Optimization makes the most effective use of a situation or resource.

- **Business**: Find the profit-maximizing rate of oil extraction from an oil well.
- **Economics**: Create models to explain inflation. Would a higher minimum wage increase unemployment?
- **Political Science**: How many lawn signs would maximize a candidate's chance to win an election?

Single-Variable Optimization Overview

- Graphing functions to find Extreme Points
- Using differential calculus to find candidates for extreme points
- The first and the second derivative test to evaluate these candidates
- Optimization in Economics, Business, and Social Sciences
- Local minimums and maximums, Inflection Points

What is an Extreme Point?

- An **Extreme Point** is where a function reaches its largest or smallest values.
- So, an extreme point can be either a max or a min.
- To find the extreme points of a function, you must first know its domain.

Review: find the domain, D, of $f(x)$

$$\underline{f(x)} = \frac{\sqrt{x}+1}{x}$$

$f(x)$ = real number

$f(x)$ must be defined

ex) $\underline{\underline{x=-1}}$ $f(-1) = \dfrac{\sqrt{-1}+1}{-1}$ But $\sqrt{-1}$ is undefined

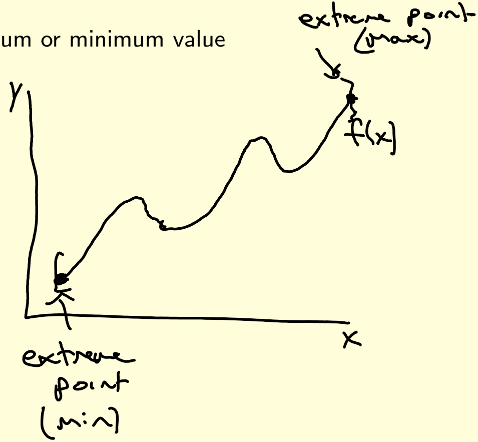
This is not in the domain

try $x=0$ $f(0) = \dfrac{\sqrt{0}+1}{0} = \dfrac{1}{0}$ is also undefined

The domain, D, of $f(x)$ is $\left[0 < x < \infty\right]$

What is an Extreme Point?

- An extreme point is the maximum or minimum value of $f(x)$ over the domain D.



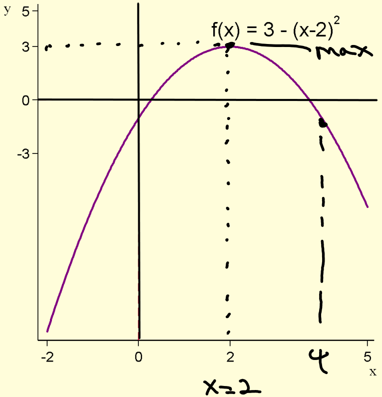
What is an Extreme Point?

- An extreme point is the maximum or minimum value of $f(x)$ over the domain D.

$$f(x) = 3 - (x-2)^2$$

- Does the above have a maximum? A minimum? Can you answer the question by simply looking at the function?

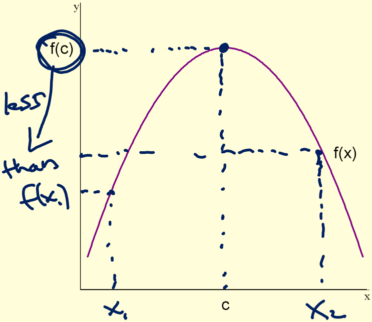
Another way to find the max/min is to graph the function:



$$f(2) = 3 - 0^2$$
$$= 3$$

$$f(4) = 3 - (4-2)^2$$
$$3 - 4$$
$$= -1$$

Generally, we can write:

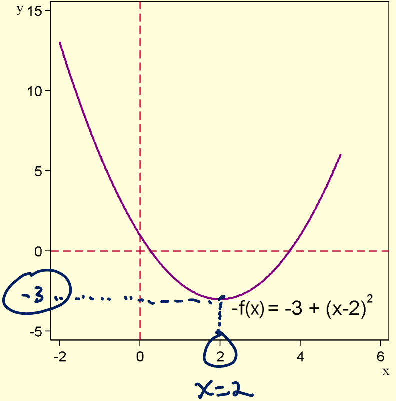


"in"

if $c \in D$ is a maximum point for f than this implies $f(x) \leq f(c)$ for all $x \in D$.

