The Loop Quantities and Bifurcations of Homoclinic Loops

Maoan Han[a*] and Huaiping Zhu[a, b †]
[a]Department of Mathematics,
Shanghai Normal University, Shanghai, China, 200030
[b]LAMPS/LIAM, Department of Mathematics and Statistics,
York University, Toronto, ON, Canada M3J 1P3

Abstract

The stability and bifurcations of a homoclinic loop for planar vector fields are closely related to the limit cycles. For a homoclinic loop of a given planar vector field, a sequence of quantities, the homoclinic loop quantities were defined to study the stability and bifurcations of the loop. Among the sequence of the loop quantities, the first nonzero one determines the stability of the homoclinic loop. There are formulas for the first three and the fifth loop quantities. In this paper we will establish the formula for the fourth loop quantity for both the single and double homoclinic loops. As applications, we present examples of planar polynomial vector fields which can have five or twelve limit cycles respectively in the case of a single or double homoclinic loop by using the method of stability-switching.

Keywords: Homoclinic loops; saddle quantities; limit cycles; stability; bifurcation.

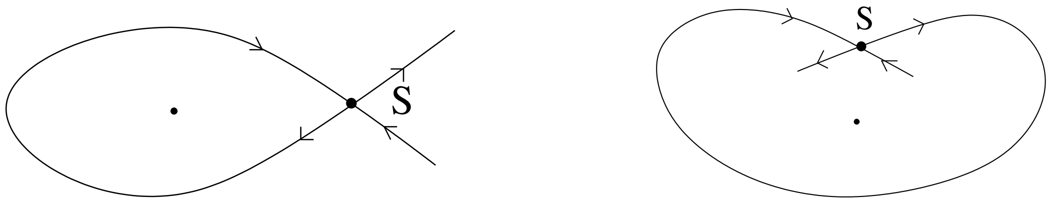
1 Introduction

Consider a planar vector field

$$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}^2 \tag{1.1}$$

[*]Research partially supported by the National Natural Science Foundation of China 10371072.
[†]Corresponding author, Email: huaiping@mathstat.yorku.ca, Research supported by Natural Science and Engineering Research Council of Canada, by Mathematics for Information Technology and Complex Systems and by Canadian Foundation of Innovation

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^∞ function. Suppose that the vector field (1.1) has a homoclinic loop consisting of a homoclinic orbit L and a hyperbolic saddle point S. For simplicity, L is also called a saddle loop or homoclinic loop. Since the saddle S is hyperbolic, we can define a Poincaré map on one and only one side of a given loop L. More precisely, if we suppose that L is oriented clockwise as shown in Fig. 1, there are two possible cases: convex (Fig. 1(a)) and concave (Fig. 1(b)). The Poincaré map is well defined near L in the interior of L, $Int.(L)$ for the convex loop (Fig. 1(a)), and in the exterior of L, $Ext.(L)$ for the concave loop (Fig. 1(b)). If there exists a neighborhood U of L such that for any point $A \in U \cap Int.L$ or $U \cap Ext.L$, the positive orbit $\gamma^+(A)$ of (1.1) starting at A approaches L, then L is said to be stable. If the negative orbit $\gamma^-(A)$ approaches L, then L is said to be unstable.



(a) Convex case (b) Concave case

Figure 1: Two cases of homoclinic loops L

The study of the stability and bifurcations of homoclinic loops can be traced back to Dulac [5], and since then great progress have been made and there have been a large amount of work published in the field. We remark that the work of Leontovich(1946), Andronove *et al* [2], Melnikov [18], Chow and Hale [3] and Roussarie [19], Joyal [14], Joyal and Rousseau [17], Dumortier and Li [6] are among the important references.

A homoclinic loop is called isolated if there is no other loop in its neighborhood. Clearly, a homoclinic loop can be either stable or unstable. A non-isolated homoclinic loop may appear as the boundary curve of period annuli [4, 6, 23]. In many cases a non-isolated homoclinic loop can generate an isolated loop under perturbations on a codimension one surfaces in the parameter space. A homoclinic loop naturally has a saddle point. The saddle point is called weak or neutral if the hyperbolicity ratio $r = -\frac{\lambda_2}{\lambda_1} = 1$, here $\lambda_2 < 0 < \lambda_1$ are the two eigenvalues of the saddle point. It was Roussarie [19, 20] who initiated a systematic study of the homoclinic loops. We point out that the studies later led to a program aiming at proving the finiteness part of Hilbert's problem for quadratic vector fields [8]. Joyal [14, 15, 16] then conducted further studies of the homoclinic loops with a weak saddle.

Joyal in [14], by using the Poincaré normal forms at the saddle point, defined a set of quantities c_1, c_2, c_3 $\cdots$ (a_i^* in the original paper [14]) to study the stability and bifurcations of homoclinic loops. It follows from [14] that a homoclinic loop L

is said to be of order k if $c_1 = \cdots = c_{k-1} = 0$ and $c_k \neq 0$. And L can generate at most k limit cycles under any C^∞ perturbations. Moreover, the k limit cycles can be obtained in a neighborhood of L by suitable perturbations. The sign of the first nonzero saddle quantities c_k also determines the stability of L. For a homoclinic loop of order k, it is stable (resp. unstable) if $c_k < 0$ (resp., $c_k > 0$).

For a given homoclinic loop of a planar vector field, the first nonzero quantity c_k is not only decided by the saddle point, but also decided by the loop of the vector field, hence we would rather call the sequence c_1, c_2, $\cdots$, c_k the *homoclinic loop quantities*. We will call the first nonzero loop quantity the homoclinic constant or homoclinic value, and accordingly, the homoclinic loop is called of order k provided $c_k \neq 0$ and $c_i = 0$, $i = 1, 2, \cdots, k-1$.

It is well known that for the center of a planar vector field, there are Lyapunov center quantities (coefficients) [2] which determine the number of limit cycles bifurcating from the center in a generalized Hopf bifurcation. Naturally, the homoclinic loop quantities play an important role in these studies for the theory of bifurcations of dynamical systems, but it is also closely related to Hilbert's sixteenth problem. As pointed out in [14] that the homoclinic loop quantities can be used to determine the number of limit cycles which can be bifurcated from a homoclinic loop. Andronov and C. E. Chaiken mentioned this analogy in [1], Joyal [14] made a theoretical study into the duality between the generalized Hopf bifurcation and generalized homoclinic bifurcations.

For a given homoclinic loop of a planar vector field, the first saddle quantity is the divergence of the vector field at the saddle point [2], the second loop quantity c_2 (assuming the first one is zero) is the integral of the divergence around the loop [2, 9]. If the first two loop quantities vanish, the third quantity was computed in [13]. The fifth loop quantity c_5 was obtained in [19] when the first four loop quantities all vanish. In this paper we fill the gap and establish the formula for the fourth loop quantity c_4. It is a more complicated integral around the homoclinic loop. All these computations are not trivial, and it seems that there is no general form for the loop quantities. In principle, these formulaes are integrals along the homoclinic loop which is similar to the Melnikov integral, but they are more difficult to compute in practice.

