What is an Exponential Function?

$f(x) = \underline{\underline{a}}^x$ ← base

What's going on with this function?

- IF $x = n$ is a positive integer

$a^n = \underbrace{a \cdot \cdots \cdot a}_{n \text{ times}}$ (e.g. $2^3 = \underbrace{2 \cdot 2 \cdot 2}_{3 \text{ times}}$)

Says:
If increase my input x by 1
Then my output is multiplied by ~~my~~ base a

- $a^0 = 1 \rightsquigarrow f(0) = 1$
- $a^{-n} = \frac{1}{a^n} \rightsquigarrow f(-n) = \frac{1}{f(n)}$

} for all exponential functions f

- If x is a rational number (i.e. $x = \frac{p}{q}$ for some integers p, q)

$a^x = a^{p/q} = a^{p \cdot 1/q}$

$= (a^p)^{1/q}$ AND $(a^{1/q})^p$ ← If $q > 0$

$\sqrt[q]{a^p}$ = $(a^p)^{1/q}$, $(\sqrt[q]{a})^p$ = $(a^{1/q})^p$

Warm-Up

Simplify $\frac{8^{-1}}{3^{-5}}$

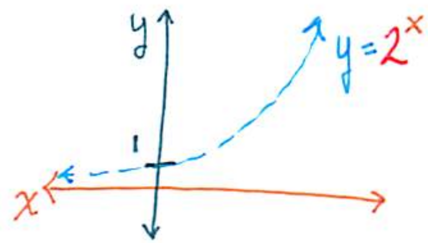
$= 8^{-1} \cdot \frac{1}{3^{-5}}$

$= \frac{1}{8} \cdot \frac{1}{\frac{1}{3^5}}$

$= \frac{1}{8} \cdot 3^5$

$= \boxed{\frac{3^5}{8}}$

- If x is irrational (not rational)
"fill in the holes"



careful!

hen Integrating:

$= \frac{1}{k} \int e^u \, du$

$= \frac{1}{k} e^u + C$

$= \frac{1}{k} e^{kx} + C$

$+C$

Careful with $\ln(u)$ and $\frac{1}{u}$!

$$\frac{d}{dx}(\ln(x)) = \frac{1}{(x)} \quad \text{AND} \quad \int \frac{1}{(x)}\,dx = \ln(|x|) + C$$

Must use chain rule to differentiate:

$$\frac{d}{dx}\left(\ln(\overbrace{1-x}^{\text{inside}})\right) = \frac{1}{(1-x)} \cdot \frac{d}{dx}(1-x)$$

$$= \frac{1}{(1-x)} \cdot (-1)$$

$$= \frac{-1}{1-x}$$

$$\boxed{\frac{d}{dx}(\ln(1-x)) = \frac{-1}{1-x}}$$

Need to use substitution when integrating

$$\int \frac{1}{\underbrace{1-x}_{u}}\,dx = \int \frac{1}{u} \cdot -du = -\int \frac{1}{u}\,du =$$

$$= -\ln(|u|) + C$$

$$= -\ln(|1-x|) + C$$

$u = 1-x$

$du = -1 \cdot dx$

$-du = dx$

$$\boxed{\int \frac{1}{1-x}\,dx = -\ln(|1-x|) + C}$$

Comparing Function Growth Rate Introduced

To compare how functions $f(x)$, $g(x)$ grow, can find $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$

Have already seen:

① $\lim\limits_{x\to\infty} \frac{2x^3-1}{1000x^2+\pi x} = \infty$

(so $f(x) = 2x^3-1$ grows faster)

② $\lim\limits_{x\to\infty} \frac{2x^3-1}{14e^2-\pi x^3} = \frac{-2}{\pi}$

(finite, so similar growth)

③ $\lim\limits_{x\to\infty} \frac{ex-e^2}{x^2-\pi} = 0$

(0, so $g(x) = x^2-\pi$ grows faster)

Q: What if when taking limit end up w/ an $\boxed{\text{indeterminate form}}$ i.e. $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\frac{\pm\infty}{\pm\infty}$?

A: l'Hôpital's Rule!

(Next time)

l'Hôpital's Rule Introduced

IF $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ gives an indefinite form i.e.
$\begin{cases} \frac{0}{0}, & \text{e.g. } \lim\limits_{x\to\infty} \frac{e^{-\pi x}}{1/x} \quad \text{OR} \\ \frac{\pm\infty}{\mp\infty}, & \text{e.g. } \lim\limits_{x\to\infty} \frac{1-x^2}{e^{5x}} = \frac{-\infty}{\infty} \quad \text{OR} \\ \frac{\pm\infty}{\pm\infty} & \text{e.g. } \lim\limits_{x\to\infty} \frac{e^x}{\ln(x)} \end{cases}$

THEN $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{\frac{d}{dx}(f(x))}{\frac{d}{dx}(g(x))}$

i.e. IF limit gives an indeterminate form

THEN can compute limit by

① Computing derivative of top & bottom SEPARATELY

(no quotient rule!!!)