What about the negative of $f(x)$?

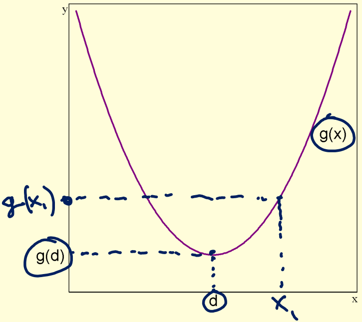
$-f(x) = -3 + (x-2)^2$



When $x = 2$ then $-f(x)$ is minimized.

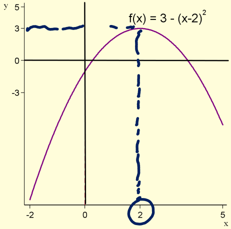
$$-f(2) = -3 + (2-2)^2$$
$$= -3$$

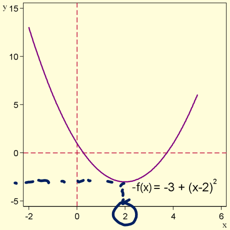
Generally, we can write:



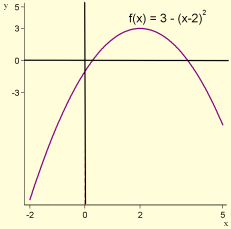
If $d \in D$ is a minimum point for g this implies $g(d) \leq g(x)$ for any $x \in D$.

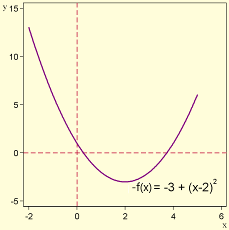
The max of $f(x)$ is equal to the min of $-f(x)$.





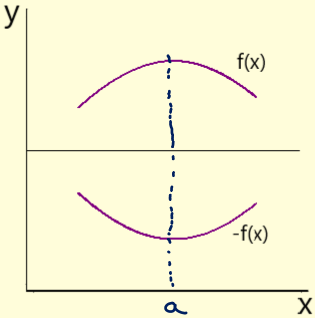
The max of $f(x)$ is equal to the min of $-f(x)$.



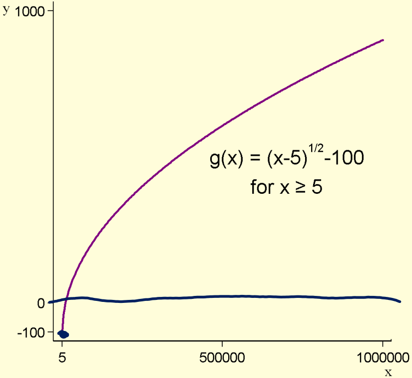


This is in general true.
If the max of $f(x)$ occurs at c, then
the min of $-f(x)$ also occurs at c.

If the max of $f(x)$ occurs at c, then the min of $-f(x)$ also occurs at c.



Find the Extreme Points for $g(x) = \sqrt{x-5} - 100$



minimized when
$x = 5$

$x < 5$, $g(x)$
is undefined

Find the Extreme Points

- Rarely are Optimization problems so easy!
- A major challenge is to correctly model a real world problem as a function. This is the goal of the economists, political scientist, and business analyst.
- Next, we introduce calculus to help us make the most effective choice.