One other motivation to compute the fourth loop quantity is the needs to study the cyclicity of a degenerate graphic of HH type [22] which is related to Hilbert's sixteenth problem for quadratic vector field. Dumortier, Roussarie and Rousseau [8] lunched a project aiming at proving the finiteness part of the Hilbert's sixteenth problem for quadratic vector fields. The project breaks down the proof into proving all the graphics (limit periodic sets) have finite cyclicity inside quadratic families. A graphic of planar vector field can be elementary or non-elementary in the sense that its singular points are elementary (hyperbolic or semihyperbolic, i.e. at least one nonzero eigenvalue) or non-elementary. The none-elementary graphics in the quadratic vector fields include graphic through a nilpotent saddle or graphic through a nilpotent elliptic point. The finite cyclicity of the nilpotent graphics of quadratic vector fields can be proved if one can prove that all the limit periodic sets of the blown-up families have finite cyclicity [24]. The limit periodic sets for the nilpotent graphics of elliptic type fall into three categories: PP, HP and HH. Zhu and Rousseau [21] have proved that all the PP-graphics with a nilpotent elliptic singularity have finite cyclicity. It turns out that in order to study the cyclicity of a HH-graphics, one need to compute the fourth loop quantity.

This paper is organized as the follows. We first give some preliminaries and summarize the formulas for the first three and the fifth order of homoclinic numbers. The main results, the formulas for the fourth order homoclinic loop number for both single and double homoclinic loops will be presented in this second section. We prove the main results in section 3. As applications of the main results, in section 4 we study the the limit cycles which can be born from the bifurcations of homoclinic loops in two special systems. To find the limit cycles near a homoclinic loop which can be bifurcated from the perturbations, the stability-switching technique [11, 13] was employed.

2 Preliminaries and Main Results

Dulac [5] was the first to give the homoclinic constant for the homoclinic loop of order one (see also [2, 3, 18]):

$$c_1 = \mathrm{tr} f_x(S) = (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2})(S) = \mathrm{div} f(S). \tag{2.1}$$

When $c_1 = 0$, the saddle point S becomes a weak saddle. It follows from [2, 9] that we have

$$c_2 = \oint_L \mathrm{div} f dt = \int_{-\infty}^{\infty} (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2})(u(t))dt \qquad \text{if } c_1 = 0 \tag{2.2}$$

where $u(t)$, $-\infty < t < \infty$ is a time parametrization of the homoclinic loop L.

To discuss the case when $c_1 = c_2 = 0$ and also for the purpose of presenting our main results, let us give the normal forms for the weak saddle and recall some known results on the saddle quantities [17].

Assume that (1.1) has a homoclinic loop with $c_1 = 0$, i.e., the saddle is weak. If we locate saddle point S at the origin and modulo a linear transformation, $f(x)$ can then be written as

$$f(x) = \begin{pmatrix} \lambda[\ x_1 + \sum\limits_{i+j \geq 2} a_{ij} x_1^i x_2^j\ ] \\ \lambda[\ -x_2 + \sum\limits_{i+j \geq 2} b_{ij} x_1^i x_2^j\ ] \end{pmatrix}, \quad \lambda > 0, \tag{2.3}$$

or

$$f(x)=\begin{pmatrix}\lambda[\ x_2+\sum\limits_{i+j\geq 2}a_{ij}x_1^ix_2^j\ ]\\ \lambda[\ x_1+\sum\limits_{i+j\geq 2}b_{ij}x_1^ix_2^j\ ]\end{pmatrix},\qquad \lambda>0. \tag{2.4}$$

In [14], vector field (1.1) was transformed into the normal form

$$\begin{cases}\dot{x_1} &= \lambda x_1[\ 1+\sum\limits_{i=1}^{n}a_i(x_1x_2)^i+(x_1x_2)^{n+1}V_1(x_1,x_2)\ ] \quad = f_1(x_1,x_2),\\ \dot{x_2} &= \lambda x_2[\ -1+\sum\limits_{i=1}^{n}b_i(x_1x_2)^i+(x_1x_2)^{n+1}V_2(x_1,x_2)\ ] \ = f_2(x_1,x_2),\end{cases} \tag{2.5}$$

where $n\geq 2$. Let

$$R_i=a_i+b_i=R_i(S),\quad i\geq 1.$$

Then R_i is called the ith saddle quantity of (1.1) at S (see [14, 17]). Formulas for saddle quantities were also given in [17]. For example, for the first saddle quantity of the vector field (1.1) with $f(x)$ given in (2.3), we have

$$R_1=a_{21}+b_{12}-a_{20}a_{11}+b_{02}b_{11}, \tag{2.6}$$

while the first saddle quantity of the vector field (1.1) with $f(x)$ given in (2.4) reads

$$\begin{aligned}R_1 \ = \ & 3(a_{30}-b_{03})+b_{21}-a_{12}+2(a_{02}b_{02}-a_{20}b_{20})\\ & +a_{11}(a_{02}-a_{20})+b_{11}(b_{02}-b_{20}).\end{aligned} \tag{2.7}$$

Consider (1.1) with $f(x)$ in the form of (2.4). One can assume that $\lambda=1$. Then (1.1) can be rewritten as

$$\begin{cases}\dot{x_1} &= x_2+P(x_1,x_2),\\ \dot{x_2} &= x_1+Q(x_1,x_2),\end{cases} \tag{2.8}$$

where

$$P(x_1,x_2)=\sum_{k+l\geq 2}a_{kl}\ x_1^k\ x_2^l,\qquad Q(x_1,x_2)=\sum_{k+l\geq 2}b_{kl}\ x_1^k\ x_2^l.$$

As in [17], letting

$$\omega=x_1+j\ x_2,\quad \bar{\omega}=x_1-j\ x_2$$

with $j^2=-1$. In terms of ω and $\bar{\omega}$, (2.8) becomes

$$\begin{cases}\dot{\omega} &= j\omega+F(\omega,\bar{\omega}),\\ \dot{\bar{\omega}} &= -j\bar{\omega}+\bar{F}(\omega,\bar{\omega}),\end{cases} \tag{2.9}$$

where

$$\begin{aligned}F(\omega,\bar{\omega})&=P(\frac{\omega+\bar{\omega}}{2},\frac{\omega-\bar{\omega}}{2j})+j\ Q(\frac{\omega+\bar{\omega}}{2},\frac{\omega-\bar{\omega}}{2j}),\\ \bar{F}(\omega,\bar{\omega})&=P(\frac{\omega+\bar{\omega}}{2},\frac{\omega-\bar{\omega}}{2j})-j\ Q(\frac{\omega+\bar{\omega}}{2},\frac{\omega-\bar{\omega}}{2j}).\end{aligned} \tag{2.10}$$

We can write

$$F(\omega, \bar{\omega}) = \sum_{k+l \geq 2} (A_{kl} + jB_{kl})\, \omega^k\, \bar{\omega}^l,$$

where A_{kl} and B_{kl} are polynomials of a_{mn} and b_{mn} with $2 \leq m + n \leq k + l$ whose coefficients are independent of j. Using (2.10) and $j^2 = -1$, we have

$$\bar{F}(\omega, \bar{\omega}) = \sum_{k+l \geq 2} (A_{kl} - jB_{kl})\, \bar{\omega}^k\, \omega^l.$$

Hence, if we define

$$\mathrm{Re}(A_{kl} + jB_{kl}) = A_{kl}, \quad \mathrm{Im}(A_{kl} + jB_{kl}) = B_{kl},$$

we can apply the formulas for the second and third Lyapunov constants V_5 and V_7 to the first equation of system (2.9) to obtain the second and third saddle quantities R_2 and R_3 respectively.

