Can you take Akemann-Weaver's $\diamondsuit$ away?

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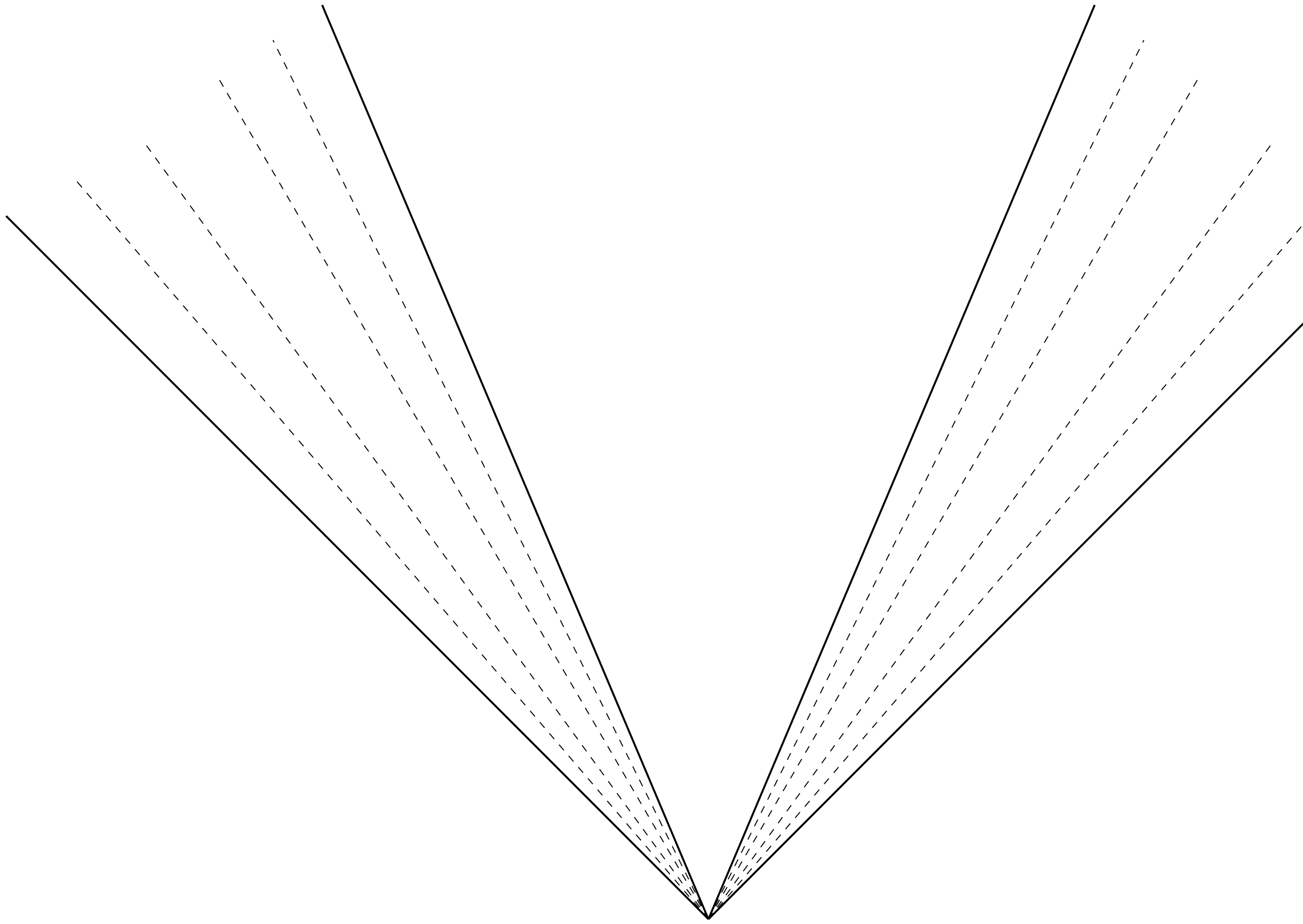
Abstract

In 2004 Akemann and Weaver showed that if $\diamondsuit$ holds, there is a C^*-algebra with a unique irreducible representation up to spatial equivalence that is not isomorphic to any algebra of compact operators. This answered, under some additional set-theoretic assumptions, an old question due to Naimark. All known counterexamples to Naimark's Problem have been constructed using a modification of the Akemann-Weaver technique and it was not known whether there exists an algebra of this kind in the absence of $\diamondsuit$. We show that it is relatively consistent with ZFC that there is a counterexample to Naimark's Problem while $\diamondsuit$ fails.



A Carlos Di Prisco en su cumpleaños número 70.

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Foreword

1 About the work

Let $\mathscr{K}(H)$ be the C*-algebra of compact operators on a complex, not necessarily separable, Hilbert space H. In 1948 Naimark proved (see [13]) that $\mathscr{K}(H)$ has a unique irreducible representation up to spatial equivalence; the identity. A few years later he asked (see [14]) whether this property characterizes $\mathscr{K}(H)$ up to isomorphism. In other words: if A is a C*-algebra with a unique irreducible representation up to spatial equivalence, is $A \cong \mathscr{K}(H)$ for some Hilbert space H?

Soon after Naimark proposed this problem, a positive answer to his question was given for some specific classes of C*-algebras, such as separable (see [15]) and type I ones (see [8]). More recently, a positive answer to the problem was proved also for certain graph C*-algebras (see [16]). On the other hand, in 2004 Akemann and Weaver showed (see [1]) that if $\diamondsuit$, a principle that is independent of the axioms of contemporary set theory, holds then there exists a counterexample to Naimark's Problem.

Besides $\diamondsuit$, the Akemann-Weaver construction uses a deep 2001 result due to Kishimoto, Ozawa, and Sakai (see [10]) that implies that if A is a separable, simple, and unital C*-algebra, and φ and ψ are two of its pure states, there exists an automorphism g of A so that $\varphi \circ g = \psi$. All known counterexamples to Naimark's Problem have been constructed by modifying the Akemann-Weaver construction (see [5] and [17]), making a truly novel construction necessary.

In this work, we show that it is relatively consistent with ZFC that there is a counterexample to Naimark's Problem while $\diamondsuit$ fails. To prove this, we isolate a combinatorial principle $\diamondsuit(\mathsf{Cohen})$ that, together with CH and $\neg\diamondsuit$, holds after adding $\aleph_1$ Cohen reals to a model of CH and the failure of $\diamondsuit$. The main result follows observing that the conjunction $\diamondsuit(\mathsf{Cohen}) + \mathsf{CH}$ implies the existence of a counterexample to Naimark's Problem.

The work is organized as follows: Act one is dedicated to introducing some basic notions about C*-algebras and their representation theory as well as a very compact presentation of some set-theoretic-flavoured ideas. In Act two we define a forcing poset which implements the same extension as the Cohen forcing when A is separable and whose generic object is an automorphism as in the Kishimoto-Ozawa-Sakai Theorem. Act three is devoted to showing some applications of the machinery developed in

Act two, as well as introducing the principle $\diamondsuit(\mathsf{Cohen})$ and finally, presenting a proof of the main result. In the Epilogue, we propose some questions that can motivate future research in the task of understanding Naimark's Problem.

Toronto, July 2019.

2 Conventions

As far as possible we use some standard set-theoretic and functional-analytic notation; sets are indexed by their cardinality, ω is the least infinite ordinal and ω_1 the least uncountable one. $\mathbb{Q}^+$ stands for $(0,1] \cap \mathbb{Q}$. If E is a normed complex vector space and $F \subseteq E$ then $\mathbb{C}F$ is the closed complex span of F, E^* is the continuous dual of E, $\mathrm{B}_E := \{x \in E : \|x\| \leq 1\}$ is its unit ball, and $\partial \mathrm{B}_E := \{x \in E : \|x\| = 1\}$ is the boundary of B_E. The set $\{e_x : x \in X\}$ stands for the canonical basis of $\ell_2(X)$.

Act one: Preparing the ground

3 A paperback on C*-algebras and its representations

Definition 3.1. A *C*-algebra* is a norm-complete complex algebra A equipped with a (often known as involution) map $^* : A \to A$ so that for all $a, b \in A$ and $\lambda \in \mathbb{C}$:

1 $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$,

2 $(a^*)^* = a$,

3 $(ab)^* = b^* a^*$ and

4 $\|a^* a\| = \|a\|^2$ (this last oftenly known as the C*-equality).

We say that A is *unital* if it contains a multiplicative identity 1.

As it is usual, once we defined our objects of interest, we are in the responsibility of defining their morphisms as well:

Definition 3.2. Let A and B be C*-algebras. An algebras homomorphism $\varphi : A \to B$ is called a **-homomorphism* if $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. If A and B are unital and $\varphi(1) = 1$ we say that φ is *unital*. A bijective *-homomorphism $\varphi : A \to B$ is called a **-isomorphism*. In this case we say that A and B are **-isomorphic* and we write $A \cong B$.

A natural question that arises after reading Definition 3.2 is why the topologies of A and B are not involved in our notion of morphism. The following result shows one (among many) strong and very desirable property of C*-algebras and their morphisms, the *automatic continuity*:

Theorem 3.3. *Every *-homomorphism is contractive and if it is injective, then it is an isometry.*

Proof. See Lemma 1.2.9 and Corollary 1.2.10 in [4] or Theorem 2.1.7 in [12]. □

Before deepening more, let us see some warm-up examples of Definition 3.1. The first family of examples that is convenient to have in mind are the *abelian C*-algebras*. In this context, the adjective "abelian" acts on the multiplication of the algebra.

Example 3.4. Recall that if X is a locally compact Hausdorff space then $C(X)$ is the set of complex-valued continuous functions defined on X. We say that $f \in C(X)$ *vanishes at infinity* if for all $\varepsilon \in \mathbb{Q}^+$ the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. We write $C_0(X)$ for the set of complex-valued continuous functions that vanishes at infinity. Since every element of $C_0(X)$ is bounded, the equation $\|f\| := \sup\{|f(x)| : x \in X\}$ defines a norm on $C_0(X)$. Thus, we can endow $C_0(X)$ with a C*-algebra structure taking pointwise addition and multiplication as the algebraic signature and pointwise conjugation as involution. Surprisingly, these examples are exhaustive in the following sense: if A is an abelian C*-algebra, there exists a locally compact Hausdorff space $\hat{A}$, called the *Gelfand spectrum of A* (see §1.3 in [4] or §1.3 in [12]), so that $A \cong C_0(\hat{A})$.

