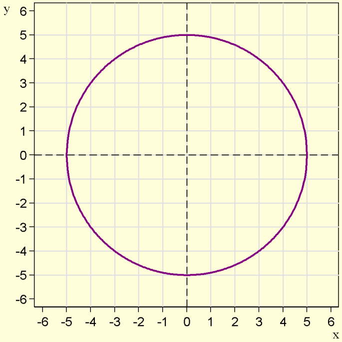
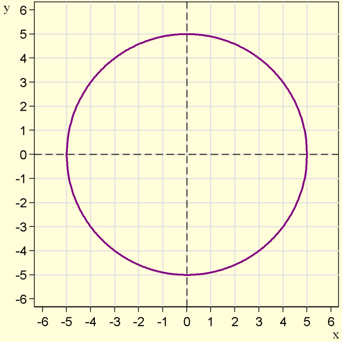
Implicit Functions



Is a circle a function? $y^2 + x^2 = 25$

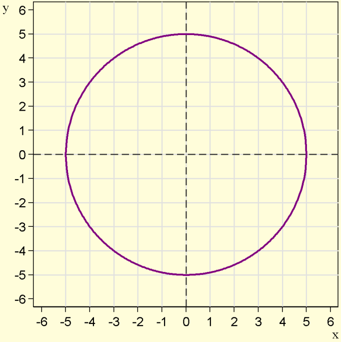
Implicit Functions



Example: $y^2 + x^2 = 25$

- Although, we cannot write y as an explicit function of x, the value of y depends on the value of x.
- So, we say the above expression defines y *implicitly* as a function of x

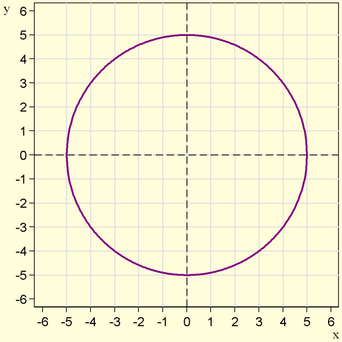
Implicit Differentiation



$$y^2 + x^2 = 25$$

1. Recognize that the value of y depends on the value of x.

Implicit Differentiation



$$y^2 + x^2 = 25$$

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2. Take the derivative with respect to (w.r.t.) x of both sides of the expression while applying the **Chain Rule** to y.

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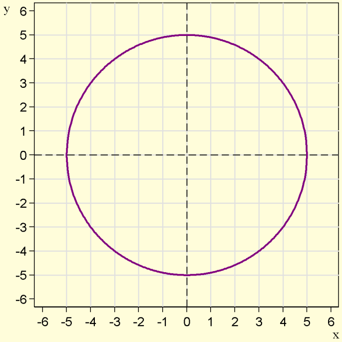

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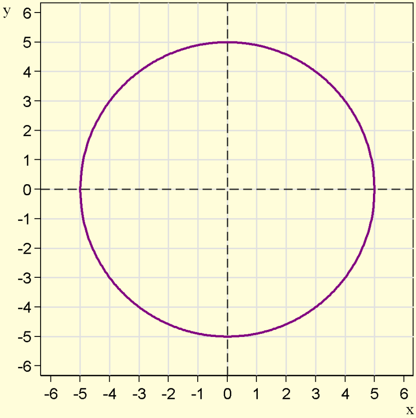
1. Recognize that the value of y depends on the value of x.
2. Take the derivative with respect to (w.r.t.) x of both sides of the expression while applying the **Chain Rule** to y.
3. Re-arrange the expression to isolate/solve for the derivative.

1. Example 1: $x^2 + y^2 = 25$
    a) Use implicit differentiation to find the slope when $x = 3, y = -4$
    b) Illustrate the slope with a tangent line on at $x = 3, y = -4$ on a graph.



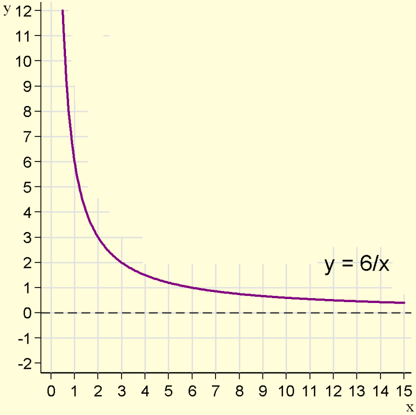
$$x^2 + y^2 = 25$$

2. Example: $2xy = 12$

   a) Find the derivative, y' using the implicit differentiation: $\left(\frac{dy}{dx}\right)$.

   b) Compute the slope when $x = 6$.

   c) Illustrate on a diagram.

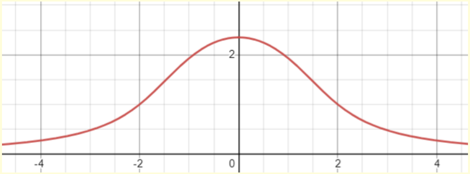

y
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
x
y = 6/x

Summary of Implicit Differentiation

If y is an implicit function of x,

1) let $y = f(x)$.
2) Apply the derivative operator to *both sides* of the expression.
3) Re-arrange the expression to isolate and solve for the derivative, $f'(x)$.
4) Replace $f'(x) = y'$.

