^3}+\frac{6}{(\tau-1)^4}}{X^{\tau-1}}\right).$$

□

### 3.3.3 Mean value theorem for Dirichlet polynomials

We require Montgomery and Vaughan's mean value theorem for Dirichlet polynomials in the form derived by Ramaré [90].

**Lemma 3.6.** *Let $(u_n)$ be a real-valued sequence. For every $T\geq 0$ we have*

$$\int_0^T\Big|\sum_{n=1}^\infty u_n n^{it}\Big|^2 dt\leq\sum_{n\geq 1}|u_n|^2(T+\pi m_0(n+1)),\tag{3.38}$$

*with*

$$m_0=\sqrt{1+\frac{2}{3}\sqrt{\frac{6}{5}}}.\tag{3.39}$$

*Let $0<T_1<T_2$. Then*

$$\int_{T_1}^{T_2}|\sum_{n=1}^\infty u_n n^{it}|^2 dt\leq\sum_{n\geq 1}|u_n|^2(T_2-T_1+2\pi m_0(n+1)).\tag{3.40}$$

*Proof.* The inequality (3.38) is [90, Lemma 6.5], and (3.40) follows by the same proof. This argument is an explicit version of Corollary 3 of [80] which makes use of the main theorem of [86]. Note that (3.40) follows from two applications of (3.38). □

### 3.3.4 Choice for the smooth weight $g$

**Lemma 3.7.** *Let $\alpha > 0$ and $\beta = 2$. Let $s = \sigma + it$ and let $g$ be as defined in (3.23):*

$$g(s) = \frac{s-1}{s} e^{\alpha\left(\frac{s}{T}\right)^2}. \tag{3.41}$$

*Let $\sigma_1 = \frac{1}{2}, \sigma_2 > 1$, and $H < T$. Define*

$$\omega_1(\sigma, T, \alpha) \quad = e^{\alpha\left(\frac{\sigma}{T}\right)^2}, \tag{3.42}$$

$$\omega_2(\sigma, T, \alpha) \quad = \left(1 - \frac{1}{H}\right) e^{\alpha\left(\frac{\sigma}{T}\right)^2 - \alpha}. \tag{3.43}$$

*Then for $\frac{1}{2} \le \sigma \le \sigma_2$, $g$ satisfies (3.19) and (3.20):*

$$|g(\sigma + it)| \le \omega_1(\sigma, T, \alpha) e^{-\alpha\left(\frac{|t|}{T}\right)^2} \qquad \text{for all } t, \tag{3.44}$$

$$\omega_2(\sigma, T, \alpha) \le |g(\sigma + it)| \qquad \text{for } H \le t \le T. \tag{3.45}$$

*Proof.* Since $\sigma \ge \frac{1}{2}$, we have $\left|\frac{s-1}{s}\right|^2 = 1 - \frac{2\sigma - 1}{\sigma^2 + t^2} \le 1$. Thus $|g(s)| \le |e^{\alpha\left(\frac{s}{T}\right)^2}| = e^{\frac{\alpha\sigma^2}{T^2}} e^{\frac{-\alpha t^2}{T^2}}$ and we have the expression for $\omega_1(\sigma, T, \alpha)$.
In addition, $|\frac{s-1}{s}| = |1 - \frac{1}{s}| \ge 1 - \frac{1}{|s|} \ge 1 - \frac{1}{|t|}$, so for all $t \in [H, T]$, we have

$$|g(s)| \ge \left(1 - |t|^{-1}\right) e^{\frac{\alpha\sigma^2}{T^2}} e^{\frac{-\alpha t^2}{T^2}} \ge (1 - H^{-1}) e^{\frac{\alpha\sigma^2}{T^2}} e^{-\alpha},$$

which gives $\omega_2(\sigma, T, \alpha)$. □

## 3.4 Proof of the Main Theorem

Unless specified in the rest of the article, we set $H_0 = 3.0610046 \cdot 10^{10}$ and we have the following conditions on the parameters $k, \sigma_1, \delta$, and $\sigma_2$:

$$k \ge \frac{10^9}{H_0},\ \sigma_1 = \frac{1}{2},\ \delta > 0, \text{ and } \sigma_2 = 1 + \frac{\delta}{\log X}. \tag{3.46}$$

### 3.4.1 Bounding $F_X(\sigma, T)$

We establish here some preliminary lemmas to estimate $F_X(\sigma, T)$ at $\frac{1}{2}$ and at $1 + \frac{\delta}{\log X}$.

#### 3.4.1.1 Bounding $F_X(\frac{1}{2}, T)$

We first need to bound the second moment of $M_X(\frac{1}{2} + it)$, where $M_X$ is defined in (3.11).

**Lemma 3.8.** *Let $T > 0$, $X \geq kH_0$, and $k$ satisfies* (3.46). *Then*

$$\int_0^T \left|M_X(\tfrac{1}{2} + it)\right|^2 dt \leq (C_1 T + C_2 X)(\log X), \tag{3.47}$$

*where*

$$C_1 = C_1(k) = \frac{6}{\pi^2} + \frac{b_2}{\log(kH_0)}, \tag{3.48}$$

$$C_2 = C_2(k) = \frac{\pi m_0 b_1}{\log(kH_0)} + \frac{6m_0}{\pi k H_0} + \frac{\pi m_0 b_2}{kH_0 \log(kH_0)}, \tag{3.49}$$

*and the $b_i$'s are defined in* (3.29) *and $m_0$ in* (3.39).

*Proof.* We apply (3.38) to $u_n = \frac{\mu(n)}{n^{\frac{1}{2}}}$:

$$\int_0^T \left|M_X\left(\tfrac{1}{2} + it\right)\right|^2 dt \leq \sum_{n \leq X} \frac{\mu^2(n)}{n}(T + \pi m_0(n+1)).$$

Since $X \geq 1700$, we apply (3.31) to $(T + \pi m_0) \sum_{n \leq X} \frac{\mu^2(n)}{n}$ and (3.30) to $(\pi m_0) \sum_{n \leq X} \mu^2(n)$ respectively. We factor $\log X$ to give

$$\begin{aligned}\int_0^T \left|M_X\left(\tfrac{1}{2} + it\right)\right|^2 dt &= (T + \pi m_0)\left(\frac{6}{\pi^2}\log X + b_2\right) + \pi m_0 b_1 X \\ &= \left(\left(\frac{6}{\pi^2} + \frac{b_2}{\log X}\right) T + \left(\frac{6m_0}{\pi X} + \frac{\pi m_0 b_2}{X \log X} + \frac{\pi m_0 b_1}{\log X}\right) X\right)(\log X),\end{aligned}$$

and use the fact that $X \geq kH_0$ to obtain the announced bound. ∎

**Lemma 3.9.** *Let $T > 0$, $X \geq kH_0$, and $k$ satisfies* (3.46)*. Then*

$$F_X(\tfrac{1}{2}, T) \leq C_4 \left( T^{\frac{1}{6}} \log T + \frac{a_2}{a_1} \right)^2 \left( T + \frac{C_2}{C_1} X \right) (\log X), \tag{3.50}$$

*where $a_1, a_2$ are defined in* (3.28)*, $C_1$ in* (3.48)*, $C_2$ in* (3.49)*, and*

$$a_3 = -\frac{6a_1}{e} + a_2, \tag{3.51}$$

$$C_3 = C_3(k) = a_3^2 C_1(k) \log(kH_0), \tag{3.52}$$

$$C_4 = C_4(k) = C_1(k) a_1^2 \left( 1 + \frac{1}{\sqrt{C_3(k)}} \right)^2. \tag{3.53}$$

*Proof.* We have from the definition of $F_X(\sigma, T)$ given as (3.17) and Minkowski's inequality that

$$\sqrt{|F_X(\tfrac{1}{2}, T)|} \leq \sqrt{\int_0^T |\zeta(\tfrac{1}{2} + it) M_X(\tfrac{1}{2} + it)|^2 dt} + \sqrt{T}.$$

To the last integral we apply Hiary's subconvexity bound (3.27) to bound zeta and (3.47) to bound the mean square of $M_X$. We let $I_0$ denote the resulting bound so that

$$I_0 = (a_1 T^{\frac{1}{6}} \log T + a_2)^2 (C_1 T + C_2 X)(\log X),$$

and thus

$$|F_X(\tfrac{1}{2}, T)| \leq \left( \sqrt{I_0} + \sqrt{T} \right)^2 = I_0 \left( 1 + \sqrt{\frac{T}{I_0}} \right)^2.$$

We note that $a_1 T^{\frac{1}{6}} \log T + a_2$ is minimized at $T = e^{-6}$ and we let $a_3$ represent this minimum. Then

$$I_0 \geq a_3^2 (C_1 T + C_2 X) \log X \geq a_3^2 C_1 T \log X.$$

We conclude with the lower bound $\frac{I_0}{T} \geq a_3^2 C_1 \log(kH_0)$, which is labeled $C_3$, and

$$I_0 = C_1 a_1^2 \left( T^{\frac{1}{6}} \log T + \frac{a_2}{a_1} \right)^2 \left( T + \frac{C_2}{C_1} X \right) (\log X),$$

which completes the proof. □

#### 3.4.1.2 Bounding $F_X(\sigma_2, T)$ at $\sigma_2 = 1 + \frac{\delta}{\log X}$

**Lemma 3.10.** *Let $T > 0$, $X \geq kH_0$ and $k, \delta, \sigma_2$ satisfy* (3.46)*. Then*

$$F_X(\sigma_2, T) \leq \left( C_5(k, \delta) + \frac{C_6(k, \delta)(T + \pi m_0)}{X} \right) (\log X)^2, \tag{3.54}$$

*where*

$$C_5(k, \delta) = \frac{\pi m_0 b_4}{2\delta} \Big( 1 + \frac{2\delta}{\log(kH_0)} \Big)^2 e^{\frac{2\delta\gamma}{\log(kH_0)}}, \tag{3.55}$$

$$C_6(k, \delta) = \frac{b_4}{5\delta e^{\delta}} \Big( 1 + \frac{\delta}{\log(kH_0)} \Big)^2 + \frac{b_3 e^{-2\delta}}{(\log(kH_0))^2}, \tag{3.56}$$

*the $b_i$'s are defined in* (3.29)*, $m_0$ in* (3.39)*, and $\gamma$ is Euler's constant.*

*Proof.* Recall that $F_X$ is defined by (3.17) and by (3.13) we have

$$F_X(\sigma_2, T) = \int_0^T |f_X(\sigma_2 + it)|^2 dt = \int_0^T \Big| \sum_{n \geq 1} \frac{\lambda_X(n)}{n^{\sigma_2 + it}} \Big|^2 dt.$$

Inequality (3.38) implies the bound

$$F_X(\sigma_2, T) \leq \pi m_0 \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2 - 1}} + (T + \pi m_0) \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2\sigma_2}}.$$

For $2\sigma_2 - 1 = 1 + \frac{2\delta}{\log X}$ and $2\sigma_2 = 2 + \frac{2\delta}{\log X}$, we apply the bounds for arithmetic sums (3.34) and (3.35) to respectively bound the two above sums. Thus

$$\sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{1 + 2\frac{\delta}{\log X}}} \leq \frac{b_4}{2\delta} \Big( 1 + \frac{2\delta}{\log X} \Big)^2 e^{\frac{2\delta\gamma}{\log X}} (\log X)^2,$$

$$\text{and} \quad \sum_{n \geq 1} \frac{\lambda_X(n)^2}{n^{2 + \frac{2\delta}{\log X}}} \leq \frac{b_4}{5\delta e^{\delta}} \Big( 1 + \frac{\delta}{\log X} \Big)^2 \frac{(\log X)^2}{X} + \frac{b_3 e^{-2\delta}}{X}.$$

We combine these results and use the fact that $X \geq kH_0$ to complete the proof. □

From here we may derive a bound for $\mathcal{M}_{g,T}(X, \sigma)$.

### 3.4.2 Explicit upper bounds for the mollifier $\mathcal{M}_{g,T}(X,\sigma)$

The results in this section are proven for a general weight $g$ satisfying the conditions described in Section 3.2.3. In [44, Theorem 7], Hardy et al. proved the following convexity estimate:

**Lemma 3.11.** *Let $\frac{1}{2} \le \sigma_1 < 1 < \sigma_2$, let $T > 0$, and $X > 1$. Then*

$$\mathcal{M}_{g,T}(X,\sigma) \le \mathcal{M}_{g,T}(X,\sigma_1)^{\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}} \mathcal{M}_{g,T}(X,\sigma_2)^{\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}}. \tag{3.57}$$

In order to obtain a bound for the mollifier $\mathcal{M}_{g,T}(X,\sigma)$ inside the strip $\frac{1}{2} \le \sigma \le 1 + \frac{\delta}{\log X}$, we need explicit bounds at the extremities $\frac{1}{2}$ and $1 + \frac{\delta}{\log X}$.

**Lemma 3.12.** *Let $T > 0$, $X > 0, \sigma \ge \frac{1}{2}$, and let $g$ satisfy conditions* (3.19) *and* (3.21). *Then*

$$\mathcal{M}_{g,T}(X,\sigma) \le 4\omega_1(\sigma,T,\alpha)^2 \alpha\beta \int_0^\infty x^{\beta-1} e^{-2\alpha x^\beta} F_X(\sigma, xT) dx. \tag{3.58}$$

*Proof.* By (3.21) and $|g(\sigma+it)| = |g(\sigma-it)|$ for $t \in \mathbb{R}$ and by an application of (3.19) to the weight $g$ in the definition (3.18) of $\mathcal{M}_{g,T}(X,\sigma)$, we have

$$\mathcal{M}_{g,T}(X,\sigma) \le 2\omega_1(\sigma,T,\alpha)^2 \int_0^\infty e^{-2\alpha\left(\frac{t}{T}\right)^\beta} |f_X(\sigma+it)|^2 dt. \tag{3.59}$$

Note that $\int_0^U |f_X(\sigma+it)|^2 dt = F_X(\sigma,U)$ with $F_X(\sigma,0) = 0$ and $\lim\limits_{U\to\infty} \left(F_X(\sigma,U) e^{-2\alpha\left(\frac{U}{T}\right)^\beta}\right) = 0$. Integrating by parts gives

$$\begin{aligned} \int_0^\infty e^{-2\alpha\left(\frac{t}{T}\right)^\beta} |f_X(\sigma+it)|^2 dt &= 2\alpha\beta \int_0^\infty \left(\frac{t}{T}\right)^\beta e^{-2\alpha\left(\frac{t}{T}\right)^\beta} F_X(\sigma,t) \frac{dt}{t} \\ &= 2\alpha\beta \int_0^\infty x^\beta e^{-2\alpha x^\beta} F_X(\sigma,xT) \frac{dx}{x}, \end{aligned}$$

by the variable change $x = \frac{t}{T}$. This combined with (3.59) yields the announced (3.58). □

#### 3.4.2.1 Bounding $\mathcal{M}_{g,T}(X, \frac{1}{2})$

Let $\alpha, \beta, A > 0$ and let $n$ be a non-negative integer. We define

$$I(A, n) = \int_0^{\infty} x^A e^{-2\alpha x^{\beta}} (\log x)^n dx. \tag{3.60}$$

In our context, $I(A, n)$ is a constant depending on parameters $A$ and $n$ and is $\mathcal{O}(1)$ in comparison with $T$. The change of variable $y = 2\alpha x^{\beta}$ leads to the identity

$$I(A, n) = (2\alpha)^{-\frac{A+1}{\beta}} \beta^{-(n+1)} \sum_{j=0}^{n} \binom{n}{j} (-\log(2\alpha))^j \Gamma^{(n-j)} \left( \frac{A+1}{\beta} \right), \tag{3.61}$$

where $\Gamma^{(j)}(z)$ denotes the $j$-th derivative of Euler's gamma function. We also define

$$\begin{aligned} \mathcal{J}(k, T) =& I(\beta + \tfrac{1}{3}, 0) + \frac{C_2}{C_1} k I(\beta - \tfrac{2}{3}, 0) + \frac{2I(\beta + \frac{1}{3}, 1) + 2\frac{C_2}{C_1} k I(\beta - \frac{2}{3}, 1)}{(\log T)} \\ &+ \frac{I(\beta + \frac{1}{3}, 2) + \frac{C_2}{C_1} k I(\beta - \frac{2}{3}, 2)}{(\log T)^2} + \frac{2a_2 \left( I(\beta + \frac{1}{6}, 0) + \frac{C_2 k}{C_1} I(\beta - \frac{5}{6}, 0) \right)}{a_1 T^{\frac{1}{6}} (\log T)} \\ &+ \frac{2a_2 \left( I(\beta + \frac{1}{6}, 1) + \frac{C_2 k}{C_1} I(\beta - \frac{5}{6}, 1) \right)}{a_1 T^{\frac{1}{6}} (\log T)^2} + \frac{a_2^2 \left( I(\beta, 0) + \frac{C_2 k}{C_1} I(\beta - 1, 0) \right)}{a_1^2 T^{\frac{1}{3}} (\log T)^2}, \end{aligned} \tag{3.62}$$

$$\mathcal{U}(\alpha, k, T) = 4\alpha\beta C_4 \omega_1(\tfrac{1}{2}, T, \alpha)^2 \mathcal{J}(k, T), \tag{3.63}$$

where $\omega_1$ and $C_4$ are respectively defined in (3.42) and (3.53). We remark that in the case of our weight $g$, we have $\beta = 2$. Thus in our calculations of $\mathcal{J}(k, T)$ we specialize to $\beta = 2$.

**Lemma 3.13.** *Let $\alpha, \beta > 0$ and $g$ be a function satisfying* (3.19) *and* (3.21)*. Let $T \geq H_0$, $X = kT$, and $k$ satisfies* (3.46)*. Then*

$$\mathcal{M}_{g,T}(X, \tfrac{1}{2}) \leq \mathcal{U}(\alpha, k, T)(\log(kT))(\log T)^2 T^{\frac{4}{3}}.$$

*Proof.* We combine the bound (3.58) for $\mathcal{M}_{g,T}$ with the bound (3.50) for $F_X(\frac{1}{2}, xT)$:

$$\mathcal{M}_{g,T}(X,\tfrac{1}{2}) \le 4\alpha\beta C_4\omega_1(\tfrac{1}{2},T,\alpha)^2(\log X)\Big\{T^{\frac{4}{3}}\int_0^\infty x^{\beta+\frac{1}{3}}(\log(xT))^2e^{-2\alpha x^\beta}dx + \frac{2a_2}{a_1}T^{\frac{7}{6}}\int_0^\infty x^{\beta+\frac{1}{6}}\log(xT)e^{-2\alpha x^\beta}dx + \frac{a_2^2}{a_1^2}T\int_0^\infty x^\beta e^{-2\alpha x^\beta}dx + \frac{C_2}{C_1}XT^{\frac{1}{3}}\int_0^\infty x^{\beta-\frac{2}{3}}(\log(xT))^2e^{-2\alpha x^\beta}dx + \frac{2a_2}{a_1}\frac{C_2}{C_1}XT^{\frac{1}{6}}\int_0^\infty x^{\beta-\frac{5}{6}}\log(xT)e^{-2\alpha x^\beta}dx + \frac{a_2^2}{a_1^2}\frac{C_2}{C_1}X\int_0^\infty x^{\beta-1}e^{-2\alpha x^\beta}dx\Big\}.$$

We also use the fact that $(\log(xT))^2 = (\log x)^2 + 2(\log x)(\log T) + (\log T)^2$ and obtain

$$\mathcal{M}_{g,T}(X,\tfrac{1}{2}) \le 4\alpha\beta C_4\omega_1(\tfrac{1}{2},T,\alpha)^2(\log X)\Big\{T^{\frac{4}{3}}\big(I(\beta+\tfrac{1}{3},2) + 2(\log T)I(\beta+\tfrac{1}{3},1) + (\log T)^2I(\beta+\tfrac{1}{3},0)\big) + \frac{2a_2}{a_1}T^{\frac{7}{6}}\big(I(\beta+\tfrac{1}{6},1) + (\log T)I(\beta+\tfrac{1}{6},0)\big) + \frac{a_2^2}{a_1^2}TI(\beta,0) + \frac{C_2}{C_1}XT^{\frac{1}{3}}\big(I(\beta-\tfrac{2}{3},2) + 2(\log T)I(\beta-\tfrac{2}{3},1) + (\log T)^2I(\beta-\tfrac{2}{3},0)\big) + \frac{2a_2}{a_1}\frac{C_2}{C_1}XT^{\frac{1}{6}}\big(I(\beta-\tfrac{5}{6},1) + (\log T)I(\beta-\tfrac{5}{6},0)\big) + \frac{a_2^2}{a_1^2}\frac{C_2}{C_1}XI(\beta-1,0)\Big\},$$

where $I$ is the integral defined in (3.60). At this point we choose $X = kT$ so as to optimize the above bound, and we factor out the main term $T^{\frac{4}{3}}(\log T)^2$:

$$\mathcal{M}_{g,T}(X,\tfrac{1}{2}) \le 4\alpha\beta C_4\omega_1(\tfrac{1}{2},T,\alpha)^2(\log(kT))(\log T)^2T^{\frac{4}{3}}\Big\{I(\beta+\tfrac{1}{3},0) + \frac{kC_2}{C_1}I(\beta-\tfrac{2}{3},0) + 2\frac{I(\beta+\frac{1}{3},1) + \frac{kC_2}{C_1}I(\beta-\frac{2}{3},1)}{(\log T)} + \frac{I(\beta+\frac{1}{3},2) + \frac{kC_2}{C_1}I(\beta-\frac{2}{3},2)}{(\log T)^2} + \frac{2a_2}{a_1}\frac{I(\beta+\frac{1}{6},0) + \frac{kC_2}{C_1}I(\beta-\frac{5}{6},0)}{(\log T)T^{\frac{1}{6}}} + \frac{2a_2}{a_1}\frac{I(\beta+\frac{1}{6},1) + \frac{kC_2}{C_1}I(\beta-\frac{5}{6},1)}{(\log T)^2T^{\frac{1}{6}}} + \frac{a_2^2}{a_1^2}\frac{I(\beta,0) + \frac{kC_2}{C_1}I(\beta-1,0)}{(\log T)^2T^{\frac{1}{3}}}\Big\}.$$

We recognize in the above term between brackets $\mathcal{J}(k,T)$ as introduced in (3.62). □

#### 3.4.2.2 Bounding $\mathcal{M}_{g,T}(X,\sigma_2)$ at $\sigma_2 = 1 + \frac{\delta}{\log X}$

**Lemma 3.14.** *Let $g$ be as defined in Lemma 3.7. Let $T \ge H_0$, $X = kT$, and $k, \delta, \sigma_2$ satisfy* (3.46)*. Then*

$$\mathcal{M}_{g,T}(X,\sigma_2) \le \mathcal{V}(\alpha,k,\delta,T)(\log(kT))^2,$$

*where*

$$\mathcal{V}(\alpha,k,\delta,T) = 8\alpha\omega_1(\sigma_2,T,\alpha)^2\mathcal{K}(k,\delta,T), \tag{3.64}$$

$$\mathcal{K}(k,\delta,T) = \left(C_5(k,\delta) + \frac{C_6(k,\delta)\pi m_0}{kT}\right)I(1,0) + \frac{C_6(k,\delta)}{k}I(2,0), \tag{3.65}$$

*and* $m_0, \omega_1, C_5, C_6,$ *and* $I$ *are respectively defined in* (3.39)*,* (3.42)*,* (3.55)*,* (3.56)*, and* (3.60)*.*

*Proof.* We combine the bound (3.58) for $\mathcal{M}_{g,T}$ with the bound (3.54) for $F_X(\sigma_2, xT)$ (since $X \geq kH_0$) to obtain

$$\mathcal{M}_{g,T}(X,\sigma_2) \leq 4\alpha\beta\omega_1(\sigma_2,T,\alpha)^2\left(\int_0^\infty x^{\beta-1}e^{-2\alpha x^\beta}\Big(C_5(k,\delta) + \frac{C_6(k,\delta)(xT+\pi m_0)}{X}\Big)(\log X)^2 dx\right). \tag{3.66}$$

Rearranging this and recalling the definition for $I$ in (3.60) we obtain

$$\mathcal{M}_{g,T}(X,\sigma_2) \leq 4\alpha\beta\omega_1(\sigma_2,T,\alpha)^2(\log X)^2\left(\left(C_5(k,\delta) + \frac{C_6(k,\delta)\pi m_0}{X}\right)I(\beta-1,0) + \frac{C_6(k,\delta)T}{X}I(\beta,0)\right).$$

We conclude by noting that $X = kT$ and for our $g$, $\beta = 2$. □

#### 3.4.2.3 Conclusion

Finally, we provide bounds for $\mathcal{M}_{g,T}$.

**Lemma 3.15.** *Let $g$ be as defined in Lemma 3.7. Let $T \geq H_0$, $X = kT$, and $k$ satisfies* (3.46)*. Assume* $\frac{1}{2} \leq \sigma \leq 1 + \frac{\delta}{\log X}$*. Then*

$$\mathcal{M}_{g,T}(X,\sigma) \leq e^{\frac{8}{3}\delta(2\sigma-1)M(k,\sigma) + \frac{4\delta(2\sigma-1)\log\log H_0}{\log(kH_0)+2\delta}}\mathcal{U}(\alpha,k,T)^{2(1-\sigma)+\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}} \times \mathcal{V}(\alpha,k,\delta,T)^{2\sigma-1-\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)}, \tag{3.67}$$

*where* $\mathcal{U}$ *and* $\mathcal{V}$ *are respectively defined in* (3.63) *and* (3.64) *and*

$$M(k,\delta) = \max\Big(\frac{\log H_0}{\log(kH_0)+2\delta}, 1\Big). \tag{3.68}$$

*Proof.* Let $\sigma_1 = \frac{1}{2}$ andf $\sigma_2 = 1 + \frac{\delta}{\log X}$ and $\sigma \in [\sigma_1, \sigma_2]$. We apply the convexity inequality (3.57) with exponents

$$a = \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} = \frac{1 + \frac{\delta}{\log X} - \sigma}{(1 + \frac{\delta}{\log X}) - \frac{1}{2}} \text{ and } b = 1 - a = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} = \frac{\sigma - \frac{1}{2}}{(1 + \frac{\delta}{\log X}) - \frac{1}{2}} \tag{3.69}$$

in combination with Lemmas 3.13, Lemma 3.14 to obtain

$$\mathcal{M}_{g,T}(X, \sigma) \le \mathcal{U}(\alpha, k, T)^a \mathcal{V}(\alpha, k, \delta, T)^b (\log(kT))^{a+2b} (\log T)^{2a} T^{\frac{4}{3}a}. \tag{3.70}$$

Next, from the definitions of (3.69) it may be checked that

$$a = 2(1-\sigma) + \frac{2\delta(2\sigma - 1)}{\log X + 2\delta}, \text{ and } b = 2\sigma - 1 - \frac{2\delta(2\sigma - 1)}{\log X + 2\delta}. \tag{3.71}$$

From these equalities it follows that $a + 2b \le 2\sigma$. Using (3.71) and the bound for $a + 2b$ (since $\log(kT) \ge \log(kH_0) \ge \log(10^9) > 1$), we have

$$\begin{aligned} \mathcal{M}_{g,T}(X, \sigma) \le e^{\frac{4}{3} \times \frac{2\delta(2\sigma-1)\log T}{\log(kT)+2\delta} + 2 \times \frac{2\delta(2\sigma-1)\log\log T}{\log(kT)+2\delta}} \mathcal{U}(\alpha, k, T)^{2(1-\sigma) + \frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}} \times \\ \mathcal{V}(\alpha, k, \delta, T)^{2\sigma - 1 - \frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}} (\log(kT))^{2\sigma} (\log T)^{4(1-\sigma)} T^{\frac{8}{3}(1-\sigma)}. \end{aligned} \tag{3.72}$$

Next we observe that the function $\frac{\log T}{\log(kT)+2\delta}$ decreases if $\log k + 2\delta < 0$ and increases if $\log k + 2\delta > 0$ and thus

$$\frac{\log T}{\log(kT) + 2\delta} \le M(k, \delta) := \begin{cases} \frac{\log H_0}{\log(kH_0)+2\delta} & \text{if } \log k + 2\delta < 0, \\ 1 & \text{if } \log k + 2\delta \ge 0 \end{cases} \tag{3.73}$$

where $M(k, \delta)$ was defined in (3.68). Furthermore, it may be checked by the conditions on $k$, that $\frac{\log\log T}{\log(kT)+2\delta}$ decreases as long as $0 < \delta < \frac{\log(H_0)(\log\log H_0 - 1)}{2}$. Using these observations in (3.72) we deduce (3.67). □

### 3.4.3 Bounding $F_X(\sigma,T) - F_X(\sigma,H)$

**Lemma 3.16.** *Let $g$ be as defined in Lemma 3.7. Let $\sigma \in [\frac{1}{2},1]$ and $\alpha > 0$. Let $T \geq H_0 \geq H > 0$, $X = kT$, $k$ satisfies (3.46), and $0 < \delta < \frac{\log(H_0)(\log\log H_0 - 1)}{2} = 26.36\ldots$. Then*

$$F_X(\sigma,T) - F_X(\sigma,H) \leq \frac{e^{\frac{8}{3}\delta(2\sigma-1)M(k,\delta)+\frac{4\delta(2\sigma-1)\log\log H_0}{\log(kH_0)+2\delta}}\mathcal{U}(\alpha,k,T)^{2(1-\sigma)+\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}\mathcal{V}(\alpha,k,\delta,T)^{2\sigma-1-\frac{2\delta(2\sigma-1)}{\log(kT)+2\delta}}}{2(\omega_2(\sigma,T,\alpha))^2} \times (\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)}, \tag{3.74}$$

*where $\omega_2, \mathcal{U}, \mathcal{V}$ are respectively defined in (3.43), (3.63), (3.64).*

*Proof.* By the assumed lower bound on $g$, (3.20), we have

$$F_X(\sigma,T) - F_X(\sigma,H) = \int_H^T |f_X(\sigma+it)|^2 dt \leq \frac{1}{(\omega_2(\sigma,T,\alpha))^2}\int_H^T |g(\sigma+it)|^2|f_X(\sigma+it)|^2 dt.$$

Since $t \to |g(\sigma+it)f_X(\sigma+it)|$ is even, it follows that

$$F_X(\sigma,T) - F_X(\sigma,H) \leq \frac{\mathcal{M}_{g,T}(X,\sigma)}{2(\omega_2(\sigma,T,\alpha))^2}$$

and we conclude by inserting the bound (3.67) for $\mathcal{M}_{g,T}(X,\sigma)$. □

### 3.4.4 Explicit upper bounds for $\int_{\sigma'}^{\mu} \arg h_X(\tau+iT)d\tau - \int_{\sigma'}^{\mu} \arg h_X(\tau+iH)d\tau$

The following Proposition and Corollary are a variant of Titchmarsh [102, Lemma, p. 213]. This proposition gives a bounds for $\arg f(\sigma+iT)$ where $f$ is a holomorphic function. The argument we use here is due to Backlund [7] in the case that $f(s) = \zeta(s)$. The cases of Dirichlet $L$-functions and Dedekind zeta functions have been worked out by McCurley [77] and by the first and third authors [60] respectively.

**Proposition 3.1.** *Let $\eta > 0$. Let $f(s)$ be a holomorphic function, for $\mathfrak{Re}(s) \geq -\eta$, real for real $s$. Assume* *there exist positive constants $M$ and $m$ such that*

$$|f(s)| \leq M \text{ for } \mathfrak{Re}(s) \geq 1+\eta, \tag{3.75}$$

$$|\mathfrak{Re} f(1+\eta+it)| \geq m > 0 \text{ for all } t \in \mathbb{R}. \tag{3.76}$$

*Let $\sigma \in (0, 1+\eta]$ and assume that $U$ is not the ordinate of a zero of $f(s)$. Then there exists an increasing sequence of natural numbers $\{N_k\}_{k=1}^{\infty}$ such that*

$$|\arg f(\sigma+iU)| \leq \frac{\pi}{\log 2}\mathscr{L}_k + \frac{\pi \log M}{2\log 2} - \frac{\pi \log m}{\log 2} + \frac{\pi}{2} + o_k(1) \tag{3.77}$$

*where*

$$\mathscr{L}_k = \frac{1}{2\pi N_k}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\left(\frac{1}{2}\sum_{j=0}^{1} |f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^{N_k}\right) d\theta \tag{3.78}$$

*and $o_k(1)$ is a term that approaches 0 as $k \to \infty$.*

*Proof of Proposition 3.1.* Let $\eta > 0$. We define $\arg f(1+\eta) = 0$, and $\arg f(s) = \arctan \frac{\mathfrak{Im} f(s)}{\mathfrak{Re} f(s)}$ for $\mathfrak{Re}(s) = 1+\eta$, since, by (3.76), $\mathfrak{Re}(f(s))$ does not vanish on $\mathfrak{Re}(s) = 1+\eta$. It follows that

$$|\arg f(1+\eta+iU)| < \frac{\pi}{2}. \tag{3.79}$$

Recall that $\arg f(\sigma+iU)$ is defined by continuous variation, moving along the line $\mathcal{C}$ from $1+\eta+iU$ to $\sigma+iU$. It follows that

$$|\arg f(\sigma+iU)| \leq |\Delta_{\mathcal{C}} \arg f(s)| + \frac{\pi}{2}. \tag{3.80}$$

We now bound the argument change on $\mathcal{C}$. Let $N \in \mathbb{N}$ and let

$$F_N(w) = \frac{1}{2}(f(w+iU)^N + f(w-iU)^N). \tag{3.81}$$

Since $f(s)$ is real when $s$ is real, the reflection principle gives $F_N(\sigma) = \mathfrak{Re}\, f(\sigma+iU)^N$ for all $\sigma$ real. Suppose $F_N(\sigma)$ has $n$ real zeros in the interval $[\sigma, 1+\eta]$. These zeros partition the interval into $n+1$ subintervals. On each of these subintervals $\arg f(\sigma+iU)^N$ can change by at most $\pi$, since $\mathfrak{Re}\, f(\sigma+iU)^N$ is nonzero on the interior of each subinterval. It follows that

$$|\Delta_{\mathcal{C}} \arg f(s)| = \frac{1}{N}|\Delta_{\mathcal{C}} \arg f(s)^N| \le \frac{(n+1)\pi}{N}. \tag{3.82}$$

We now provide an upper bound for $n$. Jensen's theorem asserts that

$$\log|F_N(1+\eta)| + \int_0^{1+2\eta} \frac{n(u)du}{u} = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta,$$

where $n(u)$ denotes the number of zeros of $F_N(z)$ in the circle centered at $1+\eta$ of radius $u$. Observe that $n(u) \ge n$ for $u \ge \frac{1}{2}+\eta$ and thus

$$n\log 2 \le \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta - \log|F_N(1+\eta)|. \tag{3.83}$$

Trivially from (3.81),

$$|F_N(1+\eta+(1+2\eta)e^{i\theta})| \le \frac{1}{2}\sum_{j=0}^{1}|f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^N,$$

so for the left part of the contour in (3.83),

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta \le \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\left(\frac{1}{2}\sum_{j=0}^{1}|f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^N\right)d\theta. \tag{3.84}$$

For the right part of the contour in (3.83), we have $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $\mathfrak{Re}(1+\eta+(1+2\eta)e^{i\theta}) \ge 1+\eta$. We apply (3.75) and obtain

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log|F_N(1+\eta+(1+2\eta)e^{i\theta}|d\theta \le \frac{N}{2}\log M. \tag{3.85}$$

To complete our bound for $n$, we require a lower bound for $\log|F_{N_k}(1+\eta)|$.
We write $f(1+\eta+iU) = re^{i\phi}$ and then choose (by Dirichlet's approximation theorem) an increasing sequence of positive integers $N_k$ tending to infinity such that $N_k\phi$ tends to 0 modulo $2\pi$. Since

$$\frac{F_{N_k}(1+\eta)}{|f(1+\eta+iU)|^{N_k}} = \frac{r^{N_k}\cos(N_k\phi)}{r^{N_k}},$$

it follows that $\lim_{k\to\infty} \frac{F_{N_k}(1+\eta)}{|f(1+\eta+iU)|^{N_k}} = 1$. Thus we derive

$$\log|F_{N_k}(1+\eta)| \geq N_k \log|f(1+\eta+iU)| + o_k(1),$$

where the term $o_k(1) \to 0$ as $k \to \infty$. Together with (3.76), we obtain

$$\log|F_{N_k}(1+\eta)| \geq N_k \log m + o_k(1). \tag{3.86}$$

Then (3.83), (3.84), (3.85), and (3.86) give

$$n\log 2 \leq \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log\Big(\frac{1}{2}\sum_{j=0}^{1} |f(1+\eta+(1+2\eta)e^{i\theta}+(-1)^j iU)|^{N_k}\Big)\,d\theta$$
$$+\frac{N_k \log M}{2} - N_k\log m + o_k(1). \tag{3.87}$$

By (3.82) it follows that

$$|\delta_{\mathcal{C}} \arg f(s)| \leq \frac{\pi}{\log 2}\mathscr{L}_k + \frac{\pi\log M}{2\log 2} - \frac{\pi\log m}{\log 2} + o_k(1),$$

where $\mathscr{L}_k$ is defined by (3.78) . We conclude by combining this with (3.80). □

We derive the following Corollary for $\arg h_X(s)$ from Proposition 3.1.

**Corollary 3.1.** *Let $\eta_0 = 0.23622\ldots$, $\eta \in [\eta_0, \frac{1}{2})$, and $X \geq 10^9$. Assume that $U \geq H \geq 1002$ and that $U$ is not the ordinate of a zero of $h_X(s)$. Then for all $\tau \in (0, 1+\eta]$,*

$$|\arg h_X(\tau+iU)| \leq \frac{(1+2\eta)}{\log 2}\log\Big(\frac{b_8(\eta,H)}{2\pi}U\Big) + \frac{\pi(1+\eta)}{\log 2}(\log X) + \frac{\pi\log b_7(k,\eta,H_0)}{2\log 2} + \frac{\pi\log b_5(\eta)}{2\log 2}$$
$$-\frac{\pi\log(1-b_6(10^9,\eta)^2)}{\log 2} + \frac{\pi}{2},$$

*where $b_5, b_6, b_7, b_8$ are defined in* (3.89), (3.90), (3.95), *and* (3.96).

*Proof of Corollary 3.1.* We apply Proposition 3.1 to $f = h_X$ as defined in (3.15):

$$h_X(s) = 1 - f_X(s)^2 = \zeta(s)M_X(s)(2-\zeta(s)M_X(s)).$$

Let $\sigma \geq \eta+1$ and $t \in \mathbb{R}$. We establish an upper bound for $|h_X(\sigma+it)|$. The triangle inequality in conjunction with $|\zeta(s)| \leq \zeta(1+\eta)$ and with $|M_X(s)| \leq \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{1+\eta}} = \frac{\zeta(1+\eta)}{\zeta(2+2\eta)}$ give

$$|h_X(s)| \leq b_5(\eta) \tag{3.88}$$

with

$$b_5(\eta) = \frac{\zeta(1+\eta)^4}{\zeta(2+2\eta)^2} + \frac{2\zeta(1+\eta)^2}{\zeta(2+2\eta)}. \tag{3.89}$$

We now give a lower bound for $|\mathfrak{Re} h_X(1+\eta+it)|$. We use the reverse triangle inequality $|h_X(s)| \geq 1-|f_X(s)|^2$. It remains to provide an upper bound for $|f_X(s)|$. Trivially from (3.12),

$$|f_X(s)| \leq \sum_{n>X} \frac{|\lambda_X(n)|}{n^{1+\eta}} \leq \sum_{n>X} \frac{d(n)}{n^{1+\eta}},$$

and by Lemma 3.5, we obtain

$$|f_X(1+\eta+it)| \leq b_6(X,\eta) = \frac{(1+\eta)(\log X)}{\eta X^{\eta}}\Big(1 + \frac{1}{\eta \log X} + \frac{\gamma}{\log X} + \frac{7\eta}{12(1+\eta)X(\log X)}\Big). \tag{3.90}$$

Note that $\frac{(\log X)}{X^{\eta}}$ decreases when $\eta > \frac{1}{\log X}$, which is the case since we assumed $\eta > \frac{1}{\log(10^9)} = 0.048254\ldots$ and $X \geq 10^9$. Thus $|f_X(s)| \leq b_6(10^9,\eta)$ and

$$|\mathfrak{Re}(h_X(s))| = |1-\mathfrak{Re}(f_X(s))^2| \geq |1-|f_X(s)|^2| \geq 1-|f_X(s)|^2 \geq 1-b_6(10^9,\eta)^2. \tag{3.91}$$

Note our assumption $\eta \geq \eta_0 = 0.23622\ldots$ ensures $1-b_6(10^9,\eta)^2 > 0$, which allows us to use it as our lower bound $m$ in Proposition 3.1.