Now we will define a family of C*-algebras in the *non-abelian framework*:

Example 3.5. Let H be a Hilbert space and let $\mathscr{B}(H)$ be the set of bounded operators on H. We know that $\mathscr{B}(H)$ has an operator norm given by

$$\|T\| := \sup\{\|T\xi\| : \xi \in \mathrm{B}_H\}.$$

We can define an algebraic structure in $\mathscr{B}(H)$ given by the pointwise addition and the composition. Finally, we can endow $\mathscr{B}(H)$ with a C*-algebra structure if we consider the involution given by the *adjoint operator*; the unique bounded operator that satisfies the implicit equation $\langle T\xi|\eta\rangle = \langle \xi|T^*\eta\rangle$ for all $\xi, \eta \in H$.

Definition 3.6. A norm-closed linear subspace I of a C*-algebra A is called a *left (right) ideal* if $ab \in I$ ($ba \in I$) for all $b \in I$ and $a \in A$. If I is both a left and a right ideal we say that it is a *two-sided ideal*. A C*-algebra is called *simple* if it contains no non-trivial two-sided ideals.

Example 3.7. Let H be a Hilbert space and $\mathscr{K}(H)$ be the set of *compact operators* on H. This is, the norm-clousure of the set of finite-rank operators. It is well known that $\mathscr{K}(H)$ is a two-sided ideal of $\mathscr{B}(H)$.

Definition 3.8. Let A be a C*-algebra. An element $a \in A$ is called *positive* if there is $b \in A$ so that $a = b^*b$. If A is unital then $u \in A$ is called *unitary* if $uu^* = u^*u = 1$. We write A^+ and $\mathrm{U}(A)$ for the sets of positive and unitary (if A is unital) elements of A respectively.

For the rest of this section fix a C*-algebra A:

Definition 3.9. A pair (H, π) is called a *representation of A* if H is a Hilbert space and $\pi : A \to \mathscr{B}(H)$ is a *-homomorphism.

Note that the representations of A are all the ways in which we can read A as some collection of operators on a Hilbert space. This implies that representations give us a tool to turn any abstract C*-algebra into some sort of concrete one. Now, to understand the representation theory of A we will see that every representation can be "sliced in smaller pieces" that we will show are closely related with certain linear functionals on A. The *direct sum* of a set of representations $\{(H_\alpha, \pi_\alpha) : \alpha < \kappa\}$ of A is defined as the representation of A whose underlying Hilbert space is the direct sum of the H_α's and its *-homomorphism is given by $a \mapsto [(\xi_\alpha : \alpha < \kappa) \mapsto (\pi_\alpha(a)\xi_\alpha : \alpha < \kappa)]$.

Definition 3.10. Let (H, π) be a representation of A. A unit vector $\xi \in \partial \mathrm{B}_H$ is called *cyclic for* (H, π) if the set $\{\pi(a)\xi : a \in A\}$ is norm-dense in H. A representation is called *cyclic* if it admits a cyclic vector. If every unit vector is cyclic for (H, π) we say that the representation is *irreducible*.

Theorem 3.11. *Every representation of A is a direct sum of cyclic representations.*

Proof. Let (H, π) be a representation of A and define $H_\xi := \mathbb{C}\{\pi(a)\xi : a \in A\}$ for each $\xi \in \partial \mathrm{B}_H$. Let $\mathbb{P} := \{S \subseteq \partial \mathrm{B}_H : (\forall \xi, \eta \in S)(\xi \neq \eta \Rightarrow H_\xi \perp H_\eta)\}$ ordered by inclusion and use Zorn's Lemma to find a maximal element $M \in \mathbb{P}$. The maximality of M implies that $H = \bigoplus\{H_\xi : \xi \in M\}$. If we define $\pi_\xi : A \to \mathscr{B}(H_\xi)$ given by $a \mapsto \pi(a)|_{H_\xi}$ we obtain that (H, π) is the direct sum of the cyclic representations (H_ξ, π_ξ). $\square$

Now we will introduce a class of functionals of A that will be extremely useful to understand its representation theory.

Definition 3.12. A functional $\varphi \in A^*$ is called *positive* if $\varphi(a) \geq 0$ for all $a \in A^+$.

Note that if φ is a positive functional, then the map $\langle \cdot | \cdot \rangle_\varphi : A \times A \to \mathbb{C}$ given by $\langle a | b \rangle_\varphi := \varphi(b^* a)$ defines a pre-inner[1] product on A. This implies that also the equation $\|a\|_\varphi := \langle a | a \rangle_\varphi^{1/2}$ for each $a \in A$, defines a seminorm. Also, we have a *Cauchy-Schwarz inequality* given by $\left|\langle a | b \rangle_\varphi\right| \leq \|a\|_\varphi \|b\|_\varphi$ for all $a, b \in A$.

Definition 3.13. We say that a positive functional φ is a *state* if it has norm 1 and we denote $\mathrm{S}(A)$ the set of states of A.

The rest of the section will be devoted to show that states are in correspondence with some specific kind of cyclic representations. Even though the GNS construction can be made for arbitrary C*-algebras (see §1.9 in [4] or §3.4 and Theorem 5.1.1 in [12]), we will concern only about the unital case as a simplification. A general approach usually uses the fact that every C*-algebra admits an *approximate unit*.

Theorem 3.14. *(The Gelfand-Naimark-Segal construction) Let $\varphi \in S(A)$ be a state of A. Then there exists a cyclic representation $(H_\varphi, \pi_\varphi, \xi_\varphi)$ so that the triangle*

$$\begin{array}{ccc} A & \xrightarrow{\quad \pi_\varphi \quad} & \mathscr{B}(H_\varphi) \\ & \searrow^{\varphi} & \big\downarrow \langle \,\cdot\, \xi_\varphi | \xi_\varphi \rangle \\ & & \mathbb{C} \end{array}$$

[1]Possibly degenerate. This means that there could be some $a \in A \setminus \{0\}$ so that $\langle a | a \rangle_\varphi = 0$.

commutes.

Before proving this, we need some auxiliary results:

Proposition 3.15. *Let $\varphi \in A^*$ be a positive functional. Then $\|\varphi\| = \varphi(1)$.*

Proof. If $a \in \mathrm{B}_A$ then, by the Cauchy-Schwarz inequality

$$|\varphi(a)|^2 = \left|\langle a|1\rangle_\varphi\right|^2 \leq \varphi(1) \cdot \varphi(a^*a) \leq \varphi(1)\|\varphi\|.$$

Being a arbitrary, $\|\varphi\| \leq \varphi(1)$. Of course, $\varphi(1) \leq \|\varphi\|$. □

Lemma 3.16. *Let $\varphi \in A^*$ be a positive functional. Then:*

1 $(\forall b \in A)\left(\|b\|_\varphi = 0 \Rightarrow (\forall a \in A)\left(\langle b|a\rangle_\varphi = 0\right)\right)$ *and*

2 $\|ab\|_\varphi \leq \|a\|\|b\|_\varphi$ *for all $a, b \in A$.*

Proof. Clearly **1** follows immediately from the Cauchy-Schwarz inequality. Now, if $\|b\|_\varphi = 0$ then using **1** we obtain that $\|ab\|_\varphi^2 = \langle ab|ab\rangle_\varphi = \varphi(b^*a^*ab) = \langle b|a^*ab\rangle_\varphi = 0$. We will assume then that $\|b\|_\varphi > 0$ and let $\psi \in A^*$ given by $\psi(c) = \langle cb|b\rangle_\varphi \|b\|_\varphi^{-2}$; which is linear and positive. By Proposition 3.15 we have that $\|\psi\| = \psi(1) = 1$ and therefore $\psi(a^*a) \leq \|a^*a\|$. This translates into

$$\frac{\|ab\|_\varphi^2}{\|b\|_\varphi^2} = \frac{\langle a^*ab|b\rangle_\varphi}{\|b\|_\varphi^2} = \psi(a^*a) \leq \|a^*a\| = \|a\|^2$$

and therefore $\|ab\|_\varphi \leq \|a\|\|b\|_\varphi$. This concludes the proof of **2**. □

Corollary 3.17. *If φ is positive then the set $L_\varphi := \{b \in A : \|b\|_\varphi = 0\}$ is a left ideal of A.*

The results above implies that if φ is positive then the map $\langle\cdot|\cdot\rangle_\varphi$ defines an inner product on the complex algebra A/L_φ. It is not necessarily the case that A/L_φ is a Hilbert space, since it may not be complete. To fix this, we will consider its Hilbert completion H_φ with respect to $\|\cdot\|_\varphi$.

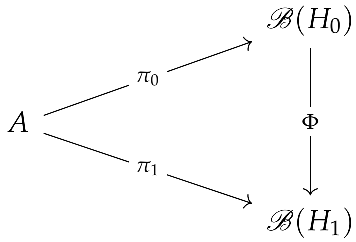
Proof of Theorem 3.14. Let H_φ be as in the discussion above and $\rho_\varphi : A \to \mathrm{End}(A/L_\varphi)$ defined by $\rho_\varphi(a)(b + L_\varphi) := ab + L_\varphi$. First of all, let us see that $\rho_\varphi(a) \in \mathscr{B}(A/L_\varphi)$ for all $a \in A$. To see this, note that if $b \in A$ then

$$\|\rho_\varphi(a)(b + L_\varphi)\|_\varphi^2 = \|ab + L_\varphi\|_\varphi^2 \leq \|a\|^2\|b + L_\varphi\|_\varphi^2$$

by Lemma 3.16. This implies that $\|\rho_\varphi(a)\| \leq \|a\|$ so $\rho_\varphi(a)$ has a unique bounded extension $\pi_\varphi(a) \in \mathscr{B}(H_\varphi)$. Now, if we choose $\xi_\varphi := 1 + L_\varphi$ we obtain that the set $\{\pi_\varphi(a)\xi_\varphi : a \in A\} = A/L_\varphi$ which is norm-dense in H_φ. Also, $\|\xi_\varphi\|_\varphi = \varphi(1) = 1$ since $\varphi \in \mathrm{S}(A)$. Finally, $\langle \pi_\varphi(a)\xi_\varphi|\xi_\varphi\rangle = \langle a + L_\varphi|1 + L_\varphi\rangle_\varphi = \varphi(a)$ as desired. □

We refer to $(H_\varphi, \pi_\varphi, \xi_\varphi)$ as the *GNS triplet associated to* φ and we say that a cyclic representation (H, π, ξ) is a *GNS representation* if there is some state $\varphi \in \mathrm{S}(A)$ so that $(H, \pi, \xi) = (H_\varphi, \pi_\varphi, \xi_\varphi)$. Clearly every GNS representation is cyclic. Now, we will see that the reciprocal is "almost true" i.e. every cyclic representation is *spatially equivalent* to a GNS representation. This will imply that to understand the representation theory of A, it is enough to understand the behaviour of its GNS representations. Moreover, since Theorem 3.14 determines a correspondence between GNS representations and states, it is enough to understand the structure of $\mathrm{S}(A)$.

