SOME INDEPENDENCE RESULTS FOR AMENABLE GROUP ACTIONS, UNIVERSAL GRAPHS, AND CONSTRUCTION SCHEMES

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Abstract

This thesis uses the technique of forcing to study consistency results in three areas:

In the first chapter, we investigate the question of whether or not an amenable subgroup of the permutation group on $\mathbb{N}$ can have a unique invariant mean on its action. In joint work with Juris Steprāns, we extend the work of Foreman in [13] and show that in the Cohen model such an amenable group with a unique invariant mean must fail to have slow growth rate and a certain weakened solvability condition.

In the second chapter, the consistency of a universal graph on ω_1 with Martin's Axiom the negation of the Continuum Hypothesis is investigated. We extend an argument of Shelah in [44] to get a partial result of the consistency of a universal graph on ω_1 with MA(Cohen), Suslin's Hypothesis, and the negation of the Continuum Hypothesis.

The third chapter is an investigation of forcing extensions answering some independence questions relating to construction schemes, which are combinatorial schemes for constructing objects with domain ω_1 introduced by Todorčević in [50]. In joint work with Fulgencio López, we show that adding ω_1 Cohen reals adds a capturing construction scheme, that it is consistent to have n-capturing construction schemes but no $(n+1)$-capturing construction schemes, and show that $\mathrm{MA}_{\omega_1}(m\text{-}Knaster)$ and n-capturing are independent if $n \leq m$ and incompatible if $n > m$.

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1 Introduction

The common theme that ties the three disparate chapters of this thesis together is the fairly general concept of independence from ZFC, and the techniques of forcing. Independence results in the foundations of mathematics started in the early 1900's as a consequence of seeking answers to Hilbert's problems. The first independence result, which provides some answers to Hilbert's second question asking if the axioms of arithmetic are consistent, is Gödel's incompleteness theorems in 1931 [16]. The incompleteness theorems show that no consistent recursively axiomatizable theory which can express arithmetic, can prove it's own consistency. Later, in 1938 [17], Gödel gave a partial answer to Hilbert's first problem which seeks an answer to the Continuum Hypothesis (or CH): whether there is no cardinality between the natural numbers and real numbers. Gödel proved that CH is consistent with Zermelo-Fraenkel set theory and the axiom of choice (or ZFC), by proving that if ZF is consistent, there is a minimal model of ZFC L, now called Gödel's constructible universe, in which CH is satisfied. Hilbert's first problem would remain unsolved until the development of forcing by Cohen in 1963 [7], whereby he constructed a model of $\neg$CH and ZFC, proving that CH is independent of ZFC. The technique of forcing became a ubiquitous and crucial technique for proving independence results throughout mathematics, and continues to be developed and fleshed out to the present.

The first major problem outside of the foundations of mathematics that was proven independent of ZFC was the Whitehead problem, which asks if every Whitehead group is free. The Whitehead problem was proven independent of ZFC by Shelah in 1974 [41]; In this paper it was proven that in L all Whitehead groups are free, but the existence of a non-free Whitehead group follows from Martins Axiom and $\neg$CH.

Martins Axiom (or MA), introduced by Martin and Solovay in 1970 [32] is the first axiom characterized as a "Forcing Axiom", which informally is an axiom that specifies the existence of limited forcing constructions for a particular class of forcings. They realized that several prior forcing constructions had similar frameworks, and were able to reduce the arguments down to following from this common axiom. MA and forcing axioms in general have many interesting consequences; For an example, if MA holds then any union of less than continuum many null sets of reals is null.

Each subject of the chapters of this thesis includes demonstrations of the consequences of MA: In the first chapter we discuss that under $\mathfrak{u} = \mathfrak{p}$, which is implied by MA, it was shown by Foreman in [13] that there is a locally finite subgroup of permutations of $\mathbb{N}$ with a unique invariant mean on it's action. Later, under the same assumptions, it was shown by Raghavan and Steprāns in [38] that there is a locally solvable subgroup of permutations of $\mathbb{N}$, all of whose elements have infinite order, with a unique invariant mean on it's action

The second chapter seeks to investigate what forcing axioms may entail about the existence of universal structures. We seek to know whether MA implies there is no universal graph on ω_1. Although I would conjecture that this is not the case, this question proves difficult to answer, and only a partial result that a universal graph on ω_1 can coexist with a smaller fragment of MA (namely MA(Cohen), Suslin's Hypothesis, and $\neg$CH) is obtained.

In the third chapter, with Fulgencio López, we relate the consistency of a hierarchy of forcing axioms, which are weakenings of MA, to capturing construction schemes. We show that $\mathrm{MA}_{\omega_1}(m\text{-}Knaster)$ and n-capturing are independent if $n \leq m$ and incompatible if $n > m$.

MA was proven consistent with ZFC and $\neg$CH by Solovay and Tennenbaum in 1971 [48]. Afterwards, forcing axioms for progressively larger classes of forcings were considered and proven consistent with ZFC until a maximal forcing axiom (in the sense that no larger class of forcings can have a consistent forcing axiom), Martins Maximum (or MM), was proven to be consistent with ZFC (by Foreman, Magidor, and Shelah in 1988 [14]) assuming the existence of a supercompact cardinal, which is stronger than assuming the consistency of ZFC. Most of the important consequences of MM follow from a weaker similar forcing axiom, the Proper Forcing Axiom (or PFA), which was proven consistent with ZFC by Baumgartner in 1983 [10], also from the existence of a supercompact cardinal.

All initial proofs of consistency of forcing axioms with ZFC required the technique of iterated forcing, which was developed to force a model of MA and $\neg$CH in [48]. In this result, Iterated forcing was applied with finite support. Later, the development of proper forcing by Shelah in 1980 [43] followed, with the result that countable iterations of proper forcings are proper.

All of the new results of this thesis that rely on iterated forcing, use finite supports: In the first chapter a finite support iteration of Cohen forcing extending a model of CH is used. This model had shown prior utility for investigating unique invariant means since Foreman's result, which shows that in this model no locally finite subgroup of permutations of $\mathbb{N}$ can have a unique invariant mean on it's action [13]. The results of this author and Steprāns in the first chapter show that in this model any amenable subgroup of permutations of $\mathbb{N}$ must fail to have slow growth rate and a certain weakened solvability condition, but questions still remain about this common forcing extension and unique invariant means.

The second chapter uses an ad hoc finite support iteration for getting the consistency of a universal graph on ω_1 with MA(Cohen), Suslin's Hypothesis, and $\neg$CH. The core portion of this forcing iteration for adding the universal graph was created by Shelah to get the consistency of a universal graph on ω_1 without CH [44]. The proper countable support iterations of Miller forcings followed by P-ideal dichotomy forcings in the work done independently by Steprāns and Shelah in [46] can also be augmented to model $\neg$CH, Suslin's Hypothesis, MA(Cohen), and "there is a universal graph on ω_1", however it is unclear whether a universal function on ω_1 exists in this model as it does for the result of the second chapter. This forcing iteration requires countable support, as it is necessary to control which reals are added, and finite support iterations always add Cohen reals at limit steps.

Finally, a result of the third chapter with Fulgencio López is that the forcing extension by iterating Cohen forcing with finite support up to at least ω_1, contains a capturing construction scheme.

For more detail on the history of forcing and independence results, the reader can see [27] or [21].

2 Uniquess of Means for Amenable Group Actions

The results of this chapter are joint work with Juris Steprāns.

2.1 Introduction

2.1.1 Overview

The study of amenability originated in the study of paradoxical decompositions of $\mathbb{R}^3$ by Banach and Tarski, based on work by Hausdorff, which showed that there can be no isometry invariant measure (finitely additive probability measure) of $\mathbb{R}^3$ which measure all subsets of $\mathbb{R}^3$ [1]. In light of this Von Neumann observed that a total measure on a group invariant with respect to the groups action on itself, can be transferred into an invariant measure on any set this group acts on, and defined this as the notion of amenability [35]. He conjectured that amenability is equivalent to containing a copy of $\mathbb{F}_2$, which was later proven false by Ol'shanskii [36].

Given an amenable group G, the question of the number of invariant measures with specific additional properties were explored in Rosenblatt and Talagrand in [40]. Rosenblatt and Talagrand note that their results extend to general actions of a group G on a set X provided that $|G| \leq |X|$. They then asked whether there may even be a unique G-invariant mean if $|G| > |X|$. As a partial answer, they also showed in [40] that this is not the case if G is nilpotent. Krasa [26] later extended this result to apply to solvable groups. For arbitrary group actions it suffices to look at subgroups of the symmetric group, and since we need to look at actions on infinite sets, the simplest such groups to investigate are amenable subgroups of the symmetric group on $\mathbb{N}$. This chapter focuses on the question of if an amenable subgroup of the symmetric group on $\mathbb{N}$ can have a unique invariant mean on its action. The existence of such a group is still open and has proven difficult to decide, and therefore could offer further interesting insights into amenable groups in addition to Rosenblatt and Talagrand's initial motivations.

Partially answering the question of Rosenblatt and Talagrand, Yang showed in [54] that, assuming the Continuum Hypothesis, there is an amenable subgroup G of the full symmetric group on $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique invariant mean. Later, Foreman showed in [13] that under various other set theoretic hypotheses weaker than the Continuum Hypothesis, there is an amenable subgroup G of the full symmetric group on $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique invariant mean. While Yang's mean attains all values in the interval $[0, 1]$, in Foreman's construction the unique invariant mean is an ultrafilter. Moreover, Foreman also showed in [13] that in the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis, there are no such groups that are locally finite. The significance of this results is that the groups constructed by both Yang and Foreman are amenable by virtue of being locally finite. Nevertheless it is natural to ask whether there is any amenable group with a faithful action on $\mathbb{N}$ that has unique invariant mean in the Cohen model. It is of interest to know whether there are amenable groups, that are not locally finite (in some non-trivial sense) acting with unique invariant mean. Raghavan and Steprāns [38] under the same hypothesis as Foreman, constructed a locally solvable subgroup of the full symmetric group on $\mathbb{N}$, with a generating set whose elements all have infinite order, and whose natural action on $\mathbb{N}$ has a unique invariant mean.

It will be shown in this chapter that in the Cohen model, any group with a faithful action on $\mathbb{N}$ and a unique invariant mean must fail to have slow growth rate and weakened solvability conditions that will be defined precisely in §2.1.3. It therefore follows from these results that any amenable group acting faithfully on $\mathbb{N}$ with a unique invariant mean, must have exponential growth and not be ω_2-solvable. Indeed it will be shown that the required growth is more than exponential.

2.1.2 Background and Preliminaries

To begin introducing this chapter in detail, first recall basic definitions for a group acting on a set:

Definition 2.1.1. Recall that a (left) action of a group G on the set X is a mapping $\cdot$ from $G \times X$ to X satisfying the associative law $g(h \cdot s) = (gh) \cdot s$ and for which the identity element e satisfies $e \cdot s = s$. A group G is called free if for all $g \in G$, $((\exists x \in X)\ gx = x) \longrightarrow g = e$, and faithful if for all $g \in G$, $g \neq e \longrightarrow ((\exists x \in X)\ gx \neq x)$. The orbit of a point $x \in X$ under the action of G is $O_x = \{gx : g \in G\}$, which clearly forms an equivalence class under $x \sim y \longleftrightarrow O_x = O_y$.

In this section, we begin the chapter with some history on the notion of amenability and it's development.

Definition 2.1.2. For a group G acting on a set X, and $A, B \subseteq X$, A and B are G-equidecomposable, written as $A \equiv_G B$, if there are partitions $A_0 \sqcup ... \sqcup A_n = A$ $B_0 \sqcup ... \sqcup B_n = B$ and $g_0, ..., g_n \in G$ such that $B_i = g_i A_i$ for each i.

Proposition 2.1.3. *Equidecomposability is an equivalence relation.*

Proof. To see equidecomposability is an equivalence relation, show $\equiv_G$ is transitive: Let $A \equiv_G B \equiv_G C$ with the first witnessed by $A_0, ..., A_n, B_0, ..., B_n, g_0, ..., g_n$, and the second witnessed by $B'_0, ..., B'_k, C_0, ..., C_k, h_0, ..., h_k$. Set $A_{i,j} = g_i^{-1}(B_i \cap B'_j),\ g_{i,j} = g_i,\ h_{i,j} = h_j$, and

$$C'_{i,j} = h_{i,j} g_{i,j} A_{i,j} = h_j(B_i \cap B'_j) = C_j \cap h_j B_i.$$

Noting that

$$C = \bigsqcup_j C_j = \bigsqcup_j C_j \cap h_j B = \bigsqcup_j \left(C_j \cap \bigsqcup_i h_j B_i \right) = \bigsqcup_{i,j} C_j \cap h_j B_i$$

the proof is done.

□

Definition 2.1.4. A set X is G-paradoxical if there are disjoint subsets $A, B \subseteq X$ with $A \equiv_G B \equiv_G X$

Proposition 2.1.5. *The group $\mathbb{F}_2$, which denotes the free group on two generators, acting freely on a set X is X-paradoxical. In particular since $\mathbb{F}_2$ acts freely on itself, it is $\mathbb{F}_2$-paradoxical.*

Proof. Let $\langle a, b \rangle = \mathbb{F}_2$, E be the set of representatives of orbits of $\mathbb{F}_2$, and for $x \in \mathbb{F}_2$ let

$$W_x = \{wr : r \in E,\ w \text{ is a word beginning with x}\}.$$

Note that by freeness, $W_a, W_{a^{-1}}, W_b, W_{b^{-1}}$ are disjoint. Set $A = W_a \sqcup W_{a^{-1}}$, $B = W_b \sqcup W_{b^{-1}}$, and finish by noting

$$X = a^{-1} W_a \sqcup W_{a^{-1}} = b^{-1} W_b \sqcup W_{b^{-1}}.$$

□

Let E be the group of isometries of $\mathbb{R}^3$.

Theorem 2.1.1 ([52] Theorem 2.1)**.** *There are two rotations which fix the origin in E which generate $\mathbb{F}_2$ as a subgroup.*

This leads us to the Hausdorff-Banach-Tarski paradox:

Theorem 2.1.2 (Hausdorff-Banach-Tarski [1])**.** *The group of isometries of $\mathbb{R}^3$ with it's natural action on $\mathbb{R}^3$ is paradoxical.*

Proof. Let $a, b \in E$ be rotations about lines that fix the origin which generate $\mathbb{F}_2$. Since the compositions of rotations are rotations, $\mathbb{F}_2$ is interpreted a countable set of rotations which fix the origin. Proposition 2.1.5 cannot be applied directly since rotations have fixed points and so $\mathbb{F}_2$ does not act freely. Let L be the countable set of fixed lines which are fixed by some rotation in $\mathbb{F}_2$, and let $K = \bigcup\{x(l) : x \in \mathbb{F}_2,\ l \in L\}$, which is a union of a countable set of lines intersecting the origin. Proposition 2.1.5 gives us that since $\mathbb{F}_2 \subseteq E$ acts freely on $\mathbb{R}^3 \setminus K$, $\mathbb{R}^3 \setminus K$ is E-paradoxical, and it suffices to show $\mathbb{R}^3 \setminus K \equiv_E \mathbb{R}^3$.

If l, l' are different lines which intersect the origin, then there are only countably many rotations r fixing the origin such that for some $n < m$, $r^n(l \cup l') \cap r^m(l \cup l') \neq \{0\}$, and thus there are only countably many rotations r fixing the origin such that for some $n < m$, $r^n(K) \cap r^m(K) \neq \{0\}$. Pick r a rotation of infinite order fixing the origin with for all $n < m$, $r^n(K) \cap r^m(K) = \{0\}$, and let $D = \bigcup_{n \in \omega} r^n(K)$. We have that $\mathbb{R}^3 \setminus K \equiv_E \mathbb{R}^3 \setminus \{0\}$ since $r(D \setminus \{0\}) = D \setminus K$ and so

$$\mathbb{R}^3 \setminus \{0\} = (D \setminus \{0\}) \sqcup (\mathbb{R}^3 \setminus D) = r(D \setminus \{0\}) \sqcup (\mathbb{R}^3 \setminus D) = \mathbb{R}^3 \setminus K.$$

Similarly, picking r a rotation of infinite order with for all $n < m$, $r^n(\{0\}) \cap r^m(\{0\}) = \emptyset$, and letting $D = \bigcup_{n \in \omega} r^n(\{0\})$, gives us $\mathbb{R}^3 \equiv_E \mathbb{R}^3 \setminus \{0\}$ since

$$\mathbb{R}^3 = D \sqcup (\mathbb{R}^3 \setminus D) = r(D) \sqcup (\mathbb{R}^3 \setminus D) = \mathbb{R}^3 \setminus \{0\}.$$

Therefore $\mathbb{R}^3 \equiv_E \mathbb{R}^3 \setminus K$, and $\mathbb{R}^3 \setminus K$ is E-paradoxical, so $\mathbb{R}^3$ is E-paradoxical.

□

The definition of paradoxical can be thought of in relation to measures.

Definition 2.1.6. A mean μ on a (discrete) set X is a finitely additive $[0, 1]$ valued nontrivial measure, measuring all subsets of X; This means that $\mu : 2^X \longrightarrow [0, 1]$ and satisfies:

1. if $A, B \subseteq X$ are disjoint then $\mu(A \cup B) = \mu(A) + \mu(B)$,
2. $\mu(X) = 1$.

A (discrete) group G is amenable if there is a mean μ on G which is (left) invariant under G. Left invariant meaning $(\forall A \subseteq G)(\forall g \in G),\ \mu(gA) = \mu(A)$.

We say that the action of G on X is an amenable action if there is a mean ν on X invariant under the action of G: $(\forall A \subseteq X)(\forall g \in G),\ \nu(gA) = \nu(A)$.

There are many other useful equivalent definitions of amenability.

Theorem 2.1.3. *The following are equivalent for a group G :*

1. *G is amenable*
2. *There is $\mu \in \ell^\infty(G)^*$ with*
   (a) *$f \geq 0 \Rightarrow \mu(f) \geq 0$*

(b) $||\mu|| = 1$

(c) μ *is G-invariant* ($\forall g \in G,\ \mu({}_gf) = \mu(f)$*, where* ${}_gf(x) = f(g^{-1}x)$*).*

3. *For all* $\varepsilon > 0$ *and all finite* $H \subseteq G$ *there is finite* $F \supseteq H$ *with* $(\forall h \in H)\ |hF \triangle F|/|F| < \varepsilon$.

4. *there is a Følner net* $(F_H)_{H\in[G]^{<\omega}}$ *of finite subsets of* G*, such that*

$$(\forall H, H',\ H \cup F_H \subseteq F_{H\cup H'})(\forall \varepsilon > 0)(\forall H)(\exists H' \supseteq H)(\forall H'' \supseteq H')(\forall h \in H)\ |hF_{H''} \triangle F_{H''}|/|F_{H''}| < \varepsilon.$$

A $\mu \in \ell^\infty(G)^*$ satisfying the properties of 2 in the above theorem, is just an equivalent interpretation of a finitely additive invariant measure on G, and so is also called a mean on G.

Note that for any amenable group G with mean μ acting on a set X, then for each $x \in X$ and $A \subseteq X$

$$\nu_x(A) = \mu(\{g \in G : gx \in A\}) \tag{2.1.1}$$

is a mean on X invariant under the action of G. Also, for any two G invariant means μ_0, μ_1 on X, and $a, b \in \mathbb{R}$ with $a + b = 1$,

$$\nu = a\mu_1 + b\mu_2 \tag{2.1.2}$$

is a G invariant mean on X.

Theorem 2.1.4 (Tarski)**.** *A group G acting on X has an amenable action if and only if it is not paradoxical.*

As a result of the above theorem applied to the Hausdorff-Banach-Tarski paradox, there are no measures on $\mathbb{R}^3$ which measure all subsets and are measure invariant with respect to the application of an isometry (or in particular, with respect to the application of some sequence of rotations). The general exploration of what groups and actions can have a total invariant measure is the study of amenability for discrete groups. Amenability is a common condition; All groups that are finite, abelian, or solvable, are amenable. Von Neumann conjectured [35] containing $\mathbb{F}_2$ is equivalent to amenability, since any group containing $\mathbb{F}_2$ cannot be amenable and all such amenable groups known did not contain $\mathbb{F}_2$, until it was proved by Ol'shanskii not to be the case in [36], where he showed that the Tarski monster group is not amenable even though it does not contain $\mathbb{F}_2$.

Notation 2.1.1. For a set X let

$$\mathrm{Sym}(X) = \{f \in X^X : f \text{ is a bijection}\}.$$

$\mathrm{Sym}(X)$ is a group under function composition, and is called the symmetric group on X. $\mathrm{Sym}(X)$ acts on X with its natural action: For $g \in \mathrm{Sym}(X), x \in X$ define $gx = g(x)$.

Note the following:

Remark 2.1.7. Every group acting faithfully on X can be isomomorphically embedded as a subgroup of $\mathrm{Sym}(X)$.

One can ask if G acting on X with an amenable action is amenable, but this is false even if we require that the action is free and transitive: If $\mathcal{U}$ is a nonprinciple ultrafilter on ω (or on any infinite set), then

$$G = \{g \in \mathrm{Sym}(\omega) : \forall A \subseteq \omega,\ g(A) \in \mathcal{U} \text{ iff } A \in \mathcal{U}\}$$

has an amenable action with mean $\mu = I_{\mathcal{U}}$. The group G cannot be amenable since $\mathbb{F}_2 \subseteq Sym(I) \subseteq G$ for any infinite $I \notin \mathcal{U}$.

An important observation about the above example is that the mean μ is the unique invariant mean under the action of G. One can ask if there can be an amenable group with a unique mean on its free action. Because of the mean defined by equation (2.1.1), such actions must also be transitive.

Given an amenable group G, the question of the number of invariant measures with specific additional properties were explored in Rosenblatt and Talagrand in [40]. Typical of the results they obtained there is the following:

Theorem 2.1.8 (Rosenblatt and Talagrand)**.** *For an infinite amenable group G acting on itself in the natural way the following are equivalent:*

1. *there is a left and right G-invariant mean of G that is not inversion invariant*
2. *there are $2^{2^{|G|}}$ such means that are mutually singular such that the failure of inversion invariance is witnessed by the same set.*

A natural extension of their results in [40] would be to the case of a group acting on an arbitrary set and Rosenblatt and Talagrand note that their results extend to general actions of a group G on a set X provided that $|G| \leq |X|$. They then asked whether there may even be a unique G-invariant mean if $|G| > |X|$. As a partial answer, they also showed in [40] that this is not the case if G is nilpotent. Krasa [26] later extended this result to apply to solvable groups. By Remark 2.1.7 for arbitrary group actions it suffices to look at subgroups of the symmetric group, and since we need to look at actions on infinite sets, the simplest such groups to investigate are amenable subgroups of $\mathrm{Sym}(\omega)$. This chapter focuses on the question of if an amenable subgroup of $\mathrm{Sym}(\omega)$ can have a unique invariant mean on its action. Partially answering the question of Rosenblatt and Talagrand, Yang showed in [54] that, assuming the Continuum Hypothesis, there is an amenable subgroup G of the full symmetric group on $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique invariant mean. Yang defined the concept of thickness:

Definition 2.1.9. Let $r \in [0, 1]$. A subset X of ω is r-thick (with respect to G) if and only if

$$(\forall H \in [G]^{<\omega})(\exists n < \omega)\ |\{h \in H : hn \in X\}|/|H| \geq r.$$

Lemma 2.1.10 ([54])**.** *If G is amenable, then a set $X \subseteq \omega$ is r-thick if and only if there is a mean μ on ω with $\mu(X) \geq r$.*

Corollary 2.1.11. *Let G be amenable and define $\mathbf{th} : \mathcal{P}(\omega) \to [0,1]$ by*

$$\mathbf{th}(X) = \sup\{r : X \text{ is } r\text{-thick}\}.$$

The following are equivalent:

1. $\mathbf{th}$ *is a mean on ω.*

2. $\mathbf{th}$ *is the unique mean on ω.*

3. *There is a unique mean on ω.*

Note that $\mathbf{th}$ is always G-invariant since r-thickness is G-invariant, so the only thing that can change is the finite additivity.

Later Foreman showed in [13] that under various other set theoretic hypotheses weaker than the Continuum Hypothesis, there is an amenable subgroup G of the full symmetric group on $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique invariant mean. While Yang's mean attains all values in the interval $[0,1]$, in Foreman's construction the unique invariant mean is an ultrafilter.

Moreover, Foreman also showed in [13] that in the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis, there are no such groups that are locally finite. The significance of this results is that the groups constructed by both Yang and Foreman are amenable by virtue of being locally finite. Nevertheless it is natural to ask whether there is any amenable group with a faithful action on $\mathbb{N}$ that has unique invariant mean in the Cohen model. Indeed, there is no model currently known in which there is no such amenable group. In this context it is of interest to know whether there are amenable groups, that are not locally finite (in some non-trivial sense) acting with unique invariant mean. The following provides some information.

Theorem 2.1.12 (Raghavan & Steprāns [38])**.** *Assuming there is an ultrafilter on $\mathbb{N}$ generated by a tower, there is a subgroup G of the full symmetric group on $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique invariant mean and that has a generating set all of whose elements have infinite order. The group is a solvable extension of a locally finite group and, hence, amenable.*

It will be shown in this chapter that in the Cohen model, any group with a faithful action on $\mathbb{N}$ and a unique invariant mean must fail to have slow growth rate and weakened solvability conditions that will be defined precisely in §2.1.3. While it is known (p. 21-22 in [37]) that locally solvable groups and groups with subexponential growth, are amenable, there are examples of amenable groups which have neither of those properties. The Basilica group [2] is an example of such a group, but since it is countable its natural action on $2^{<\omega}$ cannot have a unique invariant mean by the result of Foreman [13] that any analytic group of permutations of a countable set cannot have a unique invariant mean. However, this does not rule out the possibility that a group built using the Basilica group locally might not provide an absolute example of an action with a unique invariant mean.

2.1.3 Growth conditions

Definition 2.1.13. Let G be a group and $S \subseteq G$ a finite subset. Define $\gamma_G^S(n)$ to be the cardinality of the set

$$\{s_1 \cdot s_2 \cdot \ldots \cdot s_k \mid k \leq n \text{ and } (\forall i \leq k)\ s_i \in S\}.$$

If G is generated by S and there are d and c in $\mathbb{N}$ such that $\gamma_G^S(n) \leq cn^d$ for all n, then G is said to have polynomial growth. If $\lim_{n\to\infty}(\gamma_G^S(n))^{1/n}$ is greater than 1, then G is said to have exponential growth, and otherwise if $\lim_{n\to\infty}(\gamma_G^S(n))^{1/n} \leq 1$ (or equivalently $\lim_{n\to\infty}(\gamma_G^S(n))^{1/n} = 1$) the group is said to have subexponential growth. Define an arbitrary group to have polynomial, exponential, or subexponential growth, if all of its finitely generated subgroups have at most the corresponding growth.

If

$$\limsup_{S\in[G]^{<\omega}} \frac{\lim_{n\to\infty}(\gamma_G^S(n))^{1/n}}{|S|} > 0,$$

then G is said to have ultra-exponential growth.

It is known that finitely generated groups with subexponential growth, or solvable groups, are amenable. Since directed limits of amenable groups are amenable, it follows that a group with subexponential growth or local solvability is amenable (p. 14, 21-22 in [37]).

The above definitions of growth can be destroyed by a direct product with a countable group with large growth, hence, for uncountable groups the following is more useful.

