OPTIMAL RETIREMENT INVESTMENT STRATEGIES UNDER HEALTH SHOCKS AND JUMP-DIFFUSION PROCESSES

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Abstract

The dissertation focuses on two problems applied to personal financial management for individuals, either before or after retirement.

The first topic examines a lifetime ruin probability (LRP) model in which a jump-diffusion process drives the investment return of the agent. The value of the LRP is important to an investor who wants to find out the probability of running out of money, while maintaining a desired standard of living for the rest of his life. Our model leads to a partial-integro-differential equation (PIDE) which is solved by a numerical algorithm. Results are compared against diffusion-related LRP values that do not assume jumps by using calibrated parameters.

Retirees are often exposed to large and unpredictable medical expenses due to health shocks. The second topic examines the effect of health shocks and mortality risk on the optimal medical insurance-consumption-allocation strategy. We also derived a solution for the optimal retirement-triggering wealth in a life-cycle framework. As in the first problem, we investigated model changes, for asset return rates which obey a jump-diffusion dynamics.

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1 Introduction

1.1 Introduction and Literature Review

1.1.1 Lifetime Ruin Probability

Over the last decade or so, a number of papers in the actuarial, insurance, and financial literature have dealt with the so-called lifetime ruin probability (LRP). The LRP value can be loosely described as the probability that an investment portfolio subjected to exogenous withdrawals is depleted while the individual making the withdrawals is still alive. Note that the value of one minus the LRP (which is also between zero and one) is often referred to as the retirement sustainability quotient (RSQ) by North American practitioners in the field of retirement income planning. Algorithms, formulas and inequalities for ruin and survival probabilities have a lengthy history in the insurance and actuarial literature, going back more than a century to Lundberg in the 1900s and Cramer in the 1930s. Shaked (1985) gives an early example of ruin probability being directly calculated to evaluate financial viability of life insurers when assuming lognormal asset returns. Ruin probability is also referred to as a risk measure used in regulating capital in banking and insurance by Sherris (2006) or applied as a predetermined value in developing liquidation strategies for insurance companies by Berry-Stolzle (2008). More recently, ruin probability was addressed in the context of defined-contribution pension plans where Emms and Haberman 2008 solve an optimal allocation problem by minimizing the pensioner's expected loss over a target time horizon. Within the much narrower context of retirement income planning, the earliest scholarly paper on this topic was by Milevsky and Robinson (2000). They formulated the canonical retirement income problem in the following way: Assume that an individual retires with $1,000,000 in liquid investable wealth and they want to consume $100,000 every year for the rest of their life. They have no exogenous pension income. Let us further assume that the initial wealth is allocated to a portfolio experiencing stochastic returns and modeled in continuous time. What is the probability that the portfolio will reach zero before the individual dies? Stated differently: What is the probability that the remaining lifetime (random variable) of the portfolio, $\mathbf{T}_w$, is less than the remaining lifetime (random variable) of the retiree, $\mathbf{T}_x$? Since the original paper by Milevsky and Robinson (2000), which formulated the problem using moment matching techniques and a subsequent paper by Huang, Milevsky, and Wang (2004) which used more refined partial differential equation(PDE)-based techniques, many related papers have been published in the literature. The work of Huang, Milevsky, and Wang (2004) was essential in developing the mathematical representation of the problem for the ruin probability quantities. The jump-diffusion model details were developed through the theory of Hanson (2007), Kou (2002), and Ramezani and Zeng (1998, 2006). Once the theory was developed, numerical methods were used to solve the problem. A new moment matching technique was developed as well and the model was tested through the artificially generated return data using Monte Carlo (MC) simulations.

A number of follow-up articles have re-calibrated, refined and extended the usage of the LRP. For example, Albrecht and Maurer (2002) employed Monte Carlo simulation techniques to compute the probability that a systematic withdrawal plan (SWiP) can beat the life annuity benchmark. Gerrard, Haberman, Hojgaard, et al. (2004) examined various SWiP strategies to determine which ones have minimized a given loss function, while Orszag (2002) provided an analytic result related to deterministic plans. Another group of related research initiated by Young (2004), and then followed up by Milevsky, Moore, and Young (2005) derived strategies that would minimize the LRP using techniques from dynamic programming and solving the associated Hamilton-Jacobi-Bellman equation. The optimal dynamic ruin-minimizing strategy was then refined in a series of papers by Bayraktar and Young (2007), Wang and Young (2010) as well as Bayraktar, Hu, and Young (2011). More recently co-monotonicity techniques have been used very successfully by Weert, Dhaene, and Goovaerts (2011) to compute the LRP. All in all, the LRP appears to be an interesting and popular area of research. In fact, the LRP and related dynamic strategies that minimize this risk metric are of interest not only to scholars in the insurance and actuarial community, but the LRP is becoming increasingly important from a practitioner and policy perspective as well. Today, many consumers are formulating their retirement income (drawdown scheme) plans in light of a LRP calculation. However, most of the papers written on the LRP have assumed a rather primitive model for investment returns, namely geometric Brownian motion (GBM). And yet, in the context of asset pricing, a number of proposed alternatives to the GBM model have been listed in a survey by Kou (2007). A few of these studies have discussed the possibility of using stochastic volatility and/or stochastic interest rates (Duffie 2001; Heston 1993; Lewis 2000). The pre-eminent concern with these continuous models is that they don't capture discontinuities in the return-generating process. Indeed, after the experience in financial markets during the 2007 to 2009 period, one should be quite wary of models that do not allow for these sudden breaks, otherwise known as jumps. Of course, the crash of 1987 and the crises of 2008 did not introduce the world to jumps; which are not new to financial economics. MERTON (1976a, 1976b) was the first to introduce jump-diffusion models into the option pricing and asset allocation literature. Since his foundational work, hundreds, if not thousands, of econometric and statistical papers have been written on this topic. The objective in this dissertation is to examine how the impact of jumps in the portfolio-generating process might impact (or not) the lifetime ruin probability. There are two aspects to this question. First, the author must actually derive an expression for the probability and describe an efficient and robust solution algorithm methodology. Then, results must be compared against diffusion-related values that do not assume any jumps. Both are done in order to provide a concrete answer to the question: Will jumps increase the lifetime ruin probability? As mentioned earlier, MERTON (1976b) tried to answer the question of the impact of jumps on option values. He arrived at a volatility estimate and an option pricing formula that could be compared with the classic Black-Scholes formula derived from the diffusion-only stock price process dynamics. The magnitude of the percent error was used as a measure of comparison. The estimated variance of the logarithmic return on the stock as well as the frequency of jumps and the observation time of the time series data, all played a role in modifying the percentage error. However, the general pattern of this difference was consistent even when the model was tested with different parameters. Specifically, prices for deep in-or-out of money options were higher for the jump-diffusion versus the diffusion case but switched direction for a range close to at-the-money. While this suggests that the option pricing literature is well aware that jumps may not necessarily increase the value of an option relative to increasing volatility, it is not clear whether this is the case for the lifetime ruin probability. In this dissertation, evidence is provided that, consistent with the option pricing literature, jumps can also reduce the LRP. At first glance one would expect the LRP to be lowered, as jumps effectively increase the dispesrsion and uncertainty of a stochastic process. Jumps make hedging more difficult if not impossible, hence the general “fear” of jumps. But recall that the LRP is not a derivative price or a hedging value, it is an estimate of a long-term probability. Jumps can obviously work in both directions, possibly offsetting each other. Moreover, there are other (equivalent) ways to increase dispersion within a continuous diffusion process, namely by increasing the volatility. Either way, intuition is insufficient: ours is an empirical question that must be addressed with caution. Of course, we want to ensure we are comparing apples to apples, which is why we choose to use parameter values that lead to the same moments of the return generating distribution. Once we actually have an expression for the LRP, we employ historical data on (international) equity returns to calibrate two distinct models. The first is a standard GBM and the second is a jump-diffusion model. We locate the best-fitting parameters to these two models, and compare their respective LRP values. We find that except for very low initial wealth levels, the LRP values of the jump-diffusion model with parameters derived by moment matching techniques are actually lower, and we explain why in our work.

1.1.2 Optimal consumption-insurance-allocation portfolio problem

Many individuals plan for retirement and, in particular, are interested in strategies that result in an optimal portfolio, given the unpredictable nature of their investments or exogenous health risk. There is an increased demand for examining interactions among optimal retirement portfolio choice and savings, as defined-contribution pension plans are growing in popularity. The field of optimal portfolio problems in continuous time has been initiated by the seminal work of Merton (1969, 1971) and Samuelson and Merton (1969). More than a decade later, this area of study was further advanced by Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987), and Pliska (1986). Stochastic optimal control theory is the preferred method used in determining the optimal asset allocation and consumption strategies for this problem. This literature's common setting is that of an investor with a deterministic initial wealth level, which he must then optimally invest in either a complete or incomplete market in order to maximize the expected utility of his wealth and/or consumption up to a predetermined time. Moreover, most of this literature focuses on a market with assets with continuous sample paths, which follow a Brownian motion (i.e. diffusions). This treatment lacks the sudden jumps in value of real financial instruments. As mentioned in the first topic studied in this dissertation, Merton (1976a, 1976b, 1990) has noted this deficiency and applied discontinuous sample path Poisson processes in combination with Brownian motion processes (i.e. jump-diffusions) in order to improve the problem of pricing options. Optimal portfolio problems which consider the jump-diffusion case, have been studied more recently by Rishel (1999), who proposed a theoretical model dependent on both scheduled (deterministic) and unscheduled (stochastic) jump events. Hanson and Westman (2001) built on this model by computationally solving an optimal portfolio and consumption problem with scheduled (quasi-deterministic) jumps. In subsequent work, Hanson and Westman (2002) revisited the problem, this time around not using quasi-deterministic processes and applying a log-normal jump-amplitude distribution.

Motivated by the fact that individuals faced with more complex financial choices need to make progressively harder personal finance decisions when it comes to their retirement planning, we developed an optimal portfolio model under the framework of life-cycle models. In addition to the usual investment risk and life expectancy uncertainty, a retiree is often faced with health declines, which add the risk of health costs to their portfolio decisions. We assume that the option of buying medical insurance is available to the retiree and that he aims to use that purchase to cover unforeseen out-of-pocket health care costs. Moreover, as the agent also has the opportunity to invest in a risky asset (stock market), we extend our optimal investment problem to the jump-diffusion setting to investigate its effect.

The problem of the impact of health costs on optimal annuity demand and optimal consumption/savings decisions was previously studied by Peijnenburg, Nijman, and Werker (2011a, 2011b, 2011c) and Sinclair and Smetters (2004). Turra and Mitchell (2008) studied the impact of health status and out-of-pocket medical expenditures on annuity valuation. On the other hand, our problem investigates the demand for optimal health insurance that could cover the unforeseen health costs in retirement. Most of the literature in this direction focuses on variants of this problem but in a discrete setting, while our approach is time-continuous. Edwards (2008), in a discrete case, investigates the role of future health risk on financial risk-taking post-retirement. He concludes that health shocks prompt the investor to lower their exposure to financial risk by increasing the marginal utility of consumption. Yogo (2009) developed a life-cycle model in which the investor makes an optimal portfolio choice in retirement by considering stochastic health depreciation. Consumption, health costs and allocation of the financial capital between bonds, stocks and housing are determined through a discrete model setup. Pang and Warshawsky (2010) derive the optimal equity-bond-annuity asset portfolio for households in the retirement phase, who face stochastic capital market returns and have different exposures to mortality and uninsured health expense risks. The problem, although uses a life-cycle model, is developed in a discrete setting. French and Jones (2011) examine the importance of health insurance on retirement behavior through a discrete dynamic programming model that takes into account savings and the risk of future medical expenses.

First, we propose the development of our optimal portfolio model by considering health shocks and jump-diffusion asset return dynamics in the investor's post-retirement phase. The question we want to answer is whether, faced with these risks, a retiree should purchase health insurance protection and what the optimal amount should be. Moreover, we are interested in the optimal consumption and allocation between risky and risk-less assets. The wealth dynamics of the investor can then be divided into two parts, one describing the pre-health-shock dynamics while the other focuses on the post-health-shock process.

We further investigate how our optimal investment problem changes retirement planning by combining labour income (human capital) and financial capital over the life-cycle of the investor, this time considering both health insurance and sudden jumps in stock market returns. The problem without these considerations was investigated in the existing literature as follows. Bodie, Merton, and Samuelson (1992), Duffie, Fleming, Soner, et al. (1997), Dybvig and Liu (2005), Heaton and Lucas (2000), Koo (1998), and Viceira (2001) examined the optimal consumption and investment problem by taking human capital into consideration. Bodie, Merton, and Samuelson (1992) allow for the labour supply to be adjusted continuously and arrive at an optimal consumption, labour effort, and financial investment strategy over the investor's life cycle. As these authors note, the opportunity of continuously varying the labour income is not very realistic and therefore suggest that future research should look into the optimal portfolio problem with less flexible labour-leisure choices.

Another strand of literature, including Fischer (1973), Hakansson (1969), and Yaari (1965) has initiated the study of life insurance demand problems but without considering labour income dynamics and human capital in a life-cycle framework. More recently, Kraft, Schendel, and Steffensen (2014) investigated the optimal-insurance-investment portfolio choice problem in a life-cycle setting, when the breadwinner of a family unit faces the risk of health shocks and hence higher probability of death. In this context, the family tries to hedge this risk by purchasing life insurance. Revisions to the insurance policy can only be made while the insured breadwinner is in good health. The authors observe that, in the face of future income shocks and the subsequent need to lower insurance coverage, the demand for life-insurance protection decreases at all age levels.

In more recent portfolio choice literature, Huang, Milevsky, and Wang (2008) studied a family unit's life insurance and pension annuity demand, under constant relative risk aversion (CRRA) preferences, by considering the breadwinner's human capital. Moreover, in this life-cycle framework, they arrived at a strategy for optimal allocation between risk-free and risky assets both before and after retirement. This time-continuous problem was further extended by Huang and Milevsky (2008) to the more general case where families have more rational preferences specified through a hyperbolic absolute risk aversion (HARA) utility. Other authors like Bodie, Detemple, Otruba, et al. (2004), Chen, Ibbotson, Milevsky, et al. (2006), and Pliska and Ye (2007) developed more restrictive models for insurance demand and optimal retirement investment planning.

Farhi and Panageas (2007) extended the Bodie, Merton, and Samuelson (1992) model by considering limited flexibility in labour-leisure adjustments. In their paper, they develop an easily tractable theoretical model that addresses the question of optimal portfolio choice and optimal retirement time. The investor has a constant wage rate while still in the workforce. The labour-supply choice was modeled as an optimal stopping problem in which the agent can work for a fixed amount of time and earn a constant wage but also has the option of exiting the workforce permanently (retire) at any time he chooses. This recent research answers the questions of optimal consumption and savings in a continuous time set-up and arrives at the decision of when it is optimal to retire. The time to quit the workforce comes from a discrete jump in leisure once the agent retires. The major assumptions of the model are that the investor is not allowed to return to the workforce when retired and that the agent cannot choose retirement past a pre-specified retirement deadline. For developing a robust theoretical analysis, the authors also assume that the agent receives a constant wage income. Karatzas and Wang (2000) solved an optimal consumption problem without labor income but with discretionary stopping. Another optimal stopping problem which influenced the Farhi and Panageas (2007) model, was studied by Barone-Adesi and Whaley (1987). More recently, Huang, Milevsky, and Salisbury (2014) have developed an optimal control model in order to arrive at an optimal initiation region in an American option framework, in the context of variable annuity policies with GLWB. This approach is similar to the one we have taken in this dissertation.

We add a new model to previous literature by deriving an optimal portfolio problem which results in a solution for an optimal retirement-triggering wealth level. A working investor would find this information valuable and a guide on when to exit the workforce. We propose combining the previously developed post-retirement investment strategy with a pre-retirement phase. Our

PDE approach is derived in a life-cycle framework with infinite time horizon. Of interest to us is to study how these solutions are modified in our case, when we choose a more realistic set-up involving both health and mortality risk. We also propose an extension of our model for the case of time-dependent force of mortality. The implementation of that case is left for future work on the model.

1.2 Purpose of Dissertation and Outline

The proposed dissertation topics are applied to the personal finance management of individuals, either before or after retirement. In the first topic, we develop a lifetime ruin probability (LRP) model, by assuming that a jump-diffusion process drives the investment return of the agent. The value of the LRP is important to an investor who wants to find out the probability of running out of money, while maintaining a desired standard of living for the rest of his life. The problem of the LRP has been studied before (as seen in the Literature Review), but our objective is to examine the impact that jumps in the portfolio-generating process have on this popular risk-measure. To our knowledge, our extension to the LRP theory and its modeling has not been studied before. Its development adds value to investors as well as to practitioners and policy makers. The first topic is discussed in Chapter 2.

The second topic is discussed in Chapter 3 and it focuses on the effect of health shocks (modeled as jumps with known distribution) on optimal health insurance demand and consumption decisions. Retirees are generally exposed to large and unpredictable medical expenses. They are left with the dilemma of whether to purchase protection and what the optimal amount should be in order to maximize their investments. To our knowledge, there is a deficiency in literature on the effect of health shocks and the health insurance demand on the optimal portfolio problem. This problem is generally studied in a discrete time framework and our approach tries to fill this gap by developing a model of investment in continuous time. It can be solved numerically but we also use an explicit solution to assess the goodness of the numerical scheme for a case where the jump in health is uniformly distributed. Moreover, in this chapter we also answer the question of optimal health insurance-consumption-allocation when the risky asset returns follow jump-diffusion dynamics. This part ties in with the numerical and calibration methods developed in Chapter 2.

Once this optimal portfolio model was developed in a post-retirement setting, we applied the same techniques to the pre-retirement period. This extension will not only help the agent in making optimal decisions throughout his life, but will also predict an optimal wealth level required for entering retirement. This optimal problem is discussed in Chapter 4.

2 Lifetime Ruin Probability Problem for Retired Individuals

2.1 Introduction

We derive an expression for the lifetime ruin probability (LRP) — defined as the probability a fixed spending plan will deplete an investment portfolio prior to a retiree's time of death — but assuming the investment return driving the portfolio obeys a jump-diffusion process. Most of the previous literature assumes a continuous diffusion process for underlying returns, and thus ignores crashes and discontinuities. Our model leads to a partial-integro-differential equation (PIDE) for the LRP and some related probabilities. We then compare our PIDE solution to probabilities, which have been derived under a geometric Brownian motion (GBM) process using moment matching techniques, and have been calibrated to historical equity returns. In addition to the expression for the lifetime ruin probability (LRP), our main result is that despite the naïve intuition that crashes (i.e. jumps) increase the LRP and reduce retirement sustainability, in the context of retirement spending this is not necessarily the case. Under normal circumstances, our LRP values are lower when moment-matched to historical returns. This result is reminiscent of the option pricing literature in which jumps do not necessarily increase option values. On the other hand, under very low initial wealth values — when ruin is quite likely — crashes are more likely to ruin retirement. We will provide some further intuition in the body of this document. Overall, we believe this work should be of interest to scholars as well as practitioners who are concerned with sustainable income strategies for retirees.

2.2 Model Setup and Assumptions

The wealth process of our problem is governed by the following stochastic differential equation (SDE):

$$\begin{cases} dW_t = \underbrace{(\mu W_t - 1)dt}_{\text{drift}} + \underbrace{\sigma W_t dB_t}_{\text{diffusion}} + \underbrace{h(t, W_t, q)dP_t}_{\text{jump}} \\ \qquad = (\mu W_t - 1)dt + \sigma W_t dB_t + \int_Q h(t, W_t, q)\mathscr{P}(dt, dq), \\ W_0 = w, \end{cases} \tag{2.2.1}$$

where μ and σ are the constant drift and diffusion coefficient, Q is the Poisson mark space, dB_t is the Brownian motion driving the process, dP_t is the Poisson process driving the jump, h is the total change in wealth due to the jump and $\mathscr{P}(dt, dq)$ is the Poisson mark-time random measure. The problem was scaled by assuming a constant consumption rate of 1 with the initial value of wealth in our model being the ratio of wealth to consumption in practice.

For convenience, let the LRP and RP be denoted by P_L and P_R respectively. These two quantities of interest are formally defined as follows:

$$P_R(t, w; T, y) := \Pr\left[\inf_{t\le s\le T} W_s \le y | W_t = w\right], \tag{2.2.2}$$

$$P_L(t, w; y) := \Pr\left[\inf_{t\le s\le \mathbf{T}_x} W_s \le y | W_t = w\right]. \tag{2.2.3}$$

Throughout this dissertation, the above notations will be interchanged with a more convenient short-hand notation, $P(t, w)$, for both probabilities when this practice will not result in confusion. The random variable $\mathbf{T}_w$ can be introduced, which will capture the amount of time it takes the net wealth of the retiree to attain a wealth level of y. The remaining lifetime of the individual is denoted by the random variable $\mathbf{T}_x$ and allowed to have a Gompertz-Makeham distribution:

$$\hat{\lambda}_{x+t} = \theta + \frac{1}{b} \exp\left(\frac{x+t-m}{b}\right), \tag{2.2.4}$$

where x is the age of the individual, m is the mode of the future lifetime and b is a scale parameter of $\mathbf{T}_x$.

In this case, we think of the LRP as the probability that the net wealth amount reaches zero before the remaining lifetime of the retiree:

$$P_L(t, w; y) = Pr[\mathbf{T}_w \leq \mathbf{T}_x]. \tag{2.2.5}$$

On the other hand, we think of the RP as the probability that the net wealth reaches zero before a deterministic time T:

$$P_R(t, w; T, y) = Pr[\mathbf{T}_w \leq T]. \tag{2.2.6}$$

In an original paper which focused on the GBM model, Huang, Milevsky, and Wang (2004) have transformed the problem into an exercise in probability convolutions; one that computes the cumulative density function of the new random variable $\mathbf{T}_w - \mathbf{T}_x$.

In this case, we used the technique outlined in Hanson (2007) to arrive at the following PIDE for P_R. By Itô's lemma and applying expectations, we obtain:

$$\frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial w} + \frac{\sigma^2 w^2}{2}\frac{\partial^2 P}{\partial w^2} + \lambda \int_Q (P(t, w + h(t, w, q)) - P(t, w))\phi(q) dq = 0, \tag{2.2.7}$$

with terminal and boundary conditions

$$P(T, w; T, y) = 1 - H(w - y), \quad P(t, y; T, y) = 1, \quad P(t, \infty; T, y) = 0. \tag{2.2.8}$$

Here P represents P_R, H is the Heaviside function, λ is the jump rate and ϕ is the density function of the jump-marks.

By re-scaling the problem, we arrive at the following relationship:

$$P_L(t, w; y) = \frac{1}{1 - F_x(t)} \int_t^{\infty} P_R(t, w, \tau, y) f_x(\tau) d\tau. \tag{2.2.9}$$

Here $f_x(t)$ is the probability density function, while $F_x(t)$ is the cumulative distribution function of the $\mathbf{T}_x$. Algebraic manipulations lead to the following backward PIDE for the LRP:

$$\frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial w} + \frac{\sigma^2 w^2}{2}\frac{\partial^2 P}{\partial w^2} + \lambda \int_Q (P(t, w + h(t, w, q)) - P(t, w))\phi(q) dq = \hat{\lambda}_{x+t} P, \tag{2.2.10}$$

with terminal and boundary conditions

$$P(\infty, w_\infty; \infty, y) = 1 - H(w_\infty - y), \quad P(t, y; T, y) = 1, \quad P(t, \infty; T, y) = 0. \tag{2.2.11}$$

In the numerical computation of this quantity we truncate the infinite domain and apply equation (2.2.11) for a sufficiently large T. For both ruin probability quantities, we set $y = 0$ which means that ruin happens when the wealth is depleted.

2.2.1 The Jump-Diffusion Model

Merton (1976a, 1976b) was one of the first to try to capture realistic features of the log-stock price distribution by introducing a compound Poisson jump process. Merton's jump-diffusion model assumes that the log-stock price jump-amplitudes follow a normal distribution. Since it was proposed, many authors (Duffie 2001; Hanson and Westman 2002; Merton 1990) focused on this jump-diffusion model where log-jump-amplitudes are normally distributed (NJD). In recent years the double-exponential jump-diffusion (DEJD) model proposed by Kou (2002) has gained popularity. Some desirable properties of this model include leptokurtic and asymmetric implied returns, and the fit of implied volatility smiles. Furthermore, for certain exotic and path-dependent options, the DEJD provides analytical tractability (Kou and Wang 2004). A recent assessment of performance (Kou 2007; Ramezani and Zeng 2006) of the DEJD relative to the NJD and the GBM suggests that it matches key features of index returns better than these alternatives. This motivates our use of the DEJD model in this current problem.

Ramezani and Zeng (1998) proposed a model where a distinction is made between "good" (upward jump) and "bad" (downward jump) events which are generated by two independent Poisson processes. In their model, the jump-amplitudes are a mixture of Pareto-Beta distributions. The DEJD model proposed by Kou (2002) has only one Poisson process of fixed intensity where the jump-magnitudes are drawn from two independent exponential distributions. As shown below, the Pareto-Beta and the DEJD model are equivalent due to the fact that the parameters of one can be retrieved from the other. For a better control of the "up" and "down" jump intensities, we choose to represent the Monte Carlo simulated data by the Pareto-Beta jump-diffusion formulation (refer to the numerical results section).

Let Y_t be the *absolute jump-amplitude* such that in a time increment dt the wealth process jumps from W_t to Y_tW_t. If we assume that "good" news (upward jump in wealth) and "bad" news (downward jump in wealth) are the result of two independent Poisson processes, we draw jump-amplitudes from the Pareto and Beta distributions. We take the up-jump-amplitude (Y_t^u)

to be Pareto distributed with a probability density function:

$$f_{Y^u}(y) = \frac{\eta_1}{y^{\eta_1+1}} \quad \text{where} \quad Y^u \geq 1, \tag{2.2.12}$$

and the corresponding first two moments

$$E(Y^u) = \frac{\eta_1}{\eta_1 - 1}, \tag{2.2.13}$$

$$V(Y^u) = \frac{\eta_1}{(\eta_1 - 2)(\eta_1 - 1)^2}. \tag{2.2.14}$$

Similarly, the down-jump-amplitudes Y_t^d, are taken to be Beta distributed with the associated density function:

$$f_{Y^d}(y) = \eta_2 y^{\eta_2 - 1} \quad \text{where} \quad 0 < Y^d < 1 \tag{2.2.15}$$

and moments

$$E(Y^d) = \frac{\eta_2}{\eta_2 + 1}, \tag{2.2.16}$$

$$V(Y^d) = \frac{\eta_2}{(\eta_2 + 2)(\eta_2 + 1)^2}. \tag{2.2.17}$$

Then, the mixture of the Pareto-Beta distributions has the density function:

$$f_Y(y) = \frac{u\eta_1}{y^{\eta_1+1}} \mathbf{1}_{y>1} + (1-u)\eta_2 y^{\eta_2-1} \mathbf{1}_{0<y<1}, \tag{2.2.18}$$

where $\eta_1 > 1$, $\eta_2 > 0$ and $\lambda = \lambda_u + \lambda_d$ is the jump intensity as a combination of the up (λ_u)- and down (λ_d) intensities. The probabilities of an up and down jump are $u = \frac{\lambda_u}{\lambda}$ and $d = (1-u)$ respectively.

Then we know (Ramezani and Zeng 2006) that the *log-jump-amplitude* (jump-mark process) $Q = \ln(Y)$, is double-exponentially (Laplace) distributed:

$$Q \overset{\text{i.i.d.}}{\sim} \text{DoubleExp}\left(\frac{u}{\eta_1} - \frac{1-u}{\eta_2}, u(1-u)\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right)^2 + \left(\frac{u}{\eta_1^2} + \frac{1-u}{\eta_2^2}\right)\right), \tag{2.2.19}$$

with a density function

$$\phi(q) = u\eta_1 e^{-\eta_1 q}\mathbf{1}_{q\geq 0} + (1-u)\eta_2 e^{\eta_2 q}\mathbf{1}_{q<0}, \tag{2.2.20}$$

where $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$ represent the expected values of an up-jump-mark and a down-jump-mark respectively. This is equivalent to:

$$Q = \begin{cases} Q^u = \ln Y^u \sim \exp(\eta_1) & \text{with probability } u, \\ -Q^d = -\ln Y^d \sim \exp(\eta_2) & \text{with probability } d. \end{cases} \tag{2.2.21}$$

2.3 Numerical Results

This section describes the implementation process of the jump-diffusion model. It presents the numerical scheme which was used, the moment-matching and calibration technique, as well as a test that random return data generated synthetically results in a similarly observed pattern for the ruin probabilities.