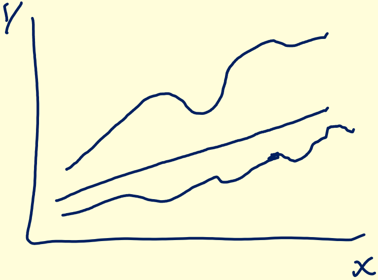
Differential Calculus in Optimization

- More complicated optimization problems require more sophisticated techniques.
- Differential calculus can help make effective use of a resource.
- Optimization with calculus works on intervals of a function/relation that are both *continuous* and *differentiable*.

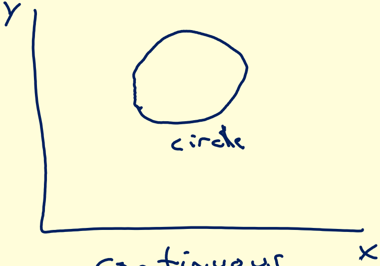
Continuous Functions

A function is **continuous** if

- You can draw the function from one end point to the other without lifting your pen from the paper.



Continuous Function

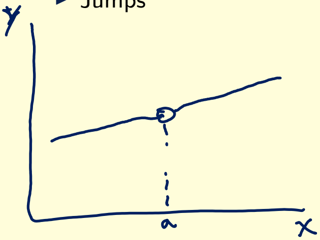


Continuous Relations (not a function)

Not Continuous Functions

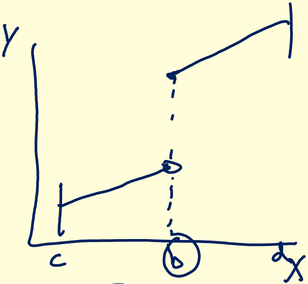
A function is not continuous if it has

- Gaps (holes)
- Jumps



Gaps/holes

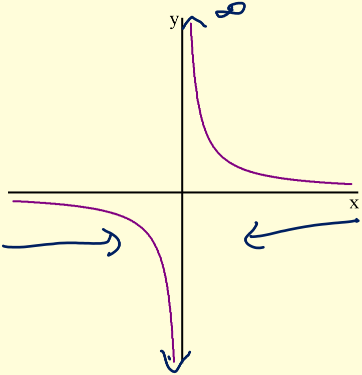
The function is discontinuous at point a.



Jump

The function is discontinuous at point b.

Not Continuous at $x = 0$



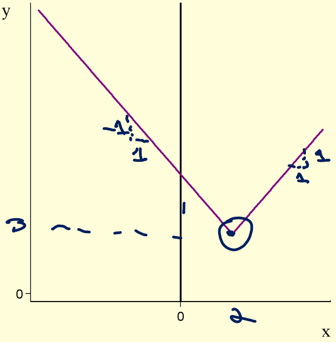
$f(x) = 1/x$ $-\infty$

$1/0$ = undefined

Non-Differentiable Points on a Continuous Function

Some functions are not differentiable at point(s):

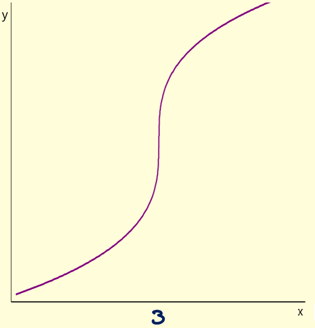
(i) $f(x) = 3 + |x - 2|$ when $x = 2$



Non-Differentiable Points on a Continuous Function

Some functions are continuous but not everywhere differentiable:

(ii) $g(x) = (x-3)^{1/3}$ if $x = 3$.



$$g(x) = (x-3)^{1/3}$$

$$g'(x) = (1/3)(x-3)^{-2/3}\ (1)$$

$$= \frac{1}{3(x-3)^{2/3}}$$

$$g'(3) = \frac{1}{(3)\,0^{2/3}}$$

$$= \frac{1}{0} \Rightarrow \text{undefined}$$

So $g'(3)$ is undefined

Interior of an Interval

- An interval is a continuous set of real numbers with two *end points* and an *interior*.
- Interval notation includes the general forms: (a, b), $[a, b]$, $[a, b)$, $(a, b]$ where $\cancel{a > b}$ $a < b$