② Then computing limit

Ex 1: $\lim\limits_{x\to\infty} \frac{e^{3x}}{2x+1} = \frac{\infty}{\infty}$ indeterminate form, so can keep going!

$= \lim\limits_{x\to\infty} \frac{\frac{d}{dx}[e^{3x}]}{\frac{d}{dx}[2x+1]}$

$= \lim\limits_{x\to\infty} \frac{3e^{3x}}{2} = \infty$

SO $\lim\limits_{x\to\infty} \frac{e^{3x}}{2x+1} = \infty$

and e^{3x} grows faster than $2x+1$!

Examples of Using l'Hôpital's Rule to Compare Function Growth

Ex 1: $f(x) = \ln(x)$, $g(x) = e - \pi x$

$$\lim_{x\to\infty} \frac{\ln(x)}{e - \pi x} = \frac{\infty}{-\infty}$$ Indeterminate form, can use LH!

$$\overset{LH}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[e - \pi x]} = \lim_{x\to\infty} \frac{1/x}{-\pi}$$

$$\overset{\text{rewrite}}{=} \lim_{x\to\infty} \frac{-1}{\pi x} = 0$$

So $\boxed{\lim_{x\to\infty} \frac{\ln(x)}{e - \pi x} = 0}$

∴ $g(x) = e - \pi x$ grows faster than $f(x) = \ln(x)$!

Ex 2: $f(x) = e^x$, $g(x) = x^2$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$$ Indeterminate form, can use LH!

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}[e^x]}{\frac{d}{dx}[x^2]} = \lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$ Ind. form, can use LH!

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}[e^x]}{\frac{d}{dx}[2x]} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

So $\boxed{\lim_{x\to\infty} \frac{e^x}{x^2} = \infty}$

∴ $f(x) = e^x$ grows faster than $g(x) = x^2$

> **Take Away:** As long as keep getting an indeterminate form, can keep applying l'Hôpital's rule!

Remark: These examples aren't coincidences!

Logarithmic Growth < Polynomial Growth < Exponential Growth

Exponential Growth & Decay Differential Equation

Set-Up: Rate of change of y wrt t is always proportional to $y(t)$,
i.e. $\frac{dy}{dt} = ky$ for some constant k

Q: Solutions to $\frac{dy}{dt} = ky$?

A: "Seperable differential eq" so can solve:

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\,dy = k\,dt$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln(|y|) + C_1 = kt + C_2$$

$$\ln(|y|) = kt + C$$

$$e^{\ln(|y|)} = e^{kt+C}$$

$$y = e^{kt}\underbrace{e^{C}}$$

constant can solve for by setting $t=0$

$$y(0) = \overset{1}{\cancel{e^{k(0)}}}\,e^{C} \rightsquigarrow e^{C} = y(0)$$

so $\boxed{y = y(0)e^{kt}}$

$\boxed{\text{If } k>0}$

Population growth an example:

For $\left\{ \begin{array}{l} t = \text{time} \\ P(t) = \text{population at time } t \end{array} \right\}$

$$\frac{dP}{dt} = kP \quad \text{w/ } k = \boxed{\text{relative growth rate}}$$

$$\boxed{P(t) = \underbrace{P_0}_{\boxed{\text{initial population}}} e^{kt}}$$

$\boxed{\text{If } k<0}$

Radioactive decay an example:

For $\left\{ \begin{array}{l} t = \text{time in years} \\ m_0 = \text{initial mass} \\ m(t) = \text{mass remaining after } t \text{ yrs} \end{array} \right\}$

$$\frac{dm}{dt} = km$$

$$\boxed{m(t) = m_0 e^{kt}}$$

Population Growth Example

Recall Eq: $P(t) = P_0 e^{kt}$

SUPPOSE $\begin{cases} \text{initial population} = 100 \\ \text{population in 1 yr is } 1000 \end{cases}$ $\left.\begin{matrix} \,\right] P_0 \\ \,\right] P(1) \end{matrix}$

Tasks: ① Population in 2 yrs] $P(2) = ?$
② Doubling time

$\boxed{1}$

Step 0: $\begin{cases} P(t) = P_0 e^{kt} \\ P_0 = 100 \\ P(1) = 1000 \end{cases}$ Will need to use to find eq to find $P(2)$