Using normal form theory and the Poincaré map near L, it was proved in [13] that if $c_1 = c_2 = 0$, for the third homoclinic loop quantity c_3 we have

$$c_3 = \begin{cases} -R_1, & \text{convex case,} \quad \text{Fig. 1(a),} \\ R_1, & \text{concave case, Fig. 1(b).} \end{cases} \tag{2.11}$$

For the higher order homoclinic loop quantities c_k with $k \geq 4$, the only known result was due to Roussarie [19]:

$$c_5 = \pm R_2, \qquad \text{if } c_1 = c_2 = c_3 = c_4 = 0. \tag{2.12}$$

However, up to now, one does not know the formula of c_4. In this paper we are going to develop the formula for c_4 and fill the gap.

To state our results, we introduce the following notations. Consider a homoclinic loop of a planar system (1.1) with a weak saddle ($c_1 = 0$). Let

$$K(L) = \int_{-\infty}^{\infty} \frac{W(u(t))}{|f(u(t))|} \exp \int_{-\infty}^{t} \mathrm{tr} f_x(u(s)) ds dt, \tag{2.13}$$

where $u(t)$, $-\infty < t < \infty$ denotes a time-parametrization of the homoclinic loop L

and

$$
\begin{aligned}
W(x) &= C(x) - A(x)B(x), \\
A(x) &= Z^T(x) f_x(x) Z(x), \\
B(x) &= \frac{1}{|f(x)|^2} f^T(x) \Big[f_x(x)Z(x) - Z_x(x) f(x) \Big], \\
C(x) &= \frac{1}{2} Z^T(x) D(x) Z(x), \\
Z(x) &= \frac{1}{|f(x)|} \begin{pmatrix} -f_2(x) \\ f_1(x) \end{pmatrix}, \\
D(x) &= \begin{pmatrix} d_{11}(x) & d_{12}(x) \\ d_{21}(x) & d_{22}(x) \end{pmatrix}, \quad d_{ij}(x) = \Big(\mathrm{grad} \frac{\partial f_i}{\partial x_j}\Big) \cdot Z \ , \ i,j = 1,2, \\
Z_x(x) &= \frac{1}{|f(x)|^2} \left[\begin{pmatrix} -\mathrm{grad}\ f_2 \\ \mathrm{grad}\ f_1 \end{pmatrix} |\ f| - \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \mathrm{grad} |f(x)| \right] \\
&= \frac{1}{|f(x)|^3} \begin{pmatrix} f_1 f_2 \ \mathrm{grad} f_1 - f_1^2 \ \mathrm{grad} f_2 \\ f_2^2 \ \mathrm{grad} f_1 - f_1 f_2 \ \mathrm{grad} f_2 \end{pmatrix}.
\end{aligned} \tag{2.14}
$$

Recall that the integral $K(L)$ is said to be convergent if the following limit exists and finite:

$$
\lim_{\substack{T_1 \to \infty \\ T_2 \to -\infty}} \int_{T_2}^{T_1} \frac{W(u(t))}{|f(u(t))|} \exp \int_{T_2}^{t} \mathrm{tr}\ f_x(u(s)) ds dt = K(L).
$$

Our main results are the following.

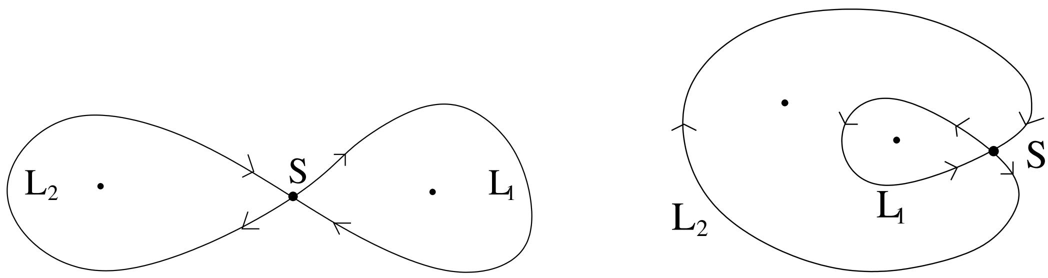
Theorem 2.1. *Let $c_1 = 0$. Then the integral $K(L)$ given by* (2.13) *is convergent if and only if the first saddle quantity $R_1 = 0$.*

Theorem 2.2. *Let $c_1 = c_2 = c_3 = 0$. Then*

$$
\begin{aligned}
c_4 &= \begin{cases} -K(L), & \text{convex case in Fig. } 1(a), \\ \ \ K(L), & \text{concave case in Fig. } 1(b), \end{cases} \\
c_5 &= R_2.
\end{aligned} \tag{2.15}
$$

Remark 2.3. *The formula for the fifth loop quantity c_5 was obtained in [19]. Here it can be obtained as a byproduct of the proof of the theorem.*

The case for the double homoclinic loop is similar to the case of a single loop since the two parts of the loop share the same saddle point. Assume that the vector field (1.1) has a double homoclinic loop $L = L_1 \bigcup L_2$ with a hyperbolic saddle point S. If we also assume the loop is oriented clockwise, in the sense of convexity similar



(a) Convex double loops (b) Concave double loops

Figure 2: Two cases of the double homoclinic loops $L = L_1 \cup L_2$

to the single loop, we also have two possible cases of double homoclinic loops as shown in Fig. 2.

Similar to the single homoclinic loop, following the work of Joyal [14] we can define the stability of double homoclinic loops L by introducing the double homoclinic loop quantities. Let c_1^*, c_2^*, $c_3^* \cdots$ be such constants that L is stable (resp,. unstable) if

$$c_1^* = c_2^* = \cdots = c_{k-1}^* = 0,\ c_k^* < 0 \text{ (resp.,} \, c_k^* > 0 \text{).}$$

As in the case for the single loop, we call c_1^*, c_2^*, $c_3^* \cdots$ double homoclinic loop quantities of L, and say L to be of order k if $c_1^* = c_2^* = \cdots = c_{k-1}^* = 0$ and $c_k^* \neq 0$. It follows from [12, 13] that

$$\begin{aligned} c_1^* &= \operatorname{div} f(0), \\ c_2^* &= \oint_L \operatorname{div} f \, dt = \sum_{i=1}^{2} \oint_{L_i} \operatorname{div} f \, dt \quad \text{(if } c_1^* = 0 \text{),} \end{aligned}$$

and L can generate at most one (resp., two) large limit cycle under C^∞ perturbations if $c_1^* \neq 0$ (resp., $c_1^* = 0$ and $c_2^* \neq 0$), where "large limit cycle" means a cycle surrounding the unique saddle point near S of the perturbed system.

It was conjectured in [12] that L can generate at most three large limit cycles under C^∞ perturbations if $c_1^* = c_2^* = 0$ and the first saddle quantity $R_1 \neq 0$. This conjecture still remains open. For the double homoclinic loop, it was proved in [12] that if $c_1^* = c_2^* = 0$ then

$$c_3^* = \begin{cases} \quad -R_1, & \text{case in Fig. 2(a)} \\ \quad R_1, & \text{case in Fig. 2(b)} \end{cases} . \tag{2.16}$$

In this paper we will establish formulas for the fourth order homoclinic loop quantity c_4^* for the double loops too.

