On t-fold Totally-Concave Polyominoes*

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October 12, 2025

Abstract

A *t-fold totally concave polyomino* (t-TCP) is an edge-wise connected collection of cells of the square lattice with t or more gaps in every row and column. We describe an efficient algorithm for counting 1-TCPs (modulo translation) by area, and comment on its extension to $t > 1$. We prove that the minimum area of a t-TCP is 21 for $t = 1$, 50 for $t = 2$, and $6(t+1)^2 - 1$ for $t > 2$. We show that the counting sequence $\kappa_t(n)$ of t-TCPs of area n satisfies $\lambda^{n+o(n)}$ as $n \to \infty$, where λ is the same growth constant as for all polyominoes. From this, we prove that the ratio of successive terms converges to λ. For each t, we obtain an explicit constant θ_t such that $\kappa_t(n) \geq n^{-\theta_t}\lambda^n$ for infinitely-many values of n, complementing the fact that $\kappa_t(n) \leq n^{-1/2}\lambda^n$ for every $n \in \mathbb{N}$. We also briefly discuss the relation of t-TCPs to similar models from statistical physics.

Keywords: Polyominoes, Ratio-limit theorem, Pattern theorem, Totally-concave, Growth constant.

1 Introduction

A polyomino is an edge-wise connected collection of unit squares in the plane. That is, given a connected subgraph G of the square lattice (with nodes at integer coordinates), the polyomino determined by G is $P_G := \bigcup_{(x,y)\in G} [x, x+1] \times [y, y+1]$. To consider only one translate of each polyomino, we use the convention that every polyomino P satisfies $P \subset [0, \infty) \times [0, \infty)$, $P \cap \{(x, y) : x = 0\} \neq \emptyset$, and $P \cap \{(x, y) : y = 0\} \neq \emptyset$. It is sometimes useful to consider a polyomino just as a subgraph of the square lattice, and sometimes equivalently as a collection of square cells. Throughout the paper, our drawings will show both, when convenient.

*Preliminary versions of this paper appeared in references [4, 5]. Research of the third author was supported in part by a Discovery Grant from NSERC Canada. Research of the fourth author was supported in part by an Undergraduate Student Research Award from NSERC Canada.

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Counting polyominoes is a long-standing problem in discrete geometry, originating in statistical physics in the context of percolation processes [13] and popularized in Golomb's pioneering book [16] and by M. Gardner's columns in *Scientific American*. The sequence $A(n)$, which lists the number of polyominoes, is currently known up to $n = 70$ [1].

The growth constant of polyominoes has also attracted much attention in the literature. Klarner [23] showed that the limit (also known as *Klarner's constant*) $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists, although its exact value is still unknown. The convergence of $A(n+1)/A(n)$ to λ, as $n \to \infty$, was proved only three decades later by Madras [27]. The best-known lower [7] and upper [8] bounds on λ are 4.0025 and 4.5252, respectively. By applying numerical methods to the known values of $A(n)$, it is widely believed that $\lambda \approx 4.06$, and the currently best published estimate of λ based on the first 56 values of the sequence $(A(n))$ is 4.0625696 ± 0.0000005 [22]. (Based on the new counts of $A(n)$ up to $n = 70$, a better estimate, 4.06256912(2), was provided to us by I. Jensen in a personal communication.)

In a *convex* polyomino, each row and column consists of exactly one maximal continuous sequence of cells. These polyominoes arise in many application domains, and they attracted a considerable amount of attention in the literature. See, for example, a discussion of the asymptotic formula for the number of convex polyominoes [10], a derivation of a rather complex generating function for the sequence that enumerates convex polyominoes [11], a method for generating random convex polyominoes [20], and an investigation of the relation between ordering and convex polyominoes [15], among many other works.

However, the complement type of polyominoes was hardly considered. A row or a column ξ of a polyomino has a *gap* if ξ contains at least two maximal sequences of consecutive cells; likewise, ξ has t gaps if it consists of at least $t+1$ maximal sequences of consecutive cells. *Totally Concave Polyominoes (TCPs)* (resp., t-fold TCPs, t-TCPs in short) are those polyominoes in which every row and every column of cells has at least one gap (resp., t gaps). For completeness, we also say that polyominoes are "0-TCPs."

The symbols A_n, κ_n, and $\kappa_{t,n}$ will denote the sets of area-n polyominoes, TCPs (1-TCPs), and t-TCPs, respectively, while $A(n)$, $\kappa(n)$, and $\kappa_t(n)$ will denote the number of these objects. In addition, the symbols $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ and $\lambda_{\kappa_t} := \lim_{n\to\infty} \sqrt[n]{\kappa_t(n)}$ will denote the growth constant of all (resp., t-TC) polyominoes.

1-TCPs were introduced in the Handbook of Discrete and Computational Geometry [3] as an extremal opposite of *convex polyominoes*, and were first investigated in a preliminary version of this paper [4]. There,

i. The minimum possible area for a 1-TCP was proved to be 21, settling on the affirmative a conjecture given in reference [3, §14, p. 369, problem 14.5.4];

ii. The number of 1-TCPs of area-n was evaluated for $21 \le n \le 35$; and

iii. The 1-TCP growth constant λ_{κ_1} was shown to exist, bounded from below, and conjectured to be equal to λ.

In this paper, for each positive t, we consider t-TCPs. Figure 4 shows a few examples which are in fact of the minimum possible sizes (see Section 4). We generalize the result on minimal examples to t-TCPs, answer the mentioned conjecture, and prove the existence of the ratio limit $\lim_{n\to\infty} \frac{\kappa(n+1)}{\kappa(n)}$ by strengthening Madras' 1999 technique, to be found in reference [27].

In addition, we use the following lexicographic order of cells on the square lattice.

Definition 1 *(Lexicographic Order) Given two cells on the square lattice,* $c_1 = [x_1, x_1 + 1] \times [y_1, y_1 + 1]$ *and* $c_2 = [x_2, x_2 + 1] \times [y_2, y_2 + 1]$*, we say that* c_1 *is lexicographically smaller than* c_2 *if* $x_1 < x_2$*, or if* $x_1 = x_2$ *and* $y_1 < y_2$*.*

The paper is organized as follows. In Section 2, we extend the introduction and provide some physical motivation to the study of totally-concave polyominoes. In Section 3, we present a nontrivial extension of Jensen's algorithm to counting TCPs, and report counts of 1-TCPs up to size 35. In Section 4, we generalize work done in reference [4] to consider the minimum possible area of t-TCPs. In Section 5, we continue this generalization to show that the growth constants of t-TCPs (for any $t > 0$), λ_{κ_t}, all equal to λ, which has long been known to exist [23]. In Section 6, we prove the existence of the ratio limit for t-TCPs, $\lim_{n\to\infty} \frac{\kappa_t(n+1)}{\kappa_t(n)}$. Along the way, we refine the old technique of Madras [27] for proving the existence of ratio limits. In Section 7, we show that $\kappa_t(n)$ (for any $t \geq 0$) has a lower bound of the form $\Omega(\lambda^n n^{-\theta_t})$ for infinitely-many values of n. We end in Section 8 with some concluding remarks and future research directions.

2 A Physical Context

Polyominoes are closely related to a model called *lattice animals* from statistical physics. Roughly speaking, a lattice is a periodic embedding in $\mathbb{R}^d$ of an infinite graph with finite degrees. An example is the hypercubic lattice, whose vertices are the points $\mathbb{Z}^d$ with integer coordinates, and whose edges join vertices that are distance 1 apart. A lattice animal is simply a finite connected subgraph of a lattice, and a lattice tree is a lattice animal with no cycles. In this context, polyominoes and their higher-dimensional analogues, polycubes, may be viewed as induced subgraphs of the dual lattice (whose vertices correspond to cells in the original lattice); in this context, they are often called *site animals*. Lattice animals, lattice trees, and polycubes may be viewed as discrete models for spatial arrangements of branched polymer molecules: a vertex (or cell) corresponds to a monomer (an atom or small cohesive group of atoms), and an edge corresponds to a chemical bond between two monomers. In each of our models (polycubes, animals, trees), the number of such arrangements (modulo translation) with n vertices describes the entropy of the branched polymers, and is generally believed to be asymptotically proportional to $n^{-\theta}\Lambda^n$ for some constants θ and Λ. The growth constant Λ depends on the model and on the details of the lattice, but remarkably the number θ is believed to depend only on the dimension of the lattice and to be the same for polycubes, animals, and trees in that dimension [25]. In particular, for dimension $d = 2$ where polyominoes live, it is believed that $\theta = 1$ [30].

Besides the common value of the exponent θ, other large scale features are believed to be shared by polyominoes, lattice animals, and lattice trees. In the terminology of renormalization group theory, all of this can be summarized by saying that these models all belong to the same *universality class*. In particular, typical members of these models are expected to display a fractal geometry in the limit as their size gets large. This is supported by non-rigorous scaling theory as well as by simulations. However, almost nothing has been proven rigorously about the asymptotic geometry of these objects (except in high dimensional space, which here means above eight dimensions [19, 33]). In two dimensions, fractal behaviour would suggest that a vertical or horizontal line intersecting a large polyomino would typically have gaps on all length scales. This seems hard to prove, but a simpler task would be to show that, for each positive integer t, there is a reasonable probability that most lines intersecting a sufficiently large polyomino would have at least t gaps. Our Theorem 10 upgrades "most" to "every," but replaces "reasonable probability" with "probability that is not exponentially small". In fact, our argument shows that this probability is only polynomially small, under the assumption that $A(n)$ is asymptotically proportional to $n^{-\theta}\lambda^n$; see Proposition 20. It is worth noting that convex polyominoes cannot be in the universality class of t-TCP's for any $t \geq 0$; see Remark 1.

Thus, we can view our Theorem 10 as a first rigorous corroboration (albeit a mild one) of the behaviour that physicists expect. But it also leads us to an intriguing conjecture. One can argue that t-TCP's are yet another model for branched polymers, not substantially different from polyominoes in qualitative large-scale behaviour. In that case, we would expect their cardinalities to satisfy

$$\kappa_t(n) \sim C_t\, n^{-\theta}\, \lambda_{\kappa_t}^n \qquad \text{as } n \to \infty$$

for some constant C_t and for the same exponent θ that is shared by polyominoes and other 2-dimensional models of branched polymers. (The notation $f(n) \sim g(n)$ means that $\lim f(n)/g(n) = 1$.) Since we expect $A(n) \sim C_0\, n^{-\theta}\lambda^n$ as $n \to \infty$ for some constant C_0, and since we now know that $\lambda_{\kappa_t} = \lambda$ (Theorem 10 below), we are led to the following conjecture.

Conjecture 1 *For each $t > 0$, the ratio $\kappa_t(n)/A(n)$ converges to a nonzero limit as $n \to \infty$. That is, the probability that a random polyomino of area n is a t-TCP is bounded away from 0 as n gets large.*

A rigorous proof of this conjecture would be a novel piece of evidence in support of the universality behaviour predicted by renormalization group theory. Note that Theorem 12 below tells us that the probability that a random polyomino of area n is a t-TCP is bounded away from 1.

3 An Efficient Counting Algorithm

An algorithm for computing $\kappa(n)$, for a given n, was stated as an open problem in reference [3, §14, p. 369, problem 14.5.5]. We describe here an efficient algorithm for counting 1-TCPs and comment about extending it to counting t-TCPs for $t > 1$.

