Module: Summations and Sequences

Module Outline

- Summation notation and properties
- Summations in Excel/Google sheets
- Sums of *arithmetic* and *geometric* series
- Sequences and Limits of sequences

Summation notation

- Total population of Canada:

$$P_{NFL} + P_{PEI} + P_{NS} + P_{NB} + P_{QC} + P_{ON}$$
$$+P_{MB} + P_{SK} + P_{AB} + P_{BC} + P_{YT} + P_{NT} + P_{NU}$$

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- Writing this succinctly: $P_1 \equiv P_{NFL}, P_2 \equiv P_{PEI}, ...$

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6$$
$$+ P_7 + P_8 + P_9 + P_{10} + P_{11} + P_{12} + P_{13}$$

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- Even more succintly?

Summation notation

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Summation notation

- $P_1 + P_2 + P_3 + ... + P_{13}$:
- Sum of $P_1, P_2, ..., P_{13}$:

$$\sum_{i=1}^{13} \mathbf{P}_i$$

Summation notation

- $P_1 + P_2 + P_3 + ... + P_{13}$:
- Sum of $P_1, P_2, ..., P_{13}$:

$$\sum_{i=1}^{13} \mathbf{P}_i$$

- Three elements: (i) $\sum$ sign **[sigma]**
  (ii) **what** are we summing over: P_i
  (iii) from where to where: i from 1 to 13 **[index]**

Summation notation: Examples

- A constituency has n districts.

  Number of votes that party L receives in district $j = v_j$

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- Total number of votes for party L in the constituency=

$$\sum_{j=1}^{n} \mathbf{v}_j$$

Summation notation: Examples

- Total number of Covid cases last year:

$$\sum_{t=1}^{12} \mathbf{n}_t$$

where n_t = number of Covid cases in month t

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- Total number of Covid cases from July-Dec:

$$\sum_{t=7}^{12} \mathbf{n}_t$$

Clicker question 1

- Which of the following expressions denotes the total number of Covid cases during the summer months of June-August?

(a) $\sum_{t=1}^{8} \mathbf{n}_t$

(b) $\sum_{t=6}^{12} \mathbf{n}_t$

(c) $\sum_{t=6}^{8} \mathbf{n}_t$

(d) $\sum_{t=1}^{6} \mathbf{n}_t$

Solution to Clicker question 1

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(c) $\sum_{t=6}^{8} \mathbf{n}_t$ **(d)** $\sum_{t=1}^{6} \mathbf{n}_t$

Summation notation

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Summation notation

- $\sum_{i=1}^{n} \mathbf{Z}_i =$ "sum of Z_i from $i = 1$ to $i = n$"
- $\sum_{i=5}^{n} \mathbf{Z}_i =$
- *Example*: In a village, there are 40 families, ranked in terms of their wealth: $W_1 < W_2 < W_3 < ...... < W_{40}$

  What is the total wealth held by the top 10 families?

Clicker question 2

- There are 24 hours in a day. C_i denotes the number of cars that pass through an intersection in hour i.

  Which of the following expressions denotes the total number of cars that pass through the intersection during a day?

(a) $\sum_{i=1}^{n} \mathbf{X}_i$

(b) $\sum_{i=1}^{12} \mathbf{C}_i$

(c) $\sum_{i=1}^{24} \mathbf{C}_i$

(d) $\mathbf{C}_{24}$

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Summation notation: Examples

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  Average height $=$ (*Sum* of every tree's *height*)$/200$

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- Total height = Sum of every tree's height $= \sum_{i=1}^{200} \mathbf{h}_i$

- Average height:

$$\left(\sum_{j=1}^{100} \mathbf{h}_i\right) / 200 = \frac{1}{200} \sum_{i=1}^{200} \mathbf{h}_i$$

Summation notation: Examples from Statistics

- Average height in a forest with n trees:

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{h}_i$$

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- *General*: **Mean** of n data points $X_1, X_2, ..., X_n$:.