2.3.1 Numerical Scheme

In general, obtaining closed form solutions to problems (2.2.7) and (2.2.10) is impossible (Barles, Buckdahn, and Pardoux 1997; Duffy 2004; Hanson 2007). Hence, we must approximate the solution using numerical techniques. In this document, we solve these second order PIDEs by applying the Crank-Nicolson scheme for the diffusion term and an explicit scheme for the integral term. Stability criteria for this scheme have been developed by Cont and Voltchkova (2005). For the first order derivative P_w[1], we use an upwind scheme. The discretization of (2.2.10) becomes:

$$\frac{P_j^{n+1} - P_j^n}{-\Delta t} = \frac{(\mu w_j - 1)}{2}\left(\frac{P_{j^*}^{n+1} - P_{j^*-1}^{n+1}}{\Delta w} + \frac{P_{j^*}^{n} - P_{j^*-1}^{n}}{\Delta w}\right)$$
$$+ \frac{\sigma^2 w_j^2}{4}\left(\frac{P_{j+1}^{n+1} - 2P_j^{n+1} + P_{j-1}^{n+1}}{(\Delta w)^2} + \frac{P_{j+1}^{n} - 2P_j^{n} + P_{j-1}^{n}}{(\Delta w)^2}\right)$$
$$+ \lambda(\mathscr{I}_{JD})_j^{n+1} - \hat{\lambda}_{x+t_n}\left(\frac{P_j^{n+1} + P_j^n}{2}\right), \quad (2.3.1)$$

where $j^* = j$ if the coefficient is negative and $j^* = j + 1$ if it is positive. Here the set $\{P_j^n, P_{j-1}^n, P_{j+1}^n\}$ contains the unknown quantities, and the indices $j = 1 : N_w$ and $n = 1 : N_t$ represent wealth and time respectively. The terminal condition is $P_j^{N_t} = 1 - H(w_j)$, while the boundary conditions are $P_0^n = 1$ and $P_{N_w}^n = 0$. For equation (2.2.7) we apply the same method. To approximate the integrals derived as a result of the Poisson noise in equations (2.2.7) and (2.2.10), we let the post-jump amplitude be given by:

$$a(t_n, w_j, q_i) = w_j + (e^{q_i} - 1)w_j \equiv e^{q_i} w_j = w_{j+l_i} + \varepsilon_i \Delta w \quad (2.3.2)$$

where the floor integer is

$$l_i = \left\lfloor \frac{(e^{q_i} - 1)w_j}{\Delta w} \right\rfloor, \quad (2.3.3)$$

and

$$\varepsilon_i = \frac{(e^{q_i} - 1)w_j}{\Delta w} - l_i. \quad (2.3.4)$$

The post-jump integral term can be approximated by:

$$\int_Q P(t_n, e^{q_i} w_j)\phi(q_i) \approx \sum_{i=1}^{N_q} \beta_i P(t_n, e^{q_i} w_j), \quad (2.3.5)$$

with N_q points q_i and N_q weights β_i. This Gauss-statistics rule for the jump-integral is expected to have a polynomial precision of $n_q = N_q - 1$. This approximation describes the global dependence

1. For convenience, we also use subscripts to denote the derivatives when this causes no confusion.

of the jump integral term due to the Poisson noise, unlike the local second order partial derivatives due to the Gaussian noise. We can also approximate the integral in a more simple way by using a Riemann sum as shown in Appendix 2.B.

If we now consider $O^2(\Delta w)$ interpolation of the term $P(t_n, e^{q_i} w_j)$ we can write:

$$P(t_n, e^{q_i} w_j) \equiv P(t_n, w_{j+l_i}) + \left(\frac{e^{q_i} w_j - w_{j+l_i}}{w_{j+1+l_i} - w_{j+l_i}}) \right) (P(t_n, w_{j+1+l_i}) - P(t_n, w_{j+l_i})). \tag{2.3.6}$$

This can be further reduced to:

$$P(t_n, e^{q_i} w_j) \approx (1 - \varepsilon_i) P(t_n, w_{j+l_i}) + \varepsilon_i P(t_n, w_{j+1+l_i}) \equiv (1 - \varepsilon_i) P^n_{j+l_i} + \varepsilon_i P^n_{j+1+l_i}, \tag{2.3.7}$$

for $q_i > 0$. Similarly, for $q_i < 0$:

$$P(t_n, e^{q_i} w_j) \approx (1 + \varepsilon_i) P^n_{j-l_i} - \varepsilon_i P^n_{j-1-l_i}. \tag{2.3.8}$$

Now, if we assume that most contributions to the integral approximation are from values of $q_i \in [a, b]$, where a is a small negative value and b is a small positive value, we see that the global dependence on the jump-integral reduces to a local dependence. We implement the case where $l_i \approx 0$ which also corresponds to $\varepsilon_i \equiv \varepsilon = |\frac{(e^q - 1) w_j}{\Delta w}| \leq 1$. To satisfy this condition for the entire state domain, we restrict the jump-marks to a finite domain:

$$\ln(1 - \frac{1}{N_w}) \leq q \leq \ln(1 + \frac{1}{N_w}). \tag{2.3.9}$$

In other words, we are not required to employ the Gauss-quadrature statistics since most jump-mark contributions are from values close to zero.

As shown by Hanson (2007), for the case of smal jump-marks, we can approximate the jump-term by using piece-wise linear interpolation:

$$(\mathscr{I})_j^{n+1} = \int_0^b [(1-\varepsilon)P_j^{n+1} + \varepsilon P_{j+1}^{n+1}]\phi dq + \int_a^0 [(-\varepsilon)P_{j-1}^{n+1} + (1+\varepsilon)P_j^{n+1}]\phi dq - \int_a^b P_j^{n+1}\phi dq, \quad (2.3.10)$$

where $\varepsilon \leq 1$.

Using formula (2.2.20), the DEJD jump-term integral approximation becomes:

$$(\mathscr{I}_{JD})_j^{n+1} = P_{j-1}^{n+1}\frac{w_j(u-1)\eta_2}{\Delta w(1+\eta_2)}\{1-e^{a(1+\eta_2)}\} + P_j^{n+1}\frac{w_j}{\Delta w}\left\{\frac{(1-u)\eta_2}{1+\eta_2}(1-e^{a(1+\eta_2)}) - \frac{u\eta_1}{1-\eta_1}(e^{b(1-\eta_1)}-1)\right\} + P_{j+1}^{n+1}\frac{w_j u\eta_1}{\Delta w(1-\eta_1)}\{(e^{b(1-\eta_1)}-1)\} \quad (2.3.11)$$

With the restriction (2.3.9) on q, we can use the infinite domain to approximate the integral and write:

$$(\mathscr{I}_{JD})_j^{n+1} = P_{j-1}^{n+1}\left[\int_{-\infty}^0 \varepsilon(u-1)\eta_2 e^{\eta_2 q}dq\right] - P_j^{n+1}\left[\int_0^\infty \varepsilon u\eta_1 e^{-\eta_1 q}dq - \int_{-\infty}^0 \varepsilon(1-u)\eta_2 e^{\eta_2 q}dq\right] + P_{j+1}^{n+1}\left[\int_0^\infty \varepsilon u\eta_1 e^{-\eta_1 q}dq\right]. \quad (2.3.12)$$

In fact, assuming that jumps are centered around small values of the jump-marks results in solving the original PIDE as a PDE. In Appendix 2.B we present a discussion on the difference between the above local approximation and the global approximation over the infinite domain of the jump-marks. The difference being small, we will use the integral approximation given by equation (2.3.12) for all the numerical results. Moreover, as noted in Hanson (2007), for the financial markets, the jump-marks are generally small, which supports our decision to implement the local approximation for the integral in order to maintain a realistic viewpoint.

2.3.2 Calibration of the Jump-Diffusion and Diffusion Process

In order to compare the models effectively, we employ moment matching techniques. This section analyzes the results of our numerical computations.

2.3.2.1 Moments for the Jump-Diffusion Model

Let

$$\begin{cases} dX_s = \mu X_s ds + \sigma X_s dB_s + h(s, X_s, q) dP_s, \\ \quad X_t = x, \end{cases} \tag{2.3.13}$$

and let $\varphi(X_s)$ be any function of X_s. By Itô's lemma we have:

$$\varphi(X_T) = \varphi(x) + \int_t^T L\varphi(X_s) ds + \int_t^T \sigma X_s \varphi'(X_s) dB_s + \int_t^T [\varphi](s, X_s) dP_s, \tag{2.3.14}$$

where $L\varphi(x) = \mu x \varphi'(x) + \frac{1}{2}\sigma^2 x^2 \varphi''(x)$. Taking expectations (Hanson 2007) of the above equation we obtain:

$$E\{\varphi(X_T)\} = \varphi(x) + \int_t^T E\{L\varphi(X_s)\} ds + \int_t^T E\{[\varphi]\}\lambda ds. \tag{2.3.15}$$

Let $p(t, x; T, y)$ be the transitional probability density function. From definition, we have:

$$E[\varphi(X_T)] = \int_0^\infty \varphi(y) p(t, x; T, y) dy. \tag{2.3.16}$$

Thus, we write:

$$\int_0^\infty \varphi(y) p(t, x; T, y) dy = \varphi(x) + \int_t^T \int_0^\infty (L\varphi(y) + E\{[\varphi]\}\lambda) p(t, x; s, y) dy ds. \tag{2.3.17}$$

Taking time derivatives of equation (2.3.17) with respect to T, we have:

$$\begin{aligned}
\int_0^\infty \varphi(y)\frac{\partial p}{\partial T}dy &= \int_0^\infty \{L\varphi(y) + \lambda E(\varphi(y+h(T,y,q)) - \varphi(y))\}pdy \\
&= \int_0^\infty \{\mu y\varphi'(y) + \frac{1}{2}\sigma^2 y^2\varphi''(y) + \lambda E(\varphi(y+h)dq - \varphi(y))\}pdy \\
&= \int_0^\infty \varphi(y)\left\{\left(\frac{\sigma^2 y^2 p}{2}\right)_{yy} - (\mu yp)_y\right\}dy + \lambda\int_0^\infty pE(\varphi(y+h)-\varphi(y))dy \\
&\quad + \mu y\varphi p|_{y=0}^{y=\infty} + \frac{\sigma^2 y^2}{2}\varphi' p|_{y=0}^{y=\infty} - \left(\frac{\sigma^2 y^2 p}{2}\right)_y \varphi'|_{y=0}^{y=\infty} \\
&= \int_0^\infty \varphi(y)\left\{\left(\frac{\sigma^2 y^2 p}{2}\right)_{yy} - (\mu yp)_y\right\}dy + \lambda\int_0^\infty pE(\varphi(y+h)-\varphi(y))dy,
\end{aligned} \tag{2.3.18}$$

as the terms

$$\mu y\varphi p|_{y=0}^{y=\infty}, \quad \frac{\sigma^2 y^2}{2}\varphi' p|_{y=0}^{y=\infty}, \quad \left(\frac{\sigma^2 y^2 p}{2}\right)_y \varphi'|_{y=0}^{y=\infty},$$

will vanish.

Consider a special case of the jump process where $h(s, X_s, q) = (\nu(q) - 1)X_s$ is the relative jump-amplitude and $\ln \nu(q)$ is the jump-mark process which is double-exponentially distributed. Thus $\varphi(y + h(s, y, q)) = \varphi(\nu y)$ and we can simplify:

$$\begin{aligned}
\int_0^\infty pE(\varphi(y+h)-\varphi(y))dy &= \int_0^\infty pE(\varphi(\nu y)-\varphi(y))dy \\
&= \int_0^\infty p\left(\int_{-\infty}^\infty \varphi(e^q y)\phi(q)dq\right)dy - \int_0^\infty p\varphi(y)dy \\
&= \int_{-\infty}^\infty \left(\int_0^\infty p(\tilde{y}e^{-q})\varphi(\tilde{y})d\tilde{y}\right)e^{-q}\phi(q)dq - \int_0^\infty p\varphi(y)dy \\
&= \int_0^\infty \varphi(y)\int_{-\infty}^\infty p(ye^{-q})e^{-q}\phi(q)dqdy - \int_0^\infty p\varphi(y)dy.
\end{aligned} \tag{2.3.19}$$

Here ϕ is the probability density function for the double-exponentially distributed jump-marks, given by equation (2.2.20). Then for any φ we obtain:

$$\int_0^\infty \varphi \frac{\partial p}{\partial T} dy = \int_0^\infty \varphi(y) \left\{ \left(\frac{\sigma^2 y^2 p}{2} \right)_{yy} - (\mu y p)_y + \lambda \left(\int_{-\infty}^\infty e^{-q} \phi(q) p(y e^{-q}) dq - p \right) \right\} dy. \tag{2.3.20}$$

This results in the following forward PIDE:

$$\frac{\partial p}{\partial T} = \left(\frac{\sigma^2 y^2 p}{2} \right)_{yy} - (\mu y p)_y + \lambda \left(\int_{-\infty}^\infty e^{-q} \phi(q) p(y e^{-q}) dq - p \right). \tag{2.3.21}$$

After expansion we have:

$$\frac{\partial p}{\partial T} + (\mu + \lambda - \sigma^2) p + (\mu - 2\sigma^2) y \frac{\partial p}{\partial y} - \frac{\sigma^2 y^2}{2} \frac{\partial^2 p}{\partial y^2} - \lambda \int_{-\infty}^\infty e^{-q} \phi(q) p(y e^{-q}) dq = 0. \tag{2.3.22}$$

Note that:

$$\begin{aligned} \int_0^\infty \left\{ \int_{-\infty}^\infty e^{-q} \phi(q) p(e^{-q} y) dq \right\} y^n dy &= \int_{-\infty}^\infty e^{-q} \phi(q) \int_0^\infty p(e^{-q} y) y^n dy dq \\ &= \int_{-\infty}^\infty e^{nq} \phi(q) \int_0^\infty p(y) y^n dy dq \\ &= I_n \int_{-\infty}^\infty e^{nq} \phi(q) dq \\ &= \xi_n I_n. \end{aligned} \tag{2.3.23}$$

Using the relation:

$$\int_{-\infty}^\infty e^{-(ax^2 + bx + c)} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2 - 4ac}{4a}}, \tag{2.3.24}$$

we simplify ξ_n

$$\xi_n = \frac{u \eta_1}{\eta_1 - n} + \frac{(1-u)\eta_2}{\eta_2 + n}, \tag{2.3.25}$$

if $\eta_1 > n$.

By definition the n^{th} moment of the state variable is given by:

$$I_n(t, x; T) = \int_0^\infty p(t, x; T, y) y^n dy.$$

We multiply (2.3.22) by y^n and integrate:

$$\frac{\partial I_n}{\partial T} + (\mu + \lambda - \sigma^2 - \lambda\xi_n)I_n + (\mu - 2\sigma^2)\int_0^\infty y^{n+1}\frac{\partial p}{\partial y}dy - \frac{\sigma^2}{2}\int_0^\infty y^{n+2}\frac{\partial^2 p}{\partial y^2}dy = 0. \quad (2.3.26)$$

Note that through integration by parts:

$$\begin{aligned}\int_0^\infty y^{n+1}\frac{\partial p}{\partial y}dy &= y^{n+1}p|_0^\infty - \int_0^\infty (n+1)y^n p dy \\ &= -(n+1)I_n, \quad (2.3.27)\\ \int_0^\infty y^{n+2}\frac{\partial^2 p}{\partial y^2}dy &= y^{n+2}\frac{\partial p}{\partial y}|_0^\infty - (n+2)y^{n+1}p|_0^\infty + (n+2)(n+1)\int_0^\infty y^n p dy \\ &= (n+2)(n+1)I_n. \quad (2.3.28)\end{aligned}$$

Finally, we obtain the following ODE for I_n:

$$\begin{cases} \frac{dI_n}{dT} + [(\mu + \lambda - \sigma^2 - \lambda\xi_n) - (n+1)(\mu - 2\sigma^2) - \frac{(n+1)(n+2)\sigma^2}{2}]I_n = 0, \\ I_n(t, x; T) = \int_0^\infty y^n \delta(y - x)dy = x^n. \end{cases} \quad (2.3.29)$$

By solving (2.3.29) we find the n^{th} moment formula:

$$I_n = x^n e^{\int_t^T \gamma_n(s)ds}, \quad (2.3.30)$$

where

$$\gamma_n = \frac{(n+2)(n+1)\sigma^2}{2} + (n+1)(\mu - 2\sigma^2) + \sigma^2 - \mu + \lambda(\xi_n - 1). \quad (2.3.31)$$

Remark 2.3.1. *If $\lambda = 0$ the jump-diffusion model reduces to the pure diffusion model. In this case:*

$$I_n = x^n e^{\int_t^T \gamma_n(s)ds}$$

and

$$\gamma_n = \frac{(n+2)(n+1)\sigma^2}{2} + (n+1)(\mu - 2\sigma^2) + \sigma^2 - \mu.$$

Remark 2.3.2. *For simplicity we have considered only the case where λ is constant. A deterministic but time dependent λ can be handled in a similar way.*

2.3.2.2 Moment Matching

A distribution G is well represented by a distribution F, if their moments match. We estimate the parameters of both the pure diffusion and the DEJD model by calibrating them to the historical data.

Consider the diffusion SDE, linear in the state process X_s with constant coefficients:

$$\begin{cases} dX_s = \mu X_s ds + \sigma X_s dB_s, \\ X_t = x, \end{cases} \tag{2.3.32}$$

where μ is called the drift or deterministic coefficient and σ is called the volatility or standard deviation of the diffusion term. First, we estimate the parameters μ and σ of the diffusion model by matching the first two moments of the GB model to the historical moments. Our data are the monthly equity returns corresponding to S&P 500, MSCI EAFE, and S&P/TSX indices. We create a new data set by using a combination of indices with equal weighting:

$$R_i^M = \frac{1}{3} R_i^{\text{S\&P 500}} + \frac{1}{3} R_i^{\text{MSCI EAFE}} + \frac{1}{3} R_i^{\text{S\&P/TSX}}. \tag{2.3.33}$$

The first two moments of the data $X_i = \ln(1 + R_i^M)$ were matched to the mean and variance of the GBM:

$$E[X_i] = (\mu - \frac{1}{2}\sigma^2)\Delta t, \tag{2.3.34}$$

$$V[X_i] = \sigma^2 \Delta t. \tag{2.3.35}$$

Using $\Delta t = \frac{1}{12}$ we obtain the annualized estimated parameters μ and σ.

Next, we match the moments of the DEJD model to the moments obtained from the historical data (January 1970 to January 2003). The n^{th} empirical moment for observations $i = 1, 2, ..., N$

and data series $Y_i = 1 + R_i^M$ is given by:

$$J_n = E[(Y_i)^n] = \frac{1}{N} \sum_{i=1}^{N=396} (Y_i)^n. \tag{2.3.36}$$

By matching the theoretical moment formula (2.3.30) with the empirical moments (2.3.36) we obtain:

$$J_n = I_n \Leftrightarrow \frac{\log(E\left[(Y_i)^n\right] x^{-n})}{\Delta t} = \gamma_n, \tag{2.3.37}$$

where γ_n is as in equation (2.3.31). Let

$$M_n = \frac{\log(J_n x^{-n})}{\Delta t} = \frac{\log(E\left[(Y_i)^n\right] x^{-n})}{\Delta t}. \tag{2.3.38}$$

We want to obtain the six parameters (μ, σ, λ, η_1, η_2 and u) by matching the first six moments. To this end we solve the following system of nonlinear equations, using $x = 1$ and $\Delta t = \frac{1}{12}$:

$$\begin{cases} M_1 = \mu + \lambda(\xi_1 - 1), \\ M_2 = 2\mu + \sigma^2 + \lambda(\xi_2 - 1), \\ M_3 = 3\mu + 3\sigma^2 + \lambda(\xi_3 - 1), \\ M_4 = 4\mu + 6\sigma^2 + \lambda(\xi_4 - 1), \\ M_5 = 5\mu + 10\sigma^2 + \lambda(\xi_5 - 1), \\ M_6 = 6\mu + 15\sigma^2 + \lambda(\xi_6 - 1). \end{cases} \tag{2.3.39}$$

The system was solved by a MATLAB nonlinear solver *lsqnonlin* and the results were summarized in Table 2.1.

Remark 2.3.3. *We have also matched moments up to order five by using a system of five nonlinear equations (see Table 2.1). These estimated parameters were used to narrow the search domain of equation (2.3.39).*

Fixed Parameter	Estimated Parameters					
	μ	σ	λ	η_1	η_2	u
	DEJD Order 6					
	0.1174	0.1328	0.0091	6.9995	10.000	0.1059
	0.1199	0.1263	0.0091	6.9417	3.9968	0.2966
	0.1202	0.1226	0.0091	6.9423	3.0476	0.5105
	0.1214	0.1164	0.0091	6.9478	2.9597	0.7511
u	DEJD Order 5					
0.10	0.1163	0.1359	0.0090	6.5912	13.338	X
0.30	0.1166	0.1344	0.0091	6.6221	2.5464	X
0.50	0.1168	0.1315	0.0098	6.4191	1.8874	X
0.75	0.1185	0.1223	0.0109	6.4553	1.7782	X
	GBM					
	0.1176	0.1372	X	X	X	X

Table 2.1: Estimated parameter values by matching the moments with historic data from January 1970 to January 2003.

Remark 2.3.4. *We had to impose constraints on the parameters of interest in order for them to be representative of real data. For example all of them were chosen to be positive. Another example considers η_1 and η_2 of the DEJD model. We have to take into consideration the fact that $\eta_1, \eta_2 \to \infty$ reduces the jump-diffusion model to the pure diffusion model. In order to avoid such a case an upper bound was selected for these parameters.*

We considered a data series of monthly total returns calculated as a weighted sum of three indices: S&P 500, MSCI EAFE, and S&P/TSX. The observation period is January 1970 to January 2003.

For the results presented here, we have used the estimated parameters listed in Table 2.1 when moments were matched up to order six. Note that we have also included estimated parameters obtained by matching moments up to order five. Once we found a solution (μ, σ , λ, η_1, η_2), we could more easily find the parameters from the sixth order moment match by narrowing the solution search domain Ω for the six degrees of freedom system. For example, knowing that $\sigma = 0.1359$ was obtained for a fixed $u = 0.1$, we set a new search domain for σ — of the six degrees system — to $\Omega = (0.1, 0.4)$, thereby ensuring a faster convergence to the solution as seen by a reduced number of iterations. For all of our computations, we also assumed that the wealth level $y = 0$ (see equations (2.2.7) and (2.2.10)), $T = 35$ years, $m = 80$, $b = 10$, $\theta = 0$ and $x = 50$.

We observed that the ruin probabilities studied are lower in the jump-diffusion model (DEJD) when compared to the GBM (Figure 2.1). Note that the estimate of the volatility of the GBM model ($\sigma = 0.1372$) is greater than the volatility under the jump-diffusion model (see Table 2.1). For clarity, we have presented some values of LRP and RP in Table 2.2.

For example, if an individual retires with an actual initial wealth of $900,000 and desires to

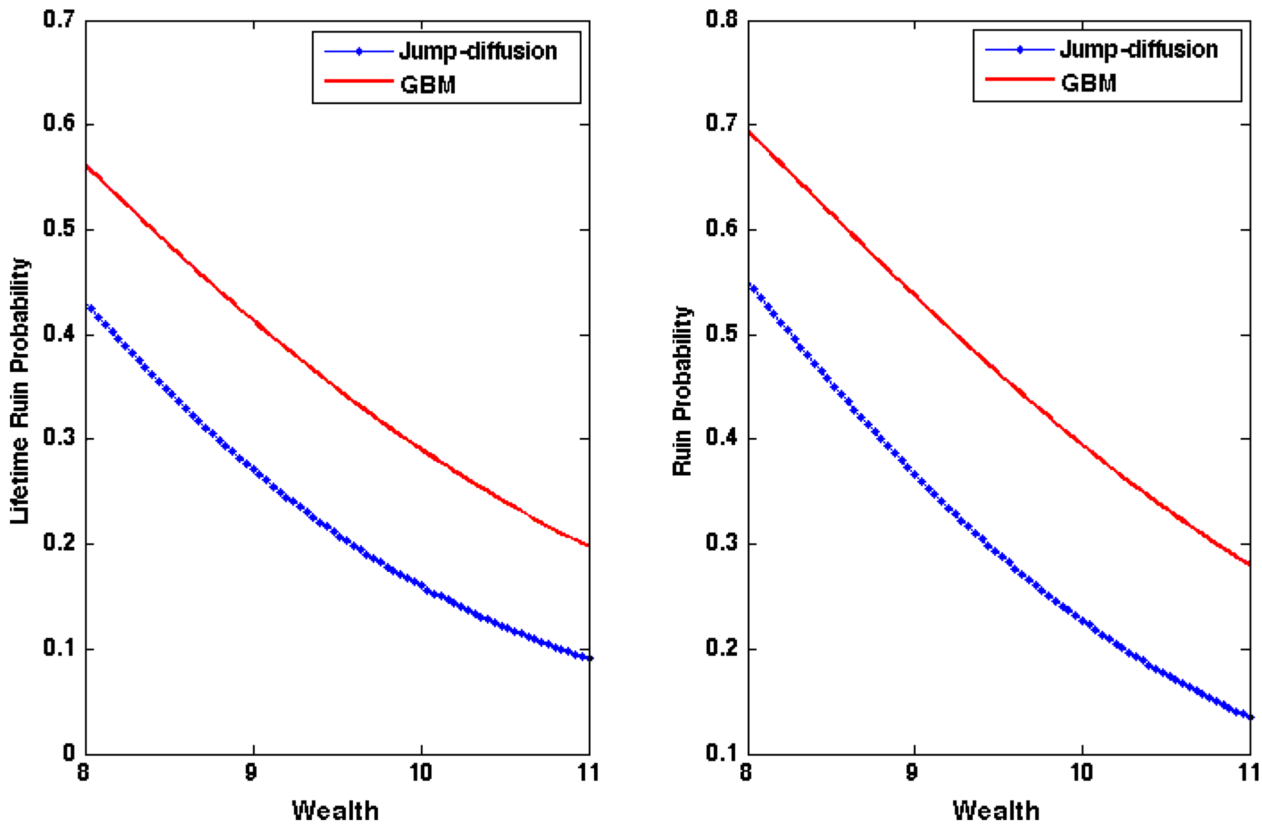


Figure 2.1: LRP and RP for $u = 0.5105$ (up-jump probability) and $w = 20$ (wealth level).

	P_L		P_R	
w	GBM	DEJD	GBM	DEJD
8	0.5619	0.4318	0.6942	0.5514
9	0.4132	0.2714	0.5367	0.3664
10	0.2901	0.1602	0.3944	0.2272
11	0.1973	0.0908	0.2793	0.1346
12	0.1312	0.0503	0.1925	0.0775
13	0.0859	0.0275	0.1299	0.0438
14	0.0554	0.0149	0.0860	0.0244
15	0.0351	0.0080	0.0558	0.0135

Table 2.2: P_L and P_R for different initial wealth levels and fix up-jump probability ($u = 0.5105$).

spend \$100,000 per year for the rest of his life, then w corresponds to 9 units of wealth. Specifically, a 50-year-old individual has a 41.32% lifetime probability of ruin if he starts with an initial wealth level of $w = 9$ units and consumes 1 unit of wealth throughout his lifetime. If this wealth process includes jumps, the same individual's lifetime ruin probability is lowered: He now has a 27.14% chance of reaching financial insolvency within his lifetime. Similarly, within a time horizon $T = 35$ years, the probability of crossing a wealth level of zero decreases from 53.67% for the pure diffusion case to 36.64% when jumps are introduced. Note that the actual initial wealth is the product of the initial wealth unit w and the desired consumption rate.