Finally, we must bound $\mathscr{L}_k$ as defined in (3.78) in the case $f = h_X$. We assume $w$ is a complex number such that $-\eta \leq \mathfrak{Re} w \leq 1+\eta$ and $|\mathfrak{Im} w| \geq U-(1+2\eta)$. Recall that by Lemma 3.1

$$|\zeta(w)| \leq 3\frac{|1+w|}{|1-w|}\Big(\frac{|w+1|}{2\pi}\Big)^{\frac{1+\eta-\mathfrak{Re} w}{2}}\zeta(1+\eta).$$

Since $\frac{|1+w|}{|1-w|} = \left|1+\frac{2}{w-1}\right| \leq 1+\frac{2}{|\mathfrak{Im}(w)|} \leq 1.002$ when $|\mathfrak{Im}(w)| \geq 1000$, then

$$|\zeta(w)| \leq 3.006\zeta(1+\eta)\Big(\frac{|w+1|}{2\pi}\Big)^{\frac{1+\eta-u}{2}} \text{ for } |\mathfrak{Im}(w)| \geq 1000. \tag{3.92}$$

From the definition (3.11), we have the trivial bound

$$|M_X(w)| \le X^{1+\eta}. \tag{3.93}$$

It follows from

$$|h_X(w)| \le |\zeta(w)M_X(w)|^2 + 2|\zeta(w)||M_X(w)|,$$

the bounds (3.92), (3.93), $\frac{|w+1|}{2\pi} > 1, -\frac{1+\eta-\mathfrak{Re}w}{2} < 0$, and $X \ge kH_0$, that

$$|h_X(w)| \le b_7(k,\eta,H_0)\Big(\frac{|w+1|}{2\pi}\Big)^{1+\eta-u} X^{2(1+\eta)} \text{ for } |\mathfrak{Im}(w)| \ge 1000, \tag{3.94}$$

with

$$b_7(k,\eta,H_0) = \left(1 + \frac{2}{3.006\zeta(1+\eta)(kH_0)^{1+\eta}}\right)(3.006\zeta(1+\eta))^2. \tag{3.95}$$

We apply this with $w = 1+\eta+(1+2\eta)e^{i\theta} \pm iU$. Since $\cos\theta \le 0$, a little calculation gives

$$|w+1| = |2+\eta+(1+2\eta)e^{i\theta} \pm iU| \le \sqrt{(2+\eta)^2 + (1+2\eta+U)^2} \le b_8(\eta,H)U,$$

with

$$b_8(\eta,H) = \sqrt{\frac{(2+\eta)^2}{H^2} + \Big(\frac{1+2\eta}{H}+1\Big)^2}. \tag{3.96}$$

In addition $1+\eta-u = 1+\eta-(1+\eta+(1+2\eta)\cos\theta) = -(1+2\eta)(\cos\theta)$, and (3.94) gives

$$|h_X(1+\eta+(1+2\eta)e^{i\theta} \pm iU)| \le b_7(k,\eta,H_0)\Big(\frac{b_8(\eta,H)}{2\pi}U\Big)^{-(1+2\eta)(\cos\theta)} X^{2(1+\eta)}, \tag{3.97}$$

since $|\mathfrak{Im}(1+\eta+(1+2\eta)e^{i\theta} \pm iU)| \ge U-(1-2\eta) \ge H-2 \ge 1000$. We use this to bound $\mathscr{L}_k$ as defined in (3.78):

$$\mathscr{L}_k \le \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\log b_7(k,\eta,H_0) - (1+2\eta)(\cos\theta)\log\Big(\frac{b_8(\eta,H)}{2\pi}U\Big) + 2(1+\eta)(\log X)\right) d\theta.$$

Calculating the integrals give

$$\mathscr{L}_k \le \frac{\log b_7(k,\eta,H_0)}{2} + \frac{(1+2\eta)}{\pi}\log\Big(\frac{b_8(\eta,H)}{2\pi}U\Big) + (1+\eta)(\log X). \tag{3.98}$$

By (3.88) and (3.91) we may take $M = b_5(\eta)$ and $m = 1 - b_6(10^9, \eta)^2$ in (3.75) and (3.76) in the case of $f(s) = h_X(s)$. Therefore by Proposition 3.1

$$|\arg h_X(\sigma + iU)| \le \frac{\pi}{\log 2}\mathscr{L}_k + \frac{\pi \log b_5(\eta)}{2\log 2} - \frac{\pi \log(1 - b_6(10^9, \eta)^2)}{\log 2} + \frac{\pi}{2} + o_k(1). \tag{3.99}$$

Inserting the upper bound for $\mathscr{L}_k$ from (3.98) and letting $k \to \infty$ we complete the proof as the $o_k(1)$ terms goes to zero. □

We are now in a position to bound the arguments.

**Lemma 3.17.** *Let $0 < H \le H_0 \le T$ and $X \le T$. Let $\eta \in (\eta_0, \frac{1}{2})$ with $\eta_0 = 0.23622\ldots$, $\sigma'$ and $\mu$ satisfying $\frac{1}{2} \le \sigma' < 1 < \mu \le 1 + \eta$. Then*

$$\Big|\int_{\sigma'}^{\mu} \arg h_X(\tau + iT)d\tau - \int_{\sigma'}^{\mu} \arg h_X(\tau + iH)d\tau\Big| \le C_7(\eta, H)\,(\mu - \sigma')(\log T), \tag{3.100}$$

*where*

$$C_7(\eta, H) = \frac{2(1+2\eta) + 2\pi(1+\eta)}{\log 2} + \frac{b_9(\eta, H)}{\log H_0}. \tag{3.101}$$

*with $b_9(\eta, H)$ defined in (3.104).*

*Proof.* Note that

$$\Big|\int_{\sigma'}^{\mu} \arg h_X(\tau + iT)d\tau - \int_{\sigma'}^{\mu} \arg h_X(\tau + iH)d\tau\Big| \le (\mu - \sigma') \max_{\tau \in (\sigma', \mu)} \Big(|\arg h_X(\tau + iT)| + |\arg h_X(\tau + iH)|\Big). \tag{3.102}$$

By Corollary 3.1 we have

$$|\arg h_X(\tau + iH)| + |\arg h_X(\tau + iT)| \le b_9(\eta, H) + \frac{(1+2\eta)}{\log 2}(\log(HT)) + \frac{2\pi(1+\eta)}{\log 2}(\log X) \tag{3.103}$$

with

$$b_9(\eta, H) = \frac{\pi \log b_7(k, \eta, H_0)}{\log 2} + \frac{\pi \log b_5(\eta)}{\log 2} - \frac{2\pi \log(1 - b_6(10^9, \eta)^2)}{\log 2} + \pi + \frac{2(1+2\eta)}{\log 2}\log\Big(\frac{b_8(\eta, H)}{2\pi}\Big) \tag{3.104}$$

where $b_7, b_5, b_6, b_8$ are defined in (3.95), (3.89), (3.90), (3.96). Factoring $\log T$ in the right hand side of

(3.103), using $H \le T$, $X \le T$, and $H_0 \le T$ yields

$$|\arg h_X(\tau+iH)| + |\arg h_X(\tau+iT)| \le (\log T)\left(\frac{2(1+2\eta)+2\pi(1+\eta)}{\log 2} + \frac{b_9(\eta,H)}{\log H_0}\right). \tag{3.105}$$

Combining (3.102) and (3.105) leads to (3.100). □

### 3.4.5 Explicit lower bounds for $\int_H^T \log|h_X(\mu+it)|dt$

First, observe that (3.90) implies for

$$\mu \ge 1+\eta_0 = 1.23622\ldots,\ \ |f_X(\mu+it)| < 1. \tag{3.106}$$

This fact is used in the next lemma.

**Lemma 3.18.** *Assume $\mu \ge 1+\eta_0$ where $\eta_0 = 0.23622\ldots$. Let $X = kT$ where $T \ge H_0$, $k$ satisfies* (3.46), *$k \le 1$, and $2\pi m_0 \le H < T$. Then*

$$-\int_H^T \log|h_X(\mu+it)|dt \le C_8(k,\mu)(\log T). \tag{3.107}$$

*with*

$$C_8(k,\mu) = b_{10}(k,\mu)\frac{(\log(kH_0))^2}{(kH_0)^{2\mu-2}}\left(\frac{4\mu b_{11}(kH_0,2\mu)}{k(2\mu-1)} + \frac{2\pi m_0(2\mu-1)b_{11}(kH_0,2\mu-1)}{(\mu-1)}\right), \tag{3.108}$$

*$b_{10}$ is defined in* (3.111)*, $b_{11}$ in* (3.113)*, and $m_0$ in* (3.39).

*Proof.* We begin by remarking that (3.90) implies $|f_X(\mu+it)| \le b_6(kH_0,\mu-1) < 1$ since $X \ge kH_0 \ge 10^9$ and $\mu \ge 1+\eta_0$. Next, observe that $|h_X(\mu+it)| \ge |1-f_X(\mu+it)^2| \ge 1-|f_X(\mu+it)|^2$ and thus

$$-\log|h_X(\mu+it)| \le -\log(1-|f_X(\mu+it)|^2). \tag{3.109}$$

Since $-\frac{\log(1-u^2)}{u^2}$ increases with $u \in (0,1)$, we have

$$-\log(1-|f_X(\mu+it)|^2) \le b_{10}(k,\mu)|f_X(\mu+it)|^2, \tag{3.110}$$

with

$$b_{10}(k,\mu) = -\frac{\log\left(1-b_6(kH_0,\mu-1)^2\right)}{b_6(kH_0,\mu-1)^2} \tag{3.111}$$

where $b_6$ is defined in (3.90). It follows from (3.109) and (3.110) that

$$-\int_H^T \log|h_X(\mu+it)|dt \le b_{10}(k,\mu)\int_H^T |f_X(\mu+it)|^2 dt. \tag{3.112}$$

We apply Lemma 3.6 and the bound $|\lambda_X(n)| \le d(n)$ with $\lambda_X(n)=0$ if $n \le X$. We obtain

$$\begin{aligned}\int_H^T |f_X(\mu+it)|^2 dt &\le \sum_{n=1}^{\infty} \frac{|\lambda_X(n)|^2}{n^{2\mu}}(T-H+2\pi m_0(n+1)) \\ &\le (T-H+2\pi m_0)\sum_{n>X}\frac{d(n)^2}{n^{2\mu}} + 2\pi m_0 \sum_{n>X}\frac{d(n)^2}{n^{2\mu-1}}.\end{aligned}$$

We appeal to (3.37) to bound the above sums:

$$\sum_{n\ge X}\frac{d(n)^2}{n^{\tau}} \le \frac{(\log X)^3}{X^{\tau-1}}\frac{2\tau b_{11}(kH_0,\tau)}{(\tau-1)},$$

since $X \ge kH_0$ where

$$b_{11}(X,\tau) = 1+\frac{3}{(\tau-1)(\log X)}+\frac{6}{(\tau-1)^2(\log X)^2}+\frac{6}{(\tau-1)^3(\log X)^3}. \tag{3.113}$$

Since $X = kT$ we deduce that

$$\int_H^T |f_X(\mu+it)|^2 dt \le \frac{(\log(kT))^3}{(kT)^{2\mu-2}}\left(\frac{4\mu b_{11}(kH_0,2\mu)}{k(2\mu-1)}+\frac{2\pi m_0(2\mu-1)b_{11}(kH_0,2\mu-1)}{(\mu-1)}\right).$$

Note that $\frac{(\log(kT))^2}{(kT)^{2\mu-2}}$ decreases with $T$ as long as $10^9 > e^{\frac{1}{\mu-1}}$ (i.e. $\mu > \mu_2 = 1.072382\ldots$). Using this and $\log(kT) \le \log T$ (since $k \le 1$) implies

$$\int_H^T |f_X(\mu+it)|^2 dt \le \frac{(\log(kH_0))^2}{(kH_0)^{2\mu-2}}\left(\frac{4\mu b_{11}(kH_0,2\mu)}{k(2\mu-1)}+\frac{2\pi m_0(2\mu-1)b_{11}(kH_0,2\mu-1)}{(\mu-1)}\right)(\log T). \tag{3.114}$$

We conclude by combining this with (3.112). □

### 3.4.6 Proof of Zero Density Result

Finally, we are able to compile our bounds to obtain an upper bound for $N(\sigma, T)$.

**Lemma 3.19.** *Assume* $\alpha > 0, d > 0, \delta > 0, \eta_0 = 0.23622\ldots, \eta \in [\eta_0, \frac{1}{2})$, *and* $\mu \in [1+\eta_0, 1+\eta]$. *Let* $H_0 = 3.0610046 \cdot 10^{10}$, $1002 \leq H \leq H_0$, $\frac{10^9}{H_0} \leq k \leq 1$, $T \geq H_0$, *and* $X = kT$ *Assume* $\sigma > \frac{1}{2} + \frac{d}{\log H_0}$, $\mathcal{U}(\alpha, k, H_0) > 1$, *and* $\mathcal{U}(\alpha, k, T)$ *decreases in* $T$. *Thus*

$$N(\sigma, T) \leq \frac{(T-H)(\log T)}{2\pi d} \log\left(1 + \mathcal{C}_1 \frac{(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)} T^{\frac{8}{3}(1-\sigma)}}{T-H}\right) + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2, \tag{3.115}$$

$$N(\sigma, T) \leq \frac{\mathcal{C}_1}{2\pi d}(\log(kT))^{2\sigma}(\log T)^{5-4\sigma} T^{\frac{8}{3}(1-\sigma)} + \frac{\mathcal{C}_2}{2\pi d}(\log T)^2, \tag{3.116}$$

*with*

$$\mathcal{C}_1 = \mathcal{C}_1(\alpha, d, \delta, k, H, \sigma) = b_{12}(H) e^{\frac{8}{3}\delta(2\sigma-1)M(k,\delta) + \frac{4\delta(2\sigma-1)\log\log H_0}{\log(kH_0)+2\delta}} \mathcal{U}(\alpha, k, H_0)^{2(1-\sigma) + \frac{2d}{\log H_0} + \frac{2\delta(2\sigma-1)}{\log(kH_0)+2\delta}} \times \tag{3.117}$$

$$\mathcal{V}(\alpha, k, \delta, H_0)^{2\sigma-1} e^{\frac{2d(2\log\log H_0 - \log\log(kH_0))}{\log H_0} + \frac{8d}{3} + 2\alpha},$$

$$\mathcal{C}_2 = \mathcal{C}_2(d, \eta, k, H, \mu, \sigma) = C_7(\eta, H)\left(\mu - \sigma + \frac{d}{\log H_0}\right) + C_8(k, \mu), \tag{3.118}$$

*and* $\mathcal{U}, \mathcal{V}, M(k, \delta), C_7, C_8$ *and* $b_{12}$ *are respectively defined in* (3.63), (3.64), (3.68), (3.101), (3.108), *and* (3.120).

*Remark.* 1. The assumptions that $U(\alpha, k, H_0) > 1$ and $U(\alpha, k, T)$ are decreasing can be removed from the theorem. However, this would overly complicate the statement of the theorem. In all instances that we apply this theorem (for various values of $\alpha$ and $k$) these conditions hold.

*Proof.* We begin by assuming that $T$ is not the ordinate of a zero of $\zeta(s)$. From (3.10), (3.16), and the definition (3.17) of $F_X$, we have for $\sigma \in [\sigma', 1]$ where $\sigma' \geq \frac{1}{2}$ and $\mu \in [1+\eta_0, 1+\eta]$

$$N(\sigma, T) \leq \frac{1}{2\pi(\sigma - \sigma')}\Big((T-H)\log\left(1 + \frac{F_X(\sigma', T) - F_X(\sigma', H)}{(T-H)}\right) + \int_{\sigma'}^{\mu} \arg h_X(\tau + iT)d\tau - \int_{\sigma'}^{\mu} \arg h_X(\tau + iH)d\tau - \int_H^T \log|h_X(\mu + it)|dt\Big).$$

We apply Lemma 3.16, Lemma 3.17, and Lemma 3.18 to achieve

$$N(\sigma,T) \le \frac{(T-H)}{2\pi(\sigma-\sigma')}\times$$
$$\log\left(1+\frac{e^{\frac{8}{3}\delta(2\sigma'-1)M(k,\delta)+\frac{4\delta(2\sigma'-1)\log\log H_0}{\log(kH_0)+2\delta}}\mathcal{U}(\alpha,k,T)^{2(1-\sigma')+\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}}\mathcal{V}(\alpha,k,\delta,T)^{2\sigma'-1-\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}}}{2(\omega_2(\sigma',T,\alpha))^2}\times \right. \tag{3.119}$$
$$\left.\frac{(\log(kT))^{2\sigma'}(\log T)^{4(1-\sigma')}T^{\frac{8}{3}(1-\sigma')}}{(T-H)}\right)+\frac{(C_7(\eta,H)\,(\mu-\sigma')+C_8(k,\mu))\,(\log T)}{2\pi(\sigma-\sigma')}.$$

We make the choice $\sigma'=\sigma-\frac{d}{\log T}$, for some $d>0$. From the definition (3.64), we note that $\mathcal{V}(\alpha,k,\delta,T)$ decreases with $T$. Since by assumption $\mathcal{U}(\alpha,k,H_0)>1$ and $T\to U(k,\alpha,T)$ decreases, it follows that $\mathcal{U}(\alpha,k,H_0)^{\frac{2d}{\log T}+\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}}$ decreases with $T$ and thus

$$\mathcal{U}(\alpha,k,T)^{2(1-\sigma')+\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}} \le \mathcal{U}(\alpha,k,H_0)^{2(1-\sigma)+\frac{2d}{\log H_0}+\frac{2\delta(2\sigma'-1)}{\log(kH_0)+2\delta}}.$$

It may be shown that for our choice of parameters $\alpha,k,\delta$ that $\mathcal{V}(\alpha,k,\delta,T)>1$ for all $T\ge H_0$ and thus

$$\mathcal{V}(\alpha,k,\delta,T)^{2\sigma'-1-\frac{2\delta(2\sigma'-1)}{\log(kT)+2\delta}} \le \mathcal{V}(\alpha,k,\delta,T)^{2\sigma'-1}.$$

In addition,

$$\begin{aligned}(\log(kT))^{2\sigma'}(\log T)^{4(1-\sigma')}T^{\frac{8}{3}(1-\sigma')} &= e^{\frac{2d}{\log T}(2\log\log T-\log\log(kT))+\frac{8d}{3}}(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)}\\ &\le e^{\frac{2d(2\log\log H_0-\log\log(kH_0))}{\log H_0}+\frac{8d}{3}}(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{\frac{8}{3}(1-\sigma)},\end{aligned}$$

since $T\ge H_0$ and $\frac{10^9}{H_0}\le k\le 1$ imply $\frac{2\log\log T-\log\log(kT)}{\log T}$ decreases in $T$. Since $\omega_2(\sigma',T,\alpha)$ as defined in (3.43) increases with $\sigma'\ge\sigma-\frac{d}{\log H_0}$ and decreases with $T$, then

$$\frac{1}{2(\omega_2(\sigma',T,\alpha))^2}\le b_{12}(H)e^{2\alpha}\text{ with }b_{12}(H)=\frac{1}{2(1-\frac{1}{H})^2}. \tag{3.120}$$

Combining the above inequalities establishes (3.115), and thus (3.116) (applying $\log(1+y) \le y$).

$$\begin{aligned}
N(\sigma,T) \le {} & \frac{(T-H)(\log T)}{2\pi d} \log\Bigg(1 + b_{12}(H) e^{\frac{8}{3}\delta(2\sigma'-1)M(k,\delta) + \frac{4\delta(2\sigma'-1)\log\log H_0}{\log(kH_0)+2\delta}} \\
& \times \mathcal{U}(\alpha,k,H_0)^{2(1-\sigma)+\frac{2d}{\log H_0}+\frac{2\delta(2\sigma'-1)}{\log(kH_0)+2\delta}} \mathcal{V}(\alpha,k,\delta,H_0)^{2\sigma-1} e^{\frac{2d(2\log\log H_0 - \log\log(kH_0))}{\log H_0}+\frac{8d}{3}+2\alpha} \\
& \times \frac{(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)} T^{\frac{8}{3}(1-\sigma)}}{(T-H)}\Bigg) \\
& + \frac{\Big(C_7(\eta,H)\,(\mu-\sigma+\frac{d}{\log H_0}) + C_8(k,\mu)\Big)(\log T)^2}{2\pi d}.
\end{aligned} \tag{3.121}$$

Since $\sigma' \le \sigma$, each remaining occurrence of $\sigma'$ may be replaced by $\sigma$. Finally, by a continuity argument these inequalities extend to the case where $T$ is the ordinate of a zero of the zeta function. □

## 3.5 Tables of Computation

For fixed values of $\sigma$, Table 3.1 provides bounds for $N(\sigma,T)$ of the shape (3.116). We fix values for $k$ in $[\frac{10^9}{H_0}, 1]$. The parameters $\alpha, d, \delta, \eta$ and $H$ are chosen to make $\frac{\mathcal{C}_1}{2\pi d}$ as small as possible with $\mathcal{C}_1(\alpha,d,\delta,k,H,\sigma)$ as defined in (3.117) . The program returns $H = H_0 - 1$ for all lines in the table. With this $H$ we minimize $C_7(\eta,H)$ which chooses $\eta = 0.25618\ldots$. Then $\mu$ is chosen to minimize $\mu C_7(\eta,H) + C_8(k,\mu)$ (as in the definition (3.118) of $\mathcal{C}_2 = \mathcal{C}_2(d,\eta,k,H,\mu,\sigma)$). We remark that there is a small bit of subtlety when considering $\mathcal{U}(\alpha,k,T)$, it is necessary to ensure all the coefficients in $\mathcal{J}(k,T)$ are positive and this is checked with each set of parameters used. This is to guarantee that $\mathcal{U}(\alpha,k,T)$ decreases with $T$.

For fixed values of $\sigma$, Table 3.2 provide bounds for $N(\sigma,H_0)$ of the shape (3.115). In this case, the choice of $H$ is essential and we choose $H = H_0 - 10^{-6}$. As a consequence the "main term" is $\frac{10^{-6}}{2\pi d}(\log H_0)\log\Big(1 + 10^6\mathcal{C}_1(\log(kH_0))^{2\sigma}(\log H_0)^{4(1-\sigma)}H_0^{\frac{8}{3}(1-\sigma)}\Big)$ which becomes insignificant in comparison to $\frac{\mathcal{C}_2(d,\eta,k,H,\mu,\sigma)}{2\pi d}(\log H_0)^2$, the term arising from the argument. We take $\alpha = 0.324$, $\delta = 0.3000$, and $k = 1$ (as we did not find any other values giving better bounds). The parameter $\eta$ is chosen to minimize $C_7(\eta,H)$, and then $\mu$ to minimize $\mu C_7(\eta,H) + C_8(k,\mu)$: $\eta = 0.2561\ldots$ and $\mu = 1.2453\ldots$.

Table 3.1: The bound $N(\sigma,T) \le A(\log(kT))^{2\sigma}(\log T)^{5-4\sigma}T^{\frac{8}{3}(1-\sigma)} + B(\log T)^2$ (3.116) for $\sigma = \sigma_0$ with $\frac{10^9}{H_0} \le k \le 1$.

| $\sigma_0$ | $k$ | $\mu$ | $\alpha$ | $\delta$ | $d$ | $A = \frac{\mathcal{C}_1}{2\pi d}$ | $B = \frac{\mathcal{C}_2}{2\pi d}$ |
|---|---|---|---|---|---|---|---|
| 0.60 | 0.5 | 1.251 | 0.288 | 0.3140 | 0.341 | 2.177 | 5.663 |
| 0.65 | 0.6 | 1.249 | 0.256 | 0.3070 | 0.340 | 2.963 | 5.249 |
| 0.70 | 0.8 | 1.247 | 0.222 | 0.3040 | 0.339 | 3.983 | 4.824 |
| 0.75 | 1.0 | 1.245 | 0.189 | 0.3030 | 0.338 | 5.277 | 4.403 |
| 0.80 | 1.0 | 1.245 | 0.160 | 0.3030 | 0.337 | 6.918 | 3.997 |
| 0.85 | 1.0 | 1.245 | 0.133 | 0.3030 | 0.336 | 8.975 | 3.588 |
| 0.86 | 1.0 | 1.245 | 0.127 | 0.3030 | 0.335 | 9.441 | 3.514 |
| 0.87 | 1.0 | 1.245 | 0.122 | 0.3030 | 0.335 | 9.926 | 3.430 |
| 0.88 | 1.0 | 1.245 | 0.116 | 0.3030 | 0.335 | 10.431 | 3.346 |
| 0.89 | 1.0 | 1.245 | 0.111 | 0.3030 | 0.335 | 10.955 | 3.262 |
| 0.90 | 1.0 | 1.245 | 0.105 | 0.3030 | 0.334 | 11.499 | 3.186 |
| 0.91 | 1.0 | 1.245 | 0.100 | 0.3030 | 0.334 | 12.063 | 3.102 |
| 0.92 | 1.0 | 1.245 | 0.095 | 0.3030 | 0.334 | 12.646 | 3.017 |
| 0.93 | 1.0 | 1.245 | 0.089 | 0.3030 | 0.333 | 13.250 | 2.941 |
| 0.94 | 1.0 | 1.245 | 0.084 | 0.3030 | 0.333 | 13.872 | 2.856 |
| 0.95 | 1.0 | 1.245 | 0.079 | 0.3030 | 0.333 | 14.513 | 2.772 |
| 0.96 | 1.0 | 1.245 | 0.074 | 0.3030 | 0.332 | 15.173 | 2.694 |
| 0.97 | 1.0 | 1.245 | 0.069 | 0.3030 | 0.332 | 15.850 | 2.609 |
| 0.98 | 1.0 | 1.245 | 0.064 | 0.3030 | 0.331 | 16.544 | 2.532 |
| 0.99 | 1.0 | 1.245 | 0.060 | 0.3030 | 0.331 | 17.253 | 2.446 |

Table 3.2: Bound (3.115) with $k = 1$

| $\sigma$ | $d$ | $\frac{1}{2\pi d}$ | $\mathcal{C}_1$ | $\frac{\mathcal{C}_2}{2\pi d}$ | $N(\sigma, H_0) \le$ |
|---|---|---|---|---|---|
| 0.60 | 2.414 | 0.066 | 2094.73 | 0.893 | 520 |
| 0.65 | 3.621 | 0.044 | 97986.60 | 0.595 | 346 |
| 0.70 | 4.828 | 0.033 | 4583580.34 | 0.447 | 260 |
| 0.75 | 6.036 | 0.027 | 214409007.32 | 0.357 | 208 |
| 0.80 | 7.243 | 0.022 | 10029544375.44 | 0.298 | 173 |
| 0.85 | 8.450 | 0.019 | 469158276689.92 | 0.255 | 148 |
| 0.86 | 8.691 | 0.019 | 1012341447042.27 | 0.248 | 144 |
| 0.87 | 8.933 | 0.018 | 2184412502812.95 | 0.242 | 140 |
| 0.88 | 9.174 | 0.018 | 4713486735514.76 | 0.235 | 136 |
| 0.89 | 9.416 | 0.017 | 10170678467214.40 | 0.229 | 133 |
| 0.90 | 9.657 | 0.017 | 21946110446020.33 | 0.224 | 130 |
| 0.91 | 9.899 | 0.017 | 47354929689448.17 | 0.218 | 126 |
| 0.92 | 10.140 | 0.016 | 102181631292174.11 | 0.213 | 123 |
| 0.93 | 10.382 | 0.016 | 220485720114084.42 | 0.208 | 120 |
| 0.94 | 10.623 | 0.015 | 475760194464125.94 | 0.203 | 118 |
| 0.95 | 10.864 | 0.015 | 1026586948666903.92 | 0.199 | 115 |
| 0.96 | 11.106 | 0.015 | 2215151194732183.30 | 0.195 | 113 |
| 0.97 | 11.347 | 0.015 | 4779814142285142.58 | 0.190 | 110 |
| 0.98 | 11.589 | 0.014 | 10313798574616601.14 | 0.186 | 108 |
| 0.99 | 11.830 | 0.014 | 22254932487167323.15 | 0.183 | 106 |

# 4 Explicit Bounds for $L$-Functions on the edge of the critical strip

## 4.1 Introduction

In analytic number theory, and increasingly in other surprising places, $L$-functions show up as a tool for describing interesting algebraic and geometric phenomena. In particular, understanding the value of $L$-functions on the 1-line has a number of applications. For example, the non-vanishing of the Riemann zeta function for $\zeta(1+it)$, $t \in \mathbb{R}$, proves the celebrated Prime Number Theorem. Additionally, understanding the value $L(1,\chi)$ for certain Dirichlet characters, provides us with insight to the order of the class group of imaginary quadratic fields through Dirichlet's Class Number Formula. Unconditionally, for any non-trivial Dirichlet character $\chi$ with conductor $q$, we have

$$\frac{1}{q^{\epsilon}} \ll |L(1,\chi)| \ll \log q.$$

In fact, we can improve the lower bound to $(\log q)^{-1}$, excluding some exceptional cases related to Landau-Siegel zeros (see [21, Chapter 14]). Louboutin [72] proves an explicit upper bound of this shape. Under the assumption of the Generalized Riemann Hypothesis (GRH), we have the much stronger bounds due to Littlewood [70]:

$$\frac{\zeta(2)(1+o(1))}{2e^{\gamma}\log\log q} \leq |L(1,\chi)| \leq (2e^{\gamma}+o(1))\log\log q,$$

where $o(1)$ tends to 0 as $q \to \infty$. Recently, Lamzouri, Li and Soundararajan gave the following explicit refinement

**Theorem 4.1.** *[65, Theorem 1.5 ] Asume GRH. Let $q$ be a positive integer and $\chi$ be a primitive character* *modulo $q$. For $q \geq 10^{10}$ we have*

$$|L(1,\chi)| \leq 2e^{\gamma}\left(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q}\right)$$

*and*

$$\frac{1}{|L(1,\chi)|} \leq \frac{12e^{\gamma}}{\pi^2}\left(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q} + \frac{14\log\log q}{\log q}\right).$$

The goal of this paper is to provide explicit upper and lower bounds for a large class of $L$-functions, including $L$-functions attached to automorphic cuspidal forms on $GL(n)$. More precisely, we bound the quantity $|L(1,f)|$, where $L$ is a degree $d \geq 1$ $L$-function and $f$ is some arithmetic or geometric object. The results will be valid under the assumption of GRH and the Ramanujan-Petersson conjecture. Additionally, we improve on the bound that comes from generalizing Littlewood's technique, which under both GRH and Ramanujan-Petersson conjecture provides

$$(1+o(1))\left(\frac{12e^{\gamma}}{\pi^2}\log\log C(f)\right)^{-d} \leq |L(1,f)| \leq (1+o(1))\Big(2e^{\gamma}\log\log C(f)\Big)^{d},$$

where $o(1)$ is a quantity that tends to 0 as $C(f) \to \infty$. Here $C(f)$ denotes the analytic conductor of the $L$-function. A precise definition of $C(f)$ along with what the term $L$-function describes will be provided after another example. Other works discussing explicit bounds for higher degree $L$-functions focus on bounding $L(\frac{1}{2}, f)$, we refer the reader to [18] for details.

We provide a degree 2 example before appealing to the precise definitions. Let $k, q \geq 1$ be integers and let $\chi$ be a Dirichlet character modulo $q$. Take $f$ to be a Hecke cusp form of weight $k$, level $q$, and character $\chi$, with the following Fourier expansion at the cusp $\infty$,

$$f(z) = \sum_{n\geq 1}\lambda_f(n)n^{(k-1)/2}e(nz),\; e(z) = e^{2\pi i z}.$$

Then

$$L(s,f) = \prod_p\left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1} = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s},$$

is a degree 2 $L$-function. By works of Deligne [22] and Deligne and Serre [23], it is known that $L(s,f)$ satisfies

Ramanujan-Petersson for all weights $k \geq 1$. In this situation, the analytic conductor is given by

$$C(f) = \frac{q}{\pi^2}\left(\frac{1+(k-1)/2}{2}\right)\left(\frac{1+(k+1)/2}{2}\right) \asymp qk^2.$$

We deduce the following corollary from our main results Thereom 4.2 and Theorem 4.3 below.

**Corollary 4.1.** *Under the assumption of GRH, if* $\log C(f) \geq 46$*, we have*

$$|L(1,f)| \leq (2e^{\gamma})^2\left((\log\log C(f))^2 - (2\log 4 - 1)\log\log C(f) + (\log 4)^2 - \log 4 + 2.51\right),$$

*and*

$$\frac{1}{|L(1,f)|} \leq \left(\frac{12e^{\gamma}}{\pi^2}\right)^2 \Big((\log\log C(f))^2 - (2\log 4 - 1)\log\log C(f) + (\log 4)^2 - \log 4 + 2.67 \\ + \frac{89.40((\log\log C(f))^2 - 2\log 4\log\log C(f) + \log^2 4)}{\log C(f)}\Big).$$

### 4.1.1 Definitions and Notation

To begin, let $d \geq 1$ be a fixed positive integer, and let $L(s,f)$ be given by the Dirichlet series and Euler product

$$L(s,f) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s} = \prod_p \prod_{j=1}^{d}\left(1 - \frac{\alpha_{j,f}(p)}{p^s}\right)^{-1},$$

where $\lambda_f(1) = 1$, and both the series and product are absolutely convergent in $\mathrm{Re}(s) > 1$. We shall assume that $L(s,f)$ satisfies the Ramanujan-Petersson conjecture which states that $|\alpha_{j,f}(p)| \leq 1$ for all primes $p$ and $1 \leq j \leq d$. Further, we define the gamma factor

$$\gamma(s,f) = \pi^{-ds/2}\prod_{j=1}^{d}\Gamma\left(\frac{s+\kappa_j}{2}\right),$$

where $\kappa_j$ are complex numbers. These $\kappa_j$ are called the local parameters at infinity and may be referred to as such throughout. In general, it is assumed that $\mathrm{Re}(\kappa_j) > -1$, in our case the Ramanujan-Petersson conjecture guarantees that $\mathrm{Re}(\kappa_j) \geq 0$. This last condition ensures that $\gamma(s,f)$ has no pole in $\mathrm{Re}(s) > 0$. Furthermore, there exists a positive integer $q(f)$ (called the conductor of $L(s,f)$), such that the completed

$L$-function,

$$\xi(s,f) = q(f)^{s/2}\gamma(s,f)L(s,f),$$

has an analytic continuation to the entire complex plane, and has finite order. This completion satisfies a functional equation

$$\xi(s,f) = \epsilon(f)\xi(1-s,\overline{f}),$$

where $\epsilon(f)$ is a complex number of absolute value 1, and $\xi(s,\overline{f}) = \overline{\xi(\overline{s},f)}$ ($\overline{f}$ is called the dual of $f$). Uniform estimates for analytic quantities associated to $L(s,f)$, when $L(s,f)$ is varying rely on a number of parameters, it is therefore convenient to state the results in terms of the analytic conductor which we define as follows: For $s \in \mathbb{C}$,

$$C(f,s) := \frac{q(f)}{\pi^d}\prod_{j=1}^{d}\left|\frac{s+\kappa_j}{2}\right|.$$

In this article we are interested in studying the value of $L(1,f)$

$$C(f) := C(f,1) = \frac{q(f)}{\pi^d}\prod_{j=1}^{d}\left|\frac{1+\kappa_j}{2}\right|.$$

We note that in [18] the author uses $C(f) = C(f,1/2)$. This definition is very similar to the one given in Iwaniec and Kowalski [52] and only differs by a constant factor to the power of the degree of the $L$-function. To help orient the reader, we give an example in the form of the analytic conductor of a Dirichlet $L$-function. Let $\chi$ be a Dirichlet character modulo $q$ then the associated $L$-function has analytic conductor:

$$C(\chi) = q\frac{1+\mathfrak{a}}{2\pi}, \text{ where } \mathfrak{a} = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1. \end{cases}$$

## 4.2 Results

Here we detail the theorems and make some remarks about how they fit into the general context of what is already known.

**Theorem 4.2.** *Let $d \geq 1$ be a fixed positive integer and let $L(s,f)$ be an $L$-function of degree $d$ with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s,f)$. Then for*

*$C(f)$ chosen such that $\log C(f) \geq 23d$ we have*

$$|L(1,f)| \leq 2^d e^{d\gamma} \left( (\log\log C(f) - \log 2d)^d + \frac{d}{2}(\log\log C(f) - \log 2d)^{d-1} + \frac{dK(d)}{4}(\log\log C(f) - \log 2d)^{d-2} \right),$$

*where*

$$K(d) = 2.31 + \frac{22.59}{d}(e^{0.31d} - 1 - 0.31d). \tag{4.1}$$

**Remark 4.1.** *This result is asymptotically better than the classical bound as the upper bound has the shape*

$$|L(1,f)| \leq (2e^{\gamma})^d \left( (\log\log C(f))^d - (d\log(2d) - \tfrac{d}{2})(\log\log C(f))^{d-1} + O_d((\log\log C(f))^{d-2}) \right),$$

*and $(d\log(2d) - \frac{d}{2}) > 0$ for all $d \geq 1$.*

**Remark 4.2.** *If we take $d = 1$, we may take $C(f) \geq 10^{10}$ and we obtain $K(1)/4 \leq 0.88$ which gives essentially Theorem 4.1*

$$|L(1,\chi)| \leq 2e^{\gamma} \left( \log\log C(f) - \log 2 + \frac{1}{2} + \frac{0.88}{\log\log C(f) - \log 2} \right).$$

**Theorem 4.3.** *Let $d \geq 1$ be a fixed positive integer and let $L(s,f)$ be an L-function of degree d with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s,f)$. Then for $C(f)$ chosen such that $\log C(f) \geq 23d$ we have*

$$\frac{1}{|L(1,f)|} \leq \left(\frac{12e^{\gamma}}{\pi^2}\right)^d \Bigg( (\log\log C(f) - \log 2d)^d + \frac{d}{2}(\log\log C(f) - \log 2d)^{d-1} + \frac{dJ_1(d)}{4}(\log\log C(f) - \log 2d)^{d-2} + \frac{d^2 J_2(d)(\log\log C(f) - \log 2d)^d}{\log C(f)} \Bigg)$$

*where*

$$J_1(d) \leq 2 + \frac{4.18}{d}(e^{0.69d} - 1 - 0.69d). \tag{4.2}$$

*and*

$$J_2(d) = 9 + \frac{16.74}{d}(e^{0.69d} - 1 - 0.69d). \tag{4.3}$$

We notice that lower bound also provides something asymptotically better as in Remark 4.1.

**Remark 4.3.** *If $d = 1$ we may take $C(f) \geq 10^{10}$ then $J_1(1)/4 \leq 0.82$ and $J_2(1) \leq 14.09$ this provides essentially Theorem 4.1*

$$\frac{1}{|L(1,\chi)|} \leq \frac{12e^\gamma}{\pi^2}\left(\log\log C(f) + \frac{1}{2} - \log 2 + \frac{0.82}{\log\log C(f) - \log 2} + \frac{14.09(\log\log C(f) - \log 2)}{\log C(f)}\right).$$

As an easy corollary to these theorems we may a bound degree $d$ $L$-functions in the $t$ aspect as follows. Let $t$ be a real number and define $L_t(1,f) := L(1+it,f)$, then the analytic conductor of $L_t(s,f)$ is given by

$$C_t(f) := \frac{q(f)}{\pi^d}\prod_{j=1}^{d}\left|\frac{1+it+\kappa_j}{2}\right| \asymp_f |t|^d.$$

**Corollary 4.2.** *Let $d \geq 1$ be a fixed positive integer and let $L(s,f)$ be an $L$-function of degree $d$ with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s,f)$. If $\log C_t(f) \geq 23d$ then*

$$|L(1+it,f)| \leq (2e^\gamma)^d\left((\log\log C_t(f) - \log 2d)^d + \frac{d}{2}(\log\log C_t(f) - \log 2d)^{d-1} + \frac{dK(d)}{4}(\log\log C_t(f) - \log 2d)^{d-2}\right),$$

*and*

$$\frac{1}{|L(1+it,f)|} \leq \left(\frac{12e^\gamma}{\pi^2}\right)^d\left((\log\log C_t(f) - \log 2d)^d + \frac{d}{2}(\log\log C_t(f) - \log 2d)^{d-1}\right.$$
$$\left.+\frac{dJ_1(d)}{4}(\log\log C_t(f) - \log 2d)^{d-2} + \frac{d^2J_2(d)(\log\log C_t(f) - \log 2d)^d}{\log C_t(f)}\right).$$

*The definitions of $K(d)$, $J_1(d)$ and $J_2(d)$ are given by equations (4.1), (4.2) and (4.3) respectively.*

## 4.3 Lemmata

In this section we will outline a number of results which are necessary for proving the final bound. Additionally, we will disclose a few more properties of the $L$-functions we are studying. First, the logarithmic derivative of $L(s,f)$ is given by

$$-\frac{L'}{L}(s,f) = \sum_{n\geq 2}\frac{a_f(n)\Lambda(n)}{n^s} \text{ for } \mathrm{Re}(s) > 1, \tag{4.4}$$

where $a_f(n) = 0$ unless $n = p^k$ is a prime power in which case $a_f(n) = \sum_{j=1}^{d} \alpha_{j,f}(p)^k$. Since $L(s,f)$ satisfies the Ramanujan-Petersson conjecture, then $|a_f(n)| \le d$. Further, let $\{\rho_f\}$ be the set of the nontrivial zeros of $L(s,f)$. Then we have the Hadamard factorization formula ([52, Theorem 5.6]),

$$\xi(s,f) = e^{A(f)+sB(f)} \prod_{\rho_f} \left(1 - \frac{s}{\rho_f}\right) e^{s/\rho_f}, \tag{4.5}$$

where $A(f)$ and $B(f)$ are constants. We note that $\mathrm{Re}B(f) = -\mathrm{Re}\sum_{\rho_f} 1/\rho_f$ and taking the logarithmic derivatives of both sides of (4.5) gives

$$\mathrm{Re}\frac{\xi'}{\xi}(s,f) = \mathrm{Re}\sum_{\rho_f} \frac{1}{s-\rho_f}. \tag{4.6}$$

### 4.3.1 Explicit Formulas for $\log|L(1,f)|$ and $|\mathbf{Re}(B(f))|$.

**Lemma 4.1.** *Let $d \ge 1$ be a fixed positive integer and let $L(s,f)$ be an $L$-function of degree $d$ with conductor $q(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s,f)$. For any $x \ge 2$ there exists a real number $|\theta| \le 1$ such that*

$$\begin{aligned}\log|L(1,f)| = \mathrm{Re}\sum_{n\le x} \frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x} + \frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right) \\ - \left(\frac{1}{\log x} - \frac{2\theta}{\sqrt{x}\log^2 x}\right)|\mathrm{Re}B(f)| + \frac{2d\theta}{x\log^2 x}.\end{aligned}$$

*Proof.* We have for any fixed $\sigma \ge 1$ combining (4.4) with the relationship given by (2.1) we have

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s+\sigma,f)\frac{x^s}{s^2}ds = \sum_{n\le x}\frac{a_f(n)\Lambda(n)}{n^\sigma}\log(\tfrac{x}{n}). \tag{4.7}$$

As described in Section 2.2 we will shift the contour to the left by constructing a rectangle and taking limits. Doing this allows us to apply the Residue Theorem (Theorem 2.4). In this contour shift we will pick up poles coming from the zeros of our $L$-function as well, which provides the following formula:

$$-\left(\frac{L'}{L}\right)'(\sigma,f) - \frac{L'}{L}(\sigma,f)\log x - \sum_{\rho_f}\frac{x^{\rho_f-\sigma}}{(\rho_f-\sigma)^2} - \sum_{j=1}^{d}\sum_{m=0}^{\infty}\frac{x^{-2m-\kappa_j-\sigma}}{(2m+\kappa_j+\sigma)^2}. \tag{4.8}$$

Thus, setting (4.7) equal to (4.8) and using that $\mathrm{Re}(\kappa_j) \geq 0$ we have

$$-\frac{L'}{L}(\sigma, f) = \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n^\sigma} \frac{\log(\frac{x}{n})}{\log x} + \frac{1}{\log x}\left(\frac{L'}{L}\right)'(\sigma, f)$$
$$+ \frac{\theta x^{\frac{1}{2}-\sigma}}{\log x} \sum_{\rho_f} \frac{1}{|\rho_f|^2} + \frac{\theta x^{-\sigma}}{\log x} \sum_{j=1}^{d} \sum_{m=0}^{\infty} \frac{x^{-2m}}{(2m+1)^2}.$$

We integrate both sides with respect to $\sigma$ from 1 to $\infty$, then take real parts to obtain

$$\log |L(1, f)| = \mathrm{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \frac{\log(\frac{x}{n})}{\log x} - \frac{1}{\log x} \mathrm{Re} \frac{L'}{L}(1, f) + \frac{\theta}{\sqrt{x} \log^2 x} \sum_{\rho_f} \frac{1}{|\rho_f|^2} + \frac{2d\theta}{x \log^2 x}.$$

We note that, since we have assumed GRH the $\mathrm{Re}(\rho_f) = \frac{1}{2}$, hence $\sum_{\rho_f} \frac{1}{|\rho_f|^2} = 2|\mathrm{Re}B(f)|$. Now we have

$$-\frac{L'}{L}(1, f) = \frac{1}{2} \log q(f) - \frac{d}{2} \log \pi + \frac{1}{2} \sum_{j=1}^{d} \frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right) - \frac{\xi'}{\xi}(1, f)$$

Hence, after taking real parts we have the desired result. □

**Lemma 4.2.** *Let $d \geq 1$ be a fixed positive integer and let $L(s, f)$ be an $L$-function of degree $d$ with conductor $q(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s, f)$. Define $0 \leq l(f) \leq d$ to be the number of $\kappa_j$ in the gamma factor of $L(s, f)$ which equal 0. For any $x > 1$ there exists a real number $|\theta| \leq 1$ such that*

$$-\frac{\xi'}{\xi}(0, \overline{f}) - \frac{1}{x}\frac{\xi'}{\xi}(0, f) + \frac{2\theta}{\sqrt{x}}|\mathrm{Re}(B(f))| =$$
$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1 - \frac{1}{x}\right) - \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n}\left(1 - \frac{n}{x}\right) + E(f, x),$$

*where*

$$E(f, x) = l(f)\left(-\log 2 - \frac{\gamma}{2}\left(1 - \frac{1}{x}\right) + \frac{\log x + 1}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)}\right)$$
$$+ \sum_{i=1}^{d-l(f)} \left(\frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1+\kappa_i}{2}) - \frac{1}{2x}\frac{\Gamma'}{\Gamma}(\frac{\kappa_i}{2}) - \sum_{n=0}^{\infty} \frac{x^{-2n-\kappa_j-1}}{(2n+\kappa_i)(2n+\kappa_i+1)}\right).$$

*In particular,* $\left(1+\frac{1}{x}+\frac{2\theta}{\sqrt{x}}\right)|\mathrm{Re}(B(f)|$ *equals*

$$\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\left(\frac{q(f)}{\pi^d}\right)+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)-\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)$$
$$-(d\theta-(1+\theta)l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}+l(f)\frac{\log x+1}{x}+\frac{(d-2l(f))\log 2}{x}$$
$$-\frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)+\frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right).$$

*In both of the above expressions, the terms inside* $\sum_{i=1}^{d-l(f)}$ *are ranging over the local parameters at infinity,* $\kappa_i \neq 0$.