open set

excludes a, b

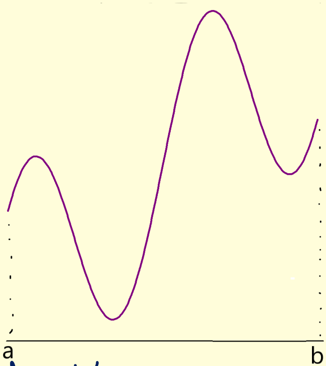
closed set

includes a, b

Interior of an Interval

- An interval is a continuous set of real numbers with two *end points* and an *interior*.
- Interval notation includes the general forms: (a, b), $[a, b]$, $[a, b)$, $(a, b]$ where $a > b$
- Extreme points are often located in the **interior of an interval**.
- For example, the interior of the interval $[a, b]$ is (a, b)

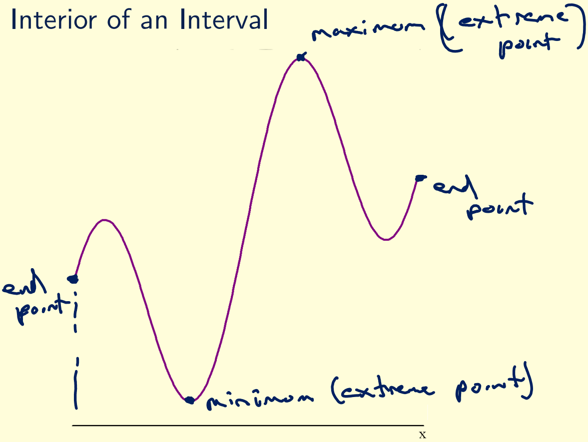
Interior of an Interval



end point   end point

The interior of the interval $[a, b]$ is (a, b) continuous

Interior of an Interval



i) Label the end points

ii) Label the extreme points

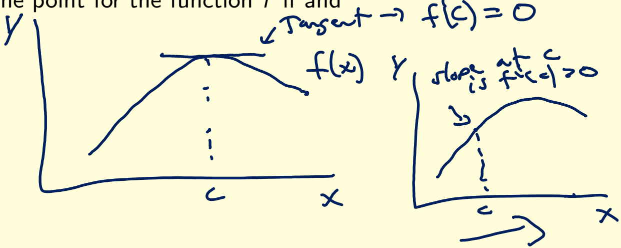
Differential Calculus in Optimization

- When the *interval* of a function or relation is *continuous* and *differentiable*, differential calculus can find the most effective use of a resource(s).
- Sometimes these effective uses are called "optimal allocations."

Interior Extreme Points and Critical Points

Restricting our analysis to continuous and differentiable intervals:

- c is an interior extreme point for the function f if and only if $f'(c) = 0$.



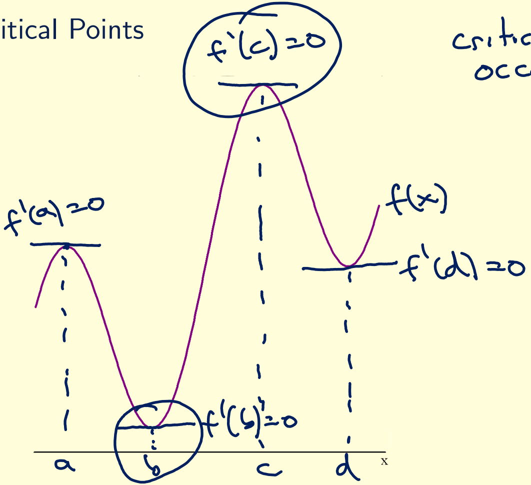
Interior Extreme Points and Critical Points

Restricting our analysis to continuous and differentiable intervals:

- c is an interior extreme point for the function f if and only if $f'(c) = 0$.
- Any x in the interior of an interval where $f'(x) = 0$ is called a **critical point**.
- For c to be an extreme point, it is necessary that $f'(c) = 0$. Otherwise, c is not an extreme point.