Definition 2.1.14. For $\gamma : \omega \longrightarrow \omega$, a finite subset H of a group G will be said to satisfy the γ-growth condition if $\gamma_G^H \leq^* \gamma$, which means that $\gamma_G^H(n) \leq \gamma(n)$ for all but finitely many $n \in \mathbb{N}$.

For functions $\gamma_j : \omega \longrightarrow \omega$ and $m \in \mathbb{N}$, an uncountable group G satisfies the m-$\{\gamma_j\}_{j\in\omega}$-$\kappa$-$\lambda$-growth condition if for every family $\{H_\xi\}_{\xi\in\kappa}$ of pairwise disjoint subsets of G of cardinality m, there is $S \in [\kappa]^\lambda$, such that for all infinite $B \in [S]^{<\lambda}$ there is some j where for all k and all $A \in [B]^k$, $\bigcup_{\xi\in A} H_\xi$ satisfies the γ_j-growth condition. An uncountable group G is said to satisfy the $\{\gamma_j\}_{j\in\omega}$-$\kappa$-$\lambda$-growth condition if it satisfies the m-$\{\gamma_j\}_{j\in\omega}$-$\kappa$-$\lambda$-growth condition for all $m \in \mathbb{N}$.

Definition 2.1.15. Recall that for a group G the derived series $G^{(\xi)}$ for ordinals ξ is defined by setting $G^{(0)} = G$, setting $G^{(\xi+1)}$ to be the commutator group $[G^{(\xi)}, G^{(\xi)}]$ and, if ξ is a limit ordinal, letting

$G^{(\xi)} = \bigcap_{\eta\in\xi} G^{(\eta)}$. A group is solvable if there is some $n \in \omega$ such that $G^{(n)}$ is trivial. For an arbitrary subset $H \subseteq G$ define $H^{[0]} = H$ and let $H^{[n+1]} = \dot{[}H^{[n]}, H^{[n]}\dot{]}$ be the set of commutators formed from $H^{[n]}$ rather than the commutator subgroup; in other words $H^{[n+1]} = \{[g,h] \mid g,h \in H^{[n]}\}$. Note that if $H \subseteq G$ then $H^{[n]} \subseteq G^{(n)}$. For any subset H of a group G the notation $\langle H\rangle$ will be used to denote the subgroup of G generated by H.

The following definition is a weakening of the notion of solvability.

Definition 2.1.16. A group G will be called (κ, λ, m)-solvable if for every family $\{H_\xi\}_{\xi\in\kappa}$ of pairwise disjoint subsets of G of cardinality m there is $S \in [\kappa]^\lambda$ such that for all $B \in [S]^{<\lambda}$ there is $A \in [B]^{\aleph_0}$ such that $\left\langle \bigcup_{\xi\in A} H_\xi \right\rangle$ is solvable. Call a group (κ, λ)-solvable if it is (κ, λ, m)-solvable for every $m \in \mathbb{N}$.

In determining if this is the correct Ramsey theoretic analogue for solvability, the following question would need to be answered:

Question 2.1.17. If G is a group of size $\aleph_2$ and

$$(\forall H \in [G]^{\aleph_2})(\exists S \in [H]^{\aleph_1})(\forall B \in [S]^{\aleph_0})(\exists A \in [B]^{\aleph_0})\ \langle A\rangle \text{ is solvable}$$

does it follow that

$$(\forall H \in [G]^{\aleph_2})(\exists S \in [H]^{\aleph_1})\ \langle S\rangle \text{ is solvable?}$$

The above question does have a positive answer if "solvable" is replaced with "abelian"; in other words, if G is such that

$$(\forall H \in [G]^{\aleph_2})(\exists S \in [H]^{\aleph_1})(\forall B \in [S]^{\aleph_0})(\exists A \in [B]^{\aleph_0})\ \langle A\rangle \text{ is abelian} \tag{2.1.3}$$

then

$$(\forall H \in [G]^{\aleph_2})(\exists S \in [H]^{\aleph_1})\ \langle S\rangle \text{ is abelian.}$$

This follows from a standard application of the Dushnik-Miller theorem [11]: From (2.1.3) it follows that for any $H \in [G]^{\aleph_2}$ there is $S \in [H]^{\aleph_1}$ such that for every infinite $B \subseteq S$ there is some infinite $A \subseteq B$ such that any two elements of A commute. Define a colouring on $[S]^2$ by sending a pair to 0 if its elements commute and to 1 otherwise. By the Dushnik-Miller Theorem there is either an uncountable homogeneous set for this colouring of colour 0 or an infinite homogeneous set of colour 1. Since the second alternative is ruled out by the choice of S, it must be the case that there is an uncountable abelian subgroup of S.

2.2 Unique means in the Cohen model

This section will amplify Foreman's argument of Theorem 4.1 from [13] showing that there are no locally finite groups acting on $\mathbb{N}$ with a unique invariant mean in the model obtained by adding $\aleph_2$ Cohen reals.

It is supposed that the ground model V satisfies the Continuum Hypothesis. Let $\mathbb{P}$ be the partial order for adding $\aleph_2$ Cohen reals represented as all finite functions from $\omega_2 \times \omega$ to 2 ordered by inclusion and let $\mathbb{G}$ be a $\mathbb{P}$ name for the generic subset of $\mathbb{P}$. Let Γ be name for $\bigcup \mathbb{G}$. The argument begins by assuming that there is a $\mathbb{P}$-name for a subgroup G of the symmetric group on $\mathbb{N}$ and a name $\mathfrak{m}$ such that

$$1 \Vdash_{\mathbb{P}} \text{“}\mathfrak{m} \text{ is the unique mean invariant under the natural action on } \mathbb{N}.\text{”}$$

Notation 2.2.1. For any set of permutations H of $\mathbb{N}$ and $n \in \mathbb{N}$ let $H\langle n \rangle = \{h(n) \mid h \in H\}$.

For each $\xi \in \omega_2$ let $c_\xi = \{i \in \omega \mid \Gamma(\xi, i) = 0\}$ be the ξ^{th} Cohen real. Either $\aleph_2$ many Cohen reals have measure less than 1 or $\aleph_2$ many of their complements do, so by symmetry it can be assumed the first case holds. Using Lemma 2.1.10 and the uniqueness of the mean, there are $\aleph_2$ many $\xi \in \omega_2$ for which there is a finite $H \subseteq G$ with $H\langle n \rangle \not\subseteq c_\xi$ for each $n \in \mathbb{N}$. Using the Continuum Hypothesis, a Δ-system argument can then be used to find $\{(D_\eta, f_\eta, H_\eta, \xi(\eta))\}_{\eta < \omega_2}$ such that for $\eta < \omega_2$:

1. D_η is a countable subset of ω_2 with $\xi(\eta) \in D_\eta$,
2. if $\mathbb{D}_\eta$ is defined to be the partially ordered subset of $\mathbb{P}$ whose conditions have support in $D_\eta \times \omega$, then $f_\eta \in \mathbb{D}_\eta$ and H_η is a $\mathbb{D}_\eta$-name,
3. there is a countable $D \subseteq \omega_2$ such that $\{D_\zeta\}_{\zeta < \omega_2}$ is a Δ-system with root D,
4. if $\mathbb{D}$ is defined to be the partially ordered subset of $\mathbb{P}$ whose conditions have support in $D \times \omega$ then there is $f \in \mathbb{D}$ such that $f_\eta \restriction D \times \omega = f$ for each η,
5. there is $T \in \mathbb{N}$ not depending on η such that $f_\eta \Vdash_{\mathbb{D}_\eta}$ “$|H_\eta| = \check{T}$”,
6. for all $n \in \mathbb{N}$, $f_\eta \Vdash_{\mathbb{D}_\eta}$ “$H_\eta\langle \check{n} \rangle \not\subseteq c_{\xi(\eta)}$”, and
7. there is a $\mathbb{D}$ name H such that $f_\eta \Vdash_{\mathbb{D}_\eta}$ “$H_\eta \cap V[\mathbb{D} \cap \mathbb{G}] = H$” for each η.

Note that $\mathbb{D}_\eta$ is the forcing for adding a single Cohen real. Without loss of generality, by arguing in the model $V[\mathbb{G} \cap \mathbb{D}]$, it can be assumed that $D = \varnothing$. Also, by adding functions to H_η we may assume T is arbitrarily large.

Definition 2.2.1. Given any $\mathbb{P}$ name for a subgroup $\bar{H}$ of G and $f \in \mathbb{P}$, let $A(\bar{H}, f, k, m)$ be the following statement:

$$(\forall a \subseteq \mathbb{N})(\forall l \in \mathbb{N}) \text{ if } |a| \leq m \text{ and } \min(a) > k \text{ then } f \not\Vdash_{\mathbb{P}} \text{“}(\exists u \in \check{a})\ \max(\bar{H}\langle u \rangle) \leq \check{l}.\text{”}$$

Lemma 2.2.2. *Given $\eta \in \omega_2$, $f_\eta \subseteq f \in \mathbb{D}_\eta$, and $m \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $A(H_\eta, f, k, m)$ holds.*

Proof. If the lemma fails for some η, f and m, then it is possible to construct a sequence $\{(k_j, l_j, a_j)\}_{j<\omega}$ such that:

(i) $N < k_j, l_j \in \mathbb{N}$,

(ii) $a_j \subseteq \mathbb{N}$ and $|a_j| \leq m$,

(iii) $k_j < \min(a_j) \leq \max(a_j) < k_{j+1}$, and

(iv) $f \Vdash$ "$(\exists u \in a_j)\ \max(H_\eta\langle u\rangle) \leq l_j$".

Let $L > T|f|$ and let d be so large that $d > \max_{j\leq L}(l_j)$. Define $g \in \mathbb{D}_\eta$ by setting

$$\mathbf{domain}(g) = (\{\xi(\eta)\} \times d) \cup \mathbf{domain}(f)$$

and letting

$$g(u,v) = \begin{cases} f(u,v) & \text{if } (u,v) \in \mathbf{domain}(f) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma \subseteq \mathbb{D}_\eta$ be generic such that $g \in \Gamma$. In $V[\Gamma]$ note that

$$\left\{h^{-1}(u)\ \mid (\xi(\eta), u) \in \mathbf{domain}(f) \text{ and } h \in H_\eta\right\}$$

has cardinality no greater than $T|f|$ and so there must be some $j \leq L$ such that $(\xi(\eta) \times H_\eta\langle u\rangle) \cap\ \mathbf{domain}(f) = \varnothing$ for each $u \in a_j$. Using (iv) and the fact that $g \supseteq f$ it follows that $V[\Gamma]$ satisfies $\max(H_\eta\langle u\rangle) \leq l_j \leq d$ for some $u \in a_j$. But

$$g \Vdash \text{“}\{u \in d\ \mid (\xi(\eta), u) \notin \mathbf{domain}(f)\} \subseteq c_{\xi(\eta)}\text{”}$$

contradicting the hypothesis (6) since $g \supseteq f_\eta$.

□

Claim 2.2.3. *Without loss of generality*

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“}\{H_\eta\}_{\eta\in\omega_2} \textit{ is a pairwise disjoint family”}.$$

Proof. Keeping in mind that we are now arguing in $V[\mathbb{G}\cap\mathbb{D}]$, let $H_{\mathbb{G}}$ be the interpreation of H in $V[\mathbb{G}\cap\mathbb{D}]$. It suffices to show that Lemma 2.2.2 is still satisfied if one replaces H_η with $H_\eta \setminus H_{\mathbb{G}}$ for each $\eta \in \omega_2$. To see that this is the case, note that by the genericity of $\mathbb{G} \cap \mathbb{D}_\eta \setminus \mathbb{D}$ over the model $V[\mathbb{G}\cap\mathbb{D}]$ it follows that there are infinitely many $n \in \mathbb{N}$ such that $H_{\mathbb{G}}\langle n\rangle \subseteq c_{\xi(\eta)}$. In other words, the elements of $H_{\mathbb{G}}$ are never used to satisfy the conclusion of Lemma 2.2.2. So henceforth it will be assumed that $H_\eta = H_\eta \setminus H_{\mathbb{G}}$. □

2.2.1 κ-solvability in the Cohen model

Lemma 2.2.4. *Suppose that*

- $i_0, ..., i_{N-1} \in \omega$
- $\bigcup_{j<N} f_{i_j} \subseteq f \in \prod_{j<N} \mathbb{D}_{i_j}$
- $2^m \leq N$.

There is $k \in \mathbb{N}$ *such that* $A((\bigcup_{j<N} H_{i_j})^{[m]}, f, k, 1)$ *holds.*

Proof. Proceed by induction on m; the case $m = 0$ is true by Lemma 2.2.2. Assume the lemma is true for m and let $2^{m+1} \leq N$, $i_0, ..., i_{N-1} \in \omega$, $\bigcup_{j<N} f_{i_j} \subseteq f \in \prod_{j<N} \mathbb{D}_{i_j}$. To see that there is some k such that $A((\bigcup_{j<N} H_{i_j})^{[m+1]}, f, k, 1)$ is true let l be given. By the inductive hypothesis, there is k_1 such that

$$A\left(\left(\bigcup_{2^m \leq j<N} H_{i_j}\right)^{[m]}, f \restriction \prod_{2^m \leq j<N} \mathbb{D}_{i_j}, k_1, 1\right) \tag{2.2.1}$$

holds. Let $n > k_1$ be arbitrary and extend $f \restriction \prod_{j<2^m} \mathbb{D}_{i_j}$ to $f' \in \prod_{j<2^m} \mathbb{D}_{i_j}$ so there is some $L \in \omega$ such that

$$f' \Vdash_{\prod_{j<2^m} \mathbb{D}_{i_j}} \text{“}(\forall i \leq \check{l}) \left(\bigcup_{j<2^m} H_{i_j}\right)^{[m]} \langle i \rangle < L\text{”}. \tag{2.2.2}$$

By the inductive hypothesis, there is k_2 such that $A((\bigcup_{j<2^m} H_{i_j})^{[m]}, f', k_2, 1)$ holds, and since (2.2.1) holds, it is possible to find $h \in (\bigcup_{2^m \leq j<N} H_{i_j})^{[m]}$, $n' \in \omega$, and $f'' \in \prod_{2^m \leq j<N} \mathbb{D}_{i_j}$ extending $f \restriction \prod_{2^m \leq j<N} \mathbb{D}_{i_j}$, such that

$$f'' \Vdash_{\prod_{2^m \leq j<N} \mathbb{D}_{i_j}} \text{“}n' = h(n) > \check{k}_2\text{”}. \tag{2.2.3}$$

Since $A((\bigcup_{j<2^m} H_{i_j})^{[m]}, f', k_2, 1)$ holds,

$$(\forall K \in \omega)\ f' \not\Vdash_{\mathbb{P}} \text{“}\max\left(\left(\bigcup_{j<2^m} H_{i_j}\right)^{[m]} \langle n' \rangle\right) \leq K\text{”}$$

so there are infinitely many possible $K \in \omega$ for which there are $g_K \in (\bigcup_{j<2^m} H_{i_j})^{[m]}$ and $f'_K \in \prod_{j<2^m} \mathbb{D}_{i_j}$ extending f' such that

$$f'_K \Vdash \text{“}g_K(n') = K. \tag{2.2.4}$$

This implies there is K as above with $f'' \not\Vdash h^{-1}(K) \leq L$, so we can extend f'' to $f''' \in \prod_{2^m \leq j<N} \mathbb{D}_{i_j}$ deciding $h^{-1}(K) > L$. Therefore combining (2.2.2), (2.2.3),and (2.2.4),

$$f'_K \cup f''' \Vdash l < g_K^{-1} h^{-1} g_K h(n) = [g_K, h](n),$$

and this proves $A((\bigcup_{j<N} H_{i_j})^{[m+1]}, f, k, 1)$ holds. □

Now it will be proven that G cannot be $(\aleph_2, \aleph_1)$-solvable.

Theorem 2.2.5. *In the $\aleph_2$ Cohen real model, every group acting faithfully on $\mathbb{N}$ with a unique invariant mean is not $(\aleph_2, \aleph_1)$-solvable.*

Proof. If G is a counterexample, let $\{(D_\eta, f_\eta, H_\eta, \xi(\eta))\}_{\eta<\omega_2}$ and T be as in (1) to (7) of §2.2. The set $\Lambda = \{\eta \mid f_\eta \in \mathbb{G}\}$ must have size $\aleph_2$. Suppose that

$$\mathbb{1} \Vdash \text{“}S \in [\Lambda]^{\omega_1} \text{ and } (\forall B \in [S]^{\aleph_0})(\exists A \in [B]^{\aleph_0}) \left\langle \bigcup_{\eta \in A} H_\eta \right\rangle \text{ is solvable”}.$$

Extend each f_η such that $f_\eta \nVdash$ “$\eta \notin S$” to $\bar{f}_\eta$ so that $\bar{f}_\eta \Vdash$ “$\eta \in S$”, and extend D_η to $\bar{D}_\eta$ so that if $\bar{\mathbb{D}}_\eta$ is defined accordingly then $\bar{f}_\eta \in \bar{\mathbb{D}}_\eta$. Let $E = \{\eta \in \omega_2 \mid \bar{f}_\eta \Vdash \eta \in S\}$. E must be uncountable, and so refine E so that $\{\text{supp}(\bar{f}_\eta)\}_{\eta\in E}$ forms a Δ-system. As in Lemma 2.2.2 it may be assumed that $\{\text{supp}(\bar{f}_\eta)\}_{\eta\in E}$ and $\{\bar{D}_\eta\}_{\eta\in E}$ are pairwise disjoint. Without loss of generality, by re-labeling the first ω indices in E, assume $\omega \subseteq E$ so that $(\forall i \in \mathbb{N})$ $f_i \Vdash i \in S$. It will be shown that $B = \omega \cap S$ satisfies that

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“}|B| = \omega \text{ and } (\forall A \in [B]^{\aleph_0}) \left\langle \bigcup_{i \in A} H_i \right\rangle \text{ cannot be solvable.”}$$

To see $|B| = \omega$, note that for any $n \in \omega, f \in \mathbb{P}$ there is i such that $n < i < \omega$ and $\text{supp}(f_i) \cap \text{supp}(f) = \varnothing$; hence, $f \cup f_i \Vdash_{\mathbb{P}}$ “$i \in B$”. Suppose $\mathbb{1} \Vdash A \in [B]^{\aleph_0}$, and let $\tilde{G} = \langle \bigcup_{i\in A} H_i \rangle$. The proof follows as a corollary from Lemma 2.2.4. Suppose $m \in \omega$ and f is some condition forcing $\tilde{G}^{[m]}$ is trivial. Let $N \in \omega$ be larger than 2^m and extend f to force that $i_0, ..., i_{N-1}$ are distinct elements of A. Since $A \subseteq \Lambda$, f must extend $\bigcup_{j<N} f_{i_j}$. Lemma 2.2.4 yields some $k \in \omega$ with the property $A((\bigcup_{j<N} H_{i_j})^{[m]}, f, k, 1)$, and so there are $g \in (\bigcup_{j<N} H_{i_j})^{[m]} \subseteq \tilde{G}^{[m]}$ and $f' \in \prod_{j<N} \mathbb{D}_{i_j}$ extending f such that

$$f' \Vdash g(k+1) > k+1.$$

In other words, the condition f' forces a contradiction since g is the identity but $g(k+1) > k+1$. □

2.2.2 Subexponential growth in the Cohen model

Notation 2.2.2. In the next two lemmas the notation n^m, for n and m elements of $\mathbb{N}$, will be used to denote both the set of all functions from $m = \{0, 1, ..., m-1\}$ to $n = \{0, 1, ..., n-1\}$, as well as the cardinality of this set of functions. However, this potential ambiguity should cause no distress to the careful reader.

Lemma 2.2.6. *Let $K, J \in \mathbb{N}$ with $K \geq JT^2$, let $\mathbb{Q} = \prod_{i\leq K} \mathbb{D}_i$, and let $q \in \mathbb{Q}$ be a condition with $q(i) \leq f_i$. There are $\{(q_n, \{(a_t, b_t)\}_{t\in K^n}, k_n^0, k_n^1)\}_{n\in\omega}$ such that for $n \in \omega$:*

1. *$q_0 = q$,*

2. $q_{n+1}(i) \supseteq q_n(i)$ *for each* $i \leq K$.

3. *the property* $A(H_0, q_n(0), k_n^0, K^n)$ *of Lemma 2.1 holds*

4. *the property* $A(H_{i+1}, q_n(i+1), k_n^1, K^n)$ *of Lemma 2.2.2 holds for* $i \in K-1$

5. $a_t, b_t \in \mathbb{N}$, *and* $h_n^i \in H_i$ *for* $i \leq K$, $t \in K^n$

6. $k_n^0 < a_t < k_{n+1}^0$ *and* $k_n^1 < b_t < k_{n+1}^1$ *for each* $t \in K^n$

7. $q_{n+1}(0) \Vdash$ "$b_t \in H_0\langle a_t \rangle$" *for each* $t \in K^n$, *and for at least* $\frac{|\{a_t\}_{t \in K^n}|}{T}$ *many* $\{a_t\}_{t \in K^n}$, $q_{n+1}(0) \Vdash$ "$b_t = h_n^0(a_t)$"

8. $q_{n+1}(i+1) \Vdash$ "$a_{t^\frown i} \in H_{i+1}\langle b_t \rangle$" *for each* $t \in K^n$ *and* $i \in K-1$, *and for at least* $\frac{|\{b_t\}_{t \in K^n}|}{T}$ *many* $\{b_t\}_{t \in K^n}$, $q_{n+1}(0) \Vdash$ "$a_{t^\frown i} = h_n^{i+1}(b_t)$"

9. *if* t *and* s *are in* K^n *and* $j \in K-1$ *then* $a_{t^\frown j} < a_{s^\frown j+1}$.

10. $|\{a_t\}_{t \in K^n}| \geq J^n$.

Proof. Proceed by induction on n. To begin, let $q_0 = q$ and use Lemma 2.2.2 to find k_0 sufficiently large that the property $A(H_i, q_0(i), k_0, 1)$ holds for each $i \leq K$ and let $k_0^0 = k_0^1 = k_0$. Let $a_\varnothing \in \mathbb{N}$ be arbitrary such that $a_\varnothing > k_0$. Then using $A(H_0, q_0(0), k_0^0, 1)$ for $l = k_0$ let $q_1(0) \supseteq q(0)$ be such that there is $h_0^0 \in H_0$ with

$$q_1(0) \Vdash \text{"}h_0^0(a_\varnothing) \text{ is decided and above } k_0\text{"}.$$

Set $b_\varnothing = h_0^0(a_\varnothing)$.

Then let $k_1^0 > a_\varnothing$ be so large that property $A(H_0, q_1(0), k_1^0, K)$ holds. Using property $A(H_1, q_0(1), k_0^1, 1)$ with $l = k_1^0$ let $q_1(1) \supseteq q_0(1)$, $h_0^1 \in H_1$, and $a_{\varnothing^\frown 0}$ be such that

$$q_1(1) \Vdash \text{"}a_{\varnothing^\frown 0} = h_0^1(b_\varnothing) \text{ and } a_{\varnothing^\frown 0} > k_1^0\text{"}.$$

Using property $A(H_2, q_0(2), k_0^1, 1)$ with $l = a_{\varnothing^\frown 0}$ let $q_1(2) \supseteq q_0(2)$, $h_0^2 \in H_2$, and $a_{\varnothing^\frown 1}$ be such that

$$q_1(2) \Vdash \text{"}a_{\varnothing^\frown 1} = h_0^2(b_\varnothing) \text{ and } a_{\varnothing^\frown 1} > a_{\varnothing^\frown 0} > k_1^0\text{"}.$$

Proceed inductively to use property $A(H_i, q_0(i), k_0^1, 1)$ with $l = a_{\varnothing^\frown i-1}$ to let $q_1(i) \supseteq q_0(i)$, $h_0^i \in H_i$, and $a_{\varnothing^\frown i-1}$ be such that

$$q_1(i) \Vdash \text{"}a_{\varnothing^\frown i-1} = h_0^i(b_\varnothing) \text{ and } a_{\varnothing^\frown i-1} > a_{\varnothing^\frown i-2}\text{"}$$

for each $i \leq K-1$. Then let $k_1^1 > b_\varnothing$ sufficiently large that $A(H_{i+1}, q_1(i+1), k_1^1, K)$ holds for $i \leq K-1$. The values of the condition $q_1(i)$ have been defined for each $i \in K$ and, noting $|\{a_i\}_{i \in K}| = K \geq J$, it is easy to check that the induction hypotheses are satisfied.

Now assume that q_m, $\{(a_t, b_t)\}_{t\in K^m}$, k_m^0 and k_m^1 are all given satisfying the induction hypotheses. Using (3) it follows that the property $A(H_0, q_m(0), k_m^0, K^m)$ holds. Note that, in the notation of Lemma 2.2.2 setting $a = \{a_t\}_{t\in K^m}$, it is the case that $a > k_m^0$. Hence it is possible to apply this property to $l = k_m^1$ and a to find $q_{m+1}(0) \supseteq q_m(0)$, such that for all $t \in K^m$ there is $g_t^0 \in H_0$ with

$$q_{m+1}(0) \Vdash \text{"}g_t^0(a_t) \text{ is decided and above } k_m^1\text{"}.$$

Set $b_t = g_t^0(a_t)$. By pigeonholing, there must be some $h_m^0 \in \{g_t^0\}_{t\in K^m}$ that is forced by $q_{m+1}(0)$ to map at least $\frac{|a|}{T}$ elements of a above k_m^1, and since h_m^0 is injective, $|\{b_t\}_{t\in K^m}| \geq \frac{|\{a_t\}_{t\in K^m}|}{T}$.

Let $k_{m+1}^0 > \max_{t\in K^m} a_t$ be so large that property $A(H_0, q_{m+1}(0), k_{m+1}^0, K^{m+1})$ holds. Using property $A(H_1, q_m(1), k_m^1, K^m)$ with $l = k_{m+1}^0$, let $q_{m+1}(1) \supseteq q_m(1)$ be such that for every $t \in K^m$ there is $g_t^1 \in H_1$ with

$$q_{m+1}(1) \Vdash \text{"}g_t^1(b_t) \text{ is decided and above } k_{m+1}^0\text{"}.$$

Set $a_{t^\frown 0} = g_t^1(b_t)$. By pigeonholing, there must be some $h_m^1 \in \{g_t^1\}_{t\in K^m}$ that is forced by $q_{m+1}(0)$ to map at least $\frac{|\{b_t\}_{t\in K^m}|}{T} \geq \frac{|\{a_t\}_{t\in K^m}|}{T^2}$ elements of $\{b_t\}_{t\in K^m}$ above k_m^1, and since h_m^1 is injective, $|\{a_{t^\frown 0}\}_{t\in K^m}| \geq \frac{|\{a_t\}_{t\in K^m}|}{T^2}$.