Step 1: $P_0 = 100 \rightsquigarrow \underline{P(t) = 100e^{kt}}$

Step 2: Use $P(1) = 1000$ to solve for k:

$$100e^{k(1)} = 1000$$
$$e^{k} = 10$$
$$\ln(e^{k}) = \ln(10)$$
$$\underline{k = \ln(10)}$$

Step 3: Substitute $k = \ln(10)$ into $P(t) = 100e^{kt}$

$$P(t) = 100e^{\ln(10)t}$$
$$P(t) = 100\left(e^{\ln(10)}\right)^{t}$$
$$\underline{P(t) = 100 \cdot 10^{t}}$$

Step 4: $P(2) = 100 \cdot 10^{(2)}$
$= 100 \cdot 100 = \boxed{10,000}$

(Always good to check now that task's finished)

$\boxed{2}$ Asking for t with $P(t) = 2\overbrace{P(0)}^{P_0}$,

i.e. $P(t) = 2 \cdot (100)$) $P(t) = 100 \cdot 10^{t}$

$$100 \cdot 10^{t} = 200$$
$$10^{t} = 2$$
$$\log_{10}(10^{t}) = \log_{10}(2)$$
$$\boxed{t = \log_{10}(2) \approx .3 \text{ yrs}}$$

It only takes 4 months for the population to double!

Comparing Exponential Growth Example

$\boxed{P(t) = P_0 e^{kt}}$ (*)

Assuming t in months, wave started w/ 1 case

Task: Approximate equation for the wave (starting in May) caused by previous variants & that caused by omicron

Other Variants:

- $\underbrace{P(0) = 1}_{P_0}$, $P(2.5) \approx 6000$
- $P_0 = 1 \xrightarrow{(*)} P(t) = e^{kt}$ (**)
- Use $P(2.5) = 6000$ to solve for k

$$e^{k(2.5)} = 6000$$
$$\ln(e^{k(2.5)}) = \ln(6000)$$
$$2.5k = \ln(6000)$$
$$k = \frac{\ln(6000)}{2.5}$$

Sub k into (**): $P(t) = e^{\left(\frac{\ln(6000)}{2.5}\right)t}$

$$P(t) = \left(e^{(\ln 6000)/2.5}\right)^t = \left(\left(\cancel{e^{\ln(6000)}}^{\,6000}\right)^{1/2.5}\right)^t$$

$$P(t) = (\underbrace{6000^{1/2.5}}_{\approx 32.5})^t \rightsquigarrow \boxed{P(t) \approx (32.5)^t}$$

So each month cases multiplied by ≈ 32.5

Omicron

- $P_0 = 1$, $P(1) \approx 10{,}000$
- $P_0 = 1 \xrightarrow{(*)} P(t) = e^{kt}$ (**)
- Use $P(1) = 10{,}000$ to solve for k

$$e^{\cancel{k(1)}\to k} = 10{,}000$$
$$\ln(e^k) = \ln(10{,}000)$$
$$k = \ln(10{,}000)$$

- Sub $k = \ln(10{,}000)$ into $P(t) = e^{kt}$

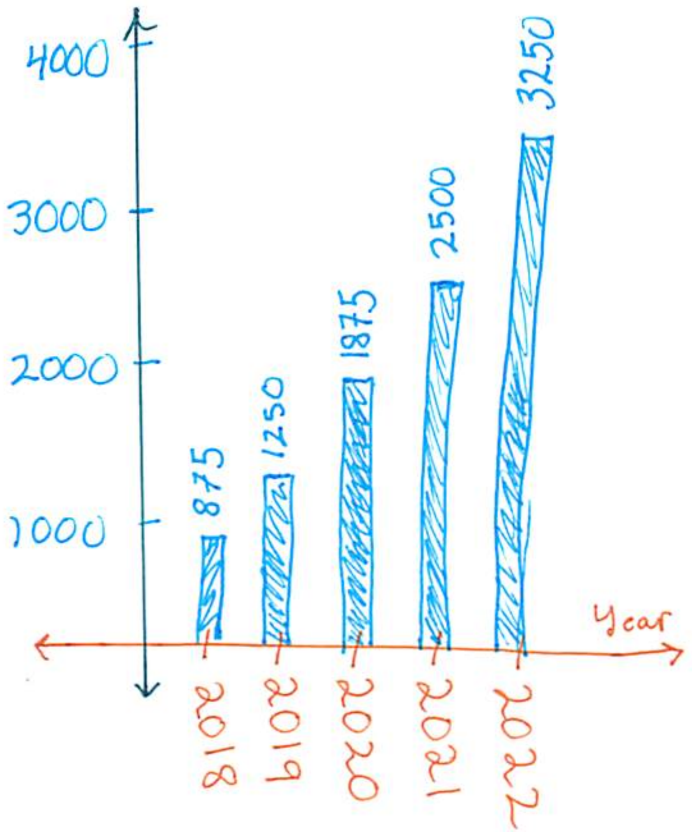
$$P(t) = e^{\ln(10{,}000)t} = \left(e^{\ln(10{,}000)}\right)^t$$

$$\boxed{P(t) = (10{,}000)^t}$$

With Omicron each month cases multiplied by $\approx 10{,}000$

Linear vs Exponential Growth Example

Cost of an Item



Q: Linear or Exponential?

