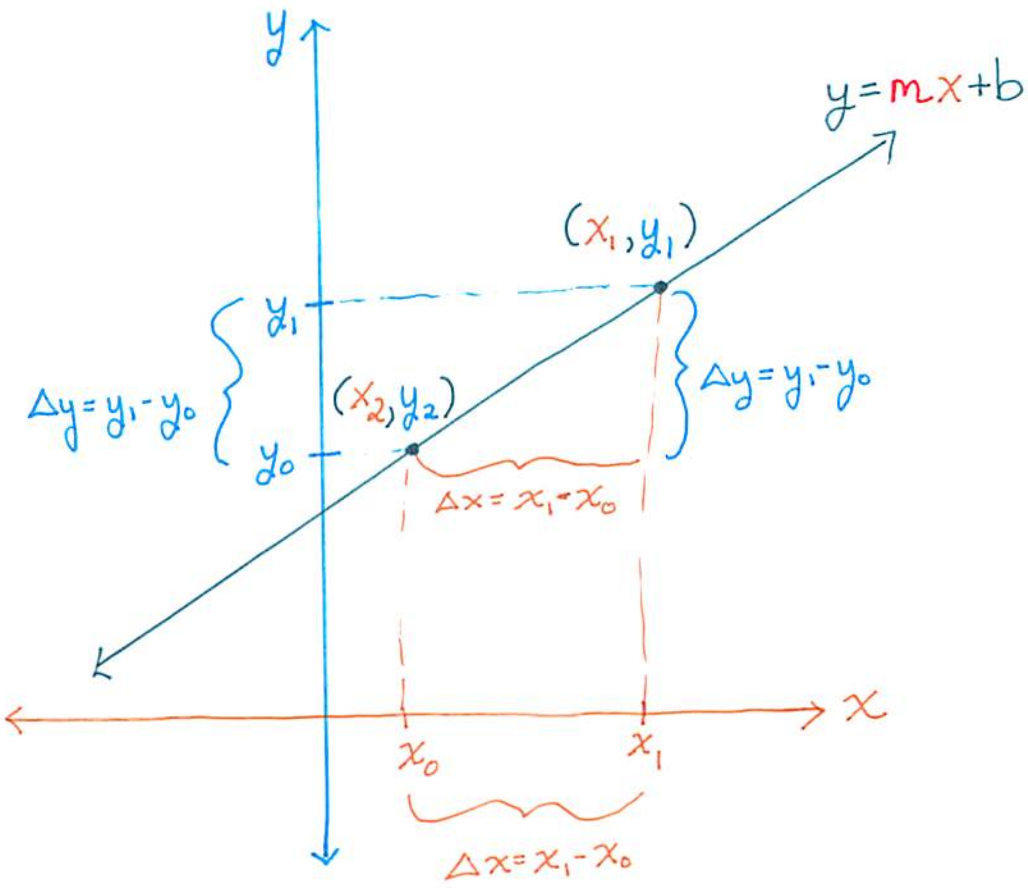
# Slope As Rate of Change for Lines

For lines, the slope is the rate of change of $y$ wrt (with respect to) $x$

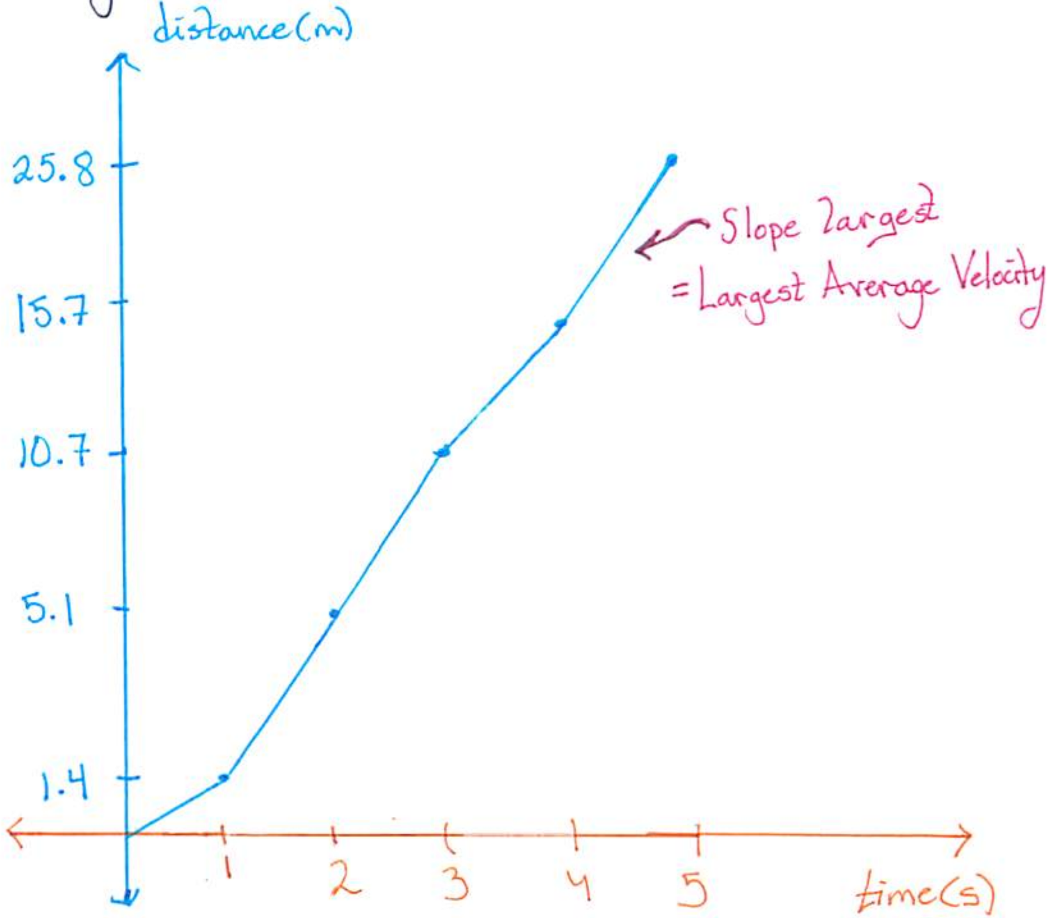



$$\text{Slope} = m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

# Example of Rate of Change from Limited Data

## Cyclist Riding on Straight Road

| time (s) | distance (m) |
|---|---|
| 0 | 0 |
| 1 | 1.4 |
| 2 | 5.1 |
| 3 | 10.7 |
| 4 | 15.7 |
| 5 | 25.8 |




Q: On which 1 second interval was she the fastest?

$$\text{Average Velocity} = \frac{\Delta \text{distance}}{\Delta \text{time}}$$

A: [4, 5]

# From Average Velocity to Instantaneous Velocity

**Set-Up:**

Cyclist riding down straight road

| time(s) | distance (m) |
|---|---|
| 0 | 0 |
| 1 | 1.4 |
| 2 | 5.1 |
| 3 | 10.7 |
| 4 | 15.7 |
| 5 | 25.8 |

"Zoom In"

**Task:** Approximate instantaneous velocity at time $t = 5$

**Method:** Zoom In

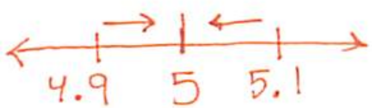
| time(s) | distance (m) |
|---|---|
| 4.8 | 23.63 |
| 4.9 | 24.85 |
| 5.0 | 25.78 |
| 5.1 | 26.04 |
| 5.2 | 26.77 |