Proceeding by induction using property $A(H_i, q_m(i), k_m^1, K^m)$ with $l = \max_{t\in K^m} a_{t^\frown i-2}$ let $q_{m+1}(i) \supseteq q_m(i)$ and $a_{t^\frown i-1}$ be such that for every $t \in K^m$ there is $g_t^i \in H_i$ with

$$q_{m+1}(i) \Vdash \text{"}a_{t^\frown i-1} = g_t^i(b_t) > \max_{t\in K^m} a_{t^\frown i-2},$$

Again there must be some $h_m^i \in \{g_t^i\}_{t\in K^m}$ that is forced by $q_{m+1}(i)$ to map at least $\frac{|\{b_t\}_{t\in K^m}|}{T} \geq \frac{|\{a_t\}_{t\in K^m}|}{T^2}$ elements of $\{b_t\}_{t\in K^m}$ above $\max_{t\in K^m} a_{t^\frown i-2}$, and since h_m^i is injective, $|\{a_{t^\frown i-1}\}_{t\in K^m}| \geq \frac{|\{a_t\}_{t\in K^m}|}{T^2}$. Let $k_{m+1}^1 > \max_{t\in K^m} b_t$ be sufficiently large that $A(H_{i+1}, q_{m+1}(i+1), k_{m+1}^1, K^{m+1})$ holds for each $i \in K-1$. Noting that

$$|\{a_t\}_{t\in K^{m+1}}| \geq \sum_{i\in K} |\{a_{t^\frown i}\}_{t\in K^m}| \geq K\frac{|\{a_t\}_{t\in K^m}|}{T^2} \geq K\frac{J^m}{T^2} \geq J^{m+1},$$

it is again routine to check that the induction hypotheses are all satisfied.

□

Corollary 2.2.7. *Given $K, J, N \in \mathbb{N}$ with $K \geq JT^2$, and $q \in \mathbb{Q} = \prod_{i\leq K} \mathbb{D}_i$ with $q(i) \leq f_i$, there is $B \subseteq K^N$ with $|B| = J^N$ and $\{a_t\}_{t\in B} \subseteq \mathbb{N}$, $a_\varnothing \in \mathbb{N}$, $\{h_t^i\}_{i\leq K, t\in K^{\leq N}}$, and $q' \in \mathbb{Q}$ extending q such that*

1. $(\forall t, s \in B)$ $t \neq s$ *then* $a_t \neq a_s$,
2. $(\forall t \in K^{\leq N})(\forall i \leq K)$ $f \Vdash$ "$h_t^i \in H_i$",
3. $(\forall t \in B)$ $q' \Vdash$ "$a_t = h_N^{t(N-1)+1} h_{N-1}^0 \circ \ldots \circ h_2^{t(1)+1} h_1^0 \circ h_1^{t(0)+1} h_0^0(a_\varnothing)$".

In particular, for $\gamma_j = n \mapsto j^n$, $\bigcup_{i\leq K} H_i$ *does not satisfy the* $\gamma_{(J-1)}$*-growth condition.*

Proof. Using Lemma 2.2.6 set $q' = q_N$, and pick J^N distinct elements

$$\{a_t\}_{t\in B} \subseteq \{a_t\}_{t\in K^N}.$$

□

Theorem 2.2.8. *In the* $\aleph_2$ *Cohen real model, every group acting faithfully on* $\mathbb{N}$ *with a unique invariant mean does not have the* $\{n \mapsto j^n\}_{j\in\omega}$*-*$\aleph_2$*-*$\aleph_1$*-growth condition.*

In particular, we prove a stronger result that every group acting faithfully on $\mathbb{N}$ *with a unique invariant mean cannot satisfy the following property: For every* m *and every family* $\{H_\xi\}_{\xi\in\aleph_2}$ *of pairwise disjoint subsets of* G *of cardinality* m*,*

$$(\exists S \in [\aleph_2]^{\aleph_1})(\forall B \in [S]^{<\aleph_1})(\exists j \in \mathbb{N})(\forall A \in [B]^{(j+1)m^2}), \bigcup_{\xi\in A} H_\xi \textit{ has } (n \mapsto j^n)\textit{-growth.}$$

Proof. If G is a counterexample, let $\{(D_\eta, f_\eta, H_\eta, \xi(\eta))\}_{\eta<\omega_2}$ and T be as in (1) to (7) of §2.2. Define the function γ_j on $\mathbb{N}$ by $\gamma_j(n) = j^n$. Let Γ be a generic filter for $\mathbb{P}$ and set $\Lambda = \{\eta : f_\eta \in \Gamma\}$, noting that this set must have size $\aleph_2$. Suppose that

$$\mathbb{1} \Vdash \text{“}S \in [\Lambda]^{\omega_1}\text{” and}$$

$$\mathbb{1} \Vdash \text{“}(\forall B \in [S]^{\aleph_0})(\exists j \in \omega)(\forall A \in [B]^{(j+1)T^2}) \bigcup_{\eta\in A} H_\eta \text{ satisfies the } \gamma_j\text{-growth condition”}.$$

As in Theorem 2.2.5, extend each f_η such that $f_\eta \nVdash$ “$\eta \notin S$” to $\bar{f}_\eta$ so that $\bar{f}_\eta \Vdash$ “$\eta \in S$”, and extend D_η to $\bar{D}_\eta$ so that if $\bar{\mathbb{D}}_\eta$ is defined accordingly then $\bar{f}_\eta \in \bar{\mathbb{D}}_\eta$. Let $E = \left\{\eta \in \omega_2 \;\middle|\; \bar{f}_\eta \Vdash \eta \in S\right\}$. E must be uncountable, and so refine E so that $\{\text{supp}(\bar{f}_\eta)\}_{\eta\in E}$ forms a Δ-system. As in Lemma 2.2.2 it may be assumed that $\{\text{supp}(\bar{f}_\eta)\}_{\eta\in E}$ and $\{\bar{D}_\eta\}_{\eta\in E}$ are pairwise disjoint. Without loss of generality, by re-labeling the first ω indices in E, assume $\omega \subseteq E$ so that $(\forall i \in \mathbb{N})$ $f_i \Vdash i \in S$. As in Theorem 2.2.5, for $B = \omega \cap S$, $\mathbb{1} \Vdash_{\mathbb{P}}$ “$|B| = \omega$”.

Suppose for some $p \in \mathbb{P}$, $J \in \omega$ that

$$p \Vdash \text{“}(\forall A \in [B]^{(J+1)T^2}) \bigcup_{\eta\in A} H_\eta \text{ satisfies the } \gamma_J\text{-growth condition”}.$$

Let $K = (J+1)T^2$. There are $f_l, ..., f_{l+K}$ such that for $i \leq K$, $\text{supp}(f_{l+i}) \cap \text{supp}(p) = \varnothing$. For $q = p \cup \bigcup_{i\leq K} f_{l+i}$ apply Corollary 2.2.7 to get $q' \leq q$ such that

$$q' \Vdash \text{“} \bigcup_{i\leq K} H_{l+i} \text{ does not satisfy the } \gamma_J\text{-growth condition”}.$$

Since $q' \Vdash l, ..., l+K \in B$ it is the case that

$$q' \Vdash \text{“} \bigcup_{i\leq K} H_{l+i} \text{ satisfies the } \gamma_J\text{-growth condition”},$$

yielding a contradiction.

□

Corollary 2.2.9. *In the $\aleph_2$ Cohen real model, every group acting faithfully on $\mathbb{N}$ with a unique invariant mean must have ultra-exponential growth.*

Proof. By negating the property in the second paragraph of theorem 2.2.8, see that theorem 2.2.8 implies that in the Cohen model a group with a unique invariant mean must satisfy

$$(\exists m)(\forall j)(\exists S \in [G]^{(j+1)m^2})\ \gamma_G^S(n) \not\leq^* (n \mapsto j^n),$$

which implies

$$(\exists m)(\forall j)(\exists S \in [G]^{(j+1)m^2})\ \lim_{n\to\infty} (\gamma_G^S(n))^{1/n} > j-1.$$

Let m satisfy the above formula, set $M = 3m^2$, and substitute the variable j with $i = j-1$ to get the formula

$$(\forall i \geq 1)(\exists S \in [G]^{iM})\ \lim_{n\to\infty} (\gamma_G^S(n))^{1/n} > i,$$

and therefore

$$(\exists M)(\forall^\infty i)(\exists S \in [G]^{\leq iM})\ \lim_{n\to\infty} (\gamma_G^S(n))^{1/n} > i,$$

which is exactly the definition of ultra-exponential growth.

□

We will finish with some closing remarks and remaining questions. The main question "Is there an amenable subgroup of $\mathrm{Sym}(\omega)$ that has a unique invariant mean on ω?" still remains unanswered, furthermore, the following special case of a negative answer also still remains unknown:

Question 2.2.10. Is it consistent that there is no locally solvable subgroup of $\mathrm{Sym}(\omega)$ that has a unique invariant mean on ω?

Recall on the other hand that in [38] it was proven that under $\mathfrak{u} = \mathfrak{p}$ there is a locally solvable subgroup of $\mathrm{Sym}(\omega)$ that has a unique invariant mean on ω.

The following definition pertains to the next question:

Definition 2.2.11. In the case that an amenable group G acts on X, say that $f \in \ell^\infty(X)$ is a cyclic vector if $\{{}_g f : g \in G\}$ spans a dense subspace of $\ell^\infty(X)$.

Rosenblatt and Talagrand in [40] note that the existence of a cyclic vector for an amenable action implies the existence of a unique invariant mean, the proof of which was not shown in [40] so it is shown here for completeness.

Proposition 2.2.12. *If an amenable group G acting on X has a cyclic vector, then it has a unique invariant mean on its action.*

Proof. Let f be a cyclic vector and suppose $\mu, \nu \in \ell^\infty(X)^*$ are distinct invariant means that differ on some $\phi \in \ell^\infty(X)$. Let $\sum_{i<n} a_i\ {}_{g_i}f$ be $\frac{\varepsilon}{2}$ close to ϕ, where $|\mu(\phi) - \nu(\phi)| > \varepsilon$. Let $\bar{a} = \sum_{i<n} a_i$ and see that $\mu(f) \neq \nu(f)$ since otherwise

$$|\mu(\phi) - \nu(\phi)| \leq |\mu(\phi) - \bar{a}\mu(f) + \bar{a}\mu(f) - \nu(\phi)| \leq |\mu(\phi) - \bar{a}\mu(f)| + |\bar{a}\nu(f) - \nu(\phi)|$$
$$\text{and } |\mu(\phi) - \bar{a}\mu(f)| + |\bar{a}\nu(f) - \nu(\phi)| = |\mu(\phi - \sum_{i<n} a_i\ {}_{g_i}f)| + |\nu(\sum_{i<n} a_i\ {}_{g_i}f - \phi)| < \varepsilon$$

Let $\sum_{i<k_n} b_{i,n}\ {}_{h_{i,n}}f$ converge to X, and set $\bar{b}_n = \sum_{i<k_n} b_{i,n}$. Since $\mu(X) = \nu(X) = 1$, for η equal to either μ or ν we have

$$|\eta(\sum_{i<k_n} b_{i,n}\ {}_{h_{i,n}}f - X)| = |\bar{b}_n\eta(f) - \eta(X)| = \eta(f)|\bar{b}_n - \frac{1}{\eta(f)}| \longrightarrow 0,$$

which means $\bar{b}_n \longrightarrow \frac{1}{\eta(f)}$, contradicting $\mu(f) \neq \nu(f)$. □

Since the existence of a cyclic vector implies the existence of a unique invariant mean, one can ask if the two are equivalent:

Question 2.2.13. Does an amenable subgroup of $\mathrm{Sym}(\omega)$ have a cyclic vector if and only if it has a unique invariant mean on ω?

Or, as asked in [40]:

Question 2.2.14. Is it consistent that there is no amenable subgroup of $\mathrm{Sym}(\omega)$ with a cyclic vector?

Note that the constructions in the Cohen model which exclude the possibility of a unique invariant mean for a group satisfying certain conditions, do not necessarily contradict a finite set of means spanning the space of invariant means (Recall equation (2.1.2), and that if $\mu_0, .., \mu_{n-1}$ are invariant means then $\mathrm{span}\{\mu_0, .., \mu_{n-1}\} = \{\sum_{i<n} a_i\mu_i : \sum_{i<n} a_i = 1\}$ is a set of invariant means). These constructions rely on ω_2-many Cohen reals or their compliments not being 1-thick, which may not be the case if for instance μ, ν are invariant, and for all Cohen reals c_ξ, $\mu(c_\xi) = \nu(c_\xi^c) = 1$.

Question 2.2.15. Is there an amenable subgroup of $\mathrm{Sym}(\omega)$ that has a finite (or even countable) set of invariant means on ω which span the space of invariant means on ω?

Since all of these questions remain unanswered even in the Cohen model (forcing ω_2 Cohen reals over a model of CH), an intermediate result would be to answer any of the previous questions, or prove they cannot be answered, in that model:

Question 2.2.16. Which of the previous questions can be answered, or provably not be answered, by the Cohen model?

3 Universal Graph Structures on ω_1

3.1 Introduction

3.1.1 Overview

To introduce the study of universality, we will first introduce saturation. Saturation was introduced by Keisler in 1961 [24], though the name saturation (for the case of ω-saturated) was introduced later by Vaught in 1961 [51]. Universal structures (see definition 3.1.5) in abstract, were introduced by Jónsson in 1957 [23], however universal elements for particular structures had been considered in individual cases prior to the general model theoretic definition, for example universal partial orders in 1956 [22] by Johnston. Saturated models are universal as we shall see, but have other useful properties which are not implied by universality, like being strongly homogeneous (see [42] Theorem 1.11 for details). More historical information about the development of these topics and other model theory can be found in [19].

Since saturation requires the assumption that there is $\kappa \geq |A|$ with $\kappa = \kappa^{<\kappa}$, one can ask if universality also requires these assumptions. This was answered first for linear orders in [43], in which Shelah obtains a model of $2^\omega = \omega_2$ with a universal linear order of size ω_1. Later in [44], the question was answered for graphs with Shelah producing a model of $2^\omega = \omega_2$ with a universal graph of size ω_1. Also, in [45] Shelah showed there is a model of $2^\kappa = \mu$ with a universal graph of size λ, for any $\kappa < \lambda \leq \mu$ with $\mu^{<\lambda} = \mu$. The question of universal graphs of successors of singular cardinals has been studied in [8] and [9]. In [33] Mekler shows that for a class of structures K which satisfy the following amalgamation condition which requires (A), are closed under taking substructures, isomorphism, or unions of chains, and in which there are only countably many structures of a specific finite size, there is a model with $2^\omega = \omega_2$ and a universal structure for this class on ω_1:

For any structures $A_0, A_1, A_2 \in K$ on elements of $\mathcal{P}(3) \setminus \{3\}$ such that for all $i < j < 3$, $A_0 \cap A_1 \cap A_2, A_i \cap A_j \in K$, we have a structure in $A \in K$ on 3 where for all $i < 3$, $A \cap A_i = A_i$. (A)

If all elements of K on two points have their single point induced substructures in K, then condition (A) is simply requiring elements of K on a subset of 3 with two or less points, to amalgamate into a structure in K. Note that this result cannot apply to triangle free graphs since they do not satisfy (A).

A different set of questions about when universals for a class of structures which omit a substructure or set of substructures can exist, has been investigated extensively (in [25], [12], [6] among others). In [25] it is shown by Komjth, Mekler, and Pach that for each n the class of graphs omitting all paths of length n, and the class of graphs omitting all circuits of length at least n, possess universal models of any infinite cardinality. In [12] it was shown by Shelah and Džamonja that there is model with $2^\omega = 2^{\omega_1} = \omega_3$ in which there is a family $\mathcal{U}$ with size $|\mathcal{U}| = \omega_2$, of triangle free graphs on ω_1, such that every triangle free graph on ω_1 will embed to some member of $\mathcal{U}$.

However, many of the natural questions are still open; For instance it is not known if a universal triangle free graph of size ω_1 can exist without CH. In the case of countable universals, it is not known when a there is a universal graph of size ω for graphs which omit some specific finite graph [6].

The goal of this chapter is to investigate the consistency of MA, $\neg$CH, and a universal function on ω_1, where universal functions are a generalization of universal graphs. We obtain a partial result towards a model of MA $\neg$CH and a universal function on ω_1, by showing that forcing with Suslin trees to destroy them can be included into the iteration in [44], and so a model of $\neg$CH, SH, MA(Cohen), and a universal function on ω_1 is obtained.

The proper countable support iterations of Miller forcings followed by P-ideal dichotomy forcings in the work done independently by Steprāns and Shelah in [46], can also be augmented to model $\neg$CH, SH, MA(Cohen), and "there is a universal graph on ω_1": One can interlace forcing Suslin trees to model SH since they do not add reals, and force a Cohen real at successor of limit ordinals to model MA(Cohen) since adding a Cohen real preserves $\sqsubseteq^{\text{Cohen}}$ (Lemma 6.3.18 in [3]). However it is unclear whether a universal function (Definition 3.1.7) exists in this model as it does in the results of this chapter. Indeed, a universal function does not exist in the $\mathbf{PT}_{f,g}$ iteration defined in [46]. Additionally, the other iterations in [46] which rely on a ground model set of full outer measure becoming universal in the extension, cannot add Cohen reals cofinally, since adding a Cohen real makes all ground model sets null.

In the other direction, it is consistent to have a model of MA and no universal graph on ω_1 (and thus also no universal function on ω_1) ([28] corollary 5.22). Furthermore, Lemma 5.16 in [28], shows that Hechler forcing $\mathbb{H}$ is such that no c.c.c. forcing in $V^{\mathbb{H}}$ can force that there was a function in V that is now universal function on ω_1. This shows that an argument using a c.c.c. forcing over a ground model that may satisfy $2^{\omega_1} = \omega_2$ and CH, to obtain a model with a universal function on ω_1, $\neg$CH, and MA, like in [44] or section 3's, cannot work. In particular this shows that the usual model of MA and $\neg$CH obtained by iterating c.c.c. forcings with finite support, has no universal function on ω_1. By a similar argument, lemma 3.2.4 shows that adding a Cohen real is such that no Knaster forcing in the Cohen extension can force that there was a graph in V that is now a universal graph on ω_1. This shows that an argument using a Knaster forcing over a ground model that may satisfy $2^{\omega_1} = \omega_2$ and CH, to obtain a model with a universal graph on ω_1, $\neg$CH, and MA(Knaster), cannot work.

Interestingly, the model augmenting the Miller iteration of [46] to model $\neg$CH, SH, MA(Cohen), and "there is a universal graph on ω_1", also cannot be augmented to model MA(Hechler), since Hechler forcing makes ground model sets meagre.

The only relative consistency question left open with MA, $\neg$CH, and the existence of a universal function or graph on ω_1, is that of a model of MA + $\neg$CH + "there is a universal on ω_1".

3.1.2 Background and Preliminaries

For the logical details of the introduction, assume unless otherwise stated that we work with a fixed countable first order language $\mathcal{L}$. For a set X, we denote by $\mathcal{L}_X$ the language $\mathcal{L}$ extended with constant symbols for every element in X. We abuse notation and use the same notation for a model and its universe. For an $\mathcal{L}_X$-model A containing X, assume that A interprets every constant symbol $x \in X$ as itself.

The notation $\phi(x)$ for a formula ϕ, where x is a tuple of variable symbols (or a single variable symbol), is used to indicate that the free variables in ϕ are among those listed by x. Similarly the notation $p(x)$ for a set of formulas p, indicates that the free variables in formulas in p are among those listed by x. By $A \models p$ we mean $(\forall \phi \in p),\ A \models \phi$.

We start with the definition of type:

Definition 3.1.1. 1. An $\mathcal{L}$ type is a set of formulas $p(x)$ where for all $\mathcal{L}$ formulas $\phi(x)$, either $\phi(x) \in p(x)$ or $\neg\phi(x) \in p(x)$. This is somtimes referred to as a complete type.

2. A type $p(x)$ is a type of A, or satisfiable by A, if for every finite $p_0(x) \subseteq p(x)$ there is some $a \in A$ such that $A \models p_0(a)$.

3. A type $p(x)$ is realised by A if there is $a \in A$ with $A \models p(a)$.

4. A model A is κ-saturated if for all $X \subseteq A$ with $|X| < \kappa$, all $\mathcal{L}_X$ types are realized by A. A model A is saturated if it is $|A|$-saturated.

For saturation it does not matter if the types considered are of a single free variable or less than n free variables for some n.

Saturated models are universal as we shall see, but have other useful properties which are not implied by universality, like being strongly homogeneous (see [42] Theorem 1.11 for details).

Definition 3.1.2. A model A is κ-universal, if for all elementarily equivalent models B (B satisfies the same sentences as A) of size $|B| = \kappa$, there is an elementary embedding from B to A. A model A is universal if it is $|A|$-universal.

Proposition 3.1.3. *If A is κ-saturated for all of its types in one free variable, then it is κ-saturated.*

Proof. If $X \subseteq A$ has $|X| < \kappa$, and A realises all satisfiable $\mathcal{L}_X$ types in one free variable, then for such type in 2 free variables $p(x_0, x_1)$ it can be shown that $p(x_0, x_1)$ is realised. Let $q(x_0) = \{\exists x_1 \phi : \phi \in p(x_0, x_1)\}$, and see that since $p(x_0, x_1)$ is satisfied by A, so is $q(x_0)$, hence there is $a \in A$ with $A \models q(a)$. For all $p_0(x_0, x_1) \subseteq p(x_0, x_1)$ finite, by completeness of $p(x_0, x_1)$ and satisfiability of $p_0(x_0, x_1)$, $\exists x_1 \bigwedge p_0(x_0, x_1) \in q(x_0)$ and so $p(a, x_1)$ is satisfiable and therefore realised by A. Applying this argument inductively proves the proposition. □

Theorem 3.1.1 ([42] Theorem 1.7)**.** *1. Every model A with $X \subseteq A$, has an elementary extension of size $|A| + 2^{|X|}$ realising any $\mathcal{L}_X$ types of A*

2. Every model A has a κ-saturated elementary extension.

3. If $|A| \leq \kappa = \kappa^{<\kappa}$, then A has a saturated elementary extension of size κ.

Proof. Without loss of generality, X is infinite. Introduce a set of constant symbols $C = \{c_\alpha\}_{\alpha < 2^{|X|}}$ to the language, and let all $\mathcal{L}_X$ types of A in at most one free variable be enumerated by $\{p_\alpha(x)\}_{\alpha < 2^{|X|}}$. The theory $T = \bigcup_\alpha p_\alpha(c_\alpha) \cup p$, where p is the set of all $\mathcal{L}_A$ sentences true in A, is finitely satisfied by A and so there is some $\mathcal{L}_{A \cup C}$ model $A_1 = A \cup C$ of T. Since p is realised in A_1, A_1 is an elementary extension of A (in $\mathcal{L}$). Also A_1 realises all of its $\mathcal{L}_X$ types: If $p(x)$ is an $\mathcal{L}_X$ type of A_1, then for $p_0(x) \subseteq p(x)$ finite, $\exists x \bigwedge p_0(x)$ is an $\mathcal{L}_X$ sentence satisfied by A and so $p(x)$ is an $\mathcal{L}_X$ type of A, which therefore is realised in A_1. This proves 1.

To prove 2, without loss of generality by increasing κ we may assume it is regular. Use the above argument to inductively obtain an increasing elementary κ sequence of models $\{A_\alpha\}_{\alpha < \kappa}$ with $A_0 = A$ and $A_{\alpha+1}$ realising all of its $\mathcal{L}_{A_\alpha}$ types. $B = \bigcup_{\alpha < \kappa} A_\alpha$ is a κ-saturated elementary extension of A.

For 3, fix a set of constant symbols $C = \{c_\alpha\}_{\alpha < \kappa}$ and a subset increasing $[A \cup C]^{<\kappa}$-cofinal function $f : \kappa \longrightarrow [A \cup C]^{<\kappa}$. Use the above argument to inductively obtain an increasing elementary κ-sequence of models $\{A_\alpha\}_{\alpha < \kappa}$ with $A_0 = A$ and $A_{\alpha+1} = A \cup f(\alpha)$ realising all of its $\mathcal{L}_{f(\alpha)}$ types. $B = \bigcup_{\alpha < \kappa} A_\alpha = A \cup C$ is a saturated elementary extension of A. □

On the other hand of Point 3 in the above theorem, we have the following, which implies that the existence of saturated extensions for an arbitrary model A is independent of ZFC, since it requires the assumption that there is $\kappa \geq |A|$ with $\kappa = \kappa^{<\kappa}$, even for the relatively simple test cases of linear orders or graphs:

Proposition 3.1.4. *There is a graph A (or linear order) such that if B is a saturated elementary extension of A of infinite size $|B| = \kappa$, then $\kappa = \kappa^{<\kappa}$.*

Proof. We will just prove the case for graphs. Without loss of generality it suffices to show the result for A with an arbitrary binary relation R, by coding R as a graph relation R' on $A \times A \cup \{\emptyset\}$ defined by extending $R'((a,b),\emptyset) = R(a,b)$ to any graph relation. Let $\mathbb{C}_\lambda$ be notation for the set of all finite partial functions from λ to λ. Let $A = \mathbb{C}_\omega$ with the function extension relation $\sqsubseteq$. Note that we can write formulas which define $x \perp y$, $x \cup y$, "x is a singleton" and note the following useful sentences are satisfied by A and thus B:

1. $(\forall x)(\exists y, z) x \sqsubseteq y \wedge x \not\sqsubseteq z$.

2. $(\forall x, y)[(\forall z) z \sqsubseteq x \longleftrightarrow z \sqsubseteq y] \longleftrightarrow x = y$.

3. $(\forall x)(\exists y) x \perp y$.

4. $(\forall x_0, ..., x_n, y_0, ..., y_m)[\bigwedge_{i,j<n}$ "x_i is a singleton" $\wedge\, x_i \perp x_j \wedge$ $\bigwedge_{i<n,j<m}$ "y_j is a singleton" $\wedge\, x_i \not\perp y_j \wedge x_i \neq y_j] \longrightarrow [(\exists z)$"$z$ is a singleton" $\wedge \bigwedge_{i<n,j<m} x_i \perp z \wedge y_j \not\perp z]$.

Let $B = \{b_\alpha\}_{\alpha<\kappa}$. We will inductively show that we can "interpret" all $[\kappa \times \kappa]^1$ as functions in B. More precisely, we mean that the type $p_{(\beta,\gamma)}(x)$ for $\gamma \leq \beta < \kappa$, defined inductively below, is realised in B. Let $\{(0,0)\}$ be interpreted as any singleton in B. Assume for $\gamma \leq \beta < \kappa$ we have inductively found interpretations of $[\beta \times \beta]^1 \cup [\{\beta\} \times \gamma]^1$ in B, then the realisation of the satisfied type (satisfied by 4.)

$$p_{(\beta,\gamma)}(x) = \{\text{“}x \text{ is a singleton”}\} \cup \{x \perp \{(\beta,\alpha)\}^B\}_{\alpha<\gamma} \cup \{x \not\perp \{(\alpha,\lambda)\}^B\}_{\alpha,\lambda<\beta}$$

gives us an interpretation of $\{(\beta,\gamma)\}$. Now given any $\beta < \kappa$ and $f : \beta \longrightarrow \kappa$, we can fix a realization b_f in B of the type

$$p_f(x) = \{\{(\alpha, f(\alpha))\}^B \sqsubseteq x\}_{\alpha<\beta} \cup \{\{(\beta,0)\}^B \not\sqsubseteq x\} \cup \{x \not\perp \{(\beta,0)\}^B\},$$

and see that $f \neq g \in \kappa^{<\kappa}$ implies $b_f \neq b_g$.

□

Since we cannot have saturated models in ZFC, one can ask whether a weakened condition can be said to hold, even for simpler cases like graphs or linear orders.

Definition 3.1.5. A model A is κ-universal, if for all elementarily equivalent models B (B satisfies the same sentences as A) of size $|B| = \kappa$, there is an elementary embedding from B to A. A model A is universal if it is $|A|$-universal.

For example, the above definition applied to the theory and language of linear orders, states that a linear order $(A, <_A)$ is universal if and only if for every linear order $(B, <_B)$, there is an order preserving injection from B to A.