Theorem 2.4. *Consider the double homoclinic loop L as shown in Fig. 2. Let $c_1^* = c_2^* = c_3^* = 0$, then*

$$\begin{aligned} c_4^* &= \begin{cases} K(L_1) + K(L_2)e^{c_{21}}, & \textit{convex case in Fig. 2(a)}, \\ -[K(L_1) + K(L_2)e^{c_{21}}], & \textit{concave case in Fig. 2(b)}, \end{cases} \\ c_5^* &= R_2, \end{aligned} \tag{2.17}$$

where

$$c_{21} = \oint_{L_1} \mathrm{div} f dt.$$

We remark that if (1.1) is centrally symmetric, then $c_1^* = c_2^* = c_3^* = 0$ imply $c_{21} = 0$ and $K(L_1) = K(L_2)$.

The paper is organized as follows. In Section 2 we present proof of our main results. In Section 3 we will give applications of Theorems 1.2 and 1.3 respectively. For the homoclinic case, we give an example of homoclinic loop of order five and prove the existence of five limit cycles near the loop in the perturbed vector field. For the double homoclinic case, we give an example of double homoclinic loop of order five but find twelve limit cycles in a neighborhood of the loop in a perturbed vector field. To find the limit cycles near L which can be born from the perturbations, we use the so-called stability-changing method [11, 13].

3 Proof of the Main Theorems

Let L be a homoclinic loop as shown in Fig. 1.1. We use a Poincaré map to study the stability of L. For the purpose, the normal form equation (2.5) plays a key role. Without loss of generality, we assume that (2.5) is a valid normal form defined in the square $Q = \left\{ x \middle| |x_1| \le 1, |x_2| \le 1 \right\}$ and that the homoclinic loop L is located mainly in the fourth quadrant. Define cross sections Σ_1 and Σ_2 as follows

$$\Sigma_1 : x_2 = -1, 0 < x_1 < 1 \ ; \quad \Sigma_2 : x_1 = 1, -1 < x_2 < 0$$

Then by using the positive orbits of (2.5), we can define a Dulac map $D : \Sigma_1 \to \Sigma_2$ and a regular map $R : \Sigma_2 \to \Sigma_1$, see Fig. 3.

Introducing points $A_1(0, -1)$ and $A_2(1, 0)$ and unit vectors $n_1 = (-1, 0)^T$ and $n_2 = (0, 1)^T$ on Σ_1 and Σ_2 respectively. Note that n_i is a directional vector of the cross sections Σ_i, $i = 1, 2$. Then any point on Σ_i can be represented as $A_i + n_i r$ with $-1 < r < 0$, and as function of r, the maps D and R satisfying

$$A_2 + n_2 D(r) \in \Sigma_2, \quad A_1 + n_1 R(r) \in \Sigma_1.$$

In next two lemmas, we develop the expressions for the Dulac map D and the regular map G respectively.

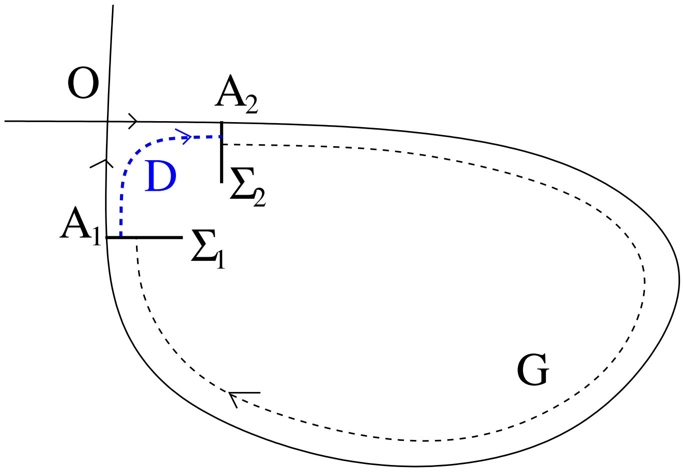


Figure 3: Dulac and regular maps near the homoclinic loop L

Lemma 3.1. *Let $c_1 = 0$. Then for $r \in (-1, 0)$ and $|r|$ small we have*

$$D(r) = r - R_1 r^2 \ln|r| \Big[1 + O(r \ln|r|) \Big] - (R_2 r^3 \ln|r| + O(r^4 \ln|r|)). \tag{3.1}$$

Proof. Noting that if $c_1 = 0$, we have by (2.5) that

$$\frac{dx_2}{dx_1} = \frac{x_2}{x_1} \Big[-1 + R_1 x_1 x_2 + R_2^*(x_1 x_2)^2 + O(|x_1 x_2|^3) \Big],$$

where $R_2^* = R_2 - a_1 R_1$. Then the expression (3.1) for the Dulac map can be obtained following the development in [19]. □

Lemma 3.2. *Let $u(t)$ be a time-parametrization of the homoclinic loop L. Let*

$$\begin{aligned} h_1(\theta) &= \frac{f(u(t_2))}{|f(u(\theta))|} \exp \int_{t_2}^{\theta} tr\ f_x(u(t)) dt, \\ h_2(t_1) &= h_1(t_1) |f(u(t_2))| \int_{t_2}^{t_1} \frac{W^*(\theta)}{|f(u(\theta))|} \exp \int_{t_2}^{\theta} tr\ f_x(u(t)) dt d\theta, \end{aligned} \tag{3.2}$$

where W^* is defined similarly as W in (2.14). Then for the regular map G we have

$$G(r) = h(t_1, r) = h_1(t_1) r + h_2(t_1) r^2 + O(r^3). \tag{3.3}$$

Proof. Let

$$z(t) = Z(u(t)) = \frac{1}{|f(u(t))|} \begin{pmatrix} -f_2(u(t)) \\ f_1(u(t)) \end{pmatrix}.$$

Then there exist unique t_1 and t_2 ($t_2 < t_1$) such that

$$u(t_i) = A_i\ , \quad z(t_i) = n_i\ , \quad i = 1, 2. \tag{3.4}$$

It follows from [13] that a change of variables of the form

$$x = u(\theta) + z(\theta)h \ , \ t_2 \le \theta \le t_1 \tag{3.5}$$

can carry (1.1) into the form

$$\begin{cases} \dot{\theta} = 1 + B^*(\theta)h + O(h^2), \\ \dot{h} = A^*(\theta)h + C^*(\theta)h^2 + O(h^3), \end{cases} \tag{3.6}$$

where

$$\begin{aligned} A^*(\theta) &= z^T(\theta) f_x(u(\theta))z(\theta) = \mathrm{tr} f_x(u(\theta)) - \frac{d}{d\theta} \ln |f(u(\theta))|, \\ B^*(\theta) &= \frac{1}{|f(u(\theta))|^2} f^T(u(\theta)) \left[f_x(u(\theta))z(\theta) - \frac{d}{d\theta} z(\theta) \right], \\ C^*(\theta) &= \frac{1}{2} z^T(\theta) \left[\frac{\partial^2}{\partial^2 h} f\Big(u(\theta) + z(\theta)h\Big) \right]_{h=0}. \end{aligned} \tag{3.7}$$

Straightforward calculations can lead to

$$A^*(\theta) = A(u(\theta)) \ , \quad B^*(\theta) = B(u(\theta)) \ , \quad C^*(\theta) = C(u(\theta)), \tag{3.8}$$

where A, B and C are defined in (2.14). Rewriting (3.6) as

$$\frac{dh}{d\theta} = A^*(\theta)h + \Big[C^*(\theta) - A^*(\theta)B^*(\theta) \Big] h^2 + O(h^3). \tag{3.9}$$