*Proof.* We consider

$$I(f)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\xi'}{\xi}(s,f)\frac{x^{s-1}}{s(s-1)}ds.$$

Pulling the contour to the left we collect the residues of the poles at $s=0,1$ and $\rho_f$ the nontrivial zeros of $L(s,f)$. Hence,

$$I(f)=\frac{\xi'}{\xi}(1,f)-\frac{1}{x}\frac{\xi'}{\xi}(0,f)+\sum_{\rho_f}\frac{x^{\rho_f-1}}{\rho_f(\rho_f-1)}.$$

Thus applying GRH we have for some $|\theta|\leq 1$

$$I(f)=-\frac{\xi'}{\xi}(0,\overline{f})-\frac{1}{x}\frac{\xi'}{\xi}(0,f)+\frac{2\theta}{\sqrt{x}}|\mathrm{Re}(B(f))|.$$

On the other hand, we can also write

$$\begin{aligned}I(f)&=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{1}{2}\log(q(f))+\frac{\gamma'}{\gamma}(s,f)+\frac{L'}{L}(s,f)\right)\frac{x^{s-1}}{s(s-1)}ds\\&=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\frac{x^{s-1}}{s(s-1)}ds+\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{1}{2}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa_j}{2}\right)\frac{x^{s-1}}{s(s-1)}ds\\&\quad+\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s,f)\frac{x^{s-1}}{s(s-1)}ds\\&=I_1+I_2+I_3.\end{aligned}$$

The contribution from $I_1$ and $I_3$ is

$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right)-\sum_{n\le x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right).$$

We rewrite $I_2$ as

$$I_2=\sum_{j=1}^{d}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa_j}{2}\right)\frac{x^{s-1}}{s(s-1)}ds.$$

Fix $j$, if $\kappa_j\neq 0$ then the $j$-th term of the summand will have simple poles at $s=0,1$ and $s=-2n-\kappa_j$ for $n\ge 0$. Thus the contribution will be

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)-\frac{1}{2x}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_j}{2}\right)-\sum_{n=0}^{\infty}\frac{x^{-2n-\kappa_j-1}}{(2n+\kappa_j)(2n+1+\kappa_j)}.$$

On the other hand, if $\kappa_j=0$ then the $j$-th term of the summand will have simple poles at $s=1$ and $s=-2n$ for $n\ge 1$, which contribute

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)-\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}.$$

Additionally, we know that

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)=-\frac{1}{s}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right),$$

so the residue of the double pole at $s=0$ is given by

$$\frac{1+\log x+\frac{1}{2}\frac{\Gamma'}{\Gamma}(1)}{x}.$$

Using the fact that $\frac{\Gamma'}{\Gamma}(1)=-\gamma$ and $\frac{\Gamma'}{\Gamma}(1/2)=-2\log 2-\gamma$ we see the overall contribution will be

$$-\log 2-\frac{\gamma}{2}\left(1-\frac{1}{x}\right)+\frac{\log x+1}{x}-\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}.$$

Let $l(f)$ be as in the statement of the lemma. Then, reordering the $\kappa_j$ so that $\kappa_1,\kappa_2,\ldots,\kappa_{d-l(f)}$ are all nonzero and summing over $j$ we get the desired expression for $E(f,x)$.

Finally, since $-\mathrm{Re}\frac{\xi'}{\xi}(0,\overline{f})=-\mathrm{Re}\frac{\xi'}{\xi}(0,f)=|\mathrm{Re}(B(f))|$, we see that taking real parts of the established identity we obtain

$$\left(1+\frac{1}{x}+\frac{2\theta}{\sqrt{x}}\right)|Re(B(f))|=\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right)-\mathrm{Re}\sum_{n\le x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)+\mathrm{Re}(E(f,x)).$$

We find an explicit expression for the right hand side as follows:

Start by noting that for $\kappa_j = 0$ we have

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) = \frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right),$$

so that

$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right) + E(f,x) = \frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right) + \sum_{j=1}^{d}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)$$
$$+ l(f)\left(\frac{\log x + 1 + \gamma/2}{x} - \sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}\right) - \sum_{i=1}^{d-l(f)}\left(\frac{1}{2x}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}\right) + \sum_{n=0}^{\infty}\frac{x^{-2n-\kappa_i-1}}{(2n+\kappa_i)(2n+\kappa_i+1)}\right).$$

We note that for some $|\theta| \le 1$

$$\sum_{n=0}^{\infty}\frac{x^{-2n-\kappa_j-1}}{(2n+\kappa_i)(2n+\kappa_i+1)} = \frac{x^{-\kappa_i}}{x\kappa_i(\kappa_j+1)} + \theta\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)},$$

hence

$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right) + E(f,x) = \frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right) + \sum_{j=1}^{d}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right) \tag{4.9}$$
$$-(d\theta - (1+\theta)l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)} + l(f)\frac{\log x + \gamma/2 + 1}{x} - \frac{1}{x}\sum_{i=1}^{d-l(f)}\left(\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}\right) + \frac{x^{-\kappa_i}}{\kappa_i(\kappa_i+1)}\right).$$

Now, from the functional equation of $\Gamma(s)$ we see that

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}\right) = \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}+1\right) - \frac{1}{\kappa_i},$$

we recall Legendre's duplication formula

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = 2^{1-2s}\log(\sqrt{\pi})\Gamma(2s),$$

so we have

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}+1\right) = -\log 2 + \frac{\Gamma'}{\Gamma}(\kappa_i+1) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right).$$

Finally, we note

$$\frac{\Gamma'}{\Gamma}(s) = -\gamma - \frac{1}{s} - \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right),$$

so that

$$\left[\frac{\Gamma'}{\Gamma}(\kappa_i + 1) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right)\right] - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right) = \frac{1}{\kappa_i+1} + \sum_{n=1}^{\infty}\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}.$$

Combinging these facts gives (4.9) as

$$\begin{aligned}
\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right) + E(f,x) &= \frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right) + \sum_{j=1}^{d}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right) \qquad (4.10)\\
&- (d\theta - (1-\theta)l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)} + l(f)\frac{\log x + \gamma/2 + 1}{x}\\
&- \frac{1}{x}\sum_{i=1}^{d-l(f)}\left(\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right) - \log 2 + \sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}\right) + \frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right).
\end{aligned}$$

Then since $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right) = \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1}{2}) = -\log 2 - \gamma/2$ when $\kappa_i = 0$ we add $\frac{-l(f)\frac{\Gamma'}{\Gamma}(\frac{1}{2}) + l(f)\frac{\Gamma'}{\Gamma}(\frac{1}{2})}{2x}$ so that the RHS of (4.10) is given by

$$\begin{aligned}
&\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\left(\frac{q(f)}{\pi^d}\right) + \sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right) - (d\theta - (1+\theta)l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)} + l(f)\frac{\log x + 1}{x}\\
&\qquad + \frac{(d-2l(f))\log 2}{x} - \frac{1}{x}\sum_{i=1}^{d-l(f)}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}\right) + \frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right).
\end{aligned}$$

Taking real parts gives the desired result. □

### 4.3.2 Bounds for the Digamma Function

The following are some technical lemmas which help to shorten the proof of the main results. The first is taken from V. Chandee.

**Lemma 4.3.** *[18, Lemma 2.3] Let $z = x + iy$, where $x \geq \frac{1}{4}$. Then*

$$\mathrm{Re}\frac{\Gamma'}{\Gamma}(z) \leq \log|z|.$$

**Lemma 4.4.** *Let $\kappa = \sigma + it$ such that $\sigma \geq 0$, then*

$$\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa+1+2n} - \frac{1}{\kappa+1+n}\right)\right) = \frac{1}{2}\log 4 + \frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2} - \frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2} + \frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right).$$

*Proof.* We take the real part inside the sum and focus on the individual partial sums given by

$$\sum_{n=1}^{N}\frac{2(\sigma+1+2n)}{(\sigma+1+2n)^2+t^2} \text{ and } \sum_{n=1}^{N}\frac{(\sigma+1+n)}{(\sigma+1+n)^2+t^2}.$$

Using partial summation we find

$$\begin{aligned}\sum_{n=1}^{N}\frac{2(\sigma+1+2n)}{(\sigma+1+2n)^2+t^2} &= \frac{2N(\sigma+1+2N)+\sigma^2+2\sigma(N+1)+2N+1+t^2}{(\sigma+1+2N)^2+t^2} - \frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2}\\ &+ \frac{1}{2}\log((\sigma+1+2N)^2+t^2) - \frac{1}{2}\log((\sigma+3)^2+t^2)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{N}\frac{(\sigma+1+n)}{(\sigma+1+n)^2+t^2} &= \frac{N(\sigma+1+N)+\sigma^2+\sigma(N+1)+\sigma+N+1+t^2}{(\sigma+1+N)^2+t^2} - \frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2}\\ &+ \frac{1}{2}\log((\sigma+1+N)^2+t^2) - \frac{1}{2}\log((\sigma+2)^2+t^2).\end{aligned}$$

Taking the limit as $N \to \infty$ we see

$$\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa+1+2n} - \frac{1}{\kappa+1+n}\right)\right) = \frac{1}{2}\log 4 + \frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2} - \frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2} + \frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right),$$

as was claimed. □

**Lemma 4.5.** *Let $\kappa = \sigma + it$ such that $\sigma \geq 0$, and $x > 1$ then*

$$\left|\frac{x^{-\kappa}-1}{\kappa(\kappa+1)}\right| \leq \frac{2\log x}{\log 3}.$$

*Proof.* We consider two cases.

First suppose $|\kappa| \geq \frac{c}{\log x}$ then we can trivially bound the norm to obtain

$$\left|\frac{x^{-\kappa}-1}{\kappa(\kappa+1)}\right| \leq \frac{2\log x}{c}.$$

If $|\kappa| < \frac{c}{\log x}$ then

$$x^{-\kappa} - 1 = \sum_{k=1}^{\infty} \frac{(-\kappa \log x)^k}{k!},$$

so that

$$\left| \frac{x^{-\kappa} - 1}{\kappa(\kappa+1)} \right| \leq \log x \sum_{k=1}^{\infty} \frac{c^{k-1}}{k!} = \frac{e^c - 1}{c} \log x.$$

The choice of $c = \log 3$ gives the desired result. $\square$

### 4.3.3 Relevant Results from [65].

Let

$$B = -\sum_{\rho} \mathrm{Re}\frac{1}{\rho} = \frac{1}{2}\log(4\pi) - 1 - \frac{\gamma}{2},$$

where the sum is taken over the non-trivial zeros of the Riemann zeta function.

**Lemma 4.6.** *[65, Lemma 2.4] Assume the Riemann Hypothesis. For $x > 1$ we have, for some $|\theta| \leq 1$,*

$$\sum_{n \leq x} \Lambda(n) n \left(1 - \frac{n}{x}\right) = \log x - (1+\gamma) + \frac{2\pi}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + 2\frac{\theta|B|}{\sqrt{x}}.$$

**Lemma 4.7.** *[65, Lemma 2.6] Assume the Riemann Hypothesis. For all $x \geq e$ and some $|\theta| \leq 1$ we have*

$$\sum_{n \leq x} \frac{\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} = \log\log x - \gamma - 1 + \frac{\gamma}{\log x} + \frac{2|B|\theta}{\sqrt{x}\log^2 x} + \frac{\theta}{3x^3 \log^2 x}.$$

We also prove the following lemma which is a slight generalization of [65, Lemma 5.1].

**Lemma 4.8.** *Assume the Ramanujan-Petersson conjecture. Then for $x \geq 100$, we have*

$$\mathrm{Re} \sum_{n \leq x} \alpha_{j,f}(n)\Lambda(n) \left(\frac{1}{n\log n} - \frac{1}{x \log x}\right) \geq \sum_{p^k \leq x} \Lambda(p^k)(-1)^k \left(\frac{1}{p^k \log p^k} - \frac{1}{x\log x}\right). \tag{4.11}$$

*In particular, we have*

$$\mathrm{Re} \sum_{n \leq x} a_f(n)\Lambda(n) \left(\frac{1}{n\log n} - \frac{1}{x \log x}\right) \geq d \sum_{p^k \leq x} \Lambda(p^k)(-1)^k \left(\frac{1}{p^k \log p^k} - \frac{1}{x\log x}\right). \tag{4.12}$$

*Proof.* Note that if $x$ is a prime power then the summand at $x$ on both sides of the inequality (4.11) contribute

0, so we assume $x$ is not a prime power. We begin by recalling that $a_f(n) = 0$ unless $n = p^k$ is a prime power in which case $a_f(n) = \sum_{j=1}^{d} \alpha_{j,f}(p)^k$. So that (4.12) follows immediately after we prove (4.11).

Fix $j$ and consider each $\alpha_{j,f}$ separately. From the definition we see $\alpha_{j,f}(n)$ is only nonzero if $n = p^k$ for some prime power. If $\alpha_{j,f}(p) = 0$ then the contribution is 0 while the value on the right hand side $< 0$. If $\alpha_{j,f}(p) \neq 0$ then, from Ramanujan-Petersson we have that $|\alpha_{j,f}(p)| \leq 1$, so we express $\alpha_{j,f}(p) = -re(\theta)$, for $0 < r \leq 1$ where $e(\theta) = e^{2\pi i\theta}$. Consider the difference of the left and right side of (4.11):

$$\log(p) \sum_{p^k \leq x} (-1)^{k-1}(1 - r^k \cos(k\theta)) \left( \frac{1}{p^k \log p^k} - \frac{1}{x \log x} \right). \tag{4.13}$$

If we establish this is non-negative, then we are finished.

Before we proceed, we note that if $x - p < 2$ then the only contribution appearing in (4.13) is the term coming from $p$, meaning only $k = 1$ contributes and the result is thus non-negative since $p < x$.

Now, suppose $x - p \geq 2$. In what follows we will require the following claim for $k \geq 1$

$$1 - r^k \cos(k\theta) \leq k^2(1 - r\cos\theta). \tag{4.14}$$

When $r = 1$ this is already known to be true. For the rest we will consider $0 < r < 1$. The case $k = 1$ is trivial, for the remaining $k \geq 2$, we need to prove

$$k^2 - 1 \geq k^2 r \cos(\theta) - r^k \cos(k\theta). \tag{4.15}$$

The claim will follow if we can show that $k^2 - 1$ is greater than the maximum achieved on the RHS. We begin with setting the derivative with respect to $\theta$ equal to 0:

$$\begin{aligned} \frac{d}{d\theta}(k^2 r \cos(\theta) - r^k \cos(k\theta)) &= kr^k \sin(k\theta) - k^2 r \sin(\theta) = 0, \\ &\Leftrightarrow r^{k-1} \sin(k\theta) - k \sin(\theta) = 0, \\ &\Leftrightarrow \sin(\theta)(r^{k-1} U_{k-1}(\cos(\theta)) - k) = 0, \end{aligned} \tag{4.16}$$

where $U_n(x)$ is a Chebyshev polynomial of the second kind. The identity giving (4.16), along with the properties we will use of these polynomials, can be found in [1, Chapter 22]. We have that $|U_n(x)| \leq 1$ on the interval $[-1, 1]$ and that they achieve their extrema at the endpoints $\pm 1$: $U_n(1) = n + 1$ and

$U_n(-1) = (-1)^n(n+1)$. Hence we see, since $0 < r < 1$, that $r^{k-1}U_{k-1}(\cos(\theta)) - k \neq 0$ for any $\theta \in [0, 2\pi)$ and the only critical points for (4.16) occur when $\sin(\theta) = 0$. Plugging these values back into the RHS of (4.15) we see the maximum occurs at $\theta = 0$ which returns $k^2 r - r^k \leq k^2 - 1$ as desired.

If $p \geq 3$, then by (4.14) we have (4.13) is greater than

$$\log(p)(1-r\cos\theta)\left(\frac{1}{p\log p} - \frac{1}{x\log x} - \sum_{j=1}^{\infty}\frac{(2j)^2}{p^{2j}\log p^{2j}}\right) = \log(p)(1-r\cos\theta)\left(\frac{1}{p\log p} - \frac{1}{x\log x} - \frac{2p^2}{\log p(p^2-1)^2}\right).$$

This will be nonnegative if $x - p \geq 2$, since

$$\left(\frac{1}{t\log t} - \frac{1}{(t+c)\log(t+c)} - \frac{2t^2}{\log t(t^2-1)^2}\right) > 0$$

if $t > 7$ and $t \geq c \geq 2$ and decreases to 0 for $c \geq 2$ fixed and $t \to \infty$ . For $p = 2$, when $k \geq 6$ we apply (4.14) again. Otherwise, when $1 \leq k \leq 5$ we compute the trigonometric polynomial exactly. A little computer computation completes the result. □

## 4.4 Proof of Theorems 4.2 and 4.3

### 4.4.1 Upper bounds for $L(1, f)$.

Let $C(f) \geq 10^{10}$ and $x \geq 132$, be a real number to be chosen later. Lemma 4.1 says

$$\begin{aligned}\log|L(1,f)| \leq &\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right)\right)\\ &+\frac{2d}{x\log^2 x} - \left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)|\mathrm{Re}B(f)|.\end{aligned}$$

Applying Lemma 4.2 with the conditions on $x$ as above, we see

$$|\mathrm{Re}B(f)| \geq \left(1+\frac{1}{\sqrt{x}}\right)^{-2} \left(\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_j+1}{2}\right)\right) - \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\right.$$
$$+l(f)\frac{\log x+1}{x} + \frac{(d-2l(f))\log 2}{x} - (d-2l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}$$
$$\left.-\frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)+\frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right)\right).$$

For $x \geq 132$ we bound

$$-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\left(l(f)\frac{\log x+1}{x}+\frac{(d-2l(f))\log 2}{x}\right.$$
$$\left.-(d-2l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)} - \frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)+\frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right)\right)+\frac{2d}{x\log^2 x}$$
$$= -\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_1+A_2-A_3-A_4-A_5)+\frac{2d}{x\log^2 x}.$$

First, we consider

$$A_4 = \frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)\right).$$

Fix $i$ and study the inner sum, writing $\kappa_i = \sigma + it$, and noting that Ramanujan-Petersson gives us $\sigma \geq 0$, we apply Lemma 4.4 so that

$$\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)\right) = \frac{1}{2}\log 4 + \frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2} - \frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2} + \frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right)$$
$$\leq \log 2 - \frac{4}{5}(\sqrt{3}-2) \leq 1.$$

The inequality comes from the following facts. First, the last term is negative. Next, taking $\sigma \geq 0$, a maple calculation finds that $\frac{4}{5}(2-\sqrt{3})$ is a global maximum for

$$\frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2} - \frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2}.$$

Thus we may combine the terms $A_2$ and $A_4$ to obtain

$$-\left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_2 - A_4) \leq \frac{(d-l(f))(1-\log 2)+l(f)\log 2}{(1+\sqrt{x})^2\log x}.$$

For $A_5$, fix $i$, then writing $\kappa_i = \sigma + it$, since we have $\sigma \geq 0$, we apply Lemma 4.5 to obtain

$$\mathrm{Re}\left(\frac{x^{-\kappa_i}-1}{\kappa_j(\kappa_i+1)}\right) \leq \frac{2\log x}{\log 3}.$$

Thus combining $A_1$ and $A_5$ we have

$$-\left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_1 - A_5) \leq \frac{(2d/\log 3 - l(f)(1+2/\log 3))}{(1+\sqrt{x})^2} - \frac{l(f)}{(1+\sqrt{x})^2\log x}.$$

Finally, for $x \geq 132$ we have

$$\begin{aligned}
&-\left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_1 + A_2 - A_3 - A_4 - A_5) + \frac{2d}{x\log^2 x}\\
&\leq \frac{1}{(1+\sqrt{x})^2}\left(2d/\log 3 - l(f)(1+2/\log 3) + \frac{(d-2l(f))(1-\log 2 + \sum_{n=1}^{\infty}\frac{x^{-2n}}{2n(2n+1)})}{\log x} + \frac{2d(1+\sqrt{x})^2}{x\log^2 x}\right)\\
&\leq \frac{2d}{(1+\sqrt{x})^2}.
\end{aligned}$$

The last inequality follows since the term

$$\frac{1}{\log 3} - l(f)\frac{(1+2/\log 3)}{2d} + \frac{(d-2l(f))(1-\log 2+\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n(2n+1)})}{2d\log x} + \frac{(1+\sqrt{x})^2}{x\log^2 x} \leq 1$$

for $x \geq 132$.

Hence,

$$\begin{aligned}
\log|L(1,f)| \leq &\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right)\right)\\
&+\left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right) + \frac{2d}{(1+\sqrt{x})^2}\\
&-\left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_j+1}{2}\right)\right).
\end{aligned}$$

Next, note that

$$0 \leq \frac{1}{2\log x} - \left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\frac{1}{2}\left(1-\frac{1}{x}\right) \leq \frac{1}{(\sqrt{x}+1)\log x}\left(1+\frac{1}{\log x}\right),$$

and Lemma 4.3 gives

$$\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_j+1}{2}\right) \leq \log\left|\frac{1+\kappa_i}{2}\right|,$$

so

$$\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right) \leq \log C(f).$$

Therefore,

$$\begin{aligned}\log|L(1,f)| \leq & \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{\log C(f)}{(\sqrt{x}+1)\log x}\left(1+\frac{1}{\log x}\right) \\ & + \left(\frac{1}{\log x} - \frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right) + \frac{2d}{(1+\sqrt{x})^2}.\end{aligned}$$

The right hand side of the above is largest when $a_f(p) = d$ for all $p \leq x$, thus

$$\log|L(1,f)| \leq d\mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{d}{\log x}\mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n}\left(1-\frac{n}{x}\right) + \frac{\log C(f)}{(\sqrt{x}+1)\log x}\left(1+\frac{1}{\log x}\right) + \frac{2d}{(1+\sqrt{x})^2}.$$

So applying Lemmas 4.6 and 4.7 and choosing $x = \frac{\log^2 C(f)}{4d^2}$ (which implies $\frac{\log C(f)}{\sqrt{x}} = 2d$ and allows us to factor $d$ from each term) we obtain

$$\log|L(1,f)| \leq d\left(\log\log x + \gamma - \frac{1}{\log x} + \frac{2}{(1+\sqrt{x})^2}\right) + \frac{\log C(f)}{\sqrt{x}\log x}\left(1+\frac{1}{\log x}\right).$$

Thus for $x \geq 132$ we have

$$\begin{aligned}\log|L(1,f)| &\leq d\left(\log\log x + \gamma + \frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{2}{(1+\sqrt{x})^2}\right) \\ &\leq d\left(\log\log x + \gamma + \frac{1}{\log x} + \frac{2.31}{\log^2 x}\right).\end{aligned}$$

Therefore,

$$|L(1,f)| \leq e^{d\gamma}\log^d x\left(1+\frac{d}{\log x} + \frac{dK(d)}{\log^2 x}\right)$$

where $K(d) = 2.31 + (1 + \frac{4.62}{\log x} + \frac{(2.31)^2}{\log^2 x})\sum_{k=0}^{\infty} \frac{d^{k+1}}{(k+2)!}(\frac{1}{\log x} + \frac{2.31}{\log^2 x})^k$. Replacing $x$ gives

$$|L(1,f)| \leq 2^d e^{d\gamma}\left((\log\log C(f) - \log 2d)^d + \frac{d}{2}(\log\log C(f) - \log 2d)^{d-1} + \frac{dK(d)}{4}(\log\log C(f) - \log 2d)^{d-2}\right),$$

which proves the result.

### 4.4.2 Lower bounds for $L(1,f)$

The argument proceeds similarly. As before we let $C(f)$ be chosen such that $x = \frac{\log^2 C(f)}{4d^2} \geq 132$, then from Lemma 4.1 we have

$$\log|L(1,f)| \geq \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x} + \frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)$$
$$-\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)|\mathrm{Re}B(f)| - \frac{2d}{x\log^2 x}.$$

Applying Lemma 4.2 we see

$$|\mathrm{Re}(B(f))| \leq \left(1 - \frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\left(\frac{q(f)}{\pi^d}\right) + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right) - \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\right.$$
$$\left.-d\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)} + l(f)\frac{\log x + 1}{x} + \frac{(d-2l(f))\log 2}{x} - \frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n} + \frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right)\right)\right).$$

For $x \geq 132$ we bound

$$-\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(l(f)\frac{\log x+1}{x} + \frac{(d-2l(f))\log 2}{x} - d\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}\right.$$
$$\left.-\frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}\right)\right) + \frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right) - \frac{2d}{x\log^2 x}$$
$$= -\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_1 + A_2 - A_3 - A_4 - A_5) - \frac{2d}{x\log^2 x}.$$

First, we consider

$$A_4 = \frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}\right)\right).$$

Fix $i$ and study the inner sum, writing $\kappa_i = \sigma + it$, and noting that Ramanujan-Petersson gives us $\sigma \geq 0$, we apply Lemma 4.4 so that

$$\begin{aligned}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)\right) &= \frac{1}{2}\log 4 + \frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2} - \frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2} + \frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right) \\ &\geq 2\log 2 - \log(3).\end{aligned}$$

The inequality comes from the following facts. First, the combination of the second and third term is positive since $\sigma \geq 0$, and the last term has a global minimum at the point $(0,0)$ which gives $\log(2/3)$. Thus we may combine the terms $A_2$ and $A_4$ to obtain

$$-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_2-A_4) \geq 1.04\frac{d(\log 2-\log 3)+l(f)\log 3}{(\sqrt{x}-1)^2\log x}.$$

For $A_5$, fix $j$, then writing $\kappa_j = \sigma + it$ and invoking Ramanujan-Petersson, we can apply Lemma 4.5 to obtain

$$\mathrm{Re}\left(\frac{x^{-\kappa_j}-1}{\kappa_j(\kappa_j+1)}\right) \geq -\frac{2\log x}{\log 3}.$$

Thus combining the terms $A_1$ and $A_5$ we have

$$-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_1-A_5) \geq -1.04\left(\frac{2d/\log 3+l(f)(2/\log(3)-1)}{(\sqrt{x}-1)^2}+\frac{l(f)}{(\sqrt{x}-1)^2\log x}\right).$$

Finally, for $x \geq 132$ we have

$$\begin{aligned}&-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_1+A_2-A_3-A_4-A_5)-\frac{2d}{x\log^2 x} \\ &\geq \frac{1}{(\sqrt{x}-1)^2}1.04\left(-(2d/\log 3-l(f)(2d/\log(3)-1))-\frac{2d}{1.04x\log^2 x}(\sqrt{x}-1)^2\right. \\ &\quad\left.+\frac{d(\log 2-\log 3)+l(f)(\log(3)-1)-d\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}}{\log x}\right) \geq \frac{-2.05d}{(\sqrt{x}-1)^2}.\end{aligned}$$

Thus, for $x \geq 132$

$$\log|L(1,f)| \geq \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x}+\frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)$$
$$-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\left(\frac{q(f)}{\pi^d}\right)+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)\right.$$
$$\left.-\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\right)-\frac{2.05d}{(\sqrt{x}-1)^2}.$$

We note that

$$\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)\left(\frac{1}{2\log x}-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{2}\left(1-\frac{1}{x}\right)\right)\right)$$
$$\geq -\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)\frac{1}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)$$
$$\geq -\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right).$$

Where the last inequality comes from Lemma 4.3. So far, we have proven

$$\log|L(1,f)| \geq \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x}-\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)$$
$$+\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)-\frac{2.05d}{(\sqrt{x}-1)^2}. \quad (4.17)$$

To continue, we see from Lemma 4.6 if $x \geq 132$ we have

$$\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\geq d(1-\log x),$$

thus as in [65, pg 18 line 11 ] we have

$$\left(\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)-\frac{1}{\log x}\right)\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\geq -\frac{2d}{\sqrt{x}}.$$

Using this in (4.17) we have

$$\begin{aligned}\log|L(1,f)| &\geq \operatorname{Re}\sum_{n\leq x} a_f(n)\Lambda(n)\left(\frac{1}{n\log n}-\frac{1}{x\log x}\right)-\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)-\frac{2d}{\sqrt{x}}-\frac{2.05d}{(\sqrt{x}-1)^2}\\ &\geq \operatorname{Re}\sum_{n\leq x} a_f(n)\Lambda(n)\left(\frac{1}{n\log n}-\frac{1}{x\log x}\right)-\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)-\frac{9d}{4\sqrt{x}}.\end{aligned}$$

We apply Lemma 4.8, thus guaranteeing that the first term in the right hand side is smallest when $a_f(p)=-d$ for every prime $p\leq x$. Therefore, we have

$$\log|L(1,f)| \;\geq\; d\sum_{p^k\leq x}\Lambda(p^k)(-1)^k\left(\frac{1}{p^k\log p^k}-\frac{1}{x\log x}\right) \;-\; \frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right) \;-\; \frac{9d}{4\sqrt{x}}.$$

Following the discussion after [65, Equation 5.3] we see that

$$\log|L(1,f)|\geq d\left(-\log\log x-\gamma+\log\zeta(2)+\frac{1}{\log x}-\frac{8}{5\sqrt{x}}\right)-\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)-\frac{9d}{4\sqrt{x}}.$$

Our choice of $x=\frac{\log^2 C(f)}{4d^2}$ gives $-\log C(f)\geq -2d\sqrt{x}-1$ so that with a little calculation one obtains

$$\log|L(1,f)|\geq d\left(-\log\log x-\gamma+\log\zeta(2)-\frac{1}{\log x}-\frac{2}{\log^2 x}-\frac{9}{2\sqrt{x}}\right).$$

Exponentiating both sides gives

$$\begin{aligned}\frac{1}{|L(1,f)|} &\leq \left(e^{\gamma}\frac{6}{\pi^2}\right)^d \log^d x\exp\left(\frac{d}{\log x}+\frac{2d}{\log^2 x}+\frac{9d}{2\sqrt{x}}\right)\\ &\leq \left(e^{\gamma}\frac{6}{\pi^2}\right)^d \log^d x\left(1+\frac{d}{\log x}+\frac{dJ_1(d)}{\log^2 x}+\frac{dJ_2(d)}{2\sqrt{x}}\right),\end{aligned}$$

where

$$J_1(d)=2+\left(1+\frac{4}{\log x}+\frac{4}{\log^2 x}\right)\sum_{k=0}^{\infty}\frac{d^{k+1}}{(k+2)!}\left(\frac{1}{\log x}+\frac{2}{\log^2 x}+\frac{9}{2\sqrt{x}}\right)^k,$$

and

$$J_2(d)=9+\left(\frac{18}{\log x}+\frac{18}{\log^2 x}+\frac{81}{2\sqrt{x}}\right)\sum_{k=0}^{\infty}\frac{d^{k+1}}{(k+2)!}\left(\frac{1}{\log x}+\frac{2}{\log^2 x}+\frac{9}{2\sqrt{x}}\right)^k.$$

Replacing $x$ we get

$$\frac{1}{|L(1,f)|} \leq \left(2e^{\gamma}\frac{6}{\pi^2}\right)^d \Bigg((\log\log C(f) - \log 2d)^d + \frac{d}{2}(\log\log C(f) - \log 2d)^{d-1}$$
$$+\frac{dJ_1(d)}{4}(\log\log C(f) - \log 2d)^{d-2} + \frac{d^2 J_2(d)(\log\log C(f) - \log 2d)^d}{\log C(f)}\Bigg)$$

Thus the theorem is proven.

# 5 The distribution of Values of $L(1,\chi_D)$ over Function Fields

## 5.1 Introduction

An interesting and important problem in number theory is to understand the size of the class group, known as the class number, for a given field. The case of quadratic extensions of $\mathbb{Q}$ has a rich history of investigation which extends back to Gauß. Let $d$ be a fundamental discriminant and $h_d$ represent the class number of the field $\mathbb{Q}(\sqrt{d})$. Describing the extreme values of $h_d$ and the distribution of these values has been widely investigated. The main line of attack in this problem is to study the moments of $L(1,\chi_d)$, with $\chi_d$ taken as the Kronecker symbol $\left(\frac{d}{\cdot}\right)$. This approach works because of Dirichlet's class number formula. Some recent notable papers discussing this problem are those of Granville and Soundararajan [40] and Dahl and Lamzouri [20]. The approach in these articles is to compare the complex moments of $L(1,\chi_d)$ to that of a random model and use the class number formula to apply this information to $h_d$.

Here we discuss the adaptation of these techniques to study the class number, denoted as $h_D$, over function fields, $\mathbb{F}_q(T)$ with $q\equiv 1 (\bmod 4)$ and $D$ a monic square free polynomial in $\mathbb{F}_q[T]$. In this context, $h_D = |\mathrm{Pic}(\mathcal{O}_D)|$, where $\mathrm{Pic}(\mathcal{O}_D)$ is the Picard group of the ring of integers $\mathcal{O}_D \subseteq \mathbb{F}_q(T)(\sqrt{D(T)})$. Since $D$ is a square free polynomial we have that $\mathrm{Pic}(\mathcal{O}_D) = \mathcal{C}l(\mathcal{O}_D)$, the class group of $\mathcal{O}_D$, which provides the justification for the name 'class number'. We refer the reader to Section 2.6 for a refresher on the preliminaries needed for this chapter.

In 1992, Hoffstein and Rosen [46] investigated this question and obtained an average result by fixing the degree of the polynomial. The result is stated as follows: let $M$ be odd and positive then

$$\frac{1}{q^M}\sum_{\substack{D \text{ monic}\\ \deg(D)=M}} h_D = \frac{\zeta_{\mathbb{F}_q[T]}(2)}{\zeta_{\mathbb{F}_q[T]}(3)} q^{(M-1)/2} - q^{-1}, \tag{5.1}$$

where

$$\zeta_{\mathbb{F}_q[T]}(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} \text{ for } \mathfrak{Re}(s) > 1,$$

is the Riemann zeta function over $\mathbb{F}_q[T]$. Here the norm of $f \in \mathbb{F}_q[T] \setminus \{0\}$ is $|f| = q^{\deg(f)}$. This result is directly comparable to Gauß's conjecture (proven by Siegel [101]) for class numbers of imaginary quadratic number fields. Finally, letting $q \to \infty$ one obtains an asymptotic formula which can be compared to the 2012 work of Andrade [4] described below.

There are two limits that can be considered when studying problems over function fields. The first fixes the degree of the polynomial and lets the number of elements in the base field go to infinity as was done by Hoffstein and Rosen. The second fixes the number of elements in the base field and allows the degree of the polynomials to go to infinity. The result of Andrade [4] considers the second perspective. His article describes the mean value of $h_D$ by averaging over $\mathcal{H}_{2g+1}$ the set of monic, square free polynomials with degree $2g+1$. Andrade proves that

$$\frac{1}{|\mathcal{H}_{2g+1}|} \sum_{D \in \mathcal{H}_{2g+1}} h_D \sim \zeta_{\mathbb{F}_q[T]}(2) \prod_{P \text{ irreducible}} \left(1 - \frac{1}{(|P|+1)|P|^2}\right) q^g \text{ as } g \to \infty. \tag{5.2}$$

We remark that (5.1) and (5.2) have the same order of magnitude in the main term as can be seen by taking $M = 2g+1$.

Now, for any monic $D \in \mathbb{F}_q[T]$ we have Dirichlet characters modulo $D$ on $\mathbb{F}_q[T]$, see Definition 2.27. The natural follow up to this is to define a Dirichlet $L$-function associated to such a character:

$$L(s, \chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s}, \text{ for } s \in \mathbb{C}.$$

Artin [5] proved a class number formula valid over function fields which links $h_D$ to $L(1, \chi_D)$ where $\chi_D(\cdot)$ is the Kronecker symbol $\left(\frac{D}{\cdot}\right)$:

$$L(1, \chi_D) = \frac{\sqrt{q}}{\sqrt{|D|}} h_D = q^{-g} h_D, \text{ for } D \in \mathcal{H}_{2g+1}. \tag{5.3}$$

To prove (5.2) Andrade makes use of an "approximate" functional equation for $L(1,\chi_D)$ to show

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}} L(1,\chi_D)\sim\zeta_{\mathbb{F}_q[T]}(2)\prod_{P\text{ irreducible}}\left(1-\frac{1}{(|P|+1)|P|^2}\right)\text{ as } g\to\infty, \tag{5.4}$$

and then applies (5.3). The main drawback to using the approximate functional equation is that is difficult to use it to calculate large moments of $L(1,\chi_D)$.

In this article, we shall investigate the distribution of $L(1,\chi_D)$ for $D\in\mathcal{H}_n$ as $n\to\infty$, where

$$\mathcal{H}_n=\{D\in\mathbb{F}_q[T]: D \text{ is monic, square free, } \deg(D)=n\}. \tag{5.5}$$

To do this we will need to compute large complex moments of the associated $L(1,\chi_D)$. We approach the computation of such moments via a random model, a technique that has been used successfully in the study of quadratic number fields.

For the remainder of the chapter the following notation will be fixed. Let $\mathbb{A}=\mathbb{F}_q[T]$ taking $q\equiv 1(\bmod 4)$ for simplicity. Here $\log_q$ denotes base $q$ logarithm, log is the natural logarithm. Finally, let $P$ represent an irreducible (prime) polynomial. We define the generalized divisor function $d_z(f)$ on its prime powers as

$$d_z(P^a)=\frac{\Gamma(z+a)}{\Gamma(z)a!}, \tag{5.6}$$

and extend it to all monic polynomials multiplicatively. Then, we can express the complex moments of $L(1,\chi_D)$ as follows.

**Theorem 5.1.** *Let $n$ be a positive integer, and $z\in\mathbb{C}$ be such that $|z|\le\frac{n}{260\log_q(n)\log\log_q(n)}$ and let $c_0>0$ be a constant. Then*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n} L(1,\chi_D)^z=\sum_{f\ monic}\frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\left(1+O\left(\frac{1}{n^{11}}\right)\right)+O\left(q^{-\frac{n}{c_0\log_q n}}\right).$$

The strategy for proving this, and a following result about the distribution of values, is to compare the distribution of $L(1,\chi_D)$ to that of a probabilistic random model: Let $\{\mathbb{X}(P)\}$ denote a sequence of independent random variables indexed by the irreducible (prime) elements $P \in \mathbb{A}$, and taking the values $0, \pm 1$ as follows

$$\mathbb{X}(P) = \begin{cases} 0 & \text{with probability } \frac{1}{|P|+1} \\ \pm 1 & \text{with probability } \frac{|P|}{2(|P|+1)}. \end{cases} \tag{5.7}$$

Let $f = P_1^{e_1} P_2^{e_2} \cdots P_s^{e_s}$ be the prime power factorization of $f$, then we extend the definition of $\mathbb{X}$ multiplicatively as follows

$$\mathbb{X}(f) = \mathbb{X}(P_1)^{e_1} \mathbb{X}(P_2)^{e_2} \cdots \mathbb{X}(P_s)^{e_s}. \tag{5.8}$$

In this article we compare the distribution of $L(1, \chi_D)$ with

$$L(1, \mathbb{X}) := \sum_{f \text{ monic}} \frac{\mathbb{X}(f)}{|f|} = \prod_{P \text{ irreducible}} \left(1 - \frac{\mathbb{X}(P)}{|P|}\right)^{-1}, \tag{5.9}$$

which converges almost surely. Further properties of this model will be discussed in Section 5.3.2.

For $\tau > 0$, define

$$\Phi_{\mathbb{X}}(\tau) := \mathbb{P}(L(1, \mathbb{X}) > e^{\gamma}\tau) \text{ and } \Psi_{\mathbb{X}}(\tau) := \mathbb{P}\left(L(1, \mathbb{X}) < \frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau}\right). \tag{5.10}$$

We prove that the distribution of $L(1, \chi_D)$ is well approximated by the distribution of $L(1, \mathbb{X})$ uniformly in a large range.

**Theorem 5.2.** *Let $n$ be large. Uniformly in $1 \le \tau \le \log_q n - 2\log_q \log_q n - \log_q \log_q \log_q n$ we have*

$$\frac{1}{|\mathcal{H}_n|} |\{D \in \mathcal{H}_n : L(1, \chi_D) > e^{\gamma}\tau\}| = \Phi_{\mathbb{X}}(\tau)\left(1 + O\left(\frac{e^{\tau}(\log_q n)^2 \log_q \log_q n}{n}\right)\right),$$

*and*

$$\frac{1}{|\mathcal{H}_n|} |\{D \in \mathcal{H}_n : L(1, \chi_D) < \frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau}\}| = \Psi_{\mathbb{X}}(\tau)\left(1 + O\left(\frac{e^{\tau}(\log_q n)^2 \log_q \log_q n}{n}\right)\right).$$

And below we describe the asymptotic behaviour of $\Phi_{\mathbb{X}}$ and $\Psi_{\mathbb{X}}$.