We say that (H_0, π_0) and (H_1, π_1) are *spatially equivalent*, and write $\pi_0 \sim \pi_1$, if there is a *-isomorphism $\Phi : \mathscr{B}(H_0) \to \mathscr{B}(H_1)$ so that the triangle



commutes. Informally, π_0 and π_1 are the same representation of A up to some relabeling of $\mathscr{B}(H)$. Now, if H_0 and H_1 are Hilbert spaces and $U : H_0 \to H_1$ is a unitary operator[2], let $\mathrm{Ad}U : \mathscr{B}(H_0) \to \mathscr{B}(H_1)$ be the **-isomorphism implemented by* U. This is, $\mathrm{Ad}U(T) := UTU^*$ for all $T \in \mathscr{B}(H_0)$. Since every *-isomorphism $\Phi : \mathscr{B}(H_0) \to \mathscr{B}(H_1)$ is implemented by a unitary operator (see Exercise 2.8.29 in [4]), then $\pi_0 \sim \pi_1$ if and only if there is a unitary $U : H_0 \to H_1$ so that $\pi_1 = \mathrm{Ad}U \circ \pi_0$.

Theorem 3.18. *Every cyclic representation is spatially equivalent to a GNS representation.*

Proof. Let (H, π, ξ) be a cyclic representation of A. Then the functional defined by $\varphi(a) := \langle \pi(a)\xi | \xi \rangle$ for each $a \in A$ is a state. Since $\xi \in \partial \mathrm{B}_A$, the map $T : A/L_\varphi \to H$ given by $a + L_\varphi \mapsto \pi(a)\xi$ defines a partial isometry from H_φ into H. Since ξ is cyclic for (H, π), we know that T has dense range thus it extends uniquely to a unitary operator $U : H_\varphi \to H$. Also, if $a, b \in A$ then

$$(U\pi_\varphi(a)U^*)(\pi(b)\xi) = (U\pi_\varphi(a))(b + L_\varphi) = U(ab + L_\varphi) = \pi(a)(\pi(b)\xi)$$

and therefore $\pi = \mathrm{Ad}U \circ \pi_\varphi$. $\square$

[2] Surjective and inner product preserving.

4 What is a pure state?

We are in shape now to introduce our objects of main interest. By the Banach-Alaoglu Theorem, $\mathrm{S}(A)$ is a convex and weak*-compact subset of A^* so the Krein-Millman Theorem implies that it has a non-empty set of extreme points[3]. We say that a state is *pure* if it is a extreme point of $\mathrm{S}(A)$ and write $\mathrm{P}(A)$ for this set. The *pure states space of* A is $\mathrm{P}(A)$ endowed with the weak* topology. Let us show the relation between pure states and representation theory.

Theorem 4.1. *A state φ is pure if and only if its GNS representation is irreducible.*

Proof. See Proposition 3.6.4 in [4] or Theorem 5.1.6 in [12]. □

The previous result shows that the GNS representations of a C*-algebra A can be thought of as "convex combinations" of its irreducible representations. This implies that the irreducible representations of A are, up to spatial equivalence, the "building blocks" for its representation theory. The following classical result due to Naimark (see [13]) classifies all the irreducible representations of $\mathscr{K}(H)$:

Theorem 4.2. *(Naimark's Theorem) Every algebra of compact operators has a unique irreducible representation up to spatial equivalence; the identity.*

A few years after Naimark proved Theorem 4.2, he asked (see [14]) whether this property characterize $\mathscr{K}(H)$ up to isomorphism:

Problem 4.3. (Naimark's Problem) Let A be a C*-algebra with a unique irreducible representation up to spatial equivalence. Is $A \cong \mathscr{K}(H)$ for some Hilbert space H?

Following the spirit of the work so far, let us see what is the pure-state-analogue of the spatial equivalence and Naimark's Problem.

If A is a non-unital C*-algebra, define $A^\dagger$ as $A \oplus \mathbb{C}$ along with

$$
\begin{aligned}
(a \oplus \mu) + (b \oplus \lambda) &:= (a + b) \oplus (\mu + \lambda), \\
(a \oplus \mu) \cdot (b \oplus \lambda) &:= (ab + \lambda a + \mu b) \oplus \mu\lambda, \\
(a \oplus \mu)^* := a^* \oplus \overline{\mu} \text{ and the norm } \|a \oplus \mu\| &:= \sup_{b \in \mathrm{B}_A} \|ab + \mu b\|.
\end{aligned}
$$

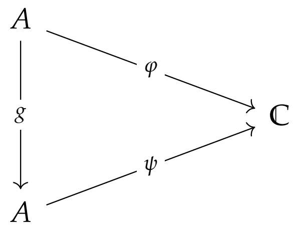
Note that $A^\dagger$ is always unital with identity $0 \oplus 1$. We define $\tilde{A}$, the *unitization of* A, as A if it is unital and $A^\dagger$ otherwise. The *unitization operation* can be viewed as the C*-algebraic analogue of the one-point compactification.

Let $\mathrm{Aut}(A)$ be the *automorphisms group* of A. We say that $g \in \mathrm{Aut}(A)$ is *inner* or *implemented by a unitary* if there exists $u \in \tilde{A}$ so that $g(a) = \mathrm{Ad}u(a) := uau^*$ for all

[3] A state φ is a *extreme point* of $\mathrm{S}(A)$ if whenever $\sigma, \tau \in \mathrm{S}(A)$ are so that $\varphi = (\sigma + \tau)/2$ then $\sigma = \tau$.

$a \in A$. We write $\mathrm{Inn}(A)$ for the subgroup of inner automorphisms of A. If g is in the pointwise-convergence-clousure of $\mathrm{Inn}(A)$, we say that it is *approximately inner*.

Two pure states $\varphi, \psi \in \mathrm{P}(A)$ are called *conjugate* if there exists $g \in \mathrm{Aut}(A)$ so that the triangle



commutes. If g can be chosen to be inner, we say that φ and ψ are *unitarily equivalent*. In this case we write $\varphi \sim \psi$. The following result shows which is the relation between unitary and spatial equivalence via the GNS representation, as well as an effective way to characterize them:

Lemma 4.4. *Let A be unital and $\varphi, \psi \in P(A)$. Then the following are equivalent:*

1 *φ and ψ are unitarily equivalent,*

2 *π_φ and π_ψ are spatially equivalent and*

3 *there is a unitary $u \in U(A)$ so that $\|\varphi \circ Adu - \psi\| < 2$.*

Proof. See Lemma 3.8.1 in [4]. □

We can state now Naimark's Problem in terms of the pure states space of A:

Problem 4.5. (Naimark's Problem) Let A be a C*-algebra with a unique pure state up to unitary equivalence. Is $A \cong \mathscr{K}(H)$ for some Hilbert space H?

If A has a non-trivial ideal, then it has irreducible representations with different kernels and therefore A cannot give a negative answer to Naimark's Problem. In the years immediately after Naimark proposed this problem, it was shown that his question has a positive answer for some specific classes of C*-algebras, such as the separable[4] (see [15]) and the *type I* ones (see [8]). More recently, a positive answer to the problem was proved also for certain graph C*-algebras (see [16]).

Recall that a C*-algebra is said to be *of type I* if $\mathscr{K}(H) \subseteq \pi[A]$ for every irreducible representation (H, π) of A. A deep result due to Glimm (see [6]) implies that every *non-type I* C*-algebra contains a subalgebra with a quotient *-isomorphic to the *CAR* (Canonical *A*nticommutation *R*elations) algebra $\bigotimes_\omega M_2(\mathbb{C})$. This implies that every

[4]Proposition 6 in [1] shows that a counterexample to Naimark's Problem cannot have density $< \mathfrak{c}$.

separable simple non-type I C*-algebra has $\mathfrak{c}$-many $\sim$-classes[5]. We also have the following classical result due to Glimm:

Lemma 4.6. *(Glimm's Lemma) If A is a simple unital non-type I C^*-algebra, then the orbit of φ under the action of $Inn(A)$, $\{\varphi \circ Adu : u \in U(A)\}$ is weak*-dense in $P(A)$ for all $\varphi \in P(A)$.*

To sum up, a counterexample to Naimark's Problem, i.e. a C*-algebra with a unique pure state up to unitary equivalence which is not *-isomorphic to any $\mathscr{K}(H)$, must be a simple non-type I C*-algebra with density character $\geq \mathfrak{c}$.

5 About Set Theory

In this section, we will introduce some set-theoretic notions that will appear later in this document. This presentation does not plan to be exhaustive, but rather give to the non-specialized reader in Set Theory a general idea of the language and the nature of the objects that will appear in later sections. A standard introduction to the topics to be mentioned here can be found in Chapters III, IV and V in [11] or Chapters 14, 16 and 27 in [7].

A subset $C \subseteq \omega_1$ is called a *club* if it is *cl*osed in the order topology and *un*bounded. If $E \subseteq \omega_1$ has a non-empty intersection with all the clubs, we say that E is stationary.

Definition 5.1. We say that $\diamondsuit$ *holds* if there is a sequence $(S_\alpha : \alpha < \omega_1)$ so that each $S_\alpha \subseteq \alpha$ and the set $\{\alpha : S \cap \alpha = S_\alpha\}$ is stationary for all $S \subseteq \omega_1$.

By an standard argument, the set $\{S \subseteq \omega_1 : S \text{ contains a club}\}$ is a filter on $\wp(\omega_1)$ and therefore $\diamondsuit$ can be thought of as the existence of a sequence of countable sets that "guesses" all the initial segments of subsets of ω_1 in a "big set of occasions".

We say that the *Continuum Hypothesis* (abbreviated CH) *holds* if $\mathfrak{c} := |\wp(\mathbb{N})| = \aleph_1$.

A *forcing notion* is a triple $(\mathbb{P}, \leq, \mathbb{1})$ where $(\mathbb{P}, \leq)$ is a partially pre-ordered[6] set (abbreviated *poset*) with maximum $\mathbb{1}$. Elements in a forcing notion are often refer to as *conditions*. A subset $D \subseteq \mathbb{P}$ is called *dense* if for all $q \in \mathbb{P}$ there is a $p \in D$ below q. A downwards closed subset of $\mathbb{P}$ is called *open*. These definitions coincide with those of being topologically dense or open when $\mathbb{P}$ is endowed with the topology generated by its cones. Two elements $p, q \in \mathbb{P}$ are called *compatible* (denoted $p \not\perp q$) if there is some $r \in \mathbb{P}$ so that $r \leq p$ and $r \leq q$ and *incompatible* otherwise. A subset $A \subseteq \mathbb{P}$ is called an *antichain* if all its elements are pairwise incompatible. A subset $\mathfrak{G} \subseteq \mathbb{P}$ is called a *filter* if it is upwards closed and given $p, q \in \mathfrak{G}$ there is $r \in \mathfrak{G}$ so that $r \leq p$ and $r \leq q$. This last restriction means that all elements in a filter are compatible and a witness of

[5] A 2017 result due to Farah and Hirshberg (see [5]) implies that this consistently fails if A is not assumed to be separable. Moreover, if $\diamondsuit$ holds, then for every $1 \leq n \leq \aleph_0$ there is a simple unital non-type I C*-algebra of density $\aleph_1$ with exactly n $\sim$-classes.