Linear if <u>adding</u> same amount each time

Exponential if <u>multiplying</u> by same amount each time

Year	Cost of an Item
2018	875
2019	1250
2020	1875
2021	2500
2022	3250

+375 ×≈1.43

+625 ×1.5

+625 ×≈1.33

+750 ×1.3

Not always adding same amount, so not linear

Multiplying by close to same amount, so roughly exponential

The Logarithmic Scale

The chart is on the logarithmic scale, so exponential growth looks linear:

exponential growth $\longrightarrow$ linear growth

$y = e^{kt} \rightsquigarrow \ln(e^{kt})$

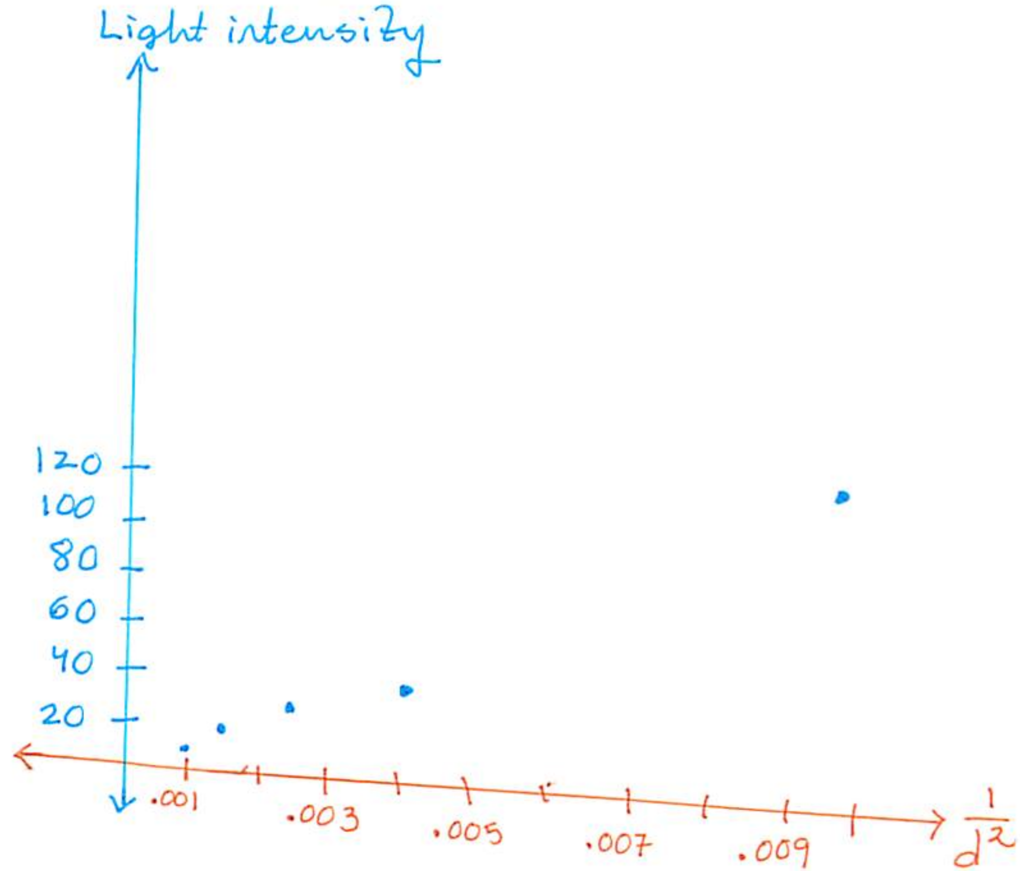
so have line $y = kt$

k growth constant $\rightsquigarrow$ now $k = $ slope

Inverse Square Law Example

Distance from light source d	Light intensity
10	120
15	54
20	30
25	17
30	13

Task: Does the inverse square law apply?



There is a proportional relationship between $\frac{1}{d^2}$ and intensity

SO inverse square law IS a good fit!