Theorem 3.1.2 ([42] Theorem 1.9)**.** *Every saturated model is universal*

Proof. Recall that an (A, B)-elementary map f means that for all $\mathcal{L}$ formulas $\phi(x_0, ..., x_n)$,

$$(\forall a_0, ..., a_n \in \text{dom}(f))\ A \models \phi(a_0, ..., a_n) \iff B \models \phi(f(a_0), ..., f(a_n)).$$

Let B be saturated, $|A|$ an elementary equivalent model with $\kappa = |A| = |B|$, and $\{a_\alpha\}_{\alpha<\kappa}$ enumerating A. It suffices to inductively construct an increasing sequence of (A,B)-elementary maps $\{f_\alpha\}_{\alpha<\kappa}$ with $\text{dom}(f_\beta) = \{a_\alpha\}_{\alpha<\beta}$. For limit stages and the final embedding $g = \bigcup_{\alpha<\kappa} f_\alpha$, note that the limit/union of an increasing sequence of elementary maps must be an elementary map. Let $f_0 = \emptyset$ and define $f_{\beta+1}$ as f_β on $\text{dom}(f_\beta)$, and on a_β as the realisation of the $\mathcal{L}_{\text{range}(f_\beta)}$ type

$$p(x) = \{\phi(x) : \phi(x) \text{ is an } L_{\text{range}(f_\beta)} \text{ formula and } B \models \phi(b)\},$$

where $b \in B \setminus \text{range}(f_\beta)$ is arbitrary. ☐

It is not the case that every universal model is saturated. For instance consider the theory of dense linear orders without endpoints, extended with constant symbols $\{c_i\}_{i<\omega}$ and a set of axioms $\{c_i < c_{i+1}\}_{i<\omega}$. We can model this with $\mathbb{Q}$, where c_i is interpreted as $\sum_{j\leq i} \frac{1}{2^j}$ (denote this model $\tilde{\mathbb{Q}}$ to avoid confusion). This model is universal but not saturated: The type $p(x) = \{c_i < x < 2\}_{i<\omega}$ is satisfied but not realised. To show it is universal, recalling that $\mathbb{Q}$ is the only countable model for a dense linear order without endpoints, fixing an interpretation of $\{c_i\}_{i<\omega}$ we can embed each interval $(-\infty, c_0], [c_i, c_{i+1}]$ to $(-\infty, 0], [\sum_{j\leq i} \frac{1}{2^j}, \sum_{j\leq i+1} \frac{1}{2^j}]$ respectively. This is an embedding if $\{c_i\}_{i<\omega}$ is unbounded in $\mathbb{Q}$, and otherwise we can embed $[\sup\{c_i\}_{i<\omega}, \infty)$ to $[2, \infty)$ if $\sup\{c_i\}_{i<\omega} \in \mathbb{Q}$, or to $[3, \infty)$ otherwise.

It turns out that we also cannot prove the existence of a universal linear order or graph of size ω_1 in ZFC, since in the Cohen model there cannot be one (see 3.2.4 for graphs and [43] for linear orders). It is a natural question whether a the existence of a universal graph or linear order of size κ is equivalent, as in the case for saturation, to $\kappa = \kappa^{<\kappa}$. This was answered first for linear orders in [43], in which Shelah obtains a model of $2^\omega = \omega_2$ with a universal linear order of size ω_1. Later in [44], the question was answered for graphs with Shelah producing a model of $2^\omega = \omega_2$ with a universal graph of size ω_1. Also, in [45] Shelah showed there is a model of $2^\kappa = \mu$ with a universal graph of size λ, for any $\kappa < \lambda \leq \mu$ with $\mu^{<\lambda} = \mu$. The question of universal graphs of successors of singular cardinals has been studied in [8] and [9]. In [33] Mekler shows that for a class of structures K which satisfy the following amalgamation condition which requires (A), are closed under taking substructures, isomorphism, or unions of chains, and in which there are only countably many structures of a specific finite size, there is a model with $2^\omega = \omega_2$ and a universal structure for this class on ω_1:

For any structures $A_0, A_1, A_2 \in K$ on elements of $\mathcal{P}(3) \setminus \{3\}$ such that for all $i < j < 3$, $A_0 \cap A_1 \cap A_2, A_i \cap A_j \in K$, we have a structure in $A \in K$ on 3 where for all $i < 3$, $A \cap A_i = A_i$. (A)

If all elements of K on two points have their single point induced substructures in K, then condition (A) is simply requiring elements of K on a subset of 3 with two or less points, to amalgamate into a structure in K. Note that this result cannot apply to triangle free graphs since they do not satisfy (A).

A different set of questions about when universals for a class of structures which omit a substructure or set of substructures can exist, has been investigated extensively (in [25], [12], [6] among others). In [25] it is shown by Komjth, Mekler, and Pach that for each n the class of graphs omitting all paths of length n, and the class of graphs omitting all circuits of length at least n, possess universal models of any infinite cardinality. In [12] it was shown by Shelah and Džamonja that there is model with $2^\omega = 2^{\omega_1} = \omega_3$ in which there is a family $\mathcal{U}$ with size $|\mathcal{U}| = \omega_2$, of triangle free graphs on ω_1, such that every triangle free graph on ω_1 will embed to some member of $\mathcal{U}$.

However, many of the natural questions are still open; For instance it is not known if a universal triangle free graph of size ω_1 can exist without CH. In the case of countable universals, it is not known when a there is a universal graph of size ω for graphs which omit some specific finite graph [6].

The goal of this chapter is to investigate the consistency of MA, $\neg$CH, and a universal function on ω_1, where universal functions are a generalization of universal graphs.

Definition 3.1.6. Let F, F' be functions $F : [G]^2 \longrightarrow \omega,\ F' : [G']^2 \longrightarrow \omega$, or graph relations on G, G' respectively. An embedding of (G, F) to (G', F') is a one to one function $f : G \longrightarrow G'$ such that

$$(\forall x, y \in G)F(x, y) = F'(f(x), f(y)).$$

Definition 3.1.7. A graph (U, R) of size κ is universal if for all graphs (G, R') of size κ, there is an embedding from (G, R') to (U, R). Similarly, a function $F : [U]^2 \longrightarrow \omega$ of size κ is universal if for all functions of size κ $F' : [G]^2 \longrightarrow \omega$, there is an embedding from (G, F') to (U, F). Universal graphs and functions on κ will both be referred to by the term "universal on κ" .

Notice that in the analysis of universals of size κ, it suffices to look at relations/functions on $[\kappa]^2$. Clearly the existence of a universal function implies the existence of a universal graph of the same size, however it is consistent that the converse is false; It is shown in [46] that in a model obtained by a countable support iteration of proper forcings over a model of CH, alternatingly adding $\mathbf{PT}_{f,g}$ reals (Definition 7.3.3 of [3]) followed by a P-ideal dichotomy forcing, there is universal graph on ω_1 but no universal function on ω_1.

One could more generally look at universal objects as being functions $F : \kappa \times \lambda \longrightarrow \alpha$, where F' embeds to F if there are functions $h : \kappa \longrightarrow \kappa,\ k : \lambda \longrightarrow \lambda$ with $(\forall x, y \in \kappa \times \lambda)\ F'(x, y) = F(h(x), k(y))$, and this more general definition is investigated in [28]. Alternatively, one could weaken the requirement of embedding so that the universal selects a set of possibilities rather than an exact value; specifically, define $U : [\kappa]^2 \longrightarrow \mathcal{A}$ to be $\mathcal{A}$-weakly universal if for every $F : [\kappa]^2 \longrightarrow \omega$ there exists a one to one function $f : \kappa \longrightarrow \kappa$ such that $(\forall x, y \in \kappa)\ F(x, y) \in U(f(x), f(y))$. This weak form of universality is investigated in [47]. The investigation of universals in this thesis, unless otherwise mentioned, will focus on universal functions on $\kappa = \omega_1$ as defined by definition 3.1.7.

3.2 Universals and Martin's Axiom

The existence of a universal graph or function on ω_1 is implied by CH, since there is a saturated model of any theory of size ω_1. However, in the Cohen model there is no universal graph ([44],[47], or see lemma 3.2.4) or linear order [43]. The question if a universal graph on ω_1 can exist in a model with the negation of CH, was answered positively in [44], with a model obtained by a c.c.c. finite support iteration. That argument can also be easily strengthened to force a universal function on ω_1. In the next two sections, the question of if a universal function or graph on ω_1 can exist in a model with the negation of CH and MA will be investigated, with some partial answers.

As a partial result towards a model of MA $\neg$CH and a universal function on ω_1, it will be shown that forcing with Suslin trees to destroy them can be included into the iteration in [44], and so a model of $\neg$CH, SH, MA(Cohen), and a universal function on ω_1 is obtained.

The proper countable support iterations of Miller forcings followed by P-ideal dichotomy forcings in the work done independently by Steprāns and Shelah in [46], can also be augmented to model $\neg$CH, SH, MA(Cohen), and "there is a universal graph on ω_1": One can interlace forcing Suslin trees to model SH since they do not add reals, and force a Cohen real at successor of limit ordinals to model MA(Cohen) since adding a Cohen real preserves $\sqsubseteq^{\text{Cohen}}$ (Lemma 6.3.18 in [3]). However it is unclear whether a universal function (Definition 3.1.7) exists in this model as it does in the results of this chapter. Indeed, a universal function does not exist in the $\mathbf{PT}_{f,g}$ iteration defined in [46]. Additionally, the other iterations in [46] which rely on a ground model set of full outer measure becoming universal in the extension, cannot add Cohen reals cofinally, since adding a Cohen real makes all ground model sets null.

In the other direction, it is consistent to have a model of MA and no universal graph on ω_1 (and thus also no universal function on ω_1) ([28] corollary 5.22). Furthermore, Lemma 5.16 in [28], shows that Hechler forcing $\mathbb{H}$ is such that no c.c.c. forcing in $V^{\mathbb{H}}$ can force that there was a function in V that is now universal function on ω_1. This shows that an argument using a c.c.c. forcing over a ground model that may satisfy $2^{\omega_1} = \omega_2$ and CH, to obtain a model with a universal function on ω_1, $\neg$CH, and MA, like in [44] or section 3's, cannot work. In particular this shows that the usual model of MA and $\neg$CH obtained by iterating c.c.c. forcings with finite support, has no universal function on ω_1. By a similar argument, lemma 3.2.4 shows that adding a Cohen real is such that no Knaster forcing in the Cohen extension can force that there was a graph in V that is now a universal graph on ω_1. This shows that an argument using a Knaster forcing over a ground model that may satisfy $2^{\omega_1} = \omega_2$ and CH, to obtain a model with a universal graph on ω_1, $\neg$CH, and MA(Knaster), cannot work.

Interestingly, the model augmenting the Miller iteration of [46] to model ¬CH, SH, MA(Cohen), and "there is a universal graph on ω_1", also cannot be augmented to model MA(Hechler), since Hechler forcing makes ground model sets meagre.

The only relative consistency question left open with MA, ¬CH, and the existence of a universal function or graph on ω_1, is that of a model of MA + ¬CH + "there is a universal on ω_1". In the next section we will modify the construction in [44] to obtain the following theorem:

Theorem 3.2.1. *Assuming the consistency of ZFC, there is model of ZFC satisfying ¬CH + SH + MA(countable) + "there is a universal function on ω_1".*

By the aforementioned Lemma 5.16 in [28], this theorem is about the best one can hope for towards model of a universal function ω_1, ¬CH, and MA with an argument using a c.c.c. forcing which forces MA_{ω_1} and makes a function on ω_1 in the ground model into a universal.

First for the sake of completeness, we will first include the proof of Lemma 5.16 in [28]. The proof uses the following fact:

Lemma 3.2.2. *There is $F : [\omega_1]^2 \longrightarrow \omega$ such that*

1. $(\forall X \in [\omega_1]^{\omega_1})(\forall n \in \omega)(\exists Y \in [\omega_1]^{\omega_1})(\forall \alpha, \beta \in Y)\ F(\alpha, \beta) > n$,
2. $(\forall \alpha \in \omega_1)\ F_\alpha : \alpha \longrightarrow \omega$ *defined by* $F_\alpha(\beta) = F(\alpha, \beta)$ *is one to one.*

Proof. Fix some set of distinct reals $\{r_\alpha\}_{\alpha<\omega_1} \subseteq \omega^\omega$, and some bijection $\phi : \omega^{<\omega} \longrightarrow \omega$. For $0 < \alpha < \omega_1$ let $\{\beta_n\}_{n<\omega} = \alpha$, and define $F(\alpha, \beta_n)$ inductively by setting $F(\alpha, \beta_n)$ to be ϕ applied to any element of $\omega^{<\omega}$ extending $r_{\beta_n} \restriction (\Delta(r_\alpha, r_{\beta_n}) + 2)$ that results in a different value than $F(\alpha, \beta_0), ..., F(\alpha, \beta_{n-1})$, where $\Delta(r_\alpha, r_\beta) = \min\{n < \omega : r_\alpha(n) \neq r_\beta(n)\}$.

It is clear from the definition that F satisfies property 2. To see that F satisfies 1, let $n < \omega$ and $X \in [\omega_1]^{\omega_1}$. Refine X to $Y \in [\omega_1]^{\omega_1}$ so that there is $m > 2 + \max\{|\phi^{-1}(i)| : i \leq n\}$ such that for all $\alpha \in Y$, $r_\alpha \restriction m$ is the same function.

□

Lemma 3.2.3 (Lemma 5.16 in [28])**.** *Let $\mathbb{H}$ denote Hechler forcing. There is no c.c.c. forcing $\mathbb{P}$ in $V^{\mathbb{H}}$ such that there is $U \in V$ on ω_1 with $\mathbb{1}_{\mathbb{P}} \Vdash$ "U is a universal function on ω_1".*

Proof. Suppose $\mathbb{P}$ and U are counterexamples to the above statement. Let r denote the generic Hechler real, and define $G : [\omega_1]^2 \longrightarrow \omega$ by

$$G(\alpha, \beta) = r(F(\alpha, \beta)).$$

Let $\mathbb{1}_{\mathbb{P}} \Vdash$ "$h : \omega_1 \longrightarrow \omega_1$ embeds G to U", and set $\{(s_\alpha, f_\alpha, p_\alpha, \delta_\alpha)\}_{\alpha<\omega_1}$, to be a family where $(s_\alpha, f_\alpha) \in \mathbb{H}$ (meaning s_α is a finite partial function from ω to ω and $f_\alpha \in \omega^\omega$), $p_\alpha \in \mathbb{P}$, $\delta_\alpha < \omega_1$, and

$$(s_\alpha, f_\alpha, p_\alpha) \Vdash h(\alpha) = \delta_\alpha.$$

Let $Y \in [\omega_1]^{\omega_1}$ be such that for all $\alpha, \beta \in Y$, s_α equals some fixed s not depending on α, and $F(\alpha, \beta) > |s|$. Define $g_\alpha : \{F_\alpha(\beta)) : \beta < \alpha\} \longrightarrow \omega$ by $g_\alpha(F(\alpha, \beta)) = U(\delta_\alpha, \delta_\beta)$, recalling that F_α is one to one. Since $\mathbb{H} * \mathbb{P}$ is c.c.c., there must be $\beta < \alpha \in Y$ with $(s, f_\alpha + g_\alpha + 1, p_\alpha)$ and $(s, f_\beta + g_\beta + 1, p_\beta)$ compatible, and therefore they have a joint extension (s', f, q) with $F(\alpha, \beta) \in \mathrm{dom}(s')$. This provides us with a contradiction since

$$(s', f, q) \Vdash G(\alpha, \beta) = r(F(\alpha, \beta)) > g_\alpha(F(\alpha, \beta)) = U(\delta_\alpha, \delta_\beta) = U(h(\alpha), h(\beta)).$$

□

It is unknown if the above lemma can be proven for universal graphs instead of functions, but a similar result holds:

Lemma 3.2.4. *Let $\mathbb{C}$ denote Cohen forcing. There is no Knaster forcing $\mathbb{P}$ in $V^{\mathbb{C}}$ such that there is $U \in V$ on ω_1 with $\mathbb{1}_{\mathbb{P}} \Vdash$ "U is a universal graph on ω_1". Note that since Hechler forcing adds a Cohen real and is Knaster, this also holds if $\mathbb{C}$ is replaced with $\mathbb{H}$.*

Proof. Suppose $\mathbb{P}$ and U are counterexamples to the above statement. Let c denote the generic Cohen real, and define $G : [\omega_1]^2 \longrightarrow 2$ by

$$G(\alpha, \beta) = c(F(\alpha, \beta)).$$

Let $\mathbb{1}_{\mathbb{P}} \Vdash$ "$h : \omega_1 \longrightarrow \omega_1$ embeds G to U", and set $\{(q_\alpha, p_\alpha, \delta_\alpha)\}_{\alpha<\omega_1}$, to be a family where $(q_\alpha, p_\alpha) \in \mathbb{C} * \mathbb{P}$, $\delta_\alpha < \omega_1$, and

$$(q_\alpha, p_\alpha) \Vdash h(\alpha) = \delta_\alpha.$$

Let $Y \in [\omega_1]^{\omega_1}$, $q \in \mathbb{C}$ be such that for all $\alpha, \beta \in Y$, $q_\alpha = q$ and $F(\alpha, \beta) > |q|$. Since $\mathbb{1}_{\mathbb{C}} \Vdash$ "$\mathbb{P}$ is Knaster", there must be a name for a set $Y' \in [Y]^{\omega_1}$ with

$$q \Vdash (\forall \{\alpha, \beta\} \in [Y']^2)\ p_\alpha \not\perp p_\beta.$$

There is a ground model $X \in [\omega_1]^{\omega_1}$ such that $q \Vdash X \subseteq Y'$. Pick any $\{\alpha, \beta\} \in [X]^2$. Let p be such that $q \Vdash p \leq p_\alpha, p_\beta$, and extend q to q' by setting $q'(F(\alpha, \beta)) = |1 - U(\delta_\alpha, \delta_\beta)|$. This provides us with a contradiction since

$$(q', p) \Vdash G(\alpha, \beta) = c(F(\alpha, \beta)) \neq U(\delta_\alpha, \delta_\beta) = U(h(\alpha), h(\beta)).$$

□

Note that this also shows that adding $\aleph_2$ Cohen reals forces a model with no universal graph on ω_1.

3.3 Universal functions, Suslin's Hypothesis, and MA(Cohen), without CH

Now we work towards proving theorem 3.2.1, proceeding in the same way as [28]. The argument in [28] is done for universal graphs and not universal functions as done in this chapter, but the argument is essentially the same for functions and graphs. Start by obtaining a ground model satisfying $2^{\omega_1} = \omega_2$ and containing a family $\{S_\beta\}_{\beta<\omega_2}$ of almost disjoint stationary subsets of ω_1. To obtain such a ground model, we first start with a model of GCH and $\Diamond_{\omega_1}$, which gives us a family $\{B_\beta\}_{\beta<\omega_2}$ of stationary sets with countable pairwise intersection: if $\{X_\beta\}_{\beta<\omega_1}$ is a diamond sequence and $\{E_\beta\}_{\beta<\omega_2}$ is an almost disjoint family subsets of ω_1, set $B_\beta = \{\alpha : X_\alpha = E_\beta \cap \alpha\}$. From this model, we then extend with a c.c.c. forcing $\mathbb{P}$ of size ω_2, where $\mathbb{P}$ is the set of all finite partial functions $f_p : \omega_2 \longrightarrow [\omega_1]^{<\omega}$ with $(\forall \alpha \in \mathrm{dom}(f_p))\ f_p(\alpha) \subseteq B_\alpha$, ordered by

$$f_p \leq f_q \iff (\forall \alpha, \beta \in \mathrm{dom}(f_q))\ f_q(\alpha) \subseteq f_p(\alpha) \text{ and } f_p(\alpha) \cap f_p(\beta) = f_q(\alpha) \cap f_q(\beta).$$

We set our ground model as the resulting extension, and $\{S_\beta\}_{\beta<\omega_2} = \{\bigcup_{p\in\Gamma} f_p(\beta)\}_{\beta<\omega_2}$ where Γ is the generic filter of $\mathbb{P}$.

We will obtain a model as in the theorem, by extending our ground model with a c.c.c. finite support forcing iteration $(\mathbb{P}_\alpha, \mathbb{Q}_\alpha)_{\alpha<\omega_2}$. We will sometimes need to refer to a finite support iteration $(\mathbb{P}'_\alpha, \mathbb{Q}'_\alpha)_{\alpha<\omega_2}$, of which $(\mathbb{P}_\alpha, \mathbb{Q}_\alpha)_{\alpha<\omega_2}$ is a suborder. We start the iteration with $\mathbb{Q}_0 = \mathbb{Q}'_0$ forcing a generic function $F_0 : [\omega_1]^2 \longrightarrow \omega$ with, finite conditions

$$p = (w_p, F_{p(0)}), w_p \in [\omega_1]^{<\omega}, F_{p(0)} : [w_p]^2 \longrightarrow \omega,$$

ordered by extension. For the rest of the iteration we alternate forcing with names for Suslin trees on odd ordinals, and forcing an embedding from a name for a function on ω_1 to the generic graph in the first coordinate, on positive even ordinals. More specifically, let $0 < \alpha < \omega_2$, and we define the iteration inductively, assuming $\mathbb{P}_\alpha$ is defined, as follows:

For even α, we are given a $\mathbb{P}_\alpha$ name $F_\alpha : [\omega_1]^2 \longrightarrow \omega$, and we let $\mathcal{M}_\alpha = \{M_{\alpha,\xi}\}_{\xi<\omega_1}$ be an increasing sequence of countable elementary submodels of some large enough H_θ, containing as elements $\mathbb{P}_\alpha, \omega_1$, the name F_α, and such that the heights $\{M_{\alpha,\xi} \cap \omega_1\}_{\xi<\omega_1} = \{\delta_{\alpha,\xi}\}_{\xi<\omega_1} \subseteq S_\alpha$ are the intersection of a club with S_α. For each $\xi < \omega_1$, define $Y_{\alpha,\xi}$ as the first ω limits of sequences in $\{\delta_{\alpha,\xi}\}_{\xi<\omega_1}$ greater than ξ. We can now define $\mathbb{Q}_\alpha$ in $V^{\mathbb{P}_\alpha}$ as the set of all conditions $p = (f_p, X_p)$ satisfying:

1. f_p is a finite partial injective function from ω_1 to ω_1 with range disjoint from X_p (we call X_p the range restriction elements).

2. $(\forall (\xi, \lambda) \in f_p)\ \lambda \in Y_{\alpha,\xi}$.

3. $(\forall (\xi, \lambda), (\xi', \lambda') \in f_p)\ F_0(\lambda, \lambda') = F_\alpha(\xi, \xi')$,

and ordered by $p \leq q \longleftrightarrow f_q \subseteq f_p, X_q \subseteq X_p$. We also denote by $\mathbb{Q}'_\alpha$ the set of all conditions satisfying all of the above, minus requirement 3, and ordered in the same way.

For odd α, we are given a $\mathbb{P}_\alpha$ name $(T'_\alpha, \leq_{T'_\alpha}) = (\omega_1, \leq_{T'_\alpha})$ for a Suslin tree, arranged so that the γ'th level of the tree is

$$\mathcal{L}_\gamma = \begin{cases} \{\omega\gamma + n : n \in \omega\} \text{ if } \gamma \geq \omega \\ \{\omega(\gamma - 1) + n : n \in \omega\} \text{ if } 0 < \gamma < \omega \\ \{\emptyset\} \text{ if } 0 = \gamma \end{cases} .$$

For each $\xi < \omega_1$, we pick $M_{\alpha,\xi}$ a countable elementary submodel of some large enough H_θ, containing containing as elements $\mathbb{P}_\alpha, (\omega_1, \leq_{T'_\alpha}), \{M_{\alpha,\lambda}\}_{\lambda<\xi}, \xi$. Denote $\mathcal{M}_\alpha = \{M_{\alpha,\lambda}\}_{\lambda<\omega_1}$ and $\{\delta_{\alpha,\lambda}\}_{\lambda<\omega_1} = \{M_{\alpha,\xi} \cap \omega_1\}_{\xi<\omega_1}$. Now working in $V^{\mathbb{P}_\alpha}$, we define a tree $(T_\alpha, \leq_{T_\alpha}) = (\{\delta_{\alpha,\lambda}\}_{\lambda<\omega_1}, \leq_{T_\alpha})$ isomorphic to $(T'_\alpha, \leq_{T'_\alpha})$, by setting

$$(\forall \xi, \lambda < \omega_1)\ \delta_{\alpha,\xi} \leq_{T_\alpha} \delta_{\alpha,\lambda} \longleftrightarrow \xi \leq_{T'_\alpha} \lambda.$$

Denote by $\mathbb{Q}'_\alpha$ the set of all finite subsets in T_α, identifying $\emptyset, \{\emptyset\}$ as the same, ordered by

$$c' \leq c \longleftrightarrow \max(c) \leq_{T_\alpha} \max(c') \text{ and } c \subseteq c'.$$

Define $\mathbb{Q}_\alpha$ as the suborder of $\mathbb{Q}'_\alpha$ in which conditions are finite chains in T_α. Note that $\mathbb{Q}_\alpha$, $\mathbb{Q}'_\alpha$, T_α, T'_α are all forcing equivalent, and are defined differently for notational convention and bookkeeping purposes.

If we show inductively at each stage α of the iteration that $\mathbb{P}_\alpha$ is c.c.c., then we could have selected every $\mathbb{P}_{\omega_2}$ name for a function on ω_1 or Suslin tree, at some stage in the iteration. In addition, since Cohen reals are added at limit stages, we add Cohen reals cofinally. This will force that $V^{\mathbb{P}_{\omega_2}}$ is a model of ¬CH + SH + MA(*countable*)+ "there is a universal function on ω_1". To finish the proof of the theorem, the rest of the chapter works toward a proof that the iteration is c.c.c. by induction. Clearly $\mathbb{P}_1$ is c.c.c., so we fix an $0 < \alpha < \omega_2$ and assume $\mathbb{P}_\alpha$ is c.c.c.. Without loss of generality α is even, since if α is odd $\mathbb{Q}_\alpha$ is forced to be c.c.c., and so $\mathbb{P}_{\alpha+1}$ is also c.c.c.. We sometimes abuse notation and identify, for $\eta < \beta \leq \alpha + 1$, $p \in \mathbb{P}_\eta$ with the condition $p' \in \mathbb{P}_\beta$ that has the same support as p and is equal to p on that support. For the following notations and definitions, let $\beta \leq \alpha + 1$.

Definition 3.3.1. Define

$$D^1_\beta = \{p \in \mathbb{P}'_\beta : p \text{ is a determined finite set in all coordinates}\},$$

and

$$D^0_\beta = \mathbb{P}_\beta \cap D^1_\beta.$$

These sets are clearly dense in their respective partial orders.

Notation 3.3.1.