**Theorem 5.3.** *For any large $\tau$ we have*

$$\Phi_{\mathbb{X}}(\tau) = \exp\left(-C_1(q^{\{\log_q \kappa(\tau)\}})\frac{q^{\tau - C_0(q^{\{\log_q \kappa(\tau)\}})}}{\tau}\left(1 + O\left(\frac{\log_q \tau}{\tau}\right)\right)\right), \tag{5.11}$$

*where $\kappa(\tau)$ is defined by* (5.29)*, $C_0(t) = G_2(t)$, $C_1(t) = G_2(t) - G_1(t)$ and $G_i(t)$ are defined in* (5.34) *and*

(5.36) *respectively. Furthermore we have*

$$-\frac{1}{\log q} + \log(\cosh(c))/c - \tanh(c) < -C_1(q^{\{\log_q \kappa(\tau)\}}) < \log(\cosh(q))/q - \tanh(q),$$

*where* $c = 1.28377....$ *In particular,* $C_1(q^{\{\log_q \kappa(\tau)\}}) > 0$. *The same results hold for* $\Psi_{\mathbb{X}}$. *Additionally, if we let* $0 < \lambda < e^{-\tau}$, *then*

$$\Phi_{\mathbb{X}}(e^{-\lambda}\tau) = \Phi_{\mathbb{X}}(\tau)(1 + O(\lambda e^{\tau})) \text{ and } \Psi_{\mathbb{X}}(e^{-\lambda}\tau) = \Psi_{\mathbb{X}}(\tau)(1 + O(\lambda e^{\tau})). \tag{5.12}$$

Our Theorem 5.3 should be compared to those of [40] and [20], both of which study the behaviour of $L(1, \chi_d)$ over quadratic number fields. The asymptotic behaviour of $\Phi_{\mathbb{X}}(\tau)$ is strikingly similar in both of these papers. In [40] the authors are studying the distribution of $L(1, \chi_d)$ over all fundamental discriminants $d$, $|d| \leq x$, comparing it to a corresponding probabilistic model $L(1, \mathbb{X})$. In [20] the authors are studying the distribution of $L(1, \chi_d)$ over fundamental discriminants of the form $d = 4m^2 + 1$, $m \geq 1$ and $d$ is square free. The restriction in [20] is used in order to study the behaviour of class numbers associated to such $d$, again comparing to a corresponding probabilistic model. In both papers $\Phi_{\mathbb{X}}(\tau) = \text{Prob}(L(1, \mathbb{X}) > e^{\gamma}\tau)$. Each obtains:

$$\Phi_{\mathbb{X}}(\tau) = \exp\left(-C_1 \frac{e^{\tau - C_0}}{\tau} + O\left(\frac{e^{\tau}}{\tau^2}\right)\right),$$

where

$$C_1 := 1 \text{ and } C_0 := \int_0^1 \frac{\tanh(t)}{t} dt + \int_1^{\infty} \frac{\tanh(t) - 1}{t} dt = 0.8187\ldots$$

Similar behaviour appears when studying the distribution of Euler-Kronecker constants of quadratic fields, see [63, Theorem 1.2] for details. As can be seen from the statement of Theorem 5.3 we observe some pathological behaviour special to function fields. We no longer achieve two constants reflected above as $C_0$ and $C_1$. In our case the value of both $C_0(q^{\{\log_q \kappa(\tau)\}})$ and $C_1(q^{\{\log_q \kappa(\tau)\}})$ varies, although they remain bounded as the argument varies between 1 and $q$. Below is a graph of $C_0(t)$ for $1 \leq t < q$ taking $q = 5$, and $q = 9$ the first moduli which satisfy the hypothesis $q \equiv 1 (\bmod 4)$. Additionally, we also notice the coefficient $C_1$ which appears in all of the theorems describing the behaviour of $\Phi_{\mathbb{X}}(\tau)$( cf. [40, 20, 63]). We find over function fields that the coefficient $C_1$ is no longer fixed, but remains bounded between $-\log(\cosh(q))/q + \tanh(q)$ and $1/\log(q) - \log(\cosh(c))/c + \tanh(c)$. Below is a graph of the behaviour of $C_1(t)$ for $1 < t < q$ with $q = 5$ and $q = 9$. The reason for this difference stems from Proposition 5.2 which is used to evaluate the natural

Figure 5.1: $C_0(t)$ for $1 < t < q$ and $q = 5$ and $q = 9$ respectively.

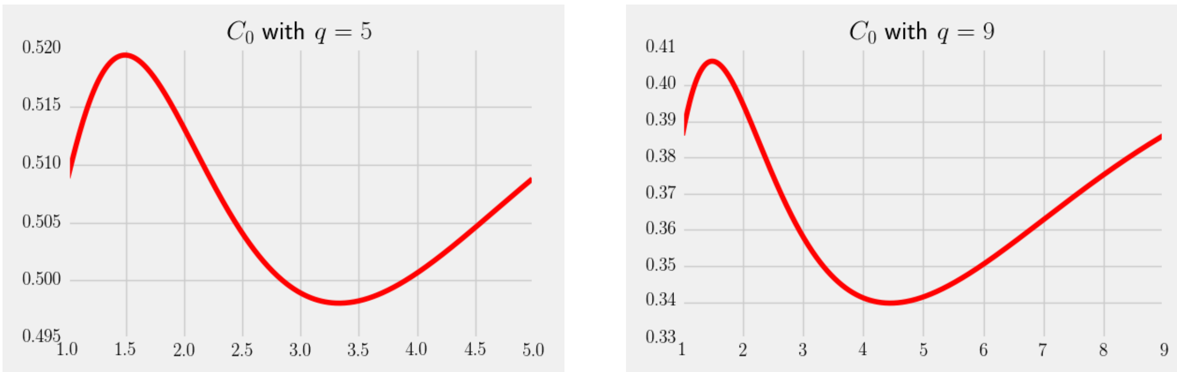
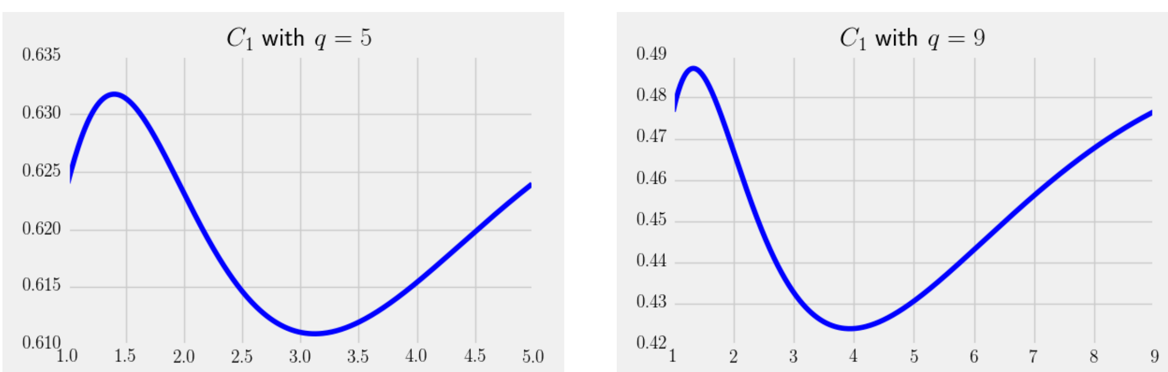



Figure 5.2: $C_1(t)$ for $1 < t < q$ and $q = 5$ and $q = 9$ respectively.



log of the real moments of our random model. In this proposition we obtain two sums over primes $G_1(t)$ and $G_2(t)$, equations (5.34) and (5.36) respectively. The corresponding sums over number fields do not have the parameter $t$ (it is always equal to 1), which in our case arises from the way that primes are measured in function fields.

Furthermore, we obtain the following unconditional bounds:

**Proposition 5.1.** *Let $F$ be a monic polynomial, and $\chi$ be a non-trivial character on $(\mathbb{A}/F\mathbb{A})^{\times}$.*

*For any complex number $s$ with $\mathrm{Re}(s) = 1$ we have*

$$\frac{\zeta_{\mathbb{A}}(2)}{2e^{\gamma}}(\log_q \log_q |F| + O(1))^{-1} \leq |L(s,\chi)| \leq 2e^{\gamma} \log_q \log_q |F| + O(1). \tag{5.13}$$

It is important to note that in this setting Weil [112] proved the Riemann Hypothesis (RH), hence these results are achieved unconditionally. We conjecture here that the true size for the extreme values of $L(1,\chi_D)$ is half as large in keeping with the expected results in the quadratic number field case.

**Conjecture 5.1.** *Let $n$ be large.*

$$\max_{D\in\mathcal{H}_n} L(1,\chi_D) = e^{\gamma}(\log_q n + \log_q \log_q n) + O(1),$$

*and*

$$\min_{D\in\mathcal{H}_n} L(1,\chi_D) = \zeta_{\mathbb{A}}(2)e^{-\gamma}(\log_q n + \log_q \log_q n + O(1))^{-1}.$$

Finally, we also unconditionally obtain $\Omega$-results which we claim are best possible, unlike in the case of number fields where the corresponding bounds for Dirichlet characters is only valid under the Generalized Riemann Hypothesis (GRH).

**Theorem 5.4.** *Let $N$ be large. There are irreducible polynomials $Q_1$ and $Q_2$ of degree $N$ such that*

$$L(1,\chi_{Q_1}) \geq e^{\gamma}(\log_q \log_q |Q_1| + \log_q \log_q \log_q |Q_1|) + O(1), \tag{5.14}$$

*and*

$$L(1,\chi_{Q_2}) \leq \zeta_{\mathbb{A}}(2)e^{-\gamma}(\log_q \log_q |Q_2| + \log_q \log_q \log_q |Q_2| + O(1))^{-1}. \tag{5.15}$$

The result (5.14) can be compared with [2, Theorem 1] a recent work discussing the size of $|L(1,\chi)|$ over a number field. The authors prove using a variant of the resonator method that for $\epsilon > 0$ and sufficiently large $d$ there is a character $\chi(\text{mod } d)$ such that

$$|L(1,\chi)| \geq e^{\gamma}(\log\log d + \log\log\log d - (1 + \log\log 4) - \epsilon).$$

This result provides an improvement over a paper of Granville and Soundararajan [39], however, the paper does not give improvements for quadratic characters $\chi_d$ where $d$ varies over fundamental discriminants in the range $|d| \leq x$ cf. [40, 61].

The result (5.15) can be compared to [40, Theorem 5a] which under the assumption of GRH proves for any $\epsilon > 0$ and all large $x$ there are $\gg x^{1/2}$ primes $d \leq x$ such that

$$L(1,\chi_d) \leq \frac{\zeta(2)}{e^{\gamma}}(\log\log d + \log\log\log d - \log\log 4 - \epsilon)^{-1}.$$

Unconditionally, for $\chi$ a Dirichlet character modulo $d$ we have the weaker results $|L(1,\chi)| \leq \frac{\zeta(2)}{e^{\gamma}}(\log\log d - O(1))^{-1}$ from [39] .

### 5.1.1 Applications

From the theorems above and in light of (5.3) if we specialize $n$ as $n = 2g+1$ and letting the genus $g \to \infty$ we can prove analogous results about the class number $h_D$ over $\mathcal{H}_{2g+1}$. This specialization is the equivalent of studying the imaginary quadratic extensions of $\mathbb{Q}$, as described by Artin. Below we state a few of the resulting corollaries for $h_D$ with $D \in \mathcal{H}_{2g+1}$.

**Corollary 5.1.** *Let $z \in \mathbb{C}$ be such that $|z| \leq \frac{g}{130\log_q(g)\log\log_q(g)}$ and let $c_0 > 0$ be a constant. Then*

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}} h_D^z = q^{gz}\sum_{f \text{ monic}} \frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\left(1+O\left(\frac{1}{g^{11}}\right)\right) + O\left(q^{-\frac{g}{c_0\log_q g}}\right).$$

This result follows from applying Artin's class number formula (5.3) to Theorem 5.1 when $n = 2g+1$. Additionally, from Theorems 5.2 and 5.3 we obtain that the tail of the distribution of large (and small) values of $h_D$ over $\mathcal{H}_{2g+1}$ is doubly exponentially decreasing:

**Corollary 5.2.** *Let $g$ be large and $1 \le \tau \le \log_q g - 2\log_q \log_q g - \log_q \log_q \log_q g$. The number of discriminants $D \in \mathcal{H}_{2g+1}$ such that*

$$h_D > e^{\gamma} \tau q^g$$

*equals*

$$|\mathcal{H}_{2g+1}| \cdot \exp\left( -C_1(q^{\{\log_q \kappa(\tau)\}}) \frac{q^{\tau - C_0(q^{\{\log_q \kappa(\tau)\}})}}{\tau} \left(1 + O\left(\frac{\log_q \tau}{\tau}\right)\right)\right),$$

*where $\kappa(\tau)$ is given by (5.29), $C_1(q^{\{\log_q \kappa(\tau)\}})$ and $C_0(q^{\{\log_q \kappa(\tau)\}})$ are positive constants depending on $\tau$ defined in Theorem 5.3. Similar estimates hold for the number of discriminants $D \in \mathcal{H}_{2g+1}$ such that*

$$h_D < \frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau} q^g.$$

Similarly Proposition 5.1 give analogous upper and lower bounds and Theorem 5.4 provides analogous Omega results for $h_D$ with $D \in \mathcal{H}_{2g+1}$.

Specializing to $n = 2g+2$, we can also make connections to the class number $h_D$ for $D \in \mathcal{H}_{2g+2}$. This case is analogous to studying a real quadratic extension of $\mathbb{Q}$ and so the class number formula changes. The analogy holds in this way since the prime at infinity splits for even degree hyperelliptic curves which does not happen in the case of odd degree hyperelliptic curves. Indeed for $D \in \mathcal{H}_{2g+2}$ Artin proves:

$$L(1, \chi_D) = \frac{q-1}{\sqrt{|D|}} h_D R_D, \tag{5.16}$$

where $R_D$ denotes the regulator of $\mathcal{O}_D$. In this case $R_D$ is defined to be $\log_q |\epsilon|_{P_\infty}$ where $\epsilon$ is a fundamental unit of $\mathcal{O}_D$, $P_\infty$ is the prime at infinity such that $\mathrm{ord}_{P_\infty}(\epsilon) < 0$ and

$$\log_q |\epsilon|_{P_\infty} = -\deg(P_\infty)\mathrm{ord}_{P_\infty}(\epsilon).$$

For more details on the regulator see [94, Chapter 14]. The case of the mean value for $L(1, \chi_D)$ taken over $\mathcal{H}_{2g+2}$ was investigated by Jung [53, 54]. Taking $n = 2g+2$ we deduce from (5.16) and Theorem 5.1:

**Corollary 5.3.** *Let $z \in \mathbb{C}$ be such that $|z| \le \frac{g}{130 \log_q(g) \log\log_q(g)}$ and let $c_0 > 0$ be a constant. Then*

$$\frac{1}{|\mathcal{H}_{2g+2}|} \sum_{D \in \mathcal{H}_{2g+2}} (h_D R_D)^z = \left(\frac{q^{g+1}}{q-1}\right)^z \sum_{f \ monic} \frac{d_z(f^2)}{|f|^2} \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} \left(1 + O\left(\frac{1}{g^{11}}\right)\right) + O\left(q^{-\frac{g}{c_0 \log_q g}}\right).$$

Of course, similar results about the distribution of $h_D R_D$, upper and lower bounds and omega results for $D \in \mathcal{H}_{2g+2}$ follow from Theorems 5.2 and 5.3, Proposition 5.1 and Theorem 5.4 respectively.

Finally, we give the outline of the paper. Section 5.2 will establish some facts about $\mathbb{A}$ and the properties $L$-functions have over this ring. Section 5.3 will connect the complex moments of $L(1,\chi_D)$ to the expectation of the complex moments of the random model and provide the proof of Theorem 5.1. Section 5.4 will be used to prove Theorem 5.3. Section 5.5 proves Theorem 5.2 and Corollary 5.2. Section 5.6 proves the $\Omega$-results of Theorem 5.4.

## 5.2 Preliminaries

For a refresher on definitions and other relevant theorems please see Section 1.4 and 2.6.

### 5.2.1 Estimates for sums over irreducible monic polynomials

Here and throughout we let $\Pi_q(n)$ be the number of monic irreducible polynomials $P$ such that $\deg P \leq n$.

**Lemma 5.1.** *Let $M$ be a large positive integer. Then we have*

$$\Pi_q(M) = \zeta_{\mathbb{A}}(2)\frac{q^M}{M}\left(1 + O\left(\frac{\log_q M}{M}\right)\right). \tag{5.17}$$

*Furthermore, we have*

$$\sum_{\deg P \leq M} \frac{1}{|P|} = \log M + O(1). \tag{5.18}$$

*Proof.* Note that

$$\Pi_q(M) = \sum_{n=1}^{M} \pi_q(n) = \sum_{n=1}^{M}\left(\frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)\right),$$

where the last equality comes from the Prime Number Theorem. The main term in this sum is $q^M/M$, and we see that if $n \leq M - \log_q M$, then $q^n/n \ll q^M/M^2$. Hence we have that

$$\Pi_q(M) = \sum_{M-\log_q M < n \leq M} \frac{q^n}{n} + O\left(\frac{q^M}{M^2}\right).$$

Then for $n \in (M - \log_q M, M]$ we have $\frac{1}{n} = \frac{1}{M}\left(1 + O\left(\frac{\log_q M}{M}\right)\right)$. Therefore,

$$\Pi_q(M) = \frac{q^M}{M}\left(1 + O\left(\frac{\log_q M}{M}\right)\right)\sum_{l<\log_q M}\frac{1}{q^l} = \zeta_{\mathbb{A}}(2)\frac{q^M}{M}\left(1 + O\left(\frac{\log_q M}{M}\right)\right).$$

We now establish (5.18). For this result, we follow a similar tactic as the previous proof. First, by the prime number theorem we have

$$\sum_{\deg(P)\leq M}\frac{1}{|P|} = \sum_{d\leq M}\frac{\pi_q(d)}{q^d} = \sum_{d\leq M}\frac{1}{d} + O\left(\sum_{d\leq M}\frac{1}{dq^{d/2}}\right).$$

The first sum is $\log(M) + O(1)$, the second is $O(1)$.

□

**Lemma 5.2.** *Let $F$ be a monic polynomial, and $\chi$ be a non-trivial character on $(\mathbb{A}/\mathbb{A}F)^\times$. For a positive integer $M$ and any complex number $s$ with $\mathrm{Re}(s) = 1$ we have*

$$\log L(s,\chi) = -\sum_{\deg P\leq M}\log\left(1 - \frac{\chi(P)}{|P|^s}\right) + O\left(\frac{q^{-M/2}}{M}\deg F\right).$$

*Proof.* Split the sum as

$$\log L(s,\chi) = -\sum_{\deg P\leq M}\log\left(1 - \frac{\chi(P)}{|P|^s}\right) - \sum_{\deg P>M}\log\left(1 - \frac{\chi(P)}{|P|^s}\right).$$

The error term follows from (2.13) as below

$$\begin{aligned}\sum_{k>M}\sum_{\deg P=k}\log\left(1 - \frac{\chi(P)}{|P|}\right) &= \sum_{k>M}\sum_{\deg P=k}\frac{\chi(P)}{|P|^s} + O\left(\sum_{\deg P>M}\frac{1}{|P|^2}\right)\\ &= \sum_{k>M}\frac{1}{q^k}\sum_{\deg P=k}\chi(P) + O(q^{-M})\\ &\ll \deg F\sum_{k>M}\frac{1}{kq^{k/2}}\\ &\ll \deg F\frac{q^{-M/2}}{M}.\end{aligned}$$

$\square$

We now prove a refined form of a Mertens' type estimate due to Rosen [93].

**Lemma 5.3.** *Let $M$ be large. Then, we have*

$$-\sum_{\deg P\leq M}\log\left(1-\frac{1}{|P|}\right)=\log M+\gamma+\frac{1}{2M}+O\left(\frac{1}{M^2}\right).$$

*Proof.* We have

$$-\sum_{\deg P\leq M}\log\left(1-\frac{1}{|P|}\right)=\sum_{\deg P\leq M}\sum_{\ell=1}^{\infty}\frac{1}{\ell|P|^{\ell}}=\sum_{k\leq M}\sum_{\ell=1}^{\infty}\frac{\pi_q(k)}{\ell q^{\ell k}}=\sum_{k\ell\leq M}\frac{\pi_q(k)}{\ell q^{\ell k}}+\sum_{\substack{k\leq M\\ k\ell>M}}\frac{\pi_q(k)}{\ell q^{\ell k}}.$$

By making the change of variables $m=k\ell$ and using (2.9), we deduce that the first sum on the right hand side of the last identity equals

$$\sum_{k\ell\leq M}\frac{\pi_q(k)}{\ell q^{\ell k}}=\sum_{m\leq M}\frac{1}{q^m m}\sum_{k|m}k\pi_q(k)=\sum_{m\leq M}\frac{1}{m}=\log M+\gamma+\frac{1}{2M}+O\left(\frac{1}{M^2}\right).$$

The result follows upon noting that

$$\sum_{\substack{k\leq M\\ k\ell>M}}\frac{\pi_q(k)}{\ell q^{\ell k}}\ll\sum_{\substack{k\leq M\\ k\ell>M}}q^{k(1-\ell)}\ll\sum_{2\leq\ell\leq M}\sum_{\frac{M}{\ell}<k\leq M}q^{k(1-\ell)}+q^{-M}\ll q^{-M}\sum_{2\leq\ell\leq M}q^{\frac{M}{\ell}}\ll Mq^{-M/2}.$$

$\square$

### 5.2.2 Proof of Proposition 5.1

*Proof of Proposition 5.1.* For $\mathrm{Re}(s)=1$, we use Lemma 5.2 and together with the choice $M=2\log_q\log_q|F|$ to get

$$\log L(s,\chi)=-\sum_{\deg P\leq M}\log\left(1-\frac{\chi(P)}{|P|^s}\right)+O\left(\frac{1}{M}\right).$$

Using this estimate together with Lemma 5.3 we deduce that

$$|L(s,\chi)|\leq\prod_{\deg P\leq M}\left(1-\frac{1}{|P|}\right)^{-1}\left(1+O\left(\frac{1}{M}\right)\right)\leq e^{\gamma}M+O(1),$$

which completes the proof of the upper bound in (5.13). To see the lower bound, note that from Lemma 5.2 we have

$$|L(s,\chi)| \geq \prod_{\deg(P)\leq M} \left(1+\frac{1}{|P|}\right)^{-1} \left(1+O\left(\frac{1}{M}\right)\right),$$

and from Lemma 5.3

$$\prod_{\deg(P)\leq M} \left(1+\frac{1}{|P|}\right)^{-1} = \prod_{\deg(P)\leq M} \frac{\left(1-\frac{1}{|P|^2}\right)^{-1}}{\left(1-\frac{1}{|P|}\right)^{-1}} \geq \frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}M+O(1)}.$$

□

### 5.2.3 Sums over $\mathcal{H}_n$.

The orthogonality relation:

**Lemma 5.4.** *Let $f$ be a monic polynomial. If $f$ is a square in $\mathbb{A}$, then*

$$\sum_{D\in\mathcal{H}_n} \chi_D(f) = |\mathcal{H}_n|\cdot\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1} + O\left(\sqrt{|\mathcal{H}_n|}\right). \tag{5.19}$$

*Furthermore, if $f$ is not a square in $\mathbb{A}$, then*

$$\sum_{D\in\mathcal{H}_n} \chi_D(f) \ll \sqrt{|\mathcal{H}_n|}\cdot 2^{\deg f}. \tag{5.20}$$

**Remark 5.1.** *We remark here that making use of [15, Lemma 3.5] the second estimate becomes*

$$\sum_{D\in\mathcal{H}_n} \chi_D(f) \ll \frac{\sqrt{|\mathcal{H}_n|}}{(q-1)^{1/2}}|f_1|^{\epsilon},$$

*for $\epsilon > 0$ and $f = f_1 f_2^2$ with $f_1$ square free. This in turn leads to a better error for the sum over the non-square terms which is done in Lemma 5.7.*

The proof of (5.19) below adapts the proof of [3, Proposition 5.2]. For a proof of (5.20) see [3, Lemma 6.4].

*Proof.* Note that all sums are taken over monic polynomials. Note first, since $f$ is a square, it can be expressed as $l^2$ for some monic $l \in \mathbb{A}$, so the sum we are studying is

$$\sum_{D \in \mathcal{H}_n} \chi_D(l^2) = \sum_{\substack{D \in \mathcal{H}_n \\ (D,l)=1}} 1 = \sum_{\substack{\deg(D)=n \\ (D,l)=1}} \sum_{h^2|D} \mu(h),$$

where the second equality comes from the definition of $\chi_D(f)$ and the last one follows from the fact that

$$\sum_{h^2|D} \mu(h) = \begin{cases} 1 & \text{if } D \text{ is squarefree} \\ 0 & \text{otherwise.} \end{cases}$$

We swap the order of summation and use an inclusion/exclusion argument to rewrite the sum as:

$$\sum_{D \in \mathcal{H}_n} \chi_D(l^2) = \sum_{\substack{\deg(h) \le n/2 \\ (h,l)=1}} \mu(h) \sum_{\substack{\deg(D)=n-2\deg(h) \\ (D,l)=1}} 1.$$

We recognize the inner most sum is giving an estimate for the size of the set $S_m = \{D \in \mathbb{A} : D \text{ monic } \deg(D) = m, (D,l) = 1\}$. Note that

$$\begin{aligned} \#S_m = \sum_{\substack{\deg(D)=m \\ (D,l)=1}} 1 &= \sum_{\deg(D)=m} \sum_{A|(D,l)} \mu(A) \\ &= \sum_{A|l} \mu(A) \sum_{\substack{\deg(D)=m \\ A|D}} 1 \\ &= \sum_{A|l} \mu(A) q^{m-\deg(A)} \\ &= q^m \frac{\varphi(l)}{|l|}. \end{aligned}$$

Hence we have that

$$\sum_{D \in \mathcal{H}_n} \chi_D(l^2) = \sum_{\substack{\deg(h) \le n/2 \\ (h,l)=1}} \mu(h) q^{n-2\deg(h)} \frac{\varphi(l)}{|l|} = q^n \frac{\varphi(l)}{|l|} \sum_{\substack{\deg(h) \le n/2 \\ (h,l)=1}} \frac{\mu(h)}{|h|^2}.$$

The proof of (5.19) is finished once we have an estimate for $\displaystyle\sum_{\substack{\deg(h)\le n/2\\(h,l)=1}} \frac{\mu(h)}{|h|^2}$, which is given below.

$$\begin{aligned}\sum_{\substack{\deg(h)\le n/2\\(h,l)=1}} \frac{\mu(h)}{|h|^2} &= \sum_{(h,l)=1} \frac{\mu(h)}{|h|^2} - \sum_{\substack{\deg(h)>n/2\\(h,l)=1}} \frac{\mu(h)}{|h|^2},\\ &= \sum_{(h,l)=1} \frac{\mu(h)}{|h|^2} + O\left(\frac{1}{q^{n/2}}\right),\\ &= \prod_{P\nmid l}\left(1-\frac{1}{|P|^2}\right) + O\left(\frac{1}{q^{n/2}}\right).\end{aligned}$$

Finally, the product term can be expressed as:

$$\prod_{P\nmid l}\left(1-\frac{1}{|P|^2}\right) = \prod_{P \text{ irreducible}}\left(1-\frac{1}{|P|^2}\right)\prod_{P|l}\left(1-\frac{1}{|P|^2}\right)^{-1} = \frac{1}{\zeta_{\mathbb{A}}(2)}\prod_{P|l}\left(1-\frac{1}{|P|^2}\right)^{-1}.$$

Putting everything together, we obtain

$$\sum_{D\in\mathcal{H}_n}\chi_D(l^2) = q^n\frac{\varphi(l)}{|l|}\left(\frac{1}{\zeta_{\mathbb{A}}(2)}\prod_{P|l}\left(1-\frac{1}{|P|^2}\right)^{-1} + O\left(\frac{1}{q^{n/2}}\right)\right).$$

The proof is complete by recognizing that $\zeta_{\mathbb{A}}(2) = q/(q-1)$ and $\varphi(l)/|l| = \prod_{P|l}\left(1-\frac{1}{|P|}\right)$ to get the desired result. □

## 5.3 Complex moments of $L(1,\chi)$

Let $D\in\mathcal{H}_n$, $z\in\mathbb{C}$ such that $|z|\ll \log_q|D|/(\log_q\log_q|D|\log\log_q\log_q|D|)$. Let $\chi_D(f) = \left(\frac{D}{f}\right)$. We we recall that $d_z(f)$ is defined as in (5.6). We will prove the following key lemma which will allow us to connect our complex moments of the random model to the complex moments of $L(1,\chi_D)$.

**Lemma 5.5.** *Let $D\in\mathcal{H}_n$. Let $A>4$ be a constant $z\in\mathbb{C}$ such that $|z|\le \frac{\log_q|D|}{10A\log_q\log_q|D|\log\log_q\log_q|D|}$, $M = A\log_q\log_q|D|$ and $c_0>0$ a constant. Then*

$$L(1,\chi_D)^z = \left(1+O\left(\frac{1}{(\log_q|D|)^B}\right)\right)\sum_{\substack{f \text{ monic}\\ |f|\le|D|^{1/3}\\ P|f\Rightarrow \deg P\le M}} \frac{\chi_D(f)d_z(f)}{|f|} + O\left(|D|^{-\frac{1}{c_0\log_q\log_q|D|}}\right),$$

*where $B = A/2 - 2$.*

Before giving the proof, we make some estimates:

**Lemma 5.6.** *Let $D \in \mathcal{H}_n$, $A > 4$ be a fixed constant and $z \in \mathbb{C}$ such that $|z| \leq \frac{\log_q |D|}{10A \log_q \log_q |D| \log \log_q \log_q |D|}$ and $M = A \log_q \log_q |D|$. Then for $c_0$ some positive constant we have*

$$\sum_{\substack{f \text{ monic} \\ P|f \Rightarrow \deg P \leq M}} \frac{\chi_D(f)}{|f|} d_z(f) = \sum_{\substack{f \text{ monic} \\ |f| \leq |D|^{1/3} \\ P|f \Rightarrow \deg P \leq M}} \frac{\chi_D(f)}{|f|} d_z(f) + O\left(|D|^{-\frac{1}{c_0 \log_q \log_q |D|}}\right), \tag{5.21}$$

*and furthermore,*

$$\sum_{\substack{f \text{ monic} \\ P|f \Rightarrow \deg P \leq M}} \frac{\chi_D(f)}{|f|} d_z(f) \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} = \sum_{\substack{f \text{ monic} \\ |f| \leq |D|^{1/3} \\ P|f \Rightarrow \deg P \leq M}} \frac{\chi_D(f)}{|f|} d_z(f) \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1}$$
$$+ O\left(|D|^{-\frac{1}{c_0 \log_q \log_q |D|}}\right). \tag{5.22}$$

*Proof.* First we prove (5.21). Let $z \in \mathbb{C}$ and let $k \in \mathbb{Z}$ such that $|z| < k$. Let $0 < \alpha < \frac{1}{2}$, then using Rankin's trick we see

$$\left| \sum_{\substack{f \text{ monic} \\ |f| > |D|^{1/3} \\ P|f \Rightarrow \deg P \leq M}} \frac{\chi_D(f)}{|f|} d_z(f) \right| \leq \sum_{\substack{f \text{ monic} \\ |f| > |D|^{1/3} \\ P|f \Rightarrow \deg P \leq M}} \frac{d_k(f)}{|f|} \leq |D|^{-\alpha/3} \sum_{\substack{f \text{ monic} \\ P|f \Rightarrow \deg P \leq M}} \frac{d_k(f)}{|f|^{1-\alpha}}$$
$$= |D|^{-\alpha/3} \prod_{\deg P \leq M} \left(1 - \frac{1}{|P|^{1-\alpha}}\right)^{-k}$$
$$= |D|^{-\alpha/3} \exp\left(k \sum_{\deg P \leq M} \frac{1}{|P|^{1-\alpha}} + O(k)\right),$$

where the last estimate comes from Taylor expansion. Choosing $\alpha = \frac{1}{M}$ we have that $|P|^\alpha = q^{\alpha \deg P} \leq q = O(1)$, so that

$$|D|^{-\alpha/3}\exp(k\sum_{\deg P\le M}\frac{1}{|P|^{1-\alpha}}+O(k))\ll |D|^{-\frac{1}{3M}}\exp\left(O\left(k\sum_{\deg P\le M}\frac{1}{|P|}\right)\right)$$
$$\ll |D|^{-\frac{1}{3M}}\exp O\left(k\log_q M\right),$$

by (5.18). Taking $k\ll\frac{\log_q|D|}{\log_q\log_q|D|\log\log_q\log_q|D|}$, and using $M=A\log_q\log_q|D|$ the expression inside of the big Oh becomes

$$k\log(A\log_q\log_q|D|)\ll\frac{\log_q|D|\log(A\log_q\log_q|D|)}{\log_q\log_q|D|\log\log_q\log_q|D|}.$$

So we have

$$|D|^{-\frac{1}{3M}}\exp O\left(k\log_q M\right)\ll |D|^{-\frac{1}{c_0\log_q\log_q|D|}},$$

for some $c_0>0$.

The proof of (5.22) follows from the previous argument since

$$\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\le 1.$$

□

*Proof of Lemma 5.5.* From Lemma 5.2 we have

$$\begin{aligned}L(1,\chi_D)^z&=\exp(z\log(L(1,\chi_D))\\&=\exp\left(-z\sum_{\deg P\le M}\log\left(1-\frac{\chi_D(P)}{|P|}\right)\right)\exp\left(O\left(\frac{q^{-M/2}}{M}\deg D|z|\right)\right).\end{aligned}$$

Here we use the fact that $M=A\log_q\log_q|D|$ implying that $q^{-M/2}=(\log_q|D|)^{-A/2}$, $\deg D=\log_q|D|$, $|z|\le\frac{\log_q|D|}{10A\log_q\log_q|D|\log\log_q\log_q|D|}$ to see that the expression inside of the big Oh has the shape

$$\frac{(\log_q|D|)^2}{(\log_q|D|)^{A/2}}\frac{1}{10A^2(\log_q\log_q|D|)^2\log\log_q\log_q|D|}=O\left(\frac{1}{(\log_q|D|)^B}\right),$$

by the assumption on $A$. Hence, we have

$$\begin{aligned} L(1,\chi_D)^z &= \prod_{\deg P\le M}\left(1-\frac{\chi_D(P)}{|P|}\right)^{-z}\left(1+O\left(\frac{1}{(\log_q|D|)^B}\right)\right) \\ &= \left(1+O\left(\frac{1}{(\log_q|D|)^B}\right)\right)\prod_{\deg P\le M}\left(\sum_{a=0}^{\infty}\frac{\chi_D(P^a)}{|P|^a}d_z(P^a)\right) \\ &= \left(1+O\left(\frac{1}{(\log_q|D|)^B}\right)\right)\sum_{\substack{f \text{ monic} \\ P|f\Rightarrow \deg P\le M}}\frac{\chi_D(f)}{|f|}d_z(f). \end{aligned}$$

Finally we apply (5.21) from Lemma 5.6. □

Using this lemma we have that

$$\begin{aligned} \sum_{D\in\mathcal{H}_n} L(1,\chi_D)^z &= \sum_{D\in\mathcal{H}_n}\left(1+O\left(\frac{1}{(\log_q|D|)^B}\right)\right)\sum_{\substack{f \text{ monic} \\ |f|\le|D|^{1/3} \\ P|f\Rightarrow \deg P\le M}}\frac{\chi_D(f)}{|f|}d_z(f) \\ &= \left(1+O\left(\frac{1}{(\log_q|D|)^B}\right)\right)\sum_{\substack{f \text{ monic} \\ |f|\le|D|^{1/3} \\ P|f\Rightarrow \deg P\le M}}\frac{d_z(f)}{|f|}\sum_{D\in\mathcal{H}_n}\chi_D(f) \\ &= \left(1+O\left(\frac{1}{(\log_q|D|)^B}\right)\right)(S_1+S_2), \end{aligned}$$

where

$$S_1 := \sum_{\substack{f \text{ monic and a square} \\ |f|\le|D|^{1/3} \\ P|f\Rightarrow \deg P\le M}}\frac{d_z(f)}{|f|}\sum_{D\in\mathcal{H}_n}\chi_D(f), \tag{5.23}$$

and

$$S_2 := \sum_{\substack{f \text{ monic and not a square} \\ |f|\le|D|^{1/3} \\ P|f\Rightarrow \deg P\le M}}\frac{d_z(f)}{|f|}\sum_{D\in\mathcal{H}_n}\chi_D(f). \tag{5.24}$$

With this separation we can use our orthogonality relation to evaluate $S_1$ and $S_2$.

### 5.3.1 Evaluating $S_2$: Contribution of the non-square terms.

**Lemma 5.7.** *Let $D \in \mathcal{H}_n$, $A > 4$ be a constant, $z \in \mathbb{C}$ be such that $|z| \leq \frac{\log_q |D|}{10A \log_q \log_q |D| \log\log_q \log_q |D|}$ and $M = A \log_q \log_q |D|$. Then*

$$S_2 \ll |D|^{1/2+(2\log_q 2)/3},$$

*with $S_2$ defined as in (5.24).*

We note that $2(\log_q 2)/3 \leq 0.29$ since $q \geq 5$. That is for all $q$ we can say $S_2 \ll |D|^{4/5}$.

**Remark 5.2.** *Making use of Remark 5.1 and following the argument from [15, page 12] we can improve this bound to*

$$S_2 \ll |D|^{1/2+\epsilon}.$$

*Proof.* By Lemma 5.4 the inner sum of $S_2$ is $O(\sqrt{|\mathcal{H}_n|}2^{\deg f})$, hence we have

$$S_2 \ll \sqrt{|\mathcal{H}_n|} \sum_{\substack{f \text{ monic and not a square} \\ |f| \leq |D|^{1/3} \\ P|f \Rightarrow \deg P \leq M}} \frac{d_z(f) 2^{\deg f}}{|f|}.$$

Now, we have that $|\mathcal{H}_n| = O(|D|)$ and $2^{\deg f} = |f|^{\log_q 2}$ Thus

$$\begin{aligned} S_2 &\ll |D|^{1/2} \sum_{\substack{f \text{ monic and not a square} \\ |f| \leq |D|^{1/3} \\ P|f \Rightarrow \deg P \leq M}} \frac{d_z(f)|f|^{\log_q 2}}{|f|}, \\ &\ll |D|^{1/2+1/3(\log_q 2+(\log_q 2)/2)} \sum_{\substack{f \text{ monic and not a square} \\ P|f \Rightarrow \deg P \leq M}} \frac{d_z(f)}{|f|^{1+(\log_q 2)/2}}, \\ &\ll |D|^{1/2+1/3(\log_q 2+(\log_q 2)/2)} (\zeta_{\mathbb{A}}(1+(\log_q 2)/2))^k, \end{aligned}$$

for some $k \in \mathbb{Z}$ such that $|z| \asymp k$. We note that $\zeta_{\mathbb{A}}(1+(\log_q 2)/2) = c$ for some constant $c$ so that

$$(\zeta_{\mathbb{A}}(1+(\log_q 2)/2))^k \ll c^{\frac{\log_q |D|}{10A \log_q \log_q |D| \log\log_q \log_q |D|}} = |D|^{\frac{\log_q c}{10A \log_q \log_q |D| \log\log_q \log_q |D|}} \ll |D|^{(\log_q 2)/6},$$

for $n$ large enough. Hence we have the desired result.

□

### 5.3.2 Evaluating $S_1$: Contribution of the square terms.

The last step is to understand the main term $S_1$. From Lemma 5.4 we have that

$$S_1 = \left(1 + O\left(\frac{1}{(\log_q |D|)^B}\right)\right)\left(\sum_{\substack{f \text{ monic} \\ |f| \le |D|^{1/3} \\ P|f \Rightarrow \deg P \le M}} \frac{d_z(f^2)}{|f|^2}\left(|\mathcal{H}_n| \cdot \prod_{P|f}\left(1 + \frac{1}{|P|}\right)^{-1} + O\left(\sqrt{|\mathcal{H}_n|}\right)\right)\right).$$

Estimating this term is where the difficulties lie, thus enters the random model $L(1, \mathbb{X})$: Let $\{\mathbb{X}(P)\}$ denote a sequence of independent random variables indexed by $P \in \mathbb{A}$ an irreducible (prime) element, which takes the values $0, \pm 1$ described as (5.7). We will use the following Lemma to help us estimate the main term. Theorem 5.1 follows immediately after combining Lemmas 5.8 and 5.9.