Let

$$h(\theta, r) = h_1(\theta)r + h_2(\theta)r^2 + O(r^3) \tag{3.10}$$

be a solution of (3.9) satisfying $h(t_2) = r$. Substituting $h(\theta, r)$ into (3.9) one can find that h_1 and h_2 satisfy

$$\begin{cases} \dfrac{dh_1}{d\theta} = A^*(\theta)h_1, & h_1(t_2) = r, \\ \dfrac{dh_2}{d\theta} = A^*(\theta)h_2 + W^*(\theta)h_1^2, & h_2(t_2) = 0, \end{cases}$$

where by (3.8) we have $W^*(\theta) = C^*(\theta) - A^*(\theta)B^*(\theta) = W(u(\theta))$. Solving the above linear equations we obtain

$$\begin{aligned} h_1(\theta) &= \exp \textstyle\int_{t_2}^{\theta} A^*(s) ds, \\ h_2(\theta) &= h_1(\theta) \textstyle\int_{t_2}^{\theta} W^*(s)h_1(s) ds. \end{aligned} \tag{3.11}$$

It follows from (3.8), (3.10) and (3.11) that we have the expression for the regular map G in (3.3). $\square$

Proof of Theorem 1.1: We now prove Theorem 1.1 by using the normal form (2.5).

Let $u(t) = (u_1(t), u_2(t))^T$. Then by (2.5) we have that

$$u_1(t) = 0 \quad \text{for} \quad t \geq t_1 \qquad \text{and} \qquad u_2(t) = 0 \quad \text{for} \quad t \leq t_2 \tag{3.12}$$

Hence, again from (2.5) we have

$$\mathrm{tr} f_x(u(t)) = 0 \qquad \text{for} \qquad t \leq t_2 \quad \text{or} \quad t \geq t_1.$$

Let $T_2 < t_2$, $T_1 > t_1$. Then we have

$$\int_{T_2}^{t} \mathrm{tr} f_x(u(s))ds = \begin{cases} 0, & \text{if} \quad T_2 \leq t \leq t_2, \\ \displaystyle\int_{t_2}^{t} \mathrm{tr} f_x(u(s))ds, & \text{if} \quad t_2 \leq t \leq t_1, \\ \displaystyle\int_{t_2}^{t_1} \mathrm{tr} f_x(u(s))ds, & \text{if} \quad t_1 \leq t \leq T_1. \end{cases}$$

Note that

$$\int_{t_2}^{t_1} \mathrm{tr} f_x(u(s))ds = \int_{-\infty}^{\infty} \mathrm{tr} f_x(u(t))dt = c_2.$$

By (2.13) and $W^*(\theta) = W(u(\theta))$ we have therefore

$$\begin{aligned} K^*(T_1, T_2) &= \int_{T_2}^{T_1} \frac{W^*(t)}{|f(u(t))|} \exp \int_{T_2}^{t} \mathrm{tr} f_x(u(s))dsdt \\ &= \int_{T_2}^{t_2} \frac{W^*(t)}{|f(u(t))|} dt + \int_{t_1}^{T_1} \frac{W^*(t)}{|f(u(t))|} e^{c_2} dt \\ &\quad + \int_{t_2}^{t_1} \frac{W^*(t)}{|f(u(t))|} \exp \int_{t_2}^{t} \mathrm{tr} f_x(u(s))dsdt. \end{aligned} \tag{3.13}$$

Using (3.12) we have

$$z(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u(t) + z(t)h = \begin{pmatrix} u_1(t) \\ h \end{pmatrix}$$

for $t \leq t_2$. Hence, by (3.7) we have

$$A^*(t) = -\lambda, \quad B^*(t) = a_1 u_1(t), \quad C^*(t) = \lambda b_1 u_1(t),$$

and

$$\frac{W^*(t)}{|f(u(t))|} = \frac{C^*(t) - A^*(t)B^*(t)}{|f(u(t))|} = \frac{\lambda u_1(t)(a_1 + b_1)}{\lambda u_1(t)} = a_1 + b_1$$

for $t \leq t_2$. Similarly for $t \geq t_1$, we have

$$\frac{W^*(t)}{|f(u(t))|} = \frac{-\lambda u_2(t)(a_1 + b_1)}{-\lambda u_2(t)} = a_1 + b_1.$$

Therefore it follows from (3.13) that

$$K^*(T_1, T_2) = (a_1 + b_1)[t_2 - T_2 + e^{c_2}(T_1 - t_1)] + K^*(t_1, t_2).$$

Hence $K(L) = \lim\limits_{\substack{T_1 \to \infty \\ T_2 \to -\infty}} K^*(T_1, T_2)$ is convergent if and only if $R_1 \equiv a_1 + b_1 = 0$. This completes the proof of Theorem 1.1. $\square$

From the above discussion, we have

$$|f(u(t_1))| = |f(u(t_2))| = \lambda$$

and

$$W^*(t) = \lambda R_1 \qquad \text{for} \qquad t \leq t_2 \quad \text{or} \quad t \geq t_1.$$

Thus, it follows from (3.2) and (3.13) that if $R_1 = 0$ we have

$$h_1(t_1) = e^{c_2}, \quad h_2(t_1) = e^{c_2}\lambda K(L).$$

Hence, by (3.3) we obtain

Corollary 3.3. *Let $c_1 = 0$. Then the regular map $G : \Sigma_2 \longrightarrow \Sigma_1$ can be written as*

$$G(r) = e^{c_2}r + \frac{1}{2}G''(0)r^2 + O(r^3), \qquad 0 < -r \ll 1, \tag{3.14}$$

where $\frac{1}{2}G''(0) = \lambda e^{c_2} K(L)$ when the first saddle quantity $R_1 = 0$.

Proof of Theorem 1.2: Consider the first case of homoclinic loop in the Fig. 1(a).

The Poincaré map $P : \Sigma_1 \longrightarrow \Sigma_1$ near L can be decomposed as

$$P(r) = (G \circ D)(r).$$

If $c_1 = 0$, it follows from Lemma 2.1 and Corollary 2.1 that we have

$$\begin{aligned} P(r) &= e^{c_2}D(r) + \frac{1}{2}G''(0)D^2(r) + O(r^3) \\ &= e^{c_2}r + \lambda e^{c_2}Kr^2 - e^{c_2}R_2r^3\ln|r| + O(r^3). \end{aligned}$$

Thus, if $c_1 = c_2 = R_1 = 0$ there holds

$$P(r) - r = \lambda K r^2 - R_2 r^3 \ln|r| + O(r^3), \qquad \text{for} \qquad 0 < -r \ll 1. \tag{3.15}$$

Note that the loop L is stable (unstable) if

$$P(r) - r > 0 \;(\; < 0\;) \qquad \text{for} \qquad 0 < -r \ll 1,$$

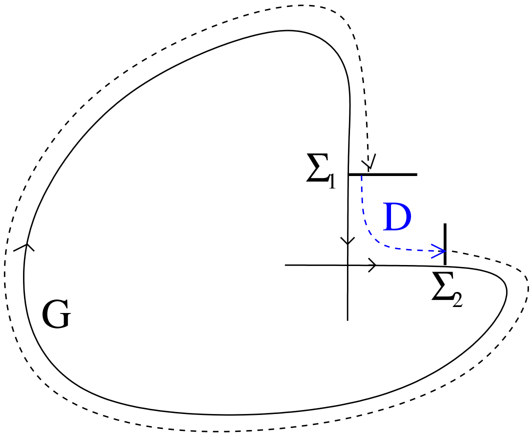


Figure 4: The Poincaré map for second case of the loop L

since l_1 has the same direction as n_1. The conclusion follows directly from (3.15).