Steps 1 and 4 are optional. Either do both steps 1 and 4 or do neither.



Graph created using Desmos.com/calculator

3. Example: $y^3 + 3x^2y = 13$
   a) Find y' when
   b) Compute y' when $x = 2$

4. Example: $y^7 - 3x = Ax^2$

Suppose A is a constant, and y is an implicit function of x.
Find the derivative of y w.r.t x.

Find y' (Leibnitz notation: $\mathrm{d}y/\mathrm{d}x$) using implicit differentiation: $2xy = 36$

A) $\mathrm{d}y/\mathrm{d}x = \quad 0$

B) $\mathrm{d}y/\mathrm{d}x = \quad \frac{x}{y}$

C) $\mathrm{d}y/\mathrm{d}x = \quad \frac{y}{x}$

D) $\mathrm{d}y/\mathrm{d}x = -\frac{x}{y}$

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5. Example: $2w^2 - 2v = w$
   Find w' ($\mathrm{d}w/\mathrm{d}v$) using implicit differentiation:

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- If you reduce c_1, you save more and c_2 rises.
- So, we can say that c_2 is an implicit function of c_1.

Utility in a Two-Period Model

How much does c_2 change if we change c_1 by a (very) small amount?

$$u(c_1) + \beta u(c_2) = \bar{U}$$

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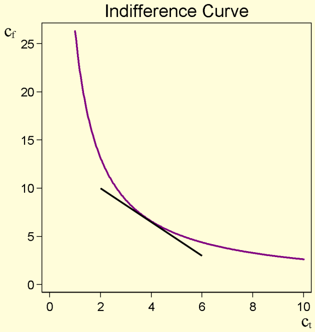
- The marginal rate of substitution (MRS) is how much additional c_2 the consumer needs to be indifferent towards giving up some c_1.
- The MRS is the slope of an indifference curve.
- In general,

$$\text{MRS} = \frac{\text{MU}_{c_1}}{\text{MU}_{c_2}}$$



Second Implicit derivative

$2xy = 10$

Second Implicit derivative

$$x^2 + y^2 = 25$$

Find the second implicit derivative, y'', and simplify so that only y, x and any constants appear on the right-hand side of your equation.

A Function within an Expression

Example 1: $4g(x) + y^3 = 2$

Find y'. Note that $g(x)$ is a function.

Functions within Functions

Example 2:

$g(h(x)) - yx - y^2 = 0$

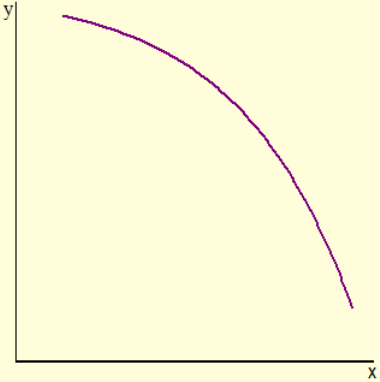
$h(x) = 2x^2 + 4$

Find y' where g, h are functions.

Functions within Functions

Example 3: $g(2x + y) - x^2 + y^2 = 0$
Find y' where g is a function

Approximating Change With the Derivative



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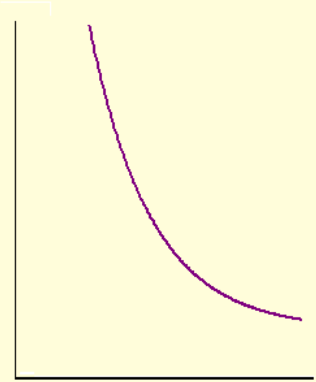

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- Stocks reward investors with dividends.
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- Fixed amounts make bond prices vulnerable to unexpected price inflation and changes in market interest rates more generally.
- **Duration** is a useful way to measure the sensitivity of a bond to a change in market interest rates. [a]

[a] Market interest rates are (partially) set by central banks like the Bank of Canada or U.S. Federal Reserve.

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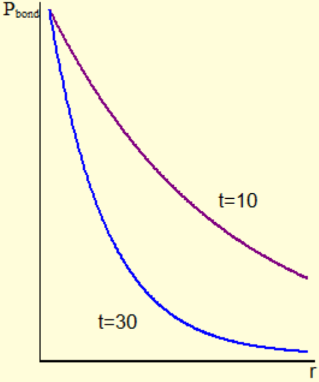
$$P = Fe^{-rt}$$

a) Find the relative rate of change in the bond price with respect to the market interest rate, r. Interpret your result.



P_{bond}
t=10
t=30
r

Example 1: Bond Prices and Interest Rate Risk

$$P = Fe^{-rt}$$

- b) Use your answer in part a to approximate the relative rate of change in the price of bond if the market interest rate increases by 1 percent.
- c) Use your answer in part a to approximate the relative rate of change in the price of bond if the market interest rate decreases by 1 percent.

Example 1: Bond Prices and Interest Rate Risk

Consider a bond that will pay \$1,000 in 10 years with a market interest rate of 2%.

d) Calculate the bond price, P.

e) Calculate the **actual** relative rate of change in the bond price if the market rate increases 1 percent.

f) Calculate the **actual** relative rate of change in the bond price if the market rate decreases 1 percent.

g) How do your answers compare with your answers in parts a,b?

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