**Lemma 5.8.** *Let $D \in \mathcal{H}_n$. Let $z \in \mathbb{C}$ be such that $|z| \le \frac{\log_q |D|}{260 \log_q \log_q |D| \log \log_q \log_q |D|}$. Then*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D \in \mathcal{H}_n} L(1, \chi_D)^z = \mathbb{E}(L(1, \mathbb{X})^z)\left(1 + O\left(\frac{1}{(\log_q |D|)^{11}}\right)\right) + O\left(|D|^{-\frac{1}{c_0 \log_q \log_q |D|}}\right).$$

The expectation of $\mathbb{X}$, $\mathbb{E}(\mathbb{X}(P))$, is zero and $\mathbb{E}(\mathbb{X}(P)^2) = \frac{|P|}{|P|+1}$. We extend the definition of $\mathbb{X}$ to all monic polynomials $f \in \mathbb{A}$ as in (5.8). Then, since $\mathbb{X}$ is independent on the primes, if $f = P_1^{e_1} P_2^{e_2} \cdots P_s^{e_s}$ we have

$$\mathbb{E}(\mathbb{X}(f)) = \prod_{i=1}^{s} \mathbb{E}(\mathbb{X}(P_i)^{e_i}).$$

We note that $\mathbb{X}(P)^{e_j} = \mathbb{X}(P)$ if $e_j \equiv 1 (\mod 2)$ and $\mathbb{X}(P)^2$ if $e_j \equiv 0 (\mod 2)$. Combining this fact with the independence of $\mathbb{X}$ we see that

$$\mathbb{E}(\mathbb{X}(P)^{e_j}) = \begin{cases} 0 & \text{if } e_j \equiv 1 \pmod 2 \\ \frac{|P|}{|P|+1} & \text{if } e_j \equiv 0 \pmod 2. \end{cases}$$

Thus we have proved:

**Lemma 5.9.**

$$\mathbb{E}(\mathbb{X}(f)) = \begin{cases} 0 & \textit{if } f \textit{ is not a square} \\ \prod_{P|f}\left(1 + \frac{1}{|P|}\right)^{-1} & \textit{if } f \textit{ is a square.} \end{cases}$$

Now, for any $z \in \mathbb{C}$, using the definition of a power of a Dirichlet series we see

$$L(1,\mathbb{X})^z = \sum_{f \text{ monic}} \frac{d_z(f)\mathbb{X}(f)}{|f|}.$$

Then, since $d_z(f)$ and $|f|$ are scalars and expectation is linear we see that

$$\mathbb{E}(L(1,\mathbb{X})^z) = \sum_{f \text{ monic}} \frac{d_z(f)\mathbb{E}(\mathbb{X}(f))}{|f|} = \sum_{f \text{ monic}} \frac{d_z(f^2)}{|f|^2} \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1}, \tag{5.25}$$

where $L(1,\mathbb{X})$ is defined in (5.9). We recognize the shape of $S_1$ from this. On the other hand, from the random Euler product definition we have

$$\mathbb{E}(L(1,\mathbb{X})^z) = \prod_{P \text{ irreducible}} E_P(z),$$

where

$$E_P(z) \quad := \quad \mathbb{E}\left(\left(1 - \frac{\mathbb{X}(P)}{|P|}\right)^{-z}\right) \quad = \quad \frac{1}{|P|+1} \; + \; \frac{|P|}{2(|P|+1)}\left(\left(1 - \frac{1}{|P|}\right)^{-z} + \left(1 + \frac{1}{|P|}\right)^{-z}\right). \tag{5.26}$$

Now, we notice if $\deg P > M$ then we can use the following Taylor expansions

$$\left(1 - \frac{1}{|P|}\right)^{-z} = 1 + \frac{z}{|P|} + O\left(\frac{|z|}{|P|^2}\right),$$

and

$$\left(1 + \frac{1}{|P|}\right)^{-z} = 1 - \frac{z}{|P|} + O\left(\frac{|z|}{|P|^2}\right).$$

That is to say, for $P$ irreducible and $\deg P > M$ we have $E_P(z) = 1 + O(|z|/|P|^2)$, so that

$$\prod_{\substack{P \text{ irreducible} \\ \deg P > M}} E_P(z) = \exp\left(O\left(|z| \sum_{\deg P > M} \frac{1}{|P|^2}\right)\right) = 1 + O\left(\frac{1}{(\log_q |D|)^B}\right),$$

this last equality follows from the relative sizes of $|z|$ and $M$, where we again note that $M = A \log_q \log_q |D|$ and we choose $A = B/2 - 2$ large enough to provide the desired error term above. Finally, we use a similar analysis as in Lemma 5.6 on $\mathbb{E}(L(1,\mathbb{X})^z)$ in order to truncate the sum in a similar fashion as $L(1,\chi_D)^z$ has been. Then we see

$$\mathbb{E}(L(1,\mathbb{X})^z) = \sum_{\substack{f \text{ monic} \\ |f|<|D|^{1/3} \\ P|f \Rightarrow \deg P \le M}} \frac{d_z(f^2)}{|f|^2} \prod_{P|f} \left(1+\frac{1}{|P|}\right)^{-1} \left(1+O\left(\frac{1}{(\log_q |D|)^B}\right)\right) + O\left(|D|^{-\frac{1}{c_0 \log_q \log_q |D|}}\right). \tag{5.27}$$

The above discussion and taking the choice $A = 26$ in Lemma 5.5 gives $B = 11$ which proves Lemma 5.8. Using the fact that $|D| = q^n$ we obtain Theorem 5.1. Corollary 5.1 also follows from this discussion by simply scaling everything appropriately via (5.3) and the fact that expectation is linear. Finally, Corollary 5.3 is obtained in the same way but instead we apply (5.16).

## 5.4 The distribution of values of $L(1,\mathbb{X})$

Here we prove results about $\Phi_{\mathbb{X}}(\tau)$ and $\Psi_{\mathbb{X}}(\tau)$. The proofs of $\Psi_{\mathbb{X}}(\tau)$ require only minor adjustments to those for $\Phi_{\mathbb{X}}(\tau)$. The discussion in this section is modelled after [20, Section 4]. These authors use a saddle point analysis to achieve their results, and we adapt that idea here. To this end, we define some useful auxiliary functions. For $z \in \mathbb{C}$ define

$$\mathcal{L}(z) := \log \mathbb{E}(L(1,\mathbb{X})^z) = \sum_{P \text{ irreducible}} \log(E_P(z)), \tag{5.28}$$

where $E_P(z)$ is defined as in (5.26). Furthermore we consider the equation

$$(\mathbb{E}(L(1,\mathbb{X})^r)(e^\gamma \tau)^{-r})' = 0 \Leftrightarrow \mathcal{L}'(r) = \log(\tau) + \gamma, \tag{5.29}$$

where the derivative is taken with respect to the real variable $r$. It follows from Proposition 5.2 that $\lim_{r\to\infty} \mathcal{L}'(r) = \infty$, and from (5.41) we see that $E''_P(r)E_P(r) > (E'_P(r))^2$ for all monic irreducible polynomials $P$, and thus $\mathcal{L}''(r) > 0$. Therefore (5.29) has a unique solution for each fixed $\tau$: we define $\kappa = \kappa(\tau)$ as this unique solution.

Finally, we define

$$f(t) := \begin{cases} \log\cosh(t) & \text{if } 0 \le t < 1 \\ \log\cosh(t) - t & \text{if } t \ge 1. \end{cases} \tag{5.30}$$

### 5.4.1 Distribution of the Random Model.

**Theorem 5.5.** *Let $\tau$ be large and $\kappa$ denote the unique solution to* (5.29). *Then, we have*

$$\Phi_{\mathbb{X}}(\tau) = \frac{\mathbb{E}(L(1,\mathbb{X})^{\kappa})(e^{\gamma}\tau)^{-\kappa}}{\kappa\sqrt{2\pi\mathcal{L}''(\kappa)}}\left(1 + O\left(\sqrt{\frac{\log_q \kappa}{\kappa}}\right)\right). \tag{5.31}$$

*Moreover, for any $0 \leq \lambda \leq 1/\kappa$ we have*

$$\Phi_{\mathbb{X}}(e^{-\lambda}\tau) = \Phi_{\mathbb{X}}(\tau)(1 + O(\lambda\kappa)). \tag{5.32}$$

We prove Theorem 5.3 from this and the following proposition which gives some estimates on the size of $\mathcal{L}$ and its first few derivatives.

**Proposition 5.2.** *Let $r$ be a real number which is not a power of $q$ and $f$ be defined by* (5.30). *Let $k \in \mathbb{Z}$ be the unique positive integer such that $q^k < r < q^{k+1}$ and let $t := \frac{r}{q^k}$. With this notation in mind for $r$ any real number large enough we have*

$$\mathcal{L}(r) = r\left(\log\log_q r + \gamma\right) + \frac{r}{\log_q r}G_1(t) + O\left(\frac{r\log_q \log_q r}{(\log_q r)^2}\right), \tag{5.33}$$

*where*

$$G_1(t) := \frac{1}{2} - \log_q t + \sum_{l=-\infty}^{\infty}\frac{f(tq^l)}{tq^l}. \tag{5.34}$$

*Furthermore, we have*

$$\mathcal{L}'(r) = \log\log_q r + \gamma + \frac{1}{\log_q r}G_2(t) + O\left(\frac{\log_q \log_q r}{(\log_q r)^2}\right), \tag{5.35}$$

*where*

$$G_2(t) := \frac{1}{2} - \log_q t + \sum_{l=-\infty}^{\infty} f'(tq^l). \tag{5.36}$$

*Moreover, for all real numbers $y$ , $x$ such that $|y| \geq c_q$ and for all $x$ such that $|y| \leq |x|$ we have*

$$\mathcal{L}''(y) \asymp \frac{1}{|y|\log|y|} \text{ and } \mathcal{L}'''(y+ix) \ll \frac{1}{|y|^2\log|y|}. \tag{5.37}$$

Combining these results gives Theorem 5.3:

*Proof of Theorem 5.3.* By Theorem 5.5 and (5.35) we have

$$\begin{aligned}\Phi_{\mathbb{X}}(\tau) &= \frac{\mathbb{E}(L(1,\mathbb{X})^{\kappa})(e^{\gamma}\tau)^{-\kappa}}{\kappa\sqrt{2\pi\mathcal{L}''(\kappa)}}\left(1+O\left(\sqrt{\frac{\log_q \kappa}{\kappa}}\right)\right)\\ &= \exp\left(\mathcal{L}(\kappa)-\kappa(\log\tau+\gamma)+O(\log_q \kappa)\right),\end{aligned}$$

where $\kappa$ is the unique solution which satisfies (5.29).

Also from (5.35) we have

$$\log\tau = \log\log_q \kappa + \frac{G_2(q^{\{\log_q \kappa\}})}{\log_q \kappa} + O\left(\frac{\log_q \log_q \kappa}{(\log_q \kappa)^2}\right). \tag{5.38}$$

Hence using (5.33) we obtain

$$\Phi_{\mathbb{X}}(\tau) = \exp\left(\kappa\frac{G_1(q^{\{\log_q \kappa\}})-G_2(q^{\{\log_q \kappa\}})}{\log_q \kappa} + O\left(\frac{\kappa\log_q\log_q \kappa}{(\log_q \kappa)^2}\right)\right).$$

We note that from (5.38) we have $\kappa \asymp q^{\tau}$ and thus

$$\log_q \kappa = \tau - G_2(q^{\{\log_q \kappa\}}) + O\left(\frac{\log_q \tau}{\tau}\right),$$

since for every $\tau$ there is a unique $\kappa$ which satisfies (5.29) and $G_2(q^{\{\log_q \kappa\}})$ is bounded for any $\kappa$. This is enough to obtain the shape of the result. It remains to prove that

$$-\frac{1}{\log q}+\frac{\log\cosh c}{c}-\tanh c < G_1(q^{\{\log_q \kappa\}})-G_2(q^{\{\log_q \kappa\}}) < \frac{\log(\cosh(q))}{q}-\tanh(q),$$

where $c = 1.28377...$ For ease of notation let $t = q^{\{\log_q \kappa\}}$, we note that $1 \le t < q$ and

$$G_1(t)-G_2(t) = \sum_{l=-\infty}^{\infty}\left(\frac{f(tq^l)}{tq^l}-f'(tq^l)\right).$$

We recall from the definition of $f$ that the shape is different depending on the size of the input. So we split the sum:

$$G_1(t) - G_2(t) = \sum_{l < -\log_q t} \left( \frac{\log\cosh(tq^l)}{tq^l} - \tanh(tq^l) \right) + \sum_{l \geq -\log_q t} \left( \frac{\log\cosh(tq^l) - tq^l}{tq^l} - (\tanh(tq^l) - 1) \right)$$
$$= \sum_{l=-\infty}^{\infty} \left( \frac{\log\cosh(tq^l)}{tq^l} - \tanh(tq^l) \right).$$

To prove the upper bound, it is enough to show that all the summands are negative and so the sum will be less than the contribution from the $l = 0$ term. We note that

$$\frac{d}{dy}\left[\frac{\log\cosh y}{y} - \tanh y\right] = \frac{\tanh y}{y} - \frac{\log\cosh y}{y^2} - \operatorname{sech}^2 y = 0$$

when $y = \pm 1.28377\ldots$ but the argument of our function is $tq^l > 0$ for all $l$ since $t, q > 0$, so we need only consider $c = 1.28377\ldots$. A simple calculation shows that this is a minimum and that $\frac{\log\cosh y}{y} - \tanh y$ is strictly decreasing on the interval $(0, c)$ and strictly increasing on the interval $(c, \infty)$. Taking the limit as $y \to 0$ and $y \to \infty$ we see these are both 0, hence all of the summands are negative since $\frac{\log\cosh c}{c} - \tanh c = -0.339834\ldots$. Therefore we have a suitable upper bound by simply evaluating $\frac{\log\cosh tq^l}{tq^l} - \tanh tq^l$ at $l = 0$, which gives $\frac{\log\cosh t}{t} - \tanh t$. As we discussed this function reaches its maximum value when $t$ does, in this case $t < q$.

In order to consider the lower bound, we compute first the integral associated to the sum.

$$\int_{-\infty}^{\infty} \frac{\log\cosh tq^y}{tq^y} - \tanh tq^y dy = \lim_{Y\to\infty} \int_{-Y}^{Y} \frac{\log\cosh tq^y}{tq^y} - \tanh tq^y dy$$
$$= \lim_{Y\to\infty} \int_{-Y}^{Y} \frac{\log\cosh tq^y}{tq^y} dy - \int_{-Y}^{Y} \tanh tq^y dy.$$

We begin with an integration by parts of the first integrand, taking $u = -\frac{\log\cosh(tq^y)}{t\log q}$ and $dv = \frac{-\log q}{q^y}$, so that

$$\int_{-Y}^{Y} \frac{\log\cosh tq^y}{tq^y} dy = -\frac{\log\cosh(tq^y)}{tq^y \log q}\Big|_{-Y}^{Y} + \int_{-Y}^{Y} \tanh tq^y dy,$$

and

$$\int_{-\infty}^{\infty} \frac{\log\cosh tq^y}{tq^y} - \tanh tq^y dy = \lim_{Y\to\infty} -\frac{\log\cosh(tq^y)}{tq^y\log q}\Big|_{-Y}^{Y} = -\frac{1}{\log q}.$$

Thus we conclude

$$-\frac{1}{\log q} \leq \sum_{l=-\infty}^{\infty} \left( \frac{\log\cosh tq^l}{tq^l} - \tanh tq^l \right) - \left( \frac{\log\cosh t}{t} - \tanh t \right).$$

and therefore,

$$\sum_{l=-\infty}^{\infty} \frac{\log\cosh tq^l}{tq^l} - \tanh tq^l \geq -\frac{1}{\log q} + \frac{\log\cosh t}{t} - \tanh t.$$

Finally, as we discussed this is minimized when $t = c$. □

### 5.4.2 Tools for proving Proposition 5.2

First we recall the following standard estimates on $f$ and $f'$.

**Lemma 5.10.** *[63, Lemma 4.5] $f$ is bounded on $[0, \infty)$ and $f(t) = t^2/2 + O(t^4)$ if $0 \leq t < 1$. Moreover, we have*

$$f'(t) = \begin{cases} t + O(t^2) & \text{if } 0 < t < 1 \\ O(e^{-2t}) & \text{if } t \geq 1. \end{cases}$$

**Lemma 5.11.** *Let $r \geq c_q$ be a real number, where $c_q$ is a positive constant depending on $q$. Then we have*

$$\log E_P(r) = \begin{cases} -r\log(1 - 1/|P|) + O(1) & \text{if } |P| \leq r^{2/3}, \\ \log\cosh\left(\frac{r}{|P|}\right) + O\left(\frac{r}{|P|^2}\right) & \text{if } |P| > r^{2/3}, \end{cases} \tag{5.39}$$

*and*

$$\frac{E_P'}{E_P}(r) = \begin{cases} -\log\left(1 - \frac{1}{|P|}\right)\left(1 + O\left(e^{-r^{1/3}}\right)\right) & \text{if } |P| \leq r^{2/3} \\ \frac{1}{|P|}\tanh\left(\frac{r}{|P|}\right) + O\left(\frac{1}{|P|^2} + \frac{r}{|P|^3}\right) & \text{if } |P| > r^{2/3}. \end{cases} \tag{5.40}$$

*Furthermore, we have that*

$$E_P''(r)E_P(r) > (E_P'(r))^2. \tag{5.41}$$

*Proof.* First we prove (5.39). Start by considering $|P| \leq r^{2/3}$. Since $|P|$ is small we have

$$E_P(r) = \frac{|P|}{2(|P|+1)}\left(1-\frac{1}{|P|}\right)^{-r}\left(1+\left(1+\frac{2}{|P|-1}\right)^{-r}+\frac{2}{|P|}\left(1-\frac{1}{|P|}\right)^{r}\right)$$
$$= \frac{|P|}{2(|P|+1)}\left(1-\frac{1}{|P|}\right)^{-r}\left(1+O(\exp(-r/(r^{2/3}-1)))\right), \quad (5.42)$$

where the bound in the big $O$ comes from taylor expansion. The final result is obtained by taking logs.

Suppose now that $|P| > r^{2/3}$, we see that

$$\left(1-\frac{1}{|P|}\right)^{-r} = e^{r/|P|}\left(1+O\left(\frac{r}{|P|^2}\right)\right) \text{ and } \left(1+\frac{1}{|P|}\right)^{-r} = e^{-r/|P|}\left(1+O\left(\frac{r}{|P|^2}\right)\right),$$

thus using the bounds $\cosh(t)-1 \ll t\cosh(t)$ and $\sinh(t) \ll t\cosh(t)$, which are valid for all $t \geq 0$ we see that

$$E_P(r) = \frac{|P|}{|P|+1}\cosh\left(\frac{r}{|P|}\right)\left(1+O\left(\frac{r}{|P|^2}\right)\right)+\frac{1}{|P|}$$
$$= \cosh\left(\frac{r}{|P|}\right)\left(1+O\left(\frac{r}{|P|^2}\right)\right). \quad (5.43)$$

Taking logs completes the proof.

For (5.40), we first see from (5.26) that

$$E_P'(r) = \frac{-|P|}{2(|P|+1)}\left(\left(1-\frac{1}{|P|}\right)^{-r}\log\left(1-\frac{1}{|P|}\right)+\left(1+\frac{1}{|P|}\right)^{-r}\log\left(1+\frac{1}{|P|}\right)\right).$$

If $|P| \leq r^{2/3}$ then (5.42) finishes the claim.

On the other hand for $|P| > r^{2/3}$ we have

$$E_P'(r) = \frac{|P|}{2(|P|+1)}\left(\frac{e^{r/|P|}}{|P|}-\frac{e^{-r/|P|}}{|P|}\right)\left(1+O\left(\frac{1}{|P|}+\frac{r}{|P^2|}\right)\right)$$
$$= \frac{1}{|P|}\sinh\left(\frac{r}{|P|}\right)\left(1+O\left(\frac{1}{|P|^2}+\frac{r}{|P|^3}\right)\right)+O\left(\frac{1}{|P|^2}\cosh\left(\frac{r}{|P|}\right)\right).$$

Combining this with (5.43) we have the desired result.

For (5.41) we first compute $E''_P(r)$, $E''_P(r)E_P(r)$ and $(E'_P(r))^2$:

$$(E'_P(r))^2 = \frac{|P|^2}{4(|P|+1)^2}\left(\left(1-\frac{1}{|P|}\right)^{-2r}\left(\log\left(1-\frac{1}{|P|}\right)\right)^2 + \left(1+\frac{1}{|P|}\right)^{-2r}\left(\log\left(1+\frac{1}{|P|}\right)\right)^2 \right.$$
$$\left. +2\log\left(1-\frac{1}{|P|}\right)\log\left(1+\frac{1}{|P|}\right)\left(1-\frac{1}{|P|^2}\right)^{-r}\right), \quad (5.44)$$

$$E''_P(r) \quad = \quad \frac{|P|}{2(|P|+1)}\left(\left(1-\frac{1}{|P|}\right)^{-r}\left(\log\left(1-\frac{1}{|P|}\right)\right)^2 + \left(1+\frac{1}{|P|}\right)^{-r}\left(\log\left(1+\frac{1}{|P|}\right)\right)^2\right),$$

and

$$E''_P(r)E_P(r) = \frac{1}{|P|+1}E''_P(r) + \frac{|P|^2}{4(|P|+1)^2}\left(\left(1-\frac{1}{|P|}\right)^{-2r}\left(\log\left(1-\frac{1}{|P|}\right)\right)^2\right.$$
$$\left. + \left(1+\frac{1}{|P|}\right)^{-2r}\left(\log\left(1+\frac{1}{|P|}\right)\right)^2 + \left(\left(\log\left(1-\frac{1}{|P|}\right)\right)^2 + \left(\log\left(1+\frac{1}{|P|}\right)\right)^2\right)\left(1-\frac{1}{|P|^2}\right)^{-r}\right). \quad (5.45)$$

Taking the difference (5.45) – (5.44) we obtain

$$\frac{1}{|P|+1}E''_P(r) + \frac{|P|^2}{4(|P|+1)^2}\left(1-\frac{1}{|P|^2}\right)^{-r}\left(\log\left(1-\frac{1}{|P|}\right) - \log\left(1+\frac{1}{|P|}\right)\right)^2 > 0,$$

as desired. □

*Proof of Proposition 5.2.* We only write the details for (5.33) and (5.35) as the argument for (5.37) follows along the same lines. For the entire proof, we recall that $k \in \mathbb{Z}$ is the unique positive integer such that $q^k \le r < q^{k+1}$ and let $t := \frac{r}{q^k}$.

We first prove the result for $\mathcal{L}(r)$. By Lemma 5.11 and Lemma 5.10 we have

$$\mathcal{L}(r) = -r\sum_{|P|\le r^{2/3}}\log\left(1-\frac{1}{|P|}\right) + \sum_{|P|>r^{2/3}}\log\cosh\left(\frac{r}{|P|}\right) + O(r^{2/3})$$
$$= -r\sum_{\deg P\le k}\log\left(1-\frac{1}{|P|}\right) + \sum_{|P|>r^{2/3}} f\left(\frac{r}{|P|}\right) + O(r^{2/3}). \quad (5.46)$$

The first summand is taken care of by recognizing Mertens' theorem, which we will apply at the end. The more interesting part of the proof comes from the second sum. First, from the prime number theorem we get

$$\sum_{|P|>r^{2/3}} f\left(\frac{r}{|P|}\right) = \sum_{n>\frac{2}{3}\log_q r} \frac{q^n}{n} f\left(\frac{r}{q^n}\right) + O\left(\sum_{n>\frac{2}{3}\log_q r} \frac{q^{n/2}}{n} f\left(\frac{r}{q^n}\right)\right).$$

The error term is

$$\begin{aligned}\sum_{n>\frac{2}{3}\log_q r} \frac{q^{n/2}}{n} f\left(\frac{r}{q^n}\right) &\ll \sum_{n\geq \log_q r} q^{n/2}\frac{r^2}{q^{2n}} + \sum_{\frac{2}{3}\log_q r<n<\log_q r} q^{n/2} \text{ by Lemma 5.10,}\\ &\ll \sqrt{r}.\end{aligned}$$

It remains to consider

$$\begin{aligned}\sum_{n>\frac{2}{3}\log_q r} \frac{q^n}{n} f\left(\frac{r}{q^n}\right) &= \sum_{n>k+\log_q k} + \sum_{n<k-\log_q k} + \sum_{k-\log_q k\leq n\leq k+\log_q k} \left(\frac{q^n}{n} f\left(\frac{r}{q^n}\right)\right)\\ &= T_1 + T_2 + T_3.\end{aligned}$$

We first bound $T_1$ and $T_2$, again referring to Lemma 5.10

$$\begin{aligned}T_1 &\ll \frac{1}{k}\sum_{n>k+\log_q k} \frac{q^n r^2}{q^{2n}} \ll \frac{r^2}{k}\sum_{n>k+\log_q k}\frac{1}{q^n}\\ &\ll \frac{r^2}{k}\frac{1}{q^{k+\log_q k}} \ll \frac{r}{(\log_q r)^2} \text{ by the choice of } k \text{ with respect to } r,\end{aligned}$$

and similarly

$$T_2 \ll \sum_{n<k-\log_q k} \frac{q^n}{n} \ll \frac{q^{k-\log_q k}}{k} \ll \frac{r}{(\log_q r)^2}.$$

For $T_3$ we notice that for $n \in [k-\log_q k, k+\log_q k]$, we have $\frac{1}{n} = \frac{1}{k}\left(1+O\left(\frac{\log k}{k}\right)\right)$. Using this, we factor out $\frac{q^k}{k}$ and do the variable change $l = k - n$ so that

$$T_3 = \left(\frac{q^k}{k} + O\left(\frac{q^k \log k}{k^2}\right)\right) \sum_{|l| \leq \log_q k} \frac{f(tq^l)}{q^l},$$

where we recall $t := \frac{r}{q^k}$. Next, we see that for $|l| > \log_q k$ the sum is small:

$$\sum_{|l| > \log_q k} \frac{f(tq^l)}{q^l} \ll \sum_{l > \log_q k} \frac{1}{q^l} + \sum_{l < -\log_q k} q^l \ll \frac{1}{q^{\log_q k}} \ll \frac{1}{k}.$$

Hence we have

$$T_3 = \frac{q^k}{k} \left(\sum_{l=-\infty}^{\infty} \frac{f(tq^l)}{q^l} + O\left(\frac{\log k}{k}\right)\right).$$

Returning this to an expression in terms of $r$ we see

$$T_3 = \frac{r}{\log_q r} \left(\sum_{l=-\infty}^{\infty} \frac{f(tq^l)}{tq^l} + O\left(\frac{\log \log r}{\log r}\right)\right).$$

Finally, we complete the bound of $\mathcal{L}(r)$ by applying Mertens' theorem to the first summand and convert everything in terms of $r$. Combining the terms which have $\frac{r}{\log_q r}$ in common we achieve the claimed result.

For $\mathcal{L}'(r)$, we again appeal to Lemma 5.11 and Lemma 5.10 giving

$$\begin{aligned}
\mathcal{L}'(r) &= -\sum_{|P| \leq r^{2/3}} \log\left(1 - \frac{1}{|P|}\right) + \sum_{|P| > r^{2/3}} \frac{\tanh\left(\frac{r}{|P|}\right)}{|P|} + O(r^{-1/3}) \\
&= -\sum_{\deg P \leq k} \log\left(1 - \frac{1}{|P|}\right) + \sum_{|P| > r^{2/3}} \frac{f'\left(\frac{r}{|P|}\right)}{|P|} + O(r^{-1/3}).
\end{aligned} \tag{5.47}$$

Applying the prime number theorem to the second sum we obtain

$$\sum_{|P| > r^{2/3}} \frac{f'\left(\frac{r}{|P|}\right)}{|P|} = \sum_{n > 2/3 \log_q r} \frac{f'(r/q^n)}{n} + O\left(\sum_{n > 2/3 \log_q r} \frac{f'(r/q^n)}{q^{n/2} n}\right).$$

The error term in this case is

$$\sum_{n>2/3\log_q r} \frac{f'(r/q^n)}{q^{n/2}n} \ll \sum_{n\geq \log_q r} \frac{r}{q^{3n/2}n} + \sum_{2/3\log_q r<n<\log_q r} \frac{e^{-2r/q^n}}{q^{n/2}n} \text{ by Lemma 5.10}$$
$$\ll \frac{1}{r^{1/3}\log_q r}.$$

As before, we split the remaining sum into 3 pieces:

$$\sum_{n>2/3\log_q r} \frac{f'(r/q^n)}{n} = \sum_{n>k+\log_q k} + \sum_{n<k-\log_q k} + \sum_{k-\log_q k\leq n\leq k+\log_q k} \left(f'\left(\frac{r}{q^n}\right)\frac{1}{n}\right)$$
$$= T_1' + T_2' + T_3'.$$

We first bound $T_1'$ and $T_2'$, referring to Lemma 5.10 gives

$$T_1' \ll \sum_{n>k+\log_q k} \frac{r}{q^n n} \ll \frac{r}{k}\sum_{n>k+\log_q k}\frac{1}{q^n}$$
$$\ll \frac{r}{k}\frac{1}{q^{k+\log_q k}} \ll \frac{1}{(\log_q r)^2} \text{by the choice of } k \text{ with respect to } r.$$

Similarly

$$T_2' \ll \sum_{n<k-\log_q k} \frac{e^{-2r/q^n}}{n} \ll \frac{1}{(\log_q r)^2}.$$

For $T_3'$ we notice that for $n \in [k-\log_q k, k+\log_q k]$, we have $\frac{1}{n} = \frac{1}{k}\left(1+O\left(\frac{\log k}{k}\right)\right)$. Using this, we factor out $\frac{q^k}{k}$ and do the variable change $l = k-n$ and recall $t := \frac{r}{q^k}$, so that

$$T_3' \ll \left(\frac{1}{k} + O\left(\frac{\log k}{k^2}\right)\right)\sum_{|l|<\log_q k} f'(tq^l).$$

Next, we show this sum is small for $|l| > \log_q k$:

$$\sum_{|l|>\log_q k} f'(tq^l) \ll \sum_{l>\log_q k} e^{-2tq^l} + \sum_{l<-\log_q k} tq^l$$
$$\ll e^{-2\log_q r} + \frac{1}{q^{\log_q k}}, \text{ the bound on the second sum follows from } 1\leq t\leq q$$
$$\ll \frac{1}{\log_q r}.$$

Hence we have

$$T_3' \ll \frac{1}{k}\left(\sum_{l=-\infty}^{\infty} f'(tq^l) + O\left(\frac{\log k}{k}\right)\right).$$

Finally, we complete the bound of $\mathcal{L}'(r)$ by applying Mertens' theorem to the first summand and convert everything in terms of $r$. Combining the terms which have $\frac{1}{\log_q r}$ in common we achieve the claimed result.

□

#### 5.4.2.1 Proof of Theorem 5.5

One of the key ingredients in the proof of Theorem 5.5 is to show that $|\mathbb{E}(L(1,\mathbb{X})^{r+it})|/\mathbb{E}(L(1,\mathbb{X})^r)$ is rapidly decreasing in $t$ when $|t| \geq \sqrt{r\log r}$. For this we prove the following lemmas.

**Lemma 5.12.** *Let $r$ is a large positive number and $c_q \geq q$ a positive constant depending on $q$. If $|P| > \frac{r}{c_q}$, then for some positive constant $b_1$ we have*

$$\frac{|E_P(r+it)|}{E_P(r)} \leq \exp\left(-b_1\left(1-\cos\left(t\log\left(\frac{|P|+1}{|P|-1}\right)\right)\right)\right),$$

*where $c_q$ is a positive constant dependent on $q$.*

*Proof.* Let $x_1$, $x_2$ and $x_3$ be positive real numbers and $\theta_2$ and $\theta_3$ be real numbers. We use the following inequality established in the proof of [40, Lemma 3.2]:

$$|x_1 + x_2e^{i\theta_2} + x_3e^{i\theta_3}| \leq (x_1+x_2+x_3)\exp\left(-\frac{x_1x_3(1-\cos\theta_3)}{(x_1+x_2+x_3)^2}\right).$$

Choosing $x_1 = \frac{|P|}{2(|P|+1)}(1+1/|P|)^{-r}$, $x_2 = \frac{1}{|P|+1}$ and $x_3 = \frac{|P|}{2(|P|+1)}(1-1/|P|)^{-r}$ with $\theta_2 = t\log(1+1/|P|)$ and $\theta_3 = t\log\left(\frac{|P|+1}{|P|-1}\right)$ provides the desired result since $|P| > \frac{r}{c_q}$. □

**Lemma 5.13.** *Let $r$ be large and let $c_q \geq q > 4$ be a positive constant dependent on $q$. Then there exists a constant $b_2 > 0$ such that*

$$\frac{|\mathbb{E}(L(1,\mathbb{X})^{r+it})|}{\mathbb{E}(L(1,\mathbb{X})^r)} \ll \begin{cases} \exp\left(-b_2\frac{t^2}{r\log r}\right) & \text{if } |t| \leq \frac{r}{c_q} \\ \exp\left(-b_2\frac{|t|}{\log|t|}\right) & \text{if } |t| > \frac{r}{c_q}. \end{cases}$$

*Proof.* Let $z = r + it$. Since $|E_P(z)| \le E_P(r)$ we obtain for any real numbers $q \le y_1 < y_2$

$$\frac{|\mathbb{E}(L(1,\mathbb{X})^z)|}{\mathbb{E}(L(1,\mathbb{X})^r)} \le \prod_{y_1 \le |P| \le y_2} \frac{|E_P(z)|}{E_P(r)}. \tag{5.48}$$

Note that $|t| \log\left(\frac{|P|+1}{|P|-1}\right) \sim 2|t|/|P|$ so that when $|t| \le \frac{|P|}{c_q}$ we have

$$1 - \cos\left(|t| \log\left(\frac{|P|+1}{|P|-1}\right)\right) \gg \frac{|t|^2}{|P|^2}.$$

If $|t| \le \frac{r}{c_q}$ then, we choose $y_1 = r$ and $y_2 = c_q r/2$. Appealing to Lemma 5.12 we have

$$\begin{aligned}\prod_{y_1 \le |P| \le y_2} \frac{|E_P(z)|}{E_P(r)} &\ll \prod_{\log r \le d \le \log(c_q r/2)+1} \exp\left(-b_1 \frac{q^d}{2d} \frac{|t|^2}{q^{2d}}\right) \\ &= \exp\left(-\frac{b_1 |t|^2}{2} \sum_{\log r \le d \le \log(c_q r/2)+1} \frac{1}{dq^d}\right) \ll \exp\left(-b_2 \frac{|t|^2}{r \log r}\right).\end{aligned}$$

In the case of $|t| > \frac{r}{c_q}$ we use a similar argument but choose $y_1 = c_q|t|$ and $y_2 = 2c_q|t|$ to complete the result. $\square$

Let $\varphi(y) = 1$ if $y > 1$ and equal to 0 otherwise. Then we have the following smooth analogue of Perron's formula:

**Lemma 5.14.** *[20, Lemma 4.7] Let $\lambda > 0$ be a real number and $N$ be a positive integer. For any $c > 0$ we have for $y > 0$*

$$0 \le \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left(\frac{e^{\lambda s} - 1}{\lambda s}\right)^N \frac{ds}{s} - \varphi(y) \le \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left(\frac{e^{\lambda s} - 1}{\lambda s}\right)^N \frac{1 - e^{-\lambda N s}}{s} ds, \tag{5.49}$$

*and*

$$0 \le \varphi(e^\lambda y) - \varphi(y) \le \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left(\frac{e^{\lambda s} - 1}{\lambda s}\right) \frac{e^{\lambda s} - e^{-\lambda s}}{s} ds. \tag{5.50}$$

*Proof of Theorem 5.5.* We first prove (5.31). Let $0 < \lambda < 1/(2\kappa)$ be a real number which we choose later.

Using (5.49) from Lemma 5.14, taking $N = 1$ we obtain

$$0 \le \int_{\kappa-i\infty}^{\kappa+i\infty} \mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{e^{\lambda s}-1}{\lambda s}\frac{ds}{s} - \Phi_{\mathbb{X}}(\tau)$$
$$\le \int_{\kappa-i\infty}^{\kappa+i\infty} \mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s}\frac{(1-e^{-\lambda s})}{s}ds. \tag{5.51}$$

Since $\lambda\kappa < 1/2$ we have $|e^{\lambda s}-1| \le 3$ and $|e^{-\lambda s}-1| \le 2$. Hence, using Lemma 5.13 along with the fact that $|\mathbb{E}(L(1,\mathbb{X})^s)| \le \mathbb{E}(L(1,\mathbb{X})^\kappa)$ we obtain, for some constant $b_3 > 0$ that

$$\int_{\kappa-i\infty}^{\kappa-i\kappa^{3/5}} + \int_{\kappa+i\kappa^{3/5}}^{\kappa+i\infty} \mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{e^{\lambda s}-1}{\lambda s}\frac{ds}{s} \ll \frac{e^{-b_3\kappa^{1/6}}}{\lambda\kappa^{3/5}}\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma\tau)^{-\kappa}, \tag{5.52}$$

and similarly,

$$\int_{\kappa-i\infty}^{\kappa-i\kappa^{3/5}} + \int_{\kappa+i\kappa^{3/5}}^{\kappa+i\infty} \mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s}\frac{(1-e^{-\lambda s})}{s}ds \ll \frac{e^{-b_3\kappa^{1/6}}}{\lambda\kappa^{3/5}}\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma\tau)^{-\kappa}. \tag{5.53}$$

Let $s = \kappa + it$. If $|t| \le \kappa^{3/5}$ then $|(e^{\lambda s}-1)(1-e^{-\lambda s})| \ll \lambda^2|s|^2$, hence the remaining part of the integral is bounded as follows

$$\int_{\kappa-i\kappa^{3/5}}^{\kappa+i\kappa^{3/5}} \mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s}\frac{(1-e^{-\lambda s})}{s}ds \ll \lambda\kappa^{3/5}\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma\tau)^{-\kappa}.$$

Combining this estimate with (5.51), (5.52) and (5.53) we obtain

$$\Phi_{\mathbb{X}}(\tau) - \frac{1}{2\pi i}\int_{\kappa-i\kappa^{3/5}}^{\kappa+i\kappa^{3/5}} \mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s^2}ds$$
$$\ll \left(\lambda\kappa^{3/5} + \frac{e^{-b_3\kappa^{1/6}}}{\lambda\kappa^{3/5}}\right)\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma\tau)^{-\kappa}. \tag{5.54}$$

On the other hand we have from (5.37) when $|t| \le \kappa^{3/5}$ then

$$\mathcal{L}(\kappa+it) = \mathcal{L}(\kappa) + it\mathcal{L}'(\kappa) - \frac{t^2}{2}\mathcal{L}''(\kappa) + O\left(\frac{|t|^3}{\kappa^2\log\kappa}\right).$$

We also note that

$$\frac{e^{\lambda s}-1}{\lambda s^2} = \frac{1}{s}(1+O(\kappa)) = \frac{1}{\kappa}\left(1 - i\frac{t}{\kappa} + O\left(\lambda\kappa + \frac{t^2}{\kappa^2}\right)\right).$$

Hence, using the fact that $\mathbb{E}(L(1,\mathbb{X})^s) = \exp(\mathcal{L}(s))$ and $\mathcal{L}'(\kappa) = \log\tau + \gamma$ we find

$$\begin{aligned}&\mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s^2}\\&= \frac{1}{\kappa}\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma\tau)^{-\kappa}\exp\left(-\frac{t^2}{2}\mathcal{L}''(\kappa)\right)\left(1 - i\frac{t}{\kappa} + O\left(\lambda\kappa + \frac{t^2}{\kappa^2} + \frac{|t|^3}{\kappa^2\log\kappa}\right)\right).\end{aligned}$$

Thus, since we have chosen $\kappa$ such that the integral involving $it/\kappa$ vanishes we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\kappa-i\kappa^{3/5}}^{\kappa+i\kappa^{3/5}}\mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s^2}ds\\&= \frac{1}{\kappa}\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma\tau)^{-\kappa}\frac{1}{2\pi}\int_{-\kappa^{3/5}}^{\kappa^{3/5}}\exp\left(-\frac{t^2}{2}\mathcal{L}''(\kappa)\right)\left(1 + O\left(\lambda\kappa + \frac{t^2}{\kappa^2} + \frac{|t|^3}{\kappa^2\log\kappa}\right)\right)dt. \quad (5.55)\end{aligned}$$

Further, from (5.37) we have $\mathcal{L}''(\kappa) \asymp 1/(\kappa\log\kappa)$, so there exists a positive constant $b_4$ such that

$$\frac{1}{2\pi}\int_{-\kappa^{3/5}}^{\kappa^{3/5}}\exp\left(-\frac{t^2}{2}\mathcal{L}''(\kappa)\right)dt = \frac{1}{\sqrt{2\pi\mathcal{L}''(\kappa)}}\left(1 + O\left(e^{-b_4\kappa^{1/6}}\right)\right),$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_{-\kappa^{3/5}}^{\kappa^{3/5}}|t|^n\exp\left(-\frac{t^2}{2}\mathcal{L}''(\kappa)\right)dt &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}|t|^n\exp\left(-\frac{t^2}{2}\mathcal{L}''(\kappa)\right)dt\\&\ll \frac{1}{(\mathcal{L}''(\kappa))^{(n+1)/2}} \ll \frac{(\kappa\log\kappa)^{n/2}}{\sqrt{2\pi\mathcal{L}''(\kappa)}}.\end{aligned}$$

Inserting these estimates into (5.55) we get

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa^{3/5}}^{\kappa+i\kappa^{3/5}}\mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s^2}ds = \frac{\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma\tau)^{-\kappa}}{\kappa\sqrt{2\pi\mathcal{L}''(\kappa)}}\left(1 + O\left(\lambda\kappa + \sqrt{\frac{\log\kappa}{\kappa}}\right)\right). \quad (5.56)$$

Finally, combining the estimates (5.54), (5.56) and choosing $\lambda = \kappa^{-2}$ we obtain the desired result.

Next we prove (5.32). To do this let $0 \le \lambda \le 1/\kappa$. Using (5.50) from Lemma 5.14, we have

$$\varphi(e^{-\lambda}\tau) - \varphi(\tau) \le \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma\tau)^{-s}\frac{(e^{\lambda s}-1)}{\lambda s}\frac{e^{\lambda s}-e^{-\lambda s}}{s}ds.$$

We write $s = \kappa + it$ and split this integral into two pieces: $|t| \le \lambda\sqrt{\kappa\log\kappa}$ and $|t| > \lambda\sqrt{\kappa\log\kappa}$.

We note that both $|(e^{\lambda s}-1)/\lambda s|$ and $|(e^{\lambda s}-e^{-\lambda s})/\lambda s|$ are less than 4. Therefore, it follows that the first part of the integral contributes $\ll \lambda\sqrt{\kappa\log\kappa}\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\lambda\tau)^{-\kappa}$. Then, from Lemma 5.13 the second portion contributes

$$\ll \lambda \mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\lambda \tau)^{-\kappa} \left( \int_{\sqrt{\kappa \log \kappa} < |t| \le \frac{\kappa}{c_q}} e^{-b_2 t^2/(\kappa \log \kappa)} + \int_{|t| \ge \frac{\kappa}{c_q}} e^{-b_2 |t|/(\log |t|)} \right)$$
$$\ll \lambda \sqrt{\kappa \log \kappa} \mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\lambda \tau)^{-\kappa}.$$

The final result follows from (5.31) and (5.37), specifically they prove:

$$\Phi_{\mathbb{X}}(\tau) \asymp \frac{\mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma \tau)^{-\kappa}}{\kappa \sqrt{\mathcal{L}''(\kappa)}} \asymp \sqrt{\frac{\log \kappa}{\kappa}} \mathbb{E}(L(1,\mathbb{X})^\kappa)(e^\gamma \tau)^{-\kappa}. \tag{5.57}$$

□

## 5.5 Proofs of Theorem 5.2 and Corollary 5.2

We begin with some notation: Let

$$\mathbb{P}(L(1,\chi_D) > e^\gamma \tau) := \frac{1}{|\mathcal{H}_n|} |\{D \in \mathcal{H}_n : L(1,\chi_D) > e^\gamma \tau\}|$$

and

$$M(z) := \frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} L(1,\chi_D)^z.$$

*Proof of Theorem 5.2.* As in section 5.4.1 let $\kappa = \kappa(\tau)$ be the unique solution to (5.29). Let $N$ be a positive integer and $0 < \lambda < \min\{1/(2\kappa), 1/N\}$ be a real value which we choose later. Finally, let $Y = b \log_q |D| / (\log_q \log_q |D| \log_q \log_q \log_q |D|)$ for some $b > 0$ small enough.