One explanation for this apparent decrease in the probability quantities involves taking a closer look at the estimated volatility. In Table 2.1 we establish that by solving the system with six degrees of freedom (DEJD), we obtain a smaller volatility estimate as compared to the volatility of the system with two degrees of freedom (GBM). It appears that the relative value of the estimated volatility has a substantial effect on the values LRP and RP. The introduction of jumps has decreased volatility and in turn has lowered the probability quantities of interest.

Next, we wanted to examine what happens to the ruin probabilities as we allow only "up" or only "down" jumps to be selected. In addition to the combination of both up and down jumps of the DEJD model, we distinguished two additional jump-diffusion cases: one that allows sampling of only up-jump-marks (up-jumps) and a second which includes only down-jump-marks (down-jumps). When looking at these alternatives, we notice that a combination of both upward and downward jumps is most effective in lowering the ruin probabilities (see Figure 2.2).

In Table 2.3, one can also see that both the LRP and RP are higher when $0.1 \leq u < 0.5$ for the up-jump case as opposed to the down-jump case. For $0.5 \leq u \leq 0.75$ the probabilities of interest are lower for the up-jump cases and higher for the down-jump case. Lower u corresponds to a high probability of downward jumps $d = (1 - u)$. In Table 2.1 one can see that a large $d = 0.8941$ ($u = 0.1049$) also corresponds to down-jump-marks which are quite small on average ($\frac{1}{\eta_2} \sim 0.1$) and vice versa. On the other hand, the average up-jump-marks remain approximately constant ($\frac{1}{\eta_1} \sim 0.14$) as u increases. A large probability of a down-jump does not guarantee an increase of the probabilities, due probably to the fact that these downward jump-amplitudes are small. However, as these downward jumps become large enough, the ruin probabilities are impacted by increasing relative to the up-jump case (see Table 2.3).

Finally, we wanted to answer the question of whether the results observed above apply to

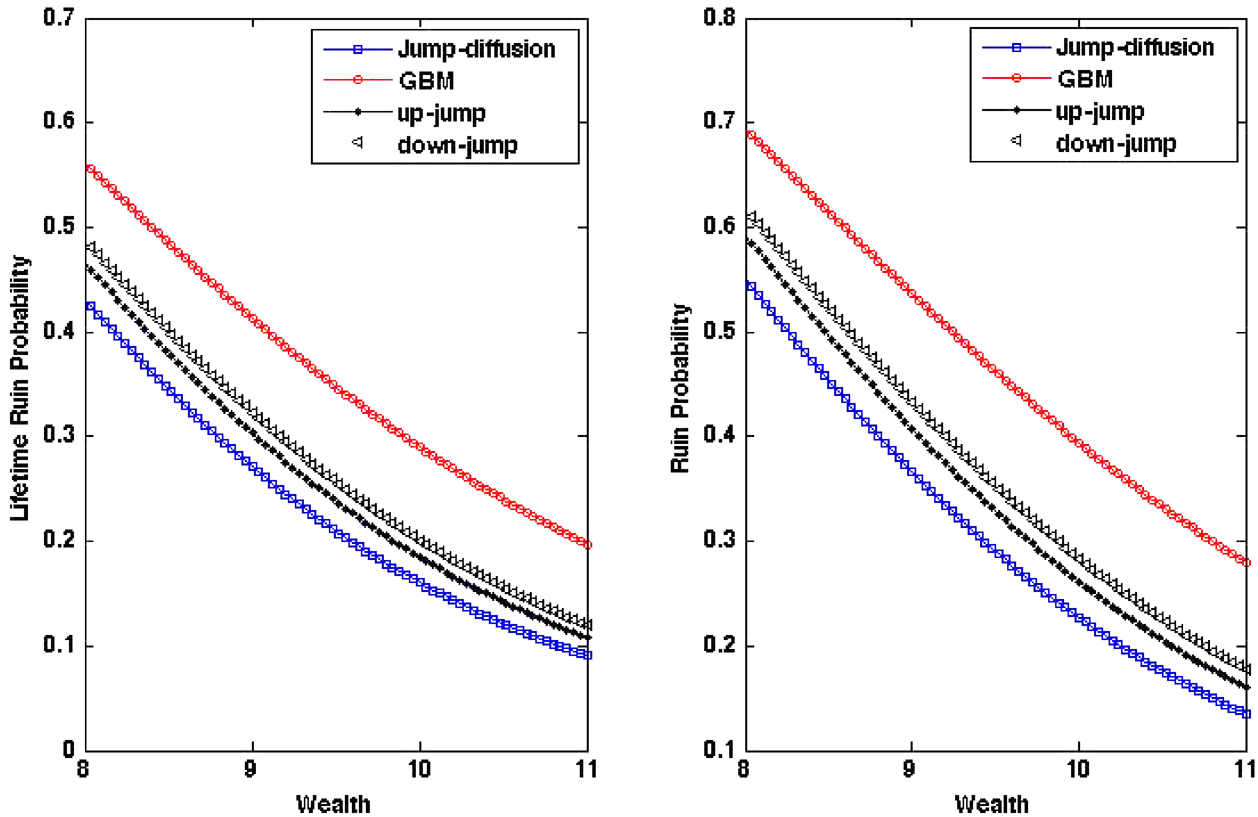


Figure 2.2: LRP and RP for $u = 0.5105$ (up-jump probability).

u	DEJD	up-jump	down-jump
P_L			
0.1059	0.3254	0.3957	0.3357
0.2988	0.2885	0.3373	0.3180
0.5105	0.2714	0.3032	0.3232
0.7511	0.2352	0.2509	0.3123
P_R			
0.1059	0.4301	0.5180	0.4432
0.2988	0.3865	0.4491	0.4246
0.5105	0.3664	0.4076	0.4335
0.7511	0.3222	0.3429	0.4235

Table 2.3: P_L and P_R for initial wealth $w = 9$.

wealth ranges where ruin becomes very likely (small initial wealth levels) or very unlikely (high initial wealth levels). The pattern remains unchanged for relatively large wealth levels, but can switch direction for small initial wealth. It appears that for low investments the ruin probability of the jump-diffusion case can exceed the one derived from the pure diffusion dynamics. For example, we present one of these results in Figure 2.3.

All other parameters are the same as those used in producing Figure 2.1. Note how for very low levels of wealth, the probabilities cross over and are actually higher in the jump-diffusion case. However, the analysis presented in Appendix 2.B suggests that, when contributions from

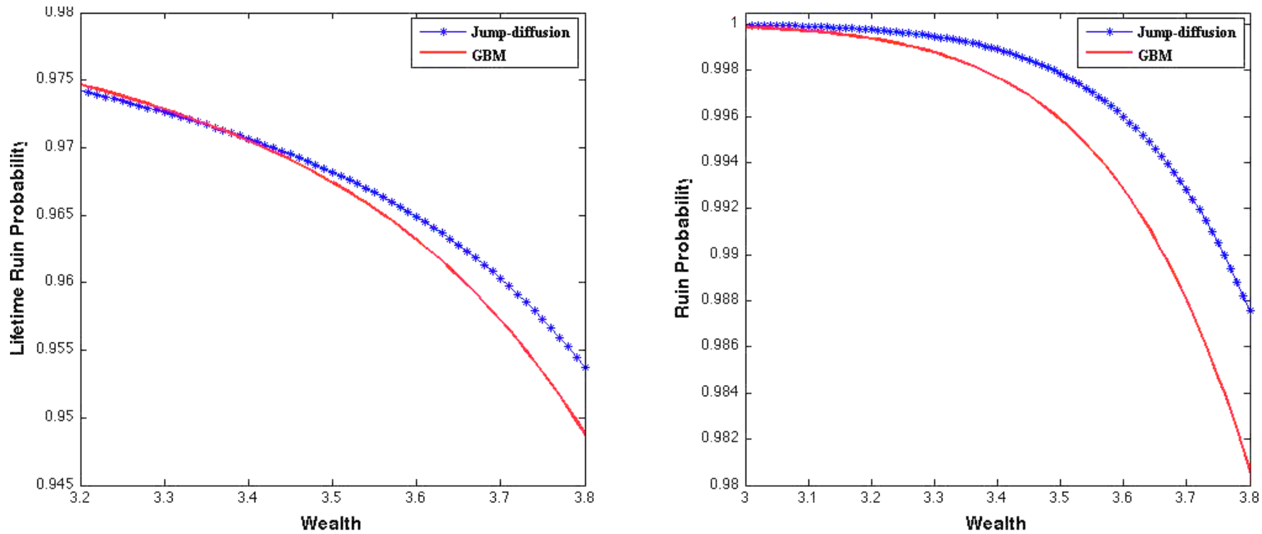


Figure 2.3: LRP and RP for $u = 0.5105$ (up-jump probability) and $w = 5$ (wealth level).

both large and small jump-marks are allowed, a reduction of the probability of ruin occurs even for small wealth levels (see Figure 2.6). There is no ambiguity that jumps have a positive impact on the investments when jump-marks can be of all sizes.

2.3.2.3 Monte Carlo Simulations

We analyzed the observed impact of the jumps on the LRP, this time using the MC simulated data to estimate the model's parameters. Our goal here was to establish whether the introduction of error in our jump-diffusiom model changes the relationship between the ruin probabilities of the GBM and DEJD models. We conclude that the ruin probabilities are once again lowered by the introduction of jumps. One example of these computations is presented in Figure 2.4. Assuming a jump-diffusion model, we generate a new set of returns by the Monte Carlo method, as follows.

Let the underlying asset price follow the SDE:

$$\begin{cases} dS_t = S_t(\mu dt + \sigma dB_t + J(Q)dN_t), \\ S_0 > 0. \end{cases} \tag{2.3.40}$$

Here B_t is the Wiener process, N_t is the standard Poisson jump counting process, $J(Q)$ is the jump amplitude and $Q = \ln(J(Q) + 1)$ is the underlying jump-mark process. We solve equation (2.3.40):

$$\begin{aligned} d(\ln S(t)) &= \frac{1}{S}dS - \frac{1}{2S^2}dS^2 \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dB(t) + \sum_{k=1}^{dN(t)}(e^{Q_k} - 1) \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dB(t) + \sum_{k=1}^{dN(t)}(Q_k + 1 - 1) \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dB(t) + QdN(t). \end{aligned} \tag{2.3.41}$$

After integration and log-inversion:

$$\ln S(t) = (\mu - \frac{\sigma^2}{2})t + \sigma B(t) + QN(t), \tag{2.3.42}$$

where

$$QN(t) = \begin{cases} \sum_{k=1}^{N(t)} Q_k, \\ 0 \quad \text{if} \quad N(t) = 0. \end{cases} \tag{2.3.43}$$

We get the solution:

$$\begin{aligned} S(t) &= S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)} e^{\sum_{k=1}^{N(t)} Q_k} \\ &= S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)} \prod_{k=1}^{N(t)} e^{Q_k}. \end{aligned} \tag{2.3.44}$$

Using solution 2.3.44, the returns can be approximated as follows:

$$
\begin{aligned}
\frac{S(t+\Delta t)-S(t)}{S(t)} &= \frac{S_0 e^{(\mu-\frac{\sigma^2}{2})(t+\Delta t)+\sigma B(t+\Delta t)} e^{\sum_{k=1}^{N(t+\Delta t)} Q_k} - S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma B(t)} e^{\sum_{k=1}^{N(t)} Q_k}}{S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma B(t)} e^{\sum_{k=1}^{N(t)} Q_k}} \\
&\approx \mu\Delta t + \sigma Z\sqrt{\Delta t} + \lambda\Delta t Q_{N(t)+1} \\
&\approx \mu\Delta t + \sigma Z\sqrt{\Delta t} + \lambda_u \Delta t Q^u_{N_u(t)+1} + \lambda_d \Delta t Q^d_{N_d(t)+1},
\end{aligned}
\tag{2.3.45}
$$

where Z is a random variable with normal distribution and $\Delta t = 1$ month. The probability of an upward jump is $u = \frac{\lambda_u}{\lambda} = \frac{\lambda_u}{\lambda_u+\lambda_d}$.

We let the size of this artificial data set be $N = 2^m n$ with $n = 400$ and $m = 1, ..., 8$. The true parameters $\mathbf{p}$ used to simulate N data points were the result of the application of the moment matching technique on the historical data set described in equation (2.3.33). In other words, these are the JD model-specific parameters calibrated to real data. From this artificially generated data we obtained a new set of parameter estimates $\hat{\mathbf{p}}_{jd}$ and $\hat{\mathbf{p}}_d$ for the jump-diffusion and diffusion cases respectively. This was done by matching the synthetic data moments to the theoretical moments. One example of these estimates is presented in Table 2.4. The error between the artificial data parameters and the JD model-parameters (true) is also presented in the table. This is presented as an L-infinity norm of the parameter vector difference. These are also the values used to produce Figure 2.4. The error in the true parameter values did not have an effect on the relationship between the jump-diffusion and diffusion ruin probabilities. It is evident from this figure and our testing that ruin probability values are always lower for the jump-diffusion case regardless of the number of data set points (ranging here from 400 to 102,400) used.

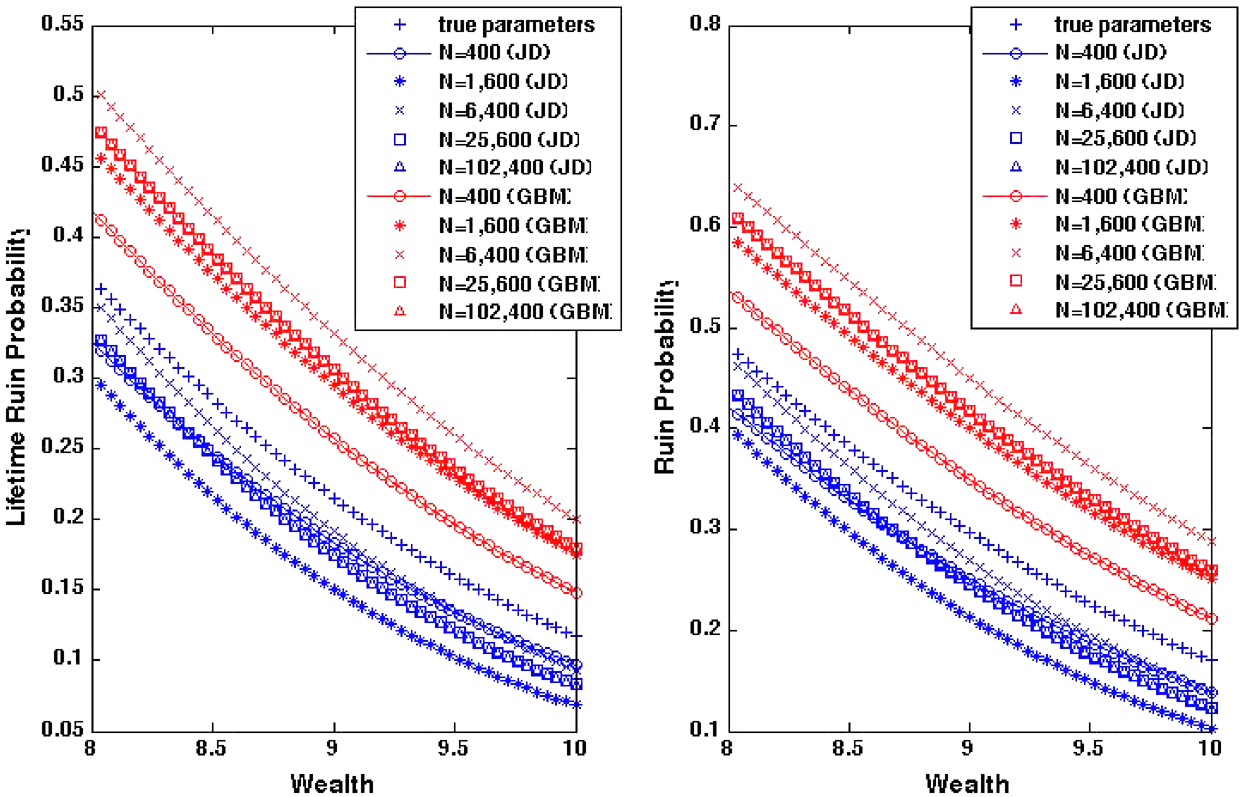


Figure 2.4: LRP and RP for given *true* parameters $\mu = 0.1210$, $\sigma = 0.1189$, $\lambda = 0.0135$, $\eta_1 = 6.9905$, $\eta_2 = 3.5896$ and $u = 0.4994$.

			N				
p		$\hat{\mathbf{p}}_{jd}$	**400**	**1,600**	**6,400**	**25,600**	**102,400**
μ	0.1210	$\hat{\mu}$	0.1301	0.1284	0.1242	0.1255	0.1253
σ	0.1189	$\hat{\sigma}$	0.1221	0.1061	0.1063	0.1067	0.1063
λ	0.0135	$\hat{\lambda}$	0.0091	0.0103	0.0099	0.0103	0.0103
η_1	6.9905	$\hat{\eta}_1$	6.9475	6.8543	6.8487	6.8801	6.8878
η_2	3.5896	$\hat{\eta}_2$	4.8500	2.1965	1.5300	2.5169	3.0912
u	0.4994	$\hat{u}$	0.5449	0.4117	0.4170	0.4035	0.3917
		$\|\|\mathbf{p} - \hat{\mathbf{p}}_{jd}\|\|$	1.2620	1.4026	2.0662	1.0827	0.5203
		$\hat{\mathbf{p}}_d$					
		$\hat{\mu}$	0.1298	0.1253	0.1206	0.1230	0.1230
		$\hat{\sigma}$	0.1201	0.1182	0.1136	0.1134	0.1134

Table 2.4: Estimated parameter values for different N.

2.4 Conclusion

In this chapter we have explained how to derive an expression for the lifetime ruin probability (LRP) — a risk metric that has been studied by numerous researchers in the retirement field — assuming the portfolio returns obey a jump-diffusion process. This is in contrast to the existing literature that for the most part assumes geometric Brownian motion (GBM) dynamics, or diffusions without jump terms. Consequently, an associated partial-integro-differential equation (PIDE) was newly derrived for the LRP and the related ruin probability (RP).

We then compared our values to GBM-based ruin probabilities and previous estimates in the literature using moment matching techniques calibrated to historical equity returns. In the chapter we provide more details on this process. Besides illustrating our method for computing the LRP, our main qualitative result was that except for the case of low initial wealth, ruin probabilities were lower than under GBM models when moments are the same. We speculate that the reason for this result is that if the first and second moment of the data-fitting distribution must be identical, then a jump — which increases the dispersion, all else being equal — must be compensated by a suitable reduction in the volatility estimate. At the same time, diffusion volatility has a greater impact on LRP values than jumps. Stated differently, the main cause of a relatively high LRP is a poor sequence of investment returns early on in the path of the portfolio process. Moreover, higher values of diffusion volatility are more likely to lead to early losses.

While our qualitative explanation may not be entirely satisfying — and our numerical results indicate that there are exceptions to the above observation that jump-diffusion processes whose moments match continuous processes lead to lower LRP values — our results do provide useful insights. We should point out that similar results have been obtained in the long-dated option pricing literature.

In summary, we believe that the most important take-away here is that practitioners who are interested in computing LRP values in the real world, should focus their energies on modelling the forward-looking equity risk premium (ERP) and its volatility, and perhaps worry less about short-term stock price movements.

2.A Appendix: Derivation of the Backward PIDE

The following is a brief outline of the derivation of the backward PIDE for the transition density and the transition probability. Let

$$\begin{cases} dW_s = (\mu_w W_s - 1)ds + \sigma_w W_s dB_s^w + h(s, W_s, q)dP_s^w, \\ W_t = w. \end{cases} \tag{2.A.1}$$

Let $\varphi(W_s)$ be any function of W_s. By Itô's lemma we have:

$$\begin{aligned} d\varphi(W_s) &= (f\varphi_w(W_s) + \frac{g^2}{2}\varphi_{ww}(W_s))ds + g\varphi_w dB_s^w + [\varphi](W_s)dP_s^w \\ &= L\varphi(W_s)ds + g\varphi_w dB_s^w + [\varphi](W_s)dP_s^w, \end{aligned} \tag{2.A.2}$$

where $f = \mu_w W_s - 1$, $g = \sigma_w W_s$, and

$$[\varphi](W_s)dP_s^w = \int_Q \{\varphi(W_s + h(s, W_s, q)) - \varphi(W_s)\}\mathscr{P}(ds, dq).$$

Here it was assumed that the Wiener process is independent of the Poisson processes and that the quadratic differential Wiener processes can be replaced by their mean square value. We also drop terms that are zero in the mean square. Then we write:

$$\varphi(W_T) = \varphi(w) + \int_t^T (L\varphi(W_s)ds + g\varphi_w dB_s^w + [\varphi](W_s)dP_s^w). \tag{2.A.3}$$

Taking expectations we obtain the Dynkin formula in integral form (Hanson 2007):

$$\begin{aligned} E\{\varphi(W_T)\} &= \varphi(w) + \int_t^T E\{(L\varphi(W_s)ds + g\varphi_w dB_s^w + [\varphi](W_s)dP_s^w)\} \\ &= \varphi(w) + \int_t^T E\{(L\varphi(W_s) + \lambda[\varphi])ds\} \\ &= \varphi(w) + \int_t^T (L\varphi(W_s) + \lambda \int_Q \{\varphi(W_s + h(s, W_s, q)) - \varphi(W_s)\}\phi dq)ds. \end{aligned} \tag{2.A.4}$$

This simplification follows from the independent increment property of Markov processes:

$$E\left[\int_t^T G(X_s)dB_s\right] = 0, \tag{2.A.5}$$

and the zero-mean jump process property

$$E\left[\int_t^T H(X_s)\mathscr{P}(ds, dq)\right] = \lambda H(X_s)\phi(q)dqds. \tag{2.A.6}$$

We conclude our derivation with the following theorem.

Theorem 2.A.1. *Let $u(t, w) = E[\varphi(W_T)|W_t = w]$. If the forward time is suppressed in favour of the backward time t, then $u(w)$ satisfies the backward PIDE:*

$$u_t(t, w) + \mathscr{B}_w[u](w) = 0, \tag{2.A.7}$$

with final time condition

$$\lim_{t \to T} u(t, w) = \varphi(w). \tag{2.A.8}$$

The backward operator is:

$$\mathscr{B}_w[u](w) = (\mu_w - 1)u_w + \frac{\sigma_w^2 w^2}{2} u_{ww} + \lambda \int_Q (u(t, w + h(t, w, q)) - u(t, w))\phi(q)dq. \tag{2.A.9}$$

Let the transition probability have the following distribution:

$$P(t, w; T, W) = Pr[W_T \leq W | W_t = w], \tag{2.A.10}$$

with probability density

$$p(t, w; T, W) = Pr[W_T \leq W + dW | W_t = w] - Pr[W_T \leq W | W_t = w]. \tag{2.A.11}$$

In terms of the transition density, the conditional expectation can be rewritten such that:

$$\begin{aligned} u(t, w) &= E[\varphi(W)|W_t = w] \\ &= \int_{-\infty}^{\infty} \varphi(W)p(t, w; T, W)dW. \end{aligned} \tag{2.A.12}$$

Thus, if we let

$$\varphi(W) = \delta(W - \xi_1), \tag{2.A.13}$$

then, by the definition of the Dirac delta function:

$$u(t, w) = p(t, w; T, \xi_1). \tag{2.A.14}$$

By Hanson (2007) and Björk (1998), the following corollaries follow:

Corollary 2.A.2. *Let $p(t, w; T, W)$ be the transition probability density. Then the backward PIDE:*

$$p_t(t, w; T, W) + \mathscr{B}_w[p](w) = 0, \tag{2.A.15}$$

with the final time condition

$$\lim_{t \to T} p(t, w; T, W) = \delta(w - W). \tag{2.A.16}$$

Corollary 2.A.3. *Assume the probability measure $P(t, w; T, W)$ has a probability density $p(t, w; T, W)$. Then we have the following PIDE:*

$$P_t(t, w; T, W) + \mathscr{B}_w[P](w) = 0, \tag{2.A.17}$$

with final condition

$$P(T, w; T, W) = \mathbf{1}_W(w) = 1 - H(W - w). \tag{2.A.18}$$

These results were applied in order to arrive at the backward equations (2.2.7) and (2.2.10).

2.B Appendix: Integral Approximation

In this section, we present a brief discussion of the comparison of the lifetime ruin probability for two different integral approximations introduced in Section 2.3.1. For convenience, we call the solution 'local' if the contribution to the solution is the result of small jumps, as is often the case with the financial market (HANSON 2007). On the other hand we call 'global' the solution obtained by utilizing the infinite domain of the jump-marks. Under the 'global' set-up, we discretize the PIDE (2.2.10) as follows:

$$\frac{P_j^n - P_j^{n+1}}{\Delta t} - T_1(P_{j^*}^n - P_{j^*}^n) - T_2(P_{j+1}^n - 2P_j^n + P_{j-1}^n) + \lambda P_j^n + \hat{\lambda} P_j^n = S_j^n + S_j^{n+1}, \tag{2.B.1}$$

where $T_1 = \frac{(\mu w_j - 1)}{\Delta w}$, $T_2 = \frac{\sigma^2 w_j^2}{2\Delta w^2}$ and $j^* = j$ if the coeffcient $T_1 < 0$ and $j^* = j + 1$ if it is positive. As before $\{P_{j-1}^n, P_j^n, P_{j+1}^n\}$ contains the unknown quantities. The integral term is approximated by S_j^n and S_j^{n+1} as follows:

$$\begin{aligned} S_j^{n+1} &= \lambda \left\{ \int_{Q_{min}}^{\infty} P^{n+1}(e^q w_j)\phi_{q>0} dq + \int_{-\infty}^{-Q_{min}} P^{n+1}(e^q w_j)\phi_{q<0} dq \right\}, \\ &\approx \lambda \sum_{k=2}^{N_q} \Delta Q_k P^{n+1}(e^{q_k} w_j)\phi_{q>0} + \lambda \sum_{-N_q}^{k=-2} \Delta Q_k P^{n+1}(e^{q_k} w_j)\phi_{q<0}, \end{aligned} \tag{2.B.2}$$

where $\Delta Q_k = q_{k+1} - q_k$. Moreover, the approximation using known values is given by:

$$\begin{aligned} S_j^n &= \lambda \left\{ \int_0^{Q_{min}} P^n(e^q w_j)\phi_{q>0} dq + \int_{-Q_{min}}^{0} P^n(e^q w_j)\phi_{q<0} dq \right\}, \\ &\approx \lambda \sum_{k=0}^{1} \Delta Q_k P^n(e^{q_k} w_j)\phi_{q>0} + \lambda \sum_{k=-1}^{0} \Delta Q_k P^n(e^{q_k} w_j)\phi_{q<0}. \end{aligned} \tag{2.B.3}$$

The term $P(e^q w)$ is further approximated by linear interpolation as described in Section 2.3.1. For small jump-mark values we have $|\frac{(e^q - 1)w}{\Delta w}| \leq 1$. This results in the restriction $Q_{min} = \ln(1 + \frac{1}{N_w})$, where N_w are the maximum wealth discretization points.

We denote by P^g and P^l, the 'global' and 'local' solutions to the PIDE respectively. Here P can represent either the lifetime ruin probability (P_L) or the ruin probability (P_R). More precisely we calculated the Root-Mean-Squared-Error (RMSE) to assess the relative error of the two approximations:

$$E_2 = \frac{1}{\sqrt{N}} \left\| \frac{P^g - P^l}{P^g} \right\|_2 . \tag{2.B.4}$$

To measure the maximum relative error, we also calculated the L^∞:

$$E_\infty = \left\| \frac{P^g - P^l}{P^g} \right\|_\infty . \tag{2.B.5}$$

We performed the computation for an investor of age $x = 50$ with parameters $\sigma = 0.12, \mu = 0.12, \lambda = 0.01, \eta_1 = 7, \eta_2 = 3, u = 0.51$ for the jump-diffusion (JD) case and $\sigma = 0.11, \mu = 0.12$ for the pure diffusion case. The market parameters were calibrated to the real data as described before. Moreover, we chose $N_t = 1000$ and $N_w = 200$, with a time horizon $T = 35$ years and $w \in [0, 20]$. As can be observed from Table 2.5, the RMSE is small and satisfies $C\frac{1}{\sqrt{N}} \leq \frac{1}{\sqrt{N}}$, where $C < 10\%$.