3.1 Algorithm

We first implemented a prototype backtracking algorithm for counting 1-TCPs. The program recursively concatenated concave columns to a growing polyomino. A branch of this procedure was abandoned when the area of the polyomino grew too large or if it was no longer possible for it to become connected with the addition of further columns. (This happened when a component of the polyomino became permanently detached.)

We then designed a much more efficient algorithm, based on Jensen's algorithm for counting all polyominoes [21, 22]. In a nutshell, Jensen's algorithm counts polyominoes within horizontal bounding strips of height h, where $1 \leq h \leq \lceil n/2 \rceil$. The algorithm considers column by column from left to right, and cell by cell from top to bottom within each column. At each cell, the algorithm considers either to have it occupied (belonging to the polyomino) or empty (not belonging). At all stages, the algorithm does not keep in memory all polyominoes but all possible *right boundaries* of polyominoes, that is, all combinations of the last h cells considered. The algorithm maintains a database whose entries have keys that are the different *signatures*, where a signature consists of a boundary plus all possible connections between cells on the boundary by cells found to the left of it. In other words, the keys reflect all possible splits of boundary cells into connected components, where the connections are to the left of the boundary. In addition, a signature also includes two bits that indicate whether or not the polyominoes associated with that entry touch the top and/or bottom of the strip. The contents of each entry in the database is statistics of all partially-built polyominoes ("partially" means that polyominoes may still consist of more than one connected component), that is, the *counts* of all polyominoes parameterized by area, having that specific signature. When the currently considered cell is chosen to be occupied, the counts of polyominoes are updated by adding the numbers of fully-built polyominoes, that is, polyominoes that consist of exactly one connected component and touch the top and bottom of the strip.

3.2 Our modifications

For counting 1-TCPs, we also need to ensure that each column and each row consists of more than one consecutive sequence of cells. This is simple to achieve for columns: At the end of processing a column, we discard from the database all entries that correspond to columns that contain less than two sequences of occupied cells.[1] (The number of such signatures is $1+\sum_{i=1}^{h}(h-i+1) = \Theta(h^2)$. This is compared to the total number of signatures whose number is $\tilde{O}(3^{h/2})$. For rows, we enhance the signatures by splitting each one into at most 4^h subsignatures: For each row, we keep a code as follows: '0' indicates that the first sequence of occupied cells has not been met yet; '1' means that we are in the middle of the first sequence; '2' states that we are between the first and second sequences; and '3' signifies that we have already entered the second sequence. (Once we reach '3,' we do not need to update this indicator any more.) Figure 1 shows an enhanced signature, in which each boundary cell is associated with two numbers: The original vertical code (left), and the

[1] Before completing a column, the boundary consists of portions of the two rightmost columns, hence we cannot rule out signatures that contain "only one" sequence of cells.

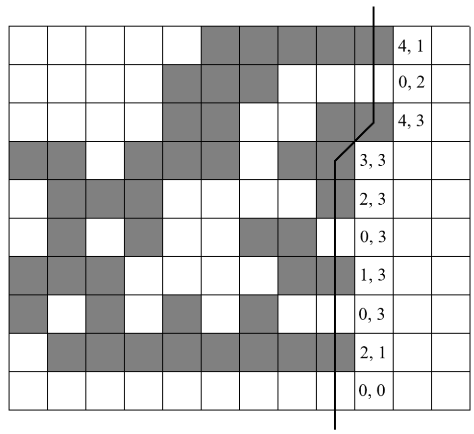


Figure 1: An enhanced boundary signature in the modified version of Jensen's algorithm.

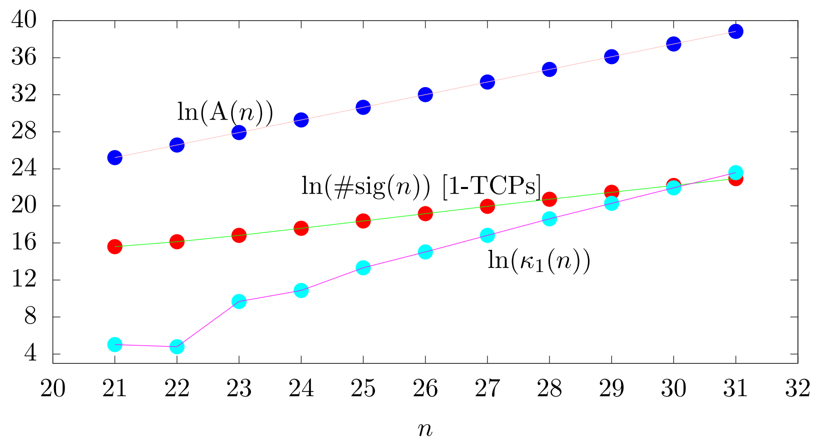


Figure 2: Plots of the number of signatures while counting 1-TCPs, and the numbers of all poyominoes and of 1-TCPs.

additional horizontal code (right). Then, we count only polyominoes with signatures whose line indicators are all '3.' Note that the indicators of the top and bottom rows make the two bits described above redundant.

Jensen's algorithm is efficient in the sense that it is the only known algorithm whose running time, $\tilde{O}(1.732^n)$ [6], is asymptotically smaller than the total number of polyominoes, $\tilde{\Theta}(\lambda^n)$. (Recall that $\lambda \approx 4.06$.) Our modification splits every signature into at most $4^{n/2} = 2^n$ subsignatures (in practice, into much less than that), thus, the running time of the modified algorithm is $\tilde{O}(3.464^n)$, which is still much smaller than the total number of polyominoes. In conclusion, our version of the algorithm is slower than the original algorithm, although we eventually count fewer polyominoes, due to the exponential growth in the number of processed signatures.

Figure 2 plots in a semi-logarithmic scale the number of distinct signatures encountered

Table 1: Counts of 1-TCPs.

n	$\kappa_1(n)$	n	$\kappa_1(n)$	n	$\kappa_1(n)$	n	$\kappa_1(n)$
1–20	0	24	52,306	28	119,309,768	32	88,476,873,440
21	152	25	606,636	29	641,447,812	33	435,921,253,072
22	120	26	3,376,528	30	3,403,173,276	34	2,113,011,155,472
23	15,820	27	20,204,672	31	17,634,751,456	35	10,065,872,407,536

by the algorithm while computing $\kappa_1(n)$ (in red circles), together with the number of 1-TCPs (cyan) and the total number of polyominoes (blue), all as functions of n, for $21 \le n \le 31$. Linear regression shows that the number of signatures is empirically similar to 2.124^n, which is much less than the theoretical upper bound (which does not take signature pruning into account) of 3.464^n.

3.3 Results

Our prototype program, implemented in Python, computed in 90 hours (elapsed time) $\kappa(n)$ up to $n = 26$ on a PC with a 64-bit system operating an i5-9400F Intel Core CPU at 2.90GHz with 12GB of RAM.

The modified version of Jensen's algorithm was implemented in `C++` and run on a 12th generation Intel(R) i9-12900KF with 128GB of RAM. Using about 41 hours of CPU, the program computed $\kappa(n)$ up to $n = 35$, obtaining the values reported in Table 1 and agreeing with all values computed by the prototype program.

3.4 Higher Values of t

Extending the algorithm to counting t-TCPs (for $t > 1$) is rather easy. However, it may no longer be competitive with an approach based on explicitly generating all polyominoes. First, instead of splitting each original code of a boundary cell c into four subcodes, we split it into $2(t{+}1)$ subcodes that indicated the number of maximal sequences of occupied cells already met in the row respective of c.

In order to guarantee that the extended algorithm counted only t-TCPs, we (1) Discarded from the database (at the end of processing a column) all entries that corresponded to columns that contained less than $t{+}1$ sequences of occupied cells; and (2) Counted only polyominoes with signatures whose line indicators were all '$2(t{+}1)$.' The first (resp., second) condition guaranteed that all columns (resp., rows) of counted polyominoes that matched the definition of a t-TCP.

Splitting the subcodes of signature cells resulted in multiplying both running time and memory consumption of the original algorithm by a factor of $(2(t{+}1))^h$ (for counting t-TCPs of height h). Since h could be as high as $n/2$, this brought the exponential factor in the complexity of the algorithm to $(\sqrt{6(t+1)})^n$, which prohibited the actual finding of t-TCPs for $t > 1$. For example, already for $t = 2$, the complexity of the algorithm was $\tilde{O}(4.243^n)$, while the minimum size of a 2-TCP is 50, which was beyond what running the algorithm

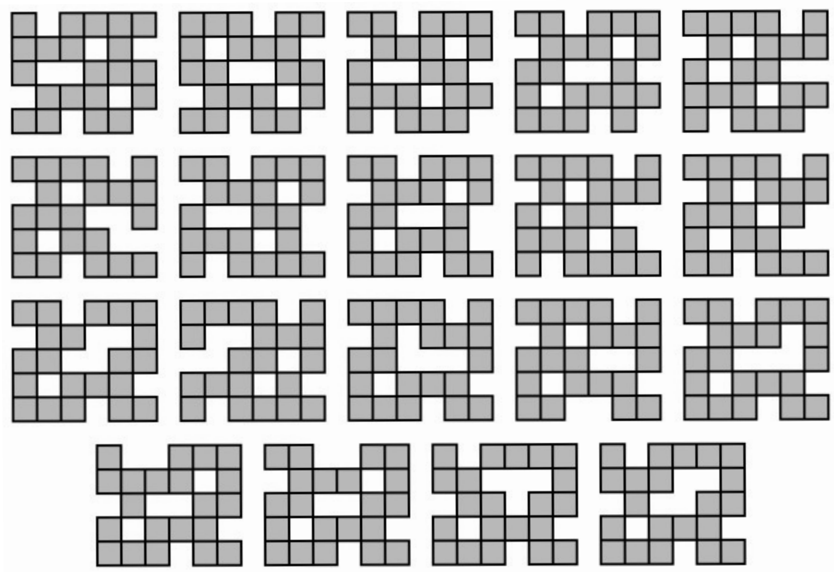


Figure 3: The 19 TC polyominoes of area 21, up to rotations and reflections.

could reach with the available time and memory.

Remark. Redelmeier's algorithm [31] generates all polyominoes of size up to n, and so runs in $\tilde{O}(\lambda^n)$ time. Checking a polyomino with n cells for t-TCness can be done in time polynomial with n, so simply filtering the output of Redelmeier's algorithm for t-TCPs also takes $\tilde{O}(\lambda^n)$ time. Assuming that $\lambda \approx 4.063$, this approach is therefore theoretically more efficient than the one we just described, barring a tighter running-time analysis for our modification of Jensen's algorithm than we gave.

4 Minimum Area of t-TCPs

We will prove a lower bound on the area of t-TCPs for all t, and construct examples for proving that it is tight, which together proves the following theorem, our first main result.

Let m_t be the minimum possible area of a t-TCP. (See sequence A385602 in the Online Encyclopedia of Integer Sequences [29].)

Theorem 2 *$m_1 = 21$, $m_2 = 50$, and $m_t = 6(t+1)^2 - 1$ for all $t > 2$.* □

The theorem above is the combination of lemmata 4 and 5 below. The value $m_1 = 21$ was confirmed by our TCP counting programs (see Section 3). Figure 3 shows representatives of the 152 TC polyominoes of area 21. (None of these polyominoes have any symmetries, hence, the polyominoes formed by the eight orientations of each of the 19 drawn polyominoes are distinct.)

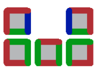
4.1 Lower Bound

We use the notion of the *minimum bounding box* of a polyomino.