If $\log_q |D|$ is large enough, then for our range of $\tau$ we have $\kappa \le Y$, which follows from (5.38). Additionally, this means Lemma 5.8 holds for all $s = \kappa + it$ as long as $|t| \le Y$ so we consider the following integrals:

$$J(\tau) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \mathbb{E}(L(1,\mathbb{X})^s)(e^\gamma \tau)^{-s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s},$$

and

$$J_M(\tau) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} M(s)(e^\gamma \tau)^{-s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s}.$$

By Lemma 5.14 we see that

$$\Phi_{\mathbb{X}}(\tau) \le J(\tau) \le \Phi_{\mathbb{X}}(e^{-\lambda N}\tau) \tag{5.58}$$

and

$$\mathbb{P}(L(1,\chi_D) > e^{\gamma}\tau) \le J_M(\tau) \le \mathbb{P}(L(1,\chi_D) > e^{\gamma-\lambda N}\tau). \tag{5.59}$$

Using that $|e^{\lambda s} - 1| \le 3$ we have

$$\int_{\kappa-i\infty}^{\kappa-iY} + \int_{\kappa+iY}^{\kappa+i\infty} \mathbb{E}(L(1,\mathbb{X})^s)(e^{\gamma}\tau)^{-s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s} \ll \frac{1}{N}\left(\frac{3}{\lambda Y}\right)^N \mathbb{E}(L(1,\mathbb{X})^{\kappa})(e^{\gamma}\tau)^{-\kappa},$$

and similarly, together with Lemma 5.8 we obtain

$$\begin{aligned}\int_{\kappa-i\infty}^{\kappa-iY} + \int_{\kappa+iY}^{\kappa+i\infty} M(s)(e^{\gamma}\tau)^{-s}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s} &\ll \frac{1}{N}\left(\frac{3}{\lambda Y}\right)^N M(\kappa)(e^{\gamma}\tau)^{-\kappa} \\ &\ll \frac{1}{N}\left(\frac{3}{\lambda Y}\right)^N \mathbb{E}(L(1,\mathbb{X})^{\kappa})(e^{\gamma}\tau)^{-\kappa}.\end{aligned}$$

For the remaining parts of the integral we have that $|t| \le Y$ so we apply Lemma 5.8 which states that $M(s) - \mathbb{E}(L(1,\mathbb{X}))^s \ll \mathbb{E}(L(1,\mathbb{X})^{\mathfrak{Res}})/(\log_q |D|)^{11}$. Then use the inequality $|(e^{\lambda s}-1)/\lambda s| \le 4$ to obtain

$$J_M(\tau) - J(\tau) \ll \frac{1}{N}\left(\frac{3}{\lambda Y}\right)^N \mathbb{E}(L(1,\mathbb{X})^{\kappa})(e^{\gamma}\tau)^{-\kappa} + \frac{Y}{\kappa}4^N \frac{\mathbb{E}(L(1,\mathbb{X})^{\kappa})}{(e^{\gamma}\tau)^{\kappa}(\log_q |D|)^{11}}.$$

Choosing $N = [\log_q \log_q |D|]$ and $\lambda = e^{10}/Y$ then (5.57) gives us that

$$J_M(\tau) - J(\tau) \ll \frac{\Phi_{\mathbb{X}}(\tau)}{(\log_q |D|)^8}. \tag{5.60}$$

On the other hand, by Theorem 5.3 in combination with our choice for $\lambda$, $N$ and $Y$ we have

$$\Phi_{\mathbb{X}}(e^{\pm\lambda N}\tau) = \Phi_{\mathbb{X}}(\tau)\left(1 + O\left(\frac{e^{\tau}(\log_q \log_q |D|)^2 \log_q \log_q \log_q |D|}{\log_q |D|}\right)\right).$$

Hence, combining (5.58), (5.59) and (5.60)

$$\begin{aligned}\mathbb{P}(L(1,\chi_D) > e^{\gamma}\tau) &\le J_M(\tau)\\ &\le J(\tau) + O\left(\frac{\Phi_{\mathbb{X}}(\tau)}{(\log_q |D|)^8}\right)\\ &\le \Phi_{\mathbb{X}}(\tau)\left(1 + O\left(\frac{e^{\tau}\log_q\log_q|D|\log_q\log_q\log_q|D|}{\log_q|D|}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(L(1,\chi_D) > e^{\gamma}\tau) &\ge J_M(e^{\lambda N}\tau)\\ &\ge J(e^{\lambda N}\tau) + O\left(\frac{\Phi_{\mathbb{X}}(\tau)}{(\log_q |D|)^8}\right)\\ &\ge \Phi_{\mathbb{X}}(\tau)\left(1 + O\left(\frac{e^{\tau}(\log_q\log_q|D|)^2\log_q\log_q\log_q|D|}{\log_q|D|}\right)\right).\end{aligned}$$

The final step is done by recalling for $D \in \mathcal{H}_n$ we have $|D| = q^n$. □

And now how to make use of Theorem 5.2 to prove the corollaries of Section 5.1.1.

*Proof of Corollary 5.2.* We note by Artin's class number formula given by (5.3) that $h_D \ge e^{\gamma}\tau\frac{\sqrt{|D|}}{\sqrt{q}}$ if and only if for $D \in \mathcal{H}_{2g+1}$ we have $L(1,\chi_D) \ge e^{\gamma}\tau$. Specializing to $n = 2g+1$ we see Theorem 5.2 proved that the number of $D$ such that $L(1,\chi_D) > e^{\gamma}\tau$ is given by

$$|\mathcal{H}_{2g+1}|\Phi_{\mathbb{X}}(\tau)\left(1 + O\left(\frac{e^{\tau}(\log_q\log_q|D|)^2\log_q\log_q\log_q|D|}{\log_q|D|}\right)\right).$$

Finally, we use Theorem 5.3 to conclude that the number of $D$ such that $h_D \ge e^{\gamma}\tau\frac{\sqrt{|D|}}{\sqrt{q}}$ is given by

$$\begin{aligned}&|\mathcal{H}_{2g+1}|\exp\left(-C_1(q^{\{\log_q\kappa\}})\frac{q^{\tau - C_0(q^{\{\log_q\kappa\}})}}{\tau}\left(1 + O\left(\frac{\log_q\tau}{\tau}\right)\right)\right) \times \left(1 + O\left(\frac{e^{\tau}(\log_q\log_q|D|)^2\log_q\log_q\log_q|D|}{\log_q|D|}\right)\right)\\ &= |\mathcal{H}_{2g+1}|\exp\left(-C_1(q^{\{\log_q\kappa\}})\frac{q^{\tau - C_0(q^{\{\log_q\kappa\}})}}{\tau}\left(1 + O\left(\frac{\log_q\tau}{\tau}\right)\right)\right),\end{aligned}$$

where the final estimate follows from the range of $\tau$. The analogous estimate for small values of $h_D$ follows along the same lines. □

## 5.6 Optimal $\Omega$-results: Proof of Theorem 5.4

For each irreducible polynomial $P \in \tilde{F}_x$, let $\delta_P \in \{-1, 1\}$. Define $\mathcal{S}_N(n, \{\delta_P\})$ to be the set of all monic irreducibles $Q \in \tilde{F}_x$ such that $\deg Q = N$ and

$$\left(\frac{P}{Q}\right) = \delta_P,$$

for all irreducibles $P$ with $\deg P \leq n$. We also let $\mathcal{P}(n)$ denote the product of all irreducible polynomials $P$ with $\deg P \leq n$.

**Lemma 5.15.** *Let $N$ be large, and $1 \leq n \leq (\log_q(N))^2$ be a real number. Then, we have*

$$|\mathcal{S}_N(n, \{\delta_P\})| = \frac{q^N}{2^{\Pi_q(n)} N} + O\left(q^{\frac{N}{2}+n}\right).$$

*Proof.* For each monic polynomial $f \in \tilde{F}_x$, define $\delta_f = \prod_{P|f} \delta_P$. Let $Q$ be an irreducible polynomial of degree $N$. Then, observe that

$$\sum_{f|\mathcal{P}(n)} \delta_f \left(\frac{f}{Q}\right) = \prod_{\deg P \leq n} \left(1 + \delta_P \left(\frac{P}{Q}\right)\right) = \begin{cases} 2^{\Pi_q(n)} & \text{if } Q \in \mathcal{S}_N(n, \{\delta_P\}), \\ 0 & \text{otherwise.} \end{cases} \tag{5.61}$$

Therefore, we deduce that

$$|\mathcal{S}_N(n, \{\delta_P\})| = \frac{1}{2^{\Pi_q(n)}} \sum_{f|\mathcal{P}(n)} \delta_f \sum_{\substack{Q \text{ irreducible} \\ \deg Q = N}} \left(\frac{f}{Q}\right).$$

Since all the divisors of $\mathcal{P}(n)$ are square-free, we obtain from (2.13) that for all $f \neq 1$ such that $f \mid \mathcal{P}(n)$, we have

$$\sum_{\substack{Q \text{ irreducible} \\ \deg Q = N}} \left(\frac{f}{Q}\right) \ll \deg(f) q^{\frac{N}{2}} \ll q^{\frac{N}{2}+n}.$$

since

$$\deg f \leq \deg \mathcal{P}(n) = \sum_{j=1}^{n} j\pi_q(j) \asymp q^n, \tag{5.62}$$

by the prime number theorem. Finally, since the number of divisors of $\mathcal{P}(n)$ is $2^{\Pi_q(n)}$ we deduce that

$$|\mathcal{S}_N(n,\{\delta_P\})| = \frac{\pi_q(N)}{2^{\Pi_q(n)}} + O\left(q^{\frac{N}{2}+n}\right)$$

which completes the proof. □

We shall deduce Theorem 5.4 from the following proposition

**Proposition 5.3.** *We have*

$$\sum_{Q\in\mathcal{S}_N(n,\{\delta_P\})} L(1,\chi_Q) = \zeta_{\mathbb{A}}(2)\frac{\pi_q(N)}{2^{\Pi_q(n)}}\prod_{\deg P\le n}\left(1+\frac{\delta_P}{|P|}\right) + O\left(N^2 q^{N/2+2n}\right). \tag{5.63}$$

*Proof.* First, it follows from (2.11) that for all $m \ge N$ we have

$$L(1,\chi_Q) = \sum_{\deg F\le m}\frac{\chi_Q(F)}{|F|}.$$

Let $A = 2N\deg\mathcal{P}(n) \ll Nq^n$ by (5.62). Then, from (5.61) we obtain

$$\begin{aligned}\sum_{Q\in\mathcal{S}_N(n,\{\delta_P\})} L(1,\chi_Q) &= \frac{1}{2^{\Pi_q(n)}}\sum_{f|\mathcal{P}(n)}\delta_f\sum_{\substack{Q \text{ irreducible}\\ \deg Q=N}}\left(\frac{f}{Q}\right)\sum_{\deg F\le A}\frac{\left(\frac{Q}{F}\right)}{|F|}\\ &= \frac{1}{2^{\Pi_q(n)}}\sum_{f|\mathcal{P}(n)}\delta_f\sum_{\deg F\le A}\frac{1}{|F|}\sum_{\substack{Q \text{ irreducible}\\ \deg Q=N}}\left(\frac{Ff}{Q}\right),\end{aligned} \tag{5.64}$$

by quadratic reciprocity from Proposition 2.6 and Remark 2.9. Since any divisor $f$ of $\mathcal{P}(n)$ is square-free, it follows that $Ff$ is a square only when $F = fh^2$, for some monic polynomial $h$. In this case, we have

$$\sum_{\substack{Q \text{ irreducible}\\ \deg Q=N}}\left(\frac{Ff}{Q}\right) = \pi_q(N) + O(\omega(F)) = \pi_q(N) + O(A),$$

where $\omega(F)$ is the number of irreducible divisors of $F$, and $\omega(F) \le \deg F \le A$.

Furthermore, if $Ff$ is not a square, then by (2.13) we get

$$\sum_{\substack{Q \text{ irreducible}\\ \deg Q=N}}\left(\frac{Ff}{Q}\right) \ll \deg(Ff)q^{\frac{N}{2}} \ll Aq^{N/2},$$

by (5.62). Inserting these estimates in (5.64), we deduce

$$\sum_{Q\in\mathcal{S}_N(n,\{\delta_P\})} L(1,\chi_Q) = \frac{\pi_q(N)}{2^{\Pi_q(n)}} \sum_{f|\mathcal{P}(n)} \frac{\delta_f}{|f|} \sum_{\deg h\le (A-\deg f)/2} \frac{1}{|h|^2} + O\left(A^2 q^{N/2}\right), \tag{5.65}$$

since

$$\sum_{\deg F\le A} \frac{1}{|F|} = \sum_{k=1}^{A} \sum_{\deg F=k} \frac{1}{q^k} = A.$$

Finally, since $\deg f \le \deg \mathcal{P}(n) \le A/2$, then for all $f \mid \mathcal{P}(n)$ we have the tail of the inner sum is very small:

$$\sum_{\deg h>(A-\deg f)/2} \frac{1}{|h|^2} \le \sum_{\deg h>A/2} \frac{1}{|h|^2} \le \sum_{k>A/2} \frac{1}{q^{2k}} \ll q^{-N}.$$

Inserting this estimate in (5.65) completes the proof. □

We finish this section by proving Theorem 5.4.

*Proof of Theorem 5.4.* We choose $n$ such that

$$\frac{N\log_q N}{10\zeta_{\mathbb{A}}(2)q} \le q^n < \frac{N\log_q N}{10\zeta_{\mathbb{A}}(2)}. \tag{5.66}$$

We choose $\delta_P = 1$ for all monic irreducibles $P$ with $\deg P \le n$. Then, it follows from Lemma 5.15 and Proposition 5.3 that

$$\frac{1}{|\mathcal{S}_N(n,\{\delta_P\})|} \sum_{Q\in\mathcal{S}_N(n,\{\delta_P\})} L(1,\chi_Q) = \zeta_{\mathbb{A}}(2) \prod_{\deg P\le n} \left(1+\frac{1}{|P|}\right) \left(1+O\left(q^{-N/6}\right)\right). \tag{5.67}$$

Furthermore, by Lemma 5.3 we have

$$\zeta_{\mathbb{A}}(2) \prod_{\deg P\le n} \left(1+\frac{1}{|P|}\right) = \prod_{\deg P\le n} \left(1-\frac{1}{|P|}\right)^{-1} \left(1+O\left(\frac{q^{-n}}{n}\right)\right) = e^{\gamma} n + O(1).$$

Combining this estimate with (5.66) and (5.67) yield the existence of a monic irreducible $Q$ of degree $N$, such that

$$L(1,\chi_Q) \ge e^{\gamma} \log_q(N\log_q N) + O(1) = e^{\gamma}\left(\log_q \log_q |Q| + \log_q \log_q \log_q |Q|\right) + O(1),$$

as desired. Finally, one can deduce (5.15) along the same lines by taking $\delta_P = -1$ for all monic irreducibles $P$ with $\deg P \le n$. $\square$

# 6 The distribution of values of $L(\sigma, \chi_D)$, for $1/2 < \sigma < 1$, over function fields

## 6.1 Introduction

This chapter will extend the results of the previous chapter from the edge of the critical strip to the inside. The relevant history associated to the number field case was outlined in Section 1.5. As was discussed, there is a rich history of study for this topic in the classical setting. There are fewer works in function fields discussing $L$-functions inside the critical strip. The first results describe the average values, see Rosen [94, Chapter 17]. We compare the following result ([94, Theorems 17.13 & 17.14]) with what we achieve in this chapter. Note that we have shortened this result to be relevant to our discussion.

**Theorem 6.1.** *Let $\sigma > 1/2$ and let $M > 0$ be a fixed odd integer. Then*

$$q^{-M} \sum_{\substack{m \text{ monic} \\ \deg(m)=M}} L(\sigma, \chi_m) = \frac{\zeta_{\mathbb{A}}(2\sigma)}{\zeta_{\mathbb{A}}(2\sigma+1)} - \left(1 - \frac{1}{q}\right) (q^{1-2\sigma})^{(M+1)/2} \zeta_{\mathbb{A}}(2\sigma). \tag{6.1}$$

*Suppose now, $M > 0$ is an even integer and that $\sigma > 1/2$ but $\sigma \neq 1$. Then*

$$q^{-M} \sum_{\substack{m \text{ monic} \\ \deg(m)=M \\ m \neq \square}} L(\sigma, \chi_m) = \frac{\zeta_{\mathbb{A}}(2\sigma)}{\zeta_{\mathbb{A}}(2\sigma+1)} - \left(1 - \frac{1}{q}\right) (q^{1-2\sigma})^{M/2} \zeta_{\mathbb{A}}(2\sigma). \tag{6.2}$$

Our goal is to obtain information about much more than the average value of $L(\sigma, \chi_D)$ for $D \in \mathcal{H}_n$, and our main result establishes a formula for the tail of the distribution of values of $\log |L(\sigma, \chi_D)|$. Using this information we will obtain evidence toward the true maximal order of our $L$-functions. As we have seen from the previous chapter a key step in the process of describing distribution of values is to establish a formula for the large complex moments of the $L$-functions. That is we need to understand $\sum_{D\in\mathcal{H}_n} L(1,\chi_D)^z$ for $z\in\mathbb{C}$ with $|z|$ large. Unfortunately, the bounds on $\log|L(\sigma,\chi_D)|$ for $\sigma\in(1/2,1)$ are, even under GRH, not strong enough to give us an asymptotic formula in the full expected range for our complex moments. To get around this issue we adapt some ideas from [64]. Below we describe their main idea and the reasoning behind it.

In [62], under the assumption of the Riemann Hypothesis (RH), Lamzouri establishes an asymptotic formula for

$$M_z(T) := \frac{1}{T}\int_T^{2T} |\zeta(\sigma+it)|^z dt, \qquad \tfrac{1}{2}<\sigma<1,$$

where $z\in\mathbb{C}$ with $|z|\ll(\log T)^{2\sigma-1}$ and conjectures that it should hold for $|z|\ll(\log T)^{\sigma}$. We note that the assumption of the Riemann Hypothesis is necessary since if $\sigma+it$ is close to a zero then $|\zeta(\sigma+it)|^r$, $r\in\mathbb{R}_{<0}$ will be very large.

On the other hand, if $\mathfrak{Re}(z)>0$ is large, then $M_z(T)$ will be heavily affected by values for $t\in[T,2T]$ which make $|\zeta(\sigma+it)|$ large. In [102, Theorem 14.5], Titchmarsh proves that RH implies, for fixed $1/2<\sigma<1$ and $t$ large, $\log|\zeta(\sigma+it)|\ll(\log t)^{2-2\sigma+o(1)}$. However, Montgomery [81] proved that $\log|\zeta(\sigma+it)| = \Omega((\log t)^{1-\sigma+o(1)})$ and further conjectured (based on probabilistic arguments) that this $\Omega$-result is optimal. However, the maximal order of $\log|\zeta(\sigma+it)|$ for $1/2<\sigma<1$ is unknown.

The authors of [64] restrict the integral in $M_z(T)$ to the following set, denoted $\mathcal{A}(T)$, to use Montgomery's $\Omega$-result above:

$$\mathcal{A}(T) := \{t\in[T,2T] : |R_Y^{\zeta}(\sigma+it)|\le(\log T)^{1-\sigma}/\log\log T\},$$

where $R_Y^{\zeta}(\sigma+it) := \sum_{p^n\le Y}\frac{1}{np^{n(\sigma+it)}}$. Note that there is a parameter $Y$ in the definition of $\mathcal{A}(T)$, if $Y$ is taken small enough then $\mathcal{A}(T)=[T,2T]$. Making use of the restriction to $\mathcal{A}(T)$, Lamzouri, Lester and Radziwiłł obtain an asymptotic formula for complex moments of $|\zeta(\sigma+it)|$ in the full conjectured range $|z|\ll(\log T)^{\sigma}$. Removing the points $t\in[T,2T]\setminus\mathcal{A}(T)$ will not affect the distribution results so long as the size of $[T,2T]\setminus\mathcal{A}(T)$ is small. Indeed, the proof of [64, Proposition 2.3] shows that $|[T,2T]\setminus\mathcal{A}(T)|$ has exponential decay as $T\to\infty$.

In this chapter we discuss the adaptation of these ideas to the case of function fields. Again, we refer the reader to Section 2.6 for a refresher on background required to understand this chapter. We study the distribution of values of $\log L(\sigma,\chi_D)$ for $\chi_D$ the quadratic Dirichlet character associated to the monic squarefree polynomial $D$. The strategy used in this chapter is to set up a probabilistic random model and compare $L(\sigma, \chi_D)$ to it. With that in mind, let $\{\mathbb{X}(P)\}$ denote a sequence of independent random variables indexed by the irreducible (prime) elements $P \in \mathbb{A}$, and taking the values $0, \pm 1$ as follows

$$\mathbb{X}(P) = \begin{cases} 0 & \text{with probability } \frac{1}{|P|+1} \\ \pm 1 & \text{with probability } \frac{|P|}{2(|P|+1)}. \end{cases} \tag{6.3}$$

Let $f = P_1^{e_1} P_2^{e_2} \cdots P_s^{e_s}$ be the prime power factorization of $f$, then we extend the definition of $\mathbb{X}$ multiplicatively as

$$\mathbb{X}(f) = \mathbb{X}(P_1)^{e_1} \mathbb{X}(P_2)^{e_2} \cdots \mathbb{X}(P_s)^{e_s}. \tag{6.4}$$

Then, we compare the distribution of $L(\sigma, \chi_D)$ with

$$L(\sigma, \mathbb{X}) := \prod_{P \text{ irreducible}} \left(1 - \frac{\mathbb{X}(P)}{|P|^\sigma}\right)^{-1}, \tag{6.5}$$

which converges almost surely for $\sigma > 1/2$.

For our purposes it is also convenient to study the series version of this model. Let $Y > 0$ be large, then

$$R_Y(\sigma, \chi) := \sum_{\substack{f \text{ monic} \\ \deg f \le Y}} \frac{\Lambda(f)\chi(f)}{\deg(f)|f|^\sigma} \qquad \text{and} \tag{6.6}$$

$$R_Y(\sigma, \mathbb{X}) := \sum_{\substack{f \text{ monic} \\ \deg f \le Y}} \frac{\Lambda(f)\mathbb{X}(f)}{\deg(f)|f|^\sigma}. \tag{6.7}$$

Finally the following notation will be fixed for the remainder of the chapter. Let $\mathbb{A} = \mathbb{F}_q[T]$ taking $q \equiv 1 (\operatorname{mod} 4)$. Here $\log_q$ denotes base $q$ logarithm, $\log$ is the natural logarithm and we denote the fractional part of $x$ by $\{x\}$.

The goal of this chapter is to describe the distribution of values of $\log L(\sigma, \chi_D)$ for $D \in \mathcal{H}_n$ with

$$\mathcal{H}_n = \{D \in \mathbb{A} : D \text{ monic, square free and } \deg(D) = n\}. \tag{6.8}$$

As was explained above in order to do this we require an asymptotic formula for

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}(L(\sigma,\chi_D))^z, \tag{6.9}$$

in a large range of $z\in\mathbb{C}$. Under the Riemann Hypothesis (see Proposition 6.1) we have

$$R_Y(\sigma,\chi_D)\ll\frac{(\log_q|D|)^{2(1-\sigma)}}{\log_q\log_q|D|}\qquad\text{for all } D\in\mathcal{H}_n, \tag{6.10}$$

which is true for all $Y$. Such a bound allows us to obtain our asymptotic formula for (6.9) with $z\in\mathbb{C}$ such that $|z|\ll(\log_q|D|)^{2\sigma-1}$, cf. Corollary 6.1. However, in accordance with the situation over number fields (see Section 1.5) we do not expect the GRH bound is optimal. Our Theorem 6.5 provides some $\Omega$-results which we believe to be the correct maximal order for these $L$-functions with $1/2<\sigma<1$ fixed. The evidence for such a claim comes after combining Theorems 6.3 and 6.4. This belief motivates the introduction of the following restricted set, which is similar to $\mathcal{A}(T)$ in the number field setting:

$$\widetilde{\mathcal{H}}_{n,g}=\left\{D\in\mathcal{H}_n\middle|R_Y(\sigma,\chi_D)\leq\frac{(\log_q|D|)^{1-g(\sigma)}}{\log_q\log_q|D|}\right\}, \tag{6.11}$$

with $2\sigma-1\leq g(\sigma)\leq\sigma$. Notice that taking $g(\sigma)=2\sigma-1$ in (6.11) returns the result of (6.10), so that $\mathcal{H}_n=\widetilde{\mathcal{H}}_{n,g}$. Taking $g(\sigma)=\sigma$ and $D\in\widetilde{\mathcal{H}}_{n,\sigma}$, we have

$$R_Y(\sigma,\chi_D)\leq\frac{(\log_q|D|)^{1-\sigma}}{\log_q\log_q|D|}.$$

Averaging $L(\sigma,\chi_D)$ over $\widetilde{\mathcal{H}}_{n,\sigma}$ generates Corollary 6.2.

Our full theorem is given as follows:

**Theorem 6.2.** *Let $n$ be large, $1/2<\sigma<1$, $B>2$ a constant be fixed. Then there exists a positive constant $b_3:=b_3(\sigma,B)$ such that for $z\in\mathbb{C}$ with $|z|\leq b_3n^{g(\sigma)}$ we have*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}L(\sigma,\chi_D)^z=\mathbb{E}(L(\sigma,\mathbb{X}))^z+O\left(\frac{\mathbb{E}(L(\sigma,\mathbb{X}))^{\mathfrak{Re}(z)}}{n^{B-(g(\sigma)+1)}}\right), \tag{6.12}$$

*where $2\sigma - 1 \leq g(\sigma) \leq \sigma$ and we have for some constants $C, C' > 0$ that*

$$|\mathcal{H}_n \setminus \widetilde{\mathcal{H}}_{n,g}| \ll |\mathcal{H}_n| \exp\left(-Cn(\sigma - g(\sigma)) - C' \frac{n}{\log_q n}\right).$$

We have the following corollaries if $g(\sigma)$ is taken at either end point of the range $2\sigma - 1 \leq g(\sigma) \leq \sigma$.

**Corollary 6.1.** *Take $g(\sigma) = 2\sigma - 1$ then under the conditions of Theorem 6.2,*

*$|z| \leq b_3 n^{2\sigma-1}$, $\widetilde{\mathcal{H}}_{n,g} = \mathcal{H}_n$ and we have*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} L(\sigma, \chi_D)^z = \mathbb{E}(L(\sigma, \mathbb{X}))^z \left(1 + O\left(\frac{1}{n^{B-2\sigma}}\right)\right). \tag{6.13}$$

**Corollary 6.2.** *Taking $g(\sigma) = \sigma$ then under the assumptions of Theorem 6.2,*

*$|z| \leq b_3 n^{\sigma}$, and we have*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \widetilde{\mathcal{H}}_{n,g}} L(\sigma, \chi_D)^z = \mathbb{E}(L(\sigma, \mathbb{X})^z) + O\left(\frac{\mathbb{E}(L(\sigma, \mathbb{X}))^{\mathfrak{Re}(z)}}{n^{B-1-\sigma}}\right), \tag{6.14}$$

*such that*

$$|\mathcal{H}_n \setminus \widetilde{\mathcal{H}}_{n,g}| \ll |\mathcal{H}_n| \exp\left(-C' \frac{n}{\log_q n}\right),$$

*where $C' > 0$ is the constant in Theorem 6.2.*

For $\tau > 0$, define

$$\Phi_{\mathbb{X},\sigma}(\tau) := \mathbb{P}(\log L(\sigma, \mathbb{X}) > \tau). \tag{6.15}$$

The following theorem proves that the distribution of values of $\log L(\sigma, \chi_D)$ are well approximated by $\Phi_{\mathbb{X},\sigma}(\tau)$.

**Theorem 6.3.** *Let $n$ be large and $1/2 < \sigma < 1$ be fixed. There exists a positive constant $b(\sigma)$ such that for $3 \leq \tau \leq b(\sigma) n^{1-\sigma} (\log_q n)^{-\frac{1}{\sigma}}$ we have*

$$\frac{1}{|\mathcal{H}_n|} |\{D \in \mathcal{H}_n : \log L(\sigma, \chi_D) > \tau\}| = \Phi_{\mathbb{X},\sigma}(\tau) \left(1 + O\left(\frac{(\tau \log_q \tau)^{\frac{\sigma}{1-\sigma}} \log_q n}{n^{\sigma}}\right)\right). \tag{6.16}$$

We note that the range of uniformity appears worse than what is achieved in Lamzouri [62, Theorem 1.6] however we obtain an asymptotic formula for $\frac{1}{|\mathcal{H}_n|} |\{D \in \mathcal{H}_n : \log L(\sigma, \chi_D) > \tau\}|$ while in [62, Theorem 1.6] only an asymptotic for log of the proportion was proved. Finally, we describe the asymptotic behaviour of $\Phi_{\mathbb{X},\sigma}(\tau)$.

**Theorem 6.4.** *Let $1/2 < \sigma < 1$ be fixed. For any large $\tau$ we have a unique $\kappa_\sigma(\tau)$ satisfying* (6.45) *so that*

$$\Phi_{\mathbb{X},\sigma}(\tau) = \exp\left(-A_{\mathbb{X}}(q^{\{\log_q \kappa_\sigma(\tau)\}}, \sigma)\tau^{\frac{1}{1-\sigma}}(\log_q \tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O\left(\frac{\log_q \log_q \tau}{(\log_q \tau)^{2-\frac{1}{\sigma}}}\right)\right)\right),$$

*where*

$$A_{\mathbb{X}}(t,\sigma) = \frac{1-\sigma}{\sigma}(G_2(t,\sigma) - G_1(t,\sigma))(G_3(t,\sigma))^{\frac{1}{\sigma}} > 0 \tag{6.17}$$

*and $G_i(t,\sigma)$ for $i = 1, 2, 3$ are defined respectively as* (6.48), (6.50) *and* (6.52).

This result should be compared to Lamzouri 2011 [62, Theorem 1.1] and Lamzouri, Lester, Radziwiłł [64, Corollary 7.7]. Both of which use a random model which has the associated random variables $\mathbb{X}$ uniformly distributed around the unit circle. These theorems describe the asymptotic behaviour of

$$\Phi^{\zeta}_{T,\sigma}(\tau) = \frac{1}{T}\text{meas}\{t \in [T, 2T] : \log|\zeta(\sigma + it)| > \tau\},$$

for which the following estimate has been proven for $1 \ll \tau \ll (\log T)^{1-\sigma}/(\log\log T)^{\frac{1}{\sigma}}$:

$$\Phi^{\zeta}_{T,\sigma}(\tau) = \exp\left(-A_{\mathbb{X}}(\sigma)\tau^{\frac{1}{1-\sigma}}(\log_q \tau)^{\frac{\sigma}{1-\sigma}}(1 + o(1))\right).$$

So it has the same general form as in our Theorem 6.4 with one major difference. The value $A_{\mathbb{X}}(\sigma)$ is independent of $\tau$. In both papers their calculations provide

$$A_{\mathbb{X}}(\sigma) = \left(\frac{\sigma^{2\sigma}}{(1-\sigma)^{2\sigma-1}g_0(\sigma)^\sigma}\right)^{\frac{1}{1-\sigma}} \qquad \text{with} \qquad g_0(\sigma) = \int_0^\infty \frac{\log I_0(u)}{u^{\frac{1}{\sigma}+1}}du,$$

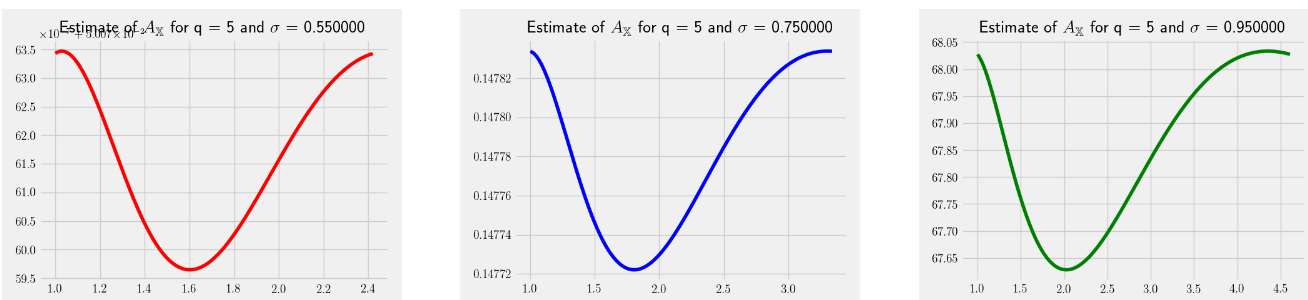
and $I_0$ is the modified Bessel function of order 0. In addition, Lamzouri [62, Theorem 1.6] describes the asymptotic behaviour associated to distribution of values for $\log L(\sigma, \chi_d)$ where $\chi_d$ is a quadratic character and $d$ goes over fundamental discriminants $|d| \leq x$. The random model used here is slightly simpler than the one we use (the random variables take values $\pm 1$ equally and is never 0). The coefficient $A_{\mathbb{X}}(\sigma)$ reflects this difference, but it is still has the same general shape and is independent of $\tau$.

As can be seen from the statement of Theorem 6.4 we observe some pathological behaviour special to function fields. The coefficient $A_{\mathbb{X}}(t,\sigma)$ does not converge to a function which is only dependent on $\sigma$ as in the previous works. The reason for this difference stems from Proposition 6.3 which is used to evaluate the natural log of the real moments of our random model along with the first few derivatives. In this proposition we obtain two sums over primes $G_1(t,\sigma)$ and $G_2(t,\sigma)$, see equations (6.48) and (6.50) respectively. The corresponding integrals over number fields do not have the parameter $t$ (it is always equal to 1), which in our case arises from the way that primes are measured in function fields.

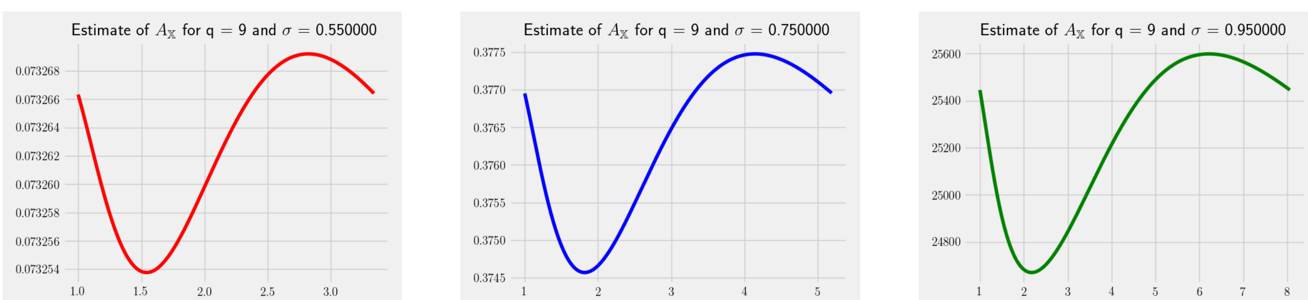
Figures 6.1a, 6.1b and 6.1c provide some estimates for how the pathological constant $A_{\mathbb{X}}(t,\sigma)$ behaves for some fixed $\sigma$ and the first three examples of valid $q$. The range of the $t$-axis is chosen since one can see from Proposition 6.3 that the coefficient is multiplicatively periodic with multiplicator $q^{\sigma}$.

Figure 6.1: $A_{\mathbb{X}}(t,\sigma)$ for $1 \leq t \leq q^{\sigma}$ with various $q$ and $\sigma$.

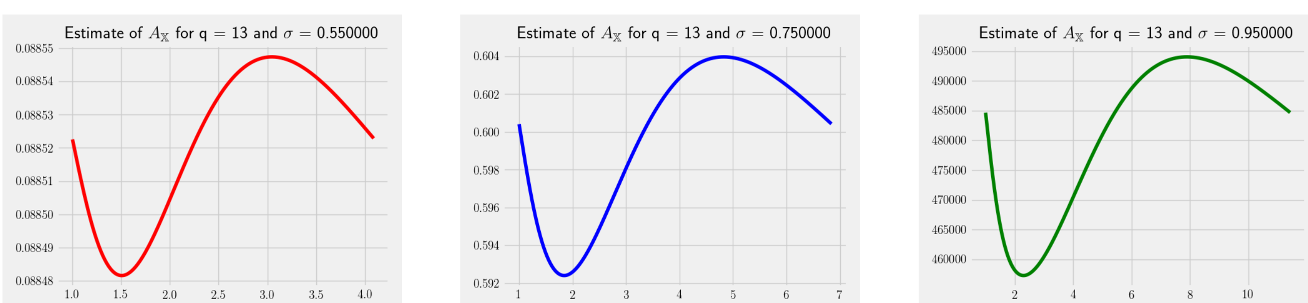
(a) $A_{\mathbb{X}}(t,\sigma)$ for $1 \leq t \leq q^{\sigma}$ with $q = 5$ for $\sigma = 0.55$, 0.75 and 0.95 respectively.




(b) $A_{\mathbb{X}}(t,\sigma)$ for $1 \leq t \leq q^{\sigma}$ with $q = 9$ for $\sigma = 0.55$, 0.75 and 0.95 respectively.




(c) $A_{\mathbb{X}}(t,\sigma)$ for $1 \leq t \leq q^{\sigma}$ with $q = 13$ for $\sigma = 0.55$, 0.75 and 0.95 respectively.



We now turn our attention to the extreme values of our $L$-functions, first an unconditional upper bound:

**Proposition 6.1.** *Let $1/2 < \sigma < 1$ be fixed. Let $F$ be a monic polynomial, and $\chi$ be a non-trivial character on $(\mathbb{A}/\mathbb{A}F)^\times$. For any complex number $s$ with $\mathfrak{Re}(s) = \sigma$ we have*

$$\log L(s,\chi) \ll_\sigma \frac{(\log_q |F|)^{2(1-\sigma)}}{\log_q \log_q |F|}. \tag{6.18}$$

In the other direction, we obtain the following $\Omega$-results, which we conjecture are best possible.

**Theorem 6.5.** *Let $N$ be large. Let $1/2 < \sigma < 1$ be fixed. There exist a constant $\beta_q(\sigma) > 0$ and irreducible polynomials $T_1$ and $T_2$ of degree $N$, such that*

$$\log L(\sigma, \chi_{T_1}) \geq \beta_q(\sigma) \frac{(\log_q |T_1|)^{1-\sigma}}{(\log_q \log_q |T_1|)^\sigma}, \tag{6.19}$$

*and*

$$\log L(\sigma, \chi_{T_2}) \leq -\beta_q(\sigma) \frac{(\log_q |T_2|)^{1-\sigma}}{(\log_q \log_q |T_2|)^\sigma}, \tag{6.20}$$

*with* $\beta_q(\sigma) = \frac{\zeta_{\mathbb{A}}(2-\sigma)}{(10\zeta_{\mathbb{A}}(2)q)^{1-\sigma}}$.

The result (6.19) can be compared with [2, Theorem 3] a recent work discussing the size of $\log |L(\sigma,\chi)|$ over a number field with $\sigma \in (1/2, 1)$. The authors prove unconditionally using a variant of the resonator method that for all sufficiently large $q$ there is a positive constant $C(\sigma)$ and a non-principal character $\chi(\bmod\ q)$ such that

$$\log |L(\sigma,\chi)| \geq C(\sigma)(\log q)^{1-\sigma}(\log\log q)^{-\sigma}.$$

Theorem 3 of [2] does not give improvements for quadratic characters $\chi_d$ where $d$ varies over fundamental discriminants in the range $|d| \leq x$. Under the assumption of GRH, [62, Theorem1.8] proves there are $\gg x^{\frac{1}{2}}$ primes $p \leq x$ such that

$$\log |L(\sigma, \chi_p)| \geq (C(\sigma) + o(1))(\log q)^{1-\sigma}(\log\log q)^{-\sigma},$$

where $\chi_p$ is the Legendre symbol. Lamzouri gives $C(\sigma) = (2\log 2)^{\sigma-1}/(1-\sigma)$. Similar results hold for comparison with (6.20) in [62, Theorem 1.8].

We close the introduction with an outline of the paper. Section 6.2 will establish some preliminary lemmas needed to complete the proofs. Section 6.3 will connect the complex moments of $L(\sigma, \chi_D)$ to the expectation of the complex moments of the random model and provide the proof of Theorem 6.2. Section 6.4 will be used to prove Theorems 6.3 and 6.4. Section 6.5 proves the Ω-results of Theorem 6.5.

## 6.2 Preliminaries

For a refresher on definitions and other relevant theorems please see Section 1.4 and 2.6.

### 6.2.1 Estimates for sums over primes

**Lemma 6.1.** *We have*

$$\sum_{\deg P \leq M} \frac{1}{|P|^{\sigma}} = \zeta_{\mathbb{A}}(2-\sigma)\frac{q^{M(1-\sigma)}}{M} + O\left(\frac{q^{M(1-\sigma)}}{M^{2-\sigma}}\right), \tag{6.21}$$

*and*

$$\sum_{\deg P > M} \frac{1}{|P|^{2\sigma}} \ll_{\sigma} \frac{q^{M(1-2\sigma)}}{M}. \tag{6.22}$$

*Proof.* To prove (6.21) we apply the prime number theorem which gives

$$\sum_{\deg(P) \leq M} \frac{1}{|P|^{\sigma}} = \sum_{d \leq M} \frac{\pi_q(d)}{q^{d\sigma}} = \sum_{d \leq M} \frac{q^{d(1-\sigma)}}{d} + O\left(\sum_{d \leq M} \frac{1}{dq^{d(\sigma-1/2)}}\right).$$

For the error term we note the exponent of $q$ falls in $(0, d/2)$, since $1/2 < \sigma < 1$ and thus

$$\sum_{d \leq M} \frac{1}{dq^{d(\sigma-1/2)}} = O\left(\sum_{d \leq M} \frac{1}{d}\right) = O(\log M).$$

Break the remaining sum into two pieces:

$$\sum_{d \leq M} \frac{q^{d(1-\sigma)}}{d} = \sum_{M-\log_q M < d \leq M} \frac{q^{d(1-\sigma)}}{d} + \sum_{d \leq M - \log_q M} \frac{q^{d(1-\sigma)}}{d}.$$

The second sum is bounded by its largest term

$$\frac{q^{(M-\log_q M)(1-\sigma)}}{M-\log_q M} = \frac{q^{M(1-\sigma)}}{M^{1-\sigma}(M-\log_q M)} = O\left(\frac{q^{M(1-\sigma)}}{M^{2-\sigma}}\right).$$

Finally note that if $d \in (M-\log_q M, M]$, we have $\frac{1}{d} = \frac{1}{M}\left(1+O\left(\frac{\log_q M}{M}\right)\right)$ so

$$\begin{aligned}\sum_{M-\log_q M<d\leq M} \frac{q^{d(1-\sigma)}}{d} &= \frac{q^{M(1-\sigma)}}{M} \sum_{M-\log_q M<d\leq M} q^{(d-M)(1-\sigma)}\left(1+O\left(\frac{\log_q M}{M}\right)\right) \\ &= \frac{q^{M(1-\sigma)}}{M}\left(1+O\left(\frac{\log_q M}{M}\right)\right)\sum_{l\leq \log_q M}\frac{1}{q^{l(1-\sigma)}} \\ &= \zeta_{\mathbb{A}}(2-\sigma)\frac{q^{M(1-\sigma)}}{M}\left(1+O\left(\frac{\log_q M}{M}\right)\right).\end{aligned}$$

Combining all the pieces we get the desired result. To prove (6.22), note that by the prime number theorem we have

$$\sum_{\deg P>M}\frac{1}{|P|^{2\sigma}} \ll \sum_{n>M}\frac{q^n}{nq^{2\sigma n}} \leq \frac{q^{(1-2\sigma)M}}{M}\sum_{d=1}^{\infty}\frac{1}{q^{d(2\sigma-1)}} \ll_\sigma \frac{q^{(1-2\sigma)M}}{M}.$$

□

**Lemma 6.2.** *Let $1/2<\sigma<1$ be fixed. Let $D$ be a monic polynomial, and $\chi$ be a non trivial character associated to $D$ on $(\mathbb{A}/\mathbb{A}D)^\times$. For a large positive integer $Y$ we have*

$$\log L(\sigma,\chi) = R_Y(\sigma,\chi) + O\left(q^{-Y(\sigma-\frac{1}{2})}\log_q|D|\right), \tag{6.23}$$

*where $R_Y(\sigma,\chi)$ is defined in equation (6.6). Furthermore, the following also holds*

$$\log L(\sigma,\chi) = -\sum_{\deg(P)\leq Y}\log\left(1-\frac{\chi(P)}{|P|^\sigma}\right) + O(q^{-Y(\sigma-\frac{1}{2})}\log_q|D|). \tag{6.24}$$

*Proof.* Let $1/2 < \sigma < 1$, then

$$\begin{aligned}\log L(\sigma,\chi) &= -\sum_{P} \log\left(1 - \frac{\chi(P)}{|P|^{\sigma}}\right)\\ &= \sum_{\substack{P\\ n\geq 1}} \left(\frac{\chi(P)}{|P|^{\sigma}}\right)^{n} \frac{1}{n}\\ &= \sum_{f \text{ monic}} \frac{\Lambda(f)\chi(f)}{\deg f |f|^{\sigma}}.\end{aligned}$$

To obtain the desired result we consider (2.12) in combination with GRH to obtain

$$\begin{aligned}\sum_{\substack{f \text{ monic}\\ k>Y}} \frac{\Lambda(f)\chi(f)}{kq^{k\sigma}} &= \sum_{k>Y} \frac{1}{kq^{k\sigma}} \sum_{\deg f = k} \Lambda(f)\chi(f)\\ &\ll \deg D \sum_{k>Y} \frac{q^{k(\frac{1}{2}-\sigma)}}{k}.\end{aligned}$$

This completes the proof.