$P_L(JD)$		$P_L(D)$	
E_2	E_∞	E_2	E_∞
6.49%	16.4%	1.9e−04%	3.0e−04%
$P_R(JD)$		$P_R(D)$	
E_2	E_∞	E_2	E_∞
5.22%	12.18%	1.8e−04%	3.0e−04%

Table 2.5: P_L and P_R relative error of the 'local' solution relative to the 'global' solution for both the diffusion (D) and jump-diffusion case (JD).

Figure 2.5 is the graphical representation of the difference between our 'local' and 'global' LRP functions for both the jump-diffusion and pure diffusion case. One can see that the largest error occurs when wealth levels are small. A similar error pattern was observed for the ruin probability.

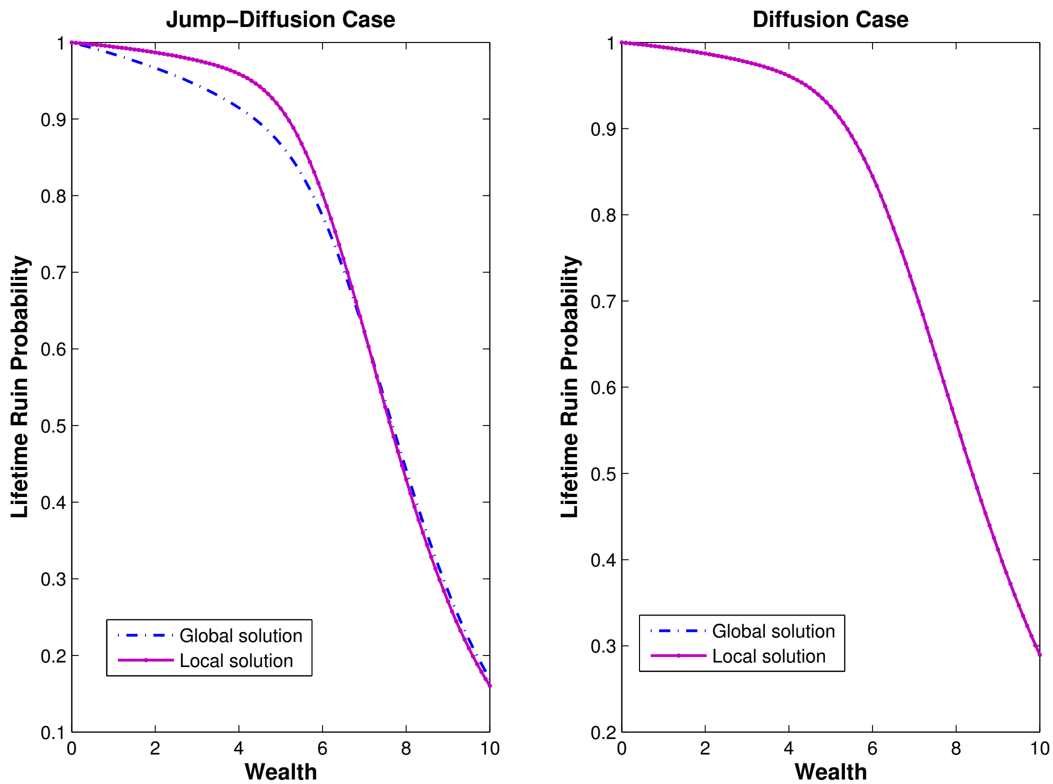


Figure 2.5: Difference between the 'local' and 'global' solutions for the jump-diffusion and diffusion cases.

We also confirm that the reduction of the lifetime ruin probability occurs when introducing jumps as seen in Figure 2.6. Moreover, the figure suggests that for small wealth levels, the probability of ruin of the 'global' approximation case is further reduced due to the introduction of jumps when compared to the 'local' approximation. An individual with relatively small investment funds seems to benefit the most from the introduction of jumps in the asset return.

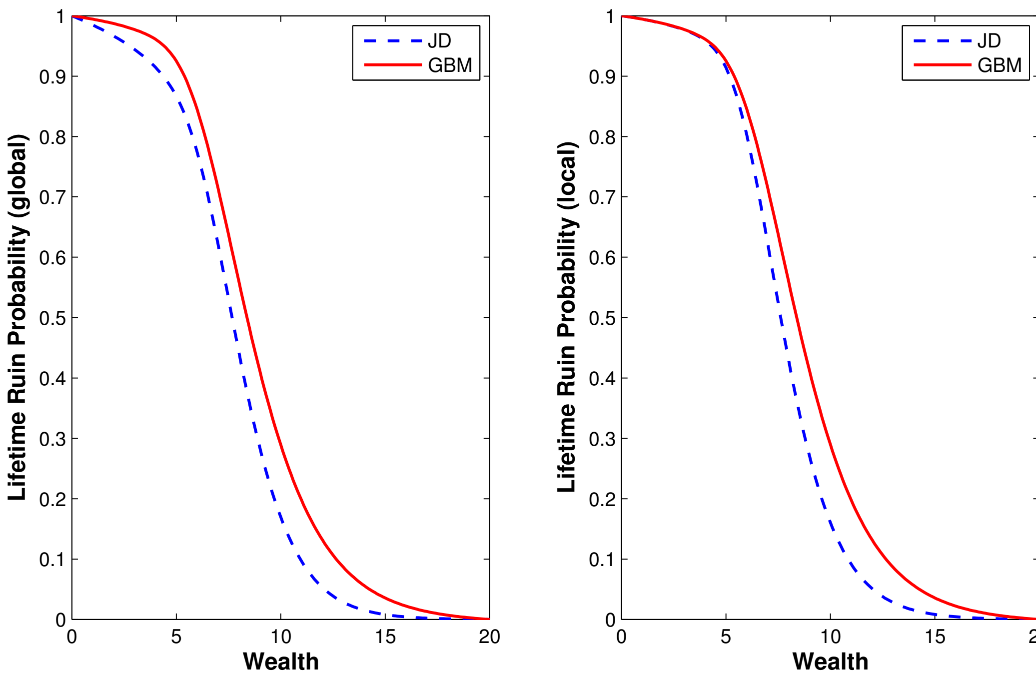


Figure 2.6: Comparison of the jump-diffusion and diffusion lifetime ruin probability for the 'global' and 'local' approximations.

3 Optimal Investment Problem Under Health Shocks for Retired Investors

3.1 Introduction

In this chapter, we develop a model of optimal consumption-allocation-medical insurance strategies for a retired individual. In addition to the usual investment risk and life expectancy uncertainty, a retiree also has to find a way to deal with out-of-pocket health care costs. We assume that the option of buying medical insurance is available to the retiree. We want to answer the question of whether he should purchase medical insurance protection and what the optimal amount should be. Moreover, we want to determine the optimal consumption-allocation schedule for this investor. We will set up the optimal control problem under the framework of life-cycle models.

3.2 Model Setup: General Notation and Assumptions

The general assumptions of the model are as follows:

- We focus only on *out-of-pocket* medical expenses. These are expenses not covered by Medicaid (US) or government health insurance (Canada).
- At the start of the investment period, the health status of the retiree is 'good'.

- For simplicity, we assume that the survival probabilities are dependent on health health status, but not on gender. Future extensions to the model could take gender into consideration.
- The instantaneous health status decline rate — the rate of transitioning from “healthy” to “unhealthy” — of an individual increases exponentially with age. It therefore follows that, the probability of maintaining good health decreases with age.
- The hazard rate is Gompertz Makeham and is health status dependent.
- Once the individual becomes sick, he remains in that state.

Assume a retiring individual currently aged x invests a fraction π_t of his wealth in a risky asset and $(1-\pi_t)$ in a risk-less asset. The following describes the dynamics of the risky and risk-less assets. Let S_t be the stock price which follows a log-normal diffusion stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dB_t), \quad S(0) = S_0. \tag{3.2.1}$$

Here, μ is the mean rate of return (constant drift coefficient), σ is the diffusive volatility and B_t is the continuous Brownian motion process. On the other hand, the risk-free asset is modelled as a bond price, continuously compounded at a rate r:

$$dN_t = rN_t dt, \quad N(0) = N_0. \tag{3.2.2}$$

A realistic problem is that of the investor starting in good health being suddenly faced with a health shock. In anticipation of such an event he purchases medical insurance I_t at the beginning of his retirement, while still in optimal health. The goal of this purchase is to provide the opportunity of offsetting future health shock expenses.

The individual's health-contingent wealth dynamics can then be divided into two phases. In particular, his portfolio wealth process at time t follows the following SDE:

$$dW_t = \begin{cases} \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt - c_t dt - I_t dt; & \tau_s > t, \\ \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt - c_t dt; & \tau_s \leq t, \end{cases} \tag{3.2.3}$$

where τ_s is the time the individual becomes ill, c_t denotes the instantaneous consumption rate per unit time, I_t is the purchased health insurance at time $t < \tau_s$. Here, B_t is the Brownian motion driving the wealth process. The wealth dynamics can be thought of as the sum of instantaneous investment returns from the allocations to the risky ($\pi \frac{dS_t}{S_t}$) and risk-less ($(1-\pi)\frac{dN_t}{N_t}$) assets, while also subtracting the consumption ($c_t dt$) and the health insurance amount ($I_t dt$). As can be seen in the above wealth process, the individual does not invest in insurance after becoming sick, but continues to consume his wealth. At the instant the individual becomes sick, $t = \tau_s$, there is a jump in the wealth process due to a percentage of the premium paid to the health insurance beneficiary as well as a downward jump in wealth due to the health costs. We will denote the proportion of the insurance premium received at the time of the health shock by α. This health insurance multiplier α is a function of time. As mentioned before, we denote by I_t the medical insurance premium payable per unit time. From the investor's perspective, the insurance premium consists of the actuarially fair premium. Similarly to life insurance, the medical insurance will induce a health benefit (face value) once the consumer becomes sick. If he purchases I_t dollars in medical insurance, he will be entitled to $\alpha I_t = \frac{I_t}{\eta_{x+t}}$ if he becomes sick at time t.

We denote the magnitude of the health costs by ν. We assume that the distribution of this jump is known. After the health decline we write the wealth process as follows:

$$W_{\tau_s^+} = W_{\tau_s^-} + \alpha I_t - \nu. \tag{3.2.4}$$

The first jump of the Poisson process driving the jump in wealth due to the health costs and the paid fraction of the health insurance occurs at the time the individual becomes sick. We also assume that the Poisson process driving the health cost shock has an intensity η_{x+t}, which can be thought of as a deterministic function of time representing the rate at which the individual gets sick.

3.2.1 Dynamic Programming Principle and the HJB

The object of the retiree is to choose a portfolio-consumption-health insurance strategy in such a way as to maximize his conditional expected current value of the discounted utility of instantaneous consumption over an infinite time horizon. We formally state the consumer's optimization problem as follows. For an individual currently aged x who is healthy and alive at time t, the optimal value function is defined as:

$$J(t,w) = \max_{\{\pi_s, c_s, I_s\}} E\left[\int_t^{\mathbf{T}_x} e^{-\rho s} u(c_s) ds | W_t = w\right], \tag{3.2.5}$$

$$= \max_{\{\pi_s, c_s, I_s\}} E\left[\int_t^{\infty} e^{-\rho s} u(c_s) 1_{\{\mathbf{T}_x > s\}} ds | W_t = w\right]. \tag{3.2.6}$$

where $\mathbf{T}_x$ is a random variable representing the remaining lifetime for an individual aged x. We also define the set on which the individual is alive by the following indicator function:

$$1_{\{\mathbf{T}_x \geq t\}} = \begin{cases} 1 & \text{when} \quad \mathbf{T}_x \geq t, \\ 0 & \text{when} \quad \mathbf{T}_x < t. \end{cases} \tag{3.2.7}$$

Assuming that there exists an optimal control law that satisfies the optimal value function (3.2.6) and that the investor aged x is healthy and alive at t, we divide the infinite time domain of the optimization problem into $[t, t+h]$ and $(t+h, \infty)$ as follows:

$$J(t,w) = \max_{\{\pi_s, c_s I_s\}} E\left[\int_t^{t+h} e^{-\rho s} u(c_s) 1_{\{\mathbf{T}_x > s\}} ds | W_t = w\right] + \max_{\{\pi_s, c_s, I_s\}} E\left[\int_{t+h}^{\infty} e^{-\rho s} u(c_s) 1_{\{\mathbf{T}_x > s\}} ds | W_t = w\right]. \tag{3.2.8}$$

Because we wish to introduce the health status risk into our problem, we define a random variable $\mathbf{T}_x^{hl}$ which will represent the remaining healthy lifetime of an individual aged x. We assume that the value function depends on the initial health status of the individual. We also define a health shock as a jump in health status from 'good' (1) to 'poor' (0). Hence, taking the random health status into consideration leads to the following optimization problem:

$$J(t,w) = \max_{\{\pi, c_t, I_t\}} E\left[\int_t^{t+h} e^{-\rho s} u(c_s) 1_{\{\mathbf{T}_x^{hl} > s\}} ds | 1_{\{t < \mathbf{T}_x^{hl} < \mathbf{T}_x\}}\right] \tag{3.2.9a}$$

$$+ \max_{\{\pi, c_t, I_t\}} E\left[\int_t^{t+h} e^{-\rho s} u(c_s) 1_{\{\mathbf{T}_x^{hl} < s\}} ds | 1_{\{t < \mathbf{T}_x^{hl} < \mathbf{T}_x\}}\right] \tag{3.2.9b}$$

$$+ \max_{\{\pi, c_t, I_t\}} E\left[E\left[\int_{t+h}^{\infty} e^{-\rho s} u(c_s) 1_{\{\mathbf{T}_x^{hl} > s\}} ds | 1_{\{t+h < \mathbf{T}_x^{hl} < \mathbf{T}_x\}}\right] | 1_{\{t < \mathbf{T}_x^{hl} < \mathbf{T}_x\}}\right] \tag{3.2.9c}$$

$$+ \max_{\{\pi, c_t, I_t\}} E\left[E\left[\int_{t+h}^{\infty} e^{-\rho s} u(c_s) 1_{\{\mathbf{T}_x^{hl} < s\}} ds | 1_{\{\mathbf{T}_x^{hl} < t+h < \mathbf{T}_x\}}\right] | 1_{\{t < \mathbf{T}_x^{h} < \mathbf{T}_x\}}\right]. \tag{3.2.9d}$$

To explain the above components of our optimization problem, suppose that there is an agent of age x. Equation (3.2.9a) represents the expectation that the individual is healthy and alive in $[t, t+h)$ given he is healthy and alive at time t (or equivalently at age $x+t$). The term (3.2.9b) shows the expectation that the individual is sick and alive in $[t, t+h)$, given he is healthy and alive at t. Term (3.2.9c) is the expectation that the individual remains healthy and alive in $[t+h, \infty)$ given he is healthy and alive at t and $(t+h)$. The term (3.2.9d) represents the expectation that the individual remains sick and alive for the interval $[t+h, \infty)$ given he is sick and alive at $(t+h)$ while healthy and alive at t.

To model the value function with uncertain health status and the remaining lifetime of the individual aged x, we also establish the following basic concepts and definitions. The conditional probability of remaining healthy for t more years provided he is healthy at age x is:

$$ {}_tg_x \ := Pr[\mathbf{T}_x^{hl} > t] = e^{-\int_0^t \eta_{x+u} du}. \tag{3.2.10} $$

The probability of remaining healthy is a decreasing function of time and η_{x+u} is the instantaneous rate of getting sick at age $x+u$. The conditional probability of getting sick before or at age $x+t$ is represented by the cumulative density function (CDF) of $\mathbf{T}_x^{hl}$ as follows:

$$ F_x(t) \ := Pr[\mathbf{T}_x^{hl} < t] = 1 - {}_tg_x \ . \tag{3.2.11} $$

Equivalently, the CDF can be written as:

$$ F_x(t) = \int_0^t f_x(u)du, \tag{3.2.12} $$

where $f_x(t)$ is the probability density function (PDF) of the random variable $\mathbf{T}_x^{hl}$. With the above definitions, we can write:

$$ f_x(t) = \frac{\partial}{\partial t}(1 - {}_tg_x) = \eta_{x+t}({}_tg_x). \tag{3.2.13} $$

Similarly, we present the definitions used for modelling the uncertain lifetime of a sick individual and a healthy investor aged x. The following are the survival probabilities if the individual is in poor health and in good health respectively:

$$ {}_tp_x^{(0)} \quad = \quad Pr[\mathbf{T}_{0,x} > t] = e^{-\int_0^t \lambda_{x+u}^{(0)} du}, \tag{3.2.14} $$

$$ {}_tp_x^{(1)} \quad = \quad Pr[\mathbf{T}_{1,x} > t] = e^{-\int_0^t \lambda_{x+u}^{(1)} du}, \tag{3.2.15} $$

where $\mathbf{T}_{0,x}$ and $\mathbf{T}_{1,x}$ are the random remaining lifetime of a sick and healthy individual aged x respectively. Moreover, the instantaneous rates of death (hazard rates) for a sick and a healthy individual aged $x+u$ are $\lambda_{x+u}^{(0)}$ and $\lambda_{x+u}^{(1)}$ respectively.

We note that the rate of becoming sick and the hazard rates of an individual aged $x + t$ are increasing functions of time. To simplify the computations, we will assume the following relationship between these rates. It is intuitive that a healthier investor has a higher probability of survival than a sick investor. There are two possible extreme cases for selecting the rate of becoming sick. One case is such that it is the same or smaller than the rate of dying of a healthy investor. On the other extreme, there is the case where the rate of becoming ill is in fact larger or equal to the hazard rate of a sick individual. The relationship can be summarized as follows:

$$\eta_{x+t} \leq \lambda^{(1)}_{x+t} \leq \lambda^{(0)}_{x+t}, \tag{3.2.16}$$

or

$$\lambda^{(1)}_{x+t} \leq \lambda^{(0)}_{x+t} \leq \eta_{x+t}. \tag{3.2.17}$$

In our numerical results discussion, we will refer to (3.2.16) as case 1 and to (3.2.17) as case 2.

Another assumption is that $\mathbf{T}^{hl}_x$, $\mathbf{T}_{0,x}$ and $\mathbf{T}_{1,x}$ are continuous random variables which follow a Gompertz-Makeham (GM) distribution. For the remainder of the chapter, we will assume for case 1 that $\eta = \hat{\beta}\lambda^{(0)}$, $\lambda^{(1)} = \beta_1\lambda^{(0)}$, $\lambda^{(0)} = \beta_0\lambda^{(0)}$, and $\hat{\beta} + \beta_1 = \beta_0 = 1$. For case 2 we have $\lambda^{(0)} = \beta_0\eta$, $\lambda^{(1)} = \beta_1\eta$, $\eta = \hat{\beta}\eta$, where $\beta_0 + \beta_1 = \hat{\beta} = 1$. Here we dropped the subscript $x + t$ to simplify the notation.

Moreover, we assume that the hazard rate is associated with the Gompertz-Makeham (GM) distribution. For a healthy individual we can write:

$$\lambda^{(1)}_{x+t} = \theta + A_1 e^{Bt}. \tag{3.2.18}$$

Here, $A_1 = \frac{1}{b}e^{\frac{x-m}{b}}$, $B = \frac{1}{b}$, m is the median life span, x is age, b is the dispersion coefficient and θ is the component of death attributable to accidents. In our numerical computations, we ignore

θ. One can see that the rate of health status decline is an increasing function of time causing the previously defined health insurance multiplier α to decrease with age.

For convenience of notation we will replace the conditional expectation $E[\cdot|W_t = w]$ by $E_{t,w}[\cdot]$. We can now arrive at our main problem's value function using the dynamic programming principle. First, after applying expectations to the indicator functions in equations (3.2.9a) to (3.2.9d), we obtain the following probabilistic expression:

$$J(t,w) \geq \int_t^{t+h} e^{-\rho s} u(c_s) \left(\int_t^s Pr(\mathbf{T}_{0,x} > s-v) Pr(\mathbf{T}_{1,x} > v-t) \eta_{x+v} Pr(\mathbf{T}_x^{hl} > v) dv \right) ds \tag{3.2.19a}$$

$$+ \int_t^{t+h} e^{-\rho s} u(c_s) Pr(\mathbf{T}_{1,x} > s-t) Pr(\mathbf{T}_x^{hl} > s-t) ds \tag{3.2.19b}$$

$$+ E_{t,w} \left[\int_{t+h}^{\infty} e^{-\rho s} u(c_s) Pr(\mathbf{T}_{1,x} > h) Pr(\mathbf{T}_x^{hl} > h) ds \right] \tag{3.2.19c}$$

$$+ E_{t,w} \left[\int_{t+h}^{\infty} e^{-\rho s} u(c_s) \left(\int_t^{t+h} Pr(\mathbf{T}_{0,x} > t+h-v) Pr(\mathbf{T}_{1,x} > v-t) \eta_{x+v} Pr(\mathbf{T}_x^{hl} > v) dv \right) ds \right]. \tag{3.2.19d}$$

Equation (3.2.19a) describes that the individual has survived $(v-t)$ while healthy and $(s-v)$ while sick with a transitional probability from healthy to sick given by the probability density function of the remaining healthy lifetime random variable $f_x(v) = \eta_{x+v}(1 - F_x(v))$. Mathematically we write:

$$\int_t^s Pr(\mathbf{T}_{0,x} > s-v) Pr(\mathbf{T}_{1,x} > v-t) \eta_{x+v} Pr(\mathbf{T}_x^{hl} > v) dv \tag{3.2.20}$$

$$= \int_t^s (e^{-\int_0^{s-v} \lambda_{x+t+u}^{(0)} du})(e^{-\int_0^{v-t} \lambda_{x+t+u}^{(1)} du}) f_x(v) dv. \tag{3.2.21}$$

Equation (3.2.19b) says that the investor remains healthy and alive throughout the interval $[t, t+$

h):

$$\int_t^s Pr(\mathbf{T}_{1,x} > v - t) Pr(\mathbf{T}_x^{hl} > v - t) dv = Pr(\mathbf{T}_{1,x} > s - t) Pr(\mathbf{T}_x^{hl} > s - t) \tag{3.2.22}$$

$$= (e^{-\int_0^{s-t} \lambda^{(1)}_{x+t+u} du})(e^{-\int_0^{s-t} \eta_{x+t+u} du}). \tag{3.2.23}$$

Equations (3.2.19c) and (3.2.19d) describe the states of the individual as it enters the time interval $[t + h, \infty)$ which is conditional on his state in the previous interval $[t, t + h)$. We then summarize the problem as follows:

$$J(t, w) \geq \int_t^{t+h} e^{-\rho s} u(c_s) \left(\int_t^s \eta_{v+x} ({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{s-v}p^{(0)}_{x+t}) dv \right) ds \tag{3.2.24a}$$

$$+ \int_t^{t+h} e^{-\rho s} u(c_s)({}_{s-t}g_{x+t})({}_{s-t}p^{(1)}_{x+t}) ds \tag{3.2.24b}$$

$$+ E_{t,w} \left[\int_{t+h}^{\infty} e^{-\rho s} u(c_s)({}_h g_{x+t})({}_h p^{(1)}_{x+t}) ds \right] \tag{3.2.24c}$$

$$+ E_{t,w} \left[\int_{t+h}^{\infty} e^{-\rho s} u(c_s) \left(\int_t^{t+h} \eta_{v+x} ({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{t+h-v}p^{(0)}_{x+t}) dv \right) ds \right]. \tag{3.2.24d}$$

In the above inequality the following states are described. The term (3.2.24a) reflects that the individual is healthy and alive at time t, becomes sick but remains alive between (t, s) and remains sick during the time interval $[s, t + h]$. The term (3.2.24b) describes the case when the individual is healthy and alive at time t and remains in that state throughout the interval $[t, t + h)$. The term (3.2.24c) states that the individual is healthy and alive between $[t, t + h)$ and remains that way for the remaining time $[t + h, \infty)$. Finally, expression (3.2.24d) is the case of a healthy and alive individual at t, becoming sick between $[t, t + h)$ and remaining sick but alive for the remaining time interval $[t + h, \infty]$. Both expressions (3.2.24a) and (3.2.24d) use the PDF of the remaining healthy lifetime defined in equation (3.2.13).

We see that the expected utility at time $(t + h)$ is given by $E_{t,w}[J^{(0)}(t + h, \hat{W}_{t+h})]$ and $E_{t,w}[J(t + h, W_{t+h})]$ if the individual is in poor and good health respectively. If the retiree starts out in poor health (superscript (0)), we assume that he does not jump back to a better health status. The optimization for this case is:

$$J^{(0)}(t, W_t) = \max_{\{\pi, c_t, I_t\}} E\left[\int_t^{\infty} e^{-\int_0^v (\lambda^{(0)}_{x+s}+\rho)ds} u(c_v)dv \middle| W_t = w\right]. \tag{3.2.25}$$

We see that the value function must satisfy:

$$\begin{aligned} J(t,w) \geq \int_t^{t+h} e^{-\rho s} u(c_s) & \left(\int_t^s \eta_{v+x} ({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{s-v}p^{(0)}_{x+t})dv\right) ds \\ & + \int_t^{t+h} e^{-\rho s} u(c_s)({}_{s-t}g_{x+t})({}_{s-t}p^{(1)}_{x+t})ds \\ & + E_{t,w}\left[J(t+h, W_{t+h})({}_h g_{x+t})({}_h p^{(1)}_{x+t})\right] \\ + E_{t,w}\left[J^{(0)}(t+h, \hat{W}_{t+h})\right. & \left.\left(\int_t^{t+h} \eta_{v+x}({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{t+h-v}p^{(0)}_{x+t})dv\right)\right]. \end{aligned} \tag{3.2.26}$$

Applying Itô's Lemma and rearranging 3.2.26:

$$\begin{aligned} J(t,w)(1 - ({}_h g_{x+t})({}_h p^{(1)}_{x+t})) \geq & \\ \int_t^{t+h} e^{-\rho s} u(c_s) & \left(\int_t^s \eta_{v+x+t}({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{s-v}p^{(0)}_{x+t})dv\right) ds \\ & + \int_t^{t+h} e^{-\rho s} u(c_s)({}_{s-t}g_{x+t})({}_{s-t}p^{(1)}_{x+t})ds \\ & + E_{t,w}\left[({}_h g_{x+t})({}_h p^{(1)}_{x+t}) \int_t^{t+h} (\mathscr{L} J_s) ds\right] \\ + E_{t,w}\left[J^{(0)}(t+h, \hat{W}_{t+h})\right. & \left.\left(\int_t^{t+h} \eta_{v+x+t}({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{t+h-v}p^{(0)}_{x+t})dv\right)\right], \end{aligned} \tag{3.2.27}$$

where

$$\mathscr{L} J_s = \partial_s J + \partial_w J[\pi w(\mu - r) + rw - c - I] + \frac{(\pi \sigma w)^2}{2} \partial_w^2 J.$$

Next, we divide both sides of equation (3.2.27) by h and take the limit as $h \to 0$. The following approximations $\lim_{h\to 0} {}_h g_{x+t} \to 1$ and $\lim_{h \to 0} {}_h p_{x+t} \to 1$ are obtained at the leading term. The

Fundamental Theorem of Calculus, $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)ds = f(t)$, is also applied. As a result, the following limits as $h \to 0$ are obtained:

$$\frac{1-({}_hg_{x+t})({}_hp^{(1)}_{x+t})}{h} \quad\to\quad \lambda^{(1)}_{x+t}+\eta_{x+t}, \tag{3.2.28a}$$

$$\frac{1}{h}\int_t^{t+h} e^{-\rho s}u(c_s)({}_{s-t}g_{x+t})({}_{s-t}p^{(1)}_{x+t})ds \quad\to\quad e^{-\rho t}u(c_t), \tag{3.2.28b}$$

$$\frac{1}{h}\int_t^{t+h} e^{-\rho s}u(c_s)\left(\int_t^s \eta_{v+x}({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{s-v}p^{(0)}_{x+t})dv\right)ds \quad\to\quad 0, \tag{3.2.28c}$$

$$\frac{1}{h}\int_t^{t+h} \eta_{v+x}({}_{v-t}g_{x+t})({}_{v-t}p^{(1)}_{x+t})({}_{t+h-v}p^{(0)}_{x+t})dv \quad\to\quad \eta_{x+t}, \tag{3.2.28d}$$

$$({}_hg_{x+t})({}_hp^{(1)}_{x+t})\frac{1}{h}\int_t^{t+h}(\mathscr{L}J_s)ds \quad\to\quad \mathscr{L}J_t. \tag{3.2.28e}$$

We note that $\hat{W}_{t+0}$ denotes the wealth level after the jump, which is assumed to be random, as the cost of health care related to the jump in health status is uncertain. However, after the jump, the wealth follows the same diffusion process.