5.0-4.9
$\Delta t = 0.1$

5.1-5.0
$\Delta t = 0.1$

$\frac{\Delta d = 25.78 - 24.85}{\Delta t = 0.1} = 9.3$

$\frac{\Delta d = 26.04 - 25.78}{\Delta t = 0.1} = 2.6$




Quite different, keep zooming in until as close as want

# Approximate Rate of Change ⇝ Rate of Change is Taking Limit

**Last Time:** Approximate instantaneous velocity at $t=5$

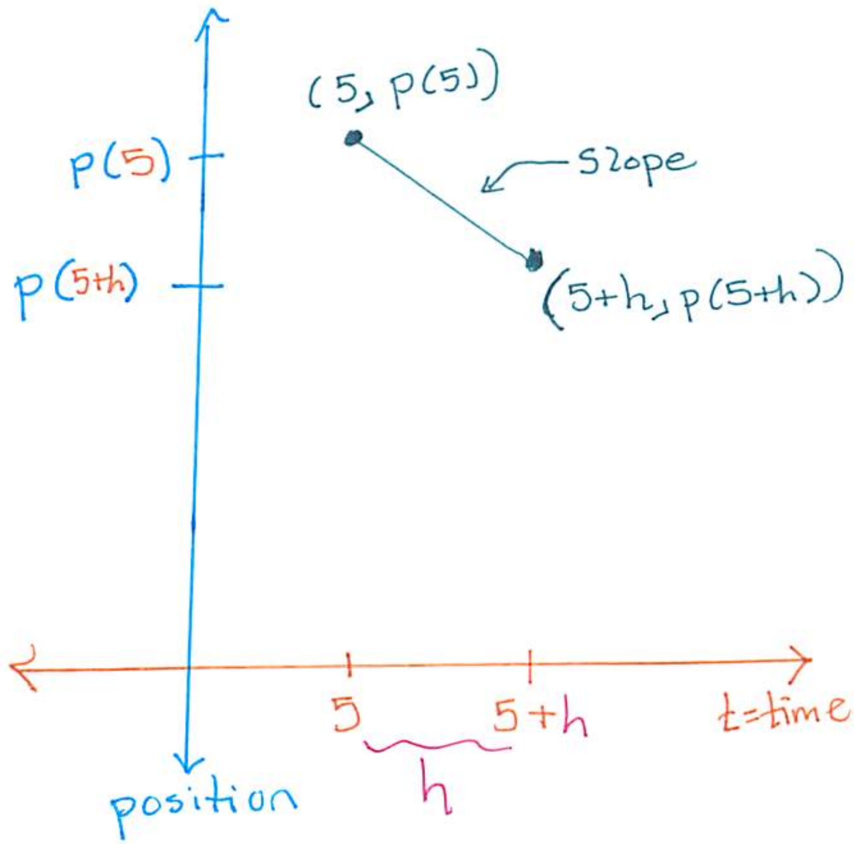
Process of Repeatedly Zooming In

$$\Big(\underbrace{[4.9,5]}_{h=0.1} \rightsquigarrow \underbrace{[4.99,5]}_{h=0.1} \rightsquigarrow \underbrace{[4.999,5]}_{h=0.1}\Big)$$

⇝ Taking a limit

$$\lim_{h\to 0} \frac{p(5+h)-p(5)}{(5+h)-(5)}$$

$$=\lim_{h\to 0} \frac{p(5+h)-p(5)}{h}$$

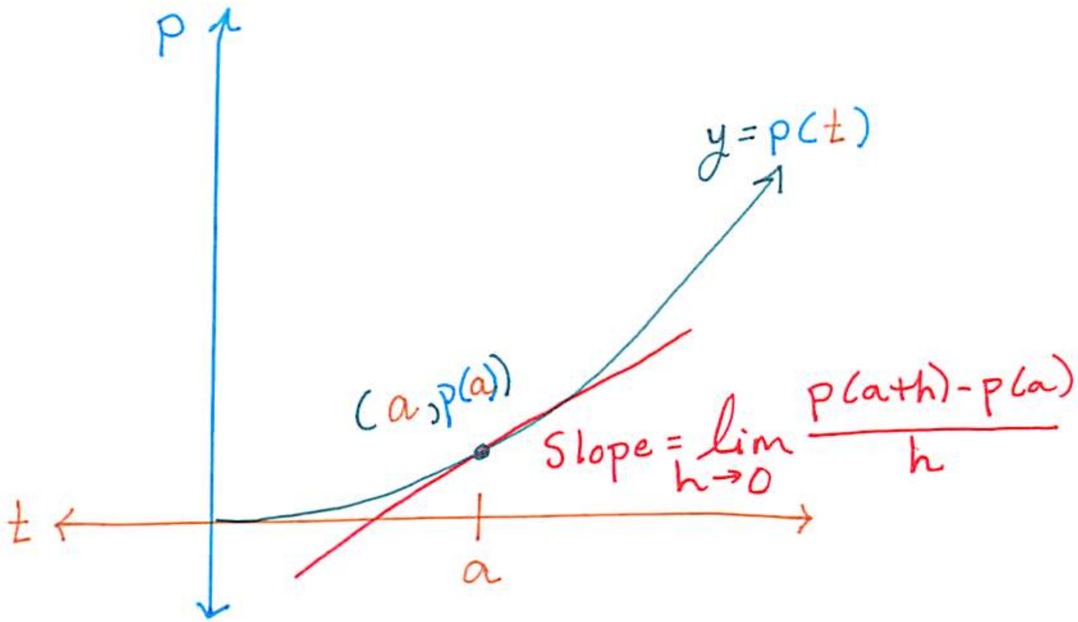


# Instantaneous Rate of Change ⇝ Slope of Tangent Line ⇝ Derivative

$$\text{Instantaneous Velocity (at time } t=5) = \lim_{h\to 0} \frac{p(\overset{a}{5}+h) - p(\overset{a}{5})}{h}$$

$t = a$

This is also the slope of the tangent line to the function $p(t)$ at the point $(a, p(a))$