For the second case of the loop in Fig. 1(b), we can define the transversal cross sections as

$$\Sigma_1 : x_2 = 1, \quad 0 < x_1 < 1 \qquad \text{and} \qquad \Sigma_2 : x_1 = 1, \; 0 < x_2 < 1$$

with end points $A_1(0, 1)$ and $A_2(1, 0)$, and directional vectors $n_1 = (1, 0)^T$ and $n_2 = (0, 1)^T$. We will consider the Dulac map $D : \Sigma_1 \longrightarrow \Sigma_2$ and the regular map $G : \Sigma_2 \longrightarrow \Sigma_1$, Fig. 4.

In this case, if $c_1 = 0$ we have

$$D(r) = r - R_1 \, r^2 \ln r[1 + O(r \ln r)] - R_2 \, r^3 \ln r + O(r^4 \ln r) \tag{3.16}$$

for $0 < r \ll 1$. Note that Corollary 2.1 remains true. The expansion (3.15) remains valid for $0 < r \ll 1$ if $c_1 = c_2 = R_1 = 0$.

Note that for the second case L is stable (unstable) if for $0 < r \ll 1$, there holds

$$P(r) - r > 0 \; (< 0 \;).$$

The conclusion then follows from (3.15). This ends the proof of Theorem 1.2. □

Proof of Theorem 1.3 Now suppose (1.1) has a double homoclinic loop $L = L_1 \cup L_2$ homoclinic to a hyperbolic saddle point S. To prove Theorem 1.3 we only consider the first case of Fig. 2(a). The proof for the second case in Fig. 2(b) is similar. As before, we work on the normal form (2.5).

Similar to the homoclinic case, we can define Dulace maps D_1 and D_2 and regular maps G_1 and G_2 in the neighborhood of the saddle point, see Fig. 5.

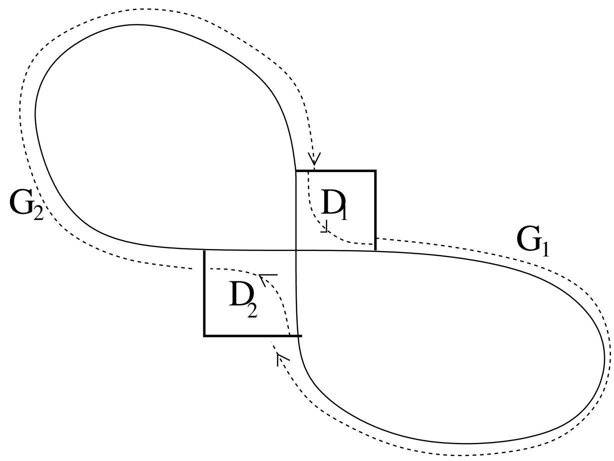


Figure 5: The Dulac maps and regular maps for the double homoclinic loop L

Let $c_1^* = R_1 = 0$. Then by Corollary 2.1 and (3.16) we have

$$\begin{aligned} G_i(r) &= e^{c_{2i}}\ r + \lambda\ e^{c_{2i}} K(L_i) r^2 + O(r^3), \quad 0 < r \ll 1\ , \\ D_i(r) &= r - R_2\ r^3 \ln r + O(r^4 \ln r), \qquad\quad 0 < r \ll 1, \end{aligned}$$

where

$$c_{2i} = \oint_{L_i} \mathrm{div} f dt, \quad i = 1, 2.$$

Thus,

$$\begin{aligned} P_i(r) &\equiv (G_i \circ D_i)(r) \\ &= e^{c_{2i}}\ r + \lambda\ e^{c_{2i}} K(L_i) r^2 - R_2 e^{c_{2i}} r^3 \ln r + O(r^3). \end{aligned}$$

Consider the Poincaré map $P = P_2 \circ P_1$. A straightforward calculation gives

$$\begin{aligned} P(r) &= e^{c_{21}+c_{22}}\ r + \lambda\ e^{c_{21}+c_{22}} \Big[K(L_1) + K(L_2) e^{c_{21}} \Big] r^2 \\ &\quad - R_2 e^{c_{21}+c_{22}} (1 + e^{2c_{21}}) r^3 \ln r \Big(1 + O(\frac{1}{\ln r}) \Big) + O(r^3). \end{aligned}$$

Therefore, if further $c_2^* = c_{21} + c_{22} = 0$, it follows that

$$P(r) - r = \begin{cases} \lambda c_4^* r^2 + O(r^3 \ln r)\ , & \text{if}\ \ c_4^* \neq 0, \\ -R_2 (1 + e^{2c_{21}}) r^3 \ln r + O(r^3)\ , & \text{if}\ \ c_4^* = 0, \end{cases}$$

where $c_4^* = K(L_1) + K(L_2) e^{c_{21}}$. Then Theorem 1.3 follows. □

4 Applications

As some applications of the main theorems, here we present examples of planar polynomial vector field which have five or twelve limit cycles in the case of a single or double homoclinic loop respectively. One other application of the formula for c_4 will appear in [22].

Consider a polynomial vector field of the form

$$\begin{cases} \dot{x} = H_y + H_x[a_0 + a_1 H + a_2 H^2], \\ \dot{y} = -H_x + H_y[b_0 + b_1 H + b_2 H^2], \end{cases} \tag{4.1}$$

where a_i and b_i $(i = 0, 1, 2)$ are all parameters, and H is a polynomial in (x, y).

First, we take

$$H = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3. \tag{4.2}$$

Then if $a_0 = b_0 = 0$, (4.1) has a homoclinic loop given by $L : H(x, y) = 0$. The loop intersects the positive x-axis at the point $A(\frac{3}{2}, 0)$.

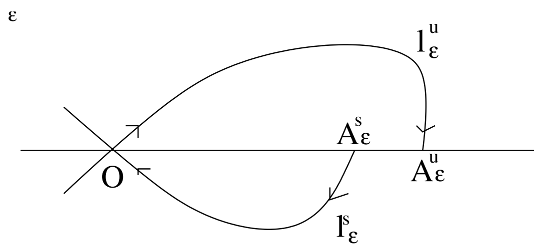


Figure 6: Breaking of the loop and the displacement

As shown in Fig. 6, let l^s and l^u denote the stable and unstable separatrices of (4.1) near L respectively. They have intersection on the positive x-axis at the points $A^s_\epsilon(x^s, 0)$ and $A^u_\epsilon(x^u, 0)$. Note that the divergence of (4.1) reads

$$\begin{aligned} \mathrm{div}(4.1) &= a_0 H_{xx} + b_0 H_{yy} + a_1 H_x^2 + b_1 H_y^2 \\ &\quad + (a_1 H_{xx} + b_1 H_{yy} + 2a_2 H_x^2 + 2b_2 H_y^2)H + (a_2 H_{xx} + b_2 H_{yy})H^2. \end{aligned} \tag{4.3}$$