Let $p \in D^1_\beta$ with $p(0) = (w, F_{p(0)})$. Let $\text{even}(p), \text{odd}(p)$ denote the positive even ordinals in $\text{supp}(p)$, and the odd ordinals in $\text{supp}(p)$, respectively. For $\eta < \beta, \delta < \omega_1$, we denote

$$\text{dom}(p(0)) = w,\ \text{range}(p(\eta)) = \text{range}(f_{p(\eta)}),\ \text{dom}(p(\eta)) = \text{dom}(f_{p(\eta)}),$$

$$p(\eta)^{(\delta)} = \begin{cases} \{(\xi, \lambda) \in f_{p(\eta)} : \lambda < \delta\} \cup \{\lambda \in X_{p(\eta)} : \lambda < \delta\}\} \text{ if } \eta \in \text{even}(p) \\ \{\gamma \in p(\eta) : \gamma < \delta\} \text{ if } \eta \in \text{odd}(p) \\ (w \cap \delta, F_{p(0)} \restriction [w \cap \delta]^2) \text{ if } \eta = 0 \end{cases}$$

$$p(\eta)^{[\delta]} = \begin{cases} \{(\xi, \lambda) \in f_{p(\eta)} : \lambda \leq \delta\} \cup \{\lambda \in X_{p(\eta)} : \lambda \leq \delta\}\} \text{ if } \eta \in \text{even}(p) \\ \{\gamma \in p(\eta) : \gamma \leq \delta\} \text{ if } \eta \in \text{odd}(p) \\ (w \cap (\delta+1), F_{p(0)} \restriction [w \cap (\delta+1)]^2) \text{ if } \eta = 0 \end{cases}$$

We then define $p^{(\delta)}, p^{[\delta]} \in D^1_\beta$ as

$$(\forall \eta < \beta)\ p^{(\delta)}(\eta) = p(\eta)^{(\delta)} \text{ and } \rho^{[\delta]}(\eta) = p(\eta)^{[\delta]}.$$

Note that for all $p \in \mathbb{P}'_\beta$, $p^{(\delta)} = p^{[\delta+1]}$. For $M \in \mathcal{M}_\beta$, denote

$$p \sqcap M$$

to be the function defined on $M \cap \beta$ as $(p \sqcap M)(\eta) = p(\eta)^{(\text{height}(M))}$. Note that $p \sqcap M \in M$, and also note that if $p \in D^0_\beta$, $p \sqcap M$ may only be in D^1_β, but in M there must be an extension of $p \sqcap M$ into D^0_β by elementarity of M.

Notation 3.3.2. For two conditions $p, q \in D^1_\beta$, denote $p \sqcup q$ to be defined on $\eta < \beta$ by

$$p \sqcup q(\eta) = \begin{cases} \big(\text{dom}(p(0)) \cup \text{dom}(q(0)), F_{p(0)} \cup F_{q(0)} \cup 1_{p(0) \times q(0)}\big) \text{ if } \eta = 0 \\ p(\eta) \cup q(\eta) \text{ otherwise} \end{cases}.$$

Note that $p \sqcup q$ is not necessarily in $\mathbb{P}_\beta$ if p, q are, but, for D_β as defined directly below, we have that if $p \in D_\beta$, and for some $\eta \leq \beta$, $q \in D_\eta$ extends $p \restriction \eta$, then $p \sqcup q \in D_\beta$ extending p, q.

Definition 3.3.2. Define

$$\begin{aligned} D_\beta = \{p \in D^0_\beta : (\forall \delta \in \omega_1)\ & p^{(\delta)} \in D^0_\beta \\ & \text{and } (\forall \eta \in \text{even}(p))\ \delta \in \text{range}(p(\eta)) \longrightarrow p^{(\delta)} \restriction \eta \text{ decides } F_\eta \restriction [\text{dom}(p(\eta)) \cap \delta]^2 \\ & \text{and } (\forall \eta \in \text{supp}(p) \setminus \{0\})\ p^{(\delta)} \restriction \eta \Vdash p(\eta)^{[\delta]} \in \mathbb{Q}_\eta\}, \end{aligned}$$

and for $p \not\perp q \in D^1_\beta$, let $D_{p,q,\beta}$, $D_{p,\beta}$ be notation for the set of all conditions in D_β which extend p and q, or extend p, respectively.

Note that since $p, q \in D^1_\beta$ we could have that $D_{p,q,\beta}$, $D_{p,\beta}$ are empty. Also, if $p \in D_\beta$, then so are $p^{(\delta)}, p^{[\delta]}$ for all $\delta < \omega_1$.

Lemma 3.3.3. *The set D_β is dense for all $\beta \leq \alpha + 1$.*

Proof. Let $p \in \mathbb{P}_\beta$, and without loss of generality we can assume $p \in D^0_\beta$ and $\mathrm{range}(p(\beta)) \neq \emptyset$. We do this by an induction on $\beta \leq \alpha + 1$. For $\beta = 0, 1$ there is nothing to show, so assume the lemma holds for $1 \leq \beta \leq \alpha$ and we will prove it holds for $\beta + 1$. Let r be in D_β extending $p \restriction \beta$.

First assume β is even. Let $0 = \delta_0 < ... < \delta_n$ enumerate $\mathrm{range}(p(\beta)) \cup \{0\}$. We will inductively get a sequence $r = r_0 \geq ... \geq r_n \in D_\beta$ such that, for $l \leq n$,

$$r_l^{(\delta_l)} \Vdash p(\beta)^{(\delta_l)} \in \mathbb{Q}_\beta, \text{ and } r_l^{(\delta_l)} \text{ decides } F_\beta \restriction [\mathrm{dom}(p(\beta)) \cap \delta_l]^2.$$

If we have this then note that $r' = r_n \cup p$ is in $D_{\beta+1}$ below p; It suffices to show $r'^{(\delta)} \in D^0_{\beta+1}$ for all $\delta \in \omega_1 \setminus \mathrm{range}(p(\beta))$, since the rest is clear. For any $\delta < \omega_1$, letting $0 < l$ be such that $\delta_l < \delta$ is maximal (if $\delta < \delta_1$ then $r'^{[\delta]}(\beta) \in \mathbb{Q}_\beta$), let $\xi < \delta_l$ be such that $f_{p(\beta)}(\xi) = \delta_l$, and noting that

$$r'^{(\delta)} \restriction \beta \leq r_l^{(\delta_l)} \Vdash p(\beta)^{(\delta_l)} \in \mathbb{Q}_\beta,$$

we just need to see that

$$(\forall (\xi', \lambda) \in f_{p(\beta)^{(\delta_l)}})\ r_n^{(\delta_l)} \Vdash F_{r_n(0)}(\lambda, \delta_l) = F_\beta(\xi, \xi'),$$

which holds because $r_n^{(\delta_l)}$ decides $F_\beta(\xi, \xi')$ in the same way as $r_n \leq p \restriction \beta$, and

$$p \restriction \beta \Vdash F_\beta(\xi, \xi') = F_{p(0)}(\lambda, \delta_l) = F_{r_n(0)}(\lambda, \delta_l).$$

For $l = 0$ there is nothing to show, so assume we have r_l and we will construct r_{l+1}. Let $M \in \mathcal{M}_\beta$ be the model with height δ_{l+1}. The set

$$D = \{r \in D_\beta : r \Vdash \text{“}p(\beta)^{(\delta_{l+1})} \in \mathbb{Q}_\beta\text{” and decides } F_\beta \restriction [\mathrm{dom}(p(\beta)) \cap \delta_{l+1}]^2, \text{ or } r \text{ has no extension with this}\}$$

is dense and in M. By our hypothesis that $\mathbb{P}_\alpha$ is c.c.c. (so $\mathbb{P}_\beta$ is also c.c.c.), $M \cap D$ is predense, so we can pick $r' \in M \cap D$ compatible with r_l, and since r_l forces $p(\beta)^{(\delta_{l+1})} \in \mathbb{Q}_\beta$, and decides $F_\beta \restriction (\mathrm{dom}(p(\beta)) \cap \delta_{l+1})$, r' does also. We can then set r_{l+1} to be any element of D_β extending r', r_l, and we are done because $r_{l+1}^{(\delta_{l+1})} \leq r'$.

Now assume β is odd, and proceed similarly to the even case. Let $\delta_0 < ... < \delta_n$ enumerate $p(\beta)$. We will inductively get a sequence $r = r_0 \geq ... \geq r_n \in D_\beta$ such that, for $l \leq n$,

$$r_l^{(\delta_l)} \Vdash p(\beta)^{[\delta_l]} \in \mathbb{Q}_\beta.$$

If we have this then note that $r' = r_n \cup p$ is in $D_{\beta+1}$ below p; let l be such that $\delta_l \leq \delta$ is maximal (if $\delta < \delta_0$ then $r'^{[\delta]}(\beta) = \mathbb{1} \in \mathbb{Q}_\beta$), and see that

$$r'^{(\delta)} \restriction \beta \leq r_l^{(\delta_l)} \Vdash r'^{[\delta]}(\beta) = p(\beta)^{[\delta_l]} \in \mathbb{Q}_\beta.$$

For $l = 0$ there is nothing to prove since $p(\beta)^{[\delta_0]} = \{\delta_0\} \in \mathbb{Q}_\beta$, so assume we have r_l and we will construct r_{l+1}. Let $M = M_{\beta,\lambda}$, and $M_{\beta,\xi}$, be our models in $\mathcal{M}_\beta$ with height δ_{l+1}, δ_l respectively. The set

$$\begin{aligned} D &= \{r \in D_\beta : r \Vdash \delta_l \leq_{T_\beta} \delta_{l+1}, \text{ or } r \text{ has no extension satisfying this}\} \\ &= \{r \in D_\beta : r \Vdash \xi \leq_{T'_\beta} \lambda, \text{ or } r \text{ has no extension satisfying this}\} \end{aligned}$$

is dense, and in M since $\xi, \lambda, (T'_\beta, \leq_{T'_\beta}) \in M$. Since $\mathbb{P}_\beta$ is c.c.c., $M \cap D$ is predense, so we can pick $r' \in M \cap D$ compatible with r_l, and because r_l forces $p(\beta)^{[\delta_{l+1}]} \in \mathbb{Q}_\beta$, it forces "$\delta_l \leq_{T_\beta} \delta_{l+1}$", and so r' must also force this by the definition of D. Now we set r_{l+1} to be any element of D_β extending r', r_l, and we are done because

$$r_{l+1}^{(\delta_{l+1})} \leq r' \Vdash \delta_l \leq_{T_\beta} \delta_{l+1} \text{ and } r_{l+1}^{(\delta_{l+1})} \leq r_l^{(\delta_l)} \Vdash \text{“}p(\beta)^{[\delta_l]} = p(\beta)^{(\delta_{l+1})} \in \mathbb{Q}_\beta\text{”}.$$

□

Definition 3.3.4. Let $\delta < \delta' < \omega_1$, $\beta \leq \alpha + 1$. We say that $p \in \mathbb{P}_\beta$ is robust in (δ, δ'), if

(1) $(\forall \eta \neq \eta' \in \text{even}(p))(\forall \lambda \in \text{dom}(p(0)))\ S_\eta \cap S_{\eta'} \subseteq \text{dom}(p(0))$ and $\lambda \in (X_{p(\eta)}) \cup (\text{range}(p(\eta)))$,

and there are elements $\gamma_\eta \in \omega_1$, for all $\eta \in \text{supp}(p)$, with

(2) $\eta \in \text{odd}(p)$, $\gamma_\eta \in p(\eta)$ and

(*i*) $\gamma_\eta \in (\delta, \delta')$ and $\gamma_\eta > \bigcup\{\gamma_{\eta'} : \eta' \in \text{supp}(p),\ \eta' > \eta\}$ or ,

(*ii*) $\gamma_\eta < \delta'$ and $p^{(\gamma_\eta)} \restriction \eta$ forces all elements of T_η in (γ_η, δ') are incompatible with γ_η,

or

(3) $\eta \in \text{even}(p)$, $\gamma_\eta \in (\delta, \delta')$, $\gamma_\eta > \hat{\gamma}_\eta = \sup(\{\gamma_{\eta'} : \eta' \in \text{supp}(p),\ \eta' > \eta\} \cup \text{dom}(p(\eta)^{[\delta']}) \cup \text{range}(p(\eta)^{(\delta')}))$ and

(*i*) $\gamma_\eta \in \{\delta_{\eta,\xi}\}_{\xi<\omega_1}$ or ,

(*ii*) $(\hat{\gamma}_\eta, \delta') \cap \{\delta_{\eta,\xi}\}_{\xi<\omega_1} = \emptyset$

Let $D_\beta^{(\delta,\delta')}$ be the set of conditions in D_β that are robust in (δ, δ').

Lemma 3.3.5. *For all $\beta \leq \alpha + 1$, $\delta < \delta' < \omega_1$, $D_\beta^{(\delta,\delta')}$ is dense. Moreover, if $p \in D_\beta$ and $\eta + 1 = \beta$ for some $\eta \in \text{even}(p)$, then there is $p' \in D_{\eta+1}^{(\delta,\delta')}$ extending p with $f_{p'(\eta)} = f_{p(\eta)}$.*

Proof. We will prove density by induction on $\beta \leq \alpha + 1$. Trivial for $\beta = 0$. Let $\beta \leq \alpha$, $p \in D_{\beta+1}$, and assume $D_\beta^{(\delta,\delta')}$ is dense.

First suppose $\beta \in \text{even}(p)$. There is some $\gamma_\beta \in (\delta, \delta')$ with $\sup(\text{dom}(p(\beta)^{[\delta']})) \cup \text{range}(p(\eta)^{(\delta')}) < \gamma_\beta < \delta'$, and either $\gamma_\beta \in \{\delta_{\beta,\xi}\}_{\xi<\omega_1}$ or $(\hat{\gamma}_\eta, \delta') \cap \{\delta_{\beta,\xi}\}_{\xi<\omega_1} = \emptyset$. Take $p' \leq p \restriction \beta$ in $D_\beta^{(\gamma_\beta,\delta')}$ and define $q \leq p' \mathbin{\underline{\cup}} p$

in $D^{(\delta,\delta')}_{\beta+1}$, by defining $q(0)$ to be any extension of $p'(0)$ to a domain including $\bigcup_{\eta\in\text{even}(p')} S_\eta \cap S_{\beta'}$, and for $0 < \eta \leq \beta$ setting

$$q(\eta) = \begin{cases} p'(\eta) \text{ if } \eta \in \text{odd}(p') \\ (f_{p'(\eta)}, X_{p'(\eta)} \cup (\text{dom}(q(0)) \setminus \text{range}(p'(\eta))) \text{ if } \eta \in \text{even}(p') \\ (f_{p(\eta)}, X_{p(\eta)} \cup (\text{dom}(q(0)) \setminus \text{range}(p(\eta))) \text{ if } \eta = \beta. \end{cases}$$

Now assume $\beta \in \text{odd}(p)$. Suppose there is an extension p' of $p \restriction \beta$ deciding some ordinal $\gamma \in (\delta, \delta')$ in T_β is comparable with all of $p(\eta)$. Let $q \in D_{\beta+1}$ be an extension of $p'^\frown(p(\beta) \cup \{\gamma\})$, and then take $p'' \in D^{(\gamma,\delta')}_\beta$ extending $q \restriction \beta$. We have $p''^\frown q(\beta)$ is in $D^{(\delta,\delta')}_{\beta+1}$ below p.

If there is no such p', then there must be a maximum level $\mathcal{L}$ of T_η strictly below δ' and finitely many elements in $T_\eta \cap [\sup(\mathcal{L}), \delta')$. In this case we can pick $p' \in D_\beta$ below $p \restriction \beta$ with some $\gamma \in T_\beta,\ \gamma < \delta'$, such that p' forces γ is comparable to all of $p(\eta)$ and no element of $T_\eta \cap (\gamma, \delta')$ is compatible with all of $p(\eta)$.

Let M be the model in $\mathcal{M}_\beta$ with height γ, $x \in M$ the element of T'_β isomorphically mapped to γ, and $X \in M$ the interval of T'_β isomorphically mapped to $(\gamma, \delta') \cap T_\beta$ ($X \in M$ since $\sup(X) < x + \omega + \omega \in M$). The set

$$\begin{aligned} D &= \{r \in D_\beta : r \Vdash \text{“everything in } (\gamma,\delta') \cap T_\beta \text{ is } \leq_{T_\beta} \text{ incompatible with } \gamma\text{”, or } r \text{ has no extension forcing this}\} \\ &= \{r \in D_\beta : r \Vdash \text{“everything in } X \text{ is } \leq_{T'_\beta} \text{ incompatible with } x\text{”, or } r \text{ has no extension forcing this}\}, \end{aligned}$$

is dense and in M. Since $\mathbb{P}_\beta$ is c.c.c, and $p' \Vdash$ “everything in $(\gamma, \delta') \cap T_\beta$ is $\leq_{T_\beta}$ incompatible with γ”, there is $p'' \in M \cap D$ comparable with p' and forcing “everything in $(\gamma, \delta') \cap T_\beta$ is $\leq_{T_\beta}$ incompatible with γ”. Let $q \in D_{\beta+1}$ be a common extension of $p''^\frown(p(\beta) \cup \{\gamma\}), p'^\frown(p(\beta) \cup \{\gamma\})$, and then take $p''' \in D^{(\max\{\gamma,\delta\},\delta')}_\beta$ extending $q \restriction \beta$. We have $p'''^\frown q(\beta)$ is in $D^{(\delta,\delta')}_{\beta+1}$ below p.

□

Definition 3.3.6. Let $p \in \mathbb{P}_\beta$. We say p is everywhere robust if, for $\delta_0 < ... < \delta_n$ enumerating $\{\delta : (\exists \eta \in \text{even}(p))\ \delta \in \text{range}(p(\eta))\}$ and $\eta_0, ..., \eta_n \in \text{even}(p)$ such that η_i is the maximal ordinal with $\delta_i \in \text{range}(p(\eta_i))$, we have

$$(\forall i < j < n)\ \eta = \min\{\eta_i, \eta_j\} \longrightarrow p \in D^{(\delta_i,\delta_j)}_{\eta+1}.$$

Lemma 3.3.7. *Everywhere robust conditions are dense in $\mathbb{P}_\beta$.*

Proof. Using lemma 3.3.5 recursively, construct $\{p_l\}_{l<\omega}, \{\beta_l\}_{l<\omega}$ with $p_0 = p,\ \beta_0 = \max(\text{even}(p))$, and for all $l < \omega$:

1. $p_{l+1} \leq p_l$.

2. $\beta_l > \beta_{l+1} = \max(\text{even}(p_{l+1}) \cap \beta_l)$ or $\beta_{l+1} = \beta_l = \min(\text{even}(p_l))$.

3. $f_{p_{l+1}}(\beta_l) = f_{p_l}(\beta_l)$.

4. $p_{l+1} \restriction (\beta_l, \beta) = p_l \restriction (\beta_l, \beta)$.

5. For $\delta_0 < ... < \delta_n$ enumerating $\{\delta : (\exists \eta \in \text{even}(p_l) \setminus \beta_l)\ \delta \in \text{range}(p_l(\eta))\}$, we have

$$(\forall i < j < n)\ \delta_i \in \text{range}(p_l(\beta_l)) \text{ or } \delta_j \in \text{range}(p_l(\beta_l)) \longrightarrow p_{l+1} \in D^{(\delta_i,\delta_j)}_{\beta_l+1}.$$

Let $l < \omega$ be the minimum l such that $\beta_{l+1} = \beta_l$. All that remains for p_{l+1} to be everywhere robust is to extend it to satisfy property (1) of the definition of robustness, which can be done simply by letting $q(0)$ be any extension of $p_{l+1}(0)$ to a domain including $\bigcup_{\eta,\eta' \in \text{even}(p_{l+1})} S_\eta \cap S_{\eta'}$, and for $0 < \eta \leq \beta$ setting

$$q(\eta) = \begin{cases} p_{l+1}(\eta) \text{ if } \eta \notin \text{even}(p_{l+1}) \\ (f_{p_{l+1}(\eta)}, X_{p_{l+1}(\eta)} \cup (\text{dom}(q(0)) \setminus \text{range}(p_{l+1}(\eta))) \text{ if } \eta \in \text{even}(p_{l+1}) \end{cases}.$$

□

Lemma 3.3.8. *Let $s \in D_\beta$, $r \in D^{(\delta,\delta')}_\beta$, and $E = \text{dom}(s(0)) \cup (\text{dom}(r(0)) \setminus \delta')$. If $s^{[\delta]} = s \leq r^{[\delta]}$, $s \not\leq r^{(\delta')}$, and*

$$\forall \eta \in \text{even}(r),\ \delta' \geq \sup(\text{range}(r(\eta))),$$

then for any function $F : [E \cup \{\delta'\}]^2 \longrightarrow \omega$ extending $F_{s(0)}, F_{r(0)}$, there is a common extension s' of s, r, such that $s'(0) \leq (E \cup \{\delta'\}, F)$, and

$$(\forall \eta \in \text{odd}(r) \cap \text{odd}(s'))\ s'(\eta) \setminus s'^{(\delta')}(\eta) \subseteq r(\eta) \text{ and } \max(s'^{(\delta')}(\eta)) \in r(\eta).$$

Proof. Without loss of generality we may assume the set

$$B = \{\eta \in \text{supp}(r) : \eta \in \text{odd}(r),\ \text{or } \eta \in \text{even}(r) \text{ and } \delta' \in \text{range}(r(\eta))\}$$

is nonempty, since if not, take $q \in D_\beta$ extending $r^{(\delta')} = r, s$, let $q'(0) \in \mathbb{P}_1$ be some common extension of $(E \cup \{\delta'\}, F), q^{(\delta')}(0), r(0)$, and set $s' = q'(0) \cup q^{(\delta')}$ to get a common extension s' of s, r such that $s'(0) \leq (E \cup \{\delta'\}, F)$.

Let $\beta_0 > ... > \beta_n > 0$ enumerate B, and for each $\beta_m \in \text{even}(r)$, define ξ_m to be the unique ordinal with $f_{r(\beta_m)}(\xi_m) = \delta'$. Let $s_0 = s$, and we will inductively construct a sequence $s_0 \geq ... \geq s_{n+1} \in D_\beta$. Let $\{\gamma_{\beta_m}\}_{m \leq n} \subseteq \delta'$ be as defined in the definition of $r \in D^{(\delta,\delta')}_\beta$, and define $\rho_{-1} = \delta+1$, $\rho_{m+1} = \max\{\rho_m, \gamma_{\beta_{m+1}}\}$, for $-1 \leq m < n$. The sequence $\{s_m\}_{m \leq n+1}$ will satisfy the following, for $0 \leq m \leq n$:

(0) s_m, s_{n+1} are compatible with $r^{(\delta')}$.

(1) $s_{m+1}^{[\rho_m]} = s_{m+1}$, and $s_{m+1}^{(\rho_m)} \upharpoonright \beta_m = s_{m+1} \upharpoonright \beta_m$.

(2) $s_{m+1} \upharpoonright (\beta_m, \beta) = s_m \upharpoonright (\beta_m, \beta)$.

(3) If β_m is odd, $\max(s_{m+1}(\beta_m)) \in r(\beta_m)$.

(4) If β_m is even, $s_{m+1} \upharpoonright \beta_m$ decides F_{β_m} on pairs of

$$w_m = \mathrm{dom}(s_m(\beta_m)) \cup \{\xi_m\} \cup \mathrm{dom}(r(\beta_m)).$$

There is nothing to show for $m = 0$ so assume s_m is given for $m \leq n$, and we will construct s_{m+1}.

First assume β_m is even. Note that since δ' is a limit in $\{\delta_{\beta_m,\xi}\}_{\xi<\omega_1}$ and $\mathrm{dom}(r(\beta_m)) = \mathrm{dom}(r(\beta_m)^{[\delta]})$, $\rho_m = \gamma_{\beta_m} > \max\{\max(w_m), \rho_{m-1}\}$. Pick $q' \in D_{r^{(\delta')},s_m,\beta}$. Let $M \in \mathcal{M}_{\beta_m}$ be of height ρ_m. The set

$$D = \{u \in D_{\beta_m} : u \text{ decides } F_{\beta_m} \text{ on } w_m\}$$

is dense and in M, so since $\mathbb{P}_\beta$ is c.c.c. there is $u \in M \cap D$ compatible with q'. Let $q \in D_{u,q',\beta_m}$, and set

$$s_{m+1} = q^{(\rho_m)} \upharpoonright \beta_m \cup\!\!\!\!\lrcorner\; s_m.$$

Since $\rho_m > \rho_{m-1}$, we have $q^{(\rho_m)} \upharpoonright \beta_m \in D_{s_m,\beta_m}$, and so $s_{m+1} \in D_\beta$. Item (4) holds since $s_{m+1} \leq u$. Item (0) holds since q extends $r^{(\delta')}, s_{m+1}$, and items $(1) - (3)$ are clear.

Now assume β_m is odd. Pick $q \in D_{r^{(\delta')},s_m,\beta}$ and define

$$s_{m+1} = (q^{(\rho_m)} \upharpoonright \beta_m {}^\frown q(\beta_m)^{[\rho_m]}) \cup\!\!\!\!\lrcorner\; s_m.$$

Since $\rho_m \geq \rho_{m-1}$ if $m > 0$, or $\rho_m > \delta$ if $m = 0$, we have that $q^{(\rho_m)} \upharpoonright \beta_m {}^\frown q(\beta_m)^{[\rho_m]} \leq s_m \upharpoonright (\beta_m + 1)$, and noting that by the definition of D_{β_m} we have $q^{(\rho_m)} \upharpoonright \beta_m {}^\frown q(\beta_m)^{[\rho_m]} \in D_{\beta_m}$, we can conclude $s_{m+1} \in D_\beta$. Item (0) holds since q extends $r^{(\delta')}, s_{m+1}$, items $(1), (2), (4)$ are clear, and we will justify why item (3) holds: Recall $\gamma_{\beta_m} \in r(\beta_m)^{(\delta')} \subseteq q(\beta_m)$ satisfies

(i) $\gamma_{\beta_m} = \rho_m$,

or

(ii) There is a maximum level of T_{β_m} strictly below δ' and

$r^{(\gamma_{\beta_m})} \upharpoonright \beta_m \Vdash$ "everything in $(\gamma_{\beta_m}, \delta') \cap T_{\beta_m}$ is $\leq_{T_{\beta_m}}$ incompatible with γ_{β_m}".

We have that either $\gamma_{\beta_m} = \rho_m = \max(q(\beta_m)^{[\rho_m]}) = \max(s_{m+1}(\beta_m))$ or everything in $(\gamma_{\beta_m}, \rho_m] \cap T_{\beta_m}$ is forced by $r^{(\gamma_{\beta_m})} \upharpoonright \beta_m$, and thus $q^{(\rho_m)} \upharpoonright \beta_m$, to be incompatible with γ_{β_m}, in which case $\gamma_{\beta_m} = \max(q(\beta_m)^{[\rho_m]}) = \max(s_{m+1}(\beta_m))$. Therefore item (3) holds.

We will now use s_{n+1} to construct a common extension s' of s, r, such that $s'(0) \leq (E \cup \{\delta'\}, F)$.

First let $q'(0) \in \mathbb{P}_1$ be a common extension of $r^{(\delta')}(0), s_{n+1}(0)$ For $0 < \eta \leq \beta$ define

$$s'(\eta) = s_{n+1}(\eta) \cup r(\eta),$$

and define $s'(0)$ to have universe

$$\text{dom}(q'^{(\delta')}(0)) \cup \{\delta'\} \cup E$$

with $F_{s'(0)}$ extending $F_{r(0)}$, $F_{q'^{(\delta')}(0)}$, and defined as follows otherwise:

$$(*) \qquad \text{For } \lambda \in E,$$

$$F_{s'(0)}(\lambda, \delta') = F(\lambda, \delta').$$

$$\text{For } m \leq n,\ \beta_m \text{ even, } f_{s_{n+1}(\beta_m)}(\xi) = \lambda,$$

$$s_{n+1} \Vdash F_{\beta_m}(\xi, \xi_m) = F_{s'(0)}(\lambda, \delta').$$

For $\lambda \in \text{dom}(q'(0)^{(\delta')}), \lambda' \in E \cup \{\delta'\}$ such that $F_{s'(0)}(\lambda, \lambda')$ is not defined from the above,

$F_{s'(0)}(\lambda, \lambda') = 0$ (could set the function to be anything).