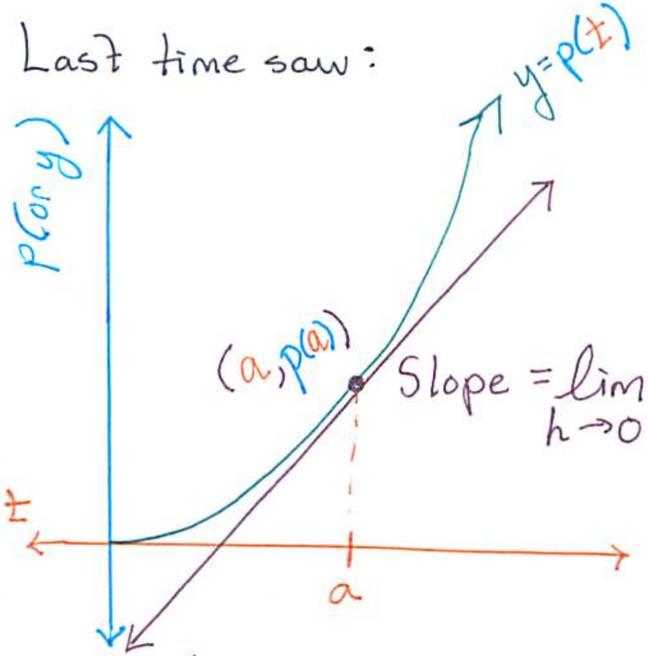

**Derivative (New Interpretation):**

The $\boxed{\text{derivative}}$ of the function $f$ at $a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

# Introduction to Tangent Lines (Useful for Approximation)

Last time saw:




$$\text{Slope} = \lim_{h \to 0} \frac{p(a+h) - p(a)}{h}$$

New! What was this line?

A: This line is the tangent line to p at t=a

Formally:

The line {with slope $p'(a)$ AND containing the point $(a, p(a))$}

SO can use the point-slope form to find the equation:

$$p'(a) = \frac{y - p(a)}{t - a} \begin{matrix} \} \text{ rise} \\ \} \text{ run} \end{matrix}$$

$$\underbrace{p'(a)(t-a)}_{}^{p'(a)t - p'(a)a} = y - p(a)$$

$$\boxed{y = p'(a)t + (p(a) - (ap'(a)))}$$

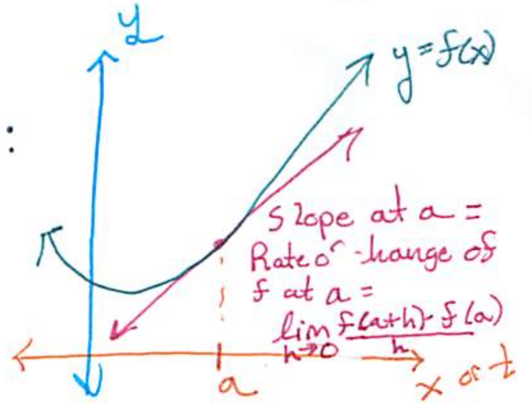
If the function is $f(x)$, the equation is:

$$\boxed{y = f'(a)t + (f(a) - (af'(a)))}$$

# The Derivative Function

We've seen: Derivative / Rate of Change / Slope "at a point" $x=a$ or $t=a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$




New: Can use, when $f'(a)$ makes sense for every input $a$, to define a $\boxed{\text{"derivative function"}}$ $f': \mathbb{R} \to \mathbb{R}$

Q: What is the value when we input $x=a$?

A: $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ (gives derivative/rate of change of $f$ at $x=a$)