Definition 2 *The* minimum bounding box *of a polyomino P is the least pair of integers (k, ℓ), such that $P \subset [0, k] \times [0, \ell]$. That is, the minimum bounding box of P is contained in any other bounding box of P.*

Lemma 3 *For a t-fold TCP with n cells in a (k, ℓ)-bounding box,*

$$(t + 1)(k + \ell) - 1 \leq n \leq k\ell - t \cdot \max\{k, \ell\} - 2t.$$

Proof: By rotating if necessary, we may assume $k \geq \ell$. For the lower bound, partition the edges of the polyomino's cells into *outside*, *inside*, and *hidden* edges, which we will say number o, i, and h, respectively. Outside edges face away from the polyomino, inside edges back into it, and hidden edges are those in between two cells. For example, the U-pentomino  has ten outside edges (red), two inside edges (blue), and eight hidden edges (green). Being the perimeter, $o = 2k + 2\ell$. By the t-TC property, $i \geq 2tk + 2t\ell$. By connectedness, every polyomino has a spanning tree with at least $n - 1$ edges, and so $h \geq 2n - 2$. The lower bound follows from this and the fact that we counted exactly $4n = o + i + h$ edges. For the upper bound, notice that we must remove at least tk cells from $[0, k] \times [1, \ell - 1]$ in order to have k-many t-fold concave columns, and a further t cells from the top and the bottom rows to guarantee their t-fold concavity. Finally, we take in the statement of the lemma the maximum of k and ℓ since their roles can be exchanged. □

These relations restrict the possible areas of t-TCPs in (k, ℓ) bounding boxes rather significantly. We see this by solving an integer non-linear program (NLP) in the general-t case using duality. Guenin *et al.* [17] provide a friendly reference for the techniques used.

Lemma 4 *$m_1 \geq 21$, $m_2 \geq 50$, and $m_t \geq 6(t + 1)^2 - 1$ for $t > 2$.*

Proof: Assume, without loss of generality, that $k \geq \ell$. For a t-TCP to exist in a (k, ℓ) bounding box, the lower and upper bounds of Lemma 3 must both hold, *i.e.*, their difference $H(k, \ell) := k\ell - (2t + 1)k - (t + 1)\ell - 2t + 1$ must be non-negative. Minimizing the lower bound of Lemma 3, we therefore have the integer NLP (1). To solve (1), we will consider two auxiliary NLPs, (2) and (3).

$$\begin{aligned} \min\ & k + \ell \\ \text{s.t.}\ & H(k, \ell) \geq 0, \\ & k - \ell \geq 0, \text{ and} \\ & k, \ell \in \mathbb{Z}^+. \end{aligned} \quad (1) \qquad \begin{aligned} \min\ & k + \ell \\ \text{s.t.}\ & H(k, \ell) \geq 0, \\ & k - \ell \geq 0, \text{ and} \\ & k, \ell \geq 0. \end{aligned} \quad (2) \qquad \begin{aligned} \min\ & k + \ell \\ \text{s.t.}\ & H(k, \ell) \geq 0, \\ & k - \ell \geq 1, \text{ and} \\ & k, \ell \geq 0 \end{aligned} \quad (3)$$

First, we solve the NLP (2). Noting the region $\{(k, \ell) : H(k, \ell) \geq 0, k, \ell > 0\}$ is convex, we may define a linear relaxation by a gradient, the LP (4). We also write its dual, the LP (5),

$$\begin{array}{ll} \min & \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}k & \ell\end{bmatrix}^T \\ \text{s.t.} & \begin{bmatrix}1 & -1 \\ 1 & -\alpha_t\end{bmatrix}\begin{bmatrix}k \\ \ell\end{bmatrix} \geq \begin{bmatrix}0 \\ \beta_t - \alpha_t\beta_t\end{bmatrix} \\ \text{where} & k, \ell \geq 0 \end{array} \quad (4) \qquad \begin{array}{ll} \max & \begin{bmatrix}0 & (\beta_t - \alpha_t\beta_t)\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix}^T \\ \text{s.t.} & \begin{bmatrix}1 & 1 \\ -1 & -\alpha_t\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} \leq \begin{bmatrix}1 \\ 1\end{bmatrix} \\ \text{where} & x, y \geq 0 \end{array} \quad (5)$$

where the number β_t is such that (β_t, β_t) is the point of intersection of the line $k = \ell$ and the hyperbola $H(k, \ell) = 0$, and α_t is the derivative of k with respect to ℓ of the hyperbola at the point $(k, \ell) = (\beta_t, \beta_t)$. Explicitly,

$$\beta_t = \frac{3}{2}t + 1 + \sqrt{\frac{9}{4}t^2 + 5t} \quad \text{and} \quad \alpha_t = \frac{t + 1 - \beta_t}{\beta_t - 2t - 1}.$$

To solve the primal-dual pair (4)-(5), notice that $(\bar{k}, \bar{\ell}) = (\beta_t, \beta_t)$ and $(\bar{x}, \bar{y}) = (\frac{1+\alpha_t}{\alpha_t - 1}, \frac{2}{1-\alpha_t})$ are feasible in (4) and (5), respectively, both with the objective value $2\beta_t$. Thus, it follows by weak duality that $(\bar{k}, \bar{\ell})$ is optimal in the LP (4). Since $(\bar{k}, \bar{\ell})$ is also feasible in the NLP (2), it is optimal there too.

We will also solve the auxiliary NLP (3), just as we solved (2). First, we find an LP relaxation of NLP (3), the LP (6), and write its dual, the LP (7),

$$\begin{array}{ll} \min & \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}k & \ell\end{bmatrix}^T \\ \text{s.t.} & \begin{bmatrix}1 & -1 \\ 1 & -\delta_t\end{bmatrix}\begin{bmatrix}k \\ \ell\end{bmatrix} \geq \begin{bmatrix}1 \\ \gamma_t - \delta_t\gamma_t + 1\end{bmatrix} \\ \text{where} & k, \ell \geq 0 \end{array} \quad (6) \qquad \begin{array}{ll} \max & \begin{bmatrix}1 & (\gamma_t - \delta_t\gamma_t + 1)\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix}^T \\ \text{s.t.} & \begin{bmatrix}1 & 1 \\ -1 & -\delta_t\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} \leq \begin{bmatrix}1 \\ 1\end{bmatrix} \\ \text{where} & x, y \geq 0 \end{array} \quad (7)$$

where γ_t and δ_t are defined analogously to α_t and β_t. The number γ_t is such that $(\gamma_t + 1, \gamma_t)$ is the point of intersection between the line $k = \ell + 1$ and the hyperbola $H(k, \ell) = 0$, and δ_t is the derivative of k with respect to ℓ of the hyperbola at the point $(k, \ell) = (\gamma_t + 1, \gamma_t)$. Explicitly,

$$\gamma_t = \frac{3}{2}t + \frac{1}{2} + \sqrt{\frac{9}{4}t^2 + \frac{11}{2}t + \frac{1}{4}} \quad \text{and} \quad \delta_t = \frac{t - \gamma_t}{\gamma_t - 2t - 1}.$$

Observe that $(\bar{k}, \bar{\ell}) = (\gamma_t + 1, \gamma_t)$ and $(\bar{x}, \bar{y}) = (\frac{1+\delta_t}{\delta_t - 1}, \frac{2}{1-\delta_t})$ are feasible in (6) and (7), respectively, both with objective value $2\gamma_t + 1$. Thus, it follows by weak duality that $(\bar{k}, \bar{\ell})$ is optimal in (6). Since $(\bar{k}, \bar{\ell})$ is feasible also in the NLP (3), it is optimal there too.

We are now ready to solve the original integer NLP (1). In the $t = 1$ case, the optimal value of (1) is at least the ceiling of the minimum of the optimal values of (4) and (6), which is 11. Since $(k, \ell) = (\gamma_1 + 1, \gamma_1) = (6, 5)$ realizes this bound, it is optimal in (1). For the $t = 2$ case, $\gamma_2 = 8$ is an integer. Hence, $(k, \ell) = (9, 8)$ is an optimal integer solution to (3). Because the only integer points feasible in (2) but not in (3) are on the line $k = \ell$, the least of which is $(k, \ell) = (\lceil \beta_2 \rceil, \lceil \beta_2 \rceil) = (9, 9)$, we have that $(k, \ell) = (8, 9)$ is optimal in (1). For $t > 2$, we observe $3t + 3 > \beta_t, \gamma_t > 3t + 2$. Since $(k, \ell) = (\gamma_t + 1, \gamma_t)$ is optimal in (3), all feasible integers $k > \ell$ have $k + \ell \geq \lceil 2\gamma_t + 1 \rceil \geq 6t + 6$. Since the least feasible

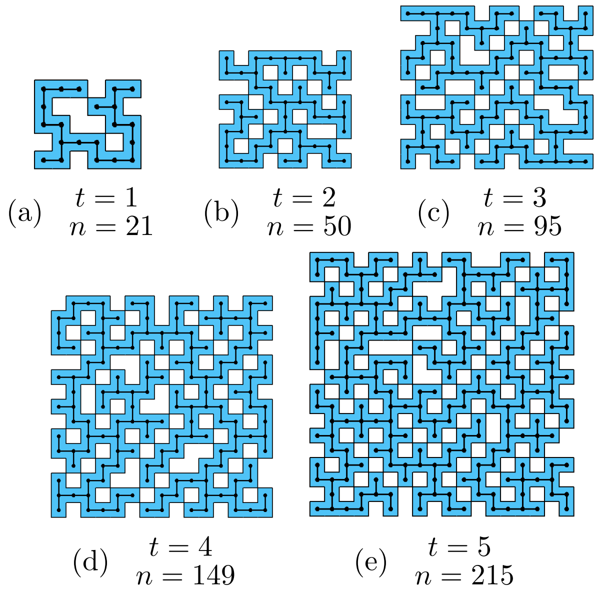


Figure 4: Minimum-area TCPs for $1 \leq t \leq 5$

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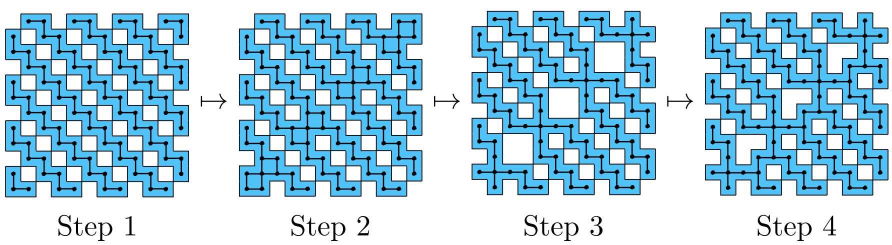


Figure 5: Constructing a t-TCP of area $6(t+1)^2 - 1$.

integer $k = \ell$ is $\lceil \beta_t \rceil = 3t + 3$, we have that $(k, \ell) = (3t + 3, 3t + 3)$ is optimal in (1). The result follows from these solutions to (1) and Bound (1) of Lemma 3. □

We remark that one could alternatively perform the above proof by finding points satisfying the Karush-Kuhn-Tucker conditions in NLPs (2) and (3).

4.2 Upper Bound

To bound the minimum area of a t-TCP from above by $n \in \mathbb{N}$ inclusive, it suffices to find a t-TCP of area n. For $t = 1, 2$, the examples given in Figures 4(a,b) are enough. For $t > 2$, we provide a general construction.

Lemma 5 *For* $t \geq 3$, $m_t \leq 6(t+1)^2 - 1$.

Proof: Consider the following construction, in four steps. It is illustrated in Figure 5 in the $t = 3$ case.