We assume that the distribution of the health cost ν is known, with PDF denoted by ϕ_ν. In this case we have:

$$E_{t,w}\left[J^{(0)}(t,\hat{W}_{t+0})\right] = \int_\nu J^{(0)}(t, w-\nu+\alpha I)\phi_\nu d\nu. \tag{3.2.29}$$

With the above approximations, equation (3.2.27) becomes:

$$(\lambda^{(1)}_{x+t}+\eta_{x+t})J \geq e^{-\rho t}u(c) + \mathscr{L}J_t + \eta_{x+t}\int_\nu J^{(0)}(t, w-\nu+\alpha I)\phi_\nu d\nu, \tag{3.2.30}$$

where

$$\mathscr{L}J_t = \partial_t J + \partial_w J[\pi w(\mu - r) + rw - c - I] + \frac{(\pi\sigma w)^2}{2}\partial_w^2 J. \tag{3.2.31}$$

The equality holds when we take the optimal control:

$$(\lambda^{(1)}_{x+t}+\eta_{x+t})J = \max_{\{c,\pi,I,\}}\left\{e^{-\rho t}u(c) + \mathscr{L}J_t + \eta_{x+t}\int_\nu J^{(0)}(t, w-\nu+\alpha I)\phi_\nu d\nu\right\}. \tag{3.2.32}$$

As we can see, the solution J of equation (3.2.32) depends on the solution $J^{(0)}$, the value function corresponding to 'poor' health. The HJB for $J^{(0)}$ can be derived in a similar fashion as:

$$\lambda^{(0)}_{x+t} J^{(0)} = \max_{\{c,\pi\}} \left\{ e^{-\rho t} u(c) + \mathscr{L} J^{(0)}_t \right\}, \tag{3.2.33}$$

where

$$\mathscr{L} J^{(0)}_t = \partial_t J^{(0)} + \partial_w J^{(0)} [\pi w(\mu - r) + rw - c] + \frac{(\pi \sigma w)^2}{2} \partial^2_w J^{(0)}. \tag{3.2.34}$$

The optimal consumption rate c^* and asset allocation π^* are given by the first-order conditions:

$$\begin{aligned} e^{-\rho t} u'(c^*) - J_w &= 0, && (3.2.35) \\ \pi^* \sigma^2 w^2 J_{ww} + (\mu - r) w J_w &= 0. && (3.2.36) \end{aligned}$$

which are applicable to both $J^{(0)}$ and J. From this point on, we will use a Constant Relative Risk-Aversion (CRRA) utility, $u(c) = c^{1-\gamma}/(1-\gamma)$. When substituting the expressions for c^* and π^* into the HJB (3.2.33), we obtain the following highly nonlinear partial differential equation (PDE):

$$\lambda^{(0)}_{x+t} J^{(0)} = \partial_t J^{(0)} + rw \partial_w J^{(0)} + \frac{\gamma}{1-\gamma} e^{-\frac{\rho t}{\gamma}} (\partial_w J^{(0)})^{1-\frac{1}{\gamma}} - \frac{(\mu - r)^2}{2\sigma^2} \frac{(\partial_w J^{(0)})^2}{\partial^2_w J^{(0)}}, \tag{3.2.37}$$

which has a terminal condition $J^{(0)}(T, T) = 0$. To find the optimal insurance I^* from equation (3.2.32), the following first order condition needs to be satisfied:

$$\alpha \eta_{x+t} \int_v J^{(0)}_w (t, w - v + \alpha I^*) \phi_v dv - J_w(t, w) = 0. \tag{3.2.38}$$

The optimal consumption and asset allocation conditions are standard. The condition for optimal insurance, on the other hand, is quite different. It is an implicit equation which involves both differentiation and integration. In general, we anticipate that the solution has to be obtained numerically. Before we develop numerical algorithms to solve the optimal control problem, the distribution of health care cost needs to be modelled.

3.2.2 Solution Methodology for $J^{(0)}(t, w)$

Under CRRA utility, we can obtain $J^{(0)}(t, w)$ of equation (3.2.33) explicitly. In particular, we assume it has the following form:

$$J^{(0)}(t, w) = h(t; T)\frac{w^{1-\gamma}}{1-\gamma}, \tag{3.2.39}$$

where $h(t; T)$ will satisfy the following ordinary differential equation (ODE):

$$\frac{h'}{h} + \gamma(e^{\rho t}h)^{\frac{-1}{\gamma}} + C - \lambda^{(0)}_{x+t} = 0, \tag{3.2.40}$$

where

$$C = \frac{(\mu - r)^2(1-\gamma)}{2\sigma^2\gamma} + r(1-\gamma), \tag{3.2.41}$$

$$\lambda^{(0)}_{x+t} = \theta + Ae^{Bt}. \tag{3.2.42}$$

As previously described, the hazard rate is associated with the Gompertz-Makeham (GM) distribution and $A = \frac{\beta_0}{b}e^{\frac{x-m}{b}}$, $B = \frac{1}{b}$, where m is the median life span, x is age, b is the dispersion coefficient and θ is the component of death attributable to accidents. The factor β_0 is the weight in case 1 (3.2.16) or case 2 (3.2.17).

We aim to find the solution of equation (3.2.40) with terminal condition $h(T; T) = 0$. We let $h = z^{-\gamma}$. Then with the substitution $h' = -\gamma z^{-\gamma-1}z$ equation (3.2.40) reduces to:

$$z' - e^{-\frac{\rho t}{\gamma}}z^2 + (\frac{-C + \lambda^{(0)}_{x+t}}{\gamma})z = 0. \tag{3.2.43}$$

Then we let $\psi = z^{-1}$ and equation (3.2.43) becomes:

$$\psi' - f(t)\psi + g(t) = 0, \tag{3.2.44}$$

where

$$g(t) = e^{-\frac{\rho t}{\gamma}}, \tag{3.2.45}$$

$$f(t) = \frac{-C + \lambda_{x+t}^{(0)}}{\gamma}. \tag{3.2.46}$$

We multiply both sides of equation (3.2.44) by the integrating factor $\zeta = e^{\int -f(t)dt} = e^{\frac{Ct}{\gamma} - \frac{Ae^{Bt}}{B\gamma}}$. We follow this operation by integrating both sides from time t to terminal time T:

$$\int_t^T d(\psi(s)e^{\frac{Cs}{\gamma} - \frac{Ae^{Bs}}{B\gamma}}) = -\int_t^T e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds. \tag{3.2.47}$$

Using the terminal condition $\psi(T;T) = 0$, we obtain an explicit solution of equation (3.2.44):

$$\psi(t;T) = \left(\int_t^T e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds\right) e^{-\frac{Ct}{\gamma} + \frac{Ae^{Bt}}{B\gamma}}. \tag{3.2.48}$$

We obtain the solution of the ODE (3.2.40) through the substitution $h(t;T) = (\psi(t;T))^\gamma$:

$$h(t;T) = \left(\int_t^T e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds\right)^\gamma e^{-Ct + \frac{Ae^{Bt}}{B}}. \tag{3.2.49}$$

Proof. We show that the solution of the ODE (3.2.40) is given by (3.2.49). It can be easily verified that by applying the Leibniz Rule we obtain:

$$\frac{d}{dt}\left(\int_t^T e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds\right) = -e^{\frac{C-\rho}{\gamma}t - \frac{Ae^{Bt}}{B\gamma}}. \tag{3.2.50}$$

After some algebraic manipulations, we also obtain:

$$\frac{h'}{h} = \frac{-\gamma e^{\frac{C-\rho}{\gamma}t - \frac{Ae^{Bt}}{B\gamma}}}{\left(\int_t^T e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds\right)} + Ae^{Bt} - C. \tag{3.2.51}$$

Moreover,

$$\gamma e^{-\frac{\rho t}{\gamma}} h^{-\frac{1}{\gamma}} = \gamma e^{-\frac{\rho t}{\gamma}} \left(\int_t^T e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds \right)^{-1} e^{\frac{Ct}{\gamma} - \frac{Ae^{Bt}}{B\gamma}}. \tag{3.2.52}$$

Substituting (3.2.51) and (3.2.52) into the ODE (3.2.40), we obtain the desired conclusion that equation (3.2.49) is indeed the correct solution.

□

Since equation (3.2.49) contains a non-elementary integral, we will approximate it numerically by Simpson's Rule. This we do by using the MATLAB defined function *quad* which approximates the integral function within an error of 10^{-6} by applying a recursive adaptive Simpson quadrature algorithm.

We can write the solution (3.2.49) in terms of the incomplete gamma function. A series of mathematical manipulations with a change of variable give the desired form. We write the nonelementary integral function as:

$$\begin{aligned}
\int_t^T e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds &= \int_t^\infty e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds - \int_T^\infty e^{\frac{C-\rho}{\gamma}s - \frac{Ae^{Bs}}{B\gamma}} ds && (3.2.53)\\
&= \int_t^\infty e^{-\hat{a}s - \hat{b}e^{Bs}} ds - \int_T^\infty e^{-\hat{a}s - \hat{b}e^{Bs}} ds && (3.2.54)\\
&= \frac{1}{B(\hat{b})^{\frac{-\hat{a}}{B}}} \left(\int_{\hat{b}e^{Bt}}^\infty w^{-\frac{\hat{a}}{B}-1} e^{-w} dw - \int_{\hat{b}e^{BT}}^\infty w^{-\frac{\hat{a}}{B}-1} e^{-w} dw \right) && (3.2.55)\\
&= B^{-1} \left(\frac{A}{B\gamma} \right)^{\frac{\rho - C}{B\gamma}} \left(\Gamma\left(\frac{\rho - C}{B\gamma}, \frac{Ae^{Bt}}{B\gamma} \right) - \Gamma\left(\frac{\rho - C}{B\gamma}, \frac{Ae^{BT}}{B\gamma} \right) \right), && (3.2.56)
\end{aligned}$$

where $\hat{a} = \frac{\rho - C}{\gamma}$ and $\hat{b} = \frac{A}{B\gamma}$. Moreover, we also used the change of variable $w = \hat{b}e^{Bs}$ and the incomplete gamma function $\Gamma(n, c) = \int_c^\infty e^{-t} t^{-n-1} dt$. Using expression (3.2.56), we rewrite

(3.2.49):

$$h(t;T) = \left(e^{-Ct+\frac{Ae^{Bt}}{B}}\right) \cdot B^{-\gamma} \left(\frac{A}{B\gamma}\right)^{\frac{\rho-C}{B}} \left(\Gamma\left(\frac{\rho-C}{B\gamma}, \frac{Ae^{Bt}}{B\gamma}\right) - \Gamma\left(\frac{\rho-C}{B\gamma}, \frac{Ae^{BT}}{B\gamma}\right)\right)^{\gamma}. \quad (3.2.57)$$

As an additional check for our calculations, we solve the following transformed ODE (3.2.40) numerically as well, with a first order backward time discretization:

$$u' + \frac{C - Ae^{Bt}}{\gamma} u + e^{-\frac{\rho t}{\gamma}} = 0; \quad u(T;T) = 0, \quad (3.2.58)$$

where $h = u^{\gamma}$. We expect the numerical scheme solution to converge to the explicit solution (3.2.49). The results are discussed in Section 3.4.3.

Note that if we ignore the instantaneous force of mortality, equation (3.2.49) reduces to the result observed by Huang and Milevsky (2008):

$$h(t;T) = \frac{e^{-\rho t}(e^{\xi(T-t)} - 1)^{\gamma}}{\xi^{\gamma}}, \quad (3.2.59)$$

where $\xi = \frac{C-\rho}{\gamma}$. This result will provide a benchmark for our solutions with or without mortality risk considerations.

Finally, the value function $J^{(0)}$ is given by:

$$\begin{aligned} J^{(0)}(t,w) &= \left(e^{-Ct+\frac{Ae^{Bt}}{B}}\right) \cdot B^{-\gamma} \left(\frac{A}{B\gamma}\right)^{\frac{\rho-C}{B}} \\ &\cdot \left(\Gamma\left(\frac{\rho-C}{B\gamma}, \frac{Ae^{Bt}}{B\gamma}\right) - \Gamma\left(\frac{\rho-C}{B\gamma}, \frac{Ae^{BT}}{B\gamma}\right)\right)^{\gamma} \frac{w^{1-\gamma}}{1-\gamma}. \end{aligned} \quad (3.2.60)$$

In the numerical results section, we compare our numerical scheme for solving equation (3.2.58) with the approximation of the explicit solution (3.2.60).

3.2.3 Solution Methodology for $J(t, w)$

In this section, we examine a special case, where the health cost amplitude is uniformly distributed. We also make the assumption that the optimal insurance is proportional to the wealth.

To illustrate how the first order condition for the health insurance can be utilized, we consider a special case where the health cost v has uniform distribution within the interval $[0, v_m]$. The probability density function (PDF) of the amplitude is $\phi_v = \frac{1}{v_m}$. We also assume that the health cost is capped by the wealth w, the first order condition for the optimal insurance premium I^* can be written:

$$\frac{\alpha\eta_{x+t}}{v_m}\int_0^w J_w^{(0)}(t, w - v + \alpha I^*)dv = J_w(t, w), \tag{3.2.61}$$

which can be simplified as

$$\frac{\alpha\eta_{x+t}}{v_m}(J^{(0)}(t, w + \alpha I^*) - J^{(0)}(t, \alpha I^*)) = J_w(t, w). \tag{3.2.62}$$

This is still an implicit equation of I^* as a function of w but is much simpler than the general condition. For other health cost distributions, however, numerical integration will be required to find I^*.

Next, we show that when $v_m = w$, a closed form solution for $J(t, w)$ can also be obtained. To this end, for CRRA utility, we assume the analytic form:

$$J(t, w) = k(t; T)\frac{w^{1-\gamma}}{1-\gamma}. \tag{3.2.63}$$

Then we have $\partial_w J = kw^{-\gamma}$, $\partial_w^2 J = -\gamma k w^{-\gamma-1}$ and $\partial_t J = k' \frac{w^{1-\gamma}}{1-\gamma}$. We also assume that the optimal insurance premium is proportional to the wealth. We can write $I^* = p^*(t)w$, where $p^*(t)$ is a function of time only, satisfying the first order condition:

$$\frac{\alpha\eta_{x+t}}{v_m}(J^{(0)}(t, w + \alpha p w) - J^{(0)}(t, \alpha p w)) = J_w(t, w). \tag{3.2.64}$$

Substituting $J^{(0)}(t, w) = h(t; T)\frac{w^{1-\gamma}}{1-\gamma}$ in equation (3.2.64) we obtain an implicit equation for $p(t)$:

$$(1 + \alpha p)^{1-\gamma} - (\alpha p)^{1-\gamma} = \frac{k(t; T)(1-\gamma)}{\alpha\eta_{x+t}h(t; T)}. \tag{3.2.65}$$

We substitute the controls $c^* = e^{-\frac{\rho t}{\gamma}} k^{-\frac{1}{\gamma}} w$, $\pi^* = \frac{\mu - r}{\sigma^2 \gamma}$ and $I^* = p^*(t) w$ into equation (3.2.32) and obtain:

$$(\lambda_{x+t}^{(1)} + \eta_{x+t}) J = e^{-\rho t} u(c^*) + \mathscr{L}^* J_t + \eta_{x+t} \int_\nu J^{(0)}(t, w - \nu + \alpha I^*) \phi_\nu d\nu, \tag{3.2.66}$$

where

$$\mathscr{L}^* J_t = \partial_t J + \partial_w J [\pi^* w(\mu - r) + rw - c^* - I^*] + \frac{(\pi^* \sigma w)^2}{2} \partial_w^2 J. \tag{3.2.67}$$

Making use of $J(t, w) = k(t; T) \frac{w^{1-\gamma}}{1-\gamma}$, equation (3.2.66) becomes:

$$\begin{aligned} \lambda_{x+t}^{(1)} + \eta_{x+t} &= \frac{k'}{k} + \gamma e^{-\frac{\rho t}{\gamma}} k^{-\frac{1}{\gamma}} + \frac{(\mu - r)^2 (1 - \gamma)}{2\sigma^2 \gamma} + (r - p)(1 - \gamma) \\ &+ \frac{\eta_{x+t} h(t; T)}{(2 - \gamma) k(t; T)} [(1 + \alpha p)^{2-\gamma} - (\alpha p)^{2-\gamma}]. \end{aligned} \tag{3.2.68}$$

Equations (3.2.65) and (3.2.68) form a system of coupled differential-algebraic equations (DAEs), which can be solved numerically for the two unknowns $k(t; T)$ and $p(t)$. By letting $k = v^\gamma$ and using the optimal controls, we rewrite the DAE system as follows:

$$\begin{cases} (\mu - r) - \gamma \pi^* \sigma^2 = 0, \\ (1 + \alpha p^*)^{1-\gamma} - (\alpha p^*)^{1-\gamma} = \dfrac{v^\gamma (1 - \gamma)}{\alpha h \eta_{x+t}}, \\ v' + v \dfrac{C_1(p^*, \pi^*) + D(p^*) - \lambda_{x+t}^{(1)} - \eta_{x+t}}{\gamma} + e^{-\frac{\rho t}{\gamma}} = 0, \quad v(T; T) = 0, \end{cases} \tag{3.2.69}$$

where:

$$D(p^*) = \frac{((1 + \alpha p^*)^{2-\gamma} - (\alpha p^*)^{2-\gamma})(1 - \gamma)}{\alpha((1 + \alpha p^*)^{1-\gamma} - (\alpha p^*)^{1-\gamma})(2 - \gamma)}, \tag{3.2.70}$$

$$C_1(p^*, \pi^*) = (\pi^*(\mu - r) + r - \frac{\gamma (\sigma \pi^*)^2}{2} - p^*)(1 - \gamma). \tag{3.2.71}$$

3.3 A Jump-Diffusion Investment Model under Health Shocks

Let S_t be the stock price which follows a log-normal jump-diffusion SDE:

$$dS_t = S_t(\mu dt + \sigma dB_t + \nu_1 dP_t), \quad S(0) = S_0. \tag{3.3.1}$$

Here, μ is the mean return rate (constant drift coefficient), σ is the diffusive volatility and B_t is the continuous Brownian motion process. The only difference between equation (3.3.1) and the simple diffusion stock price dynamics considered previously, is the random jump-amplitude $\nu_1(q)$ and the discontinuous one dimensional Poisson process P_t with a constant jump-rate λ_J. We assume that the random jump-amplitude $\nu_1(q)$ depends on the distribution of the jump-mark variable q which is assumed to be normally distributed with mean μ_q and variance σ_q^2. Before discussing our choice of mark distribution we will transform the stock price SDE to an SDE of log-returns:

$$d(\ln S_t) = \mu dt + \sigma dB_t + \nu_1 dP_t. \tag{3.3.2}$$

By letting $F(S_t) = \ln S_t$ and assuming that it is twice differentiable in S_t, we apply the following Itô stochastic chain rule for jump-diffusion processes:

$$\begin{aligned} dF(S_t) &= (S_t\mu\partial_s F(S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_s^2 F(S_t))dt + (S_t\sigma\partial_s F(S_t))dB_t \\ &+ (F(S_t + \nu_1 S_t) - F(S_t))dP_t. \end{aligned} \tag{3.3.3}$$

Since $\partial_s F(S_t) = \frac{1}{S_t}$ and $\partial_s^2 F(S_t) = -\frac{1}{S_t^2}$, we obtain:

$$dF(S_t) = (\mu - \frac{\sigma^2}{2})dt + \sigma dB_t + \ln(1 + \nu_1(q)). \tag{3.3.4}$$

The log-return jump-amplitude is then the logarithm of the relative post-jump amplitude given by $q = \ln(1 + \nu_1(q))$. The inverse is given by $\nu_1(q) = e^q - 1$. The probability density function for the jump-marks q is then given by:

$$\phi_q = \frac{e^{\frac{-(q-\mu_q)^2}{2\sigma_q^2}}}{\sqrt{2\pi\sigma_q^2}}. \tag{3.3.5}$$

The retiree also invests in the risk-free asset which is modelled as before:

$$dN_t = rN_t dt, \quad N(0) = N_0. \tag{3.3.6}$$

In turn, the wealth dynamics is given by:

$$dW_t = \begin{cases} \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt + v_1 W_t dP_t - c_t dt - I_t dt; & \tau_s > t, \\ \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt + v_1 W_t dP_t - c_t dt; & \tau_s \le t, \end{cases} \tag{3.3.7}$$

Moreover, if v_2 is the wealth jump-amplitude due to the health shock, we recall that the change in wealth is given by:

$$W_{\tau_s^+} = W_{\tau_s^-} + \alpha I_t - v_2. \tag{3.3.8}$$

Deriving the HJB equation for this problem is done as in Section 2.2 with the exception that we now apply the Itô stochastic chain rule as in Hanson (2007). We obtain the following:

$$(\lambda_{x+t}^{(1)} + \eta_{x+t})J = e^{-\rho t}u(c^*) + \mathscr{A}^* J_t + \eta_{x+t} \int_{v_2} J^{(0)}(t, w - v_2 + \alpha I^*)\phi_{v_2} dv_2, \tag{3.3.9}$$

where

$$\begin{aligned} \mathscr{A}^* J_t &= \partial_t J + \partial_w J[\pi^* w(\mu - r) + rw - c^* - I^*] + \frac{(\pi^* \sigma w)^2}{2}\partial_w^2 J \\ &+ \lambda_J \int_{-\infty}^{\infty} \{J(t, (1 + v_1\pi^*)w) - J(t, w)\}\phi_q dq. \end{aligned} \tag{3.3.10}$$

As before, the solution to equation (3.3.9) depends on the value function of the sick individual given by:

$$(\lambda_{x+t}^{(0)})J^{(0)} = e^{-\rho t}u(c^*) + \mathscr{A}_0^* J_t^{(0)}, \tag{3.3.11}$$

where

$$\begin{aligned}\mathscr{A}_0^* J_t^{(0)} &= \partial_t J^{(0)} + \partial_w J^{(0)}[\pi^* w(\mu - r) + rw - c^*] + \frac{(\pi^* \sigma w)^2}{2} \partial_w^2 J^{(0)} \\ &+ \lambda_J \int_{-\infty}^{\infty} \{J^{(0)}(t, (1 + \nu_1 \pi^*)w) - J^{(0)}(t, w)\} \phi_q dq. \end{aligned} \tag{3.3.12}$$

We require that the wealth $w > 0$, which means that for $0 < \pi < 1$, the jump-amplitude is $1 + \nu_1 \geq 0$. We will choose $-1 < \nu_1 < \infty$ such that a jump does not cause the underlying to vanish. Since $\nu_1 > -1$ we can let the jump-mark process be the log-return jump-amplitude $q = \ln(1 + \nu_1(q))$. We proceed in solving the value function of the sick individual as follows in the next section.

3.3.1 Solution Methodology for $J^{(0)}(t, w)$: Jump-Diffusion Setup

We assume as before that $J^{(0)} = h(t; T)\frac{w^{1-\gamma}}{1-\gamma}$, which leads to the following optimal controls:

$$c^* = e^{-\frac{\rho t}{\gamma}} h^{-\frac{1}{\gamma}} w, \tag{3.3.13}$$

$$\gamma \pi^* \sigma^2 - \lambda_J \int_{-\infty}^{\infty} \nu_1 (1 + \nu_1 \pi^*)^{-\gamma} \phi_q dq = (\mu - r). \tag{3.3.14}$$

As we showed, the optimal insurance allocation fraction π^* is the solution of an implicit integro-algebraic equation (3.3.14). To find the value function we now need to solve the following system of coupled integro-differential-algebraic equations (IDAEs):

$$\begin{cases} (\mu - r) - \gamma \pi^* \sigma^2 + \lambda_J \int_{-\infty}^{\infty} \nu_1 (1 + \nu_1 \pi^*)^{-\gamma} \phi_q dq = 0, \\ \frac{h'}{h} + \gamma e^{-\frac{\rho t}{\gamma}} h^{-\frac{1}{\gamma}} + \zeta(\pi^*) + \lambda_J \int_{-\infty}^{\infty} [(1 + \nu_1 \pi^*)^{1-\gamma} - 1] \phi_q dq - \lambda_{x+t}^{(0)} = 0, \end{cases} \tag{3.3.15}$$

where

$$\zeta(\pi^*) = [\pi^*(\mu - r) + r](1 - \gamma) - \frac{\gamma(1 - \gamma)\sigma^2 (\pi^*)^2}{2}. \tag{3.3.16}$$

First, we apply the following change of variable $z = \frac{q-\mu_q}{\sqrt{2}\sigma_q}$ in the integrals. We rewrite them as follows:

$$I_1(\pi^*) = \int_{-\infty}^{\infty} \nu_1(1+\nu_1\pi^*)^{-\gamma}\phi_q dq \tag{3.3.17}$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(z,\pi^*)e^{-z^2}dz, \tag{3.3.18}$$

where $f(z,\pi^*) = [(1-\pi^*)+\pi^* e^{\mu_q+\sigma_q\sqrt{2}z}]^{-\gamma}(e^{\mu_q+\sigma_q\sqrt{2}z}-1)$. It is to be noted here that in this section π in equation (3.3.18) should not be confused with the allocation ratio π^*. We also write:

$$I_2(\pi^*) = \int_{-\infty}^{\infty}[(1+\nu_1\pi^*)^{1-\gamma}-1]\phi_q dq \tag{3.3.19}$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} g(z,\pi^*)e^{-z^2}dz, \tag{3.3.20}$$

where $g(z,\pi^*) = [(1-\pi^*)+\pi^* e^{\mu_q+\sigma_q\sqrt{2}z}]^{1-\gamma}-1$.

The new integral forms allow us to apply a variation of the Gauss-Hermite quadrature with three nodes $\{z_1,z_2,z_3\} = \{-\sqrt{3},0,\sqrt{3}\}$ and weights $\{w_1,w_2,w_3\} = \{\frac{1}{6},\frac{2}{3},\frac{1}{6}\}$. In Hanson and Westman (2002) it was observed that this choice of nodes and weights results in a fifth degree polynomial precision. The final integral approximations are:

$$I_1(\pi^*) \approx \sum_i^3 w_i f(z_i,\pi^*), \tag{3.3.21}$$
$$I_2(\pi^*) \approx \sum_i^3 w_i g(z_i,\pi^*). \tag{3.3.22}$$

After applying the three point Gauss-Hermite quadrature we reduce the system (3.3.15) to:

$$\begin{cases} (\mu-r)-\gamma\pi^*\sigma^2+\frac{\lambda_J}{\sqrt{\pi}}\{\frac{1}{6}f(-\sqrt{3},\pi^*)+\frac{2}{3}f(0,\pi^*)+\frac{1}{6}f(\sqrt{3},\pi^*)\} & = 0, \\ \frac{h'}{h}+\gamma e^{-\frac{\rho t}{\gamma}}h^{-\frac{1}{\gamma}}+\zeta(\pi^*)+\frac{\lambda_J}{\sqrt{\pi}}\{\frac{1}{6}g(-\sqrt{3},\pi^*)+\frac{2}{3}g(0,\pi^*)+\frac{1}{6}g(\sqrt{3},\pi^*)\}-\lambda_{x+t}^{(0)} & = 0. \end{cases} \tag{3.3.23}$$

A further transformation $h = u^{\gamma}$ takes us to the following system, which was then implemented numerically with a first order forward discretization of time:

$$\begin{cases} (\mu - r) - \gamma \pi^* \sigma^2 + \dfrac{\lambda_J}{\sqrt{\pi}} F(\pi^*) = 0, \\ \\ u' + uR(\pi^*, t) + e^{-\frac{\rho t}{\gamma}} = 0, \quad u(T;T) = 0, \end{cases} \tag{3.3.24}$$

where

$$F(\pi^*) = \frac{1}{6} f(-\sqrt{3}, \pi^*) + \frac{2}{3} f(0, \pi^*) + \frac{1}{6} f(\sqrt{3}, \pi^*), \tag{3.3.25}$$

$$R(\pi^*, t) = \frac{\zeta(\pi^*) + \frac{\lambda_J}{\sqrt{\pi}}\{\frac{1}{6} g(-\sqrt{3}, \pi^*) + \frac{2}{3} g(0, \pi^*) + \frac{1}{6} g(\sqrt{3}, \pi^*)\} - \lambda^{(0)}_{x+t}}{\gamma}. \tag{3.3.26}$$

We note that the optimal risky asset allocation does not depend on the health status, mortality or health risk of the investor. We solved the optimal allocation for a sick individual by both the Gauss-Hermite approximation (π^*) of the integral and by the MATLAB function *quadgk*, which is an adaptive Gauss-Kronrod quadrature approximation (π^*_{gk}) . The results are comparable and are summarized in Table 3.1. We notice that the pattern of investment is maintained i.e. the more risk-averse an individual is, the smaller the risky asset allocation. For a more risk-tolerant investor it is always optimal to invest all the wealth in the risky asset ($\pi^* = \pi^*_{gk} = 1$). For comparison purposes, we also included π^*_d, which is the allocation fraction when the returns follow a GBM process. Subscript d denotes diffusion. Here we took $\lambda_J = 0.006$ ($\lambda_J = 0$ for the pure diffusion case) and fixed $\mu = 0.07$, $\sigma = 0.2$ and $r = 0.03$.