Critical Points

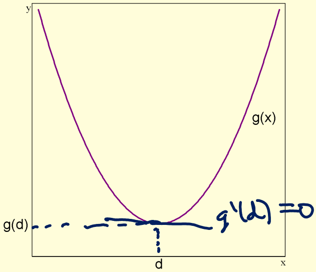
critical point
occurs wherever $f'(x) = 0$



i) Find all critical points a, b, c, d

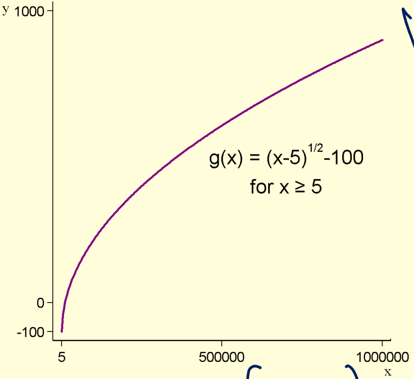
ii) Find all extreme points b, c → Both are interior extreme points

Finding an interior Extreme Point



- For d to be the maximum or minimum value, the slope of g must be zero at d: $g'(d) = 0$.

Is there an interior extreme point?



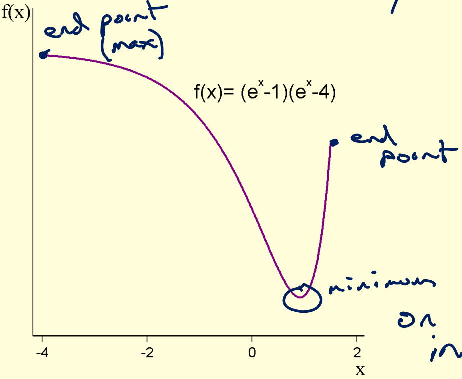
$x \in [5, \infty)$

no, there is an extreme point at the end point $x=5$ (minimum)

There is no maximum

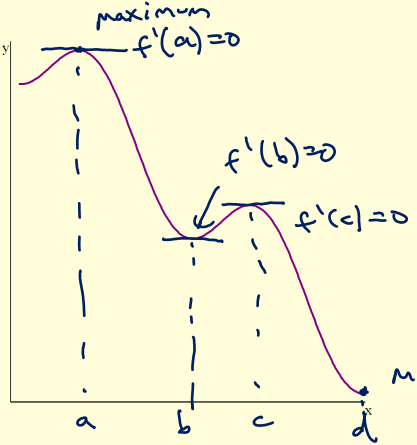
Is there an interior extreme point?

yes



on the interior

Find all critical points and Extreme Points



<u>Critical Points</u>

a, b, c

<u>Extreme Points</u>

a, d

Example: the concentration of drugs in a person's bloodstream t minutes after injection is given by the function:

$$c(t) = \frac{t}{t^2 + 4}$$

Find the time after injection at which the concentration is highest.

$c'(t) = 0$ for any interior point to be an extreme point.

$$c'(t) = \frac{(1)(t^2+4) - 2t(t)}{(t^2+4)^2} = 0$$

$$\frac{t^2 + 4 - 2t^2}{\cancel{(t^2+4)^2}} = 0$$

$$t^2 + 4 - 2t^2 = 0$$

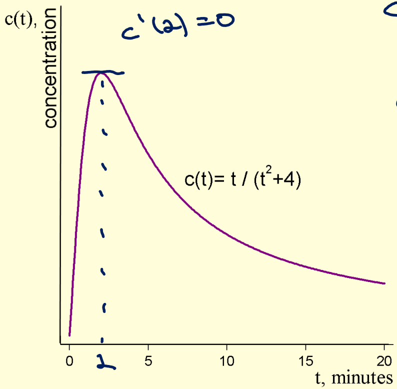
$$-t^2 + 4 = 0$$

$$t^2 = 4$$

$$t^* = 2$$

After 2 minutes, the drug in your bloodstream is at its maximum.