1. Create a collection of cells, placing a cell about a point $(x, y) \in \{0, 1, 2, \cdots, 3t+2\} \times \{0, 1, 2, \cdots, 3t+2\}$ if and only if the sum $(x+y)$ is not congruent to 2 modulo 3. This collection has $6(t+1)^2$ cells, and $2(t+1)$ connected components. Each row and column has at least t gaps, and some have more.

2. There are $(t+1)$ columns (resp., rows) with $(t+1)$ gaps, at $x \equiv 2 \mod 3$ (resp., $y \equiv 2 \mod 3$). Place more cells about points of the form $(3i+1, 3i+1)$, for $0 \leq i \leq t$. The resulting collection of cells has $6(t+1)^2 + (t+1)$ cells, and $(t+1)$ connected components. Each row and column retains at least t gaps.

3. Step 2 created multiply connected components. Therefore, we may remove the $2t+2$ cells about points of the form $(3i+1 \pm 1, 3i+1 \pm 1)$ for $0 \leq i \leq t$ without creating more connected components. There are still at least t gaps in each column and row, since $(3i, 3i)$ is adjacent to $(3i+1, 3i)$ and $(3i, 3i+1)$, while $(3i+2, 3i+2)$ is adjacent to $(3i+2, 3i+1)$ and $(3i+1, 3i+2)$. The result is a collection of $6(t+1)^2 - (t+1)$ cells.

4. We can connect the remaining $(t+1)$ connected components to each other with t additional cells, centered about points of the form $(3i, 3i-1)$ for $1 \leq i \leq t$. No such addition breaks t-fold concavity, since $(3i, 3i-1)$ is not adjacent to either $(3i+1, 3i+1)$ or $(3i, 3i)$, each removed in Step 3. We are left with a t-TCP of area $6(t+1)^2 - 1$.

Since this construction works for all $t > 2$, the claim is proved. □

Note that the examples produced by the above construction are different from the ones given in Figure 4.

Corollary 6 *For $t \geq 1$, there are at least 2^t-many t-TCPs of area $6(t+1)^2 - 1$.*

Proof: In Step 4, we could have just as well placed a cell about the point $(3i, 3i+1)$ whenever we placed one about $(3i+1, 3i)$, giving t binary choices. □

4.3 Structure of Minimum-Area TCPs

Figure 4 shows some minimum-area t-TCPs for $1 \leq t \leq 5$. Lemma 7 below characterizes the bounding boxes of minimal t-TCPs. The existence of non-square minimal t-TCP for $t > 3$ is an open question. The construction in the proof of Theorem 5 shows that there exists a minimal t-TCP in a $(3t+3, 3t+3)$ bounding box for all $t > 2$. However, the 3-TCP with a (13, 11) bounding box shown in Figure 4(c) is currently the only example of a minimal t-TCP that does not have a square bounding box for $t > 2$. The following results also relate the dimension of a t-TCPs bounding box to its connectivity and its concavity.

Lemma 7 *If a t-TCP has a (k, ℓ) bounding box and its area is $(t+1)(k+\ell) - 1$, then*
(i) It is a tree; and
(ii) It has exactly t gaps in every row and every column.

Proof: We prove the contrapositive. If P is an area-n t-TCP that is not a tree, the bound for the number of hidden edges (see the proof of Lemma 3) becomes $h \geq 2n$. If P has more than t gaps in some row or column, the bound on the number of inside edges becomes $i \geq 2t(k+\ell)+2$. In either case, we get $n \geq (t+1)(k+\ell)$ given that $i+o+h=4n$, hence $n \neq (t+1)(k+\ell)-1$. The claim follows. □

We now present our second main result.

Theorem 8 *Suppose that P is a minimum-area t-TCP of area n whose bounding box is $B=(k,\ell)$ (for $k \geq \ell$). If $t=1$, then $B=(6,5)$. If $t=2$, then $B=(9,8)$. Otherwise, if $t \geq 3$, then B is either $(3t+3,3t+3)$ or $(3t+4,3t+2)$. Moreover, P is a tree and it has exactly t gaps in every row and every column.*

Proof: Let P be a t-TCP with area m_t in a (k,ℓ) bounding box. We claim that

$$(t+1)(k+\ell)-1=m_t. \tag{8}$$

By Lemma 3, we have that $m_t \geq (t+1)(k+\ell)-1$. The pair (k,ℓ) is feasible in the integer NLP (1) because P is a t-TCP. Since m_t is equal to the lower bound given by Lemma 4, m_t is the minimum of $(t+1)(k+\ell)-1$ for feasible (k,ℓ) pairs, that is, $m_t \leq (t+1)(k+\ell)-1$. Hence, Equation (8) holds, and by Lemma 7 we have that all minimal t-TCPs are trees and have exactly t gaps everywhere.

It is easy to check that the unique solutions that satisfy Equation (8) and the relations in Lemma 3 are $(k,\ell)=(6,5)$ in the $t=1$ case and $(k,\ell)=(9,8)$ in the $t=2$ case. A manual inspection of all 1-TCPs (provided in reference [4]) is also available for $t=1$.

For $t \geq 3$, notice that the solutions to Equation (8) with $k \geq \ell$ take the form $(k,\ell)=(3t+3+\Delta,3t+3-\Delta)$ for some non-negative integer $\Delta \geq 0$. Expanding and rearranging the relations in Lemma 3 in this case give $t(1-\Delta)+4-\Delta^2 \geq 0$, which is possible only if $\Delta=0$ or $\Delta=1$. □

5 Equality of the Growth Constants of t-TCPs to λ

Bender [10], and independently Klarner and Rivest [24], showed that the number of convex polyominoes of size n is asymptotically $t\gamma^n$, for $\gamma \approx 2.3091$ and $t \approx 2.6756$, that is, the growth constant of convex polyominoes is roughly 2.3091. In this section, we investigate λ_{κ_t}, the growth constants of t-TCPs.

It is straightforward to prove the existence of λ_{κ_t} via a concatenation argument and supermultiplicativity, as was done previously for 1-TCPs [4] and is common for other families of lattice animals [2, 9, 23].

Theorem 9 *For all $t>0$, $\lambda_{\kappa_t} := \lim_{n\to\infty} \sqrt[n]{\kappa_t(n)}$ exists and is finite. Moreover, $\lambda_{\kappa_t} := \sup_n \sqrt[n]{\kappa_t(n)}$, which implies that $\kappa_t(n) \leq \lambda_{\kappa_t}^n$ for all $k,n \in \mathbb{N}$.*

Proof: Since every t-TCP is a polyomino, $\sqrt[n]{\kappa_t(n)} \leq \sqrt[n]{A(n)}$. In addition, we know that $\sqrt[n]{A(n)} \to \lambda$ as $n \to \infty$. We conclude that the sequence $(\sqrt[n]{\kappa_t(n)})_{n>0}$ is bounded from above. We will also show that for all $t, n, m > 0$, $\kappa_t(n) \cdot \kappa_t(m) \leq \kappa_t(n+m)$, *i.e.*, $(\kappa_t(n))$ is supermultiplicative. The result then follows from Lemma 1 of reference [23] which states precisely the existence of the hypothesized limits for supermultiplicative sequences that are bounded as above.

To see the supermultiplicative relation, concatenate two t-TCPs of sizes n and m. Given $P_1 \in \kappa_{t,n}$ and $P_2 \in \kappa_{t,m}$, let P_3 be the union of P_1 and the translation of P_2 such that its lexicographically smallest cell lies immediately to the right of the biggest cell of P_1. Then, P_3 uniquely determines an element of $\kappa_{t,n+m}$ since all rows and columns still have at least t gaps, and the original pair P_1, P_2 can be determined uniquely from P_3 by separating the n lexicographically-smallest from the m lexicographically-biggest cells of P_3.

For proving that $\lambda_{\kappa_t} = \sup_n \sqrt[n]{\kappa_t(n)}$, observe that for each n, the supermultiplicative relation implies that the subsequence $(\sqrt[n2^m]{\kappa_t(n2^m)})_{m\geq 0}$ is increasing. Indeed, its limit must be λ_{κ_t}, so $\sqrt[n]{\kappa_t(n)} \leq \lambda_{\kappa_t}$ for all $n > 0$. □

Theorem 9 can be used for obtaining lower bounds on λ_{κ_t}. If $\kappa_t(n) \geq x$, then we have that $\lambda_{\kappa_t} \geq \sqrt[n]{x}$. That is how the best known lower bound on λ_{κ_1}, which is 2.4474, was found [4]. However, the following construction does better. Compare the previous known bound to the one given in Corollary 11 below.

The following is our third main result.

Theorem 10 *The growth constant for t-TCPs, λ_{κ_t}, equals λ for all $t > 0$.*

Proof: We partition the set of all n-ominoes into a polynomial number of subsets. Given any polyomino P, we define the following quantities.

$$
\begin{aligned}
X_{span}(P) &= \max\,\{x : (x,y) \in P \text{ for some } y\},\\
Y_{span}(P) &= \max\,\{y : (x,y) \in P \text{ for some } x\},\\
X_0^-(P) &= \min\,\{x : (x,0) \in P\},\\
X_0^+(P) &= \min\,\{x : (x, Y_{span}(P)) \in P\},\\
Y_0^-(P) &= \min\,\{y : (0,y) \in P\},\\
Y_0^+(P) &= \min\,\{y : (X_{span}(P), y) \in P\}.
\end{aligned}
$$

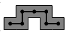
Note that if the span of a polyomino P in either of the axes is d, then the coordinates of cells of P along that axis are in the range $[0, d-1]$. Then, the set $P_n[a,b,c,d,e,f]$ is defined as

$$
\begin{aligned}
P_n[a,b,c,d,e,f] = \{P \in A_n :\ & X_{span}(P) = a,\\
& Y_{span}(P) = b,\ X_0^-(P) = c,\ X_0^+(P) = d,\\
& Y_0^-(P) = e,\ Y_0^+(P) = f\}.
\end{aligned}
$$

See Figure 6(a) for an illustration of a typical member of $P_n[a,b,c,d,e,f]$. It follows from the connectedness of P that $P_n[a,b,c,d,e,f] = \emptyset$ if any of a,b,c,d,e,f are greater than n, hence,

$$A_n = \bigcup_{\substack{0<a,b\leq n\\ 0\leq c,d,e,f<n}} P_n[a,b,c,d,e,f]. \tag{9}$$

Therefore, there exist $a^\circ, b^\circ, c^\circ, d^\circ, e^\circ, f^\circ$ for which $|P_n[a^\circ, b^\circ, c^\circ, d^\circ, e^\circ, f^\circ]| \geq \left(\frac{A(n)}{n^6}\right)$. For each $i, j \in \{0, 1, \ldots, t\}$, let $\pi_{i,j}$ be any element of $P_n[a^\circ, b^\circ, c^\circ, d^\circ, e^\circ, f^\circ]$. There are at least $\left(\frac{A(n)}{n^6}\right)^{(t+1)^2}$ choices for $\{\pi_{i,j}\}_{0\leq i,j\leq t}$.

We now construct to each choice of $\{\pi_{i,j}\}$ a unique t-TCP. First, we define the polyomino B (), and its 90° clockwise-rotated version, $B^{\circlearrowright}$. We use the notation P_G as defined in the introduction. We set $B := P_G$, where $G := \{(0,0),(1,0),(1,1),(2,1),(3,1),(3,0),(4,0)\}$, and $B^{\circlearrowright} := P_{G^{\circlearrowright}}$, where $G^{\circlearrowright} := \{(0,0),(0,1),(1,1),(1,2),(1,3),(0,3),(0,4)\}$.