3.3.2 Solution Methodology for $J(t, w)$: Jump-Diffusion Setup

In a similar fashion as the derivation of the value function of a sick individual, we can derive a system of IDAEs for a healthy investor. The only difference is due to the additional implicit

γ	8	5	3	1.5	0.5	0.2
π^*	0.1580	0.2500	0.4180	0.8380	1.0000	1.0000
π^*_{gk}	0.1575	0.2520	0.4200	0.8396	0.9991	1.0000
π^*_d	0.1562	0.2500	0.4167	0.8328	0.9990	1.0000

Table 3.1: Comparison of optimal allocation for the Gauss-Hermite and Gauss-Kronrod methods.

equation for the optimal health insurance. The optimal consumption is standard:

$$c^* \equiv c^*(t, w) = e^{-\frac{\rho t}{\gamma}} k^{-\frac{1}{\gamma}} w. \tag{3.3.27}$$

The system of IDAEs is more complex due to the introduction of the medical health insurance into the algebraic equation:

$$\begin{cases} (\mu - r) - \gamma \pi^* \sigma^2 + \lambda_J \int_{-\infty}^{\infty} \nu_1 (1 + \nu_1 \pi^*)^{-\gamma} \phi_q dq = 0, \\ (1 + \alpha p^*)^{1-\gamma} - (\alpha p^*)^{1-\gamma} = \frac{k(1-\gamma)}{h\alpha\eta_{x+t}}, \\ \frac{k'}{k} + \gamma e^{-\frac{\rho t}{\gamma}} k^{-\frac{1}{\gamma}} + \chi(\pi^*, p^*) + \lambda_J \int_{-\infty}^{\infty} [(1 + \nu_1 \pi^*)^{1-\gamma} - 1] \phi_q dq - \lambda^{(1)}_{x+t} - \eta_{x+t} = 0, \end{cases} \tag{3.3.28}$$

where

$$\chi(\pi^*, p^*) = [\pi^*(\mu - r) + r - \frac{\gamma \sigma^2 (\pi^*)^2}{2} - p^*](1 - \gamma) + \frac{\eta_{x+t} h}{(2 - \gamma) k} [(1 + \alpha p^*)^{2-\gamma} - (\alpha p^*)^{2-\gamma}]. \tag{3.3.29}$$

Furthermore, we make the substitution $k = v^\gamma$ and use the Gauss-Hermite approximation of the integrals, to arrive at a more manageable form which we later implement numerically:

$$\begin{cases} (\mu - r) - \gamma \pi^* \sigma^2 + \frac{\lambda_J}{\sqrt{\pi}} F(\pi^*) = 0, \\ C_3(p^*) = \frac{v^\gamma (1-\gamma)}{\alpha h \eta_{x+t}}, \\ v' + v \left(\frac{C_1(p^*, \pi^*) + \frac{C_2(p^*)(1-\gamma)}{C_3(p^*)\alpha(2-\gamma)} - \lambda^{(1)}_{x+t} - \eta_{x+t}}{\gamma} \right) + e^{-\frac{\rho t}{\gamma}} = 0, \quad v(T;T) = 0, \end{cases} \tag{3.3.30}$$

where

$$
\begin{aligned}
F(\pi^*) &= \frac{1}{6}f(-\sqrt{3},\pi^*) + \frac{2}{3}f(0,\pi^*) + \frac{1}{6}f(\sqrt{3},\pi^*), && (3.3.31)\\
C_1(p^*,\pi^*) &= [\pi^*(\mu - r) + r - \frac{\gamma\sigma^2(\pi^*)^2}{2} - p^*](1-\gamma) \\
&+ \frac{\lambda_J}{\sqrt{\pi}}\{\frac{1}{6}g(-\sqrt{3},\pi^*) + \frac{2}{3}g(0,\pi^*) + \frac{1}{6}g(\sqrt{3},\pi^*)\}, && (3.3.32)\\
C_2(p^*) &= (1+\alpha p^*)^{2-\gamma} - (\alpha p^*)^{2-\gamma}, && (3.3.33)\\
C_3(p^*) &= (1+\alpha p^*)^{1-\gamma} - (\alpha p^*)^{1-\gamma}. && (3.3.34)
\end{aligned}
$$

We solve for the optimal control variables numerically as shown in the following section.

3.4 Numerical Results

This section describes the numerical schemes applied in order to assess the solution of equation (3.2.60). We also describe the numerical methodology for finding the optimal insurance allocation proportion in the DAE (3.2.69), as well as the optimal consumption and allocation to risky investments.Moreover, we include the calibration method used to determine parameters for the case where the asset returns follow a GBM or a jump-diffusion process.

3.4.1 Numerical Scheme

In this subsection we assess the accuracy of our numerical schemes and other related results.

We start by presenting the case for a sick investor with GM mortality risk and the convergence of the time-dependent solution of the value function to the desired explicit solution. For the time domain $[0, T]$, where $T = 120 - x$, we calculate the time step $\Delta t = \frac{T}{N}$, where N is an arbitrarily large number. For the following numerical computations we chose $N = 1000$.

Equation (3.2.58) is an ODE which we solve with a semi-implicit discretization scheme for $n = 1, ..., N$:

$$u^n \;\; = \;\; u^{n+1}\left(\frac{\frac{1}{\Delta t} + \frac{C - Ae^{Bt(n+1)}}{2\gamma}}{\frac{1}{\Delta t} + \frac{C - Ae^{Bt(n+1)}}{2\gamma}}\right) + \left(\frac{1}{\frac{1}{\Delta t} + \frac{C - Ae^{Bt(n+1)}}{2\gamma}} e^{-\frac{\rho t(n+1)}{\gamma}}\right), \tag{3.4.1}$$

where $u = \frac{u^{n+1}+u^n}{2}$. We approximated the time derivative by a first order accurate backward finite difference numerical scheme. The unknown u^n is calculated from the known values u^{n+1}. The time-dependent function (3.2.49) is obtained by $h^n = (u^n)^\gamma$. To verify the numerical scheme , we compare the accuracy and convergence of the numerical result to the analytical solution. The error is computed as $Error = ||h_{explicit} - h_{analytic}||_\infty = \max|h_{explicit} - h_{analytic}|$. We observed that the numerical solution converged as we decreased the time step size by increasing N. We fixed the market parameters to typically observed values $\mu = 0.07$, $\sigma = 0.2$, $r = 0.01$. The mortality risk constants were taken to be $m = 86.3$, $b = 9.5$, $\beta_0 = 1.2$. The subjective discount rate was chosen to be $\rho = 0.02$. We only displayed one case for a risk preference of $\gamma = 3$, but similar results were observed for other degrees of risk aversion. Results are summarized in Table 3.2.

N	$Error$
10,000	0.1458
100,000	0.0147
200,000	0.0073

Table 3.2: Comparison between the numerical and analytic results.

Next, we turn our focus to the solution of the problem for a healthy investor with both mortality and health decline risk. We obtained numerical results for the optimal medical insurance allocation proportion p^* by solving the DAE system (3.2.69). First, we changed the time variable to $\tau = T - t$. The desired solution of the ODE is v^{n+1} for v^n given. Finally, one can obtain the time-dependent function $k(t, T)$ by applying the transformation $k^{n+1} = (v^{n+1})^\gamma$.

By making the above change of variable, we have transformed the backward ODE to a forward ODE. The semi-implicit numerical scheme can be summarized as follows:

$$\frac{v^{n+1} - v^n}{d\tau} = v^{n+1}\tilde{A}(T - \tau(n)) + \tilde{B}(T - \tau(n)), \tag{3.4.2}$$

where $\tau \in [\tau(n), \tau(n+1]$ for $n = 1, ..., N$. We initialized the optimum insurance allocation $p^*(1) = \epsilon \frac{1}{\alpha(T-\tau(1))}$ (for $\epsilon < 1$) and solved the discretized equation for $n = 2, ..., N$. The functions $\tilde{A}$ and $\tilde{B}$ are discretized functions of $p(n)^*$ and $T - \tau(n)$ appearing in the ODE. At time-nodes $n > 1$, we solved for $p^*(n+1)$ of the DAE by applying the MATLAB function *fzero*:

$$(1 + \alpha(n+1)p^*(n+1))^{1-\gamma} - (\alpha(n+1)p^*(n+1))^{1-\gamma} - \frac{(v^{n+1})^\gamma(1-\gamma)}{h^{n+1}} = 0, \tag{3.4.3}$$

where h^{n+1} is the time-dependent component of the sick investor's value function and was computed as described above.

For the results presented in the following analysis, the consumer was currently of age $x = 65$. We applied case 1 (3.2.16) and chose the force of mortality for the healthy investor to be $\lambda^{(1)} = 0.8\lambda^{(0)}$ and the rate of becoming sick to be $\eta = 0.2\lambda^{(0)}$. In other words, we are saying that the healthy consumer has a slower rate of dying and has a lower rate of becoming sick as compared to the rate of dying of a sick investor.

To examine the validity of our numerical solution, we chose to look at a case where αp is fixed as a constant smaller than the initial value $\alpha p(1) = 1$. The time-dependent solution k for this choice is expected to be different from the optimal solution. As can be seen in Table 3.3, for an initial choice of $\alpha p = 0.3$, the solution is smaller as compared to the optimal value for $\gamma < 1$, and larger than the optimal solution for $\gamma > 1$. We display only one result of this observation.

	$optimal$	$non-optimal$
$\gamma = 0.8$	9.086	9.084
$\gamma = 5$	$1.18e7$	$1.57e7$

Table 3.3: Assessment of the numerical optimal solution $k(t, T)$.

To be consistent with how insurance is discussed by industry practitioners, we chose to display (see Table 3.4) the actual face value proportion of the medical insurance αp^* relative to wealth. We present the results for different ages of the investor and two degrees of risk aversion.

$\sigma = 0.20$, $\mu = 0.07$, $\rho = 0.02$, $r = 0.03$				
Age	65	75	85	95
$\gamma = 0.5$				
αp^*	0.4658	0.4578	0.4500	0.4440
$\gamma = 5$				
αp^*	0.5946	0.5597	0.5220	0.4787

Table 3.4: Comparison of optimal medical insurance benefit proportion (αp^*) for $\gamma = 0.5$ and $\gamma = 5$.

3.4.2 Calibration of the Jump-Diffusion Process

We fit the jump-diffusion model to historical data by using the moment matching methodology developed in the previous chapter and unless otherwise specified, we keep the notation consistent.

As described in Section 3.3 , the jump-mark variable q is normally distributed, which differs from the distribution assumed in Chapter 2. Recall that the n^{th} moment formula is given by:

$$I_n = y^n e^{\int_t^T \gamma_n(s)ds}, \tag{3.4.4}$$

where

$$\gamma_n = \frac{(n+2)(n+1)\sigma^2}{2} + (n+1)(\mu - 2\sigma^2) + \sigma^2 - \mu + \lambda(\xi_n - 1). \tag{3.4.5}$$

Since we have assumed that the jump-marks are normally distributed, we have:

$$\xi_n = \int_{-\infty}^{\infty} e^{nq}\phi_q dq \tag{3.4.6}$$

$$= e^{\frac{n^2\sigma_q^2}{2}+n\mu_q}, \tag{3.4.7}$$

where $\phi_q = \frac{e^{\frac{-(q-\mu_q)^2}{2\sigma_q^2}}}{\sqrt{2\pi\sigma_q^2}}$. Without loss of generality we let $y = 1$.

We considered a data series of daily S&P 500 stock index prices from Bloomberg. We have chosen this more recent data set in order to capture the financial crash of 2007–2008. The observation period is December 2007 to February 2014 (1618 data points). We calibrate the GBM (pure diffusion process) as follows. We let $X_i = \ln\{\frac{SP_{i+1}}{SP_i}\}$, where SP_i denotes the daily index price. We know that:

$$E[X_i] = \left(\frac{\mu - \sigma^2}{2}\right)\Delta t, \tag{3.4.8}$$

$$V[X_i] = \sigma^2\Delta t. \tag{3.4.9}$$

In order to obtain annualized GBM parameters μ and σ, we consider 252 days per year and $\Delta t = \frac{1}{252}$. This number is typically the average number of trading days in one year. Next, we match the moments of the normally distributed jump-marks to the moments obtained from our historical data, by applying the moment formula for jump-diffusion processes given in equation (3.4.4). We let $Y_i = \{\frac{SP_{i+1}}{SP_i}\}$, where $i = 1, ..., N$. We obtain the desired parameters $\{\sigma, \mu, \sigma_q, \mu_q, \lambda_J\}$ by solving for:

$$M_n = \gamma_n, \quad \text{for} \quad n = 1, ..., 5, \tag{3.4.10}$$

where $M_n = \frac{\ln E[(Y_i)^n]}{\Delta t}$. This leads to solving the following system of five nonlinear equations:

$$\begin{cases} M_1 = \mu + \lambda(e^{\frac{\sigma_q^2}{2}+\mu_q} - 1), \\ M_2 = 2\mu + \sigma^2 + \lambda(e^{2\sigma_q^2+2\mu_q} - 1), \\ M_3 = 3\mu + 3\sigma^2 + \lambda(e^{\frac{9\sigma_q^2}{2}+3\mu_q} - 1), \\ M_4 = 4\mu + 6\sigma^2 + \lambda(e^{8\sigma_q^2+4\mu_q} - 1), \\ M_5 = 5\mu + 10\sigma^2 + \lambda(e^{\frac{25\sigma_q^2}{2}+5\mu_q} - 1). \end{cases} \tag{3.4.11}$$

The empirical moments from the historical data are:

$$M_1, M_2, M_3, M_4, M_5 = -0.0080, 0.0410, 0.1473, 0.3115, 0.5343$$

The results of the calibration are summarized in Table 3.5.

3.4.3 Numerical Results and Discussion

In this subsection we present numerical results derived from the theoretical methodology described in the chapter. We discuss results related to the control variables for the sick and healthy investors. We examined the behaviour of the following control variables: the optimal health insurance premium proportion $p^* = \frac{I^*}{w}$, the optimal consumption proportion $\frac{c^*}{w}$ and the optimal

Estimated Parameters				
μ	σ	λ	σ_q	μ_q
JD: Normal Distribution				
0.0100	0.2164	0.1000	0.001	0.001
GBM				
0.0100	0.1449	X	X	X

Table 3.5: Estimated parameter values by matching the moments with the historical data.

risky asset allocation π^*. We do not present numerical results for the impact of jumps in the risky asset returns on the control variables $\frac{c^*}{w}$ and $\frac{I^*}{w}$. This part was developed theoretically in the chapter and can be easily implemented in the future. In Table 3.1, we already examined the impact of jumps in asset returns on the risky asset allocation.

The first case we analyzed was that of the optimal solution for the sick or healthy investors with or without mortality risk. The relationship between the GM hazard rates was assumed to be $\lambda^{(1)} = 0.8\lambda^{(0)}$. Our main focus in this part, was the optimal consumption to wealth ratio ($\frac{c^*}{w}$). Figure 3.2, Figure 3.3 and Figure 3.4, do not include health decline risk. For clarity of the results, we display $\log(\frac{c^*}{w})$. Figure 3.2 shows the benchmark case, for an investor of undefined health status with no mortality risk. One can see that it is optimal for a risk-averse investor to allocate a larger proportion to consumption for $\gamma < 1$ when compared to the risk-tolerant consumer. A switch in the consumption strategy occurs for a degree of risk aversion $\gamma > 1$. This pattern is also evident by studying Figure 3.3 of a sick, young investor with mortality risk or Figure 3.4 of a healthy, young investor with mortality risk. However, at a more advanced age, we noticed a reversal of the trend observed for the case without mortality ($\gamma < 1$). This result is explained by the fact that older investors have a stronger force of mortality than at a younger age. At a young age, the consumer has a larger probability of survival, similar to the case without mortality force, which explains the observed match in trends for $\gamma < 1$. However, as the mortality pull increases with age, there is a cross-over between the consumption strategy curves, causing the opposite effect in the consumption proportion. A similar switch of the result for risk aversion degrees smaller than one can be obtained if the constant discount rate ρ is increased in the case without mortality. This test was done to further support these results but is not displayed here. A common trend in the above figures is that the consumption proportion spikes at the terminal time, since the investor has no bequest motives and aims at consuming all of his wealth as he approaches the terminal time.

Next, we turned our focus to analyzing the case of a healthy investor who is allowed to experience both health-decline and mortality risks. For this agent the rate of becoming sick was $\eta = 0.2\lambda^{(0)}$ (see case 1 (3.2.16)). An investor who started out in good health but became sick, receives a lump-sum fraction αp^* from the insurance provider. The medical insurance multiplier decreases with age as seen in Figure 3.1. The following figures applied the same parameters as the ones used to produce solutions for Table 3.4. Results are displayed for the optimal consumption ($\frac{c^*}{w}$) and medical insurance face value (αp^*) proportions for $\gamma = 0.5$ and $\gamma = 5$. In Figure 3.5 we displayed the optimal health insurance benefit proportion. We observed that risk-averse ($\gamma = 5$) retirees purchases more insurance premium than the risk-tolerant investor. The result is intuitive, as these investors are expected to avoid the risk of health-decline-related expenses by investing in medical insurance. On the other hand, it is evident from Figure 3.6 that risk tolerant agents ($\gamma = 0.5$) are more likely to consume a larger portion of their wealth when compared to risk-averse ($\gamma = 5$) retirees. In Figure 3.7 and Figure 3.8 we displayed results for the optimal consumption and insurance benefit proportions for case 2 ($\eta > \lambda^{(0)} > \lambda^{(1)}$). In particular, we chose $\lambda^{(1)} = 0.2\eta$ and $\lambda^{(0)} = 0.8\eta$. The results show the same trend as observed for case 1. However, the control parameters values are smaller than for case 1, suggesting that the hazard rate of the sick individual has a stronger impact on the amount allocated to health insurance than the rate of becoming sick.

Table 3.6 summarizes and compares the control variables of a risk-averse investor ($\gamma = 5$) for $x = 65$. Moreover, we choose a rate of becoming sick of $\eta = 0.2Ae^{Bt}$ and hazard rates $\lambda^{(1)} = \lambda^{(0)} = 0$ for the case without mortality. We wanted to quantify the effect of mortality on the demand for health insurance and consumption. It's evident from the results that including both mortality and health risk increases the demand for health insurance and the consumption proportion of the agent. Our results are intuitive, since in the case without mortality, the individual has a probability of survival of one and is expected to consume less than the consumer with higher probability of dying. Similarly, the retiree with higher probability of survival will allocate a smaller proportion to health insurance.

Medical Insurance, Consumption and Stock Allocation								
Age 65								
	with mortality risk				**no mortality risk**			
Risk Aversion	$\frac{I^*}{w}$	$\frac{\alpha I^*}{w}$	$\frac{c^*}{w}$	π^*	$\frac{I^*}{w}$	$\frac{\alpha I^*}{w}$	$\frac{c^*}{w}$	π^*
$\gamma = 5.0$	0.0013	0.5946	0.0375	0.299	0.0009	0.4250	0.0287	0.299

Table 3.6: Comparison of control variables with and without mortality risk.

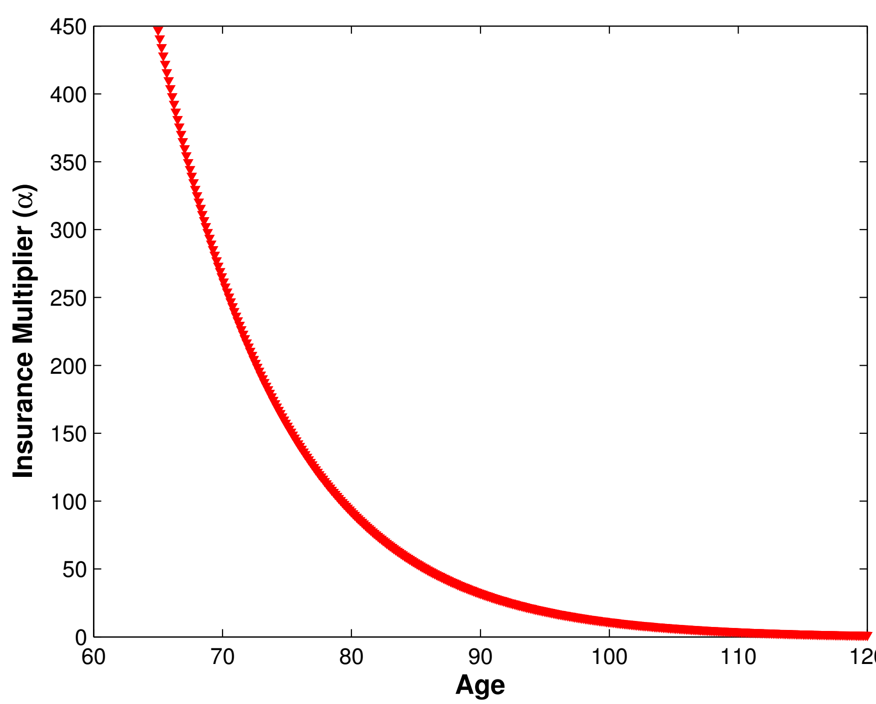


Figure 3.1: Health insurance benefit multiplier (α) as a function of time for age $x = 65$ and $\eta = 0.2\lambda^{(0)}$.

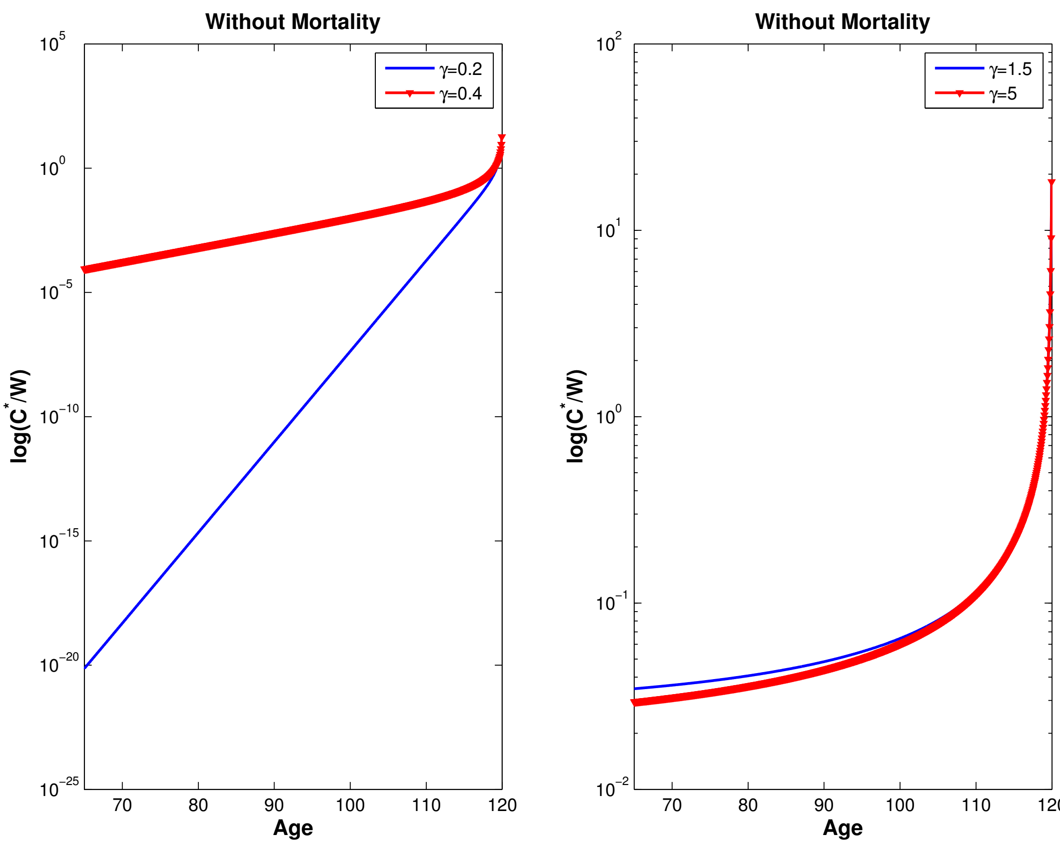


Figure 3.2: $\frac{c^*}{w}$ for an investor without GM mortality or health decline risk.

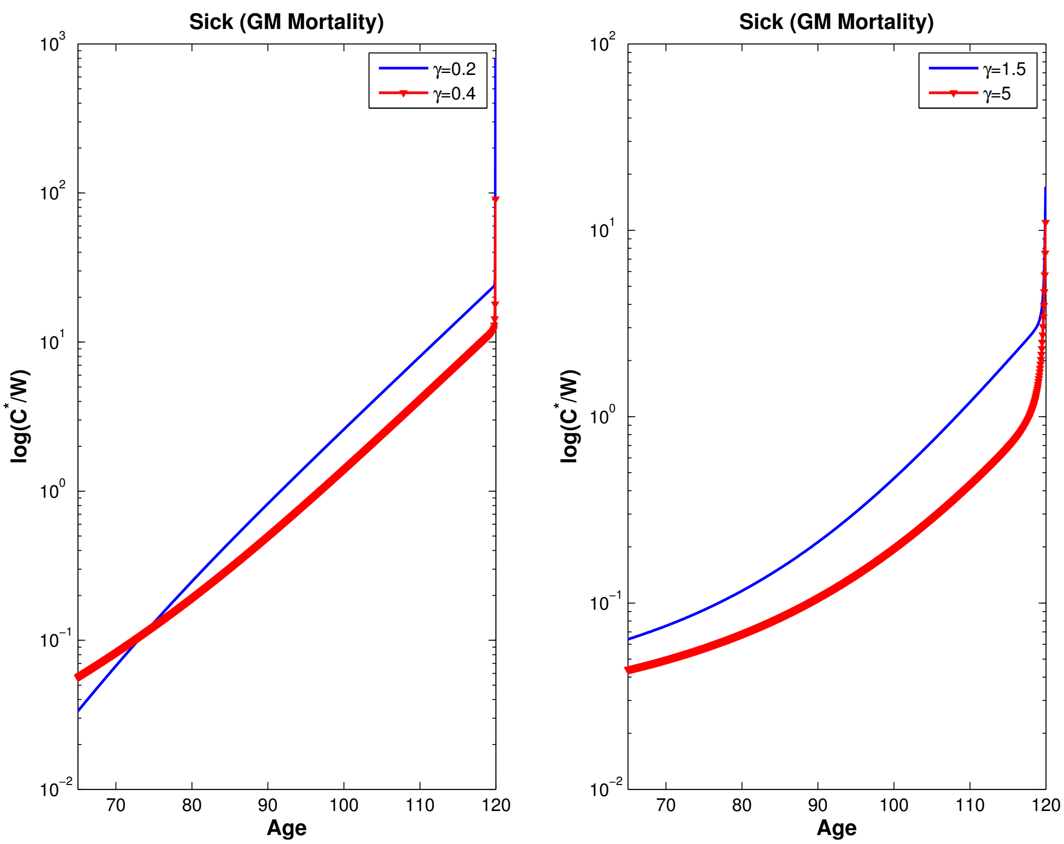


Figure 3.3: $\frac{c^*}{w}$ for a sick investor with GM mortality and without health decline risk.