Finding this derivative $f'$ is called $\boxed{\text{"differentiating"}}$

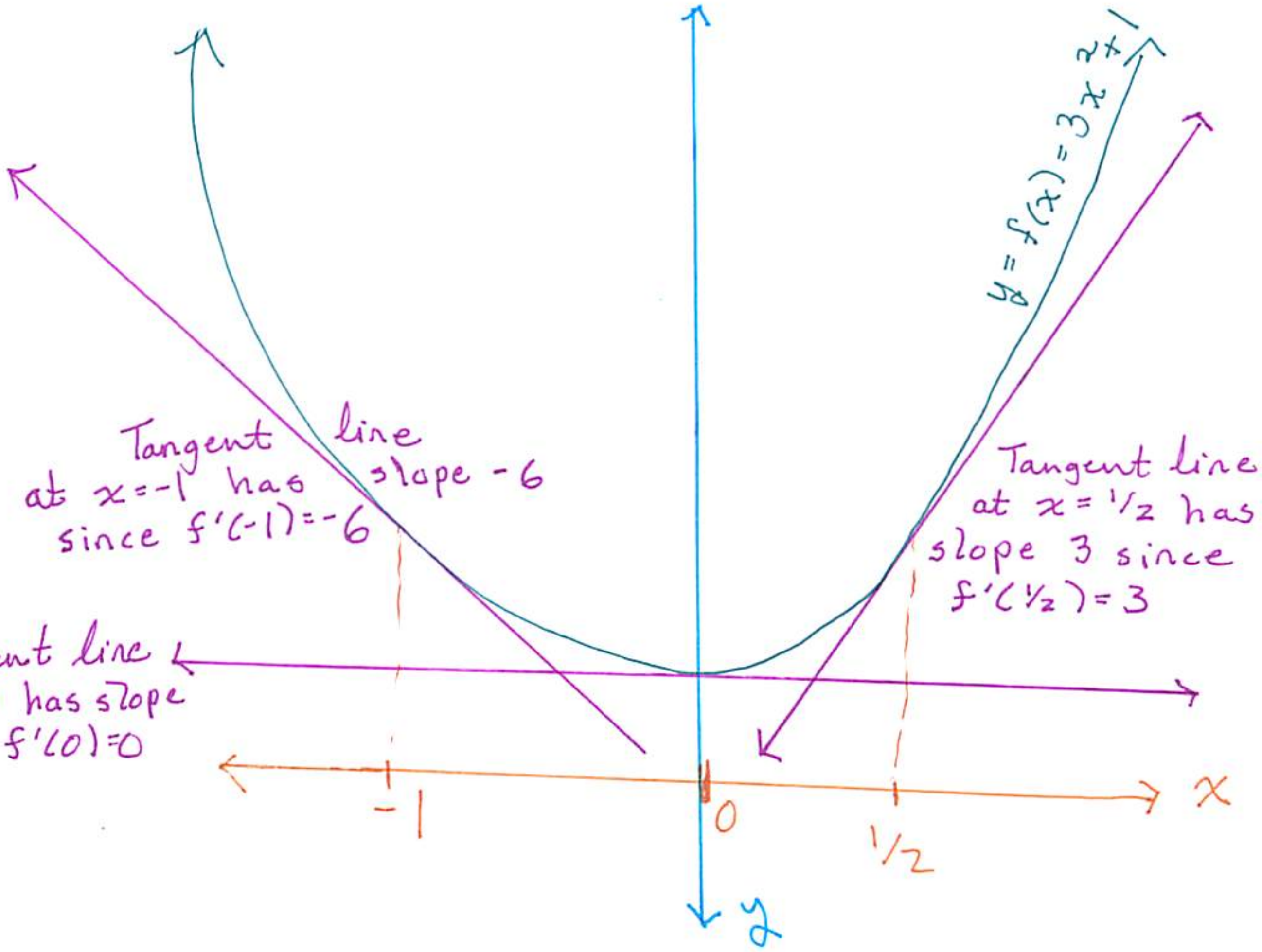
Formally: Differentiating is finding $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

# Visual Example for the Derivative Function

Example: $f(x) = 3x^2 + 1$

Will see $f'(x) = 6x$

$$\text{so} \begin{cases} f'(-1) = 6(-1) = -6 \\ f'(0) = 6(0) = 0 \\ f'(1/2) = 6(1/2) = 3 \end{cases}$$




Take Aways:

* The derivative function is a function, $f'$, so gives a (usually different) output for each input.
  These outputs give the slope/rate of change/derivative at the input points.
* For the same function get different tangent lines at different points!

# Derivative language: "with respect to"

Need to know when differentiating what differentiating $\boxed{\text{"with respect to"}}$ $\boxed{\text{(wrt)}}$
i.e. looking at

- how do the output values for $f$ change as $x$ changes

e.g. For $f(x) = 3x^2 + 1$ write

$\boxed{\frac{d}{dx}(f)}$ or $\boxed{\frac{df}{dx}}$ or $f'(x)$

so know rate of change wrt $x$

OR

- how the output values for $g$ change as $t$ changes

e.g. For $g(t) = e^t - \log(t)$ write

$\boxed{\frac{d}{dt}(g)}$ or $\boxed{\frac{dg}{dt}}$ for $g'(t)$

so gives the rate of change wrt $t$

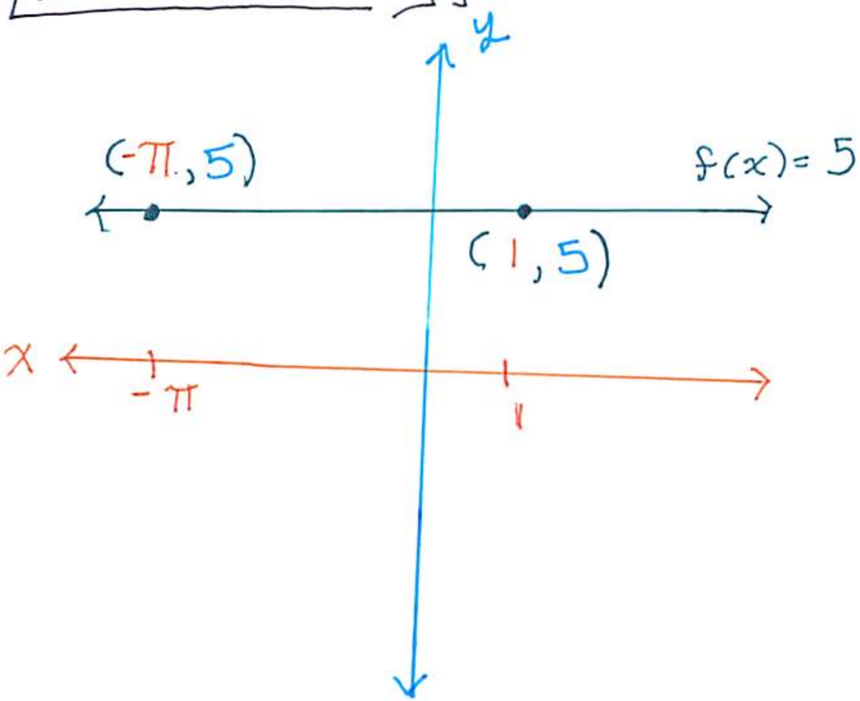
⚠ $\frac{df}{dx}$ and $\frac{dg}{dt}$ are still functions