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- Average height in a forest with n trees:

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{h}_i$$

- *General*: **Mean** of n data points $X_1, X_2, \ldots.., X_n$:.

- $$\overline{X} = \frac{\sum_{i=1}^{n}\mathbf{X}_i}{n} \equiv \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i$$

Summation notation: Examples from Statistics

- **Mean** of n data points $X_1, X_2, ..., X_n$:.

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i$$

Summation notation: Examples from Statistics

- **Mean** of n data points $X_1, X_2, ..., X_n$:.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$$

- **Variance**:

$$\frac{1}{n}\sum_{i=1}^{n}(\mathbf{X}_i - \overline{X})^2$$

Summation notation: More Applications

- *Economics:* Jim's portfolio consists of

  10 Air Canada stocks, valued at \$30 each,

  50 Ford stocks, valued at \$12 each,

  20 TD stocks, valued at \$80 each,

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- What is the value of his portfolio?

Summation notation: More Applications

- *More generally:* Jim's portfolio consists of

  $\mathbf{x}_1$ stocks of Company 1, valued at p_1 each,

  $\mathbf{x}_2$ stocks of Company 2, valued at p_2 each,

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- $p_1 x_1 + p_2 x_2 + ... + p_n x_n = \sum_{i=1}^{n} p_i x_i$

Summation notation: More Applications

- There are 5 assignments in a course

  $\mathbf{w}_1$: weightage on Assignment 1, score: s_1

  $\mathbf{w}_2$: weightage on Assignment 2, score: s_2......

  What is your overall score for the course?

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- $w_1 s_1 + w_2 s_2 + ... + w_5 s_5 = \sum_{i=1}^{5} w_i s_i$ **[weighted average]**

Summation: Properties

- $10 + 20 + ... + 100 =$

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- $10x_1 + 10x_2 + ... + 10x_n = 10(x_1 + x_2 + ... + x_n) =$

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- $10 + 20 + ... + 100 =$

- $10x_1 + 10x_2 + ... + 10x_n = 10(x_1 + x_2 + ... + x_n) =$

- *More generally:*
  $cx_1 + cx_2 + ... + cx_n = c(x_1 + x_2 + ... + x_n) = c\sum_{i=1}^{n} x_i$

Summation: Properties

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- $10x_1 + 10x_2 + ... + 10x_n = 10(x_1 + x_2 + ... + x_n) =$

- *More generally:*
  $cx_1 + cx_2 + ... + cx_n = c(x_1 + x_2 + ... + x_n) = c\sum_{i=1}^{n} x_i$

- $$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

Clicker question 3

- Aya measured the heights of trees in a forest in metres and found that the total height of all the trees was 9000 m. Maya measured the same, but in feet.

  Which of the following expressions denotes the total height that Maya found? Note that $1\ m = 3.3\ ft$.

  (a) 9000 *ft* **(b)** 3.3 X 9000 *ft*

  (c) 9000/3.3 *ft* **(d)** 33000 *ft*

Solution to Clicker question 3

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Summation: Properties

- *Adding up two sets of numbers*:

  v_i = number of violent crimes in province i

  z_i = number of non-violent crimes in province i

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- Total number of crimes in province i:

Summation: Properties

- *Adding up two sets of numbers*:

  v_i = number of violent crimes in province i

  z_i = number of non-violent crimes in province i

- Total number of crimes in province i:
- Total number of crimes across all provinces:

$$\sum_{i=1}^{n}(v_i + z_i) = \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} z_i$$

Summation: Properties

- *General property*:

$$\sum_{i=1}^{n}(a_i + b_i + c_i + d_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} d_i$$

Summation in Excel/Google Sheets

How to do summation in **Excel/Google Sheets**?