[6] Not necessarily antisymmetric.

compatibility can be found in the filter itself. If $\mathcal{D}$ is a family of dense (or dense open) subsets of $\mathbb{P}$, we say that a filter $\mathfrak{G}$ is $\mathcal{D}$-generic if $\mathfrak{G} \cap D \neq \varnothing$ for all $D \in \mathcal{D}$.

Now, if V is a countable transitive model of ZFC with $\mathbb{P} \in V$[7], we say that a filter $\mathfrak{G}$ on $\mathbb{P}$ is *V-generic* if $\mathfrak{G} \cap D \neq \varnothing$ for all dense (or dense open) set $D \in V$. A standard application of the Baire Category Theorem shows that generic filters over countable models always exists. A condition $p \in \mathbb{P}$ is said to *force* a sentence φ in the set theory language (denoted $p \Vdash \varphi$) if $V[\mathfrak{G}] \models \varphi$[8] for all V-generic filter $\mathfrak{G}$ on $\mathbb{P}$ so that $p \in \mathfrak{G}$. Two fundamental lemmas in the theory of forcing, namely Lemma IV.2.24 and Lemma IV.2.25 in [11], states that a sentence holds in $V[\mathfrak{G}]$ if and only if there is some $p \in \mathfrak{G}$ that forces it and that the forcing relation is definable in V.

Forcing axioms are far-reaching extensions of the Baire Category Theorem that enable one to apply the forcing technique without worrying about metamathematical issues. The simplest, and most popular of them, is *Martin's Axiom*. Recall that a forcing notion has the *c*ountable *c*hain *c*ondition (abbreviated *ccc*) if all its antichains are countable.

Definition 5.2. Let κ be an infinite cardinal. We say that MA_κ *holds* if for all ccc forcing poset $\mathbb{P}$ and all family $\mathcal{D}$ of dense (or dense open) subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there exists a $\mathcal{D}$-generic filter. We say that *Martin's Axiom* (abbreviated MA) *holds* if MA_κ holds for all $\omega \leq \kappa < \mathfrak{c}$.

As a final remark, it is worth mentioning that if there exists a model of ZFC then there exists a model of ZFC + CH + $\neg\diamondsuit$ (see [2]).

[7] This means that $(\mathbb{P}, \leq, \mathbb{1}) \in V$.

[8] $V[\mathfrak{G}]$ stands for the minimal transitive model of ZFC that contains V as a subset and $\mathfrak{G}$ as an element.

Act two: Homogeneity of the pure states space

6 Forcing an approximately inner automorphism

Given two inequivalent pure states $\varphi \nsim \psi$ of a simple unital C*-algebra A, we want to define a forcing notion $\mathbb{E} = \mathbb{E}_A(\varphi, \psi)$ whose generic object is an approximately inner automorphism $\mathfrak{g}$ of A so that $\varphi \circ \mathfrak{g} = \psi$.

Fix A as described above and by now, a little bit of notation:

We denote $\mathrm{P}_m(A)$ the set of tuples $\vec{\varphi} \in \mathrm{P}(A)^m$ of pairwise unitarily inequivalent pure states of A. If $G \subset A$ is finite and $\delta \in \mathbb{Q}^+$ we write

$$\vec{\varphi} \approx_{G,\delta} \vec{\psi} \equiv \max_{b \in G} \left(\max_{i<m} |\varphi_i(b) - \psi_i(b)| \right) < \delta.$$

Also, if $g \in \mathrm{Aut}(A)$ then $\vec{\varphi} \circ g := (\varphi_i \circ g : i < m)$.

Definition 6.1. Let $\vec{\varphi} \in \mathrm{P}_m(A)$, $F \subset A$ finite, $\varepsilon \in \mathbb{Q}^+$ and $u \in \mathrm{U}(A)$. We say that a pair (G, δ) with $G \subset A$ finite and $\delta \in \mathbb{Q}^+$ is *$(\vec{\varphi}, F, \varepsilon, u)$-good* if for all $\vec{\theta} \in \mathrm{P}_m(A)$ the following holds: if $\vec{\varphi} \approx_{G\,\delta} \vec{\theta}$ then for all $K \subset A$ finite and every $\gamma \in \mathbb{Q}^+$ there exists a unitary $v \in \mathrm{U}(A)$ such that $\vec{\varphi} \circ \mathrm{Ad}v \approx_{K\,\gamma} \vec{\psi}$ and $\|b - \mathrm{Ad}v(b)\| < \varepsilon/3$ for all $b \in F \cup \mathrm{Ad}u[F]$.

The Kishimoto-Ozawa-Sakai Theorem (see [10]) implies that if A is simple and separable, then all its pure states are conjugate. A key result (see Lemma 2.2 in [10]) to show the Kishimoto-Ozawa-Sakai Theorem also implies that $(\vec{\varphi}, F, \varepsilon, u)$-good couples always exists. Fix $\varphi \nsim \psi$ two pure states of A:

Definition 6.2. Let $\mathbb{E} = \mathbb{E}_A(\varphi, \psi)$ be the set of tuples $q = (F_q, G_q, \varepsilon_q, \delta_q, v_q, w_q)$ so that:

1 F_q and G_q are finite subsets of A,

2 δ_q and ε_q are elements of $\mathbb{Q}^+$,

3 $v_q, w_q \in \mathrm{U}(A)$,

4 (G_q, δ_q) is a $(\varphi, F_q, \varepsilon_q, v_q)$-good couple, and

5 $\left|(\varphi \circ \mathrm{Ad}v_q)(b) - (\psi \circ \mathrm{Ad}w_q)(b)\right| < \delta_q$ for all $b \in G_q$.

We order $\mathbb{E}$ by $p \le q$ if:

a $F_p \supseteq F_q$,

b $G_p \supseteq G_q$,

c $\varepsilon_p \le \varepsilon_q$,

d $\delta_p \le \delta_q$ and

e for all $b \in F_q$ and $i \in \{1, *\}$

$$\max\left\{\left\|\mathrm{Ad}v_p^i(b) - \mathrm{Ad}v_q^i(b)\right\|, \left\|\mathrm{Ad}w_p^i(b) - \mathrm{Ad}w_q^i(b)\right\|\right\} \le \varepsilon_q - \varepsilon_p.$$

The remainder of this section will be dedicated to showing that if $\mathfrak{G}$ is a generic enough filter on $\mathbb{E}$, then $(\mathrm{Ad}v_p w_p^* : p \in \mathfrak{G})$ is a pointwise Cauchy net so that if $\mathfrak{g}(b) := \lim_{\mathfrak{G}} \mathrm{Ad}v_p w_p^*(b)$, then $\varphi \circ \mathfrak{g} = \psi$.

Lemma 6.3. *For all $F, G \subset A$ finite and $\varepsilon, \delta \in \mathbb{Q}^+$ the set*

$$D_{FG}^{\varepsilon\delta} := \left\{p : F \subseteq F_p,\ G \subseteq G_p,\ \varepsilon_p \le \varepsilon \ \&\ \delta_p \le \delta\right\}$$

is a dense open subset of $\mathbb{E}$.

Proof. Let $q \in \mathbb{E}$ and let (G_1, δ_1) be a $(\psi \circ \mathrm{Ad}w_q, F_q, \varepsilon_q, w_q^*)$-good pair. Recall that (G_q, δ_q) is $(\varphi, F_q, \varepsilon_q, v_q)$-good and $\varphi \circ \mathrm{Ad}v_q \approx_{G_q \delta_q} \psi \circ \mathrm{Ad}w_q$. Taking $K = G_1$ and $\gamma = \delta_1$, let $v \in \mathrm{U}(A)$ be such that $\varphi \circ \mathrm{Ad}v \approx_{G_1 \delta_1} \psi \circ \mathrm{Ad}w_q v_q^*$ and $\|b - \mathrm{Ad}v(b)\| < \varepsilon_q/3$ for all $b \in F_q \cup \mathrm{Ad}v_q[F_q]$. Now set $\varepsilon_p := \min\{\varepsilon, \varepsilon_q/3\}$, $F_p := F_q \cup F$, $v_p := v v_q$, and let (G_2, δ_2) be a $(\varphi, F_p, \varepsilon_p, v_p)$-good pair. Also let $G_p := G \cup G_q \cup G_2$ and $\delta_p := \min\{\delta, \delta_q, \delta_2\}$. Taking now $K = G_p$ and $\gamma = \delta_p$ let $w \in \mathrm{U}(A)$ be so that $\varphi \circ \mathrm{Ad}v_p \approx_{G_p \delta_p} \psi \circ \mathrm{Ad}w_q w$ and $\|b - \mathrm{Ad}w(b)\| < \varepsilon_q/3$ for all $b \in F_q \cup \mathrm{Ad}w_q^*[F_q]$. Define w_p as $w_q w$ and set p to be $(F_p, G_p, \varepsilon_p, \delta_p, v_p, w_p)$. Clearly $F \subseteq F_p$, $G \subseteq G_p$, $\varepsilon_p \le \varepsilon$ and $\delta_p \le \delta$. Also, since (G_p, δ_p) is $(\varphi, F_p, \varepsilon_p, v_p)$-good, $p \in \mathbb{E}$. Note that if $b \in F_q$ then

$$\|\mathrm{Ad}v_q^*(b) - \mathrm{Ad}v_p^*(b)\| = \|b - \mathrm{Ad}v(b)\| < \varepsilon_q/3 < \varepsilon_q - \varepsilon_q/3 \le \varepsilon_q - \varepsilon_p$$

and $\|\mathrm{Ad}v_q(b) - \mathrm{Ad}v_p(b)\| = \|\mathrm{Ad}v_q(b) - \mathrm{Ad}v(\mathrm{Ad}v_q(b))\| < \varepsilon_q - \varepsilon_p$. The calculations concerning w_p and w_q are analogous and therefore $p \le q$. □

If a forcing poset $\mathbb{P}$ adds reals to some countable transitive model V of ZFC, then it also could add Cauchy sequences to A. Because of this, in a forcing extension $V[\mathfrak{G}]$ we identify A with its completion and pure states of A with their unique continuous (thus pure state) extensions to the completion of A.