Let $P + \vec{v}$ denote the translation of a polyomino P by a vector $\vec{v} \in \mathbb{Z}^2$. We are now ready to define our constructed t-TC polyomino, $\mathrm{TCP}(\{\pi_{i,j}\}_{0\leq i,j\leq t})$. Define $\{\pi'_{i,j}\}_{0\leq i,j\leq t}$, $\{\hat{\pi}_{i,j}\}_{0\leq i,j\leq t}$, $\{B_{i,j}\}_{0\leq i<t,\,0\leq j\leq t}$, and $\{B^{\circlearrowright}_{i,j}\}_{0\leq i\leq t,\,0\leq j<t}$, by the following rules.

$$\pi'_{i,j} = \begin{cases} \pi_{i,j} & i,j \text{ even} \\ \text{reflection of } \pi_{i,j} \text{ through the line } y = b^\circ/2 & i \text{ even, } j \text{ odd} \\ \text{reflection of } \pi_{i,j} \text{ through the line } x = a^\circ/2 & i \text{ odd, } j \text{ even} \\ \text{reflection of } \pi_{i,j} \text{ through } y = \frac{b^\circ}{2} \text{ and } x = \frac{a^\circ}{2} & i,j \text{ odd} \end{cases}$$

$$\hat{\pi}_{i,j} = \pi'_{i,j} + (i(a^\circ+5), j(b^\circ+5))$$

$$B_{i,j} = \begin{cases} B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+f^\circ) & i,j \text{ even} \\ B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+b^\circ-f^\circ-1) & i \text{ even, } j \text{ odd} \\ B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+e^\circ) & i \text{ odd, } j \text{ even} \\ B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+b^\circ-e^\circ-1) & i,j \text{ odd} \end{cases}$$

$$B^{\circlearrowright}_{i,j} = \begin{cases} B^{\circlearrowright} + (i(a^\circ+5)+d^\circ,\ j(b^\circ+5)+b^\circ) & i,j \text{ even} \\ B^{\circlearrowright} + (i(a^\circ+5)+c^\circ,\ j(b^\circ+5)+b^\circ) & i \text{ even, } j \text{ odd} \\ B^{\circlearrowright} + (i(a^\circ+5)+a^\circ-d^\circ-1,\ j(b^\circ+5)+b^\circ) & i \text{ odd, } j \text{ even} \\ B^{\circlearrowright} + (i(a^\circ+5)+a^\circ-c^\circ-1,\ j(b^\circ+5)+b^\circ) & i,j \text{ odd} \end{cases}$$

Thus, $B_{i,j}$ intersects each of $\hat{\pi}_{i,j}$ and $\hat{\pi}_{i+1,j}$ in one edge, and $B^{\circlearrowright}_{i,j}$ intersects each of $\hat{\pi}_{i,j}$ and $\hat{\pi}_{i,j+1}$ in one edge. Finally, define

$$TCP(\{\pi_{i,j}\}_{0\leq i,j\leq t}) = \left(\bigcup_{0\leq i,j\leq t} \hat{\pi}_{i,j}\right) \cup \left(\bigcup_{i=0}^{t-1}\bigcup_{j=0}^{t} B_{i,j}\right) \cup \left(\bigcup_{i=0}^{t}\bigcup_{j=0}^{t-1} B^{\circlearrowright}_{i,j}\right).$$

See Figure 6(b) for an illustration of this construction. It is evident that $\mathrm{TCP}(\{\pi_{i,j}\})$

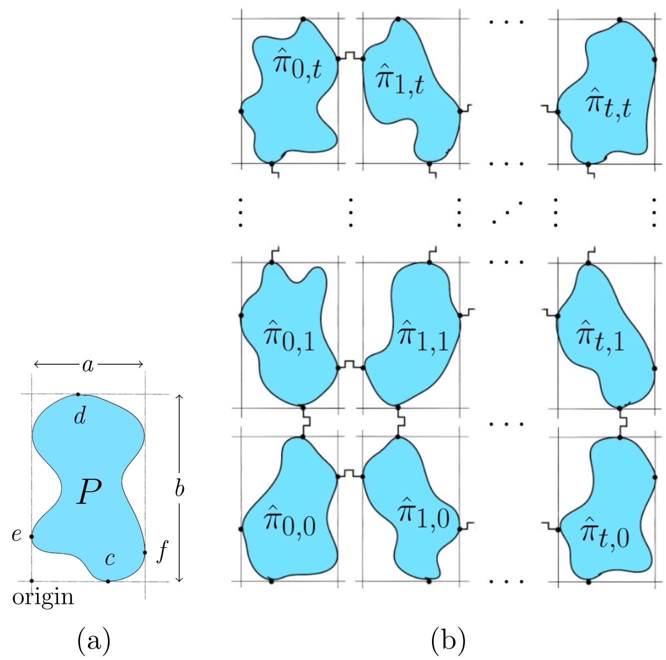
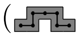


Figure 6: The construction for the proof of Theorem 10. (a) A typical member of $P_n[a, b, c, d, e, f]$; (b) A "blob" representation of the polyomino TCP($\{\hat{\pi}_{i,j}\}_{0 \leq i,j \leq t}$). The blobs represent $\hat{\pi}_{i,j}$s, and the "squiggles" between them the $B_{i,j}$s and $B^{\circlearrowright}_{i,j}$s. Notice how the $\hat{\pi}_{i,j}$s are flipped throughout the construction to match along their boundaries.

uniquely determines $\{\pi_{i,j}\}$. It is easily verified by inspecting rows and columns that TCP($\{\pi_{i,j}\}$) is indeed t-TC, although we must be careful with the following special case. If e° or f° is $b^\circ-1$ or 0, then we must redefine $B_{i,t}$ for $0 \le i < t$, replacing B () in the above with its vertical reflection () in the line $y = \frac{1}{2}$. Without this hack, the top row of TCP($\{\pi_{i,j}\}$) may not be t-TC, consisting only of the $B_{i,t}$ kinks. The same applies to $B^{\circlearrowright}_{t,j}$ for the case in which c° or d° is $a^\circ-1$ or 0, where we must replace $B^{\circlearrowright}$ with its horizontal reflection in the line $x = \frac{1}{2}$.

The constructed t-TC polyomino TCP($\{\pi_{i,j}\}$) has $\phi := (t+1)^2 n + 14t(t+1)$ cells. The $14t(t+1)$ term comes from the 7 cells in each of $B_{i,j}$ and $B^{\circlearrowright}_{i,j}$. Therefore, we have that

$$\kappa_t(\phi) \ge \left(\frac{A(n)}{n^6}\right)^{(t+1)^2}, \quad \text{and hence} \quad \sqrt[\phi]{\kappa_t(\phi)} \ge \sqrt[\phi]{\left(\frac{A(n)}{n^6}\right)^{(t+1)^2}}. \qquad (10)$$

We now let $n \to \infty$. Note that the indices ϕ define a subsequence of the sequence $\kappa_t(n)$. Since the sequence $\sqrt[n]{\kappa_t(n)}$ converges to λ_{κ_t}, so does the subsequence, and to the same limit. The right side of the final inequality shows that the limit is at least λ. However, the limit cannot exceed λ, because t-TCPs are a proper subset of all polyominoes. Therefore, it must be that $\lambda_{\kappa_t} = \lambda$. □

Corollary 11 *For all $t > 0$, we have that $4.0025 \le \lambda_{\kappa_t} \le 4.5252$.*

Proof: These are just the best known lower [7] and upper [8] bounds on λ. □

Remark 1 *We can now observe that t-TCPs (like ordinary polyominoes) are in a different universality class from convex polyominoes. As noted at the beginning of this section, the number of convex polyominoes of size n is asymptotically proportional to $n^{-\theta_c}\gamma^n$ with $\theta_c = 0$ [10, 24]. But $\kappa_t(n)$ cannot be asymptotically proportional to $\lambda^n_{\kappa_t}$ because*

$$\kappa_t(n) \le A(n) \le \frac{1}{\sqrt{2n}}\lambda^{n+2} = \frac{\lambda^2_{\kappa_t}}{\sqrt{2n}}\lambda^n_{\kappa_t} \quad \textit{for every } n \ge 1, \qquad (11)$$

where the middle relation is from Theorem 1.1(iii) of reference [26]. Indeed, if it is true that $\kappa_t(n) \sim C_t n^{-\theta_t}\lambda^n$, then θ_t must be at least $1/2$.

Theorem 10 says that t-TCPs are not exponentially rare in the polyominoes. The next theorem, our fourth main result, shows that they are not overwhelmingly common, *i.e.*, the proportion of polyominoes that are t-TCPs (out of all polyominoes) is bounded away from 1.

Theorem 12 $\limsup_{n\to\infty} \frac{\kappa_t(n)}{A(n)} \le \frac{\lambda}{\lambda+4(t+1)}$ *for all $t > 0$.*

Proof: We map every t-TCP of size $n-1$ to all elements of A_n obtained by attaching one cell immediately to the right (resp., left) of any cell in its rightmost (resp., leftmost) column, or above (resp., below) any cell in its topmost (resp., bottommost) row. This way, every polyomino in $\kappa_{t,n-1}$ is mapped to at least $4(t+1)$ polyominoes in A_n, all images are distinct, and all images are not t-TCPs. Hence, we have that $A(n) \geq \kappa_t(n) + 4(t+1)\kappa_t(n-1)$, that is, $A(n)/\kappa_t(n) \geq 1 + 4(t+1)\kappa_t(n-1)/\kappa_t(n)$. Therefore, after we prove the Ratio Limit Theorem for t-TCPs (Theorem 13 in Section 6), it will follow that

$$\limsup \frac{\kappa_t(n)}{A(n)} \leq \lim_{n\to\infty} \frac{1}{1 + \frac{4(t+1)\kappa_t(n-1)}{\kappa_t(n)}} = \frac{1}{1 + \frac{4(t+1)}{\lambda}} = \frac{\lambda}{\lambda + 4(t+1)}.$$

□

Note that this does *not* imply that $\kappa_t(n)/A(n)$ (t fixed) converges to a positive value (as a function of n), or converges at all, when $n \to \infty$. For example, if $A(n) \sim c_0 n^{\theta_0} \lambda^n$ (which is widely believed) and $\kappa_t(n) \sim c_t n^{\theta_t} \lambda^n$, where $\theta_t \leq \theta_0$, then $\lim_{n\to\infty} \kappa_t(n)/A(n)$ would be 0 if $\theta_t < \theta_0$ but nonzero if $\theta_t = \theta_0$. However, since $\lim_{t\to\infty} \lambda/(\lambda + 4(t+1)) = 0$, we conclude that the limiting fraction of t-TCPs out of all polyominoes vanishes as t tends to ∞.

6 A Ratio-Limit Theorem

The existence of ratio limits is sought after in essentially all lattice models, since they are the "true" growth constants. Results similar to the following have been obtained for self-avoiding walks, bridges, and polygons [28, §7], as well as for many polyomino variants [27]. We will now prove our fifth main result, namely, the existence of the ratio limit for the sequence enumerating t-TCPs, for any fixed $t \geq 1$.

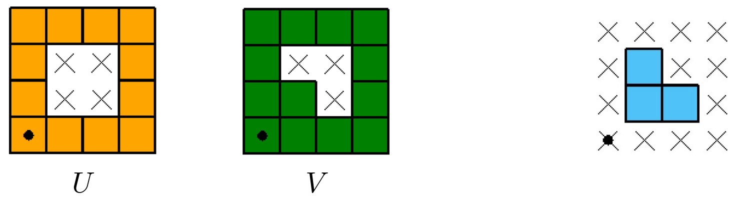
Theorem 13 *(Ratio Limit Theorem for t-TCPs)* $\lim_{n\to\infty} \frac{\kappa_t(n+1)}{\kappa_t(n)}$ *exists and equals the constant* λ_{κ_t}. □

We define *proper patterns* as in reference [27].