We will see that s' is a well defined condition to finish the proof. Note the three possible problems in the above definition:

1. For some $\eta \in \text{odd}(r)$, $s' \restriction \eta \not\Vdash s'(\eta) \in \mathbb{Q}_\eta$: We don't have this contradiction since for all $\eta \in \text{odd}(r) \cap \text{odd}(s_{n+1})$, $\max(s_{n+1}(\eta)) \in r(\eta)$, and s' extends r, s_{n+1}, so $s' \restriction \eta \Vdash s'(\eta) \in \mathbb{Q}_\eta$.

2. $F_{r(0)}$ is already defined on (λ, δ') differently than $F_{s'(0)}$ in $(*)$: If so, λ could not be in E, and must be mentioned in all even(r) coordinates by robustness. We have $f_{s_{n+1}(\beta_m)}(\xi) = \lambda$ implies $(\xi, \lambda) \in f_{r^{(\delta')}(\beta_m)} \subseteq f_{s(\beta_m)}$, and so since $s_{n+1} \not\perp r^{(\delta')}$, s_{n+1} decided $F_{\beta_m}(\xi, \xi_m)$ in the same way as $r^{(\delta')} \restriction \beta_m$ ($r^{(\delta')} \restriction \beta_m$ decided $F_{\beta_m}(\xi, \xi_m)$ by the definition of D_β), which is the same way as $F_{r(0)}, F_{s'(0)}$ on (λ, δ').

3. There is $m \neq m' \leq n$ with $f_{s_{n+1}(\beta_{m'})}(\xi') = \lambda$ and $f_{s_{n+1}(\beta_m)}(\xi) = \lambda$ but $s_{n+1} \Vdash F_{\beta_m}(\xi, \xi_m) \neq F_{\beta_{m'}}(\xi', \xi_{m'})$: By robustness the common possibilities for embedding ranges $S_{\beta_m} \cap S_{\beta_{m'}}$ where already a part of $\text{dom}(r(0)), r(\beta_{m'}), r(\beta_m)$, so $\{\lambda, \delta'\} \subseteq \text{dom}(r(0)), f_{r(\beta_{m'})}(\xi') = \lambda, f_{r(\beta_m)}(\xi) = \lambda$ and $r \Vdash F_{\beta_m}(\xi, \xi_m) \neq F_{\beta_{m'}}(\xi', \xi_{m'})$ which is impossible.

□

Lemma 3.3.9. *Let $r \in D^{(\delta,\delta')}_{\beta+1}$, and let $\delta = \delta_0 < ... < \delta_n < \delta'$ enumerate $\{\lambda : (\exists \eta \in \text{even}(r))\ \lambda \in \text{range}(r(\eta)) \setminus \delta_0\}$. Assume there is $\beta = \eta_0 > ... > \eta_n \in \text{even}(r)$ with for $0 < i \leq n$, $\delta_i \in \text{range}(r(\eta_i)), \delta' \in \text{range}(r(\beta))$. If $s_0, ..., s_n \in D_{\beta+1}$ are such that for $i \leq n$:*

1. $s_i^{[\delta_i]} = s_i = s_i \restriction (\eta_i + 1)$,

2. $s_i \leq r^{[\delta_i]} \restriction (\eta_i + 1)$, *and if* $i < n$, $s_{i+1} \leq s_i \restriction (\eta_{i+1} + 1)$,

3. $(\forall \eta \in \mathrm{odd}(r) \cap \mathrm{odd}(s_i))\ \max(s_i(\eta)) \in r(\eta)$,

and $E = \mathrm{dom}(s_n(0)) \cup \mathrm{dom}(r(0))$, *then for any function* $F : [E \cup \{\delta'\}]^2 \longrightarrow \omega$ *extending* $F_{s_n(0)}, F_{r(0)}$, *there is a common extension* s' *of* $s_0, ..., s_n, r$, *such that* $s'(0) \leq (E \cup \{\delta'\}, F)$, *and*

$$(\forall \eta \in \mathrm{odd}(r) \cap \mathrm{odd}(s'))\ s'(\eta) \setminus s'^{(\delta')}(\eta) \subseteq r(\eta) \text{ and } \max(s'^{(\delta')}(\eta)) \in r(\eta).$$

Proof. Note that each γ_{η_i} generated by the definition of $r \in D^{(\delta,\delta')}_{\beta+1}$ has $\gamma_{\eta_i} > \delta_i$. The proof is exactly the same as the proof of lemma 3.3.8, except replace B with $B \cup \{\eta_i\}_{i \leq n}$, s with s_0, and replace "$r^{(\delta')}$" anywhere it is mentioned with the condition "v" defined as follows: Let $q' = (\underline{\bigcup}_i s_i) \underline{\cup} r^{(\delta)}$, and extend to $v \in D^{(\delta,\delta')}_{\beta+1}$ by letting $v(0)$ be any extension of $q'(0)$ to a domain including $\bigcup_{\eta,\eta' \in \mathrm{even}(q')} S_\eta \cap S_{\eta'}$, and for $0 < \eta \leq \beta$ setting

$$v(\eta) = \begin{cases} q'(\eta) \text{ if } \eta \notin \mathrm{even}(q') \\ (f_{q'(\eta)}, X_{q'(\eta)} \cup (\mathrm{dom}(v(0)) \setminus \mathrm{range}(q'(\eta))) \text{ if } \eta \in \mathrm{even}(q') \end{cases}.$$

□

Theorem 3.3.10. *The partial order* $\mathbb{P}_{\alpha+1}$ *is c.c.c.*

Proof. Without loss of generality, we may assume α is even. Suppose $A = \{p_\eta\}_{\eta<\omega_1} \subseteq \mathbb{P}_{\alpha+1}$ is an antichain and work towards a contradiction. Without loss of generality assume all conditions in A are everywhere robust. Also we may assume there are $p, r \in A$ and $\delta_0 < \omega_1$ with

$$p^{[\delta_0]} = p \leq r^{[\delta_0]} \text{ and } p \restriction \alpha \not\leq r \restriction \alpha.$$

We can do this by first refining A so that $\{\mathrm{dom}(p_\eta(\alpha))\}_{\eta<\omega_1}$ is a Δ-system where the functions agree on the root (by the definition of the partial order there are only countably many allowable function values on any given element of ω_1), then picking arbitrary $p, r \in A$ that are compatible up to α (they exist since $\mathbb{P}_\alpha$ is c.c.c.), where δ_0 satisfies $p^{[\delta_0]} = p$ and $r(\alpha)^{[\delta_0]} \subseteq p(\alpha)^{[\delta_0]}$. Finally, replace p with $p'^{[\delta_0]}$ where $p' \restriction \alpha \leq p \restriction \alpha, r \restriction \alpha$, p' is everywhere robust, and $f_{p'(\alpha)} = f_{p(\alpha)}$. Without loss of generality $\delta_0 \in \mathrm{range}(r(\alpha))$.

Let $\delta_0 < ... < \delta_n$ enumerate $\mathrm{range}(r(\alpha))$, and for $i < n$, $\delta_i = \delta_{i,0} < ... < \delta_{i,n_i}$ enumerate $\{\delta < \delta_{i+1} : (\exists \eta \in \mathrm{even}(r))\ \delta \in \mathrm{range}(r(\eta)) \setminus \delta_i\}$ with $\eta_{i,0}, ..., \eta_{i,n_i} \in \mathrm{even}(r)$ such that $\eta_{i,j}$ is the maximal ordinal with $\delta_{i,j} \in \mathrm{range}(r(\eta_{i,j}))$. Note that $p \not\leq r^{(\delta_1)}$. Set $\delta_{n,1} = \delta_{n+1} = \omega_1$ and we will inductively apply lemma 3.3.8/3.3.9 to obtain $p = s_0 \geq ... \geq s_n \in D_{\alpha+1}$ with for all $i \leq n$:

1. $s_i^{[\delta_i]} = s_i \leq r^{[\delta_i]}$,

2. $s_i \not\leq r^{(\delta_{i,1})}$ (or if $\delta_{i,1}$ does not exist $s_i \not\leq r^{(\delta_{i+1})}$).

3. $(\forall \eta \in \mathrm{odd}(r) \cap \mathrm{odd}(s_i))\ \max(s_i(\eta)) \in r(\eta)$,

This will finish the proof since $r = r^{(\omega_1)} \not\perp s_n$ and $s_n \leq p$.

We may assume (3) and the base case of the induction is given, by extending p with lemma 3.3.8 as follows, and re-labeling $\{\delta_i\}_{i \leq n}$ accordingly:

Let $q \in D_{\alpha+1}$ extend p and $r^{(\delta_1)}$, let $\gamma = \gamma_\alpha$ be as in the definition of $r \in D_{\alpha+1}^{(\delta_0,\delta_1)}$, and let $M \in \mathcal{M}_\alpha$ have height γ. The set

$$D = \{u \in D_\alpha : u \text{ decides } F_\alpha \text{ on } \mathrm{dom}(r(\alpha)^{(\gamma)}) \cup \mathrm{dom}(q(\alpha)^{[\gamma]})\},$$

is dense and in M, and so since $\mathbb{P}_\alpha$ is c.c.c. there is $u \in D \cap M,\ u \not\perp q \restriction \alpha$. Let $u' \in D_\alpha$ extend $u, q \restriction \alpha$, set $s = (u' \sqcup q)^{[\gamma]}$. Noting that $r \in D_{\alpha+1}^{(\gamma,\delta_1)}$, setting $E = \mathrm{dom}(s(0)) \cup (\mathrm{dom}(r(0)^{(\delta_{1,1})}) \setminus \delta_1)$, and letting F be any function extending $F_{s(0)}, F_{r(0)}$ on E with

$$s \Vdash f_{s(\alpha)}^{-1} \restriction E \text{ is an embedding of } F \text{ into } F_\alpha,$$

we can apply lemma 3.3.8 with $\gamma, \delta_1, s, r^{(\delta_{1,1})}, E, F$ in place of $\delta, \delta', s, r, E, F$ respectively, to obtain $s' \leq s, r^{(\delta_{1,1})}$ so that we can replace $p = s_0$ with $s'^{[\delta_1]}$.

Assume the inductive hypothesis for $m < n$. First assume that $\eta_{m,0} > ... > \eta_{m,n_m}$, and our goal will be to apply lemma 3.3.9.

Use lemma 3.3.8 to inductively build $s_m = s_{m,0}, ..., s_{m,n_m} \in D_{\alpha+1}$ such that for $i \leq n_m$:

1. if $i < n_m$, $s_{m,i} \not\perp r^{(\delta_{m,i+1})} \restriction (\eta_{m,i} + 1)$,
2. $s_{m,i}^{[\delta_{m,i}]} = s_{m,i} = s_{m,i} \restriction (\eta_{m,i} + 1)$,
3. $s_{m,i} \leq r^{[\delta_{m,i}]} \restriction (\eta_{m,i} + 1)$, and if $i < n_m$, $s_{m,i+1} \leq s_{m,i} \restriction (\eta_{m,i+1} + 1)$,
4. $(\forall \eta \in \mathrm{odd}(r) \cap \mathrm{odd}(s_{m,i}))\ \max(s_{m,i}(\eta)) \in r(\eta)$.

The base case of the induction is given, so assume the inductive hypothesis for $i < n_m$.

Let $q \in D_{\eta_{m,i+1}+1}$ extend $s_{m,i}$ and $r^{(\delta_{m,i+1})}$, let $\gamma = \gamma_{\eta_{m,i+1}}$ be as in the definition of $r \in D_{\eta_{m,i+1}+1}^{(\delta_{m,i},\delta_{m,i+1})}$, and let $M \in \mathcal{M}_{\eta_{m,i+1}}$ have height γ. The set

$$D = \{u \in D_{\eta_{m,i+1}} : u \text{ decides } F_{\eta_{m,i+1}} \text{ on } \mathrm{dom}(r(\eta_{m,i+1})^{(\gamma)}) \cup \mathrm{dom}(q(\eta_{m,i+1})^{[\gamma]})\},$$

is dense and in M, and so since $\mathbb{P}_{\eta_{m,i+1}}$ is c.c.c. there is $u \in D \cap M,\ u \not\perp q \restriction \eta_{m,i+1}$. Let $u' \in D_{\eta_{m,i+1}}$ extend $u, q \restriction \eta_{m,i+1}$, set $s = (u' \sqcup q)^{[\gamma]}$. Noting that $r \in D_{\eta_{m,i+1}+1}^{(\gamma,\delta_{m,i+1})}$, setting $E = \mathrm{dom}(s(0)) \cup (\mathrm{dom}(r(0)^{(\delta_{m,i+2})}) \setminus \delta_{m,i+1})$, and letting F be any function extending $F_{s(0)}, F_{r(0)}$ on E with

$$s \Vdash f_{s(\eta_{m,i+1})}^{-1} \restriction E \text{ is an embedding of } F \text{ into } F_{\eta_{m,i+1}},$$

we can apply lemma 3.3.8 with $\gamma, \delta_{m,i+1}, s, r^{(\delta_{m,i+2})}, E, F$ in place of $\delta, \delta', s, r, E, F$ respectively, to obtain $s' \leq s, r^{(\delta_{m,i+2})}$ so that $s_{m,i+1} = s'^{[\delta_{m,i+1}]} \restriction (\eta_{m,i+1} + 1)$ satisfies the inductive hypotheses for $i + 1$.

Now that we have $s_{m,0}, ..., s_{m,n_m}$, setting $E = \text{dom}(s_{m,n_m}(0)) \cup \text{dom}(r(0)^{(\delta_{m+1,1})})$, $s = (\underline{\bigcup}_i s_{m,i}) \underline{\cup} r^{[\delta_{m+1}]}$, and letting F be any function extending $F_{s_{m,n_m}(0)}, F_{r(0)}$ on E with

$$s \Vdash f^{-1}_{s(\alpha)} \restriction E \text{ is an embedding of } F \text{ into } F_\alpha,$$

clearly we can apply lemma 3.3.9 with $\delta_{m+1}, \delta_{m,0}, ..., \delta_{m,n_m}, \eta_{m,0}, ..., \eta_{m,n_m}, s_{m,0}, ..., s_{m,n_m}, r^{(\delta_{m+1,1})}, E, F$ in place of $\delta', \delta_0, ..., \delta_n, \eta_0, ..., \eta_n, s_0, ..., s_n, r, E, F$ respectively, to obtain $s' \leq s_m, r^{(\delta_{m+1,1})}$ so that $s_{m+1} = s'^{[\delta_{m+1}]}$ satisfies the inductive hypotheses for $m + 1$.

To finish, note that assuming $\eta_{m,0} > ... > \eta_{m,n_m}$ during the inductive hypothesis for $m < n$, by a telescoping argument, did not lose generality: Let $\alpha = \xi_0 > ... > \xi_l$ and $\delta_m = \lambda_0 < ... < \lambda_l < \delta_{m+1}$ be defined by $\lambda_i = \max \text{range}(r(\xi_i)^{(\delta_{m+1})})$ and $\xi_{i+1} = \max\{\eta_{m,j} < \xi_i : j \leq n_m$ and, there is $\lambda \in \text{range}(r(\eta_{m,j})) \setminus \lambda_i\}$. Note "$\eta_{m,0} > ... > \eta_{m,n_m}$", if and only if "$l = n_m$ and for all i, $\xi_i = \eta_{m,i}, \lambda_i = \delta_{m,i}$", which is true if and only if "$l = n_m$". If $l \neq n_m$, carry out this the and the previous arguments climbing up within successive subintervals $\lambda_i < \lambda_{i+1}$ as if $\alpha = \xi_{i+1}$, $\delta_m = \lambda_i$, $\delta_{m+1} = \lambda_{i+1}$.

$\square$

4 Construction Schemes on ω_1

The results of this chapter are joint work with Fulgencio López.

4.1 Introduction

4.1.1 Overview

Following some forcing constructions in [5] and [31], Todorčević [50] introduced the concept of capturing construction schemes. Construction schemes are used to construct objects of domain ω_1 by recursive amalgamations of coherent finite substructures. In [50], capturing construction schemes are shown to exist if you assume $\diamondsuit$, are used to construct a Banach space of the form $C(K)$ without biorthogonal sequences, and used to construct other objects from [31] without forcing. These results are characterized for the recursive nature of the proofs which makes building counterexamples more intuitive. In [29] López proved that some of the more complex examples of Banach spaces from [31] also follow from the existence of a capturing construction scheme. In [30] López and Todorčević proved that Suslin trees and T-gaps exist if we assume there are capturing construction schemes. These latest results only require a weaker version of capturing, namely 3-capturing.

In this chapter we study the consistency of capturing construction schemes and the weaker versions of capturing. Throughout this chapter, when we talk about the consistency of capturing, we mean for every type $(m_k, n_k, r_k)_k$ there is a capturing construction scheme $\mathcal{F}$ of that type (see below for definitions), and analogously when we talk about the consistency of n-capturing.

We start by studying the consistency of n-capturing and m-Knaster (for $2 \leq n, m$) in Section 4.2. We show the consistency of n-capturing with no $(n+1)$-capturing construction scheme, and relate this with the m-Knaster Hierarchy. Recall that a forcing notion $\mathbb{P}$ is said to be m-Knaster (K_m) (for $2 \leq m$) if for every uncountable $W \subset \mathbb{P}$ there is $W_0 \subset W$ uncountable such that for every $p_1, \ldots, p_m \in W_0$ there is $p \in \mathbb{P}$ with

$p \leq p_1, \ldots, p_m$. $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$ is the forcing axiom for $\aleph_1$ dense sets on K_m forcings. The main result of this Section is:

Theorem 4.1.1. *n-capturing is independent of $MA_{\omega_1}(K_m)$ if $n \leq m$, and they are incompatible if $n > m$. Also capturing is independent of MA_{ω_1}(precaliber $\aleph_1$).*

The proof of this Theorem gives an alternative argument to the well known fact that $\mathrm{MA}_{\omega_1}(\mathrm{K}_m) \not\Leftarrow \mathrm{MA}_{\omega_1}(\mathrm{K}_{m+1})$.

Section 4.3 is dedicated to proving the consistency of capturing using forcing.

Theorem 4.1.2. *Adding $\kappa \geq \aleph_1$ Cohen reals also adds a capturing construction scheme.*

In Section 4.4 we show other versions of capturing are also consistent. They are forced by adding $\aleph_1$ Cohen reals.

4.1.2 Background and Preliminaries

We follow standard notation in Set Theory (see for example Kunen [27]). When we refer to a Δ-System $(s_\alpha : \alpha < \omega_1)$ we mean s_α are finite subsets of ω_1, and for every $\alpha < \beta < \omega_1$ we have $s_\alpha \cap s_\beta = s$ for some $s \subset \omega_1$ fixed, and $\max(s_\alpha) < \min(s_\beta)$.

Let us introduce the main concept of this work.

Definition 4.1.3. Let $(m_k)_{k<\omega}, (n_k)_{1\leq k<\omega}$ and $(r_k)_{1\leq k<\omega}$ be sequences of natural numbers such that $m_0 = 1$, $m_{k-1} > r_k$ for all $k > 0$, $n_k > k$ and for every $r < \omega$ there are infinitely many k's with $r_k = r$. If for every $k > 0$ we have

$$m_k = n_k(m_{k-1} - r_k) + r_k$$

we say that $(m_k, n_k, r_k)_{k<\omega}$ forms a *type*.

Definition 4.1.4. We say that $\mathcal{F}$ is a *construction scheme of type* $(m_k, n_k, r_k)_{k<\omega}$, if $\mathcal{F} \subset [\omega_1]^{<\omega}$ is a family of finite subsets of ω_1, partitioned into levels $\mathcal{F} = \bigcup_{k<\omega} \mathcal{F}_k$, such that for every $F \in \mathcal{F}$, there is $R(F) \sqsubset F$ with:

1. For every $A \subset \omega_1$ finite, there is $F \in \mathcal{F}$ such that $A \subset F$.

2. $\forall F \in \mathcal{F}_k$, $|F| = m_k$ and $|R(F)| = r_k$.

3. For all $F, E \in \mathcal{F}_k$, $E \cap F \sqsubset F, E$.

4. $\forall F \in \mathcal{F}_k$, there are unique $F_0, \ldots, F_{n-1} \in \mathcal{F}_{k-1}$ with

$$F = \bigcup_{i<n} F_i$$

Furthermore $n = n_k$ and $(F_i)_{i<n_k}$ forms an increasing Δ-system with root $R(F)$, i.e.,

$$R(F) < F_0 \setminus R(F) < \ldots < F_{n_k-1} \setminus R(F)$$

We call the sequence $(F_i)_{i<n_k}$ of (4) the *canonical decomposition* of F.

To avoid confusion we will use m_k, n_k, and r_k as above, and we will omit reference to the type of a construction scheme. For $F \in \mathcal{F}$ and $F = \bigcup_{i<n_k} F_i$ the canonical decomposition of F, we write $\varphi_i : F_0 \to F_i$ for the unique order preserving bijection between F_0 and F_i, or for $E, F \in \mathcal{F}_k$ write $\varphi_{F,E} : F \to E$ for the unique order preserving bijection between F and E. Analogously, if f is a function on F_0 we can define the function $\varphi_i(f) = f \circ \varphi_i^{-1}$ in F_i.

The following lemmas and corollaries give us properties that tell us more about the structure of a construction scheme.

Lemma 4.1.5. *For $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$, with $l \leq k$ we have $E \cap F \sqsubseteq E$.*

Proof. Given $l \in \omega$, we prove the lemma by induction on $k \geq l$. If $l = k$ the result follows by the properties of $\mathcal{F}$. Assume the result holds for l and $k - 1 \geq l$. Let F as above and let $F = \bigcup_{i<n_k} F_i$ be its canonical decomposition. Since the F_i's are in $\mathcal{F}_{k-1}$ we can apply our hypothesis and $E \cap F_i \sqsubseteq E$ for every $i < n_k$. If $E \cap (F \setminus R(F)) = \emptyset$ then the result follows, otherwise let $i < n_k$ be minimal such that $E \cap (F_i \setminus R(F)) \neq \emptyset$ then $E \cap F = E \cap F_i$. Because if not, there is $i < j < n_{k+1}$ with $E \cap F_j \not\sqsubseteq E$. Thus we have $E \cap F = E \cap F_i \sqsubset E$ and the result follows. □

Corollary 4.1.6. *For $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$ and $F = \bigcup_{i<n_k} F_i$ the canonical decomposition of F. If $E \subset F$ and $l < k$ then there is some $i < n_k$ with $E \subset F_i$. In particular, if $l = k - 1$ we have $E = F_i$.*

Corollary 4.1.7. *Let $E, F \in \mathcal{F}_k$, then $\varphi_{E,F}(\mathcal{F} \upharpoonright E) = \mathcal{F} \upharpoonright F$, where $\mathcal{F} \upharpoonright F = \{L \in \mathcal{F} : L \subset F\}$.*

Lemma 4.1.8. *For all $A \in [\omega_1]^{<\omega}$ and all $l < \omega$ there is $k \geq l$ and $F = \bigcup_{i<n_k} F_i \in \mathcal{F}_k$ with $r_l = 0$ and $A \subseteq F_0$.*

Proof. Suppose there is $A \in [\omega_1]^{<\omega}$, $l < \omega$, where for all $k > l$ with $r_k = 0$ there is no $F = \bigcup_{i<n_k} F_i \in \mathcal{F}_k$, with $A \subseteq F_0$. We will get a contradiction by inductively defining an infinite decreasing sequence $\{\alpha_n\}_{n<\omega} \subseteq \omega_1$. Pick some $F^0 \in \mathcal{F}$ with $A \subseteq F^0$, and set $\alpha_0 = \min(F^0)$. Assume for $n < \omega$, $\{\alpha_m\}_{m<n}$, $\{F^m\}_{m<n}$, and $\{l_m\}_{m<n}$ have been constructed satisfying $\forall m < m' < n$:

- $\alpha_{m'} < \alpha_m = \min(F^m)$,

- $l_m < l_{m'}$,
- $F^m \in \mathcal{F}_{l_m}$,
- $F^m \subseteq F^{m'}$.

Pick any $l_{n+1} > l_n$ with $r_l = 0$. Let F, l be such that $l > l_{n+1}$, with $F^n \subseteq F = \bigcup_{i<n_l} F_i \in \mathcal{F}_l$. Using corollary 4.1.6 we have that there is some $F^{n+1} = \bigcup_{i<n_{l_{n+1}}} (F^{n+1})_i \subseteq F$ at rank l_{n+1}, and some $i < n_{l_{n+1}}$, with $F^n \subseteq (F^{n+1})_i$. Let $\alpha_{n+1} = \min(F^{n+1})$. We have that $i > 0$, otherwise $A \subseteq F^0 \subseteq F^n \subseteq (F^{n+1})_0$, and so $\alpha_{n+1} = \min(F^{n+1}) < \min(F^n) = \alpha_n$. □

For $F \in \mathcal{F}$, when $\mathcal{F}$ is understood we now abuse notation and refer to the structure $(F, \mathcal{F} \restriction F, <)$ by just the set F. This means that we say that F, E are copies or isomorphic, when $(F, \mathcal{F} \restriction F, <)$ is isomorphic to $(E, \mathcal{F} \restriction E, <)$.

Lemma 4.1.9. *For $F \in \mathcal{F}_k$, $E \in \mathcal{F}_l$ and $E \subset F$ (in particular $l \leq k$). For every $\mu \in E$ there is a copy E^* of E in F such that*

1. *$E^* \cap (\mu + 1) = E \cap (\mu + 1)$.*
2. *$E^* \setminus \mu$ is an interval of F.*

Proof. Given $l \in \omega$, we prove the lemma by induction on $k \geq l$. If $l = k$ the result follows by the properties of $\mathcal{F}$. Assume the result holds for l and $k - 1 \geq l$. Take $F = \bigcup_{i<n_k} F_i$, the canonical decomposition of F. By Corollary 4.1.6 there is $i < n_k$ such that $E \subset F_i$. By the induction hypothesis there is E^{**} a copy of E in F_i such that the conclusion holds. If $\mu \notin R(F)$ then $E^* = E^{**}$ works, otherwise, let $E^* = \varphi_{F_i,F_0}(E^{**})$. By Corollary 4.1.7, E^* is a copy of E and $E^* \setminus \mu$ is an interval of F_0. Since $\mu \in R(E)$ then (1) holds, and (2) holds because F_0 is an initial interval of F. □

Theorem 4.1.10 ([50])**.** *For every type $(m_k, n_k, r_k)_{k<\omega}$, there is a construction scheme $\mathcal{F}$ of that type in ZFC. Additionally, the family $\mathcal{F}^\omega = \mathcal{F} \cap \mathcal{P}(\omega)$ is construction scheme restricted to ω (i.e, $\mathcal{F}^\omega$ is a family of finite subsets of ω and it is cofinal in $[\omega]^{<\omega}$, and (2),(3),(4), as in Definition 4.1.4). This is implicit in the proof of the consistency with $\diamondsuit$ of [50], but we will include the construction here which will require a definition and claim internal to the proof of this fact.*

Proof. $\mathcal{F}^\omega$ can be defined by recursively constructing $\mathcal{F}^\omega \restriction m_k$ with $m_k \in \mathcal{F}^\omega \restriction m_k$:
Start with $\mathcal{F}^\omega \restriction m_0 = \{\{0\}\}$. Assuming $\mathcal{F}^\omega \restriction m_k$ has been constructed, let

$$F = m_{k+1} = r_{k+1} \cup I_0 \cup ... \cup I_{n_{k+1}-1}$$

where each I_i is an interval of cardinality $m_k - r_{k+1}$, and let $F_i = r_{k+1} \cup I_i$. Set

$$\mathcal{F}^\omega \restriction m_{k+1} = \{F\} \cup \{\phi_{F_0,F_i}(E) : E \in \mathcal{F}^\omega \restriction m_k\}.$$

Assume for $\delta \in \mathrm{Lim}(\omega_1)$ that $\mathcal{F}^\delta \subseteq [\delta]^{<\omega}$ has been constructed to be cofinal in $[\delta]^{<\omega}$ and satisfies (2),(3),(4), as in Definition 4.1.4

Define $\mathbb{Q}_\delta$ to be the set of all $p = (D_p, \mathcal{F}_p, <)$ where $D_p \setminus \delta$ is an initial interval of $[\delta, \delta+\omega)$, and with some fixed isomorphic $(D_p^\delta, \mathcal{F}^\delta \restriction D_p^\delta, <)$ with $D_p^\delta \in \mathcal{F}^\delta$ end extending $D_p \cap \delta$. Order $\mathbb{Q}_\delta$ by $p \leq q$ if D_q is a substructure of D_p. Let $D_p^{\delta+}$ denote the set $D_p^\delta \setminus D_p \cap \delta$.