Theorem 6.4. *Let V be a countable transitive model of* ZFC *and $\mathfrak{G}$ be a V-generic filter on $\mathbb{E}$. Then $\mathfrak{g}(b) := \lim_{\mathfrak{G}} Adv_p w_p^*(b)$ for each $b \in A$, is a well-defined approximately inner automorphism of A in $V[\mathfrak{G}]$ so that $\varphi \circ \mathfrak{g} = \psi$.*

Proof. By Lemma 6.3, the nets $(\mathrm{Ad}v_p : p \in \mathfrak{G})$ and $(\mathrm{Ad}v_p^* : p \in \mathfrak{G})$ are point-norm Cauchy in $\mathrm{Aut}(A)$. This implies that $\mathfrak{g}_\varphi$ defined pointwise as $\lim_{\mathfrak{G}} \mathrm{Ad}v_p$ is a well-defined endomorphism of A, and its inverse is given by the equation

$$\mathfrak{g}_\varphi^{-1}(b) = \lim_{\mathfrak{G}} \mathrm{Ad}v_p^*(b)$$

for each $b \in A$. An analogous argument shows that $\mathfrak{g}_\psi(b) := \lim_{\mathfrak{G}} \mathrm{Ad}w_p(b)$ for each $b \in A$ is an approximately inner automorphism of A. Now let $b \in A$ and $\varepsilon \in \mathbb{Q}^+$. Again using Lemma 6.3 choose $p \in \mathfrak{G}$ so that $b \in F_p \cap G_p$ and $\max\{\varepsilon_p, \delta_p\} < \varepsilon/3$. Then

$$\begin{aligned}
&|\varphi \circ \mathfrak{g}_\varphi(b) - \psi \circ \mathfrak{g}_\psi(b)| \\
&\quad \leq \|\mathfrak{g}_\varphi(b) - \mathrm{Ad}v_p(b)\| + |\varphi \circ \mathrm{Ad}v_p(b) - \psi \circ \mathrm{Ad}w_p(b)| + \|\mathfrak{g}_\psi(b) - \mathrm{Ad}w_p(b)\| \\
&\quad < 2\varepsilon_p + \delta_p < \varepsilon.
\end{aligned}$$

Now define $\mathfrak{g} := \mathfrak{g}_\varphi \circ \mathfrak{g}_\psi^{-1}$. Since ε was arbitrary, $\varphi \circ \mathfrak{g} = \psi$. □

As a concluding remark, note that taking all the objects of interest in a norm-dense subset of A and replacing all ocurrence of ε by one of $\varepsilon/2$, we can assume that $|\mathbb{E}| = d(A)$. In particular, if A is separable, then $\mathbb{E}$ is countable and therefore implements the same generic extension as the Cohen forcing[9].

[9] In words of Ilijas Farah, "$\mathbb{E}$ is just the Cohen forcing in a tuxedo."

Act three: The main construction

7 The Unique Extension Property of pure states

The Akemann-Weaver construction of a counterexample to Naimark's Problem has two main ingredients. The first step is, given a separable, simple, unital, non-type I (from now on *pertinent*) C^*-algebra A and $\varphi, \psi \in P(A)$ with $\varphi \not\sim \psi$ to find a pertinent C^*-algebra $A^\sharp$ so that $A \hookrightarrow A^\sharp$, both φ and ψ have a unique pure state extension to $A^\sharp$, and this extensions are unitarily equivalent in $A^\sharp$. Secondly, if we build a continuous increasing ω_1-chain of pertinent C^*-algebras so that $A_{\alpha+1} = A^\sharp_\alpha$ then $\diamondsuit$ is used to "guess" all the pure states of $A_{\omega_1} := \varinjlim A_\alpha$, which is not separable any more and turns out to be the desired object.

If $A^\sharp$ can be chosen so that all pure states of A extends uniquely, then the same ideas used by Akemann and Weaver can be used to construct a counterexample to Naimark's Problem under CH. Nevertheless, we do not know of the existence of such an algebra. The aim of this section is to show that this strong form of $A^\sharp$, where all pure states extend uniquely, can be mimic in a ccc forcing extension.

Note that if $\mathbb{P}$ is an adding-reals forcing poset, then $\mathbb{P}$ adds a pure state on the CAR algebra. To see this, define $\delta_0, \delta_1 \in P(M_2(\mathbb{C}))$ by $\delta_i(T) = T_{ii}$. Now, if $r \in 2^\omega$ was added after forcing with $\mathbb{P}$, let φ_r be the unique continuous linear extension to $\bigotimes_\omega M_2(\mathbb{C})$ of the functional defined in the finitely supported tensors by

$$\bigotimes_{n\in\mathbb{N}} T_n \mapsto \prod_{n\in\mathbb{N}} \delta_{r(n)}(T_n).$$

Clearly r can be recovered from φ_r by evaluation and therefore φ_r cannot be a ground model pure state. Now, since every non-type I C^*-algebra contains a subalgebra that maps onto the CAR algebra, we obtain that if A is a non-type I C^*-algebra and $\mathbb{P}$ adds a real, then it adds a pure state of A.

Let V be a countable transitive model of ZFC. Fix $\varphi \not\sim \psi$ two pure states of a simple, unital, non-type I (and not necessarily separable) C^*-algebra A. We will always use the letter $\mathfrak{G}$ to refer to a V-generic filter on $\mathbb{E}$, $V[\mathfrak{G}]$ is its respective generic extension, and $\mathfrak{g}$ is the approximately inner automorphism of A implemented by $\mathfrak{G}$. The following is probably the most important seasoning in the proof of Theorem 8.5, and describes how the $\mathfrak{g}$-orbit of a ground model pure state interacts with $P(A)^V$.

Theorem 7.1. *If ρ and σ are pure states on A and $n \in \mathbb{N}$, then the following are equivalent:*

1 *The poset $\mathbb{E}(\varphi, \psi)$ forces $\rho \sim \sigma \circ \mathfrak{g}^n$.*

2 *Either $n = 0$ and $\rho \sim \sigma$ or $n = 1$, $\rho \sim \varphi$, and $\sigma \sim \psi$.*

Definition 7.2. Let $n \in \mathbb{N}$. We say that:

- $\mathsf{D}(n)$ *holds* if for all $\rho, \sigma \in \mathrm{P}(A)^V$ and $\zeta > 0$, if $\rho \nsim \varphi$ then $E^{n\,\zeta}_{\rho\,\sigma}$ defined as

$$\left\{ p : (\exists a \in (F_p)_{\leq 1}) \left(p \Vdash \left| \left(\rho \circ \mathfrak{g}^n \circ \mathrm{Ad} v_p w_p^* \right)(a) - \sigma(a) \right| > 2 - \zeta \right) \right\}$$

  is a dense subset of $\mathbb{E}$.

- $\mathsf{F}(n)$ *holds* if for all $\rho, \sigma \in \mathrm{P}(A)^V$, if $\rho \nsim \varphi$ then $\Vdash_{\mathbb{E}} \rho \circ \mathfrak{g}^n \nsim \sigma$.

Note that Theorem 7.1 is equivalent to the statement $(\forall n \geq 1)(\mathsf{F}(n)$ holds$)$. To prove that $\mathsf{F}(n)$ is true for all $n \geq 1$, we will proceed by induction showing that $\mathsf{D}(0)$ holds, that $\mathsf{D}(n) \Rightarrow \mathsf{F}(n+1)$, and that $\mathsf{F}(n) \Rightarrow \mathsf{D}(n)$ for all $n \geq 1$.

Lemma 7.3. *Let $q \in \mathbb{E}$, $G \subset A$ finite and $\delta \in \mathbb{Q}^+$. Then there exists $r \leq q$ such that $G \subseteq G_r$, $\delta_r \leq \delta$, $F_r = F_q$ and $\varepsilon_q < 3\varepsilon_r$.*

Proof. Let (G_1, δ_1) be a $(\psi \circ \mathrm{Ad} w_q, F_q, \varepsilon_q, w_q^*)$-good pair. Taking $K = G_1$ and $\gamma = \delta_1$, let $v \in \mathrm{U}(A)$ be such that $\varphi \circ \mathrm{Ad} v \approx_{G_1\,\delta_1} \psi \circ \mathrm{Ad} w_q v_q^*$ and $\|b - \mathrm{Ad} v(b)\| < \varepsilon_q/3$ for all $b \in F_q \cup \mathrm{Ad} v_q[F_q]$. Let now $\varepsilon_r \in \mathbb{Q}^+$ be such that $\varepsilon_q < 3\varepsilon_r < 2\varepsilon_q$. Define $F_r := F_q$ and $v_r := v v_q$. Let (G_2, δ_2) be a $(\varphi, F_r, \varepsilon_r, v_r)$-good pair and define $G_r := G \cup G_q \cup G_2$ and $\delta_r := \min\{\delta, \delta_q, \delta_2\}$. Setting now $K = G_r$ and $\gamma = \delta_r$, choose some $w \in \mathrm{U}(A)$ in a way such that $\varphi \circ \mathrm{Ad} v_r \approx_{G_r\,\delta_r} \psi \circ \mathrm{Ad} w_q w$ and $\|b - \mathrm{Ad} w(b)\| < \varepsilon_q/3$ if $b \in F_q \cup \mathrm{Ad} w_q^*[F_q]$. Set w_r as $w_q w$ and $r := (F_r, G_r, \varepsilon_r, \delta_r, v_r, w_r)$. Clearly the restrictions in the statement are fulfilled. Since $G_2 \subseteq G_r$ and $\delta_r \leq \delta_2$ then (G_r, δ_r) is a $(\varphi, F_r, \varepsilon_r, v_r)$-good pair and therefore $r \in \mathbb{E}$. It only remains to check that $r \leq q$. Note that if $b \in F_q$ then

$$\|\mathrm{Ad} v_q^*(b) - \mathrm{Ad} v_r^*(b)\| = \|b - \mathrm{Ad} v(b)\| < \varepsilon_q - 2\varepsilon_q/3 < \varepsilon_q - \varepsilon_r$$

and $\|\mathrm{Ad} v_q(b) - \mathrm{Ad} v_r(b)\| = \|\mathrm{Ad} v_q(b) - \mathrm{Ad} v(\mathrm{Ad} v_q(b))\| < \varepsilon_q - \varepsilon_r$. Again, the calculations concerning w_r and w_q are analogous and therefore $r \leq q$. □

To prove that $\mathsf{D}(0)$ holds, we will introduce a technique that will appear in some arguments where dealing with more than one condition at a time is necessary. Let $q \in \mathbb{E}$ and suppose that we want to find some $p \leq q$ with a certain property (e.g. being in some potentially dense set). In this case we begin finding some $r \leq q$ as in Lemma 7.3, so that r "is not to different" from q and the couple (G_r, δ_r) bring $\varphi \circ \mathrm{Ad} v_r$ and $\psi \circ \mathrm{Ad} w_r$ "close enough". This will allow us to perturbate the condition $r \rightsquigarrow p$ in a way that p satisfies the desired property and still below q.