Finally, (6.24) follows easily using the same argument as in Lemma 5.2. □

In particular, choosing $Y = A \log_q \log_q |D|$ where $A$ is chosen such that

$$A = B/(\sigma - \tfrac{1}{2}) \text{ for some } B > 1, \tag{6.25}$$

we obtain that

$$\log L(\sigma,\chi) = R_Y(\sigma,\chi) + O\left(\frac{1}{\log_q |D|^{B-1}}\right).$$

We now have enough to prove proposition 6.1.

*Proof of Proposition 6.1.* Let $s = \sigma + it$ and take $1/2 < \sigma < 1$ to be fixed. By Lemma 6.2 and (6.21) we obtain that

$$\log L(s,\chi) \ll \sum_{\substack{f \text{ monic}\\ \deg f \leq M}} \frac{\Lambda(f)}{\deg(f)|f|^{\sigma}} \ll_{\sigma} \sum_{\deg P \leq M} \frac{1}{|P|^{\sigma}} \ll_{\sigma} \frac{q^{(1-\sigma)M}}{(1-\sigma)M}.$$

Choosing $M = 2\log_q \log_q |F|$ yields (6.18). □

### 6.2.2 Sums over $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_{n,g}$.

First, the orthogonality relation.

**Lemma 6.3.** *Let $f$ be a monic polynomial. If $f$ is a square in $\mathbb{A}$, then*

$$\sum_{D\in\mathcal{H}_n}\chi_D(f)=|\mathcal{H}_n|\cdot\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}+O\left(\sqrt{|\mathcal{H}_n|}\right).$$

*Furthermore, if $f$ is not a square in $\mathbb{A}$, then for $\epsilon>0$ and $f=f_1f_2^2$ where $f_1$ is square free then*

$$\sum_{D\in\mathcal{H}_n}\chi_D(f)\ll_\epsilon\frac{\sqrt{|\mathcal{H}_n|}}{\sqrt{q-1}}|f_1|^\epsilon.$$

*Proof.* The first estimate follows from Proposition 5.2 of [3], while the second follows from Lemma 3.5 of [15]. □

Next, the large sieve estimate:

**Lemma 6.4.** *Let $1/2<\sigma\le 1$ be fixed. Let $n$ be large and $1\le m_1\le m_2$ be integers. Then for all positive integers $k$ such that $k\le n/(6m_2)$ and $\epsilon>0$ we have*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{m_1\le\deg P\le m_2}\frac{\chi_D(P)}{|P|^\sigma}\right)^{2k}\ll\frac{(2k)!}{2^kk!}\left(\sum_{m_1\le\deg P\le m_2}\frac{1}{|P|^{2\sigma}}\right)^k+O_\epsilon\left(|\mathcal{H}_n|^{-1/3+\epsilon}\right).$$

*Proof.* First, observe that

$$\sum_{D\in\mathcal{H}_n}\left(\sum_{m_1\le\deg P\le m_2}\frac{\chi_D(P)}{|P|^\sigma}\right)^{2k}=\sum_{\substack{P_1,P_2,\cdots,P_{2k}\\ m_1\le\deg P_j\le m_2}}\frac{1}{|P_1P_2\cdots P_{2k}|^\sigma}\sum_{D\in\mathcal{H}_n}\chi_D(P_1P_2\cdots P_{2k}).\tag{6.26}$$

The diagonal terms of the inner sum on the left hand side correspond to $P_1P_2\cdots P_{2k}$ being a perfect square. Therefore, by Lemma 6.3 the contribution of the diagonal is

$$\ll\frac{(2k)!}{2^kk!}\left(\sum_{m_1\le\deg P\le m_2}\frac{1}{|P|^{2\sigma}}\right)^k|\mathcal{H}_n|.$$

When $P_1P_2\cdots P_{2k} = Q_1Q_2^2$ with $|Q_1|>1$ square free then Lemma 6.3 gives

$$\sum_{D\in\mathcal{H}_n} \chi_D(P_1P_2\cdots P_{2k}) \ll_\epsilon \frac{\sqrt{|\mathcal{H}_n|}}{\sqrt{q-1}}|Q_1|^\epsilon.$$

Therefore, following a similar argument as in (6.21) the contribution of the terms $P_1,\cdots,P_{2k}$ such that $P_1P_2\cdots P_{2k}$ is a not a square, is

$$\ll \frac{\sqrt{|\mathcal{H}_n|}}{\sqrt{q-1}}\left(\sum_{m_1\le \deg P\le m_2} \frac{1}{|P|^{\sigma-\epsilon}}\right)^{2k} \ll \frac{\sqrt{|\mathcal{H}_n|}}{\sqrt{q-1}} q^{2(1+\epsilon-\sigma)km_2} \ll_\epsilon |\mathcal{H}_n|^{2/3+\epsilon},$$

by our assumption on $k$. □

**Lemma 6.5.** *Let $\frac{1}{2}<\sigma<1$ be fixed, $n$ be a large integer. Let $D\in\mathcal{H}_n$ and $Y = A\log_q\log_q|D|$ where $A$ satisfies* (6.25) *and $2\sigma-1\le g(\sigma)\le\sigma$. Then, for some constants $C, C'>0$, we have*

$$\#\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g} \ll \#\mathcal{H}_n \exp\left(-C\log_q|D|(\sigma-g(\sigma)) - C'\frac{\log_q|D|}{\log_q\log_q|D|}\right).$$

*Proof.* Let $1/2<\sigma<1$ be fixed and let $k\in\mathbb{Z}_+$ be chosen such that $k\le \frac{\log_q|D|}{12A\log_q\log_q|D|}$. Consider the sum

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\deg(f)\le Y}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^\sigma}\right)^{2k}.$$

Note that

$$\begin{aligned}\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\deg(f)\le Y}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^\sigma}\right)^{2k} &\ge \frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}}\left(\sum_{\deg(f)\le Y}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^\sigma}\right)^{2k}\\ &\ge \frac{|\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}|}{|\mathcal{H}_n|}\left(\frac{(\log_q|D|)^{1-g(\sigma)}}{\log_q\log_q|D|}\right)^{2k},\end{aligned} \tag{6.27}$$

where the last inequality follows from the definition of $\widetilde{\mathcal{H}}_{n,g}$.

On the other hand we have

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\deg(f)\le Y}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^{\sigma}}\right)^{2k}\le 2^k\left(\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\deg(f)\le\log_q k+\log_q\log_q k}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^{\sigma}}\right)^{2k}\right.$$
$$\left.+\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\log_q k+\log_q\log_q k\le\deg(f)\le Y}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^{\sigma}}\right)^{2k}\right).\tag{6.28}$$

The first of these sums is bounded using (6.21) giving

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\deg(f)\le\log_q k+\log_q\log_q k}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^{\sigma}}\right)^{2k}\ll\left(\zeta_{\mathbb{A}}(2-\sigma)\frac{k^{1-\sigma}}{(\log_q k)^{\sigma}}\right)^{2k},\tag{6.29}$$

The second we use Lemma 6.4 together with Striling's formula and (6.22) to obtain

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\log_q k+\log_q\log_q k\le\deg(f)\le Y}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^{\sigma}}\right)^{2k}\ll\left(\frac{2k}{e}\frac{k^{1-2\sigma}}{(\log_q k)^{2\sigma}}\right)^{k}.\tag{6.30}$$

Putting together (6.27), (6.28), (6.29) and (6.30) we see

$$\frac{|\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}|}{|\mathcal{H}_n|}\left(\frac{(\log_q|D|)^{1-g(\sigma)}}{\log_q\log_q|D|}\right)^{2k}\ll\left(\left(2\zeta_{\mathbb{A}}^2(2-\sigma)+\frac{4}{e}\right)\frac{k^{2-2\sigma}}{(\log_q k)^{2\sigma}}\right)^{k}.$$

Let $c=c(\sigma)=2\zeta_{\mathbb{A}}^2(2-\sigma)+\frac{4}{e}$. Thus we have

$$|\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}|\ll|\mathcal{H}_n|\left(c\frac{k^{2-2\sigma}(\log_q\log_q|D|)^2}{(\log_q k)^{2\sigma}(\log_q|D|)^{2(1-g(\sigma))}}\right)^{k}$$

Taking $k=\log_q|D|/c_1\log_q\log_q|D|$, with $c_1=12Ac^{1/(1-\sigma)}$ we obtain

$$|\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}|\ll|\mathcal{H}_n|\exp\left(\frac{\log_q|D|}{c_1\log_q\log_q|D|}\log\left(\frac{c}{c_1^{2(1-\sigma)}}(\log_q|D|)^{2(g(\sigma)-\sigma)}\right)\right),$$

with $\frac{c}{c_1^{2(1-\sigma)}}=\frac{1}{(12A)^{2(1-\sigma)}c}<1$, thus providing the result we want. □

In particular, we have

**Corollary 6.3.** *Let the assumptions of Lemma 6.5 hold and take $g(\sigma)=\sigma$. Then we have for some constant $C>0$*

$$\#\mathcal{H}_n \setminus \widetilde{\mathcal{H}}_{n,g} \ll \#\mathcal{H}_n \exp\left(-\frac{C\log_q |D|}{\log_q \log_q |D|}\right).$$

## 6.3 Complex moments of $L(s,\chi)$

Let $D\in\mathcal{H}_n$, $z\in\mathbb{C}$ such that $|z|\ll(\log_q|D|)^\sigma$ and set $\chi_D(f)=\left(\frac{D}{f}\right)$. We prove the following key lemma which will allow us to connect the large complex moments of $L(\sigma,\chi_D)$ to our random model $L(\sigma,\mathbb{X})$.

**Lemma 6.6.** *Let $1/2<\sigma<1$, $B\geq 2$ a constant and $A$ a constant satisfying the condition* (6.25) *be fixed. Let $D\in\mathcal{H}_n$, set $Y=A\log_q\log_q|D|$ and $z\in\mathbb{C}$ such that $|z|\ll(\log_q|D|)^\sigma$. Then we have*

$$\sum_{D\in\mathcal{H}_n} L(\sigma,\chi_D)^z = \sum_{D\in\mathcal{H}_n}\exp(zR_Y(\sigma,\chi_D))\left(1+O\left(\frac{1}{(\log_q|D|)^{B-1-\sigma}}\right)\right). \tag{6.31}$$

*Proof.* From Lemma 6.2 choosing $Y=A\log_q\log_q|D|$ with $B\geq 2$ and $A$ satisfying (6.25) we have

$$\begin{aligned}\sum_{D\in\mathcal{H}_n} L(\sigma,\chi_D)^z &= \sum_{D\in\mathcal{H}_n}\exp(z\log L(\sigma,\chi_D))\\ &= \sum_{D\in\mathcal{H}_n}\exp\left(z\left(R_Y(\sigma,\chi_D)+O\left(\frac{1}{\log_q|D|^{B-1}}\right)\right)\right)\\ &= \sum_{D\in\mathcal{H}_n}\exp(zR_Y(\sigma,\chi_D))\exp\left(O\left(\frac{|z|}{\log_q|D|^{B-1}}\right)\right),\end{aligned}$$

and the result follows using Taylor expansion and the assumptions on $z$ and $B$. ☐

### 6.3.1 Properties of the Random Model

In order to evaluate the moments of $L(\sigma,\chi_D)$ we consider the following random model: Let $\{\mathbb{X}(P)\}$ denote a sequence of independent random variables indexed by the irreducible (prime) elements $P\in\mathbb{A}$, and taking the values $0,\pm1$ as described in (6.3).

The goal of this section is to prove the following proposition, after which the proof of Theorem 6.2 follows from combining Proposition 6.2 with Lemma 6.2 and equation (6.32).

**Proposition 6.2.** *Let $1/2 < \sigma < 1$ and $A$ satisfying the condition* (6.25) *be fixed. Let $D \in \widetilde{\mathcal{H}}_{n,g}$ and set $Y = A\log_q \log_q |D|$. Then there exists positive constant $b_i = b_i(\sigma, A)$ for $i = 1, 2$ such that for all $z \in \mathbb{C}, |z| \leq b_1(\log_q |D|)^{g(\sigma)}$ we have*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} \exp(zR_Y(\sigma,\chi_D)) = \mathbb{E}(\exp(zR_Y(\sigma,\mathbb{X}))) + O\left(\exp\left(-b_2\frac{(\log_q |D|)^{g(\sigma)+1-\sigma}}{\log_q \log_q |D|}\right)\right).$$

In order to achieve this we recall that

$$L(\sigma,\mathbb{X}) = \prod_{P \text{ irreducible}} \left(1 - \frac{\mathbb{X}(P)}{|P|^\sigma}\right)^{-1},$$

converges almost surely for $\sigma > 1/2$ and the definition of $R_Y(\sigma, \mathbb{X})$ given by equation (6.7) is

$$R_Y(\sigma,\mathbb{X}) := \sum_{\substack{f \text{ monic} \\ \deg f \leq Y}} \frac{\Lambda(f)\mathbb{X}(f)}{\deg(f)|f|^\sigma}.$$

For the calculations we will require a Lemma 5.9 from the previous chapter, which we write here for ease of recall:

**Lemma 6.7.**

$$\mathbb{E}(\mathbb{X}(f)) = \begin{cases} 0 & \textit{if } f \textit{ is not a square} \\ \prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1} & \textit{if } f \textit{ is a square.}\end{cases}$$

**Lemma 6.8.** *Let $1/2 < \sigma < 1$, $B \geq 2$ a constant and $A$ a constant satisfying the condition* (6.25) *be fixed. Let $D \in \mathcal{H}_n$, set $Y = A\log_q \log_q |D|$ and let $z \in \mathbb{C}$ satisfying the assumption $|z| \ll (\log_q D)^\sigma$ that*

$$\mathbb{E}(L(\sigma,\mathbb{X})^z) = \mathbb{E}(\exp(zR_Y(\sigma,\mathbb{X})))\left(1 + O\left(\frac{1}{(\log_q |D|)^{B-1-\sigma}}\right)\right). \tag{6.32}$$

*Proof.* We begin by expressing

$$\mathbb{E}\left(L(\sigma,\mathbb{X})^z\right) = \mathbb{E}\left(\exp\left(zR_Y(\sigma,\mathbb{X}) + z\sum_{\deg(f)>Y}\frac{\Lambda(f)\mathbb{X}(f)}{\deg(f)|f|^\sigma}\right)\right).$$

Next we need to bound the contribution from the $f$ such that $\deg(f) > Y$, hence we write:

$$\sum_{\deg(f)>Y} \frac{\Lambda(f)\mathbb{X}(f)}{\deg(f)|f|^\sigma} = \sum_{\deg(P)>Y} \frac{\mathbb{X}(P)}{|P|^\sigma} + \sum_{n\geq 2} \sum_{n\deg(P)>Y} \frac{\mathbb{X}(P^n)}{n|P|^{n\sigma}}.$$

We show this second sum is small:

$$\sum_{n\geq 2} \sum_{n\deg(P)>Y} \frac{\mathbb{X}(P^n)}{n|P|^{n\sigma}} \ll \sum_{n\geq 2} \sum_{\deg(P)>Y/n} \frac{1}{n|P|^{n\sigma}} \ll \frac{q^{Y/2(1-2\sigma)}}{Y},$$

with the choice of $Y$ we have this $O((\log_q |D|)^{-B})$. So far, making use of the assumption on $z$, we have seen that

$$\mathbb{E}\left(L(\sigma,\mathbb{X})^z\right) = \mathbb{E}\left(\exp\left(zR_Y(\sigma,\mathbb{X}) + z\sum_{\deg(P)>Y} \frac{\mathbb{X}(P)}{|P|^\sigma}\right)\right)\left(1 + O\left(\frac{1}{\log_q |D|^{B-\sigma}}\right)\right). \tag{6.33}$$

Now, the remaining sums do not share any primes in common, hence we have

$$\mathbb{E}\left(L(\sigma,\mathbb{X})^z\right) = \mathbb{E}(\exp(zR_Y(\sigma,\mathbb{X})))\mathbb{E}\left(\exp\left(z\sum_{\deg(P)>Y} \frac{\mathbb{X}(P)}{|P|^\sigma}\right)\right)\left(1 + O\left(\frac{1}{\log_q |D|^{B-\sigma}}\right)\right).$$

We note that

$$\mathbb{E}\left(\exp\left(z\sum_{\deg(P)>Y} \frac{\mathbb{X}(P)}{|P|^\sigma}\right)\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!}\mathbb{E}\left(\left(\sum_{\deg(P)>Y} \frac{\mathbb{X}(P)}{|P|^\sigma}\right)^k\right).$$

We first give the following bound,

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{\deg(P)>Y} \frac{\mathbb{X}(P)}{|P|^\sigma}\right)^{2k}\right) &= \sum_{\deg(P_1),\ldots,\deg(P_{2k})>Y} \frac{\mathbb{E}\left(\prod_{j=1}^{2k} \mathbb{X}(P_j)\right)}{\prod_{j=1}^{2k} |P_j|^\sigma} \\
&= \frac{(2k)!}{2^k k!} \sum_{\deg(P_1),\ldots,\deg(P_k)>Y} \frac{\prod_{i=1}^{k}\left(1+\frac{1}{|P_i|}\right)^{-1}}{\prod_{j=1}^{k} |P_j|^{2\sigma}} \text{ by Lemma 6.7,} \\
&\ll \left(k\sum_{\deg(P)>Y} \frac{1}{|P|^{2\sigma}}\right)^k \text{ since } \prod_{i=1}^{k}\left(1+\frac{1}{|P_i|}\right)^{-1} < 1 \text{ and Stirling's approximation,} \\
&\ll \left(\frac{k}{(\log_q |D|)^{2B}}\right)^k \text{ by (6.22) and assumption on } Y.
\end{aligned}$$

By Cauchy-Schwarz we have

$$\left|\sum_{k=0}^{\infty}\frac{z^k}{k!}\mathbb{E}\left(\left(\sum_{\deg(P)>Y}\frac{\mathbb{X}(P)}{|P|^{\sigma}}\right)^k\right)\right| \leq \sum_{k=0}^{\infty}\frac{|z|^k}{k!}\left(\mathbb{E}\left(\left(\sum_{\deg(P)>Y}\frac{\mathbb{X}(P)}{|P|^{\sigma}}\right)^{2k}\right)\right)^{\frac{1}{2}}$$
$$\ll \sum_{k=0}^{\infty}\frac{|z|^k}{k!}\left(\frac{k}{(\log_q|D|)^{2B}}\right)^{\frac{k}{2}}$$
$$= \sum_{0\leq k\leq \log_q|D|}\frac{|z|^k}{k!}\left(\frac{k}{(\log_q|D|)^{2B}}\right)^{\frac{k}{2}} + \sum_{k>\log_q|D|}\frac{|z|^k}{k!}\left(\frac{k}{(\log_q|D|)^{2B}}\right)^{\frac{k}{2}} \tag{6.34}$$

For the second sum, using the assumption on $|z|$ and Stirling's approximation we have

$$\sum_{k>\log_q|D|}\frac{|z|^k}{k!}\left(\frac{k}{(\log_q|D|)^{2B}}\right)^{\frac{k}{2}} \ll \sum_{k>\log_q|D|}\left(\frac{e}{\sqrt{k}(\log_q|D|)^{B-\sigma}}\right)^k$$
$$\ll \sum_{k>\log_q|D|}\left(\frac{e}{(\log_q|D|)^{B-\sigma+1/2}}\right)^k \ll \frac{1}{(\log_q|D|)^{B-\sigma+1/2}}. \tag{6.35}$$

For the first one using the assumption on $|z|$ we have

$$\sum_{0\leq k\leq \log_q|D|}\frac{|z|^k}{k!}\left(\frac{k}{(\log_q|D|)^{2B}}\right)^{\frac{k}{2}} \ll \sum_{0\leq k\leq \log_q|D|}\frac{1}{k!}\left(\frac{|z|}{(\log_q|D|)^{B-1/2}}\right)^k$$
$$\ll \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{(\log_q|D|)^{B-\sigma-1/2}}\right)^k = \exp\left(\frac{1}{(\log_q|D|)^{B-\sigma-1/2}}\right). \tag{6.36}$$

Combining equations (6.33),(6.34), (6.35) and (6.36) we conclude

$$\mathbb{E}\left(L(\sigma,\mathbb{X})^z\right) = \mathbb{E}(\exp(zR_Y(\sigma,\mathbb{X})))\left(1+O\left(\frac{1}{(\log_q|D|)^{B-\sigma-1}}\right)\right),$$

as desired. □

With this in mind we can complete the proof of Proposition 6.2 based on the next two lemmata.

**Lemma 6.9.** *Let $1/2<\sigma<1$ and $A\geq 1$ be fixed. Let $D\in\mathcal{H}_n$, let $Y = A\log_q\log_q|D|$, and $k$ a positive* *integer such that* $k \leq \frac{\log_q |D|}{12A \log_q \log_q |D|}$. *Then, there exists a constant* $a(\sigma) > 0$ *such that*

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} (R_Y(\sigma, \chi_D))^{2k} \ll \left( \frac{a(\sigma) k^{1-\sigma}}{(\log_q k)^\sigma} \right)^{2k}.$$

*Furthermore, for any integer* $k \geq 2$ *we have*

$$\mathbb{E}((R_Y(\sigma, \mathbb{X}))^{2k}) \ll \left( \frac{a(\sigma) k^{1-\sigma}}{(\log_q k)^\sigma} \right)^{2k}.$$

*Proof.* We only prove the first assertion as the second one follows with a similar argument. We begin by writing

$$\sum_{D \in \mathcal{H}_n} (R_Y(\sigma, \chi_D))^{2k} \leq 2^k \left( \sum_{D \in \mathcal{H}_n} \left( \sum_{\deg(f) \leq \log_q k + \log_q \log_q k} \frac{\chi_D(P)}{|P|^\sigma} \right)^{2k} \right.$$
$$\left. + \sum_{D \in \mathcal{H}_n} \left( \sum_{\log_q k + \log_q \log_q k < \deg(P) \leq Y} \frac{\chi_D(P)}{|P|^\sigma} \right)^{2k} \right) + O(|\mathcal{H}_n| \log_q^{2k} L(2\sigma, \chi_D)). \quad (6.37)$$

From Lemma 6.4, (6.22) and Stirling's formula we have

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left( \sum_{\log_q k + \log_q \log_q k < \deg(P) \leq Y} \frac{\chi_D(P)}{|P|^\sigma} \right)^{2k} \ll \frac{(2k)!}{2^k k!} \left( \sum_{\log_q k + \log_q \log_q k < \deg(P) \leq Y} \frac{1}{|P|^{2\sigma}} \right)^k + |\mathcal{H}_n|^{-1/3+\epsilon}$$
$$\ll \frac{(2k)!}{2^k k!} \left( \frac{q^{(1-2\sigma)(\log_q k + \log_q \log_q k)}}{\log_q k} \right)^k$$
$$\ll \left( \frac{2}{e} \frac{k^{1-\sigma}}{(\log_q k)^\sigma} \right)^{2k}.$$

Then from (6.21) we have

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \left( \sum_{\deg(P) \leq \log_q k + \log_q \log_q k} \frac{\chi_D(P)}{|P|^\sigma} \right)^{2k} \ll \left( \sum_{\deg(P) \leq \log_q k + \log_q \log_q k} \frac{1}{|P|^\sigma} \right)^{2k}$$
$$\ll \left( \zeta_{\mathbb{A}}(2-\sigma) \frac{k^{(1-\sigma)}}{(\log_q k)^\sigma} \right)^{2k}$$

Combing these estimates give the desired result. ☐

**Lemma 6.10.** *Let $1/2 < \sigma < 1$, $B \geq 2$ and $A$ satisfying condition (6.25) be fixed. Let $D \in \mathcal{H}_n$ and set $Y = A\log_q \log_q |D|$. Then for any positive integer $k \leq \frac{\log_q |D|}{12A\log_q \log_q |D|}$ and $\epsilon > 0$ we have*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}(R_Y(\sigma,\chi_d))^{2k} = \mathbb{E}((R_Y(\sigma,\mathbb{X}))^{2k}) + O\left(\frac{(q^{Y(1-\sigma+\epsilon)})^{2k}}{Y^{2k}\sqrt{|\mathcal{H}_n|}}\right).$$

*Proof.* Expanding the inner sum gives

$$\begin{aligned}
&\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\deg(f)\leq Y}\frac{\Lambda(f)\chi_D(f)}{\deg(f)|f|^\sigma}\right)^{2k} = \frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\left(\sum_{\deg(P^j)\leq Y}\frac{\chi_D(P^j)}{j|P|^{j\sigma}}\right)^{2k}\\
&= \sum_{\substack{\deg(P_1^{m_1}),\ldots,\deg(P_k^{m_k})\leq Y\\ \deg(Q_1^{n_1}),\ldots,\deg(Q_k^{n_k})\leq Y}}\frac{1}{\prod_{i=1}^k m_i|P_i|^{m_i\sigma}\prod_{j=1}^k n_j|Q_j|^{n_j\sigma}}\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\chi_D(P_1^{m_1}\cdots P_k^{m_k}Q_1^{n_1}\cdots Q_k^{n_k})\\
&= \Sigma_D + \Sigma_O,
\end{aligned}$$

where $\Sigma_D$ equals the sum over the terms where $P_1^{m_1}\cdots P_k^{m_k}Q_1^{n_1}\cdots Q_k^{n_k}$ is a square and $\Sigma_O$ is the remaining terms. By Lemma 6.3 and Lemma 6.7 we have

$$\Sigma_D = \mathbb{E}((R_Y(\sigma,\mathbb{X}))^{2k}).$$

Now for the remaining terms, let $P_1^{m_1}\cdots P_k^{m_k}Q_1^{n_1}\cdots Q_k^{n_k} = f_1 f_2^2$ with $|f_1| > 1$ square free then for $\epsilon > 0$ Lemma 6.3 says

$$\begin{aligned}
\Sigma_O &\ll_\epsilon \frac{1}{\sqrt{|\mathcal{H}_n|(q-1)}}\sum_{\substack{\deg(P_1^{m_1}),\ldots,\deg(P_k^{m_k})\leq Y\\ \deg(Q_1^{n_1}),\ldots,\deg(Q_k^{n_k})\leq Y}}\frac{|f_1|^\epsilon}{m_1\cdots m_k n_1\cdots n_k|f_1|^\sigma|f_2|^{2\sigma}}\\
&\ll_\epsilon \frac{1}{\sqrt{|\mathcal{H}_n|(q-1)}}\left(\sum_{\deg P\leq Y}\frac{1}{|P|^{\sigma-\epsilon}}\right)^{2k}.
\end{aligned}$$

Using a similar argument as in equation (6.21)

$$\Sigma_O \ll_\epsilon \frac{(q^{Y(1-\sigma+\epsilon)})^{2k}}{Y^{2k}\sqrt{|\mathcal{H}_n|}}.$$

□

*Proof of Proposition 6.2.* We begin with the taylor expansion for $\exp(x)$:

$$\begin{aligned}\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}\exp(zR_Y(\sigma,\chi_D)) &= \frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}\sum_{l=0}^{\infty}\frac{(zR_Y(\sigma,\chi_D))^l}{l!}\\ &= \frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}\left(\sum_{l<N}\frac{(zR_Y(\sigma,\chi_D))^l}{l!}+E_1\right).\end{aligned}\tag{6.38}$$

To estimate $E_1$ we use the fact that $D\in\widetilde{\mathcal{H}}_{n,g}$ so that $R_Y(\sigma,\chi_D)\le(\log_q|D|)^{1-g(\sigma)}/\log_q\log_q|D|$ and Stirling's approximation to obtain

$$\begin{aligned}\sum_{l\ge N}\frac{(zR_Y(\sigma,\chi_D))^l}{l!} &\ll \sum_{l\ge N}\frac{|z|^l}{l!}\left(\frac{(\log_q|D|)^{1-g(\sigma)}}{\log_q\log_q|D|}\right)^l\\ &\ll \sum_{l\ge N}\left(\frac{3|z|(\log_q|D|)^{1-g(\sigma)}}{N\log_q\log_q|D|}\right)^l.\end{aligned}$$

Taking $N=\log_q|D|/b_0\log_q\log_q|D|$, $b_0$ is taken large enough so that setting $k=2N$, $k$ satisfies the assumptions of Lemma 6.9, then there exists a $b_1=b_1(\sigma,A)$ such that for all $|z|\le b_1(\log_q|D|)^{g(\sigma)}$ the above calculation gives

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}E_1\ll\sum_{l>N}(3b_1b_0)^l\ll e^{-N}.$$

In order to apply Lemma 6.3 we must complete the sum, and so we describe below the error which comes from those $D\in\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}$. So if, $l<N$, where $N$ is chosen as described in the analysis of $E_1$, we may apply Lemmas 6.9 and 6.5 in combination with Cauchy-Schwarz inequality to see

$$\begin{aligned}\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}}(R_Y(\sigma,\chi_D))^l &\le \left(\frac{|\mathcal{H}_n\setminus\widetilde{\mathcal{H}}_{n,g}|}{|\mathcal{H}_n|}\right)^{1/2}\left(\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}(R_Y(\sigma,\chi_D))^{2l}\right)^{1/2}\\ &\ll \exp\left(\frac{C\log_q|D|}{2\log_q\log_q|D|}(2(g(\sigma)-\sigma)\log\log_q|D|-C')\right)\left(\frac{a(\sigma)l^{1-\sigma}}{(\log_q(l+2))^\sigma}\right)^l\\ &\ll \exp\left(\frac{-C''\log_q|D|}{\log_q\log_q|D|}\right)\left(\frac{a(\sigma)l^{1-\sigma}}{(\log_q(l+2))^\sigma}\right)^l,\end{aligned}$$

where $C,C'$ and $a(\sigma)$ are positive constants and the last line follows since $2(g(\sigma)-\sigma)\le 0$, so that for some $C''>0$ we can make the claimed simplification.

Inserting this into (6.38) we get

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}\exp(zR_Y(\sigma,\chi_D)) = \frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\sum_{l<N}\frac{(zR_Y(\sigma,\chi_D))^l}{l!} + E_2, \tag{6.39}$$

where

$$\begin{aligned} E_2 &\ll \exp\left(\frac{-C''\log_q|D|}{\log_q\log_q|D|}\right)\sum_{l<N}\frac{|z|^l}{l!}\left(\frac{a(\sigma)l^{1-\sigma}}{(\log_q(l+2))^\sigma}\right)^l + e^{-N} \\ &\ll \exp\left(\frac{-C''\log_q|D|}{\log_q\log_q|D|}\right)\sum_{l<N}\frac{|z|^l}{l!}\left(\frac{a(\sigma)N^{1-\sigma}}{(\log_q(N+2))^\sigma}\right)^l + e^{-N} \\ &\ll \exp\left(\frac{-C''\log_q|D|}{\log_q\log_q|D|}\right)\exp\left(\frac{a(\sigma)|z|N^{1-\sigma}}{(\log_q(N+2))^\sigma}\right) + e^{-N} \\ &\ll \exp\left(\frac{-C'''(\log_q|D|)^{g(\sigma)+1-\sigma}}{2\log_q\log_q|D|}\right) + e^{-N}, \end{aligned} \tag{6.40}$$

with $N$ and $z$ chosen as above, that is for $b_0$ large enough and $b_1$ small enough.

Now, for all $l < N$, we have by our choice of $N$ that Lemma 6.10 applies and we obtain

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}(R_Y(\sigma,\chi_D))^l = \mathbb{E}((R_Y(\sigma,\mathbb{X}))^l) + O\left(\frac{(q^{Y(1-\sigma+\epsilon)})^l}{Y^l|\mathcal{H}_n|^{1/2}}\right).$$

We note that $(q^Y)^N \ll |\mathcal{H}_n|^{1/12}$ as long as $b_0$ is taken suitably large. Hence, combining this with (6.39) and (6.40) we find

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}\exp(zR_Y(\sigma,\chi_D)) = \sum_{l<N}\frac{z^l}{l!}\mathbb{E}((R_Y(\sigma,\mathbb{X}))^l) + O\left(\exp\left(\frac{-C''\log_q|D|}{2\log_q\log_q|D|}\right)\right). \tag{6.41}$$

Furthermore, for every $l \geq 2$ we have by Lemma 6.9 that

$$\mathbb{E}((R_Y(\sigma,\mathbb{X}))^l) \ll \left(a(\sigma)\frac{l^{1-\sigma}}{\sqrt{\log_q l}}\right)^l.$$

Therefore, the main term on the RHS of (6.41) equals

$$\mathbb{E}(\exp(zR_Y(\sigma,\mathbb{X}))) + E_3,$$

where

$$E_3 \ll \sum_{l\geq N} \frac{|z|^l}{l!}\left(a(\sigma)\frac{l^{1-\sigma}}{\sqrt{\log_q l}}\right)^l \ll \sum_{l\geq N}\left(a(\sigma)\frac{|z|}{l^{\sigma}\sqrt{\log_q l}}\right)^l \ll \sum_{l\geq N}\left(a(\sigma)\frac{|z|}{N^{\sigma}\sqrt{\log_q N}}\right)^l \ll e^{-N}.$$

This completes the proof. □

*Proof of Theorem 6.2.* Using Lemma 6.2, let $B \geq 2$ and choose $Y = A\log_q\log_q|D|$ with $A$ satisfying (6.25). Then we have

$$\begin{aligned}
\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} L(\sigma,\chi_D)^z &= \frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} \exp(z\log L(\sigma,\chi_D)) \\
&= \frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} \exp\left(z\left(R_Y(\sigma,\chi_D)+O\left(\frac{1}{\log_q|D|^{B-1}}\right)\right)\right) \\
&= \frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} \exp(zR_Y(\sigma,\chi_D)) + E_4, \qquad (6.42)
\end{aligned}$$

where

$$\begin{aligned}
E_4 &\ll \frac{1}{|\mathcal{H}_n|(\log_q|D|)^{B-1-g(\sigma)}}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} \exp(zR_Y(\sigma,\chi_D)) \\
&\ll \frac{1}{(\log_q|D|)^{B-1-g(\sigma)}}\mathbb{E}(\exp(\mathfrak{Re}(z)R_Y(\sigma,\mathbb{X}))) \\
&\ll \frac{1}{(\log_q|D|)^{B-1-g(\sigma)}}\mathbb{E}(L(\sigma,\mathbb{X})^{\mathfrak{Re}(z)})
\end{aligned}$$

follows from Proposition 6.2 and (6.32). Applying these to the main term we obtain

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} L(\sigma,\chi_D)^z = \mathbb{E}(L(\sigma,\mathbb{X})^z) + O\left(\frac{\mathbb{E}(L(\sigma,\mathbb{X})^{\mathfrak{Re}(z)})}{(\log_q|D|)^{B-1-g(\sigma)}}\right),$$

as desired. □

*Proof of Corollaries 6.2 and 6.1.* For the first, take $g(\sigma)=\sigma$ in Theorem 6.2 and the simplified expression for the bound on $|\mathcal{H}_n \setminus \widetilde{\mathcal{H}}_{n,g}|$ from Corollary 6.3. For the second, take $g(\sigma) = 2\sigma - 1$ in Theorem 6.2 and use the Riemann hypothesis to show $\mathcal{H}_n = \widetilde{\mathcal{H}}_{n,g}$. □

## 6.4 The distribution of values of $L(\sigma, \mathbb{X})$

For $\tau > 0$, recall

$$\Phi_{\mathbb{X},\sigma}(\tau) := \mathbb{P}(\log L(\sigma, \mathbb{X}) > \tau).$$

For $z \in \mathbb{C}$ and $\frac{1}{2} < \sigma < 1$ define

$$\mathcal{L}_\sigma(z) := \log \mathbb{E}(L(\sigma, \mathbb{X})^z) = \sum_{P \text{ irreducible}} \log(E_{P,\sigma}(z)), \tag{6.43}$$

where $E_{P,\sigma}(z)$ is defined as

$$E_{P,\sigma}(z) := \mathbb{E}\left(\left(1 - \frac{\mathbb{X}(P)}{|P|^\sigma}\right)^{-z}\right) = \left(\frac{1}{|P|+1} + \frac{|P|}{2(|P|+1)}\left(\left(1 - \frac{1}{|P|^\sigma}\right)^{-z} + \left(1 + \frac{1}{|P|^\sigma}\right)^{-z}\right)\right). \tag{6.44}$$

Furthermore, for $\sigma$ fixed consider the equation

$$(\mathbb{E}(L(\sigma, \mathbb{X})^r)e^{-\tau r})' = 0 \Leftrightarrow \mathcal{L}'_\sigma(r) = \tau, \tag{6.45}$$

where the derivative is taken with respect to the real variable $r$. It follows from Proposition 6.3 that $\lim_{r\to\infty} \mathcal{L}'_\sigma(r) = \infty$, one can check that $E''_{P,\sigma}(r)E_{P,\sigma}(r) > (E'_{P,\sigma}(r))^2$ for all monic irreducible polynomials $P$, and thus $\mathcal{L}''_\sigma(r) > 0$. Therefore (6.45) has a unique solution: we define $\kappa := \kappa_\sigma(\tau)$ to represent this unique solution.

### 6.4.1 Distribution of the random model.

**Theorem 6.6.** *Let $\tau$ be large and $\kappa$ denote the unique solution to* (6.45)*. Then, we have*

$$\Phi_{\mathbb{X},\sigma}(\tau) = \frac{\mathbb{E}(L(\sigma, \mathbb{X})^\kappa)e^{-\tau\kappa}}{\kappa\sqrt{2\pi\mathcal{L}''_\sigma(\kappa)}}\left(1 + O\left(\kappa^{1-\frac{3}{2\sigma}}(\log_q \kappa)^{3/2}\right)\right). \tag{6.46}$$

**Proposition 6.3.** *Let $\frac{1}{2} < \sigma < 1$ be fixed. Let $k \in \mathbb{Z}$ be the unique positive integer such that $q^k \leq r < q^{k+1}$*

*and let* $t := \frac{r}{q^k}$. *We have for any real number* $r$ *large enough, with* $c_q$ *a positive constant depending on* $q$ *that*

$$\mathcal{L}_\sigma(r) = G_1(t,\sigma)\frac{r^{\frac{1}{\sigma}}}{\lfloor \log_q r \rfloor} + O_\sigma\left(\frac{r^{\frac{1}{\sigma}}}{(\log_q r)^{3-\frac{1}{\sigma}}} + \frac{r^{\frac{1}{\sigma}}}{(\log_q r)^{\frac{1}{\sigma}}}\right), \tag{6.47}$$

*where*

$$G_1(t,\sigma) := \sum_{l=-\infty}^{\infty} \frac{\log\cosh\left(q^{l\sigma}t\right)}{t^{\frac{1}{\sigma}}q^l}, \tag{6.48}$$

*and*

$$\mathcal{L}'_\sigma(r) = G_2(t,\sigma)\frac{r^{\frac{1}{\sigma}-1}}{\lfloor \log_q r \rfloor} + O_\sigma\left(\frac{r^{\frac{1}{\sigma}-1}}{(\log_q r)^{3-\frac{1}{\sigma}}} + \frac{r^{\frac{1}{\sigma}-1}}{(\log_q r)^{\frac{1}{\sigma}}}\right), \tag{6.49}$$

*where*

$$G_2(t,\sigma) := \sum_{l=-\infty}^{\infty} \frac{\tanh(q^{l\sigma}t)}{t^{\frac{1}{\sigma}-1}q^{l(1-\sigma)}}. \tag{6.50}$$

*Furthermore, we have*

$$\mathcal{L}''_\sigma(r) \asymp_\sigma \frac{r^{\frac{1}{\sigma}-2}}{\log_q r} \text{ and } \mathcal{L}'''_\sigma(r+it) \ll_\sigma \frac{r^{\frac{1}{\sigma}-3}}{\log_q r}, \tag{6.51}$$

*for all* $|t| \leq r$.

**Remark.** *For each fixed* $\sigma \in (1/2, 1)$ *we have* $G_i(t,\sigma)$ *for* $i = 1, 2$ *are multiplicatively periodic with multiplicator* $q^\sigma$. *Indeed, we see for* $n \in \mathbb{Z}$

$$G_1(t,\sigma) := \sum_{l=-\infty}^{\infty} \frac{\log\cosh\left(q^{l\sigma}t\right)}{t^{\frac{1}{\sigma}}q^l} \text{ and } G_1(tq^{n\sigma},\sigma) := \sum_{l=-\infty}^{\infty} \frac{\log\cosh\left(q^{(l+n)\sigma}t\right)}{t^{\frac{1}{\sigma}}q^{l+n}},$$
$$G_2(t,\sigma) := \sum_{l=-\infty}^{\infty} \frac{\tanh\left(q^{l\sigma}t\right)}{t^{\frac{1}{\sigma}-1}q^{l(1-\sigma)}} \text{ and } G_1(tq^{n\sigma},\sigma) := \sum_{l=-\infty}^{\infty} \frac{\tanh\left(q^{(l+n)\sigma}t\right)}{t^{\frac{1}{\sigma}-1}q^{(l+n)(1-\sigma)}},$$

*Now, since the sum is over all* $l \in (-\infty, \infty)$ *we see* $G_1(t,\sigma) = G_1(tq^{n\sigma},\sigma)$ *and* $G_2(t,\sigma) = G_2(tq^{n\sigma},\sigma)$. *If we consider only the series portion of the* $G_i(t)$ *defined in Proposition 5.2 we observe similar behaviour except with mulitplicator* $q$.