Definition 3 *A* pattern $P = (\mathfrak{M}, \mathfrak{F})$ *consists of one region of mandatory cells* $\mathfrak{M}$ *and one region of forbidden cells* $\mathfrak{F}$*. For a polyomino* Q *and a vector* $\vec{x} \in \mathbb{Z}^2$*, we say that* Q *contains the translate* $P + \vec{x}$ *if* $c + \vec{x} \in Q \ \forall c \in \mathfrak{M}$ *and* $c + \vec{x} \notin Q \ \forall c \in \mathfrak{F}$*. The pattern* P *is* proper *if for every* $n > 0$*, there exists a polyomino of area at least* n *that contains some translation of* P*.*

Figure 7 illustrates this definition. Figure 7(a) shows two proper patterns, while Figure 7(b) shows an improper pattern.

Let $A_n[k, P]$ denote the set of area-n polyominoes containing fewer than k translates of the pattern P, and let $\kappa_{t,n}[k, P]$ denote the same for t-TCPs. The following theorem says, roughly, that every proper pattern appears frequently in all but exponentially few polyominoes.



(a) A pair U-V of proper patterns (b) An improper pattern

Figure 7: Some patterns. Mandatory sites, $\mathfrak{M}$, are marked with coloured squares. Forbidden sites, $\mathfrak{F}$, are crossed out. The relative location of the origin is marked with a dot. (a) A pair U-V of proper patterns used for proving the Ratio Limit Theorem for t-TCPs (Thm. 13); (b) An improper pattern that cannot be contained in any polyomino of area greater than 3.

Theorem 14 *[27, Thm. 2.1] (Pattern Theorem for Polyominoes) For any proper pattern P, there exists an $\varepsilon > 0$ such that*

$$\limsup_{n\to\infty} \sqrt[n]{|A_n[\varepsilon n, P]|} < \lambda.$$

□

The next theorem is a consequence of the equality of the growth constants λ_{κ_t} and the Pattern Theorem for Polyominoes. In Appendix A, we follow closely the method of reference [27] for proving this theorem for the case $t = 1$. Here, we take advantage of the relationship between t-TCPs and regular polyominoes and prove the pattern theorem for any value of t.

Theorem 15 *(Pattern Theorem for t-TCPs) For any proper pattern P, there exists an $\varepsilon >$ 0 such that*

$$\limsup_{n\to\infty} \sqrt[n]{|\kappa_{t,n}[\varepsilon n, P]|} < \lambda_{\kappa_t}.$$

Proof: Since every t-TCP is also a polyomino, we deduce that

$$\limsup_{n\to\infty} \sqrt[n]{|\kappa_{t,n}[\varepsilon n, P]|} \leq \limsup_{n\to\infty} \sqrt[n]{|A_n[\varepsilon n, P]|} < \lambda = \lambda_{\kappa_t}.$$

□

We can now prove the Ratio Limit Theorem for t-TC polyominoes. First, we state Theorem 2.2 of reference [27] in the special case of t-TCPs. We also assume that the patterns $U = (\mathfrak{M}_U, \mathfrak{F}_U)$ and $V = (\mathfrak{M}_V, \mathfrak{F}_V)$ are such that $\mathfrak{M}_U \cup \mathfrak{F}_U = \mathfrak{M}_V \cup \mathfrak{F}_V$. This holds for the pair of patterns in Figure 7(a). It says, roughly, that with what we already have proved about t-TCPs, we can obtain a ratio limit theorem if we can find a pair of proper patterns U and V such that changing any occurrence of U (resp., V) contained in a t-TCP to a V (resp., U) preserves the t-TCP property and increases (resp., decreases) the area of the polyomino by 1. The proof relies on bijective bounds on the number of animals (specifically, t-TCPs) containing fixed numbers of translates of U and V.

Theorem 16 *[27, Thm. 2.2] Suppose that* $\kappa_{t,n} \cap \kappa_{t,m} = \emptyset \; \forall n \neq m$, λ_{κ_t} *exists, the Pattern Theorem for t-TCPs holds, and* $\exists \Upsilon > 0$ *such that* $\kappa_t(n+1) \geq \Upsilon \kappa_t(n)$ *for n sufficiently large.*[2] *Suppose also that there exist proper patterns U and V such that, if* ζ *denotes either U or V and* $\hat{\zeta}$ *denotes the other, for all* $Q \in \kappa_{t,n}$ *and for all* $\vec{x}$ *such that Q contains* $\zeta + \vec{x}$, *we have that*

(i) $[Q \setminus (\mathfrak{M}_\zeta + \vec{x})] \cup (\mathfrak{M}_{\hat{\zeta}} + \vec{x})$ *is a t-TCP;*

(ii) $\hat{Q}$ *has exactly one more occurrence of* $\hat{\zeta}$ *than Q, and exactly one fewer occurrence of* ζ *than Q; and*

(iii) If $\zeta = U$, *then* $\hat{Q} \in \kappa_{t,n+1}$.

Then, $\lim_{n\to\infty} \frac{\kappa_t(n+1)}{\kappa_t(n)}$ *exists and equals* λ_{κ_t}. □

In Theorem 16, the condition $\kappa_{t,n} \cap \kappa_{t,m} = \emptyset \; \forall n \neq m$ holds by definition. The growth constant λ_{κ_t} exists by Theorem 9. The conditions on U and V essentially say that changing any U to a V in a t-TCP produces another t-TCP with one more V, one fewer U, and area increased by 1 (and similarly for changing a V to a U).

Theorem 17 *For each t, there exists a constant* $\Upsilon > 0$, *such that* $\kappa_t(n+1) \geq \Upsilon \kappa_t(n)$ *for sufficiently-large n.*

Proof: Let patterns U, V be defined as in Figure 7. For this proof, we will call a TCP "nice" if it contains a translate of U that lexicographically precedes every translate of V. We claim that a positive fraction of t-TCPs are nice.

By the t-TCP Pattern Theorem 15, there exists an $\varepsilon > 0$ such that at least half of all area-n t-TCPs have at least εn translates of U and also at least εn translates of Vs, for sufficiently large n. Let $\kappa_{t,n}[\alpha, \beta]$ denote the set of t-TCPs with exactly α translates of U and β translates of V. Then, noting that no t-TCP of area n can contain more than n translates of either U or V, we have that

$$\sum_{i=\lceil \varepsilon n \rceil}^{n} \sum_{j=\lceil \varepsilon n \rceil}^{n} |\kappa_{t,n}[i,j]| \geq \frac{1}{2} \kappa_t(n).$$

Since the result of interchanging U and V patterns in a t-TCP is also a t-TCP, any of the $\binom{i+j}{i}$ arrangements of the i-many Us and the j-many Vs in a t-TCP in $\kappa_{t,n}[i,j]$ is also in $\kappa_{t,n}[i,j]$. Among all these permutations, consider those that yield a nice t-TCP, which number $\binom{i+j-1}{i-1}$. Then, recalling the identity

$$\binom{p-1}{q-1} = \frac{q}{p} \binom{p}{q},$$

[2] It may look like Theorem 2.2 in reference [27] actually requires that $\kappa_t(n+1) \geq \kappa_t(n)$, but this is not the case. Comparison with the preprint posted at arxiv:math/9902161 shows that the symbol Υ was omitted in three places from Proposition 3.5 in the journal article, presumably because of a typesetting difficulty.

we find that the restriction of our set to nice t-TCPs results in a set with cardinality

$$\sum_{i=\lceil \varepsilon n \rceil}^{n} \sum_{j=\lceil \varepsilon n \rceil}^{n} \frac{i}{i+j} |\kappa_{t,n}[i,j]| \geq \frac{\varepsilon}{2} \sum_{i=\lceil \varepsilon n \rceil}^{n} \sum_{j=\lceil \varepsilon n \rceil}^{n} |\kappa_{t,n}[i,j]| \geq \frac{\varepsilon}{4} \kappa_t(n).$$

Therefore, the nice t-TCPs number at least $\frac{\varepsilon}{4}\kappa_t(n)$ for large-enough n, so the claim follows. From the subset of nice t-TCPs, we can construct $\frac{\varepsilon}{4}\kappa_t(n)$ uniquely determined (that is, different) area-$(n+1)$ t-TCPs by changing the first occurrence of U to V, increasing the area of each t-TCP by one. We have thus shown that $\kappa_t(n+1) \geq \frac{\varepsilon}{4}\kappa_t(n)$ for sufficiently-large n. □

With Theorems 9, 15, and 17, as well as the patterns shown in Figure 7, we see all the hypotheses of Theorem 16 are satisfied, proving Theorem 13, the Ratio Limit Theorem for t-TCPs. If we are being precise, there are some conditions we have ignored because the results of reference [27] actually apply to weighted sums. Our adaptation is trivial if we assign every t-TCP a weight of one.

Remark 2 *The method used for proving Theorem 17 can be used in a more general setting. That is, we can prove the existence of the necessary Υ from the Pattern Theorem and the existence of the hypothesized U-V pair (N. Madras and J. Peters, in preparation). Therefore, this proof eliminates the hypothesis that such an Υ exists from the statement of Theorem 2.2 in reference [27].*

7 A Lower Bound on Infinitely-Many Values of $A(n)$ and $\kappa_t(n)$

It was recently shown [14] that there exist constants $c, \theta > 0$, such that $A(n) \geq c\lambda^n n^{-\theta \log n}$ for all $n \in \mathbb{N}$. Moreover, he showed that under the unproven (but widely believed) assumption that the ratio sequence $A(n+1)/A(n)$ is monotone increasing, we obtain the better lower bound $c\lambda^n n^{-\theta}$ for all $n \in \mathbb{N}$. This is actually believed, as stated above, to be the asymptotic behavior of $A(n)$.

In this section, we make one more step in this direction by showing that the number of polyominoes (0-TCPs), as well as the number of t-TCPs (for a fixed value of $t > 0$) is $\Omega(\lambda^n n^{-\theta_t})$, where θ_t depends only on t, for infinitely-many values of n.

In what follows, we use the symbol z to denote a nonnegative real number. Define

$$\chi(z) := \sum_{n=1}^{\infty} n^2 A(n) z^n.$$

Let $z_c = 1/\lambda$, the inverse of the growth constant of polyominoes (and of TCPs). This is the radius of convergence of the power series χ, so $\chi(z)$ diverges for $z > z_c$ and converges for $z < z_c$. The following theorem is essentially known in the statistical physics literature.

Theorem 18 *The power series $\chi(z)$ diverges at $z = z_c$.* □

The essence of the proof is due to Bovier, Fröhlich, and Glaus [12] for lattice trees, and also holds for bond animals [19, 33] with minor adaptations. The proof for polyominoes is given in Appendix B.

The theorem above implies our sixth main result.

Corollary 19 *Let ε be any positive number. Then,*

(a) $A(n) \geq \lambda^n n^{-3-\varepsilon}$ for infinitely-many values of n; and

(b) For any natural number t, $\kappa_t(m) \geq \lambda^m m^{-9(t+1)^2-\varepsilon}$ for infinitely-many values of m.

Proof:

(a) To obtain a contradiction, assume that $A(n) < \lambda^n n^{-3-\varepsilon}$ for all but finitely-many values of n. Then, $n^2 A(n) z_c^n \leq 1/n^{1+\varepsilon}$ for all but finitely-many values of n. This implies that $\chi(z_c)$ converges, which contradicts Theorem 18.

