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A NOVEL FINITE STRAIN VISCO-HYPERELASTICITY BASED CONSTITUTIVE MODEL FOR ELASTOMERS

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Abstract—A novel three-dimensional finite strain visco-hyperelastic constitutive model is proposed to capture the strain rate dependency of rubber-like materials. The overall material behavior is defined by cumulative description of hyperelasticity and non-linear viscoelasticity. The hyperelastic part is based on exponential logarithmic Hart-Smith strain energy function and the viscous part comprises of a fading integral which links the current stresses to the applied strain history. The derived analytical framework is verified with respect to experimental data. The potential of the proposed model has been constituted by an excellent fit between proposed model and considered test data.

***Keywords*—Rubber-like materials, Strain rate, Visco-hyperelasticity, Constitutive modeling**

I. INTRODUCTION

Rubber-like materials undergo large deformations at relatively low stresses and recover their initial shape upon removal of the load. This exceptional ability makes them suitable and without a doubt irreplaceable materials in vital engineering applications. This uncommon behavior is caused by the underlying coiled long chain polymer molecules. These chains straighten when the material is stretched and recoil upon removal of the load. This substantial aspect contributes towards their extraordinary mechanical behavior and makes rubbers a strain rate dependent material. This rate dependency is associated with the rearrangement of the molecular chains [1].

Rubber viscoelasticity can be described by two distinct theories: a) Linear viscoelasticity b) Non-linear viscoelasticity. To summaries, linear viscoelasticity models are constituted by considering parallel and/or series arrangements of elastic springs and linear viscous dash-pots. These rheological analogies are superimposed by Boltzman's principle and the relaxation function is approximated with a prony series formulation[2], [3], [4]. Whereas, the non-linear viscoelastic constitutive relations are formulated by employing generalized fading history integral function [5], [6]. In this approach, an approximation of matrix stress functional defines the strain history on stresses.

In order to prescribe time dependent response of the elastomers, quasi-linear viscoelastic frame work requires to be represented by 4-6 term prony series approximations. As a result, large number of material constants are needed to be determined [7], [8], [9]. In contrast, the number of material constants can be reduced by articulating nonlinear viscoelasticity. Moreover, numerical implementation of such formulations is a straightforward procedure and the material parameter identification is comparatively less time consuming task. For that reason, a novel power-exponential strain history functional has been proposed in the nonlinear finite strain visco-hyperelastic framework to capture rate dependency in various types of elastomers. For the purpose of validation, derived constitutive relations are compared with the literature based high strain rate experimental data for polyurea [10].

II. CONSTITUTIVE MODEL DEVELOPMENT

A. Hyperelasticity

Hyperelastic constitutive laws are derived from strain energy density W. It expresses the stored elastic strain energy in material per unit reference volume as a function of the principle strain or stretch invariants.

$$W = f(I_1, I_2, I_3), \tag{1}$$

Where I_1, I_2 and I_3 are the invariants of left Cauchy-Green tensor ($B = FF^T$) defined as:

$$\begin{aligned} I_1 &= tr(B) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= (trB)^2 - trB^2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ I_3 &= J = \sqrt{det(F)} = \sqrt{(\lambda_1^2\lambda_2^2\lambda_3^2)} \end{aligned} \tag{2}$$

Cauchy stress in an incompressible hyperelastic continua is given by[11]:

$$\sigma^e = -p^e I + 2\left[\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)B - \frac{\partial W}{\partial I_2}B^2\right] \tag{3}$$

where where p^e is a hydrostatic pressure term which needs to be determined from boundary conditions and I is an identity tensor. Time independent behavior of elastomers is characterized using Hart-Smith strain energy [12] as it provides distinctive advantages in modeling finite strain hyperelastic materials due to its exponential form[13], [14], [15], [16].

$$W_{HS} = C_1 \int exp\left(C_3(I_1-3)^2\right) dI_1 + 3C_2 ln\frac{I_2}{3} \quad (4)$$

Here C_1, C_2 and C_3 are model parameters and need to be determined from experiments.