- Use SUM[.....*Range*....] for Summation
- Use AVERAGE[.....*Range*....] for Average
- Use VAR[.....*Range*....] for Variance

Common sums: Arithmetic Series

- $1 + 2 + 3 + 4 + 5 + ... + 100?$

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Common sums: Arithmetic Series

- $1 + 2 + 3 + 4 + 5 + ... + 100?$
- $1 + 2 + 3 + ... + 100 = \sum_{i=1}^{100} i$
- $1 + 2 + 3 + 4 + ... + 100 = 50 \text{ X } 101 = 5050$

Common sums: Gauss' formula

- $1 + 2 + 3 + 4 + \ldots\ldots + n = \sum_{i=1}^{n} i$?

Common sums: Gauss' formula

- $1 + 2 + 3 + 4 + \ldots\ldots + n = \sum_{i=1}^{n} i$?
- $$1 + 2 + 3 + 4 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

Common sums: Gauss' formula

- $1 + 2 + 3 + 4 + ... + n = \sum_{i=1}^{n} i$?
- $$1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$$
- $1 + 2 + 3 + 4 + ... + 1000 =$

Common sums: Gauss' formula

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  Quick proof:

Common sums: Gauss' formula

- $$1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$$

  Quick proof:

- What if $51 + 52 + 53 + ... + 78 + 79 + 80$?

Common sums: Gauss' formula

- $51 + 52 + \ldots\ldots + 79 + 80 = \sum_{i=51}^{80} i$?

Common sums: Gauss' formula

- $51 + 52 + ... + 79 + 80 = \sum_{i=51}^{80} i$?
- *General formula for arithmetic series*:

$$a_1 + a_2 + a_3 + ... + a_n = \frac{n(a_1 + a_n)}{2}$$

where $a_{i+1} = a_i + 1$.

Gauss' formula: Applications

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  Every year his salary goes up by \$1.
  If he works for 40 years, what is lifetime earnings?

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- $501 + 502 + 503 + \dots\dots + 540 =$

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- $33 + 36 + \ldots\ldots + 60?$

Clicker question 4

- What is $4 + 8 + 12 + ... + 100$?.

  (a) 325

  (b) 400

  (c) 800

  (d) 1300

Solution to Clicker question 4

- What is $4 + 8 + 12 + \ldots.. + 100$?.

  (a) 325

  (b) 400

  (c) 800

  (d) 1300

Common sums: Geometric Series

- [Model of infections]
  $T{=}1$: 1 $\quad$ $T{=}2$: R $\quad$ $T{=}3$: R^2....... $T{=}10$: R^9

Common sums: Geometric Series

- [Model of infections]
  T=1: 1 *T=2*: R *T=3*: R^2....... *T=10*: R^9
- Total number of infections till day 10 :

$$1 + R + R^2 + ... + R^9 = ?$$

Common sums: Geometric Series

- [Model of infections]
  T=1: 1 *T=2*: R *T=3*: R^2....... *T=10*: R^9
- Total number of infections till day 10 :

$$1 + R + R^2 + \ldots\ldots + R^9 = ?$$

- $\sum_{i=0}^{9} R^i = ?$

Common sums: Geometric Series

- *Geometric series*: **multiply by a constant** to get next term

$$1 + R + R^2 + \ldots\ldots + R^{n-1} = \frac{1 - R^n}{1 - R}$$

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Common sums: Geometric Series

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- If $R = 2$, $\quad \sum_{i=0}^{9} R^i =$

- General formula:

$$a + aR + aR^2 + ... + aR^{n-1} = a\frac{1 - R^n}{1 - R}$$

Geometric Series: Applications

- *Chess board story*: Put 1 grain of rice on square 1

  Put 2 grains on square 2

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- How many grains in total?

$$1 + 2 + 4 + 8 + 16 + ...?$$

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$$1 + 2 + 4 + 8 + 16 + \ldots\ldots?$$

- $1 + 2 + 2^2 + 2^3 + 2^4 \ldots. + 2^{63} =$

Geometric Series: Other Applications

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  year 1 : 100 crimes

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- year 2 : $100(1 + 0.05)$

  year 3 : $100(1 + 0.05)(1 + 0.05)$........