We don't get values (for the slope or rate of change) until we input some $x = a$ or $t = a$

# Using the Derivative Definition to Compute the Derivative of a Constant

Computing the Derivative from the Definition Example: $f(x) = 5$, $\frac{df}{dx} = ?$

## Brain Orienting




Output of $f(x) = 5$ is always 5:

$f(-\pi) = 5$, $f(2^3) = 5$

## Computation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{5 - 5}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= \lim_{h \to 0} 0 = 0$$

$\boxed{\text{so } \frac{df}{dx} = 0}$

This is the zero function

# Computing the Derivative of $g(x) = x^2$ from the Definition

Computing the Derivative from the Definition Example: $g(x) = x^2$, $g'(x) = ?$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h}$$

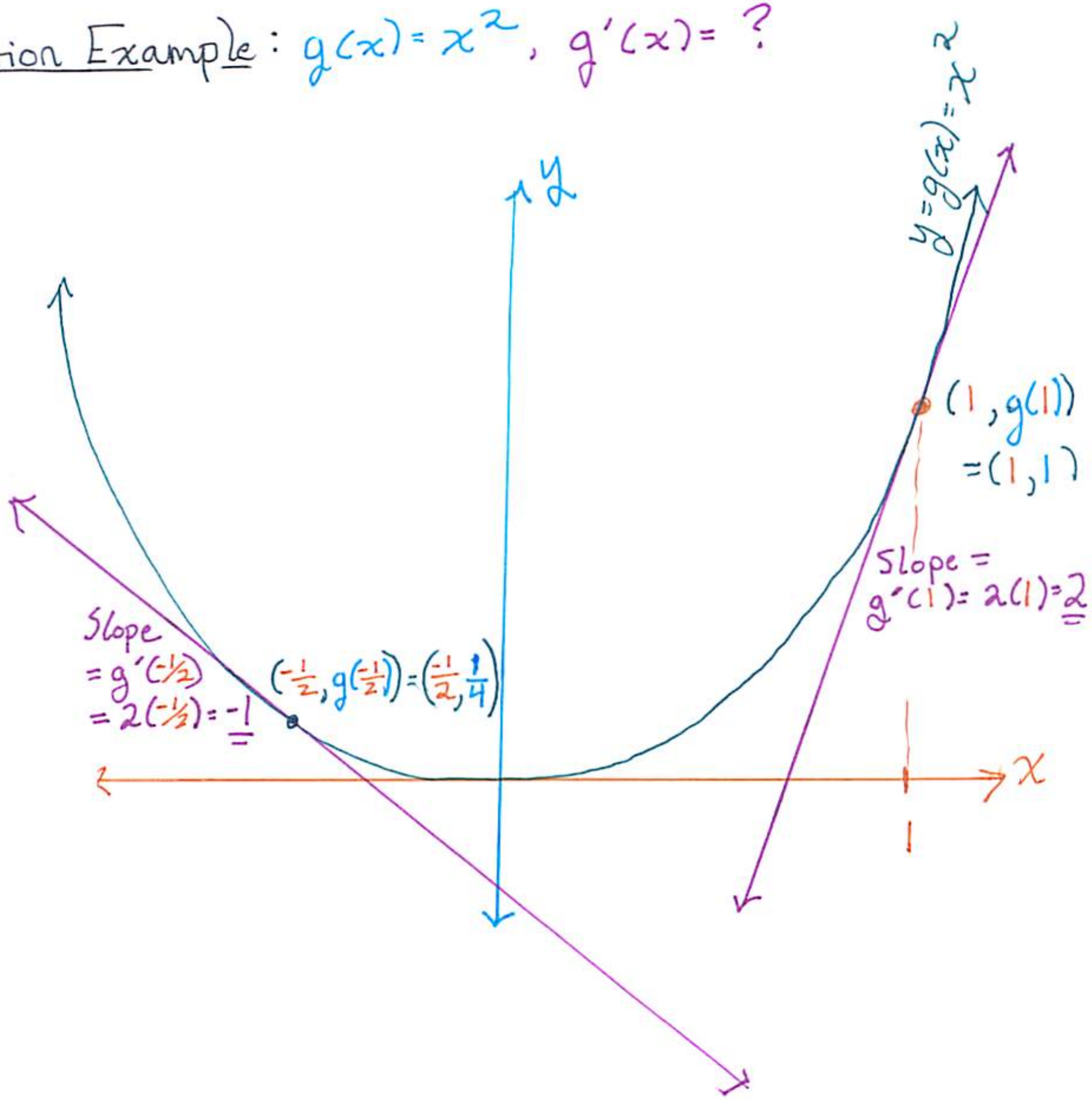
$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - (x)^2}{h}$$