Lemma 7.4. *The principle* D *holds at* $n = 0$.

Proof. Fix $q \in \mathbb{E}$, let $\vec{\varphi} := (\varphi, \rho)$ and let (G, δ) be a $(\vec{\varphi}, F_q, \varepsilon_q, v_q)$-good pair. By Lemma 7.3 choose $r \leq q$ such that $G \subseteq G_r$, $\delta_r \leq \delta$, $F_r = F_q$ and $\varepsilon_q < 3\varepsilon_r$. By Glimm's Lemma, choose $\tau \in \mathrm{P}(A)$ such that $\psi \not\sim \tau$, $\sigma \not\sim \tau$ and $\vec{\varphi} \approx_{G_r \delta_r} (\psi, \tau) \circ \mathrm{Ad} w_r v_r^*$. Since $\sigma \not\sim \tau$ choose $a \in A_{\leq 1}$ such that $|\tau(a) - \sigma(a)| > 2 - \zeta/2$. Recall that (G_r, δ_r) is also a $(\vec{\varphi}, F_q, \varepsilon_q, v_q)$-good pair and let $\vec{\psi} := (\psi, \tau)$.

Let now (G_1, δ_1) be a $(\vec{\psi} \circ \mathrm{Ad} w_r, F_q, \varepsilon_q, w_q^*)$-good pair and taking $K = G_1$ and $\gamma = \delta_1$ choose $v \in \mathrm{U}(A)$ such that $\vec{\varphi} \circ \mathrm{Ad} v \approx_{G_1 \delta_1} \vec{\psi} \circ \mathrm{Ad} w_r v_r^*$ and $\|b - \mathrm{Ad} v(b)\| < \varepsilon_q/3$ for all $b \in F_q \cup \mathrm{Ad} v_q[F_q]$. Let $F_p := F_q \cup \{a\}$ and since $\varepsilon_q < 3\varepsilon_r$ choose $\varepsilon_p > 0$ such that $\varepsilon_p < \varepsilon_r - \varepsilon_q/3$. Define v_p as $v v_r$ and let (G_2, δ_2) be a $(\varphi, F_p, \varepsilon_p, v_p)$-good pair. Set also $G_p := G_2 \cup G_q \cup \{a\}$ and $\delta_p := \min\{\delta_q, \delta_2, \zeta/2\}$. Taking now $K = G_p$ and $\gamma = \delta_p$ choose $w \in \mathrm{U}(A)$ such that $\vec{\varphi} \circ \mathrm{Ad} v_p \approx_{G_p \delta_p} \vec{\psi} \circ \mathrm{Ad} w_r w$ and $\|b - \mathrm{Ad} w(b)\| < \varepsilon_q/3$ for all $b \in F_q \cup \mathrm{Ad} w_q^*[F_q]$. Set $w_p := w_r w$ and $p = (F_p, G_p, \varepsilon_p, \delta_p, v_p, w_p)$. Clearly $p \in \mathbb{E}$ and also $|(\rho \circ \mathrm{Ad} v_p w_p^*)(a) - \tau(a)| < \delta_p \leq \zeta/2$ being $a \in G_p$. To check that $p \leq q$ let $b \in F_q$ and note that

$$\begin{aligned} \|\mathrm{Ad} v_p^*(b) - \mathrm{Ad} v_q^*(b)\| &\leq \|\mathrm{Ad} v_p^*(b) - \mathrm{Ad} v_r^*(b)\| + \|\mathrm{Ad} v_r^*(b) - \mathrm{Ad} v_q^*(b)\| \\ &< \varepsilon_q - \varepsilon_r + \|b - \mathrm{Ad} v(b)\| \\ &< \varepsilon_q - (\varepsilon_r - \varepsilon_q/3) \leq \varepsilon_q - \varepsilon_p. \end{aligned}$$

Analogously

$$\begin{aligned} \|\mathrm{Ad} v_p(b) - \mathrm{Ad} v_q(b)\| &\leq \|\mathrm{Ad} v v_r(b) - \mathrm{Ad} v v_q(b)\| + \|\mathrm{Ad} v v_q(b) - \mathrm{Ad} v_q(b)\| \\ &< \varepsilon_q - (\varepsilon_r - \varepsilon_q/3) \leq \varepsilon_q - \varepsilon_p. \end{aligned}$$

Similar calculations concerning w_p and w_q show that $p \leq q$ as desired. $\square$

Lemma 7.5. *For all* $n \in \mathbb{N}$, $\mathsf{D}(n) \Rightarrow \mathsf{F}(n+1)$.

Proof. Let $\mathfrak{G}$ be a V-generic filter on $\mathbb{E}$, $\zeta > 0$ and using $\mathsf{D}(n)$ choose $p \in \mathfrak{G} \cap E^{n\,\zeta/2}_{\rho\,\sigma}$ and $a \in (F_p)_{\leq 1}$ witnessing it. Assume also that $\varepsilon_p < \zeta/2$.

Since $p \Vdash \|\mathfrak{g}(a) - \mathrm{Ad} v_p w_p^*(a)\| \leq \varepsilon_p$, then

$$\begin{aligned} p \Vdash 2 - \zeta/2 &\leq \left| \left(\rho \circ \mathfrak{g}^n \circ \mathrm{Ad} v_p w_p^*\right)(a) - \sigma(a) \right| \\ &\leq \left| \left(\rho \circ \mathfrak{g}^{n+1}\right)(a) - \sigma(a) \right| + \|\mathfrak{g}(a) - \mathrm{Ad} v_p w_p^*(a)\| \end{aligned}$$

and therefore $p \Vdash 2 - \zeta \leq \|\rho \circ \mathfrak{g}^{n+1} - \sigma\|$. Since ζ was arbitrary, we obtain that $V[\mathfrak{G}] \models \|\rho \circ \mathfrak{g}^{n+1} - \sigma\| = 2$. Now let $u \in \mathrm{U}(A)^{V[\mathfrak{G}]}$, $\varkappa > 0$ and choose $u' \in \mathrm{U}(A)^V$

such that $\|u - u'\| < \varkappa/4$. Choose also $b \in A_{\leq 1}$ such that

$$\begin{aligned}2 - \varkappa/2 &\leq \left|\left(\rho \circ \mathfrak{g}^{n+1}\right)(b) - \left(\sigma \circ \mathrm{Ad}u'\right)(b)\right| \\ &\leq \left|\left(\rho \circ \mathfrak{g}^{n+1}\right)(b) - \sigma \circ \mathrm{Ad}u(b)\right| + \|\mathrm{Ad}u(b) - \mathrm{Ad}u'(b)\| \\ &\leq \left|\left(\rho \circ \mathfrak{g}^{n+1}\right)(b) - \sigma \circ \mathrm{Ad}u(b)\right| + 2\|u - u'\|.\end{aligned}$$

We obtain that $2 - \varkappa \leq \|(\rho \circ \mathfrak{g}^{n+1}) - \sigma \circ \mathrm{Ad}u\|$. Finally, since $\varkappa$ was arbitrary, $V[\mathfrak{G}] \models \rho \circ \mathfrak{g}^{n+1} \not\sim \sigma$. Also, since $\mathfrak{G}$ was arbitrary, then $\Vdash_{\mathbb{E}} \rho \circ \mathfrak{g}^{n+1} \not\sim \sigma$. □

The following auxiliary lemma relativize the existence of good pairs to V.

Lemma 7.6. *Let $\vec{\varphi}$ be an $\mathbb{E}$-name for an element of $P_m(A)$, $F \Subset A^V$, $\varepsilon \in \mathbb{Q}^+$ and $u \in U(A)^V$. Then there exists $(G, \delta) \in V$ such that for all $\vec{\psi} \in P_m(A)^V$ the following holds: if $\Vdash_{\mathbb{E}} \vec{\varphi} \approx_{G\,\delta} \vec{\psi}$ then for all K that is an $\mathbb{E}$-name for a finite subset of A and $\gamma \in \mathbb{Q}^+$ there exists some $v \in U(A)^V$ such that $\Vdash_{\mathbb{E}} \vec{\varphi} \circ Adv \approx_{K\,\gamma} \vec{\psi}$ and $\|b - Adv(b)\| < \varepsilon/3$ for all $b \in F \cup Adu[F]$.*

Proof. Let $\mathfrak{G}$ be a V-generic filter on $\mathbb{E}$ and $(G', \delta') \in V[\mathfrak{G}]$ be a $(\vec{\varphi}, F, \varepsilon/2, u)$-good pair. For each $a' \in G'$ choose $a \in A^V$ such that $\|a - a'\| < \delta'/3$. Let $G = \{a : a' \in G'\}$ and $\delta = \delta'/3$. If $\vec{\psi} \in \mathrm{P}_m(A)^V$ is such that $\vec{\varphi} \approx_{G\,\delta} \vec{\psi}$ then $\vec{\varphi} \approx_{G'\,\delta'} \vec{\psi}$ thus for all $K \subset A^{V[\mathfrak{G}]}$ finite and $\gamma > 0$ there exists a unitary $v' \in \mathrm{U}(A)^{V[\mathfrak{G}]}$ such that $\vec{\varphi} \circ \mathrm{Ad}v' \approx_{K\,\gamma/2} \vec{\psi}$ and $\|b - \mathrm{Ad}v'(b)\| < \varepsilon/6$ for all $b \in F \cup \mathrm{Ad}u[F]$. Let $C := \max\{\|b\| : b \in K \cup F \cup \mathrm{Ad}u[F]\}$ and choose $v \in \mathrm{U}(A)^V$ such that $\|v - v'\| < \min\{\varepsilon, \gamma\}/12C$. Then $\vec{\varphi} \circ \mathrm{Ad}v \approx_{K\,\gamma} \vec{\psi}$ and $\|b - \mathrm{Ad}v(b)\| < \varepsilon/3$ for all $b \in F \cup \mathrm{Ad}u[F]$. □

We refer to a couple as in the previous lemma as a $(\vec{\varphi}, F, \varepsilon, u)$-*good couple in* V. Note that the completion of the ground model CAR algebra in a forcing extension is *-isomorphic to the CAR algebra as calculated in the such an extension. This implies that being non-type I is preserved by forcing.