Let us consider an in-compressible {i.e. $J = 1$ in eq.(3)} material deformation under uniaxial extension mode. If the stretch in the loading direction is denoted by λ then the deformation gradient F for this loading becomes:

$$F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-0.5} & 0 \\ 0 & 0 & \lambda^{-0.5} \end{bmatrix} \quad (5)$$

Consequently the invariants are depicted as:

$$I_1 = \lambda^2 + 2\lambda^{-1}, I_2 = \lambda^{-2} + 2\lambda \quad (6)$$

From eq. (3) and (4), we arrived to uniaxial stress-stretch relation as:

$$\sigma_{11}^e = -p^e + 2\lambda^2 \left(\frac{I_1 C2}{I_2} + C1\, exp^{C3\,(I_1-3)^2} - \frac{C2\,\lambda^2}{I_2}\right) \quad (7)$$

Then after, the hydrostatic pressure term p^e is calculated by applying uniaxial loading condition $\sigma_{22}^e = \sigma_{33}^e = 0$

$$\sigma_{22}^e = 0 = -p^e + \frac{2}{\lambda}\left(\frac{I_1 C2}{I_2} + C1\, exp^{C3\,(I_1-3)^2} - \frac{C2}{\lambda\, I_2}\right) \quad (8)$$

Finally, by substituting the expression for p^e into eq(7); the we derived incompressible uniaxial hart-smith stress-strain law as:

$$\sigma_{11}^e = 2\,(\lambda^2 - \lambda^{-1})\left[\frac{I_1 C2}{I_2} + C1\, exp^{C3\,(I_1-3)^2} - \frac{(\lambda^2+\lambda^{-1})C2}{\lambda\, I_2}\right] \quad (9)$$

B. Viscoelasticity

The generalized constitutive relation for non-linear incompressible viscoelastic material behavior is given by[17]:

$$\sigma^v = -p_v I + F(t) \cdot \prod_{\tau=-\infty}^{t} \{\mathrm{C}(\tau)\} \cdot F(t)^t \quad (10)$$

where σ^v is time dependent cauchy stress tensor, p is the hydro-static pressure term also called Lagrange Multiplier and $\prod$ is a strain history function given by:

$$\prod_{\tau=-\infty}^{t} \{\mathrm{C}(\tau)\} = \int_{-\infty}^{t} \Phi(I_1, I_2) m(t-\tau) \dot{E(\tau)} d\tau \quad (11)$$

In equation (11), $\Phi(I_1, I_2)$ is the function depending on invariants of strain tensor C and the strain rate $\dot{E(\tau)}$ is defined as:

$$\dot{E} = \frac{1}{2}\,(\dot{F}^T F + F^T \dot{F}) \quad (12)$$

$m(t)$ is a relaxation function which is represented by a series of exponential series functions as

$$m(t) = \sum_{i=1}^{N} exp\left(-\frac{t-\tau}{\theta_i}\right) \quad (13)$$

where θ_i is the relaxation time. Several researchers have assigned N values greater than/equal to 2 in their work[18]. In order to reduce the number of material constants with single relaxation time scheme i.e. $N = 1$, we are proposing a novel suitable representation of the function $\Phi(I_1, I_2)$ as:

$$\Phi(I_1, I_2) = \dot{E(\tau)}^{A_1} \cdot A_2 \cdot exp(I_2 - 3)^{A_3} \quad (14)$$

The starting point for the time integration is at the instant when loading commences. It is assumed that the effect of deformation history for $\tau < 0$ on the stress at time $t > 0$ is negligible. Thus, the period of deformation history which is considered to effect the stress response and hence the limits of integration in the second term on the right-hand side of Eq.11 becomes $[0, t]$ rather than $[-\infty, t]$.

Substituting Eqs. 14 and 13 with N=1 into Eq. 11 results in the following proposed integral approximation for $\prod$

$$\prod = \int_0^t \dot{E}^{A_1} \cdot A_2 \cdot exp(I_2 - 3)^{A_3} exp\left(-\frac{t-\tau}{\theta_1}\right) \dot{E} d\tau \quad (15)$$

Substitution of eq.(15) into eq.(10) yields a finite strain viscoelastic model for incompressible materials as

$$\sigma^v = -p_v I + F \cdot \left[\int_0^t \dot{E}^{A_1} \cdot A_2 \cdot exp(I_2-3)^{A_3} \cdot exp\left(-\frac{t-\tau}{\theta_1}\right) \dot{E(\tau)} d\tau\right] \cdot F^t \quad (16)$$

C. Visco-hyperelasticity

Visco-hyperelastic behavior is generally introduced by aggregating hyperelasticity and viscocity i.e. $\sigma = \sigma^e + \sigma^v$ [18], [19].