Concentration of drugs in a person's bloodstream



$$c(t) = \frac{t}{t^2+4}$$

$$c(2) = \frac{2}{2^2+4}$$

$$= \frac{2}{8}$$

$$= 1/4$$

Example

$$f(x) = (e^x - 1)(e^x - 4)$$

$$0 = (e^x - 1)(e^x - 4)$$

Find:

1. the values of x for which $f(x) = 0$
2. $f'(x)$
3. $\lim_{x \to -\infty} f(x)$

① $e^x - 1 = 0$

$e^x = 1$

$x = \ln 1$

$x = 0$

$e^x - 4 = 0$

$e^x = 4$

$x = \ln 4$

$x = 2\ln 2$

② $f'(x) = e^x(e^x - 4) + e^x(e^x - 1)$

$f'(x) = e^x[e^x - 4 + e^x - 1]$

$= e^x[2e^x - 5]$

Interior Extreme point? $\Rightarrow f'(x) = 0$

e^x can never be zero

$2e^x - 5 = 0 \Rightarrow e^x = 5/2$

$x = \ln(5/2) = \ln 5 - \ln 2$

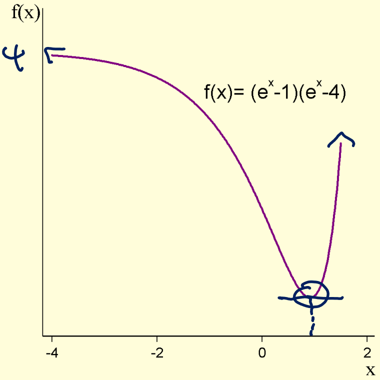
③ $\lim_{x \to -\infty} f(x) = (e^x - 1)(e^x - 4)$

$= (e^{-\infty} - 1)(e^{-\infty} - 4)$

$= (0 - 1)(0 - 4)$

$= 4$

Find the Extrema



How to Find the Global Extrema of a Function

Finite

$f(x) \leq A$

$f(x) \geq B$

[ ]

Let the function f be differentiable on the closed and bounded set $[a, b]$,

$x \in [a, b]$

How to Find the Global Extrema of a Function

Let the function f be differentiable on the closed and bounded set $[a, b]$,

1. Find all critical points in (a, b). $f'(x) = 0$

How to Find the Global Extrema of a Function

Let the function f be differentiable on the closed and bounded set $[a, b]$,

1. Find all critical points in (a, b).
2. Evaluate f at the end points, a and b.
3. Identify the largest value and smallest value of f in $[a, b]$.

calculate $f(a)$ and $f(b)$

if c is a critical point calculate $f(c)$

Find the maximum and minimum values, $x \in [-1, 3]$ of

$f(x) = (1/4)x^4 - x^3 + x^2 + 1$

① Find all critical points

$f'(x) = 0 = x^3 - 3x^2 + 2x + 0$

$0 = x(x^2 - 3x + 2)$

$0 = x(x-2)(x-1)$

$x = 0, \quad x = 2, \quad x = 1$

② $f(-1)$ and $f(3)$

$x = 0$, $x = 2$, $\boxed{x = 1}$

$f(x) = \frac{x^4}{4} - x^3 + x^2 + 1$

② $f(-1)$, $f(3) \leftarrow$ end points

Max

$f(-1) = \frac{(-1)^4}{4} - (-1)^3 + (-1)^2 + 1 = 3.25$

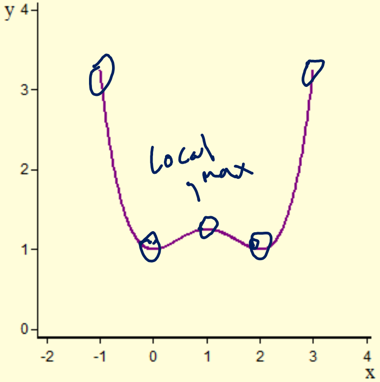
$f(3) = \frac{3^4}{4} - 3^3 + 3^2 + 1 = 3.25$