$$= \lim_{h \to 0} \frac{\overbrace{\cancel{(x^2 - x^2)}}^{0} + h(2x+h)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x + h) = 2x + 0 = 2x$$

SO $\boxed{g'(x) = 2x}$



# Differentiability

Not every function has a derivative on its domain!

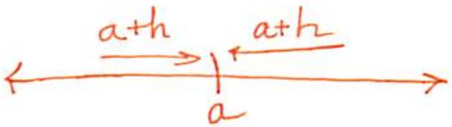
## Formal Definition (Differentiable):

$f$ is $\boxed{\text{differentiable}}$ at $a$ when $f'(a)$ exists, i.e.

$f'(a)$ exists, i.e.

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \text{ exists, i.e.}$$

$$\lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$$




## 1st Examples
(of differentiable functions):

1. all polynomials
2. $e^x$, $\ln(x)$
3. $\sin(x)$, $\cos(x)$

## Fun Facts

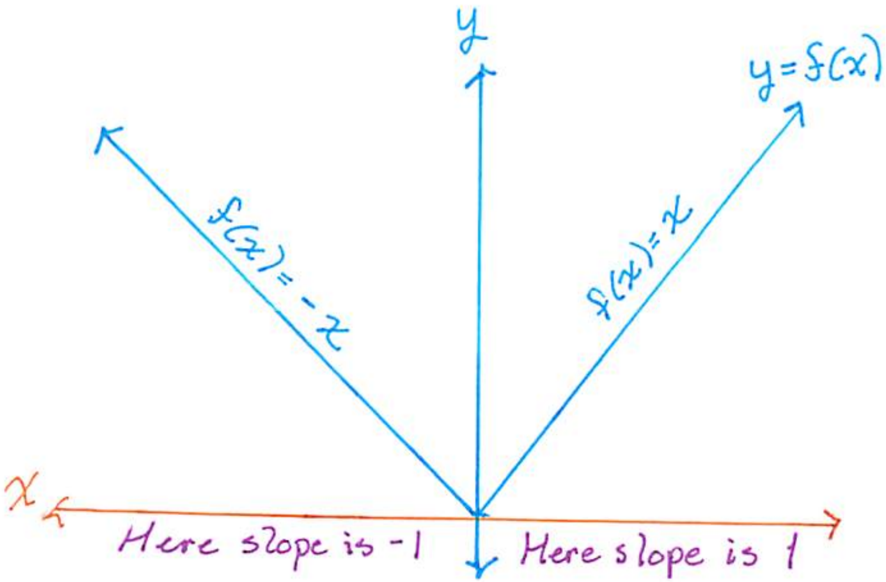
1. IF $f$ is differentiable at $a$, THEN $f$ is continuous at $a$
2. BUT NOT EVERY CONTINUOUS FUNCTION IS DIFFERENTIABLE!

If $f$ is differentiable at __each__ $a$, then $\boxed{f \text{ is differentiable}}$

# Example of Showing a Function is Differentiable

**Example:** $f(x) = |x|$, $f'(0) = ?$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$




$f(x)$ is differentiable where $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$ exists

for which $x=a$

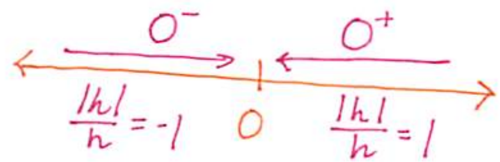
Here $\lim\limits_{h \to 0} \dfrac{|x+h|-|x|}{h}$ exists (particularly for $x=0$)

where $\underbrace{\lim\limits_{h \to 0} \dfrac{|x+h|-|x|}{h}}_{\text{Need to understand}}$