$$\sigma = -pI + 2\left[\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right) B - \frac{\partial W}{\partial I_2} B^2\right] + F \cdot \left[\int_0^t \dot{E}^{A_1} \cdot A_2 \cdot exp(I_2-3)^{A_3} \cdot exp\left(-\frac{t-\tau}{\theta_1}\right) \dot{E(\tau)} d\tau\right] \cdot F^t \quad (17)$$

where σ is the total stress and p is the total hydrostatic pressure incorporating static and viscoelastic components. Inserting $\dot{E}_{11} = \lambda\dot{\lambda}$ into eq.(17) concludes the uniaxial stress as:

$$\sigma_{11} = -p^v + \sigma_{11}^e + \lambda^3 \int_0^t (\lambda\dot{\lambda})^{A_1} \cdot A_2 \cdot exp(I_2-3)^{A_3} \cdot exp\left(-\frac{t-\tau}{\theta_1}\right) \dot{\lambda} d\tau \quad (18)$$

By applying uniaxial loading condition that $\sigma_{11} = 0$ into eq.(18), p^v is formulated as:

$$p^v = -\frac{\lambda^{-1}}{2}\left[\int_0^t \lambda^{-2}(\lambda\dot{\lambda})^{A_1}\cdot A_2 \cdot exp(I_2-3)^{A_3}\cdot exp\left(-\frac{t-\tau}{\theta_1}\right)\dot{\lambda}d\tau\right] \tag{19}$$

Replacing p^v in eq(18), we derived the uniaxial stress-deformation expression as:

$$\sigma_{11} = \sigma_{11}^e + \lambda^3 \int_0^t (\lambda\dot{\lambda})^{A_1}\cdot A_2\cdot exp(I_2-3)^{A_3}\cdot exp\left(-\frac{t-\tau}{\theta_1}\right)\dot{\lambda}d\tau + -\frac{\lambda^{-1}}{2}\left[\int_0^t \lambda^{-2}(\lambda\dot{\lambda})^{A_1}\cdot A_2\cdot exp(I_2-3)^{A_3}\cdot exp\left(-\frac{t-\tau}{\theta_1}\right)\dot{\lambda}d\tau\right] \tag{20}$$

III. Verification of proposed model

We have considered experimental data reported by Roland et. al [10] for verifying derived analytical results. They have performed uniaxial tensile test on polyurea from moderate to high strain rates using drop weight apparatus. The hyperelastic

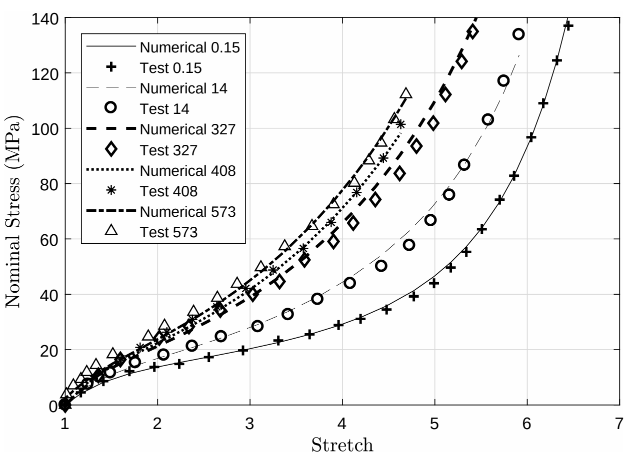

Fig. 1: Comparison of Roland et. al [10] data with proposed model

Hart-Smith material parameters are determined with respect to quasi-static (in this case 0.15 S^{-1}) experimental data. The detailed procedure for parameter identification is described in [13]. Where as the material parameters for the viscoelastic part has been derived by considering test data for the strain rates of 14 S^{-1} and 573 S^{-1}. These required material constants have been optimized using non-linear least square function "Fmincon" in MATLAB (see Table I). Fig. (**??**) demonstrates an excellent agreement between the proposed model results and the experimental data. Notably, the model is adequate to predict the material response for the rates of 327 S^{-1} and 408 S^{-1} accurately even though the relevant experimental data have not been considered for the material parameters identification.

TABLE I: Optimized material parameters

Hyperelastic model			**Viscoelastic model**			
C_1	C_2	C_3	θ_1	A_1	A_2	A_3
0.4474	12.79	8.1E-4	5.4E-3	-0.7355	259.89	0.05

IV. Conclusion and future work

To summarize, a novel visco-hyperelastc material law has been proposed to capture rate dependency in elastomers by adopting generalized non-linear viscoelasticity. Analytical stress-strain relations are derived for uniaxial tension mode. An excellent agreement between numerical results and experimental data reported by Roland et. al [10] has been obtained. Also, the predictive capability of the model has been demonstrated. Following the promising findings, the conducted work will be extended by considering broader spectrum of experimental data. The developed model will be implemented in commercial finite element code via user defined subroutine to analyze complex geometries and ladings.

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