③ $f(0) = \frac{0^4}{4} - 0^3 + 0^2 + 1 = 1.00$ Min

$f(2) = \frac{2^4}{4} - 2^3 + 2^2 + 1 = 1.00$

$f(1) = \frac{1^4}{4} - 1^3 + 1^2 + 1 = 1.25$

$$f(x) = (1/4)x^4 - x^3 + x^2 + 1, \quad x \in [-1, 3]$$

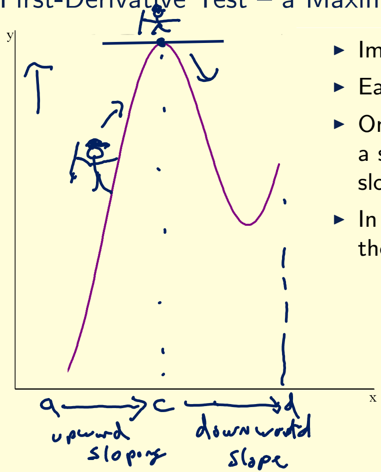


To test whether an extreme point is a maximum or minimum, use either the:

1. First-Derivative Test, or
2. Second Derivative Test

Economists traditionally prefer the second derivative test – even if the first derivative test is often easier.

First-Derivative Test – a Maximum



- Imagine you are walking up a hill.
- Each step takes you up the hill.
- Once you reach the top of the hill, a step no longer takes you up – the slope of the hill is zero.
- In fact, it might even take you down the hill.

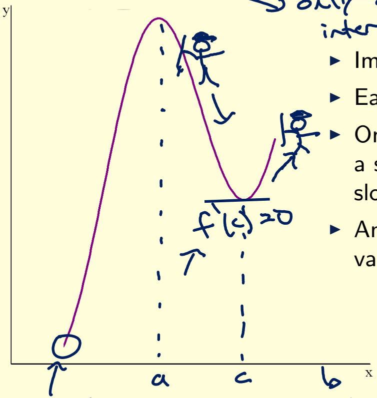
if $x < c$ if $x > d$

$f'(x) > 0$ $f'(x) < 0$

then we know that c is a max.

First-Derivative Test – a Minimum

→ only applies to critical points interior of the interval



Global minimum → end point not a critical point.

- Imagine you are walking down into a valley.
- Each step takes you down the valley-side.
- Once you reach the bottom of the valley, a step no longer takes you down – the slope of the valley-side is zero.
- An additional step takes you up the valley-side.

if for any $x \in (a, c)$, $f'(x) < 0$ and for any $x \in (c, b)$ $f'(x) > 0$ then c is a minimum.

$$c(t) = \frac{t}{t^2+4}$$ Quotient

Find the max using the first derivative test and a sign diagram.

$$c'(t) = 0 = \frac{(1)(t^2+4) - (2t)(t)}{(t^2+4)^2}$$

$$= \frac{t^2+4-2t^2}{(t^2+4)^2} = \frac{-t^2+4}{(t^2+4)^2}$$

$$0 = \frac{(2-t)(2+t)}{(t^2+4)^2} \qquad \left.\begin{array}{l} t=-2 \\ t=2 \end{array}\right\} \text{critical points}$$

Draw a **sign diagram** for the **First Derivative Test** of critical points $x = -2, 2$ where $c'(t) = \frac{(2-t)(2+t)}{(t^2+4)^2}$

t Interval from $-\infty$ to ∞

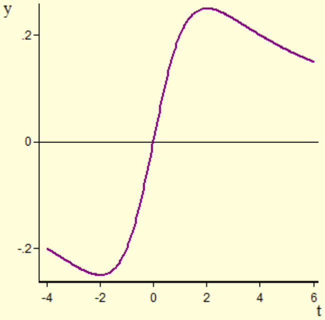
$-\infty$ $\quad -2 \quad$ 2 $\quad \infty$