Let $d(a_0, a_1, a_2, b_0, b_1, b_2)$ be the distance from $A^u_\epsilon(x^u, 0)$ to $A^s_\epsilon(x^s, 0)$. Then

$$d(a_0, a_1, a_2, b_0, b_1, b_2) = x^u - x^s = \frac{M(a_0, b_0)}{|H_x(A)|} + O(|a_0, b_0|^2), \tag{4.4}$$

where

$$\begin{aligned} M(a_0, b_0) &= \oint_L H_y(b_0 + b_1 H + b_2 H^2)dx - H_x(a_0 + a_1 H + a_2 H^2)dy \\ &= a_0 N_1(a_1, b_1) + b_0 N_2(a_1, b_1), \end{aligned}$$

where

$$
\begin{aligned}
N_1 &= \oint_L H_x^2 \exp\left(-\int_0^t (a_1 H_x^2 + b_1 H_y^2) ds\right) dt, \\
N_2 &= \oint_L H_y^2 \exp\left(-\int_0^t (a_1 H_x^2 + b_1 H_y^2) ds\right) dt.
\end{aligned}
\tag{4.5}
$$

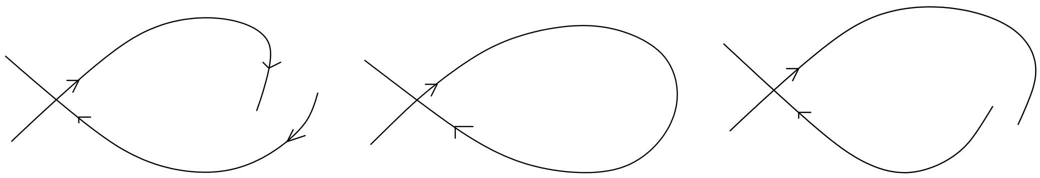
Applying the implicit function theorem to (3.4), there exists a unique function

$$\varphi_1(a_0, a_1, a_2, b_1, b_2) = -\frac{N_1}{N_2} a_0 + O(a_0^2)$$

such that for (a_0, b_0) near $(0, 0)$,

$$d(a_0, a_1, a_2, b_0, b_1, b_2) \geq 0 \qquad \text{if and only if} \qquad b_0 \geq \varphi_1. \tag{4.6}$$

This is also explained in Fig. 7.



(a) $b_0 < \phi_1$ (b) $b_0 = \phi_1$ (c) $b_0 > \phi_1$

Figure 7: Phase portraits of (4.1) near L

Thus, for $|a_0| + |b_0|$ small (3.1) has a homoclinic loop L^* near L if and only if $b_0 = \varphi_1$.

Proposition 4.1. *Consider planar vector fields* (4.1) *with* $H(x, y)$ *given in* (4.2). *It has a homoclinic loop* L *defined by* $H(x, y) = 0$. *Then for* $|a_0| + |b_0|$ *small, system* (4.1) *has a homoclinic loop* L^* *near* L *of order* k *if and only if the following* k*th condition is satisfied* ($k = 1, 2, 3, 4, 5$)*:*

(1) $b_0 = \varphi_1, a_0 \neq 0,$

(2) $b_0 = a_0 = 0, b_1 + \frac{5}{7} a_1 \neq 0,$

(3) $b_0 = a_0 = 0, b_1 = -\frac{5}{7} a_1 \neq 0,$

(4) $b_0 = a_0 = b_1 = a_1 = 0, b_2 + \frac{5}{7} a_2 \neq 0,$

(5) $b_0 = a_0 = b_1 = a_1 = 0, b_2 = -\frac{5}{7} a_2 \neq 0.$

Proof. Let $b_0 = \varphi_1$. Then by (3.3) we have

$$c_1 = \operatorname{div}(4.1)\Big|_{(0,0)} = -a_0\left(1 + \frac{N_1}{N_2}\right) + O(a_0^2).$$

Hence, L^* has order 1 if and only if $b_0 = \varphi_1, a_0 \neq 0$.

Let $a_0 = 0, b_0 = \varphi_1 = 0$. Then $L^* = L$, and by (4.3) we have

$$c_2 = \oint_{L^*} \mathrm{div}(4.1)dt = \oint_L (a_1 H_x^2 + b_1 H_y^2)dt.$$

Straightforward calculations lead to the following:

$$\begin{aligned} &\oint_L H_y^2 dt = \oint_L y dx = \frac{6}{5}, \\ &\oint_L H_x^2 dt = \oint_L (x - x^2)dy = \oint_L (2x-1)ydx = \frac{6}{7}. \end{aligned} \tag{4.7}$$

Hence $c_2 = \frac{6}{5}(b_1 + \frac{5}{7}a_1)$. It follows that L^* has order 2 if and only if $b_0 = a_0 = 0, b_1 + \frac{5}{7}a_1 \neq 0$.

Let $b_0 = a_0 = 0, b_1 = -\frac{5}{7}a_1$. We can rewrite (4.1) as

$$\begin{cases} \dot{x} = y + \frac{a_1}{2}(x^3 - xy^2) + O(|(x,y)^T|^4), \\ \dot{y} = x - x^2 + \frac{5}{14}a_1(x^2 y - y^3) + O(|(x,y)^T|^4). \end{cases}$$

Then by (2.7) we have

$$c_3 = -R_1 = -\frac{24}{7}a_1$$

which implies that L^* is of order 3 if and only if $b_0 = a_0 = 0, b_1 = -\frac{5}{7}a_1 \neq 0$.

Further, let $b_0 = a_0 = b_1 = a_1 = 0$. In this case, system (4.1) becomes

$$\begin{cases} \dot{x} = H_y + a_2 H_x H^2 \equiv f_1(x,y) \\ \dot{y} = -H_x + b_2 H_y H^2 \equiv f_2(x,y). \end{cases} \tag{4.8}$$

It is straightforward that

$$\begin{aligned} &\frac{\partial f_1}{\partial x} = a_2(H_{xx}H^2 + 2H_x^2 H)\ , \quad \frac{\partial f_1}{\partial y} = 2a_2 H_x H_y H + H_{yy}\ , \\ &\frac{\partial f_2}{\partial x} = -H_{xx} + 2b_2 H_x H_y H\ , \quad \frac{\partial f_2}{\partial y} = b_2(H_{yy}H^2 + 2HH_y^2)\ , \end{aligned}$$

and

$$\begin{aligned} &(\mathrm{grad}\frac{\partial f_1}{\partial x})\Big|_L = (2a_2 H_x^3\ ,\ 2a_2 H_x^2 H_y)\ , \\ &(\mathrm{grad}\frac{\partial f_1}{\partial y})\Big|_L = (2a_2 H_x^2 H_y\ ,\ 2a_2 H_x H_y^2)\ , \\ &(\mathrm{grad}\frac{\partial f_2}{\partial x})\Big|_L = (-H_{xxx} + 2b_2 H_x^2 H_y\ ,\ 2b_2 \varepsilon H_x H_y^2)\ , \\ &(\mathrm{grad}\frac{\partial f_2}{\partial y})\Big|_L = (2b_2 H_x H_y^2\ ,\ 2b_2 H_y^3)\ . \end{aligned}$$

Then according to notations and formulas in (2.9) we have

$$
\begin{aligned}
\frac{\partial Z}{\partial(x,y)}\bigg|_L &= \frac{1}{|\text{grad}H|^2}\begin{pmatrix} H_y^2 H_{xx} & -H_yH_xH_{yy} \\ -H_xH_yH_{xx} & H_x^2 H_{yy}\end{pmatrix}, \\
D(x,y)\bigg|_L &= \frac{H_x^2+H_y^2}{|\text{grad}H|}\begin{pmatrix} 2a_2H_x^2 & 2a_2H_xH_y \\ -\dfrac{H_xH_{xxx}}{H_x^2+H_y^2}+2b_2H_xH_y & 2b_2H_y^2\end{pmatrix}, \\
A(x,y)\bigg|_L &= \frac{1}{|\text{grad}H|^2}H_xH_y(H_{yy}-H_{xx}), \\
B(x,y)\bigg|_L &= \frac{1}{|\text{grad}H|^5}(H_{xx}-H_{yy})(H_x^4-H_y^4), \\
C(x,y)\bigg|_L &= \frac{1}{2|\text{grad}H|^3}\Big[2(H_x^2+H_y^2)^2(a_2H_x^2+b_2H_y^2)-H_x^2H_yH_{xxx}\Big].
\end{aligned}
$$