Lemma 7.7. *For all $n \in \mathbb{N}$, $\mathsf{F}(n) \Rightarrow \mathsf{D}(n)$.*

Proof. Let $q \in \mathbb{E}$ and let $\vec{\varphi}$ be an $\mathbb{E}$-name for $(\varphi, \rho \circ \mathfrak{g}^n)$. By $\mathsf{F}(n)$ we have that $\mathbb{E}$ forces that $\varphi \not\sim \rho \circ \mathfrak{g}^n$. Choose (G, δ) to be a $(\vec{\varphi}, F_q, \varepsilon_q, v_q)$-good pair in V. By Lemma 7.3 choose $r \leq q$ such that $G \subseteq G_r$, $\delta_r \leq \delta$, $F_r = F_q$ and $\varepsilon_q < 3\varepsilon_r$. By Glimm's Lemma, there exists $\tau \in \mathrm{P}(A)^V$ such that $\Vdash_{\mathbb{E}}$ '$\psi \not\sim \tau$, $\sigma \not\sim \tau$ and $\vec{\varphi} \approx_{G_r\,\delta_r} (\psi, \rho) \circ \mathrm{Ad}w_r v_r^*$'. Since $\sigma \not\sim \tau$ let $a \in A^V_{\leq 1}$ be such that $|\sigma(a) - \tau(a)| > 2 - \zeta/2$. Recall that (G_r, δ_r) is also a $(\vec{\varphi}, F_q, \varepsilon_q, v_q)$-good pair in V and let $\vec{\psi} = (\psi, \tau)$.

Let now (G_1, δ_1) be a $(\vec{\psi} \circ \mathrm{Ad}w_r, F_q, \varepsilon_q, w_q^*)$-good pair in V and taking $K = G_1$ and $\gamma = \delta_1$ choose $v \in \mathrm{U}(A)^V$ such that $\mathbb{E}$ forces '$\vec{\varphi} \circ \mathrm{Ad}v \approx_{G_1\,\delta_1} \vec{\psi} \circ \mathrm{Ad}w_r v_r^*$ and $\|b - \mathrm{Ad}v(b)\| < \varepsilon_q/3$ for all $b \in F_q \cup \mathrm{Ad}v_q[F_q]$'.

Let $F_p := F_q \cup \{a\}$ and since $\varepsilon_q < 3\varepsilon_r$ choose $\varepsilon_p > 0$ so that $\varepsilon_p < \varepsilon_r - \varepsilon_q/3$. Define also v_p as vv_r. Now let (G_2, δ_2) be a $(\varphi, F_p, \varepsilon_p, v_p)$-good pair in V and set

$G_p := G_q \cup G_2 \cup \{a\}$ and $\delta_p := \min\{\delta_2, \delta_q, \zeta/2\}$. Taking $K = G_p$ and $\gamma = \delta_p$ choose $w \in \mathrm{U}(A)^V$ such that $\mathbb{E}$ forces '$\vec{\varphi} \circ \mathrm{Ad} v_p \approx_{G_p \delta_p} \vec{\psi} \circ \mathrm{Ad} w_r w$ and $\|b - \mathrm{Ad} w(b)\| < \varepsilon_q/3$ for all $b \in F_q \cup \mathrm{Ad} w_q^*[F_q]$'. Finally, set $w_p := w_r w$ and $p = (F_p, G_p, \varepsilon_p, \delta_p, v_p, w_p)$. Since $a \in G_p$ then $p \Vdash |(\rho \circ \mathfrak{g}^n \circ \mathrm{Ad} v_p w_p^*)(a) - \tau(a)| < \delta_p \leq \zeta/2$ and therefore $p \in E_{\rho\sigma}^{n\,\zeta}$. To check that $p \leq q$ let $b \in F_q$ and note that

$$\begin{aligned}\|\mathrm{Ad} v_p^*(b) - \mathrm{Ad} v_q^*(b)\| &\leq \|\mathrm{Ad} v_p^*(b) - \mathrm{Ad} v_r^*(b)\| + \|\mathrm{Ad} v_r^*(b) - \mathrm{Ad} v_q^*(b)\| \\ &< \varepsilon_q - \varepsilon_r + \|b - \mathrm{Ad} v(b)\| \\ &< \varepsilon_q - (\varepsilon_r - \varepsilon_q/3) \leq \varepsilon_q - \varepsilon_p.\end{aligned}$$

Analogously

$$\begin{aligned}\|\mathrm{Ad} v_p(b) - \mathrm{Ad} v_q(b)\| &\leq \|\mathrm{Ad} v v_r(b) - \mathrm{Ad} v v_q(b)\| + \|\mathrm{Ad} v v_q(b) - \mathrm{Ad} v_q(b)\| \\ &< \varepsilon_q - (\varepsilon_r - \varepsilon_q/3) \leq \varepsilon_q - \varepsilon_p.\end{aligned}$$

Similar calculations concerning w_p and w_q show that $p \leq q$. □

This concludes the proof of Theorem 7.1.

Now we will proceed, given a pertinent C*-algebra A, to define $A^\sharp$ as described at the beginning of this section. Using the Hahn-Banach Extension Theorem, one can show that the direct sum over all the GNS representations of A, oftenly known as the *universal representation of A*, is *faithful*[10] (see Theorem 3.4.1 in [12]). Let $\pi : A \to \mathscr{B}(H)$ be any faithful representation of A. Now, if g is an automorphism of A, we have a faithful representation $\pi_g : A \to \mathscr{B}(H \otimes \ell_2(\mathbb{Z}))$ determined by the equation

$$\pi_g(a)(\xi \otimes e_m) = \pi(g^{-m}(a))(\xi) \otimes e_m.$$

On the other hand, consider the homomorphism $\lambda : \mathbb{Z} \to \mathrm{U}(\mathscr{B}(\ell_2(\mathbb{Z})))$ such that $\lambda(m)$ is the unique continuous linear extension of the right shift $\lambda(m)(e_n) = e_{m+n}$. This induces a homomorphism $\lambda_H : \mathbb{Z} \to \mathrm{U}(\mathscr{B}(H \otimes \ell_2(\mathbb{Z})))$ given by $\lambda_H(m) = 1 \otimes \lambda(m)$. The *reduced crossed product of* $(A, g, \mathbb{Z})$, denoted by $A \rtimes_g \mathbb{Z}$, is the C*-algebra generated by $\pi_g[A] \cup \lambda_H[\mathbb{Z}]$ inside $\mathscr{B}(H \otimes \ell_2(\mathbb{Z}))$.

First of all, note that $A \hookrightarrow A \rtimes_g \mathbb{Z}$ via π_g. This implies that if A is separable unital and non-type I, then also is $A \rtimes_g \mathbb{Z}$. Now, if $a \in A$ then for all $\xi \in H$ and $m \in \mathbb{Z}$

$$\pi_g(g(a))(\xi \otimes e_m) = \pi\left(g^{1-m}(a)\right)(\xi) \otimes e_m = \left(\lambda_H(1)\pi_g(a)\lambda_H(1)^*\right)(\xi \otimes e_m)$$

therefore g becomes inner in $A \rtimes_g \mathbb{Z}$ and is implemented by the unitary $\lambda_H(1)$. In particular, this implies that if two pure states of A are conjugate by g and extend uniquely to $A \rtimes_g \mathbb{Z}$ then their pure state extensions are unitarily equivalent.

[10] A representation (H, π) is said to be *faithful* if π is an injective *-homomorphism.

Theorem 7.8. *Let A be a unital C^*-algebra, $g \in Aut(A)$ and $\rho \in P(A)$. Then ρ has a unique pure state extension to $A \rtimes_g \mathbb{Z}$ if and only if $\rho \nsim \rho \circ g^n$ for each $n > 0$.*

Proof. See Theorem 2 in [1]. □

Conjunction of Theorem 7.1 and the previous result allow us to show the following:

Corollary 7.9. *Let V be a countable transitive model of* ZFC*, A be a pertinent C^*-algebra, φ and ψ two inequivalent pure states of A, and let $\mathfrak{G}$ be a V-generic filter on $\mathbb{E}$. Then every $\rho \in P(A)^V$ has a unique pure state extension to $(A \rtimes_{\mathfrak{g}} \mathbb{Z})^{V[\mathfrak{G}]}$.*

Corollary 7.10. *Let $\kappa < \mathfrak{c}$ be an infinite cardinal and suppose that* $\mathsf{MA}(\kappa)$ *holds. Then for all $X \subseteq P(A)$ with $|X| \leq \kappa$ and for all pure states φ and ψ on A there exists $g \in Aut(A)$ so that $\varphi \circ g = \psi$ and every $\rho \in X$ has a unique pure state extension to $A \rtimes_g \mathbb{Z}$.*

Note that the fact that a separable C*-algebra A has a proper two-sided ideal can be expressed by the $\Sigma^1_1(0_A, 1_A)$ sentence

$$(\exists x)(\forall n)(\forall a_0, \dots, a_n)(\forall b_0, \dots, b_n)\left(x \neq 0_A \wedge \left\| 1_A - \sum_{i \leq n} a_i x b_i \right\| \geq 1\right).$$

where $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ are some fixed countable norm-dense subsets of A. By the Shoenfield's Absoluteness Theorem, if A is simple in V then it still simple in $V[\mathfrak{G}]$. Finally, using an argument due to Kishimoto (see [9]), this implies that $A \rtimes_{\mathfrak{g}} \mathbb{Z}$ is simple in $V[\mathfrak{G}]$ thus $A^\sharp := (A \rtimes_{\mathfrak{g}} \mathbb{Z})^{V[\mathfrak{G}]}$ is as desired.

8 Finale

Definition 8.1. A chain $(M_\alpha : \alpha < \omega_1)$ is a $\diamondsuit(\mathsf{Cohen})$-*chain* if:

1 each M_α is a (not necessarily countable) transitive model of ZFC,

2 for every $X \subseteq \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha \in M_\alpha\}$ includes a club, and

3 for every $\alpha < \omega_1$ some real in $M_{\alpha+1}$ is Cohen-generic over M_α.

We say that $\diamondsuit(\mathsf{Cohen})$ *holds* if there is a $\diamondsuit(\mathsf{Cohen})$-chain.

Lemma 8.2. *It is relatively consistent with* ZFC *that* $\diamondsuit(\mathsf{Cohen}) + \mathsf{CH}$ *holds while* $\diamondsuit$ *fails.*

Proof. Let M_0 be a countable transitive model of a large enough fragment of ZFC + CH in which $\diamondsuit$ fails. Let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_1)$ be a finite support iteration of nontrivial ccc forcings each of which has cardinality at most $\aleph_1$. Let $\mathfrak{G} \subseteq \mathbb{P}_{\omega_1}$ be an M_0-generic filter. By the ccc-ness, $\diamondsuit$ fails in $M_0[\mathfrak{G}]$ (see Excersice IV.7.57 in [11]) and the standard counting of names argument shows that CH holds in $M_0[\mathfrak{G}]$.