$2-t$	positive	positive	negative
$2+t$	negative	positive	positive
$(t^2+4)^2$	positive	positive	positive
$c'(t)$ First Derivative Test	negative	positive	negative

Minimum $t = -2$

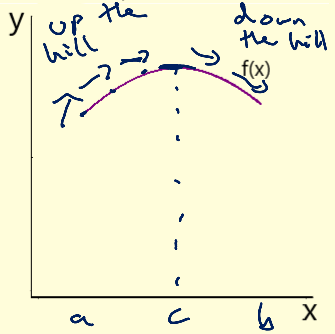
Maximum $t = 2$

$c(t) = \frac{t}{t^2+4}$,



Second-Order Conditions

Suppose that f is a function defined in an interval I and that c is a critical point for f in the interior of I.

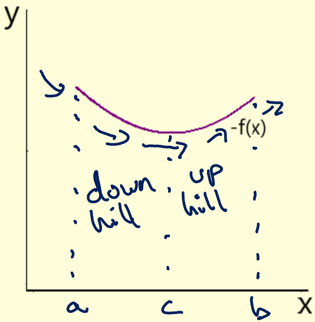


A concave function is increasing or decreasing at a decreasing rate.

Therefore, if a critical point occurs on a concave interval of the function, the critical point must be a maximum.

Second-Order Conditions

Suppose that f is a function defined in an interval I and that c is a critical point for f in the interior of I.



A convex function is increasing or decreasing at a increasing rate.

Therefore, if a critical point occurs on a convex interval of a function, the critical point must be a minimum.

Example: Consider a policy choice in which some decision maker chooses some policy x to maximize:
$h(x) = -(a-x)^2$. $h(x) = -(a-x)^2$

1. Find the critical points.
2. Determine if each critical point is a maximum or minimum using the second derivative test.

① $h'(x) = -2(a-x)(-1) = 0$

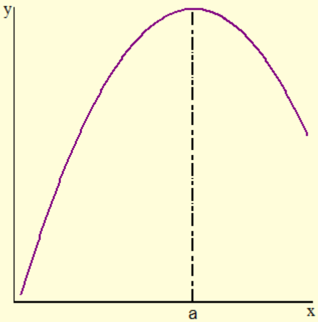
$= 2(a-x) = 0$

$x = a$

② $h'(x) = 2(a-x)$

$h''(x) = -2 \rightarrow$ The 2^{nd} derivative is negative so the function is increasing or decreasing at a decreasing date, so $x=a$ is a maximum.

$$h(x) = -(a - x)^2$$



Example: a firm's profit depends on how many microchips they produce, ~~q~~: x

profit

$$\pi(x) = -\frac{x^3}{100} + x^2 + 20x$$

1. Find the critical points.
2. Determine if each critical point is a maximum or minimum using the second derivative test.
3. Sketch the graph.

$$\pi'(x) = 0 = \underbrace{-\frac{3x^2}{100}}_{a} + \underbrace{2}_{b}x + \underbrace{20}_{c}$$

Quadratic Formula to find x

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = \frac{-3}{100}$ $b = 2$ $c = 20$

$x = -8.9$ $x = 75.5$

① $\pi'(x) = \frac{-3x^2}{100} + 2x + 20$ $\quad x = -8.9 \quad$ $x = 75.5$

② $\pi''(x) = \frac{-6}{100}x + 2 = \frac{-3x + 100}{50} = \frac{100 - 3x}{50}$

$\pi''(-8.9) = \frac{-3(-8.89) + 100}{50}$ = positive value

so $x = -8.9$ is a minimum

$\pi''(75.5) = \frac{100 - 3(75.5)}{50}$ = negative value

so $x = 75.5$ is a maximum.

$$\pi(x) = -\frac{x^3}{100} + x^2 + 20x$$