(b) Let $\delta = \varepsilon/(2(t+1)^2)$. Let I be the set of positive integers n for which $A(n) \geq \lambda^n n^{-3-\delta}$ holds. By part (a), the set I is infinite. From Equation (10) in the proof of Theorem 10, we know that for every $n \geq 1$, we have that $\kappa_t(\phi_n) \geq \left(\frac{A(n)}{n^6}\right)^{(t+1)^2}$, where $\phi_n = n(t+1)^2 + 14t(t+1)$.

Therefore, for $n \in I$, we have that

$$\kappa_t(\phi_n) \geq \frac{\lambda^{n(t+1)^2}}{n^{9(t+1)^2+\delta(t+1)^2}} = \frac{\lambda^{\phi_n - 14t(t+1)}}{n^{9(t+1)^2+\varepsilon-\varepsilon/2}} \geq \frac{\lambda^{\phi_n}}{\phi_n^{9(t+1)^2+\varepsilon}} \frac{n^{\varepsilon/2}}{\lambda^{14t(t+1)}}.$$

Hence, the relation asserted in part (b) holds whenever $m = \phi_n$ for $n \in I$, such that $n \geq \lambda^{28t(t+1)/\varepsilon}$.

□

Remark. References [12, 19, 33] actually prove the stronger result that there is a constant c such that $\chi(z) \geq c(z_c - z)^{-1/2}$ for all $z \in (z_c/2, z_c)$. As in Corollary 19(a), this inequality implies that $A(n) \geq \lambda^n n^{-5/2-\varepsilon}$ for infinitely-many values of n, which in turn allows one to replace $9(t+1)^2$ by $8.5(t+1)^2$ in the exponent in Corollary 19(b).

To close this section, we show that if a lower bound of the form $c\lambda^n n^{-\theta}$ holds for $A(n)$, as is generally believed, then a similar bound holds for t-TCPs.

Proposition 20 *Suppose that the relation $A(n) \geq c_0 n^{-\theta_0} \lambda^n$ holds for every n for some constants $c_0 > 0$ and θ_0. Let $t \geq 1$. Then, there exists a constant $c_t > 0$ such that $\kappa_t(m) \geq c_t \lambda^m m^{-(6+\theta_0)(t+1)^2}$ for all sufficiently-large m, and hence that $\kappa_t(m)/A(m) = \Omega\left(m^{-(6+\theta_0)(t+1)^2}\right)$.*

Proof: The argument proving part (b) of Corollary 19 shows that

$$\kappa_t(\phi_n) \geq \frac{\lambda^{\phi_n}}{\phi_n^{(6+\theta_0)(t+1)^2}} \left(\frac{c_0}{\lambda^{14}}\right)^{(t+1)^2}$$

for all $n \geq 1$ and $\phi_n = n(t+1)^2 + 14t(t+1)$. When n is sufficiently large, Theorem 17 implies that $\kappa_t(\phi_n + j) \geq \Upsilon^j \kappa_t(\phi_n)$ for every integer $j \geq 0$. Since every integer m can be written as $m = \phi_n + j$ for some n and some $j \in [0, (t+1)^2)$, we see that the relation claimed in the proposition holds with $c_t = (\min\{\Upsilon, 1\} c_0/\lambda^{14})^{(t+1)^2}$. The claim for $\kappa_t(m)/A(m)$ follows because $\lambda^m \geq A(m)$ [23]. □

8 Conclusion

In this paper, we make a natural generalization of 1-TCPs (previosuly just called "TCPs") to t-TCPs, generalized previous results on 1-TCPs to t-TCPs, and proved new results on both 1-TCPs and t-TCPs.

In Section 3, we counted 1-TCPs with up to 35 cells by extending Jensen's algorithm. Unfortunately, the running-time requirements of this approach prohibited counting t-TCPs for $t > 1$. Moreover, it was not clear whether our approach is faster than the simpler alternative based on Redelmeier's algorithm mentioned in Section 3.4.

In Section 4, we characterized t-TCPs of minimal area. Our first main result is that the minimum area of t-TCPs is 21, 50, or $6(t+1)^2 - 1$ when t is 1, 2, or greater than 2, respectively. Our second main result is that minimum-area t-TCPs are tree-like, have exactly t gaps in each row and column, and have a bounding box with dimensions $(3t+3, 3t+2)$ or $(3t+2, 3t+3)$ when $t \leq 2$, and $(3t+3, 3t+3)$ or $(3t+3 \pm 1, 3t+3 \mp 1)$ when $t > 2$. We have not found any minimal t-TCPs that have a non-square bounding box for any $t > 3$.

In Section 5, we investigated the abundance of t-TCPs among the polyominoes as a whole, *i.e.*, the asymptotics of the ratio $\kappa_t(n)/A(n)$. Our third main result is that both $\kappa_t(n)$ and $A(n)$ grow to the same exponential base, hence $\kappa_t(n)/A(n)$ does not decay exponentially, and our fourth is that $\kappa_t(n)/A(n)$ is bounded away from one. We conjectured that $\kappa_t(n)/A(n)$ converges to a positive constant, and gave some theoretical justification for this claim in Section 2. One might obtain empirical evidence for this conjecture by taking Monte Carlo samples of polyominoes, along the lines of reference [34]. Later, In Section 7, we proved a stronger asymptotic lower bound $\kappa_t(n)/A(n) = \Omega(n^{-\Delta_t})$ (for some constant $\Delta_t \geq 0$) under the unproven but widely believed assumption that $A(n) = \Omega(n^{-\theta_0}\lambda^n)$ for some $\theta_0 \geq 0$. Without this assumption, we could only prove that $\kappa_t(n)/A(n) \geq n^{-\Delta_t}$ for infinitely-many values of n.

In Section 6, we proved the ratio limit and pattern theorems for t-TCPs. Figure 8 plots the values of $A(n)/A(n-1)$ and $\kappa_1(n)/\kappa_1(n-1)$ for $22 \leq n \leq 35$. It may seem like $\kappa_1(n)/\kappa_1(n-1)$ is decreasing, but this cannot be the case. Indeed, if it were decreasing, then for some n_0 we would have $\kappa_t(n)/\kappa_t(n-1) \geq \lambda$ for all $n \geq n_0$ by Theorems 13 and 10. Simple algebra then shows that the sequence $\kappa_t(n)/\lambda^n$ would be increasing for

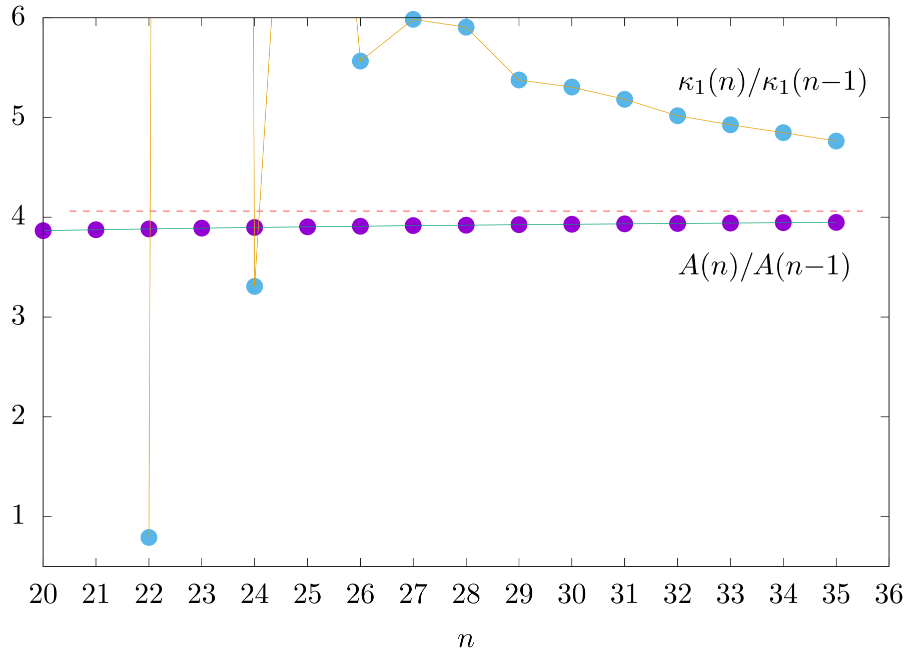


Figure 8: Ratio sequences for all polyominoes and for 1-TCPs. The red dashed line marks the estimated value $\lambda \approx 4.06$.

$n \geq n_0 - 1$. However, Equation (11) implies that this sequence converges to zero, which is a contradiction. A (maybe non-rigorous) reason for this behavior would be interesting, especially since the analogous sequences for other types of lattice animals seems to increase, see the appendix of reference [18] for data.

We also notice that the definitions of *minimality* (not to be confused with *minimum area*), *primitivity*, and *saturation* given in reference [4] alongside total concavity have clear t-fold generalizations, and questions 3–6 therein can be posed in the t-fold case too. Finally, we give three potential definitions for d-dimensional ($d > 2$) t-TC polycubes. Let P be a d-dimensional polycube.

1. The intersection of P with any line parallel to a coordinate axis is empty or consists of at least t connected components.

2. The intersection of P with any $(d-1)$-dimensional hyperplane perpendicular to a coordinate axis is either empty or a $(d-1)$-dimensional t-TCP.

3. The intersection of P with any $(d-1)$-dimensional hyperplane perpendicular to a coordinate axis is either empty or consists of at least t connected components.

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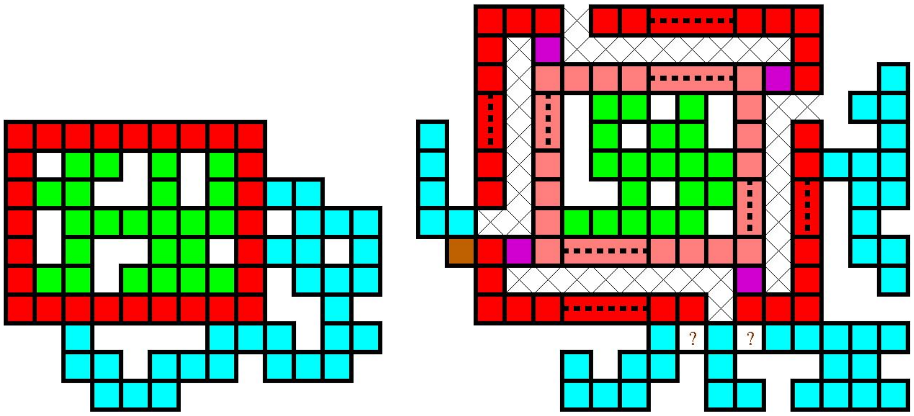

(a) Regular polyominoes (b) TC polyominoes

Figure 9: Illustration of Cluster Axiom 4 of Madras's Ratio Limit Theorem.

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Appendix A: Existence of Ratio Limit for 1-TCPs

One can alternatively prove Theorem 15 directly by Theorem 2.1 of Madras [27]. The following discussion shows that cluster axiom 4 (CA4) holds, which is the only hypothesis of Theorem 2.1 that we have not yet proved.

CA4 requires that any part of any cluster can be changed locally so as to create an occurrence of a translated copy of any proper pattern. Such a change is guaranteed to exist within the set of all polyominoes, and, for our purpose, we need to show that it maintains the property (hereafter called the "gap property") that if all rows and columns of the polyomino contain at least two sequences of occupied cells before the change, then they also do so after the change. To establish this, we need to make the details of the change much more delicate.