**Corollary 6.4.** *Let* $\tau$ *be a large number and let* $\kappa = \kappa_\sigma(\tau)$ *be defined as the unique solution to* (6.45). *Then*

$$\kappa = G_3(q^{\{\log_q \kappa\}},\sigma)(\tau \log_q \tau)^{\frac{\sigma}{1-\sigma}}\left(1 + O_\sigma\left(\frac{\log_q \log_q \tau}{(\log_q \tau)^{2-\frac{1}{\sigma}}}\right)\right),$$

*where*

$$G_3(t,\sigma) := \left(\frac{\sigma}{(1-\sigma)G_2(t,\sigma)}\right)^{\frac{\sigma}{1-\sigma}}. \tag{6.52}$$

Combining the results in this section we obtain our Theorem 6.4.

*Proof of Theorem 6.4.* By Theorem 6.6 and (6.51) we have

$$\begin{aligned}\Phi_{\mathbb{X},\sigma}(\tau) &= \frac{\mathbb{E}(L(\sigma,\mathbb{X})^{\kappa})e^{-\kappa\tau}}{\kappa\sqrt{2\pi\mathcal{L}_\sigma''(\kappa)}}\left(1+O(\kappa^{1-\frac{3}{2\sigma}}(\log_q \kappa))^{3/2}\right)\\ &= \exp\left(\mathcal{L}_\sigma(\kappa)-\tau\kappa+O\left(\frac{\kappa^{\frac{1}{\sigma}-1}}{\log_q \kappa}\right)\right)\end{aligned}$$

where $\kappa$ is the unique solution to (6.45). From (6.49) we have

$$\tau = G_2(q^{\{\log_q \kappa\}},\sigma)\frac{\kappa^{\frac{1}{\sigma}-1}}{\lfloor \log_q \kappa\rfloor}+O_\sigma\left(\frac{\kappa^{\frac{1}{\sigma}-1}}{(\log_q \kappa)^{3-\frac{1}{\sigma}}}\right).$$

Hence applying (6.47) we have

$$\Phi_{\mathbb{X},\sigma}(\tau) = \exp\left((G_1(q^{\{\log_q \kappa\}},\sigma)-G_2(q^{\{\log_q \kappa\}},\sigma))\frac{\kappa^{\frac{1}{\sigma}}}{\lfloor \log_q \kappa\rfloor}+O\left(\frac{\kappa^{\frac{1}{\sigma}}}{(\log_q \kappa)^{3-\frac{1}{\sigma}}}\right)\right).$$

Now by Corollary 6.4 we have

$$\begin{aligned}\kappa^{\frac{1}{\sigma}} &= \left(G_3(q^{\{\log_q \kappa\}},\sigma)(\tau\log_q \tau)^{\frac{\sigma}{1-\sigma}}\left(1+O\left(\frac{\log_q\log_q \tau}{(\log_q \tau)^{2-\frac{1}{\sigma}}}\right)\right)\right)^{\frac{1}{\sigma}}\\ &= G_3(q^{\{\log_q \kappa\}},\sigma)^{\frac{1}{\sigma}}(\tau\log_q \tau)^{\frac{1}{1-\sigma}}\left(1+O_\sigma\left(\frac{\log_q\log_q \tau}{(\log_q \tau)^{2-\frac{1}{\sigma}}}\right)\right).\end{aligned}$$

This implies that

$$\Phi_{\mathbb{X},\sigma}(\tau) = \exp\left(-A_{\mathbb{X}}(q^{\{\log_q \kappa\}},\sigma)\tau^{\frac{1}{1-\sigma}}(\log_q \tau)^{\frac{\sigma}{1-\sigma}}\left(1+O_\sigma\left(\frac{\log_q\log_q \tau}{(\log_q \tau)^{2-\frac{1}{\sigma}}}\right)\right)\right),$$

with

$$A_{\mathbb{X}}(t,\sigma) = \frac{1-\sigma}{\sigma}\left(G_2(t,\sigma)-G_1(t,\sigma)\right)\left(G_3(t,\sigma)\right)^{\frac{1}{\sigma}}$$

and $G_i(t,\sigma)$ for $i=1,2,3$ are defined respectively as (6.48), (6.50) and (6.52). Thus we have the shape of the result as claimed. It remains to prove that $A_{\mathbb{X}}(t,\sigma)$ is nonnegative. We have that $G_3(t,\sigma)^{\frac{1}{\sigma}}\geq 0$ so it is enough to show that $G_2(t,\sigma)-G_1(t,\sigma)\geq 0$. The analysis of this is very similar to the discussion of

$G_2(t) - G_1(t)$ in the proof of Theorem 5.3. We apply the definitions of the $G_i(t,\sigma)$ for $i = 1, 2$ to see that

$$G_2(t,\sigma) - G_1(t,\sigma) = \sum_{l=-\infty}^{\infty} \frac{tq^{l\sigma}\tanh(q^{l\sigma}t) - \log\cosh(q^{l\sigma}t)}{q^l t^{\frac{1}{\sigma}}},$$

thus we can prove $G_2(t,\sigma) - G_1(t,\sigma) \geq 0$ by showing $tq^{l\sigma}\tanh(q^{l\sigma}t) - \log\cosh(q^{l\sigma}t) \geq 0$ for all $l$. Let $y = tq^{l\sigma}$, then since $l \in (-\infty, \infty)$ we have $y \in (0, \infty)$ and

$$\lim_{y\to 0} y\tanh y - \log\cosh y = 0, \text{ and } \frac{d}{dy} y\tanh y - \log\cosh y = y\text{sech}^2 y > 0 \text{ for all } y > 0,$$

these two facts prove the summands are nonnegative as desired. □

#### 6.4.1.1 Tools for proving Proposition 6.3

The proof of Proposition 6.3 depends on two preliminary Lemmas, some standard estimates for $\log\cosh(t)$ and $\tanh(t)$ and a generalization of Lemma 5.11.

**Lemma 6.11.** *We have* $\log\cosh(t) = t + O(1)$ *on* $[0,\infty)$ *and* $\log\cosh(t) = t^2/2 + O(t^4)$ *if* $0 \leq t < 1$. *Moreover, we have* $\tanh(t)$ *is bounded on* $[0,\infty)$ *and* $\tanh(t) = t + O(t^2)$ *for* $0 \leq t < 1$.

*Proof.* This follows directly from [63, Lemma 4.5]. □

**Lemma 6.12.** *Let* $r \geq c_q > 4$ *be a real number, where* $c_q$ *is a positive constant depending on* $q$. *Let* $\frac{1}{2} < \alpha < 1$. *Then for* $\frac{1}{2} < \sigma < 1$ *fixed we have*

$$\log E_{P,\sigma}(r) = \begin{cases} -r\log(1 - 1/|P|^\sigma) + O(1) & \text{if } |P| \leq r^\alpha, \\ \log\cosh\left(\frac{r}{|P|^\sigma}\right) + O\left(\frac{r}{|P|^{2\sigma}}\right) & \text{if } |P| > r^\alpha. \end{cases} \tag{6.53}$$

*Furthermore,*

$$\frac{E'_{P,\sigma}}{E_{P,\sigma}}(r) = \begin{cases} -\log\left(1 - \frac{1}{|P|^\sigma}\right)\left(1 + O\left(e^{-r^\alpha}\right)\right) & \text{if } |P| \leq r^\alpha \\ \frac{1}{|P|^\sigma}\tanh\left(\frac{r}{|P|^\sigma}\right) + O\left(\frac{1}{|P|^{2\sigma}} + \frac{r}{|P|^{3\sigma}}\right) & \text{if } |P| > r^\alpha. \end{cases} \tag{6.54}$$

The proof of this lemma is exactly the same as Lemma 5.11.

*Proof of Proposition 6.3.* We only prove (6.47) and (6.49) as the proofs for (6.51) follow the same structure.

For the entire proof, we recall that $k \in \mathbb{Z}$ is the unique positive integer such that $q^k \le r < q^{k+1}$ and let $t := \frac{r}{q^k}$. First apply Lemma 6.12 and (6.22) to obtain

$$\mathcal{L}_\sigma(r) = -r \sum_{|P| \le r^\alpha} \log\left(1 - \frac{1}{|P|^\sigma}\right) + \sum_{|P| > r^\alpha} \log\cosh\left(\frac{r}{|P|^\sigma}\right) + O(r^\alpha) + O(r^{1-\alpha(2\sigma-1)}),$$

taking $\alpha = \frac{1}{2\sigma}$ balances the error terms above. For the first sum, we write using the PNT and (6.21) we have

$$\begin{aligned}
-r \sum_{|P| \le r^{\frac{1}{2\sigma}}} \log\left(1 - \frac{1}{|P|^\sigma}\right) &= r \sum_{|P| \le r^{\frac{1}{2\sigma}}} \sum_{n=1}^{\infty} \frac{1}{n|P|^{n\sigma}} = r \sum_{d \le \frac{\log_q r}{2\sigma}} \sum_{n=1}^{\infty} \frac{\pi_q(d)}{nq^{nd\sigma}} \\
&= r \sum_{dn \le \frac{\log_q r}{2\sigma}} \frac{\pi_q(d)}{nq^{nd\sigma}} + r \sum_{\substack{d \le \frac{\log_q r}{2\sigma} \\ dn > \frac{\log_q r}{2\sigma}}} \frac{\pi_q(d)}{nq^{nd\sigma}} \\
&= r \sum_{m \le \frac{\log_q r}{2\sigma}} \frac{1}{mq^{m\sigma}} \sum_{d|m} d\pi_q(d) + r \sum_{\substack{d \le \frac{\log_q r}{2\sigma} \\ dn > \frac{\log_q r}{2\sigma}}} \frac{\pi_q(d)}{nq^{nd\sigma}} \\
&= r \sum_{m \le \frac{\log_q r}{2\sigma}} \frac{q^{m(1-\sigma)}}{m} + O\left( r \sum_{\substack{d \le \frac{\log_q r}{2\sigma} \\ dn > \frac{\log_q r}{2\sigma}}} \frac{q^d}{dnq^{dn\sigma}} \right) \\
&= r\zeta_{\mathbb{A}}(2-\sigma) \frac{q^{\frac{\log_q r}{2\sigma}(1-\sigma)}}{\frac{\log_q r}{2\sigma}} + O\left( r \frac{q^{(\frac{\log_q r}{2\sigma})(1-\sigma)}}{(\frac{\log_q r}{2\sigma})^{2-\sigma}} \right).
\end{aligned}$$

So that finally we have

$$-r \sum_{\deg P \le \frac{\log_q r}{2\sigma}} \log\left(1 - \frac{1}{|P|^\sigma}\right) = 2\sigma\zeta_{\mathbb{A}}(2-\sigma) \frac{r^{\frac{1}{2}+\frac{1}{2\sigma}}}{\log_q r} + O\left( \frac{r^{\frac{1}{2}+\frac{1}{2\sigma}}}{(\log_q r)^{2-\sigma}} \right).$$

this is an error term when compared to the result of the second sum, which provides a more interesting calculation.

$$\sum_{|P| > r^{\frac{1}{2\sigma}}} \log\cosh\left(\frac{r}{|P|^\sigma}\right) = \sum_{n > \frac{1}{2\sigma}\log_q r} \frac{q^n}{n} \log\cosh\left(\frac{r}{q^{n\sigma}}\right) + O\left( \sum_{n > \frac{1}{2\sigma}\log_q r} \frac{q^{n/2}}{n} \log\cosh\left(\frac{r}{q^{n\sigma}}\right) \right).$$

We note that from Lemma 6.11 we have $\log\cosh t = t^2 + O(t^4)$ when $0 \le t < 1$ and $\log\cosh t = t + O(1)$ if

$t > 1$, so that the error term becomes

$$\sum_{n > \frac{1}{2\sigma}\log_q r} \frac{q^{n/2}}{n} f\left(\frac{r}{q^{n\sigma}}\right) \ll \sum_{n \geq \frac{1}{\sigma}\log_q r} q^{n/2}\frac{r^2}{q^{2n\sigma}} + \sum_{\frac{1}{2\sigma}\log_q r < n < \frac{1}{\sigma}\log_q r} \frac{q^{n/2} r}{n q^{n\sigma}} \ll r$$

It remains to consider

$$\sum_{n > \frac{1}{2\sigma}\log_q r} \frac{q^n}{n} \log\cosh\left(\frac{r}{q^{n\sigma}}\right) = \sum_{n > \beta(k+\log_q k)} + \sum_{n < \beta(k-\log_q k)} + \sum_{k-\log_q k \leq \frac{n}{\beta} \leq k+\log_q k} \left(\frac{q^n}{n}\log\cosh\left(\frac{r}{q^{n\sigma}}\right)\right) = T_1+T_2+T_3.$$

For $T_1$ we note that for an appropriate choice of $\beta$ we have $r/q^{n\sigma}$ is less than 1, thus Lemma 6.11 gives

$$T_1 = \frac{r^2}{2k}\sum_{n>\beta(k+\log_q k)} \frac{1}{q^{n(2\sigma-1)}} + O\left(\frac{r^4}{k}\sum_{n>\beta(k+\log_q k)}\frac{1}{q^{n(4\sigma-1)}}\right) = \frac{r^{2+\beta(1-2\sigma)}}{2(\log_q r)^{\beta(2\sigma-1)+1}} + O\left(\frac{r^{4-\beta(4\sigma-1)}}{(\log_q r)^{\beta(4\sigma-1)+1}}\right).$$

For $T_2$ we see that $r/q^{n\sigma}$ in $f$ is greater than 1 so by Lemma 6.11 we have

$$T_2 \ll r \sum_{n<\beta(k-\log_q k)} \frac{q^{n(1-\sigma)}}{n} \ll \frac{r^{\beta(1-\sigma)+1}}{(\log_q r)^{\beta(1-\sigma)+1}}.$$

We choose $\beta$ in order to balance the error term in $T_1$ with $T_2$. Setting the exponents of $r$ equal provides $\beta = \frac{1}{\sigma}$.

For the remaining term, we note that since $\frac{n}{\beta} \in [k - \log_q k, k + \log_q k]$ then $\frac{\beta}{n} = \frac{\beta}{k}\left(1 + O\left(\frac{\log_q k}{k}\right)\right)$. We factor out $q^{\beta k}/k$ and use the variable change $l = \beta k - n$ to obtain

$$T_3 = \frac{q^{\beta k}}{k}\left(1 + O\left(\frac{\log_q k}{k}\right)\right)\sum_{|l| \leq \beta\log_q k} \frac{1}{q^l}\log\cosh\left(\frac{q^{l\sigma} r}{q^{\beta\sigma k}}\right) = \frac{q^{\frac{k}{\sigma}}}{k}\left(1 + O\left(\frac{\log_q k}{k}\right)\right)\sum_{|l| \leq \frac{\log_q k}{\sigma}} \frac{1}{q^l}\log\cosh\left(q^{l\sigma} t\right).$$

The last equality follows from the choice of $\beta$ and the definiton of $t$. Finally, we show that the remaining sum is small when $|l| > \beta \log_q k$. Applying Lemma 6.11 we have

$$\sum_{|l| > \beta\log_q k} \frac{1}{q^l}\log\cosh\left(q^{l\sigma}t\right) \ll \sum_{l > \beta\log_q k}\frac{1}{q^{l(1-\sigma)}} + \sum_{l < -\beta\log_q k} q^{l(2\sigma-1)} \ll 1.$$

Hence, we have

$$\sum_{n>\frac{1}{2\sigma}\log_q r} \frac{q^n}{n} \log\cosh\left(\frac{r}{q^{n\sigma}}\right) = \frac{q^{k/\sigma}}{k}\left(1+O\left(\frac{\log_q k}{k}\right)\right)\sum_{l=-\infty}^{\infty}\frac{1}{q^l}\log\cosh\left(q^{l\sigma}t\right).$$

Putting everything in terms of $r$ we get the desired result.

For $\mathcal{L}'_\sigma(r)$ we again use Lemma 6.12 to obtain

$$\mathcal{L}'_\sigma(r) = -\sum_{|P|\le r^\alpha}\log\left(1-\frac{1}{|P|^\sigma}\right) + \sum_{|P|>r^\alpha}\frac{1}{|P|^\sigma}\tanh\left(\frac{r}{|P|^\sigma}\right) + O(r^{-\alpha(2\sigma-1)}).$$

Choosing $\alpha = \frac{1}{2\sigma}$ as before makes the error term have size $r^{\frac{1}{2\sigma}-1}$.

Then the same calculation as was done above for the first sum provides

$$2\sigma\zeta_{\mathbb{A}}(2-\sigma)\frac{r^{\frac{1}{2\sigma}-\frac{1}{2}}}{\log_q r} + O\left(\frac{r^{\frac{1}{2\sigma}-\frac{1}{2}}}{\log_q r^{2-\sigma}}\right).$$

As in the analysis of $\mathcal{L}_\sigma(r)$ this term is an error when compared to the second sum. The ideas follow a similar pattern as above. We first apply the prime number theorem to obtain

$$\sum_{|P|>r^{1/2\sigma}}\frac{\tanh\left(\frac{r}{|P|^\sigma}\right)}{|P|^\sigma} = \sum_{n>\frac{1}{2\sigma}\log_q r}\frac{q^n}{n}\frac{\tanh\left(\frac{r}{q^{n\sigma}}\right)}{q^{n\sigma}} + O\left(\sum_{n>\frac{1}{2\sigma}\log_q r}\frac{q^{n/2}}{n}\frac{\tanh\left(\frac{r}{q^{n\sigma}}\right)}{q^{n\sigma}}\right).$$

By by Lemma 6.11 the error term has the size

$$\begin{aligned}\sum_{n>\frac{1}{2\sigma}\log_q r}\frac{q^{n/2}}{n}\frac{\tanh\left(\frac{r}{q^{n\sigma}}\right)}{q^{n\sigma}} &= \sum_{n>\frac{1}{\sigma}\log_q r}\frac{\tanh\left(\frac{r}{q^{n\sigma}}\right)}{nq^{n(\sigma-\frac{1}{2})}} + \sum_{\frac{1}{2\sigma}\log_q r<n\le\frac{1}{\sigma}\log_q r}\frac{\tanh\left(\frac{r}{q^{n\sigma}}\right)}{nq^{n(\sigma-\frac{1}{2})}}\\ &\ll \sum_{n>\frac{1}{\sigma}\log_q r}\frac{r}{nq^{n(2\sigma-1/2)}} + \sum_{\frac{1}{2\sigma}\log_q r<n\le\frac{1}{\sigma}\log_q r}\frac{1}{nq^{n(\sigma-1/2)}}\\ &\ll \frac{r^{\frac{1}{2\sigma}-1}}{\log_q r} + 1 \ll 1.\end{aligned}$$

For the remaining term, we again split it into 3 regions:

$$\sum_{n>1/2\sigma\log_q r}\tanh\left(\frac{r}{q^{n\sigma}}\right)\frac{q^{n(1-\sigma)}}{n}=\sum_{n>\beta(k+\log_q k)}+\sum_{n<\beta(k-\log_q k)}+\sum_{k-\log_q k<\frac{n}{\beta}<k+\log_q k}\tanh\left(\frac{r}{q^{n\sigma}}\right)\frac{q^{n(1-\sigma)}}{n}$$
$$=T_1'+T_2'+T_3'.$$

For the sum $T_1'$, with an appropriate choice of $\beta$ we have $\frac{r}{q^{n\sigma}}<1$ so we apply Lemma 6.11 to obtain

$$\sum_{n>\beta(k+\log_q k)}\tanh\left(\frac{r}{q^{n\sigma}}\right)\frac{q^{n(1-\sigma)}}{n}\ll r\sum_{n>\beta(k+\log_q k)}\frac{q^{n(1-2\sigma)}}{n}$$
$$=\frac{r^{1+\beta(1-2\sigma)}}{(\log_q r)^{\beta(2\sigma-1)+1}}$$

Similarly for $T_2'$ with an appropriate choice of $\beta$ we have $r/q^{n\sigma}>1$ so by Lemma 6.11 we have

$$\sum_{n<\beta(k-\log_q k)}\tanh\left(\frac{r}{q^{n\sigma}}\right)\frac{q^{n(1-\sigma)}}{n}\ll\sum_{n<\beta(k-\log_q k)}\frac{q^{n(1-\sigma)}}{n}\ll\frac{r^{\beta(1-\sigma)}}{(\log_q r)^{\beta(1-\sigma)+1}}.$$

Following a similar analysis as before our $T_3'$ becomes

$$\sum_{k-\log_q k<\frac{n}{\beta}<k+\log_q k}\tanh\left(\frac{r}{q^{n\sigma}}\right)\frac{q^{n(1-\sigma)}}{n}=\frac{q^{k\beta(1-\sigma)}}{k}\left(1+O\left(\frac{\log_q k}{k}\right)\right)\sum_{l=-\infty}^{\infty}\frac{\tanh\left(\frac{q^{l\sigma}r}{q^{k\beta\sigma}}\right)}{q^{l(1-\sigma)}}.$$

Balancing the terms in $T_1'$ and $T_2'$ provides that we should take $\beta=\frac{1}{\sigma}$. Putting everything back in terms of $r$ provides the desired result. □

### 6.4.2 Proof of Theorem 6.6 and Theorem 6.3

One of the key ingredients in the proof of Theorem 6.6 is to show that $|\mathbb{E}(L(\sigma,\mathbb{X})^{r+it})|/\mathbb{E}(L(\sigma,\mathbb{X})^r)$ is rapidly decreasing in $t$ when $|t|\geq(r\log r)^\alpha$ for some $1/2<\alpha<1$. We require two lemmas:

**Lemma 6.13.** *Let $1/2<\sigma<1$ be fixed, let $r$ be a large positive number and $c_q\geq q$ a positive constant depending on $q$. If $|P|>\frac{r^{\frac{1}{\sigma}}}{c_q}$, then for some positive constant $b_1$ we have*

$$\frac{|E_{P,\sigma}(r+it)|}{E_{P,\sigma}(r)}\leq\exp\left(-b_1\left(1-\cos\left(t\log\left(\frac{|P|^\sigma+1}{|P|^\sigma-1}\right)\right)\right)\right).$$

The proof follows in an identical fashion as Lemma 5.13. Next:

**Lemma 6.14.** *Let $1/2 < \sigma < 1$ be fixed, $r$ be a large real number and let $c_q \geq q > 4$ be a positive constant dependent on $q$. Then there exists a constant $b_2 > 0$ such that*

$$\frac{|\mathbb{E}(L(\sigma,\mathbb{X})^{r+it})|}{\mathbb{E}(L(\sigma,\mathbb{X})^{r})} \ll_\sigma \begin{cases} \exp\left(-b_2 \frac{t^2}{r^\sigma \log r}\right) & \text{if } |t| \leq \frac{r^\sigma}{c_q} \\ \exp\left(-b_2 \frac{|t|^{2-\sigma}}{\log|t|}\right) & \text{if } |t| > \frac{r^\sigma}{c_q}. \end{cases}$$

*Proof.* Let $z = r + it$. Since $|E_{P,\sigma}(z)| \leq E_{P,\sigma}(r)$ we obtain for any real numbers $q \leq y_1 < y_2$

$$\frac{|\mathbb{E}(L(\sigma,\mathbb{X})^{z})|}{\mathbb{E}(L(\sigma,\mathbb{X})^{r})} \leq \prod_{y_1 \leq |P| \leq y_2} \frac{|E_{P,\sigma}(z)|}{E_{P,\sigma}(r)}. \tag{6.55}$$

Note that $|t| \log\left(\frac{|P|^\sigma+1}{|P|^\sigma-1}\right) \sim 2|t|/|P|^\sigma$ so that when $|t| \leq \frac{|P|^\sigma}{c_q}$ we have

$$1 - \cos\left(|t| \log\left(\frac{|P|+1}{|P|-1}\right)\right) \gg \frac{|t|^2}{|P|^{2\sigma}}.$$

So if $|t| \leq \frac{r^\sigma}{c_q}$ then, we choose $y_1 = r^\sigma$ and $y_2 = c_q r^\sigma/2$. Appealing to Lemma 6.13 we have

$$\begin{aligned}\prod_{y_1 \leq |P| \leq y_2} \frac{|E_{P,\sigma}(z)|}{E_{P,\sigma}(r)} &\ll \prod_{\sigma \log r \leq d \leq \log(c_q r^\sigma/2)+1} \exp\left(-b_1 \frac{q^d}{2d} \frac{|t|^2}{q^{2d\sigma}}\right) \\ &= \exp\left(-\frac{b_1|t|^2}{2} \sum_{\sigma \log r \leq d \leq \log(c_q r^\sigma/2)+1} \frac{1}{dq^{d(2\sigma-1)}}\right) \ll_\sigma \exp\left(-b_2 \frac{|t|^2}{r^\sigma \log r}\right),\end{aligned}$$

since $2\sigma - 1 \in (0,1)$. In the case of $|t| > \frac{r^\sigma}{c_q}$ we use a similar argument but choose $y_1 = c_q|t|^\sigma$ and $y_2 = 2c_q|t|^\sigma$ to complete the result. □

Let $\varphi(y) = 1$ if $y > 1$ and equal to 0 otherwise. Then we have the following smooth analogue of Perron's formula.

**Lemma 6.15.** *[64, Lemma 7.2] Let $\lambda > 0$ be a real number and $N$ be a positive integer. For any $c > 0$ we have for $y > 0$*

$$0 \leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s} - \varphi(y) \leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{1-e^{-\lambda N s}}{s} ds,$$

*and*

$$0 \le \varphi(e^{\lambda} y) - \varphi(y) \le \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left( \frac{e^{\lambda s}-1}{\lambda s} \right) \frac{e^{\lambda s} - e^{-\lambda s}}{s} ds. \tag{6.56}$$

*Proof of Theorem 6.6.* Let $0 < \lambda < \frac{1}{2\kappa}$ be a real number to be chosen later. Using Lemma 6.15 with $N = 1$ we obtain

$$0 \le \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left( \frac{e^{\lambda s}-1}{\lambda s} \right) \frac{ds}{s} - \Phi_{\mathbb{X},\sigma}(\tau)$$
$$\le \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left( \frac{e^{\lambda s}-1}{\lambda s} \right) \frac{1-e^{-\lambda s}}{s} ds \tag{6.57}$$

By assumption, $\lambda\kappa \le 1/2$ so we have $|e^{\lambda s} - 1| \le 3$ and $|e^{-\lambda s} - 1| \le 2$. Now applying Lemma 6.14 we have

$$\int_{\kappa-i\infty}^{\kappa-i\kappa} + \int_{\kappa+i\kappa}^{\kappa+i\infty} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left( \frac{e^{\lambda s}-1}{\lambda s} \right) \frac{ds}{s} \ll \exp\left( -b_2 \frac{\kappa^{2-\sigma}}{\log \kappa} \right) \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}\lambda\kappa} \tag{6.58}$$

and similarly,

$$\int_{\kappa-i\infty}^{\kappa-i\kappa} + \int_{\kappa+i\kappa}^{\kappa+i\infty} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left( \frac{e^{\lambda s}-1}{\lambda s} \right) \frac{1-e^{-\lambda s}}{s} ds \ll \exp\left( -b_2 \frac{\kappa^{2-\sigma}}{\log \kappa} \right) \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}\lambda\kappa}. \tag{6.59}$$

Now, if $|t| \le \kappa$ then $|(e^{\lambda s} - 1)(e^{-\lambda s} - 1)| \ll \lambda^2 |s|^2$, so that

$$\int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left( \frac{e^{\lambda s}-1}{\lambda s} \right) \frac{1-e^{-\lambda s}}{s} ds \ll \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}} \lambda\kappa.$$

Combining this previous line with (6.57), (6.58) and (6.59) we have

$$\Phi_{\mathbb{X},\sigma}(\tau) - \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left( \frac{e^{\lambda s}-1}{\lambda s} \right) \frac{ds}{s} \ll \left( \lambda\kappa + \frac{\exp(-b_2\kappa^{2-\sigma}/\log\kappa)}{\lambda\kappa} \right) \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}}.$$

On the other hand, if $|t| \le \kappa$ equation (6.51) gives us that

$$\mathcal{L}_\sigma(\kappa + it) = \mathcal{L}_\sigma(\kappa) + it\mathcal{L}'_\sigma(\kappa) - \frac{t^2}{2}\mathcal{L}''_\sigma(\kappa) + O\left( \frac{\kappa^{\frac{1}{\sigma}-3}|t|^3}{\log \kappa} \right).$$

Additionally, we have

$$\frac{e^{\lambda s}-1}{\lambda s^2} = \frac{1}{\kappa}\left( 1 - \frac{it}{\kappa} + O\left( \lambda\kappa + \frac{t^2}{\kappa^2} \right) \right).$$

So, using the facts $\mathbb{E}(L(\sigma,\mathbb{X})^s)=\exp(\mathcal{L}_\sigma(s))$ and $\mathcal{L}_\sigma'(\kappa)=\tau$ we find

$$\frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}}\left(\frac{e^{\lambda s}-1}{\lambda s^2}\right) \quad = \quad \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}\kappa}\exp\left(-\frac{t^2}{2}\mathcal{L}_\sigma''(\kappa)\right)\left(1-\frac{it}{\kappa}+O\left(\lambda\kappa+\frac{t^2}{\kappa^2}+\frac{\kappa^{\frac{1}{\sigma}-3}|t|^3}{\log\kappa}\right)\right).$$

Now, by design, the integral involving $it/\kappa$ vanishes, thus we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\kappa-i\kappa}^{\kappa+i\kappa}&\frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\frac{ds}{s}\\ &=\frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}\kappa}\frac{1}{2\pi}\int_{-\kappa}^{\kappa}\exp\left(-\frac{t^2}{2}\mathcal{L}_\sigma''(\kappa)\right)\left(1+O\left(\lambda\kappa+\frac{t^2}{\kappa^2}+\frac{\kappa^{\frac{1}{\sigma}-3}|t|^3}{\log\kappa}\right)\right)dt.\end{aligned}$$

Furthermore, we find

$$\frac{1}{2\pi}\int_{-\kappa}^{\kappa}\exp\left(-\frac{t^2}{2}\mathcal{L}_\sigma''(\kappa)\right)dt=\frac{1}{\sqrt{2\pi\mathcal{L}_\sigma''(\kappa)}}\left(1+O\left(\exp(-\kappa^2\mathcal{L}_\sigma''(\kappa)/2)\right)\right),$$

and

$$\frac{1}{2\pi}\int_{-\kappa}^{\kappa}|t|^n\exp\left(-\frac{t^2}{2}\mathcal{L}_\sigma''(\kappa)\right)dt\ll\frac{1}{\mathcal{L}_\sigma''(\kappa)^{(n+1)/2}}.$$

So that

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa}^{\kappa+i\kappa}\frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}}\left(\frac{e^{\lambda s}-1}{\lambda s}\right)\frac{ds}{s}=\frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)e^{-\tau\kappa}}{\kappa\sqrt{2\pi\mathcal{L}_\sigma''(\kappa)}}\left(1+O\left(\lambda\kappa+\frac{1}{\kappa^2\mathcal{L}_\sigma''(\kappa)^{3/2}}\right)\right).$$

Using (6.51) and making the choice $\lambda=\kappa^{-3}$ we obtain the desired result. □

*Proof of Theorem 6.3.* As before, let $\kappa=\kappa_\sigma(\tau)$ be the unique solution to (6.45). Let $N$ be a positive integer and let $\lambda\in\mathbb{R}$ be a parameter to be chosen later which satisfies $0<\lambda<\min\{\frac{1}{2\kappa},\frac{1}{N}\}$.

Let $B=12$ and take $b_3$ and $g(\sigma)=\sigma$ as in Corollary 6.2 and take $Y=b_3(\log_q|D|)^\sigma/q$. Note that if $|D|$ is large enough then by Corollary 6.4 and the assumption on $\tau$ we have $\kappa\le Y$. Now, choose $s\in\mathbb{C}$ such that $\mathfrak{Re}(s)=\kappa$ and $|\mathfrak{Im}(s)|\le Y$, then by Corollary 6.2 we have

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}}L(\sigma,\chi_D)^s=\mathbb{E}(L(\sigma,\mathbb{X})^s)+O\left(\frac{\mathbb{E}(L(\sigma,\mathbb{X})^{\mathfrak{Re}(s)})}{(\log_q|D|)^{10}}\right),\tag{6.60}$$

by our choice of $B$. Define

$$J(\sigma,\tau) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s},$$

and

$$J_n(\sigma,\tau) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} L(\sigma,\chi_D)^s\right) e^{-\tau s} \left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s}.$$

Then, from Lemma 6.15 we have

$$\Phi_{\mathbb{X},\sigma}(\tau) \le J(\sigma,\tau) \le \Phi_{\mathbb{X},\sigma}(\tau-\lambda N) \tag{6.61}$$

and

$$\mathbb{P}(\log_q L(\sigma,\chi_D) > \tau) + O(\delta(n)) \le J_n(\sigma,\tau) \le \mathbb{P}(\log_q L(\sigma,\chi_D) > \tau-\lambda N) + O(\delta(n)), \tag{6.62}$$

with

$$\delta(n) = \exp(-c_0(\sigma)n/\log_q n),$$

where $c_0$ is a positive constant, by Lemma 6.5.

Using the fact that $|e^{\lambda s}-1| \le 3$ we have

$$\int_{\kappa-i\infty}^{\kappa-iY} + \int_{\kappa+iY}^{\kappa+i\infty} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^s)}{e^{\tau s}} \left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s} \ll \left(\frac{3}{\lambda Y}\right)^N \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}}, \tag{6.63}$$

and similarly, together with (6.60) we have

$$\int_{\kappa-i\infty}^{\kappa-iY} + \int_{\kappa+iY}^{\kappa+i\infty} \left(\frac{1}{|\mathcal{H}_n|}\sum_{D\in\widetilde{\mathcal{H}}_{n,g}} L(\sigma,\chi_D)^s\right) e^{-\tau s} \left(\frac{e^{\lambda s}-1}{\lambda s}\right)^N \frac{ds}{s} \ll \left(\frac{3}{\lambda Y}\right)^N \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}}, \tag{6.64}$$

Combining (6.60), (6.63) and (6.64) with $|\frac{e^{\lambda s}-1}{\lambda s}| \le 3$ gives

$$J_n(\sigma,\tau) - J(\sigma,\tau) \ll \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}} \left(\frac{3^N Y}{(\log_q|D|)^{10}} + \left(\frac{3}{\lambda Y}\right)^N\right).$$

Choosing $N = [\log_q \log_q |D|]$ and $\lambda = \frac{e^{10}}{Y}$ we have

$$J_n(\sigma,\tau) - J(\sigma,\tau) \ll \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}(\log_q |D|)^7}. \tag{6.65}$$

Furthermore, it follows from Theorem 6.6 and Proposition 6.3 that

$$\Phi_{\mathbb{X},\sigma}(\tau) \asymp_\sigma \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{\kappa e^{\tau\kappa}\sqrt{\mathcal{L}''_\sigma(\kappa)}} \asymp_\sigma \frac{\sqrt{\log_q \kappa}}{\kappa^{\frac{1}{2\sigma}}} \frac{\mathbb{E}(L(\sigma,\mathbb{X})^\kappa)}{e^{\tau\kappa}}.$$

On the other hand, by Theorem 6.4, Corollary 6.4 and the choices of $\lambda$, $N$ and $Y$ we have

$$\begin{aligned}\Phi_{\mathbb{X},\sigma}(\tau \pm \lambda N) &= \Phi_{\mathbb{X},\sigma}(\tau)\exp(\lambda N(\tau\log_q \tau)^{\frac{\sigma}{1-\sigma}}) \\ &= \Phi_{\mathbb{X},\sigma}(\tau)\left(1 + O\left(\frac{(\tau\log_q\tau)^{\frac{\sigma}{1-\sigma}}\log_q\log_q|D|}{(\log_q|D|)^\sigma}\right)\right).\end{aligned}$$

Combining this with (6.61), (6.62) and (6.65) we have

$$\begin{aligned}\mathbb{P}(\log L(\sigma,\chi_D) > \tau) &\le J_n(\sigma,\tau) + O(\delta(n)) \\ &\le J(\sigma,\tau) + O\left(\frac{\Phi_{\mathbb{X},\sigma}(\tau)}{n^7} + \delta(n)\right) \\ &\le \Phi_{\mathbb{X},\sigma}(\tau)\left(1 + O\left(\frac{(\tau\log_q\tau)^{\frac{\sigma}{1-\sigma}}\log_q n}{n^\sigma}\right)\right) + O(\delta(n))\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(\log L(\sigma,\chi_D) > \tau) &\ge J_n(\sigma,\tau+\lambda N) + O(\delta(n)) \\ &\ge J(\sigma,\tau+\lambda N) + O\left(\frac{\Phi_{\mathbb{X},\sigma}(\tau)}{n^7} + \delta(n)\right) \\ &\ge \Phi_{\mathbb{X},\sigma}(\tau)\left(1 + O\left(\frac{(\tau\log_q\tau)^{\frac{\sigma}{1-\sigma}}\log_q n}{n^\sigma}\right)\right) + O(\delta(n)).\end{aligned}$$

Thus the result follows given that $\Phi_{\mathbb{X},\sigma}(\tau) \gg \sqrt{\delta(n)}$ for our range of $\tau$. □

## 6.5 Optimal $\Omega$-results: Proof of Theorem 6.5

For each irreducible polynomial $P \in \mathbb{A}$, let $\delta_P \in \{-1, 1\}$. Define $\mathcal{S}_N(n, \{\delta_P\})$ to be the set of all monic irreducibles $Q \in \mathbb{A}$ such that $\deg Q = N$ and

$$\left(\frac{P}{Q}\right) = \delta_P,$$

for all irreducibles $P$ with $\deg P \le n$. We also let $\mathcal{P}(n)$ denote the product of all irreducible polynomials $P$ with $\deg P \le n$.

We shall deduce Theorem 6.5 from the following proposition.

**Proposition 6.4.** *Let $1/2 < \sigma < 1$ be fixed. Then, we have*

$$\sum_{Q \in \mathcal{S}_N(n, \{\delta_P\})} \log L(\sigma, \chi_Q) = \frac{\pi_q(N)}{2^{\Pi_q(n)}} \sum_{\deg P \le n} \frac{\delta_P}{|P|^\sigma} + O\left(\frac{\pi_q(N)}{2^{\Pi_q(n)}} + q^{N/2+2n}\right). \tag{6.66}$$

*Proof.* Let $1/2 < \sigma < 1$ be fixed. Let $Q$ be a monic irreducible, and let $M = \log_q N/(\sigma - 1/2)$. First, by Lemma 6.2 equation (6.24) we have

$$\log L(\sigma, \chi_Q) = \sum_{\deg P \le M} \log\left(1 - \frac{\chi_Q(P)}{|P|^\sigma}\right) + O\left(q^{-M(\sigma - 1/2)} N\right) = \sum_{\deg P \le M} \frac{\chi_Q(P)}{|P|^\sigma} + O(1).$$

This shows that

$$\sum_{Q \in \mathcal{S}_N(n, \{\delta_P\})} \log L(\sigma, \chi_Q) = \sum_{Q \in \mathcal{S}_N(n, \{\delta_P\})} \sum_{\deg P \le n} \frac{\chi_Q(P)}{|P|^\sigma} + O\left(|\mathcal{S}_N(n, \{\delta_P\})|\right). \tag{6.67}$$

Furthermore, it follows from (5.61) that

$$\begin{aligned}\sum_{Q \in \mathcal{S}_N(n, \{\delta_P\})} \sum_{\deg P \le M} \frac{\chi_Q(P)}{|P|^\sigma} &= \frac{1}{2^{\Pi_q(n)}} \sum_{f | \mathcal{P}(n)} \delta_f \sum_{\substack{Q \text{ irreducible} \\ \deg Q = N}} \left(\frac{f}{Q}\right) \sum_{\deg P \le M} \left(\frac{Q}{P}\right) \frac{1}{|P|^\sigma} \\ &= \frac{1}{2^{\Pi_q(n)}} \sum_{f | \mathcal{P}(n)} \delta_f \sum_{\deg P \le M} \frac{1}{|P|^\sigma} \sum_{\substack{Q \text{ irreducible} \\ \deg Q = N}} \left(\frac{Pf}{Q}\right),\end{aligned} \tag{6.68}$$

by the law of quadratic reciprocity from Theorem 2.6. Since any divisor $f$ of $\mathcal{P}(n)$ is square-free, it follows that $Pf$ is a square only when $f = P$. Hence, if $f \neq P$, then by (2.13) we get

$$\sum_{\substack{Q \text{ irreducible} \\ \deg Q = N}} \left(\frac{Pf}{Q}\right) \ll \deg(fP) q^{\frac{N}{2}} \ll n q^{\frac{N}{2}+n},$$

by (5.62). Inserting this estimate in (6.68), and using Lemma 6.1 we deduce that

$$\sum_{Q \in \mathcal{S}_N(n, \{\delta_P\})} \sum_{\deg P \leq M} \frac{\chi_Q(P)}{|P|^\sigma} = \frac{\pi_q(N)}{2^{\Pi_q(n)}} \sum_{\deg P \leq n} \frac{\delta_P}{|P|^\sigma} + O\left(q^{\frac{N}{2}+2n}\right).$$

Combining this estimate with (6.67) and Lemma 5.15, and using the prime number theorem (2.10) completes the proof of (6.66). □

We finish this section by proving Theorem 6.5.

*Proof of Theorem 6.5.* We start by proving (6.19). We choose $n$ such that

$$\frac{N \log_q N}{10\zeta_{\mathbb{A}}(2) q} \leq q^n < \frac{N \log_q N}{10\zeta_{\mathbb{A}}(2)}. \tag{6.69}$$

Then, it follows from (5.17) that $2^{\Pi_q(n)} < q^{N/4}$. Let $\delta_P = 1$ for all monic irreducibles $P$ with $\deg P \leq n$. Then, it follows from Lemma 5.15 and Proposition 6.4 that

$$\frac{1}{|\mathcal{S}_N(n, \{\delta_P\})|} \sum_{Q \in \mathcal{S}_N(n, \{\delta_P\})} \log L(\sigma, \chi_Q) = \sum_{\deg P \leq n} \frac{1}{|P|^\sigma} + O(1).$$

Thus, by (6.21) we deduce

$$\max_{Q \in \mathcal{S}_N(n, \{\delta_P\})} \log L(\sigma, \chi_Q) \geq \sum_{\deg P \leq n} \frac{1}{|P|^\sigma} + O(1) \geq \beta_q(\sigma) \frac{N^{1-\sigma}}{(\log_q N)^\sigma},$$

where

$$\beta_q(\sigma) = \frac{\zeta_{\mathbb{A}}(2-\sigma)}{(10\zeta_{\mathbb{A}}(2) q)^{1-\sigma}}.$$

This completes the proof of (6.19). The proof of (6.20) follows along the same lines by taking $\delta_P = -1$ for all monic irreducibles $P$ with $\deg P \leq n$.

□

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