Given $F \subseteq \delta, \gamma < \delta$ define $p(F,\gamma) \in \mathbb{Q}_\delta$ to be the set with $p(F,\gamma) \cap \gamma = F \cap \gamma$, $p(F,\gamma) \setminus \gamma$ an initial interval of $[\delta, \delta+\omega)$, and such that the structure of $p(F,\gamma)$ is isomorphic to F. We remark that if $p \in \mathbb{Q}_\delta$, $D_p^\delta \subseteq F$, and $D_p^{\delta+}$ is an interval of F, then $p(F, \min(D_p^{\delta+})) \leq p$, and using this fact combined with lemma 4.1.9 ensures that if $D_p^\delta \subseteq F$ then $p(F, \min(D_p^{\delta+})) \leq p$

Claim 4.1.11. *If $\mathcal{F}^\delta$ satisfies*

For every $A \subseteq \delta$ finite and $a < \delta$, there is $F = \bigcup_{i<n_k} F_i \in \mathcal{F}^\omega$ of arbitrarily high rank such that $A \subseteq F_0$ and $R(F) = F_0 \cap a$. $(*_\delta)$

Then for every $A \subseteq \delta+\omega$ finite and $a < \delta+\omega$, the set $\mathcal{D}_{A,a} = \{p \in \mathbb{Q}_\delta : (\exists F \in \mathcal{F}_p)\ A \subseteq F_0, R(F) = F_0 \cap a\}$ is dense.

Proof. Let $p \in \mathbb{Q}_\delta$, $A \subseteq \delta+\omega$ finite and $a < \delta+\omega$.

First show that we can assume $B = A \cup \{a\} \subseteq D_p$ by using $(*_\delta)$ to get some $F \in \mathcal{F}^\delta$ with

$$D_p^\delta \cup (B \cap \delta) \cup (\max(D_p^\delta), \max(D_p^\delta) + |\max(B \setminus \delta) - \delta|] \subseteq F,$$

and see that $B \subseteq p(F, \min(D_p^{\delta+})) \leq p$.

First assume $a < \delta$: We can use $(*_\delta)$ to get $E = \bigcup_{i<n_k} E_i \in \mathcal{F}^\delta$ of arbitrarily high rank with

$$D_p^\delta \subseteq E_0 \text{ and } R(E) = E_0 \cap a$$

so that $q = p(E, \min D_p^{\delta+}) \in \mathcal{D}_{A,a}$.

Now assume $\delta \leq a < \delta+\omega$, and set $b = (\Phi^p)^{-1}(a)$. Use $(*_\delta)$ to get $E = \bigcup_{i<n_k} E_i \in \mathcal{F}^\delta$ of arbitrarily high rank with

$$D_p^\delta \subseteq E_0 \text{ and } R(E) = E_0 \cap b$$

so that $q = p(E, \min D_p^{\delta+}) \in \mathcal{D}_{A,a}$.

∎

The family of dense sets $\{\mathcal{D}_{A,a} : A \subseteq \delta+\omega \text{ is finite and } a < \delta+\omega\}$ is countable and so we can pick some filter $G \subseteq \mathbb{Q}_\delta$ generic for this family. Define

$$\mathcal{F}^{\delta+\omega} = \bigcup_{p\in G} \mathcal{F}^{\delta+\omega} \restriction D_p = \{\phi_{D^\delta_p, D_p}(F) : F \in \mathcal{F}^\delta,\ F \subseteq D^\delta_p,\ p \in G\}.$$

It is easy to see that $\mathcal{F}^{\delta+\omega} \subseteq [\delta+\omega]^{<\omega}$ is cofinal in $[\delta+\omega]^{<\omega}$, satisfies (2),(3),(4), as in Definition 4.1.4, and satisfies $(*_\delta)$. Therefore by induction on $\delta \in \mathrm{Lim}(\omega_1)$ we have proved the theorem.

□

The following concept is useful to construct many other results as mentioned above.

Definition 4.1.12. Let $\mathcal{F}$ be a construction scheme, and $2 \leq n$. We say that $\mathcal{F}$ is *n-capturing* if for every uncountable Δ-system $(s_\xi)_{\xi<\omega_1}$ of finite subsets of ω_1 with root s, there are $\xi_0 < \ldots < \xi_{n-1} < \omega_1$, and $F \in \mathcal{F}$ with canonical decomposition $F = \bigcup_{i<n_k} F_i$, such that

$$\begin{aligned} & s \subset R(F) \\ \text{for every } i < n, \quad & s_{\xi_i} \setminus s \subset F_i \setminus R(F), \\ \text{for every } i < n, \quad & \varphi_i(s_{\xi_0}) = s_{\xi_i}. \end{aligned}$$

We say that $\mathcal{F}$ is *capturing* if $\mathcal{F}$ is n-capturing for every $n < \omega$.

It is a useful fact that the notion of capturing uncountable Δ-systems is the same as the notion of capturing uncountable sets:

Lemma 4.1.13 (Lemma 7.1 [50])**.** *Let $\mathcal{F}$ be a construction scheme. If $\mathcal{F}$ is n-capturing for uncountable sets (meaning $\mathcal{F}$ n-captures any uncountable Δ-system of singletons), then it is n-capturing.*

4.2 The Hierarchies of n-Knaster and n-capturing

Recall that $\mathrm{MA}_{\omega_1}(K_n)$ implies $\mathrm{MA}_{\omega_1}(K_m)$ for every $m \geq n$, whereas n-capturing implies m-capturing for every $m \leq n$. Thus, we have the following two hierarchies:

$$\mathrm{MA}_{\omega_1}(\mathrm{K}_2) \Longrightarrow \cdots \Longrightarrow \mathrm{MA}_{\omega_1}(\mathrm{K}_n) \Longrightarrow \mathrm{MA}_{\omega_1}(\mathrm{K}_{n+1}) \Longrightarrow \cdots \Longrightarrow \mathrm{MA}_{\omega_1}(\text{precaliber } \aleph_1)$$

$$\text{2-capturing} \Longleftarrow \cdots \Longleftarrow n\text{-capturing} \Leftarrow (n+1)\text{-capturing} \Longleftarrow \cdots \Longleftarrow \text{capturing}$$

The main result of this section give us a relation between this two types of axioms and shows that none of the implications above can be reversed.

Theorem 4.1.1. *n-capturing is independent of $MA_{\omega_1}(K_m)$ if $n \leq m$, and they are incompatible if $n > m$. Also capturing is independent of MA_{ω_1}(precaliber $\aleph_1$).*

We start the analysis of n-capturing with the following Preservation Lemma.

Lemma 4.2.1. *n-capturing is preserved by K_n forcing notions. Let $\mathbb{P}$ be a K_n forcing notion and let $\mathcal{F}$ be a n-capturing construction scheme on V. If $G \subset \mathbb{P}$ is a generic filter for $\mathbb{P}$, then $\mathcal{F}$ is a n-capturing construction scheme on $V[G]$. In particular capturing is preserved by precaliber $\aleph_1$ forcing notions.*

Proof. Let $\mathbb{P}$ be a K_n forcing notion and Γ an uncountable subset of ω_1 in the extension. Let $W \subset \omega_1$ and $p_\alpha \in \mathbb{P}$, $\alpha \in W$ such that

$$p_\alpha \Vdash \alpha \in \Gamma$$

for every $\alpha \in W$. Since $\mathbb{P}$ is K_n there is n-linked $W_0 \subset W$ uncountable. Recall $\mathcal{F}$ is n-capturing in V, therefore there are $\alpha_0 < \ldots < \alpha_{n-1}$ in W_0 which are captured by $\mathcal{F}$. We find now $q \in \mathbb{P}$ with $q \leq p_{\alpha_0}, \ldots, p_{\alpha_{n-1}}$, then

$$q \Vdash \alpha_0, \ldots, \alpha_{n-1} \in \Gamma, \text{ and they are captured by } \mathcal{F}.$$

□

Consider the following property

$(\bigstar)_m$ For every $\Gamma \subset \omega^\omega$ there is $\Gamma_0 \subset \Gamma$ uncountable such that Γ_0 has no $g_0, \ldots, g_m$ and $k < \omega$ with $g_0 \upharpoonright k = \ldots = g_m \upharpoonright k$, and $|\{g_0(k), \ldots, g_m(k)\}| = m+1$.

Recall the following result of Todorčević implicit in [49]

Theorem 4.2.2 ([49])**.** *$MA_{\omega_1}(K_m)$ implies $(\bigstar)_m$.*

The following result proves the first half of Theorem 4.1.1

Theorem 4.2.3. *Let $\mathcal{F}$ be a $(m+1)$-capturing construction scheme. Then $(\bigstar)_m$ fails.*

Proof. Let $\mathcal{F}$ be as above. For every $F \in \mathcal{F}_l$ we construct, inductively on l, $(f^F_\alpha : (l+1) \to N_l)_{\alpha<\omega_1}$ such that $N_0 = 1$ and

1. for $E, F \in \mathcal{F}_l$ and $\varphi : E \to F$ the increasing bijection between E and F, for every $\alpha \in E$, if $\beta = \varphi(\alpha)$ then $f^F_\beta = f^E_\alpha$.
2. for $E \in \mathcal{F}_{l_0}$ and $F \in \mathcal{F}_{l_1}$, $l_0 < l_1$, if $\alpha \in E \cap F$ then $f^F_\alpha \upharpoonright (l_0+1) = f^E_\alpha$.

Let $F \in \mathcal{F}_k$ with canonical decomposition $F = \bigcup_{i<n_k} F_i$ and suppose $(f^{F_i}_\alpha : \alpha \in F_i)$ is defined for all $i < n_k$ satisfying (1) and (2) above. Let $f^F_\alpha = \emptyset$ if $\alpha \notin F$.

For $\alpha \in R(F)$ let $f^F_\alpha(k) = N_{k-1}$ and $f^F_\alpha \upharpoonright k = f^{F_0}_\alpha$.

For $\alpha_0 \in F_0 \setminus R(F)$ and $\alpha_i = \varphi_i(\alpha)$, $i < n_k$. We let $f^F_{\alpha_i} \upharpoonright k = f^{F_i}_{\alpha_i}$,

$$f^F_{\alpha_i}(k) = N_{k-1} + i + 1,$$

and let $N_k = N_{k-1} + n_k + 1$.

It is easy to see that (1) and (2) hold, and so $f_\alpha = \bigcup_{F \in \mathcal{F}} f_\alpha^F$ is a well defined function. Then $\Gamma = \{f_\alpha : \alpha < \omega_1\}$ is a witness to the failure of $(\bigstar)_m$. To see this suppose $\Gamma_0 = \{f_\alpha : \alpha \in W\}$ where $W \subset \omega_1$ is uncountable. Since $\mathcal{F}$ is $(m+1)$-capturing there are $\xi_0 < \ldots < \xi_m$ in W captured by some $F \in \mathcal{F}_k$. This implies $f_{\xi_0} \restriction k = \ldots = f_{\xi_m} \restriction k$ and $|\{f_{\xi_0}(k), \ldots, f_{\xi_m}(k)\}| = m+1$, and hence $(\bigstar)_m$ fails as we wanted to show. □

Proof of Theorem 4.1.1. Start by assuming $n \leq m$. To see n-capturing is independent of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$, note that any model of MA_{ω_1} is also a model of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$ and contains no n-capturing construction scheme for any $2 \leq n < \omega$ (see [30] for $n > 2$, and see Proposition 4.2.7 of this paper for $n = 2$). Thus, it is consistent to have $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$ and no n-capturing construction schemes. To show the other direction, start with a model V that has a capturing construction scheme $\mathcal{F}$. Let $\mathbb{K}_m$ be the K_m poset that forces $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$. Then $\mathcal{F}$ remains m-capturing on the extension by Lemma 4.2.1 hence it is n-capturing provided $n \leq m$.

Suppose now $n > m$ and V is a model of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$, then $(\bigstar)_m$ holds on V. By Theorem 4.2.3 we know V contains no $(m+1)$-capturing construction scheme, otherwise $(\bigstar)_m$ fails which is a contradiction. Thus V has no n-capturing construction scheme for $n > m$, as we wanted to show.

To see MA_{ω_1}(precaliber $\aleph_1$) and capturing are independent we proceed in the same manner. Any model of MA_{ω_1} satisfies MA_{ω_1}(precaliber $\aleph_1$) and has no capturing construction scheme. Finally, let V be a model that contains a capturing construction scheme. Let $\mathbb{K}$ be a forcing notion with precaliber $\aleph_1$ that forces MA_{ω_1}(precaliber $\aleph_1$). Since $\mathbb{K}$ has precaliber $\aleph_1$, $\mathcal{F}$ remains capturing in the extension. This finishes the proof. □

It is interesting to find a K_n forcing notion that kills $(n+1)$-capturing in an obvious way. Suppose $\mathcal{F}$ is a capturing construction scheme. Let $\mathcal{F}$ be fixed.

Definition 4.2.4. Let $P \in \mathbb{P}_n^{\mathcal{F}}$ if $P \in [\omega_1]^{<\omega}$ and $\mathcal{F}$ does not capture $\big\{\{\xi_i\} : i \leq n\big\}$ for any $\xi_0 < \ldots < \xi_n$ in P. We say $P \leq Q$ if $Q \subset P$.

Lemma 4.2.5. $\mathbb{P}_n^{\mathcal{F}}$ *defined as above is* K_n*.*

Proof. Take $(P_\alpha : \alpha < \omega_1) \subset \mathbb{P}_n^{\mathcal{F}}$. We can find $D_\alpha \in \mathcal{F}_{k_\alpha}$ such that $P_\alpha \subset D_\alpha$.

Find $\Gamma \subset \omega_1$ uncountable, and $k < \omega$ such that

1. $(D_\alpha : \alpha \in \Gamma)$ forms a Δ-System,
2. $k_\alpha = k$ for all $\alpha \in \Gamma$, and
3. for every $\alpha < \beta$ in Γ, we have $\varphi_{D_\alpha, D_\beta}(P_\alpha) = P_\beta$.

Note that (2) and (3) imply that for all $\alpha < \beta$, $\xi \in D_\alpha \cap D_\beta$, then $\xi \in P_\alpha$ if and only if $\xi \in P_\beta$.

We show $(P_\alpha : \alpha \in \Gamma)$ is n-linked. Take $\alpha_0 < \ldots < \alpha_{n-1}$ in Γ. Let $Q = \bigcup_{i<n} P_{\alpha_i}$. Suppose $\xi_0 < \ldots < \xi_n$ are in Q and $F \in \mathcal{F}_\ell$ captures $\big\{\{\xi_i\} : i \leq n\big\}$. Take $F = \bigcup_{i<n_\ell} F_i$ the canonical decomposition of F. We must have

$$\begin{gathered} \xi_i \in F_i \setminus R(F) \\ \big(F_i \setminus R(F) : i < n_l\big) \text{ are pairwise disjoint} \end{gathered} \tag{4.2.1}$$

Let us get a contradiction.

Case $l \leq k$: Let $j < n$ with $\xi_n \in P_{\alpha_j}$. Applying Proposition 4.1.5, $F \cap D_{\alpha_j} \sqsubseteq F$. Therefore $\xi_0, \ldots, \xi_n \in D_{\alpha_j}$ which implies $\xi_0, \ldots, \xi_n \in P_{\alpha_j}$. But $\mathcal{F}$ captures $\big\{\{\xi_i\} : i \leq n\big\}$ and this is a contradiction because $P_{\alpha_j} \in \mathbb{P}_n^{\mathcal{F}}$.

Case $l > k$: There is some $j < n$ and $i_0 < i_1 \leq n$ such that $\xi_{i_0}, \xi_{i_1} \in P_{\alpha_j}$. Then $F_{i_1} \in \mathcal{F}_{\ell-1}$, and $F_{i_1} \cap D_{\alpha_j} \sqsubseteq D_{\alpha_j}$ by Proposition 4.1.5, but this implies $\xi_{i_0} \in F_{i_1}$. This contradicts (4.2.1)

We conclude that for every $\xi_0 < \ldots \xi_n$ in Q, $\mathcal{F}$ does not capture $\big\{\{\xi_i\} : i \leq n\big\}$. Hence $Q \in \mathbb{P}_n^{\mathcal{F}}$. It is clear that $Q \leq P_{\alpha_i}$ for $i < n$. This finishes the proof. □

Lemma 4.2.6. *$\mathbb{P}_n^{\mathcal{F}}$ kills $(n+1)$-capturing.*

Proof. It suffices to show the set

$$\mathcal{D}_\xi = \{P \in \mathbb{P}_n^{\mathcal{F}} : (\exists \eta > \xi)\ \eta \in P\}$$

is dense for all $\xi < \omega_1$. Let $P \in \mathbb{P}_n^{\mathcal{F}}$ and use lemma 4.1.8 to get $E = \bigcup_{i<n_l} E_i \in \mathcal{F}$ with $P \cup \{\xi\} \subseteq E_0$, $R(E) = \emptyset$, and $l > 2$ so that $n_l > 2$. Set $\eta = \min(E_2) > \xi$, $Q = P \cup \{\eta\}$, and suppose $F = \bigcup_{i<n_k} F_i \in \mathcal{F}$ captures some $\xi_0 < \ldots < \xi_{n-1} < \eta$ in Q. By lemma 4.1.5 and since $E_2 \cap F \not\sqsubseteq F$, $l \leq k$. We have a contradiction, in the case that $l = k$ from $\xi_0 \in E_0, E_1 \cap \{\xi_0, \ldots, \xi_{n-1}, \eta\} = \emptyset$, and in the case that $l < k$ by lemma 4.1.5 and since $E \cap F_1 \not\sqsubseteq E$. □

From the above lemma we have an explicit proof that $\mathrm{MA}_{\aleph_1}(\mathrm{K}_n)$ is incompatible with m-capturing for $m > n$.

Assume $m > 2$ and note that the model obtained in the proof of Theorem 4.1.1, which starts with a capturing construction scheme and then forces $\mathrm{MA}_{\omega_1}(\mathrm{K}_m)$, shows the consistency of

$$\mathrm{MA}_{\omega_1}(\mathrm{K}_m) + m\text{-capturing } + \neg(m+1)\text{-capturing } + \neg\mathrm{MA}_{\omega_1}(\mathrm{K}_{m-1})$$

this gives us an alternative proof of $\mathrm{MA}_{\omega_1}(\mathrm{K}_m) \not\Rightarrow \mathrm{MA}_{\omega_1}(\mathrm{K}_{m+1})$ showing that the hierarchy of m-Knaster forcing axioms is strict.

To get that MA_{ω_1} implies there are no 2-capturing construction schemes, we prove the following:

Proposition 4.2.7. *If $\mathcal{F}$ is 2-capturing, then $\mathbb{P}_1^{\mathcal{F}}$ is c.c.c.*

Proof. Suppose $(P_\alpha : \alpha < \omega_1) \subset \mathbb{P}_1^{\mathcal{F}}$ forms an uncountable antichain, and refine this family so that it forms a Δ-system. Since $\mathcal{F}$ is 2-capturing, we can recursively construct a family $(D_\alpha : \alpha \in \Gamma) \subseteq \mathcal{F}$ and refine it so that $(D_\alpha : \alpha \in \Gamma) \subseteq \mathcal{F}_k$ forms an uncountable Δ-System, and for all $\alpha \in \Gamma$, D_α captures some $(P_{\alpha'}, P_{\alpha''})$. Again, since $\mathcal{F}$ is 2-capturing, there are some $F \in \mathcal{F}$, $\alpha < \beta \in \Gamma$, such that F captures (D_α, D_β).

We claim that $P_{\alpha'} \cup P_{\beta''} \in \mathbb{P}_1^{\mathcal{F}}$, which finishes the proof with a contradiction. Suppose $\xi_0 < \xi_1 \in P_{\alpha'} \cup P_{\beta''}$ are captured by some $E \in \mathcal{F}_l$. Note that since $P_{\alpha'}, P_{\beta''} \in \mathbb{P}_1^{\mathcal{F}}$, $\xi_0 \in P_{\alpha'} \setminus P_{\beta''}$, $\xi_1 \in P_{\beta''} \setminus P_{\alpha'}$, and so $\xi_0 \in D_\alpha \setminus D_\beta$, $\xi_1 \in D_\beta \setminus D_\alpha$. Let $E = \bigcup_{i<n_l} E_i$, $D_\beta = \bigcup_{i<n_k} (D_\beta)_i$, $D_\alpha = \bigcup_{i<n_k} (D_\alpha)_i$ be the respective canonical decompositions.

Case $l \le k$: Applying Proposition 4.1.5, $E \cap D_\beta \sqsubseteq E$. Therefore $\xi_1 \in E \cap D_\beta$ gives $\xi_0 \in D_\beta$, and this is a contradiction.

Case $l > k$: Recall that $\phi_{E_0,E_1}(\mathcal{F} \restriction E_0) = \mathcal{F} \restriction E_1$, and D_α capturing $(P_{\alpha'}, P_{\alpha''})$ implies $\xi_0 \in (D_\alpha)_0 \setminus R(D_\alpha)$. Since there is some $E' \subseteq E$ with $\xi_0 \in E' \in \mathcal{F}_k$, and $\xi_0 \in (D_\alpha)_0 \in \mathcal{F}_{k-1}$, we get that ξ_0 must be in the 0'th component of the canonical decomposition of E', and hence $\phi_{E_0,E_1}(\xi_0) = \xi_1$ must be in the 0'th component of the canonical decomposition of some element in $\mathcal{F}_k \restriction E_1$, which contradicts $\xi_1 \in (D_\beta)_1 \setminus R(D_\beta)$.

□

4.3 Capturing Construction Schemes in the Cohen Model

We dedicate this section to the proof of the following result.

Theorem 4.1.2. *Adding $\kappa \ge \aleph_1$ Cohen reals also adds a capturing construction scheme.*

Assume first that $\kappa = \aleph_1$. We start by fixing $\mathcal{F}^\omega$, a construction scheme on ω, where for any $r < \omega$ we have $r_k = r$ infinitely often, and with the following property:

For every $A \subset \omega$ finite, $N < \omega$, and $a < \omega$ there is $F \in \mathcal{F}^\omega$ with $n_k > N$ and canonical decomposition $\bigcup_{i<n_k} F_i$, such that $A \subset F_0$ and $R(F) = a$. (4.3.1)

As in [50] §8, $\mathcal{F}^\omega$ can be defined by recursively constructing $\mathcal{F}^\omega \restriction m_k$ with $m_k \in \mathcal{F}^\omega \restriction m_k$: Start with $\mathcal{F}^\omega \restriction m_0 = \{\{0\}\}$. Assuming $\mathcal{F}^\omega \restriction m_k$ has been constructed, let

$$F = m_{k+1} = r_{k+1} \cup I_0 \cup \dots \cup I_{n_{k+1}-1}$$

where each I_i is an interval of cardinality $m_k - r_{k+1}$, and let $F_i = r_{k+1} \cup I_i$. Set

$$\mathcal{F}^\omega \restriction m_{k+1} = \{F\} \cup \{\phi_{F_0,F_i}(E) : E \in \mathcal{F}^\omega \restriction m_k\}.$$

Definition 4.3.1. Let $p \in \mathbb{P}$ if and only if $p = (D_p, \Gamma_p, f_p)$, where $D_p \in \mathcal{F}^\omega_{k_p}$, $\Gamma_p \subset \mathrm{Lim}(\omega_1)$ finite, and $f_p : \Gamma_p \to D_p$ increasing.

We say $p \leq q$ if $\Gamma_q \subset \Gamma_p$, say $\Gamma_p = \{\delta_0 < \ldots < \delta_n\}$ and $\Gamma_q = \{\delta_{\ell_0} < \ldots < \delta_{\ell_m}\}$ for some $\ell_i \leq n$, and there is $W \in \mathcal{F}^\omega_{k_q}$, $W \subseteq D_p$ such that:

1. $f_p \restriction \Gamma_q = \varphi_{D_q,W} \circ f_q$, in particular $f_p \restriction \Gamma_q$ takes values in W,

2. $W \cap f_p(\delta_0) = W \cap f_p(\delta_{\ell_0}) = D_q \cap f_q(\delta_{\ell_0})$, in particular $f_p(\delta_0)$ is at least $f_q(\delta_{\ell_0})$,

3. $W \cap [f_p(\delta_{\ell_i}), f_p(\delta_{\ell_i+1})) = W \cap [f_p(\delta_{\ell_i}), f_p(\delta_{\ell_{i+1}}))$ and is an interval of D_p for every $i \leq m$, and

4. $W < f_p(\delta)$ for every $\delta_{\ell_m} < \delta \in \Gamma_p$. And $W \setminus f_p(\delta_{\ell_m})$ is an interval of D_p.

We say $p \sim q$ if $p \leq q$ and $q \leq p$.

Proposition 4.3.2. $\mathbb{P}$ *is transitive.*

Proof. Let $p, q, r \in \mathbb{P}$ with $p \leq q$, $q \leq r$, let $\Gamma_r = \{\delta_0 < \ldots < \delta_k\}$, $\delta_{\max} = \max(\Gamma_q)$, and let $\ell_0, \ell_1 < \omega$ be such that $D_r \in \mathcal{F}^\omega_{\ell_0}$, $D_q \in \mathcal{F}^\omega_{\ell_1}$. There are $W_0 \in \mathcal{F}^\omega_{\ell_0}$ and $W_1 \in \mathcal{F}^\omega_{\ell_1}$ where $W_0 \subset D_q$ and $W_1 \subset D_p$ are the witnesses for $q \leq r$ and $p \leq q$ respectively.