For $x=0$: $\dfrac{|x+h|-|x|}{h} = \dfrac{|x+h|-|x|}{h} = \dfrac{|h|}{h}$

$$\frac{|x+h|-|x|}{h} = \frac{|h|}{h} = \begin{cases} \frac{h}{h} = 1 & \text{if } h > 0 \\ \frac{-h}{h} = -1 & \text{if } h < 0 \end{cases}$$

For $\lim\limits_{h \to 0}$ to exist, we need $\lim\limits_{h \to 0^+} = \lim\limits_{h \to 0^-}$




$\lim\limits_{h \to 0^+} \dfrac{|h|}{h} = 1$

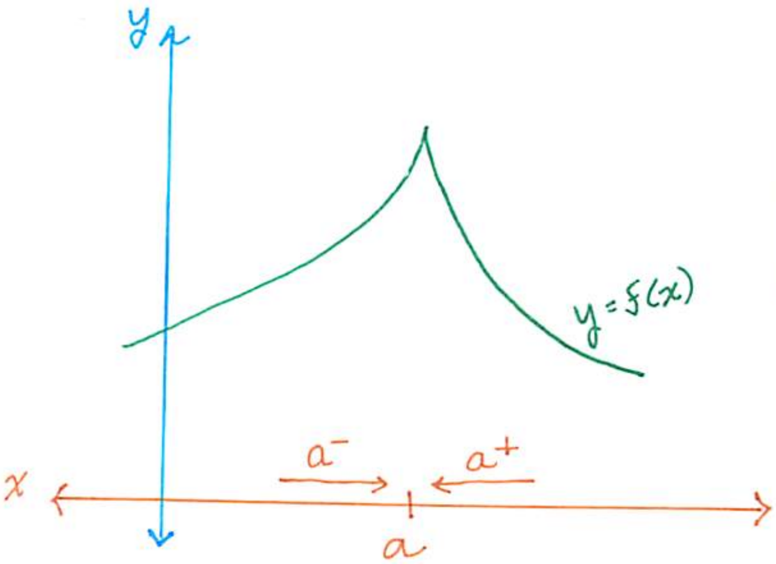
$\lim\limits_{h \to 0^-} \dfrac{|h|}{h} = -1$

Not equal, so $\lim\limits_{h \to 0} \dfrac{|x+h|-|x|}{h}$ DNE

$$\boxed{\text{SO } f(x) = |x| \text{ NOT DIFFERENTIABLE at } x = 0}$$

# Not "Differentiable at a" Pictures

## Corner

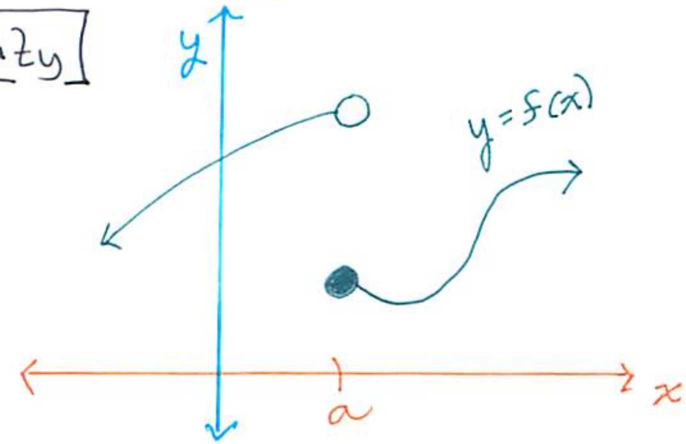



Problem:

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \neq \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

So the limit does not exist!

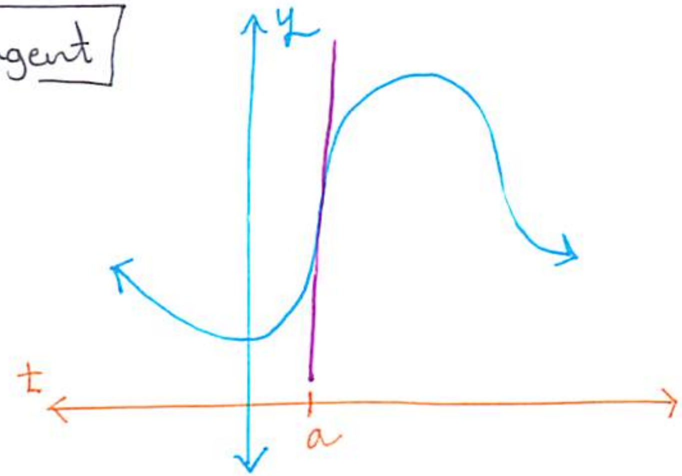
## Discontinuity




g is not continuous at a,

so g is not differentiable at a

## Vertical Tangent




Problem:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \pm \infty$$