Therefore,

$$
\frac{W(x,y)}{|\text{grad}H|}\bigg|_L = (a_2H_x^2+b_2H_y^2)+W_0(x,y), \tag{4.9}
$$

where

$$
W_0(x,y) = \frac{H_xH_y}{(H_x^2+H_y^2)^3}\Big[(H_{yy}-H_{xx})^2(H_x^2-H_y^2)-\frac{1}{2}H_xH_{xxx}(H_x^2+H_y^2)\Big]. \tag{4.10}
$$

It follows from (4.2) we have that along L, $y^2 = x^2 - \frac{2}{3}x^3$, hence we can write

$$
W_0(x,y) = yg(x), \qquad x > 0,
$$

for some C^∞ function $g(x)$ on $x > 0$. Thus, with $\text{div}(f_1,f_2)\Big|_L = 0$, we have by (4.7) that

$$
\begin{aligned}
K(L) &= \oint_L (a_2H_x^2+b_2H_y^2)dt + \oint_L g(x)ydt \\
&= \frac{6}{5}(b_2+\frac{5}{7}a_2) + \oint_L g(x)dx .
\end{aligned}
$$

Let $\delta \in (0,1)$ and $\eta = \eta(\delta) > 0$ satisfying $H(\delta, \pm\eta) = 0$. Then

$$
\oint_L g(x)dx = \lim_{\delta\to 0}\int_{(\delta\ ,\ \eta)}^{(\delta\ ,-\eta)} g(x)dx = \lim_{\delta\to 0} 0 = 0.
$$

Therefore, we have

$$
c_4 = -K(L) = -\frac{6}{5}(b_2+\frac{5}{7}a_2)
$$

and hence L^* has order 4 if and only if $b_0 = a_0 = b_1 = a_1 = 0, b_2 + \frac{5}{7}a_2 \neq 0$.

Finally, let $b_0 = a_0 = b_1 = a_1 = 0, b_2 = -\frac{5}{7}a_2$. Then (4.8) can be written as

$$\begin{cases} \dot{x} = y - \dfrac{a_2}{4}x(x^2 - y^2)^2 + O(|(x, y)^T|^6), \\ \dot{y} = x - x^2 - \dfrac{5}{28}a_2 y(x^2 - y^2)^2 + O(|(x, y)^T|^6). \end{cases}$$

Let $\omega = x + jy$, $\bar{\omega} = x - jy$. We obtain

$$\dot{\omega} = j\omega + F(\omega, \bar{\omega}),$$

where

$$\begin{aligned} F(\omega, \bar{\omega}) &= \frac{-a_2}{28}(6\omega + \bar{\omega})(\omega\bar{\omega})^2 - \frac{1}{4}(\omega + \bar{\omega})^2 \\ &= -\frac{1}{4}(\omega^2 + 2\omega\bar{\omega} + \bar{\omega}^2) - \frac{a_2}{28}(6\omega^3\bar{\omega}^2 + \omega^2\bar{\omega}^3). \end{aligned}$$

Using the formula for V_5 given in [10] we have up to a positive constant that

$$c_5 = R_2 = 6(-\frac{a_2}{28} \times 6) = -\frac{9}{7}a_2.$$

□

Note that by (4.5) and (4.7) we have

$$\varphi_1(a_0, a_1, a_2, b_1, b_2) = a_0\Big(-\frac{5}{7} + O(|a_0, a_1, b_1|)\Big).$$

We obtain the following by changing the stability of L^* in turn and breaking it finally.

Proposition 4.2. *Consider planar vector fields* (4.1) *with* $H(x, y)$ *given in* (4.2). *Then exists a function* $\varphi_1 = a_0(-\frac{5}{7} + O(|a_0, a_1, b_1|))$ *such that for given* $a_2 > 0$, *system* (4.1) *has five limit cycles near* L *if*

$$0 < \varphi_1 - b_0 \ll a_0 \ll b_1 + \frac{5}{7}a_1 \ll a_1 \ll -(b_2 + \frac{5}{7}a_2) \ll 1.$$

Now consider (4.1) again but with

$$H(x, y) = \frac{1}{2}(y^2 - x^2) + \frac{1}{4}x^4. \tag{4.11}$$

Then for $a_0 = b_0 = 0$, system (4.1) has a double homoclinic loop $L = L_1 \cup L_2$ where $L_i = \{H = 0 \,\Big|\, (-1)^i x > 0\}, i = 1, 2$. Similar to (4.5) and (4.7), we have

$$\begin{aligned} &\oint_{L_2} H_y^2 dt = \frac{4}{3} \quad , \qquad &&\oint_{L_2} H_x^2 dt = \frac{28}{35} \\ &N_1(L_2) = \frac{28}{35} + O(|a_1, b_1|) \ , \quad &&N_2(L_2) = \frac{4}{3} + O(|a_1, b_1|). \end{aligned}$$

And, there is a function $\varphi_1 = -\frac{N_1}{N_2}a_0 + O(a_0^2)$ such that for $|a_0| + |b_0|$ small (4.1) has a double homoclinic loop $L^* = L_1^* \cup L_2^*$ if and only if $b_0 = \varphi_1$. When $b_0 = \varphi_1$ we have from (4.3) that

$$c_1^* = \mathrm{div}(3.1)\Big|_{(0,0)} = -(1 + \frac{N_1}{N_2})a_0 + O(a_0^2).$$

Hence, as the previous case, by using (4.3), (4.8)-(4.10) we can obtain

$$\begin{aligned}
&c_2^* = \frac{4}{3}(b_1 + \frac{7}{5}a_1), && \text{as} \quad a_0 = b_0 = 0;\\
&c_3^* = R_1 = \frac{24}{5}a_1, && \text{as} \quad a_0 = b_0 = b_1 + \frac{7}{5}a_1 = 0;\\
&c_4^* = K(L_2) = \frac{4}{3}(b_2 + \frac{7}{5}a_2), && \text{as} \quad a_0 = b_0 = a_1 = b_1 = 0.\\
&c_5^* = R_2 = -\frac{9}{5}a_2 && \text{as} \quad a_0 = b_0 = a_1 = b_1 = b_2 + \frac{7}{5}a_2 = 0.
\end{aligned}$$

Then similar to Proposition 3.2 we have

Proposition 4.3. *Consider the planar system* (4.1) *with* $H(x, y)$ *given in* (4.11)*. It has a double homoclinic loop* L *defined by* $H(x, y) = 0$*. Then there exists a function* $\varphi_1 = a_0(-\frac{7}{5} + O(|a_0, a_1, b_1|))$ *such that for given* $a_2 > 0$ *system* (4.2) *has twelve limit cycles with two large cycles surrounding ten small cycles near* L *if*

$$0 < \varphi_1 - b_0 \ll a_0 \ll b_1 + \frac{7}{5}a_1 \ll a_1 \ll -(b_2 + \frac{7}{5}a_2) \ll 1 \ .$$

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