Take $W = \varphi_{D_q,W_1}(W_0)$. Since $\varphi_{D_r,W} = \varphi_{D_q,W_1} \circ \varphi_{D_r,W_0}$, we have

$$\varphi_{D_r,W}(f_r(\delta_i)) = \varphi_{D_q,W_1}\Big(\varphi_{D_r,W_0}(f_r(\delta_i))\Big) = \varphi_{D_q,W_1}(f_q(\delta_i)) = f_p(\delta_i), \text{ for every } i \leq k.$$

Let $\delta_{\min} = \min(\Gamma_p)$, and $\delta_\mu = \min(\Gamma_q)$. Note that $\delta_{\min} \leq \delta_\mu \leq \delta_0$. Since $p \leq q$ we have $W_1 \cap f_p(\delta_{\min}) = D_q \cap f_q(\delta_\mu)$ which implies that $W \cap f_p(\delta_{\min}) = W_0 \cap f_q(\delta_\mu)$. We also know that $W_0 \cap f_q(\delta_\mu) = D_r \cap f_r(\delta_0)$ because $q \leq r$, therefore

$$W \cap f_p(\delta_{\min}) = D_r \cap f_r(\delta_0).$$

Let $\Gamma_q \cap [\delta_i, \delta_{i+1}] = \{\delta_i = \gamma_0 < \ldots < \gamma_t = \delta_{i+1}\}$. Since $q \leq r$, we have $W_0 \cap [f_q(\delta_i), f_q(\gamma_1)) = W_0 \cap [f_q(\delta_i), f_q(\delta_{i+1}))$ is an interval of D_q. This implies $W \cap [f_p(\delta_i), f_p(\delta_{i+1})) = W \cap [f_p(\delta_i), f_p(\gamma_1))$ is an interval of $W_1 \cap [f_p(\delta_i), f_p(\gamma_1))$, which is an interval of D_p. Also, $p \leq q$ implies that for $\delta \in \Gamma_p$ with $\delta_i < \delta \leq \gamma_1$ we have $W_1 \cap [f_p(\delta_i), f_p(\gamma_1)) = W_1 \cap [f_p(\delta_i), f_p(\delta))$ is an interval of D_p, and so $W \cap [f_p(\delta_i), f_p(\delta_{i+1})) = W \cap [f_p(\delta_i), f_p(\delta))$ is an interval of D_p.

Similar arguments show that $W \setminus f_p(\delta_k)$ is an interval of D_p below any $f_p(\delta)$ for $\delta_k < \delta \in \Gamma_p$. This shows that W is a witness for $p \leq r$ as we wanted to see. □

Now let $p \in \mathbb{P}$, $D_p = \{x_0 < \ldots < x_m\}$, $\Gamma_p = \{\delta_0, ..., \delta_n\}$, and $f_p(\Gamma_p) = \{a_0 < \ldots < a_n\}$. We define $\Phi^p : D_p \to \omega_1$ as:

$$\Phi^q(x_j) = \begin{cases} x_j & x_j < a_0 \\ \delta_i + (j - j^*) & a_i = x_{j^*} \leq x_j < a_{i+1} \\ \delta_n + (j - j') & x_j \geq a_n = x_{j'} \end{cases}.$$

Finally, for p as above we define

$$\mathbf{F}_p = \Phi^p\Big(\mathcal{F}^\omega \restriction D_p\Big).$$

Let $G \subset \mathbb{P}$ be a generic filter, we define $\mathcal{F}$ in $V[G]$ as

$$\mathcal{F} = \bigcup_{p \in G} \mathbf{F}_p$$

Now we can see that points (2),(3),(4) in the definition of the partial order are to insure that when W witnesses $p \leq q$, then $\mathbf{F}_q = \Phi^p(W)$ and so $\mathbf{F}_q \subseteq \mathbf{F}_p$.

We shall see now the definition of $\mathbb{P}$ is isomorphic to a finite support iteration of length ω_1 of countable forcings, and therefore is isomorphic to the forcing for adding ω_1 Cohen reals, $\mathbb{C}_{\omega_1}$.

Definition 4.3.3. Fix $\delta \in \mathrm{Lim}(\omega_1)$ and let $\mathbb{P}_\delta = \{p \in \mathbb{P} : \Gamma_p \leq \delta\}$. Let G be $\mathbb{P}_\delta$ generic. Define $q \in \mathbb{Q}_\delta$ if $q = \mathbb{1}_{\mathbb{Q}_\delta}$, or $q = (D_q, a_q)$ where for some $p_q \in G$, $\delta \leq a_q \in \Phi^{p_q}(D_{p_q}) = D_q$. Order $\mathbb{Q}_\delta$ by $(D_q, a_q) \leq (D_{q'}, a_{q'})$ if and only if in $\mathbb{P}$

$$(D_{p_q}, \Gamma_{p_q} \cup \{\delta+\omega\}, f_{p_q} \cup \{(\delta+\omega, (\Phi^{p_q})^{-1}(a_q))\}) \leq (D_{p_{q'}}, \Gamma_{p_{q'}} \cup \{\delta+\omega\}, f_{p_{q'}} \cup \{(\delta+\omega, (\Phi^{p_{q'}})^{-1}(a_{q'}))\}).$$

Proposition 4.3.4. *For every $\delta \leq \omega_1$, $\mathbb{P}_{\delta+\omega}$ and $\mathbb{P}_\delta * \mathbb{Q}_\delta$ are forcing equivalent, and therefore $\mathbb{P} = \mathbb{P}_{\omega_1}$ is forcing equivalent to a finite support iteration $(\mathbb{P}'_\delta, \mathbb{Q}'_\delta)_{\delta \in \mathrm{Lim}(\omega_1)}$ such that $\mathbb{P}'_\delta$ is forcing equivalent to $\mathbb{P}_\delta$. In particular, the forcing $\mathbb{P}$ is forcing equivalent to $\mathbb{C}_{\omega_1}$.*

Proof. Clearly the function defined on $\mathbb{P}_{\delta+\omega}$ by

$$e(p) = \begin{cases} (p, \mathbb{1}_{\mathbb{Q}_\delta}) \text{ if } p \in \mathbb{P}_\delta \\ (p_0, (\Phi^p(D_p), \Phi^p(f_p(\delta+\omega)))) \text{ otherwise} \end{cases},$$

where $p_0 = (D_p, \Gamma_{p_0}, f_p \restriction \Gamma_{p_0})$ for $\Gamma_{p_0} = \Gamma_p \setminus \{\delta+\omega\}$, has $p \leq q \longleftrightarrow e(p) \leq e(q)$. To see it is dense, let $(p, (D_q, a_q)) \in \mathbb{P}_\delta * \mathbb{Q}_\delta$ be a condition such that (D_q, a_q) is decided in the ground model. Note that $p \leq p_q$ is implied. Let $W \subseteq D_p$ be the witness to $p \leq p_q$, and let $a' = \phi_{D_{p_q}, W}((\Phi^{p_q})^{-1}(a_q))$. We have

$$e(D_p, \Gamma_p \cup \{\delta+\omega\}, f_p \cup \{(\delta+\omega, a')\}) \leq (p, (D_q, a_q))$$

since

$$(D_p, \Gamma_p \cup \{\delta+\omega\}, f_p \cup \{(\delta+\omega, a')\}) \leq e^{-1}(p_q, (D_q, a_q)).$$

□

Lemma 4.3.5. *For every $p \in \mathbb{P}$ with $\Gamma_p = \{\delta_0 < \ldots < \delta_n\}$, there is $p^* \in \mathbb{P}$, $p \sim p^*$ such that:*

1. $f_{p^*}(\delta_0) = f_p(\delta_0)$,

2. $D_{p^*} \cap f_p(\delta_0) = D_p \cap f_p(\delta_0)$, *and*

3. $D_{p^*} \setminus f_p(\delta_0)$ *is an interval of* ω.

Proof. Let $M = \max(D_p)$, $a_0 = f_p(\delta_0)$, and find $F \in \mathcal{F}^\omega$ such that $D_p \cup [a_0, M] \subset F$. Apply Lemma 4.1.9 to find $D_{p^*} \subset F$ such that $D_{p^*} \cap (a_0+1) = D_p \cap (a_0+1)$ and $D_{p^*} \setminus a_0$ is an interval of F. Note that $D_{p^*} \setminus a_0$ is an interval of ω since $[a_0, M] \subset F$. Define

$$f_{p^*}(\delta_i) = \varphi_{D_p, D_{p^*}}(f_p(\delta_i))$$

for $i \leq n$. This defines $p^* \in \mathbb{P}$ as $p^* = (D_{p^*}, \Gamma_p, f_{p^*})$. □

Lemma 4.3.6. *Given* $p \in \mathbb{P}$ *and* $\xi < \omega_1$ *there is* $q \leq p$ *and* $x < \omega$ *such that* $\Phi^q(x) = \xi$.

Proof. Let $\xi < \omega_1$ and $p \in \mathbb{P}$, we assume that p is as in the conclusion of Lemma 4.3.5. Consider $\Gamma_p = \{\delta_0 < \ldots < \delta_n\}$ and $f_p(\delta_i) = a_i$ with $D_p \in \mathcal{F}^\omega_k$.

We want to find $q \leq p$ and $x < \omega$ such that $\xi = \Phi^q(x)$.

Take $\delta < \omega_1$ limit such that $\delta \leq \xi < \delta + \omega$. Write $\xi = \delta + \ell$ where $\ell < \omega$. If $\delta \geq \delta_n$ or $\xi < \omega$ it is easy to find q, we leave the reader to work out the details, so we are left with the following cases:

Case 1: $\delta = \delta_j$ for some $j < n$. Find $F \in \mathcal{F}^\omega$ with canonical decomposition $\bigcup_{i<n_k} F_i$, such that $D_p \cup [a_j, a_j + \ell] \subset F_0$ and $R(F) = F_0 \cap a_{j+1}$. Take $W = \varphi_1(D_p)$, and $a_i^* = \varphi_{D_p,W}(a_i)$ for $j < i \leq n$.

Define $q \in \mathbb{P}$ by $\Gamma_q = \Gamma_p$, $D_q = F$, and

$$f_q(\delta_i) = \begin{cases} a_i & \text{for } i < j, \\ a_j & \text{for } i = j \\ a_i^* & \text{for } i > j. \end{cases}$$

Note that W witnesses $q \leq p$. By construction $\Phi^q(a_j + \ell) = \xi$ so we let $x = a_j + \ell$.

Case 2: $\delta \notin \Gamma_p$ and j is the minimum $j \leq n$ with $\delta < \delta_j$. Set $a = \max(D_p) + 1$ and apply (4.3.1) to find $F \in \mathcal{F}^\omega$ with canonical decomposition $\bigcup_{i<n_k} F_i$, such that $D_p \cup [a, a+\ell] \subset F_0$ and $R(F) = F_0 \cap a_j$. Note that $D_p \cup [a, a+\ell] \setminus a_0$ is an interval of ω. Now let $W = \varphi_1(D_p)$ and $a_i^* = \varphi_{D_p,W}(a_i)$ for $j \leq i \leq n$.

Define $q \in \mathbb{P}$ with $\Gamma_q = \Gamma_p \cup \{\delta\}$, $D_q = F$, and such that

$$f_q(\gamma) = \begin{cases} a_i & \text{for } \gamma = \delta_i, i < j. \\ a & \text{for } \gamma = \delta, \\ a_i^* & \text{for } \gamma = \delta_i, i \geq j. \end{cases}$$

It's clear that $q \leq p$ as witnessed by W, and $\Phi^q(a + \ell) = \xi$ by construction. Take $x = a + \ell$. □

Claim 4.3.7. $\mathcal{F}$ *as above is a construction scheme on* $V[G]$.

Proof. To see property (1) of a Construction Scheme let $A = \{\xi_1, \ldots, \xi_n\} \subset \omega_1$ finite and $p \in \mathbb{P}$. Successively apply lemma 4.3.6 to get $p_n \leq \ldots \leq p_1 \leq p$ with $\xi_i \in \bigcup \mathbf{F}_{p_i}$. Since $\mathbf{F}_p \subseteq \mathbf{F}_{p_1} \ldots \subseteq \mathbf{F}_{p_n}$, $A \subseteq \Phi^{p_n}(D_{p_n}) \in \mathbf{F}_{p_n}$ and $p_n \Vdash \mathbf{F}_{p_n} \subseteq \mathcal{F}$.

Property (2) is clear from the construction, and properties (3) and (4) are straightforward consequences of:

1. $(\forall p \in \mathbb{P})$ Φ^p is an order/structure preserving map, and
2. $(\forall p, q \in \mathbb{P})$ $p \leq q \longrightarrow \mathbf{F}_q \subseteq \mathbf{F}_p$.

□

Based on the above, to prove theorem 4.1.2, it remains to show $\mathcal{F}$ is capturing.

Theorem 4.1.2. *Adding $\kappa \geq \aleph_1$ Cohen reals also adds a capturing construction scheme.*

Proof. To show $\mathcal{F}$ is capturing, let $S = (s_\alpha : \alpha < \omega_1)$ be an uncountable Δ-System in $V[G]$. Assume for simplicity $|s_\alpha| = 1$ and $S = (\{\xi\} : \xi \in \Gamma)$ where Γ in $V[G]$ is uncountable subset of ω_1, the proof is similar for the general case. Let $n < \omega$ be given. Take $\Omega \subset \omega_1$ uncountable and $p_\alpha \in \mathbb{P}$ for $\alpha \in \Omega$ such that

$$p_\alpha \Vdash \alpha \in \Gamma \tag{4.3.2}$$

Let $\Gamma_{p_\alpha} = \{\delta_{\alpha,0}, \ldots, \delta_{\alpha,d-1}\}$ for $\delta_{\alpha,0} < \ldots < \delta_{\alpha,d-1} \in \mathrm{Lim}(\omega_1)$. We can assume without loss of generality that the p_α's have the form of the conclusion to lemma 4.3.5 and there is $x_\alpha < \omega$ above $f_{p_\alpha}(\delta_{\alpha,0})$ such that $\Phi^{p_\alpha}(x_\alpha) = \alpha$.

Find $\Omega_0 \subset \Omega$ uncountable, $D \in \mathcal{F}^\omega$, $a_0 < \ldots < a_{d-1} < \omega$, and $x < \omega$ such that:

1. $(\Gamma_{p_\alpha} : \alpha \in \Omega_0)$ form a Δ-System with root $\{\delta_{\alpha,0}, \ldots, \delta_{\alpha,r-1}\}$,
2. $D_{p_\alpha} = D$,
3. $f_{p_\alpha}(\delta_{\alpha,i}) = a_i$ for every $i < d$,
4. $x \in D$ with $\Phi^{p_\alpha}(x) = \alpha$, and there is fixed $j_0 \geq r$ with $x \geq a_{d-1}$ and $j_0 = d - 1$, or $a_{j_0} \leq x < a_{j_0+1}$.

Pick $\alpha_0 < \ldots < \alpha_{n-1}$ in Ω_0. We want to find $q \in \mathbb{P}$ such that

$$q \Vdash \alpha_0, \ldots, \alpha_{n-1} \in \Gamma \text{ and } \mathcal{F} \text{ captures } \alpha_0, \ldots, \alpha_{n-1}. \tag{4.3.3}$$

Apply (4.3.1) to find $F \in \mathcal{F}^\omega$ with $n_k \geq n$ such that $F = \bigcup_{i<n_k} F_i$ is the canonical decomposition of F, $D \subset F_0$, and $R(F) = F_0 \cap a_r$.

Note that $D \setminus a_0$ is an interval of F_0. Let $\varphi_i : F_0 \to F_i$ be the increasing bijection between F_0 and F_i. Take $W_i = \varphi_i(D)$, $a_{i,j} = \varphi_i(a_j)$, and $x_i = \varphi_{j_0}(x)$ for $i < n$, $r \leq j < n$.

It is easy to check that

$$W_i \setminus a_{i,r} \text{ and } W_i \cap a_{i,r} \text{ are intervals of } F \tag{4.3.4}$$

We define $q \in \mathbb{P}$ with $\Gamma_q = \{\delta_{\alpha_i,j} : i < n, j < d\}$ and $D_q = F$. Note now that $\delta_{\alpha,i}$ does not depend on α for $i < r$.

$$f_q(\delta_{\alpha_i,j}) = \begin{cases} a_j & \text{for } j < r \\ a_{i,j} & \text{for } j \geq r \end{cases}$$

With this definition, we have $\Phi^q(x_i) = \alpha_i$ by property (4) and our construction, and W_i witnesses $q \leq p_{\alpha_i}$ for every $i < n$, by (4.3.4). This implies $q \Vdash \alpha_i \in \Gamma$ for every $i < n$ because of (4.3.2).

Finally, let $E = \Phi^q(F)$. Then $q \Vdash E \in \mathcal{F}$, and by the construction of $(x_i : i < n)$ we have q forces E captures $\alpha_0, \ldots, \alpha_{n-1}$. Therefore (4.3.3) holds which is what we wanted to prove.

Suppose now $\kappa > \aleph_1$. Let $\mathbb{C}_\kappa$ be the forcing for adding κ Cohen reals. It is well known (see for example Theorem 8.2.1 of Kunen [27]) that $\mathbb{C}_\kappa = \mathbb{C}_{\kappa \setminus \omega_1} * \mathbb{C}_{\omega_1}$, and therefore forcing with $\mathbb{C}_\kappa$ adds capturing construction schemes.

□

4.4 Fully Capturing and Capturing with Partitions

There is a generalization of capturing that proves useful in some examples of [50]. We present it here for completeness.

Definition 4.4.1. Let $\mathcal{F}$ be a construction scheme. We say that $\mathcal{F}$ is *fully capturing* if for every uncountable Δ-system $(s_\xi)_{\xi<\omega_1}$ of finite subsets of ω_1 with root s, and every $k^* < \omega$ there are $F \in \mathcal{F}_k$ with $k > k^*$, canonical decomposition $F = \bigcup_{i<n_k} F_i$, and $\xi_0 < \ldots < \xi_{n_k-1} < \omega_1$, such that

$$\begin{aligned} & s \subset R(F) \\ \text{for every } i < n_k, \quad & s_{\xi_i} \setminus s \subset F_i \setminus R(F), \\ \text{for every } i < n_k, \quad & \varphi_i(s_{\xi_0}) = s_{\xi_i}. \end{aligned}$$

Definition 4.4.2. Let $\omega = \bigcup_{\ell<\omega} P_\ell$ be a partition of ω into infinite components and let $\vec{P} = (P_\ell : \ell < \omega)$. Suppose (m_k, n_k, r_k) forms a type such that for every $\ell < \omega$, and every $r < \omega$ there are infinitely many k's in P_ℓ with $r_k = r$. Then we say $(m_k, n_k, r_k)_k$ forms a *$\vec{P}$-type*.

Definition 4.4.3. Let $\mathcal{F}$ be a construction scheme with type $(m_k, n_k, r_k)_k$, and $2 \geq n$. We say $\mathcal{F}$ is *n-$\vec{P}$-capturing* if $(m_k, n_k, r_k)_k$ forms a $\vec{P}$-type, and for every uncountable Δ-system $(s_\xi)_{\xi<\omega_1}$ of finite subsets of ω_1 with root s, and every $\ell < \omega$, there are $\xi_0 < \ldots < \xi_{n-1} < \omega_1$, $k \in P_\ell$ and $F \in \mathcal{F}_k$ with canonical decomposition $F = \bigcup_{i<n_k} F_i$, such that

$$\begin{aligned} & s \subset R(F) \\ \text{for every } i < n, \quad & s_{\xi_i} \setminus s \subset F_i \setminus R(F), \\ \text{for every } i < n, \quad & \varphi_i(s_{\xi_0}) = s_{\xi_i}. \end{aligned}$$

We say $\mathcal{F}$ is *$\vec{P}$-capturing* if $\mathcal{F}$ is n-$\vec{P}$-capturing for every $n < \omega$.

We can also define *$\vec{P}$-fully capturing* in the obvious manner. What makes this version interesting is that it allows for different amalgamations. For example, the existence of a 2-$\vec{P}$-capturing construction scheme implies there are Suslin trees and T-gaps. This can be shown following [30] where the same objects are build using 3-capturing.

It is clear that n-$\vec{P}$-capturing implies n-capturing and $\vec{P}$-capturing implies capturing, however we do not know if any of the implications can be reversed. Analogously, fully capturing implies capturing but we do not know if it is consistent to have capturing without fully capturing.

We prove the following Theorem about the consistency of other forms of capturing.

Theorem 4.4.4. *Adding $\kappa \geq \aleph_1$ Cohen reals implies there are Fully capturing construction schemes, $\vec{P}$-capturing construction schemes, and $\vec{P}$-fully capturing construction schemes.*

Proof. The proof is similar to Theorem 4.1.2 therefore we only give a sketch for a $\vec{P}$-fully capturing construction scheme.

Let $\vec{P}$ be a partition of ω and let $(m_k, n_k, r_k)_{k<\omega}$ be a given $\vec{P}$-type.

It is easy to see, using the fact that $(m_k, n_k, r_k)_{k<\omega}$ is a $\vec{P}$-type, that there is a Construction Scheme $\mathcal{F}^\omega$ on ω such that:

For every $\ell < \omega$, $A \subset \omega$ finite, and $a < \omega$, there is $k \in P_\ell$ and $F \in \mathcal{F}_k$ with canonical decomposition $\bigcup_{i<n_k} F_i$, such that $A \subset F_0$ and $R(F) = F_0 \cap a$. (4.4.1)

Suppose now $\mathcal{F}$ is defined as in Theorem 4.1.2 and $S = (s_\alpha : \alpha < \omega_1)$ is an uncountable Δ-System on $V[G]$. We assume that $|s_\alpha| = 1$ and $S = (\{\xi\} : \xi \in \Gamma)$ where Γ in $V[G]$ is uncountable subset of ω_1, the argument is the same for the general case. Let $\ell < \omega$ and $k^* < \omega$ be given.

Find $\Omega \subset \omega_1$ uncountable and $p_\alpha \in \mathbb{P}$ as in Lemma 4.3.5 for $\alpha \in \Omega$ such that

$$p_\alpha \Vdash \alpha \in \Gamma \tag{4.4.2}$$

And $\alpha \in \Phi^{p_\alpha}(D_{p_\alpha})$. Let $\Gamma_{p_\alpha} = \{\delta_{\alpha,0} < \ldots < \delta_{\alpha,d-1}\}$, there is $D \in \mathcal{F}^\omega$, $a_0 < \ldots < a_{d-1} < \omega$, and $x < \omega$ such that:

1. $(\Gamma_{p_\alpha} : \alpha \in \Omega)$ form a Δ-System with root $\{\delta_{\alpha,0}, \ldots, \delta_{\alpha,r-1}\}$,

2. $D_{p_\alpha} = D$,

3. $f_{p_\alpha}(\delta_{\alpha,i}) = a_i$ for every $i < d$,

4. $x \in D$ with $\Phi^{p_\alpha}(x) = \alpha$, and there is fixed $j_0 \geq r$ with $x \geq a_{d-1}$ and $j_0 = d-1$, or $a_{j_0} \leq x < a_{j_0+1}$.

Apply (4.4.1) to find $k \in P_\ell$ with $k > k^*$, and $F^* \in \mathcal{F}_k^\omega$ such that $F^* = \bigcup_{i<n_k} F_i^*$ is the canonical decomposition of F^*, $D \subset F_0^*$, and $R(F^*) = F_0^* \cap a_r$.

Pick arbitrary $\alpha_0 < \ldots < \alpha_{n_k-1}$ in Ω. We construct $q \in \mathbb{P}$, such that

$$q \Vdash \alpha_i \in \Gamma, \exists F \in \mathcal{F}_k \text{ captures } \alpha_0, \ldots, \alpha_{n_k-1}. \tag{4.4.3}$$

For $i < d$, note $a_i \in D \subset F_0^*$. Let $\varphi_i : F_0^* \to F_i^*$ be the increasing bijection between F_0^* and F_i^*. Define $W_i = \varphi_i(D)$, and $a_{i,j} = \varphi_{D,W_i}(a_j)$ for $i < n$, $j < d$, and $x_i = \varphi_{D,W_i}(x)$ for $i < n_k$.

Since $D \setminus a_0$ is an interval of ω and φ_i is an increasing bijection we have that $a_{i,j} = a_j$ for $j < r$, these are the elements that are on the root, and $W_i \setminus a_{i,r}$ is an interval of F_i^*.

We define $q \in \mathbb{P}$ with $D_q = F^*$, $\Gamma_q = \cup_{i<n_k} \Gamma_{p_{\alpha_i}} = \{\delta_{\alpha_i,j} : i < n_k, j < d\}$. Note now that $\delta_{\alpha,j}$ does not depend on α for $j < r$.

$$f_q(\delta_{\alpha_i,j}) = \begin{cases} a_j & \text{for } j < r \\ a_{i,j} & \text{for } j \geq r \end{cases}$$

and let $q = (D_q, \Gamma_q, f_q)$. With this definition, we have

$$\Phi^q(x_i) = \Phi^{p_{\alpha_i}} \circ \varphi_i^{-1}(x_i) = \Phi^{p_{\alpha_i}}(x) = \alpha_i \tag{4.4.4}$$

and $(W_i : i < n_k)$ is a witness to $q \leq p_{\alpha_i}$ for every $i < n_k$. This implies $q \Vdash \alpha_i \in \Gamma$ for every $i < n_k$ because of (4.4.2).

Finally, let $F = \Phi^q(F^*)$. Then $q \Vdash F \in \mathcal{F}$, and by the construction of $(x_i : i < n_k)$ and (4.4.4) we have q forces F captures $\alpha_0, \ldots, \alpha_{n_k-1}$. Therefore (4.4.3) holds which is what we wanted to prove. ☐

We also have the following results related to the consistency of n-$\vec{P}$-capturing.

Theorem 4.4.5. *Let $\vec{P}$ be a partition of ω as above. Then n-$\vec{P}$-capturing and $MA_{\omega_1}(K_m)$ are independent if $n \leq m$ and they are incompatible if $n > m$. Also $\vec{P}$-capturing, $\vec{P}$-fully capturing, and fully capturing are all independent of MA_{ω_1}(precaliber $\aleph_1$).*

The proof follows the arguments in Section 4.2. Recall that the proof of the analogous theorem 4.1.1 follows from these points:

1. If $\mathcal{F}$ is a $(m+1)$-capturing construction scheme then $(\bigstar)_m$ fails.

2. $\mathrm{MA}_{\omega_1}(K_m)$ implies $(\bigstar)_m$.

3. n-capturing is preserved by K_n forcing notions.

4. There is a c.c.c forcing $\mathbb{P}_n^{\mathcal{F}}$ which destroys n-capturing for a construction scheme $\mathcal{F}$.

Points 1 and 4 follow for "-$\vec{P}$-capturing" directly from the case of "-capturing", solely because n-$\vec{P}$-capturing implies n-capturing. The proof of point 3 for n-$\vec{P}$-capturing is exactly as in the n-capturing case, but we include it below for completeness:

Lemma 4.4.6. *n-$\vec{P}$-capturing is preserved by K_n forcing notions.*

Proof. Let $\mathbb{P}$ be a K_n forcing notion, $l \in \omega$, $\mathcal{F}$ an n-$\vec{P}$-capturing family, and Γ an uncountable subset of ω_1 in the extension. Let $W \subset \omega_1$ and $p_\alpha \in \mathbb{P}$, $\alpha \in W$ such that

$$p_\alpha \Vdash \alpha \in \Gamma$$

for every $\alpha \in W$. Since $\mathbb{P}$ is K_n there is n-linked $W_0 \subset W$ uncountable. Recall $\mathcal{F}$ is n-$\vec{P}$-capturing in V, therefore there are $\alpha_0 < \ldots < \alpha_{n-1}$ in W_0 which are captured by $\mathcal{F}$ with some $F \in \mathcal{F}_k$, $k \in P_l$. We find now $q \in \mathbb{P}$ with $q \leq p_{\alpha_0}, \ldots, p_{\alpha_{n-1}}$, then

$$q \Vdash \alpha_0, \ldots, \alpha_{n-1} \in \Gamma, \text{ and they are captured by } F.$$

$\square$

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