For $\alpha < \omega_1$ let $M_\alpha := M_0[\mathfrak{G} \cap \mathbb{P}_{\omega\cdot\alpha}]$ be the respective intermediate forcing extension. By the ccc-ness, no reals are added at stages of uncountable cofinality, and therefore every real in $M_0[\mathfrak{G}]$ belongs to some M_α for $\alpha < \omega_1$. Since an infinite finite support iteration of nontrivial ccc forcings adds a Cohen real, the model $M_{\alpha+1}$ contains a real that is Cohen over M_α.

Fix a name for a subset $X \subseteq \omega_1$. By the ccc-ness and the standard closing off argument, there is a club $C \subseteq \omega_1$ such that for every $\alpha \in C$ the forcing $\mathbb{P}_\alpha$ adds $X \cap \alpha$. Therefore $X \cap \alpha \in M_\alpha$ for club many α and $\mathbb{P}_{\omega_1}$ forces that $\diamondsuit(\mathsf{Cohen})$ holds. □

We are now in shape to show that the conjunction $\diamondsuit(\mathsf{Cohen}) + \mathsf{CH}$ implies the existence of a counterexample to Naimark's Problem. This together with Lemma 8.2, concludes our sailing through Naimark's Problem (by now).

Definition 8.3. Let $\{A_\alpha : \alpha < \omega_1\}$ be an increasing chain of C*-algebras and fix countable ordinals $\alpha \leq \beta < \omega_1$. We say that $\varphi \in \mathrm{S}(A_\beta)$ is *pure on* $a \in A_\alpha$ if for all $\varphi_0, \varphi_1 \in \mathrm{S}(A_\beta)$: if $\varphi = (\varphi_0 + \varphi_1)/2$ then $\varphi_0(a) = \varphi_1(a)$.

The following proposition is due to Akemann and Weaver (see Lemma 4 in [1]):

Proposition 8.4. *Let $\{A_\alpha : \alpha < \omega_1\}$ be a continuous increasing chain of separable and unital C*-algebras and let A_{ω_1} be its direct limit. Then for all $\varphi \in P(A_{\omega_1})$ the set*

$$\mathcal{C}_\varphi := \{\alpha < \omega_1 : \varphi|_{A_\alpha} \in P(A_\alpha)\}$$

is a club.

Proof. To see that $\mathcal{C}_\varphi$ is closed let $\{\alpha(n) : n \in \mathbb{N}\}$ be some set of ordinals so that $\varphi|_{A_{\alpha(n)}}$ is a pure state of $A_{\alpha(n)}$ for each $n \in \mathbb{N}$ and set $\alpha := \sup\{\alpha(n) : n \in \mathbb{N}\}$. If $\varphi|_{A_\alpha}$ is not pure then there are two different states $\varphi_0, \varphi_1 \in \mathrm{S}(A_\alpha)$ so that $\varphi|_{A_\alpha} = (\varphi_0 + \varphi_1)/2$. By the continuity of the chain, $\bigcup\{A_{\alpha(n)} : n \in \mathbb{N}\}$ is norm-dense in A_α thus there is some N so that $\varphi_0|_{A_{\alpha(N)}} \neq \varphi_1|_{A_{\alpha(N)}}$, a contradiction.

Let us see now that it is unbounded. Clearly $\varphi|_{A_\xi} \in \mathrm{S}(A_\xi)$ for all $\xi < \omega_1$. Let $\alpha < \omega_1$. First of all we claim that for each $b \in A_\alpha$ there is $\xi(b) < \omega_1$ so that $\varphi|_{A_\xi}$ is pure on b for all $\xi \geq \xi(b)$. Suppose, to reach a contradiction, that this is not the case and let $a \in A_\alpha$, X an unbounded subset of ω_1 and $\varepsilon \in \mathbb{Q}^+$ so that for all $\xi \in X$ there are states $\varphi_{\xi 0}, \varphi_{\xi 1} \in \mathrm{S}(A_\xi)$ so that $\varphi|_{A_\xi} = (\varphi_{\xi 0} + \varphi_{\xi 1})/2$ but $|\varphi(a) - \varphi_{\xi 0}(a)| \geq \varepsilon$. Let now $\mathcal{U} \in \beta X$ be an ultrafilter containing all the tails of X and define states of A_{ω_1} by $\psi_0(b) := \lim_{\mathcal{U}} \varphi_{\xi 0}(b)$ and $\psi_1(b) := \lim_{\mathcal{U}} \varphi_{\xi 1}(b)$ for each $b \in A_{\omega_1}$. Then

$$\varphi = (\psi_0 + \psi_1)/2 \quad \text{but} \quad |\varphi(a) - \psi_0(a)| \geq \varepsilon$$

which is absurd being φ pure. Now let $\{a_n : n \in \mathbb{N}\}$ be a norm-dense subset of A_α and let $\beta(0) := \sup\{\xi(a_n) : n \in \mathbb{N}\} < \omega_1$. Then $\varphi|_{A_{\beta(0)}}$ is pure on a for all $a \in A_\alpha$. Iterating this argument we can find countable ordinals $\{\beta(n) : n \in \mathbb{N}\}$ so that $\varphi|_{A_{\beta(n+1)}}$ is pure on a for all $a \in A_{\beta(n)}$. By the continuity of the chain we have that if $\beta := \sup\{\beta(n) : n \in \mathbb{N}\}$ then $\varphi|_{A_\beta}$ is pure on all $a \in \varinjlim A_{\beta(n)} = A_\beta$, thus pure. □

Theorem 8.5. *It is relatively consistent with* ZFC *that there is a counterexample to Naimark's Problem while* $\diamondsuit$ *fails.*

Proof. Let $(M_\alpha : \alpha < \omega_1)$ be a $\diamondsuit(\mathsf{Cohen})$-chain. We will recursively define, for each $\alpha < \omega_1$, a separable, simple, unital, non-type I C*-algebra $A_\alpha \in M_\alpha$ and an enummeration $\{\varphi^\alpha_\xi : \xi < \omega_1\}$ of all pure states of A_α in M_α. At each step of the construction we will ensure that for $\beta < \alpha < \omega_1$:

1 A_β is a C*-subalgebra of A_α and

2 every pure state of A_β in M_β extends uniquely to a pure state of A_α in M_α.

Let A_0 be any separable, simple, unital, non-type I C*-algebra in M_0 and using CH fix a well-ordering $\{\varphi^0_\xi : \xi < \omega_1\}$ of the pure states of A_0 in M_0. Fix (in the real world) a surjection $f : \omega_1 \to \omega_1 \times \omega_1$ such that if $f(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$. If we already defined all the objects described above for all $\delta \leq \alpha$ and $f(\alpha) = (\beta, \gamma)$ let φ^β_γ be the γth pure state of A_β in M_β and $\tilde{\varphi}^\beta_\gamma$ its unique extension in M_α to A_α. If $\tilde{\varphi}^\beta_\gamma$ is equivalent to $\tilde{\varphi}^0_0$ let $A_{\alpha+1} = A_\alpha$ and set $\{\varphi^{\alpha+1}_\xi : \xi < \omega_1\}$ some enummeration of the pure states of $A_{\alpha+1}$ in $M_{\alpha+1}$.

Otherwise, since $\mathbb{E}_{A_\alpha}(\tilde{\varphi}^0_0, \tilde{\varphi}^\beta_\gamma)$ is countable, we can use the Cohen-generic real over M_α to find a M_α-generic filter on $\mathbb{E}_{A_\alpha}(\tilde{\varphi}^0_0, \tilde{\varphi}^\beta_\gamma)$ and obtain an automorphism $\mathfrak{g}_\alpha$ of A_α such that $\tilde{\varphi}^0_0 \circ \mathfrak{g}_\alpha = \tilde{\varphi}^\beta_\gamma$. By Corollary 7.9, the C*-algebra $A_{\alpha+1} := A_\alpha \rtimes_{\mathfrak{g}_\alpha} \mathbb{Z} \in M_{\alpha+1}$ has the property that every pure state of A_α that belongs to M_α has a unique state extension to $A_{\alpha+1}$. Finally, fix an enummeration $\{\varphi^{\alpha+1}_\xi : \xi < \omega_1\}$ of all pure states of $A_{\alpha+1}$ in $M_{\alpha+1}$. If β is a limit ordinal, take $A_\beta := \varinjlim_\beta A_\alpha$. If $A_\beta \in M_\beta$ then fix an enumeration of pure states of A_β that belong to M_β as above. Finally, let $A_{\omega_1} := \varinjlim_{\omega_1} A_\alpha$.

By a suitable coding (see the introduction to §3 in [5] or §7.1 and §7.2 in[4]), we can assume that if $\psi \in \mathrm{P}(A_{\omega_1})$ then the set $\{\alpha < \omega_1 : \psi|_{A_\alpha} \in M_\alpha\}$ includes a club. Since the set $\{\alpha < \omega_1 : \psi|_{A_\alpha}$ is pure$\}$ is a club, there exists $\beta < \omega_1$ such that $\psi | A_\beta$ is a pure state of A_β in M_β. Let $\gamma < \omega_1$ so that $\psi | A_\beta = \varphi^\beta_\gamma$. Since f is a surjection, there is some $\alpha < \omega_1$ so that $f(\alpha) = (\beta, \gamma)$ and therefore in the step $\alpha + 1$ of the recursion we ensured that ψ is equivalent to $\tilde{\varphi}^0_0$.

Finally, since A_{ω_1} is infinite-dimensional and unital, it cannot be isomorphic to any algebra of compact operators. $\square$

Epilogue

9 Questions and remarks

Finally, we state some questions and remarks that naturally arise from this work:

Question 9.1. Is there always a counterexample to Naimark's Problem in a forcing extension obtained after adding one Cohen real to a model of CH?

Question 9.2. Is there a model of ZFC with a counterexample to Naimark's Problem but with no Suslin tree?

Question 9.3. Does CH implies that there is a counterexample to Naimark's Problem?

This question is tightly related with the following:

Question 9.4. Consistently, is there a unital counterexample to Naimark's Problem with density character $\aleph_2$?

Note that to reproduce the argument used at §8, but this time with length ω_2, it will be required that $\mathbb{E}_A(\varphi, \psi)$ still being "nice" even if A is not separable anymore. Sadly, this will not be possible in general. A 2010 result due to Farah (see [3]) implies that there is a simple C*-algebra $\mathcal{F}$ with $d(\mathcal{F}) = \aleph_1$ and pure states $\varphi \nsim \psi$ so that $\mathbb{E}_{\mathcal{F}}(\varphi, \psi)$ collapses $\aleph_1$.

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