For general polyominoes, Madras enclosed the new pattern by a frame whose width was one cell (see Figure 9(a)). The purpose of the frame was to ensure a connection between the new pattern (inside the frame, colored in green) and the rest of the polyomino (outside the frame, colored in cyan). The size of the frame was chosen so as to contain both parts of the pattern. (Actually, Madras used a square frame whose size was the maximum of the needed height and length, but this was not essential.)

Hence, if the new pattern is feasible (that is, it does not describe a collection of cells which is impossible in a valid polyomino, *e.g.*, an isolated cell surrounded by a ring of cells but not touching it), the new polyomino will be connected and hence valid.

However, as noted above, we also need to ensure that the gap property is still maintained in the new polyomino. We do this by using two nested frames, as is shown in Figure 9. Let us explain the logic behind this construction.

- The frame construction, including occupied and empty cells, possibly overrides existing parts of the polyomino that contains the pattern.
- The interface between the new pattern (green) and the rest of the polyomino (cyan) is made of two nested frames: An inner one (colored in pink) and an outer one (colored in red). All the crossed-out cells are empty. With the modifications described below, the two frames will ensure that every line crossing a row or a column of the pattern, or of the frame construction, will have the gap property. We do not need to worry about rows and columns outside the frame since the original polyomino is already TC. (See also below the description of brown cells.)
- The inner size of the combined frame is determined by the size of the pattern, the width of the double frame is 3, and up to four additional cells are made occupied in the layer around the outer frame, as specified below.
- Every horizontal or vertical line crossing the pattern satisfies the gap property as it meets at least once the outer frame and at least once the combination of the pattern and the inner frame.
- The same applies for any horizontal or vertical line going through the sides of the inner frame or the gap between the two frames.
- For maintaining the connectivity between the two frames, we add the four cells colored in magenta in between the two frames. These cells do not interfere anywhere with the gap property.
- In order to maintain the gap property along the sides of the outer frame, we break it by omitting one cell from each side of it, as shown in the figure.
- Some parts of the polyomino that are outside of the frame (colored in cyan), *e.g.*, the part to the right of the frame, touch the frame and are therefore connected to the rest of the polyomino.
- Some other parts, *e.g.*, the part to the left of the frame, are aligned precisely and only towards the single empty cell in one side of the frame. In such a case, one of the empty neighboring cells (colored in brown) is made occupied, thereby connecting that part to the polyomino.

- Potentially, adding such a cell, *e.g.*, for the middle part below the frame, will close a gap. In such a case, we locate the brown cell in the other empty neighboring cell described in the previous item. As shown is the figure, that other cell may also close a gap (see the cells marked with '?'). However, in fact, there is no problem at all in this case since there are already two gaps along the respective row or column, hence we can safely locate the brown cell in either empty cell without violating the gap property.
- The inner frame (colored in pink) is continuous, thus, it will always touch the new pattern. As explained above, the outer frame touches the rest of the polyomino, hence, we remain overall with a valid 1-TCP.

The proof above can be extended to any fixed value of t by using a suitable construction of frames.

Appendix B: Proof of Theorem 18

Let $|P|$ denote the area of a polyomino P. Note that in this appendix, we shall frequently consider polyominoes that have cells outside the first quadrant.

We begin with a general connectivity lemma.

Lemma 21 *Let P and Q be two polyominoes, such that $Q \subset P$. Let C be a connected component of $P \setminus Q$. Then $P \setminus C$ is connected.*

Proof: Assume first that C does not touch Q, that is, no cell $c \in C$ has a common edge with any cell $q \in Q$. Any other component of $P \setminus Q$ (if exists) must touch Q in order to form one component within P. However, whether or not $P \setminus Q$ has other components than C, we conclude that C is disconnected from Q, hence this case is not possible.

Therefore, C must touch Q. In addition, any other connected component of $P \setminus Q$ (if exists) must also touch Q, otherwise P would not be a valid (connected) polyomino. Whether or not $P \setminus Q$ has other components than C, we conclude that $P \setminus C$ is connected. □

THEOREM 18. *The power series $\chi(z)$ diverges at $z = z_c$.*

The proof closely follows the presentation of Slade [32, §7], where the claim is proved for lattice trees.

Proof:

For a given (large) positive integer R, let Λ_R be the set of cells of the square lattice inside the large square $[-R, R]^2$, and Λ_∞ be the set of all cells. We shall also use the letters u, v, x, y to label cells of Λ_R, and denote the cell at the origin by O.

For $0 < R \leq \infty$ and cells x and y (not necessarily distinct) in Λ_R, define

$$G_R(x,y|z) = \sum_{P \subseteq \Lambda_R : x,y \in P} z^{|P|},$$

i.e., the sum of $z^{|P|}$ over all polyominoes P contained in Λ_R that contain the cells x and y. Observe that

$$\lim_{R\to\infty} G_R(x,y|z) = \sup_{R<\infty} G_R(x,y|z) = G_\infty(x,y|z). \tag{12}$$

For any fixed cell x, we have

$$\sum_{y\in\Lambda_\infty} G_\infty(x,y|z) = \sum_y \sum_{P\subseteq\Lambda_\infty : x,y\in P} z^{|P|} = \sum_{n=1}^{\infty} \sum_{P\in A_n} n^2 z^n = \chi(z) \tag{13}$$

(since each polyomino of area n permits n choices for x and n choices for y).

For a finite R and $x, y, v \in \Lambda_R$, we also define

$$G_R^{(3)}(x,y,v|z) = \sum_{P\subseteq\Lambda_R : x,y,v\in P} z^{|P|}\,, \tag{14}$$

$$\chi_R(x|z) = \sum_{y\in\Lambda_R} G_R(x,y|z)\,, \quad \text{and} \tag{15}$$

$$\overline{\chi}_R(z) = \max_{x\in\Lambda_R} \chi_R(x|z)\,. \tag{16}$$

Observe that

$$\lim_{R\to\infty} \chi_R(O|z) = \chi(z) = \lim_{R\to\infty} \overline{\chi}_R(z) \tag{17}$$

(the first equation follows from Equations (12) and (13), while the second follows from $\chi_R(O|z) \leq \overline{\chi}_R(z) \leq \chi(z)$).

A key step is the following inequality:

$$G_R^{(3)}(x,y,v|z) \leq z^{-2} \sum_{u\in\Lambda_R} G_R(x,u|z)\, G_R(y,u|z)\, G_R(v,u|z)\,. \tag{18}$$

To prove this inequality, consider a polyomino P in the sum defining the left hand side, *i.e.*, $x, y, v \in P \subseteq \Lambda_R$. Then, there is a path π of cells in P from x to y, and another path π_v from v to a cell $u \in \pi$, such that $\pi \cap \pi_v = \{u\}$. Let π_x (resp., π_y) be the part of π from x (resp., y) to u. Deleting $\pi_v \setminus \{u\}$ (that is, all the cells of π_v except u) from P leaves one or more connected components; let C^* be the component containing π. Let $D_v = (P \setminus C^*) \cup \{u\}$. Then, by Lemma 21, D_v is a polyomino in Λ_R containing π_v. Similarly, let D_x be the component of $C^* \setminus (\pi_y \setminus \{u\})$ that contains x, and let $D_y = (C^* \setminus D_x) \cup \{u\}$. One can check that D_x (resp., D_y) is a polyomino containing π_x (resp., π_y), and that the intersection of any two of D_x, D_y, and D_z is the single cell u. Since D_x is one of the polyominoes in the sum for $G_R(x,u|z)$ (and analogously for D_y and D_v), we obtain that

$$\begin{aligned} \sum_{P\subseteq\Lambda_R : x,y,v\in P} z^{|P|} &\leq \sum_u \sum_{\substack{D_x, D_y, D_v \subseteq \Lambda_R : \\ x,u\in D_x,\, y,u\in D_y,\, v,u\in D_v}} z^{|D_x|+|D_y|+|D_v|-2} \\ &= z^{-2} \sum_u G_R(x,u|z)\, G_R(y,u|z)\, G_R(v,u|z). \end{aligned}$$

This proves Equation (18).

Next, we claim that for a finite R and for each x in λ_R, we have that

$$\frac{d\,\chi_R(x|z)}{dz} \;\leq\; z^{-3}\,\overline{\chi}_R(z)^3. \tag{19}$$

To prove the claim, we calculate

$$\begin{aligned}
\frac{d\,\chi_R(x|z)}{dz} &= \sum_{y\in\Lambda_R}\;\sum_{P\subseteq\Lambda_R: x,y\in P} |P|\,z^{|P|-1} \\
&= \sum_{P\subseteq\Lambda_R: x\in P}\;\sum_{y\in P} |P|\,z^{|P|-1} \\
&= \sum_{P\subseteq\Lambda_R: x\in P}\;\sum_{y,v\in P} z^{|P|-1} \\
&= \sum_{y,v\in\Lambda_R}\;\sum_{P\subseteq\Lambda_R: x,y,v\in P} z^{|P|-1} \\
&= z^{-1}\sum_{y,v\in\Lambda_R} G^{(3)}(x,y,v|z) \\
&\leq z^{-3}\sum_{y,v,u\in\Lambda_R} G_R(x,u|z)\,G_R(y,u|z)\,G_R(v,u|z) \\
&\qquad\text{(by Equation (18))} \\
&= z^{-3}\sum_{y,u\in\Lambda_R} G_R(x,u|z)\,G_R(y,u|z)\,\chi_R(u|z) \\
&= z^{-3}\sum_{u\in\Lambda_R} G_R(x,u|z)\,\chi_R(u|z)^2 \\
&\leq z^{-3}\,\overline{\chi}_R(z)^3.
\end{aligned}$$

This proves the claimed Equation (19).

Since Λ_R is a finite set, the function $\overline{\chi}_R$ is the maximum of finitely many polynomials $\chi_R(x|\cdot)$. Therefore, $\overline{\chi}_R$ is continuous and there is a finite set Θ, such that for each $z \in (0,\infty)\setminus\Theta$, the derivative of $\overline{\chi}_R$ exists and equals the derivative of $\chi_R(x|\cdot)$ for some $x\in\Lambda_R$ (depending on z). Hence, Equation (19) tells us that

$$\frac{d\overline{\chi}_R(z)}{dz} \leq z^{-3}\,\overline{\chi}_R(z)^3 \quad \text{for all } z\in(0,\infty)\setminus\Theta.$$

Let $f_R(z) = 1/\overline{\chi}_R(z)^2$. Then, f_R is a positive continuous decreasing function on $(0,\infty)$ that satisfies

$$-\frac{df_R(z)}{dz} \leq \frac{2}{z^3} \leq \frac{16}{z_c^3} \quad \text{for all } z\in(z_c/2,\infty)\setminus\Theta.$$

Therefore, since Θ is a finite set, we can integrate the above inequality and conclude that

$$|f_R(z_1) - f_R(z_2)| \leq L|z_1 - z_2| \quad \text{for all } z > z_c/2, \text{ where } L = \tfrac{16}{z_c^3}. \tag{20}$$

Since $\lim_{R\to\infty} f_R(z) = 1/\chi(z)^2$ for all $z > 0$ (by Equation (17)), we see from Equation (20) that $1/\chi(z)^2$ is continuous on $(z_c/2, \infty)$ (with Lipschitz constant L, in fact). Since $\chi(z)$ diverges for all $z > z_c$, we see that $1/\chi(z)^2 = 0$ for all $z > z_c$; hence, we have that $1/\chi(z_c)^2 = 0$ by continuity. This proves that $\chi(z_c)$ diverges.

□