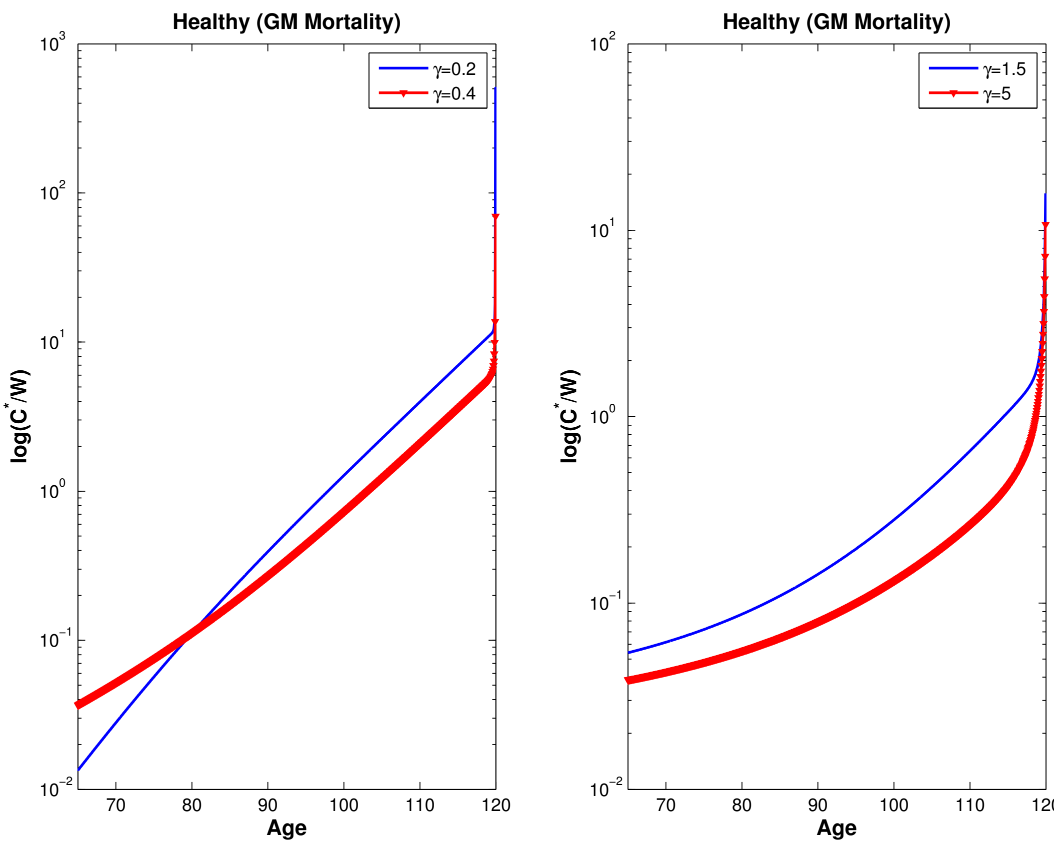
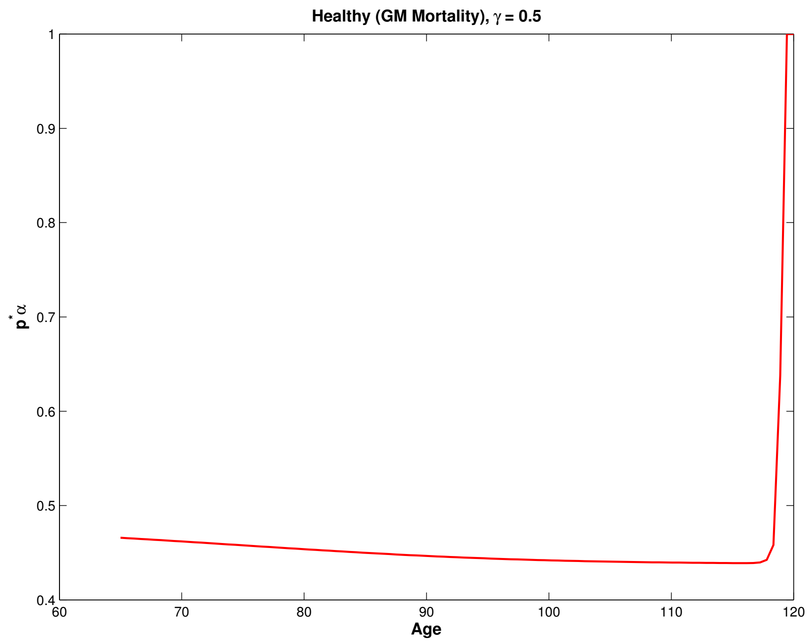
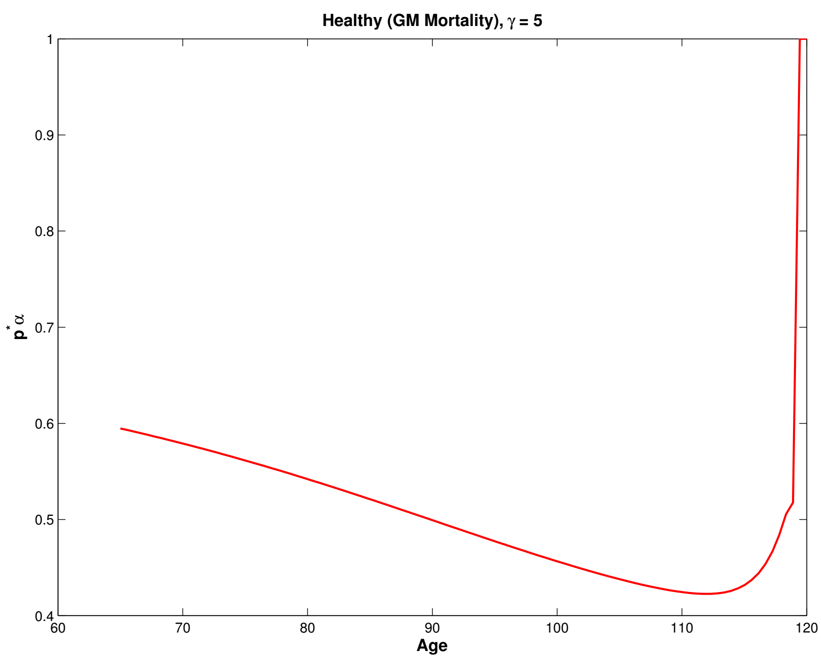


Figure 3.4: $\frac{c^*}{w}$ for a healthy investor with GM mortality and without health decline risk.

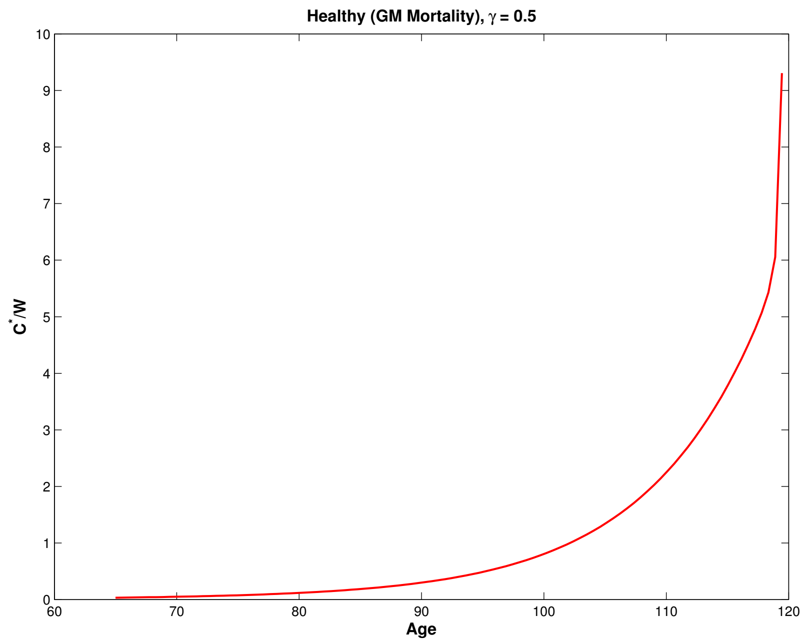


(a)

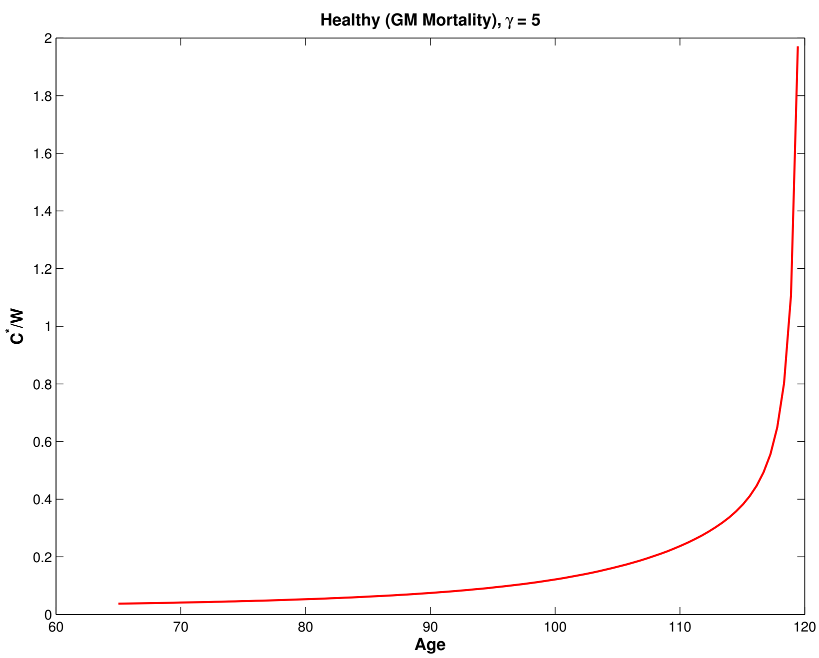


(b)

Figure 3.5: Optimal health benefit proportion αp^* for risk-tolerant (a) and risk-averse (b) investors for $\eta < \lambda^{(1)} < \lambda^{(0)}$.

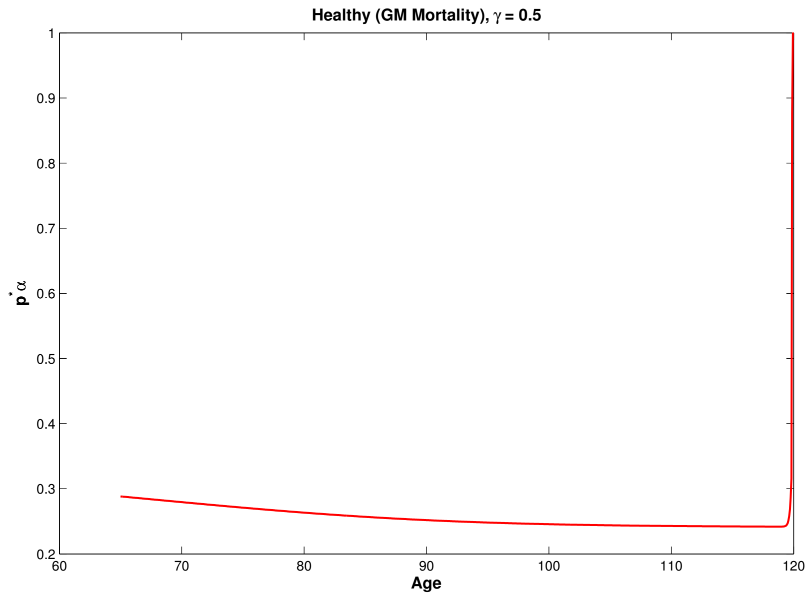


(a)

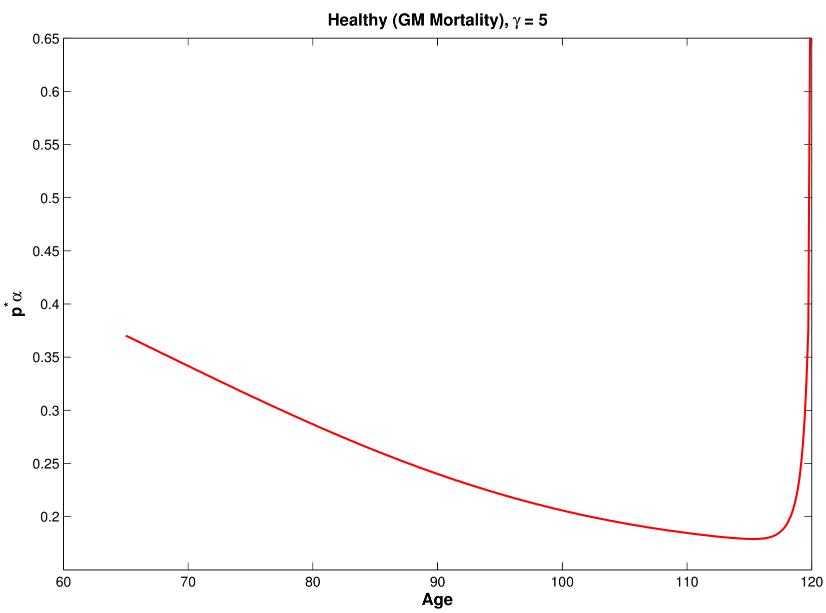


(b)

Figure 3.6: Optimal consumption proportion $\frac{c^*}{w}$ for risk-tolerant (a) and risk-averse (b) investors for $\eta < \lambda^{(1)} < \lambda^{(0)}$.

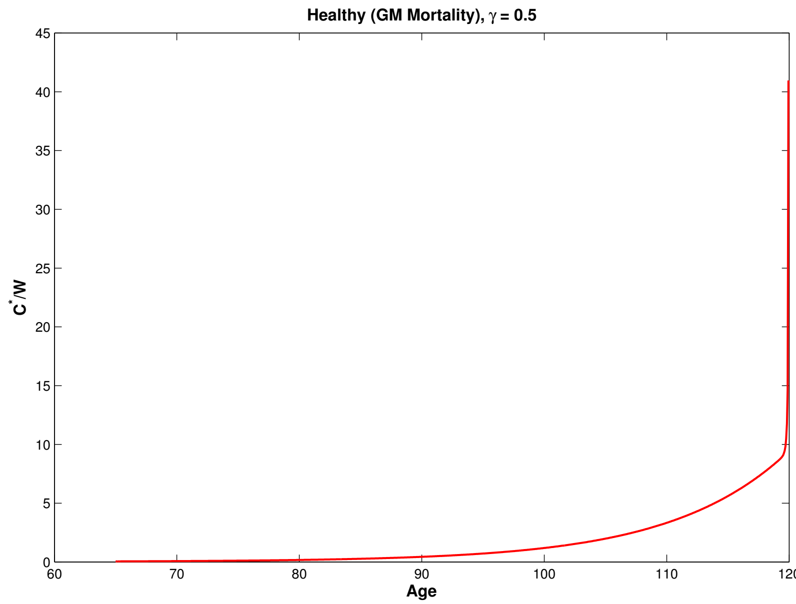


(a)

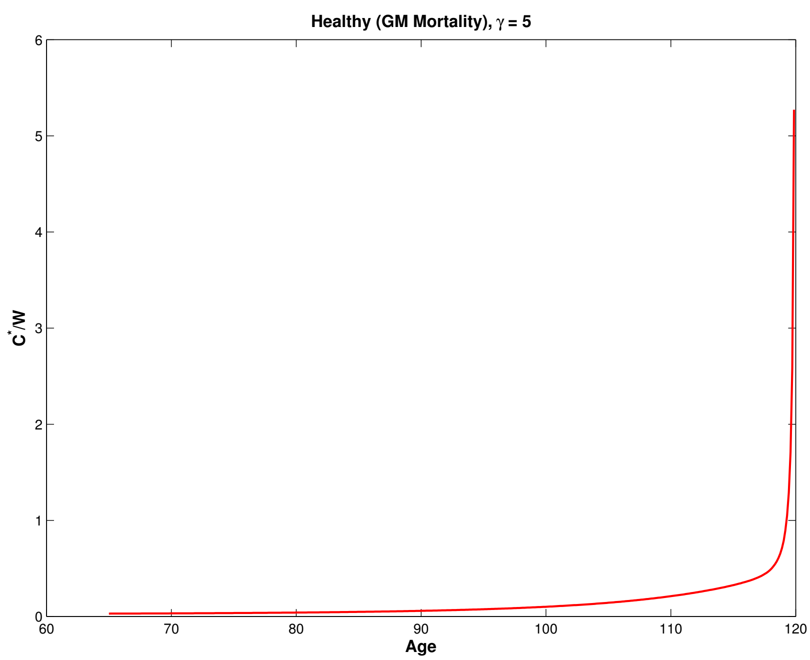


(b)

Figure 3.7: Optimal health benefit proportion αp^* for risk-tolerant (a) and risk-averse (b) investors for $\eta > \lambda^{(0)} > \lambda^{(1)}$.



(a)



(b)

Figure 3.8: Optimal consumption proportion $\frac{c^*}{w}$ for risk-tolerant (a) and risk-averse (b) investors for $\eta > \lambda^{(0)} > \lambda^{(1)}$.

3.5 Conclusion

In this chapter we have derived an optimal investment strategy for a retiree who in addition to mortality risk, experiences a health decline risk. This model predicted the optimal medical insurance and consumption proportions, as well as the optimal allocation proportion to risky-assets. The investor was assumed healthy and was able to choose to purchase out-of-pocket insurance to hedge the potential health costs. Results were obtained for the case where the health cost distribution is uniformly distributed and the utility function is of CRRA form. Results for the optimal health insurance allocation were presented in Figure 3.5 for degrees of risk aversion smaller and larger than one. The model-derived optimal strategy suggests that it is optimal for more risk-averse consumers to allocate a larger proportion to health insurance, thereby hedging the health-decline risk. Knowing that risk-averse agents avoid the risk of high health-related costs in the eventuality of a health shock, this result is as expected. The results for the optimal consumption proportion were displayed in Figures 3.6 and 3.8. The results are again intuitive, as they show that risk-averse retirees are more likely to consume a smaller portion of their wealth.

When we assessed the cases with and without mortality risk, we found that the increased probability of dying before the terminal time, increases the demand for medical insurance and also raises the optimal consumption levels . Meanwhile, the absence of force of mortality results in a lower demand for health insurance or motivation to consume wealth before the imposed terminal time.

Finally, jumps were introduced in the asset returns in the theoretical part of the chapter but were not implemented numerically for the health insurance and consumption values. On the other hand, jumps in the asset return process were shown to have a positive impact on the risky asset allocation π^*, which does not depend on health or mortality risk and could be easily studied. The impact was measured for data calibrated to historical data from 2007 to 2008. An increase in the fraction of wealth allocated to risky assets was observed. We attribute this observation to the fact, that although jumps add risk to the investment model, they can also provide the opportunity for higher financial gains. From the calibrated data we observed that jump-marks have small mean and volatilities and by our jump-term approximation will contribute in a positive manner to the risky-asset allocation fraction. As in the case without jumps, we observed that the more risk averse the investor, the less he allocates to stocks, which is an intuitive result. Further tests can be performed in the future to measure the impact of the jumps in the risky asset on other optimal control values.

4 Optimal Control Problem in the Life-cycle Framework

4.1 Introduction

The objective of this chapter is to derive an optimal strategy over the lifecycle of an investor currently of age x. In this framework we consider both the pre-retirement and the post-retirement phases of the investor's life-cycle. The investor will retire when a critical retirement-triggering wealth is reached. Hence, the methodology of finding when it is optimal to retire requires investigating the optimal retirement-triggering wealth. Some notation and assumptions of this chapter are motivated by the tractability of the Farhi and Panageas (2007) model as well as the theory developed by Huang and Milevsky (2008). Other authors, such as Huang, Milevsky, and Salisbury (2014), have more recently retrieved an optimal initiation region in an American option framework, in the context of variable annuity policies with GLWB.

Two general assumptions of our model, are that the investor is not allowed to reverse the process of retirement and that throughout his working years his income is deterministic. Once we develop the model, we will maintain both assumptions and those introduced in the previous chapter. When introducing health risk and out-of-pocket health insurance, we assume that the health shock is uniformly distributed as in the post-retirement treatment of Chapter 3. We develop the optimal control problem in stages. First we provide the background on the assumptions and notation by following the model developed by FARHI and PANAGEAS (2007). Their model ignores risk factors such as mortality and health and follows an option-valuation approach. In our model, we are able to consider special cases with constant mortality and health-decline rates as well as the case when asset rates of return follow a jump-diffusion process.

The chapter is divided as follows. Section 4.2 will introduce the general notation and assumptions. In Section 4.3 we introduce a new approach for solving for the optimal retirement-triggering wealth. For a large part of this model we applied a similar methodology as that in Chapter 3.

4.2 General Notation and Assumptions

The general notation and assumptions used by the FARHI and PANAGEAS (2007) model are summarized in this section. This will allow for a better comparison between the different approaches and these authors' work in future work. In the framework of the current chapter, the working years of the agent are included. We assume that the agent receives deterministic wages. The income rate is chosen to be constant. We write:

$$y_0 = y(\bar{\ell} - \ell_1) > 0, \tag{4.2.1}$$

where y is the constant wage rate of the working agent and is chosen so that the above inequality is satisfied. We also assume that the consumer is endowed with $\bar{\ell}$ units of leisure in retirement and ℓ_1 units if he is working. We assume as in FARHI and PANAGEAS (2007) that the utility function is of the form:

$$u(\ell_t, c_t) = \frac{(\ell_t^{1-\epsilon} c_t^{\epsilon})^{1-\gamma^*}}{\epsilon(1-\gamma^*)}; \quad 0 < \epsilon < 1, \quad \gamma^* > 0, \tag{4.2.2}$$

and normalize the leisure such that:

$$\ell_t = \begin{cases} \ell_1 = 1; & \text{while} \quad \text{working}, \\ \bar{\ell} > 1; & \text{while} \quad \text{retired}. \end{cases} \tag{4.2.3}$$

After making the change of variable $\gamma = 1 - \epsilon(1 - \gamma^*)$, we obtain a variant of the CRRA utility function which is used in the remainder of this chapter:

$$u(\ell, c) = \begin{cases} u_1(1, c) = \dfrac{c^{1-\gamma}}{1-\gamma}; & \text{while} \quad \text{working}, \\ u_2(\bar{l}, c) = \dfrac{c^{1-\gamma}}{1-\gamma}(\bar{\ell})^{(1-\epsilon)(1-\gamma^*)}; & \text{while} \quad \text{retired}. \end{cases} \tag{4.2.4}$$

Note that the change of variable implies the following equivalent relations:

$$\gamma^* > 1 \quad \Leftrightarrow \quad \gamma > 1, \tag{4.2.5}$$

$$0 < \gamma^* < 1 \quad \Leftrightarrow \quad 0 < \gamma < 1. \tag{4.2.6}$$

To simplify the notation, we let $\ell = (\bar{\ell})^{(1-\epsilon)(1-\gamma^*)}$. One can see that for $\bar{\ell} > 1$, we have $\ell < 1$ for $\gamma > 1$ and $\ell > 1$ for $\gamma < 1$.

4.3 Optimal Retirement Control Problem

In this section we focus on presenting the theory of an optimal consumption-allocation problem for an investor still in the working phase. We are interested in deriving a critical wealth level that would make exiting the workforce possible. A similar problem was developed by Farhi and Panageas (2007) in a complete market setting by using a stochastic discount factor. Our methodology differs from the Farhi and Panageas (2007) option-valuation approach. The optimal control problem will result in an explicit value for the optimal-retirement wealth levels for the case of constant force of mortality.

4.3.1 Model Setup

Let the problem's value function be formally represented by:

$$\hat{J}(t,w) = \max_{\{c_t,\pi_t,\tau\}} E\left[\int_t^\tau D_t(s)u_1(c_s)ds + \int_\tau^\infty D_t(s)u_2(c_s)ds | W_t = w\right], \tag{4.3.1}$$

where u_1 and u_2 are the utility function for the working and retirement phases, respectively. The discount factor is:

$$D_t(s) = e^{-\int_t^s(\rho+\lambda_{x+v})dv}, \tag{4.3.2}$$

where ρ is the discount rate and λ_{x+t} is the mortality rate for an individual of age x at $t = 0$.

The budget constraint of the investor is:

$$dW_t = \begin{cases} \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt - c_t dt + dM_t; & \text{for} \quad \tau > t, \\ \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt - c_t dt; & \text{for} \quad \tau \leq t, \end{cases} \tag{4.3.3}$$

where the instantaneous deterministic income stream is $dM_t = y_0 dt$, π_t is the portion of the wealth invested in the risky asset, μ and σ are the drift and volatility of the risky asset respectively, c_t is the consumption rate, y_0 is the wage income for $t < \tau$ and $y_0 = 0$ when $t > \tau$ and τ is the retirement time. To follow Farhi and Panageas (2007) , we assume that the income stream is constant, defined above as y_0 and dependent on a constant wage rate y.

Remark 4.3.1. *It is more realistic to choose a time-dependent income stream, as wages often increase with time. That case would involve using* $dM_t = M_0 e^{y_0 t} dt$.

The consumer is allowed to invest in the money market, where he receives a fixed, strictly positive interest rate $r > 0$. Then, during the working years until a terminal time T, the discounted cash flow generated by the human wage process (the value of the human capital) is given by:

$$HC(t;T) = \int_t^T y_0 e^{-r(s-t)} ds = \frac{y_0}{r}(1 - e^{-r(T-t)}). \tag{4.3.4}$$

Remark 4.3.2. *If the investor is allowed to work for an infinite amount of years, the value of the human capital asymptotically approaches* $\frac{y_0}{r}$.

Remark 4.3.3. *If the income stream is time-dependent, the human capital is* $HC(t;T) = \frac{M_0}{y_0-r}(e^{(y_0-r)(T-t)} - 1)$, *where* M_0 *is the income earned in the first year of working.*

As the relationships

$$\hat{J}(t,w) \geq \Phi(t,w) := \max_{\{c_t,\pi_t\}} E\left[\int_t^{\infty} D_t(s)u_2(c_s)ds | W_t = w\right], \tag{4.3.5}$$

and

$$\hat{J}(t,w) \geq \max_{\{c_t,\pi_t\}} E\left[\int_t^{\tau} D_t(s)u_1(c_s)ds + \int_{\tau}^{\infty} D_t(s)u_2(c_s)ds | W_t = w\right] \tag{4.3.6}$$

are true, we can rewrite (4.3.1) as follows:

$$\hat{J}(t,w) = \max\left\{\max_{\{c_t,\pi_t,\tau\}} E\left[\int_t^{\tau} D_t(s)u_1(c_s)ds + \int_{\tau}^{\infty} D_t(s)u_2(c_s)ds | W_t = w\right], \Phi(t,w)\right\}. \tag{4.3.7}$$

4.3.2 Solution Methodology

For the *post-retirement* phase $t > \tau$, $\Phi(t,w)$ is simply the solution of an HJB equation:

$$\frac{\partial \Phi}{\partial t} + \max_{\{c_t,\pi_t\}}\left\{u_2(c_t) + \left[(\pi_t(\mu - r) + r)w - c_t\right]\frac{\partial \Phi}{\partial w} + \frac{(\pi_t w\sigma)^2}{2}\frac{\partial^2 \Phi}{\partial w^2}\right\} = (\rho + \lambda_{x+t})\Phi. \tag{4.3.8}$$

The solution can be worked out as $\Phi(t,w) = u_2(w)h(t)$ and $h(t)$ satisfies an ODE with $h(\infty) = 0$. For the *pre-retirement* phase, we find the value function in three stages.