Geometric Series: Other Applications

- *Crime rate grows by 5% every year*:

  year 1 : 100 crimes

- year 2 : $100(1 + 0.05)$

  year 3 : $100(1 + 0.05)(1 + 0.05)$........

- How many crimes over 10 years?

$$100 + 100(1.05) + 100(1.05)^2 + ...100(1.05)^9 =$$

Geometric Series Application: Annuities

- *Annuity:* a stream of N equal cash flows paid at regular intervals (years, months, quarters)

  Examples: lottery payouts, retirement benefits, car loan payments,.......

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- The PV of receiving C N years from now: $\frac{C}{(1+r)^N}$

Annuities: Present value calculation

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Annuities: Present value calculation

- *Annuity*: receive C every year for years $1, 2, ..., N$
- **Present value (PV):**

$$\frac{C}{(1+r)} + \frac{C}{(1+r)^2} + ... + \frac{C}{(1+r)^N} = ?$$

- Use Geometric series formula:
  $1 + R + R^2 + ... + R^{n-1} = \frac{1-R^n}{1-R}$

PV of an Annuity

- **PV** of an annuity:

$$PV = \frac{C}{r}[1 - \frac{1}{(1+r)^N}]$$

PV of an Annuity

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- Say $C = 1000$, $N = 10$, $r = 3\%$; $PV =$

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- Suppose $r = 5\%$; $PV =$

PV of an Annuity

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$$PV = \frac{C}{r}[1 - \frac{1}{(1+r)^N}]$$

- Say $C = 1000$, $N = 10$, $r = 3\%$; $PV =$
- Suppose $r = 5\%$; $PV =$
- Upfront payment or annuity?

PV of a Perpetuity

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- PV of a perpetuity:

$$PV = \frac{C}{r}$$

PV of a Perpetuity

- PV of an annuity: $\frac{C}{r}[1 - \frac{1}{(1+r)^N}]$
- **Perpetuity**: Receive C forever i.e. $N = \infty$
- PV of a perpetuity:

$$PV = \frac{C}{r}$$

- Say $C = 1000$, $r = 3\%$; PV of perpetuity $=$

Sequences

- *Sequence*: Numbers in a particular pattern

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- (Previous example) *Kim's salary*: $501, 502, 503, 504 \ldots\ldots$

  Infections: $1, R_0, R_0^2, R_0^3, \ldots\ldots$

Sequences

- *Sequence*: Numbers in a particular pattern
- (Previous example) *Kim's salary*: $501, 502, 503, 504........$

  Infections: $1, R_0, R_0^2, R_0^3, ...$
- More general: $a_1, a_2, a_3, ..., a_n, ...$

Some common sequences

- *Arithmetic sequence* pattern:

  next number is found by adding a constant: $a_{n+1} = a_n + c$

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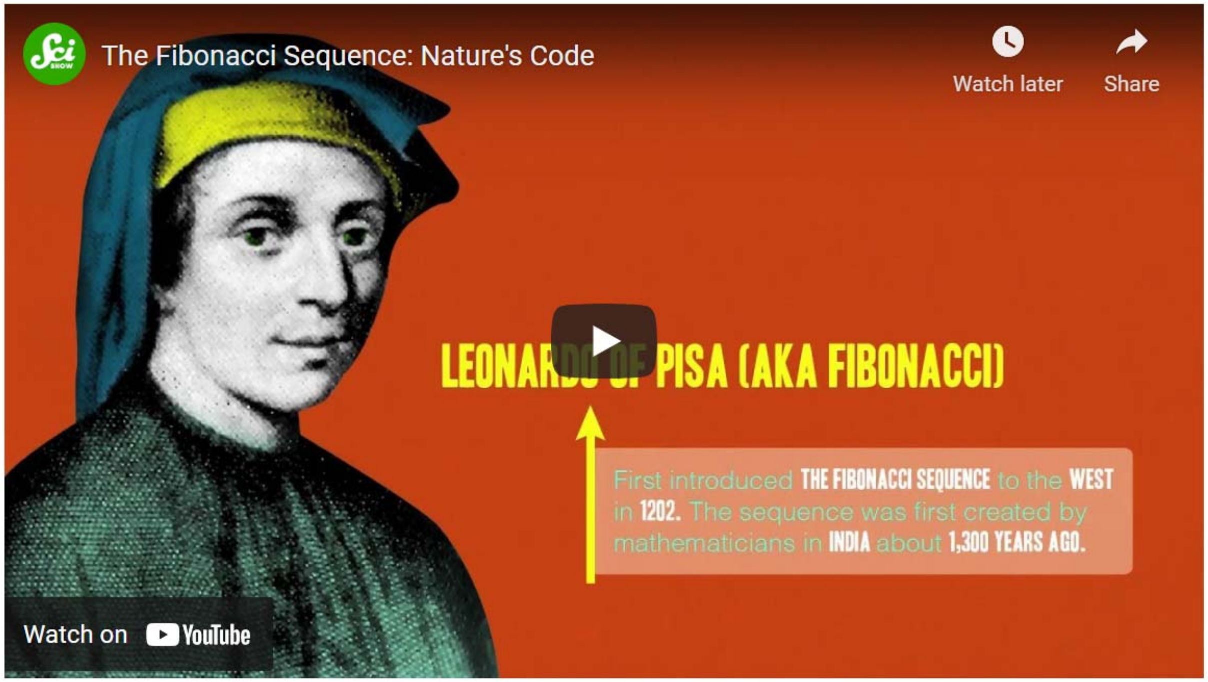
  next number is found by multiplying by a constant: $a_{n+1} = a_n r$

- *Fibonacci sequence* pattern:

  next number is found by adding previous two: $a_{n+1} = a_n + a_{n-1}$

[The Fibonacci Sequence: Nature's Code – YouTube](https://www.youtube.com/watch?v=wTlw7fNcO-0) :
https://www.youtube.com/watch?v=wTlw7fNcO-0



Limits of sequences

- *Limit of a sequence*: $\lim_{n \to \infty} a_n$

  where does the number go toward when n becomes very large?

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  where does the number go toward when n becomes very large?

- *Some common limits*:

- Kim: $501, 502, 503, ...500 + n, ... \to$

Some common limits of sequences

- $1, 2, 3, \ldots\ldots., n, \ldots. \rightarrow$

Some common limits of sequences

- $1, 2, 3, ..., n, ... \to$
- $1, 0, -1, -2, -3, ..., -n, ... \to$

Some common limits of sequences

- $1, 2, 3, \ldots\ldots., n, \ldots. \longrightarrow$
- $1, 0, -1, -2, -3, \ldots\ldots., -n, \ldots. \longrightarrow$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots. \longrightarrow$

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- Implications for other sequences: $\lim_{n\to\infty} \frac{10+n}{n}$?
- $\lim_{n\to\infty} \frac{n}{n+1}$? : $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ... \longrightarrow$

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- Not all sequences have limits: $2, -2, 3, -3, 4, -4, \ldots\ldots.$

Clicker question 5

- What is the limit of the following sequence:
  $7, \frac{9}{2}, \frac{11}{3}, \frac{13}{4}, \ldots\ldots\ldots, \frac{2n+5}{n}, \ldots.$

  (a) 0

  (b) 2

  (c) 5

  (d) ∞

Solution to Clicker question 5

- What is the limit of the following sequence:
  $7, \frac{9}{2}, \frac{11}{3}, \frac{13}{4}, \ldots\ldots\ldots, \frac{2n+5}{n}, \ldots.$

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