1. We solve the following HJB:

$$\frac{\partial J}{\partial t} + \max_{\{c_t,\pi_t\}}\left\{u_1(c_t) + \left[(\pi_t(\mu - r) + r)w + y_0 - c_t\right]\frac{\partial J}{\partial w} + \frac{(\pi_t w\sigma)^2}{2}\frac{\partial^2 J}{\partial w^2}\right\} = (\rho + \lambda_{x+t})J. \tag{4.3.9}$$

2. The individual retires when $J(t,w) \leq \Phi(t,w)$. The critical wealth $\bar{w}(t)$ is given by $J(t,\bar{w}) = \Phi(t,\bar{w})$. The individual stays working when $J(t,w) > \Phi(t,w)$. The critical retirement-triggering wealth $\bar{w}(t)$ is a free-boundary that separates the retirement and working regions.

3. Finally, we update the objective function by taking the maximum of the two value functions as follows:

   $\hat{J}(t,w) = \max\{\Phi(t,w), J(t,w)\}$. We note that this step is applied for every t and w.

Remark 4.3.4. *Unlike for Φ, in general we will not be able to find a separable form for $J(t,w)$ since the critical wealth is time dependent. Therefore, we will solve the HJB for $J(t,w)$ numerically. In this case we will truncate the time domain such that $t \leq T$, while $x + T = 120$, and we will assume that it is optimal to retire at $t = \tau$. We can solve the HJB backwards to find $\Phi(t,w)$ and $J(t,w)$ by following the three steps outlined above. The solution of this problem is left for future work. In this dissertation we have focused on the more analytically tractable case with constant mortality and income stream.*

4.3.3 Constant Force of Mortality

In this dissertation we consider a special case with a constant hazard rate, $\lambda_{x+t} = \lambda$. In this case, the solutions of the HJBs are independent of time. As a result, the critical wage for optimal retirement is also independent of time, which can be worked out explicitly. As mentioned before, we let the leisure be represented by $\ell = (\bar{\ell})^{(1-\epsilon)(1-\gamma^*)}$, where $\bar{\ell} > 1$ while retired.

4.3.3.1 Post-retirement phase

For the post-retirement phase, the optimal consumption rate and asset allocation strategy are given by the first-order conditions:

$$c_t^* = \Phi_w^{-\frac{1}{\gamma}} \ell^{\frac{1}{\gamma}}; \quad \pi_t^* = -\frac{\mu - r}{w\sigma^2} \frac{\Phi_w}{\Phi_{ww}}. \tag{4.3.10}$$

Using $\Phi = u_2(w)k(t)$ with CRRA utility:

$$u_2(c) = \frac{\ell c^{1-\gamma}}{1-\gamma}, \tag{4.3.11}$$

we have

$$c_t^* = wk^{-\frac{1}{\gamma}}; \quad \pi_t^* = \frac{(\mu - r)}{\gamma\sigma^2}, \tag{4.3.12}$$

where $k(t)$ satisfies the equation

$$\frac{1}{\gamma}\frac{dk}{dt} + k^{1-\frac{1}{\gamma}} - \Theta k = 0 \tag{4.3.13}$$

with

$$\Theta = \frac{\rho + \lambda - (1-\gamma)\left(r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right)}{\gamma}. \tag{4.3.14}$$

Since all the coefficients are constants and we are solving the problem on an infinite time horizon, k is independent of time, and we have:

$$k = \Theta^{-\gamma}. \tag{4.3.15}$$

The value function is given by $\Phi(w) = \frac{w^{1-\gamma}}{1-\gamma}\Theta^{-\gamma}\ell$.

Remark 4.3.5. *For a normalized leisure value ($\ell = 1$) we have already seen a similar equation to 4.3.15 in Chapter 3 for the case with constant mortality risk, including the case of $\lambda = 0$. We also* *know that for a terminal time T we can obtain the closed form time-dependent component of the value function as in* Huang *and* Milevsky *(2008):*

$$f(t;T) = \frac{e^{-\rho t}(e^{\xi(T-t)} - 1)^{\gamma}}{\xi^{\gamma}},$$

where $\xi = \frac{C-\rho-\lambda}{\gamma}$ *and* $C = (1-\gamma)\left(r + \frac{(\mu-r)^2}{2\gamma\sigma^2}\right)$*. As in the* Farhi *and* Panageas *(2007) model, which allows the agent to live and work for an infinite number of years, we fix the time* $t = 0$ *and take the limit as* $T \to \infty$*:*

$$\begin{aligned} f(0;\infty) &= \lim_{T\to\infty}\{\frac{(e^{\xi(T)} - 1)^{\gamma}}{\xi^{\gamma}}\} && (4.3.16)\\ &= \left(\frac{1}{\Theta}\right)^{\gamma}, && (4.3.17)\end{aligned}$$

where $\xi = -\Theta$*. For consistency of the value function we also require that* $\Theta = [(r + \frac{(\mu-r)^2}{2\sigma^2\gamma})\frac{(\gamma-1)}{\gamma} + \frac{\rho+\lambda}{\gamma}] > 0$.

Remark 4.3.6. *We would like to point out that in our current approach we solve the objective function on an infinite time horizon. However, by including a non-zero force of mortality and survival probabilities in the model, this means that the investor is not alive for the entirety of the infinite horizon time domain.*

We can write the value function as $\Phi(w) = \Omega\frac{w^{1-\gamma}}{1-\gamma}$. Then, as we see that $\Omega = (\bar{\ell})^{(1-\epsilon)(1-\gamma^*)}\left(\frac{1}{\Theta}\right)^{\gamma}$ and observe that for $\bar{\ell} > 1$, we have the following relations:

$$\Omega^{\frac{1}{\gamma}} > \frac{1}{\Theta} \quad \text{for} \quad \gamma < 1, \qquad (4.3.18)$$

$$\Omega^{\frac{1}{\gamma}} < \frac{1}{\Theta} \quad \text{for} \quad \gamma > 1. \qquad (4.3.19)$$

We can define the consumption jump experienced by the investor when he enters retirement. We write:

$$\frac{c_\tau^+}{c_\tau^-} = \ell^{\frac{1}{\gamma}} = \Omega^{\frac{1}{\gamma}}\Theta. \qquad (4.3.20)$$

It is obvious that $\frac{c_\tau^+}{c_\tau^-} > 1$ for $\gamma < 1$ and $\frac{c_\tau^+}{c_\tau^-} < 1$ for $\gamma > 1$. To put it into words, consumption is expected to experience a down-jump for $\gamma > 1$ and an up-jump for $\gamma < 1$ at the optimal retirement time.

4.3.3.2 Pre-retirement phase

For the pre-retirement phase, the optimal consumption rate and asset allocation strategy are given by the first-order conditions:

$$c_t^* = J_w^{-\frac{1}{\gamma}}; \quad \pi_t^* = -\frac{\mu - r}{w\sigma^2}\frac{J_w}{J_{ww}}. \tag{4.3.21}$$

Using $J = u_1(\hat{w})h(t)$ with $\hat{w} = w + \beta(t)$ and the CRRA utility

$$u_1(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{4.3.22}$$

we have $J_t = \frac{\hat{w}^{1-\gamma}}{1-\gamma}\frac{dh}{dt} + \hat{w}^{-\gamma}\frac{d\beta}{dt}h$, $J_w = \hat{w}^{-\gamma}h$ and $J_{ww} = -\gamma\hat{w}^{-\gamma-1}h$. The first-order conditions can then be written as:

$$c_t^* = J_w^{-\frac{1}{\gamma}} = \hat{w}h^{-\frac{1}{\gamma}}; \quad \pi_t^* = -\frac{(\mu - r)}{w\sigma^2}\frac{J_w}{J_{ww}} = \frac{(\mu - r)}{\gamma\sigma^2}\frac{\hat{w}}{w}. \tag{4.3.23}$$

After substituting the optimal controls into the HJB (4.3.9) we obtain:

$$\frac{\hat{w}^{1-\gamma}}{1-\gamma}h' + \gamma\hat{w}^{1-\gamma}h^{1-\frac{1}{\gamma}} + \{\frac{(\mu - r)^2\hat{w}^{1-\gamma}}{2\gamma\sigma^2} + r(\hat{w} - \beta)\hat{w}^{-\gamma} + y_0\hat{w}^{-\gamma}\}h + \hat{w}^{-\gamma}\beta' h = 0. \tag{4.3.24}$$

We can see that $h(t)$ and β will satisfy the following equations:

$$\frac{1}{\gamma}\frac{dh}{dt} + h^{1-\frac{1}{\gamma}} - \Theta h = 0 \tag{4.3.25}$$

and

$$\frac{d\beta}{dt} - r\beta + y_0 = 0, \tag{4.3.26}$$

with Θ as defined earlier. In economic terms, β represents the human capital generated by the constant wages. Since all the coefficients are constants, both h and β are independent of time on an infinite time horizon. Therefore, we obtain:

$$h = \Theta^{-\gamma}; \quad \beta = \frac{y_0}{r}. \tag{4.3.27}$$

4.3.3.3 Critical retirement-triggering wealth

The critical wealth and regions for the working and retirement periods can be obtained by comparing the value functions J and Φ for the pre- and post-retirement phases, which are given by:

$$J = \frac{(w+\beta)^{1-\gamma}\Theta^{-\gamma}}{1-\gamma}, \tag{4.3.28}$$

$$\Phi = \frac{w^{1-\gamma}\Theta^{-\gamma}\ell}{1-\gamma}. \tag{4.3.29}$$

This approach results in an approximate critical wealth. Since the problem is time independent, this means that we only need to solve the time independent version of the HJBs for Φ and J, both of which are dependent on wealth. The exact solution is given by joining the two solutions by applying the smooth pasting condition at the critical wealth level. However, for simplicity, we will only consider the approximate solution.

With the notation established in Section 4.2 we can obtain the critical retirement-triggering wealth for both $\gamma > 1$ and $\gamma < 1$. We note that for $\gamma > 1$ both J and Φ are negative, while for $0 < \gamma < 1$ both J and Φ are positive. We expect $J < \Phi$ for $w > \bar{w}$ and $J > \Phi$ for $w < \bar{w}$, where the critical retirement-triggering wealth is given by $J = \Phi$. The explicit solution is given by:

$$\bar{w} = \frac{\beta}{\ell^{\frac{1}{1-\gamma}} - 1}. \tag{4.3.30}$$

It is optimal to retire when $w > \bar{w}$. We recall that $\ell < 1$ for $\gamma > 1$ and that $\ell > 1$ for $\gamma < 1$. Therefore, the critical wealth $\bar{w} > 0$.

Remark 4.3.7. *Note that if we scale the critical wealth by the human capital $\frac{\bar{w}}{\beta}$, and apply the leisure function defined previously in terms of $\bar{\ell}$, we obtain an expression independent of other risk parameters and the degree of risk aversion. This is probably due to the fact that our critical wealth is an approximate value as remarked earlier.*

We can explore the effects of leisure on the critical wealth. Larger consumption jump-ratios correspond to larger leisure values. In this case, we expect the critical wealth to decrease. This is intuitive, since investors who value leisure more, are expected to enter retirement sooner due to lower critical wealth levels.

4.3.4 Extension 1: Retirement-triggering Wealth under a Jump-diffusion Investment Setup

In the previous subsection we developed a methodology for finding the critical retirement wealth by assuming that the asset rates of return follow a simple GBM diffusion process. In the current subsection, we are interested in the more complex case, in which the risky asset returns are in fact the result of a jump-diffusion process similar to the one explored in the previous chapter.

The critical retirement-triggering wealth is derived by following the technique in the previous subsection. The main difference in the results will be due to the introduction of jumps and the integral term associated with them.

The budget constraint of the investor has now changed and it is:

$$dW_t = \begin{cases} \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt + v_1 W_t P_t - c_t dt + dM_t; & \text{for} \quad \tau > t, \\ \pi_t(\mu dt + \sigma dB_t)W_t + (1-\pi_t)rW_t dt + v_1 W_t dP_t - c_t dt; & \text{for} \quad \tau \leq t, \end{cases} \tag{4.3.31}$$

where v_1 is the random wealth jump-amplitude and P_t is the Poisson process.

4.3.4.1 Post-retirement phase

By following a similar approach to that taken in Chapter 3, we can derive the HJB for the case with jumps in the asset rate of return process. For this post-retirement period $t > \tau$, $\Phi(t,w)$ satisfies:

$$\frac{\partial \Phi}{\partial t} + \max_{\{c_t,\pi_t\}} \left\{ u_2(c_t) + \left[(\pi_t(\mu - r) + r)w - c_t\right] \frac{\partial \Phi}{\partial w} + \frac{(\pi_t w\sigma)^2}{2} \frac{\partial^2 \Phi}{\partial w^2} \right.$$
$$\left. + \lambda_J \int_{-\infty}^{\infty} \{\Phi(t,(1+v_1\pi)w) - \Phi(t,w)\}\phi_q dq \right\} = (\rho + \lambda_{x+t})\Phi. \tag{4.3.32}$$

The solution can be worked out for $\Phi(t,w) = u_2(w)h(t)$, where $h(t)$ satisfies an ODE with $h(\infty) = 0$.

The first-order optimality conditions are:

$$c_t^* = \Phi_w^{-\frac{1}{\gamma}} \ell^{\frac{1}{\gamma}}, \tag{4.3.33}$$

$$(\mu - r)w\Phi_w + \frac{(\pi_t^* w\sigma)^2}{2}\Phi_{ww} + \lambda_J \int_{-\infty}^{\infty} \{\Phi(t,(1+v_1\pi_t^*)w) - \Phi(t,w)\}\phi_q dq = 0. \tag{4.3.34}$$

Substituting the optimal control c_t^* into the HJB and assuming we have constant instantaneous force of mortality $\lambda_{x+t} = \lambda$, we know that h will satisfy the following ODE:

$$\frac{h'}{h} + \gamma h^{-\frac{1}{\gamma}} + (1-\gamma)\{\frac{\pi_t^*(\mu - r)}{2} - \frac{\pi_t^* \lambda_J}{2} \int_{-\infty}^{\infty} v_1(1+v_1\pi^*)^{-\gamma}\phi_q dq + r\} - \lambda - \rho$$
$$+ \lambda_J \int_{-\infty}^{\infty} [(1+v_1\pi^*)^{1-\gamma} - 1]\phi_q dq = 0. \tag{4.3.35}$$

Since all the coefficients of the ODE are constants and we work on an infinite time horizon, we know that solution h is time independent and that it satisfies:

$$h^{-\frac{1}{\gamma}} = \frac{1}{\gamma}\left\{\lambda + \rho - (1-\gamma)\left(\frac{\pi_t^*(\mu - r)}{2} + r\right)\right.$$
$$\left. -\lambda_J \int_{-\infty}^{\infty} [(1 + \nu_1 \pi_t^*)^{-\gamma}\left(1 + (1+\gamma)\frac{\pi_t^* \nu_1}{2}\right) - 1]\phi_q dq\right\}. \quad (4.3.36)$$

The resulting value function of this case is $\Phi(w) = \frac{w^{1-\gamma}}{1-\gamma} h\ell$.

4.3.4.2 Pre-retirement phase

Similarly, for the pre-retirement period $t < \tau$, the value function $J(t, w)$ satisfies:

$$\frac{\partial J}{\partial t} + \max_{\{c_t, \pi_t\}}\left\{u_1(c_t) + \left[(\pi_t(\mu - r) + r)w - c_t + y_0\right]\frac{\partial J}{\partial w} + \frac{(\pi_t w \sigma)^2}{2}\frac{\partial^2 J}{\partial w^2}\right.$$
$$\left. +\lambda_J \int_{-\infty}^{\infty} \{J(t, (1 + \nu_1 \pi)w) - J(t, w)\}\phi_q dq\right\} = (\rho + \lambda_{x+t})J. \quad (4.3.37)$$

As in the previous subsection, we assume that $J(t, w) = u_1(\hat{w})k(t)$ and $\hat{w} = w + \beta(t)$. Since we only analyze the constant mortality case, we can write equivalently $J(w) = \frac{(w+\beta)^{1-\gamma}}{1-\gamma}k$. Then $J_w = \hat{w}^{-\gamma}k$ and $J_{ww} = -\gamma \hat{w}^{-\gamma-1}k$. The first-order optimality condition for consumption,

$$c_t^* = \hat{w}k^{-\frac{1}{\gamma}}, \quad (4.3.38)$$

is the same as calculated in the pure diffusion setup. The optimal allocation fraction is dependent on the jump in the value function. For this calculation, we denote $\tilde{w} = (1 + \nu_1 \pi_t)w + \beta$. The first-order condition is given by:

$$(\mu - r)\pi_t w J_w + \pi_t^* \sigma^2 w^2 J_{ww} + \lambda_J \int_{-\infty}^{\infty} \frac{\partial J}{\partial \tilde{w}}\frac{\partial \tilde{w}}{\partial \pi_t} = 0. \quad (4.3.39)$$

After some algebraic manipulation, the optimal allocation fraction should satisfy:

$$w(\mu - r)\hat{w}^{-\gamma}k - \gamma\pi_t^*\sigma^2 w^2\hat{w}^{-\gamma-1}k + \int_{-\infty}^{\infty}((1+v_1\pi_t^*)w+\beta)^{-\gamma}v_1 wk\phi_q dq = 0. \tag{4.3.40}$$

We let $\pi_t^* w = \hat{\pi}_t^*$ and obtain:

$$(\mu - r) - \frac{\gamma\sigma^2\hat{\pi}_t^*}{\hat{w}} + \lambda_J\int_{-\infty}^{\infty}\left(1+\frac{v_1\hat{\pi}_t^*}{\hat{w}}\right)^{-\gamma}v_1\phi_q dq = 0. \tag{4.3.41}$$

With the change of variable $\tilde{\pi}_t^* = \frac{\hat{\pi}_t^*}{w+\beta}$, we arrive at:

$$(\mu - r) - \gamma\sigma^2\tilde{\pi}_t^* + \lambda_J\int_{-\infty}^{\infty}(1+v_1\tilde{\pi}_t^*)^{-\gamma}v_1\phi_q dq = 0. \tag{4.3.42}$$

We can see that $\tilde{\pi}_t^*$ is equivalent to the optimal asset allocation proportion in the post-retirement period and that it is independent of the wealth level.

For a constant force of mortality, we rewrite the HJB:

$$\begin{aligned} u_1(c_t^*) - c_t^* J_w + (\frac{\tilde{\pi}_t^*(\mu - r)}{2} + rw + y_0)J_w \\ + \lambda_J\int_{-\infty}^{\infty}[J(w+v_1\tilde{\pi}_t^*) - J(w) - J_w v_1 w]\phi_q dq = (\lambda+\rho)J. \end{aligned} \tag{4.3.43}$$

By further simplification we obtain:

$$\begin{aligned} (\lambda+\rho)\frac{\hat{w}^{1-\gamma}}{1-\gamma} &= \frac{\gamma}{1-\gamma}\hat{w}^{1-\gamma}k^{-\frac{1}{\gamma}} + \hat{\pi}_t^*(\mu - r)\hat{w}^{-\gamma} + rw\hat{w}^{-\gamma} + y_0\hat{w}^{-\gamma} \\ &- \frac{\gamma(\hat{\pi}_t^*\sigma)^2\hat{w}^{-\gamma-1}}{2} + \frac{\lambda_J}{1-\gamma}\int_{-\infty}^{\infty}[(w+v_1\hat{\pi}_t^*+\beta)^{1-\gamma} - \hat{w}^{1-\gamma}]\phi_q dq. \end{aligned} \tag{4.3.44}$$

By the change of variable $\hat{\pi}_t^* = \tilde{\pi}_t^*\hat{w}$ and simplification, we obtain:

$$\begin{aligned} (\lambda+\rho)\hat{w}^{1-\gamma} &= \gamma\hat{w}^{1-\gamma}k^{-\frac{1}{\gamma}} + \tilde{\pi}_t^*(\mu - r)\hat{w}^{1-\gamma}(1-\gamma) + r(1-\gamma)\hat{w}^{1-\gamma} + (1-\gamma)\left\{y_0\hat{w}^{-\gamma} - r\beta\hat{w}^{-\gamma}\right\} \\ &- \frac{\gamma(1-\gamma)(\tilde{\pi}_t^*\sigma^2)\hat{w}^{1-\gamma}}{2} + \lambda_J\hat{w}^{1-\gamma}\int_{-\infty}^{\infty}[(1+v_1\tilde{\pi}_t^*)^{1-\gamma} - 1]\phi_q dq. \end{aligned} \tag{4.3.45}$$

As seen before, the human capital is $\beta = \frac{y_0}{r}$. By using the first-order condition for the risky-asset optimal allocation fraction, we obtain the final equation for the solution:

$$k^{-\frac{1}{\gamma}} = \frac{1}{\gamma}\left\{\lambda + \rho - (1-\gamma)\left(\frac{\tilde{\pi}_t^*(\mu - r)}{2} + r\right)\right.$$
$$\left. -\lambda_J \int_{-\infty}^{\infty} [(1+\nu_1\tilde{\pi})^{-\gamma}\left(\frac{\tilde{\pi}_t^*\nu_1}{2}(1+\gamma)+1\right) - 1]\phi_q dq\right\}. \quad (4.3.46)$$

We note here that solutions (4.3.36) and (4.3.46) are the same. Therefore $h = k$, $J(w) = \frac{\hat{w}^{1-\gamma}}{1-\gamma}h$ and $\Phi(w) = \frac{w^{1-\gamma}}{1-\gamma}\ell h$. It follows that the critical wealth for the pure diffusion setup is the same as represented in equation (4.3.30) .

4.3.5 Extension 2: Retirement-triggering Wealth under Health Shock Setup

In this subsection we further extend our model to include health-cost-related shocks in the wealth process. This is a realistic case analyzed previously in Chapter 3 for an investor in the post-retirement phase. The problem is as follows. The individual is currently in good health, but can face a health shock both before and after retirement is triggered. We are still interested in deriving an expression for the critical retirement-triggering wealth. Mathematical details for the derivation of the HJB with health shocks were presented in Chapter 3 and can be referred to.

As before, we can divide the problem into two phases as follows. We let $\Phi(t, w)$ and $\Phi^{(0)}(t, w)$ be the post-retirement value functions for a healthy and sick investor respectively. For the general case with time-dependent force of mortality, the value functions for the post-retirement period will satisfy:

$$
\begin{aligned}
(\lambda_{x+t}^{(0)} + \rho)\Phi^{(0)} &= \frac{\partial \Phi^{(0)}}{\partial t} + \max_{\{c_t,\pi_t\}}\{u_2(c_t) - c_t\frac{\partial \Phi^{(0)}}{\partial w}\} \\
&+ \max_{\{c_t,\pi_t\}}\{(\pi_t(\mu - r) + r)w\frac{\partial \Phi^{(0)}}{\partial w} + \frac{(\pi_t \sigma w)^2}{2}\frac{\partial^2 \Phi^{(0)}}{\partial w^2}\}; \quad t > \tau_s
\end{aligned} \tag{4.3.47}
$$

and

$$
\begin{aligned}
(\lambda_{x+t}^{(1)} + \rho)\Phi &= \frac{\partial \Phi}{\partial t} + \max_{\{I_t,c_t,\pi_t\}}\{u_2(c_t) - c_t\frac{\partial \Phi}{\partial w}\} \\
&+ \max_{\{I_t,c_t,\pi_t\}}\{[(\pi_t(\mu - r) + r)w - I_t]\frac{\partial \Phi}{\partial w} + \frac{(\pi_t \sigma w)^2}{2}\frac{\partial^2 \Phi}{\partial w^2} \\
&+ \eta_{x+t}\int_\nu \Phi(t, w - \nu + \alpha I_t)\phi_s d\nu\}; \quad t < \tau_s,
\end{aligned} \tag{4.3.48}
$$

where, as in the previous chapter, τ_s is the time when the individual becomes sick, ν is the magnitude of the health cost, ϕ_s is the distribution of the health jump and αI_t is the health benefit received at the time the investor becomes sick.

For the pre-retirement period of the investor's life-cycle we formulate the problem as follows. We let $J(t, w)$ and $J^{(0)}(t, w)$ be the value functions for the pre-retirement period associated with a healthy and sick state respectively. The value functions for this case will satisfy:

$$
\begin{aligned}
(\lambda_{x+t}^{(0)} + \rho)J^{(0)} &= \frac{\partial J^{(0)}}{\partial t} + \max_{\{c_t,\pi_t\}}\{u_1(c_t) + (y_0 - c_t)\frac{\partial J^{(0)}}{\partial w}\} \\
&+ \max_{\{c_t,\pi_t\}}\{(\pi_t(\mu - r) + r)w\frac{\partial J^{(0)}}{\partial w} + \frac{(\pi_t \sigma w)^2}{2}\frac{\partial^2 J^{(0)}}{\partial w^2}\}; \quad t > \tau_s
\end{aligned} \tag{4.3.49}
$$

and

$$
\begin{aligned}
(\lambda^{(1)}_{x+t} + \rho)J &= \frac{\partial J}{\partial t} + \max_{\{I_t, c_t, \pi_t\}} \{u_1(c_t) + (y_0 - c_t)\frac{\partial J}{\partial w}\} \\
&+ \max_{\{I_t, c_t, \pi_t\}} \{[(\pi_t(\mu - r) + r)w - I_t]\frac{\partial J}{\partial w} + \frac{(\pi_t \sigma w)^2}{2}\frac{\partial^2 J}{\partial w^2} \\
&+ \eta_{x+t}\int_v J(t, w - v + \alpha I_t)\phi_s dv\}; \quad t < \tau_s.
\end{aligned}
\tag{4.3.50}
$$

As described in Section 4.3.2, $J(t,w) = \max\{J(t,w), \Phi(t,w)\}$ at any (t,w) and the critical retirement-triggering wealth is obtained from the equality $J(t,\bar{w}(t)) = \Phi(t,\bar{w}(t))$. This is the general setup and the wealth would have to be solved numerically.

In the following, we only consider the case in which the forces of mortality are constants: $\lambda^{(0)}_{x+t} = \lambda^{(0)}$ and $\lambda^{(1)}_{x+t} = \lambda^{(1)}$. We also choose a constant rate of becoming sick $\eta_{x+t} = \eta$. In this case, the solution we seek is time-independent on an infinite time horizon. The solution methodology is decribed in the section below.

4.3.5.1 Post-retirement phase

For the sick individual the solution is taken as $\Phi^{(0)}(w) = \frac{w^{1-\gamma}}{1-\gamma}\ell h$, where both h and ℓ are constants. The utility function is $u_2(c) = \frac{c^{1-\gamma}}{1-\gamma}\ell$. This results in the optimal control variables:

$$
c^*_t = wh^{-\frac{1}{\gamma}}; \quad \pi^*_t = \frac{\mu - r}{\gamma\sigma^2}. \tag{4.3.51}
$$

Following the pure diffusion methodology as before, we obtain:

$$
h^{-\frac{1}{\gamma}} = \frac{1}{\gamma}\left\{\lambda^{(0)} + \rho - (1-\gamma)[\frac{(\mu - r)^2}{2\gamma\sigma^2} + r]\right\}. \tag{4.3.52}
$$

For the healthy investor the procedure is similar, with the difference being that we have to consider the health shock. We let $\Phi(w) = \frac{w^{1-\gamma}\ell}{1-\gamma}k$ and write:

$$c_t^* = wk^{-\frac{1}{\gamma}}; \quad \pi_t^* = \frac{\mu - r}{\gamma\sigma^2}; \quad I_t^* = p^* w. \tag{4.3.53}$$

The following system of implicit algebraic equations needs to be satisfied:

$$\begin{cases} \alpha\eta h\{(1+\alpha p^*)^{1-\gamma} - (\alpha p^*)^{1-\gamma}\} = (1-\gamma)k, \\ \qquad k^{-\frac{1}{\gamma}} = \frac{1}{\gamma}\{\lambda^{(1)} + \rho - (1-\gamma)\left(\frac{(\mu-r)^2}{2\gamma\sigma^2} + r - \alpha p^*\right) \\ \qquad\qquad - \frac{\eta h}{(2-\gamma)k}[(1+\alpha p^*)^{2-\gamma} - (\alpha p^*)^{2-\gamma}]\}. \end{cases} \tag{4.3.54}$$

Note that one can also find the solution when I_t is not optimized by choosing αp as a constant. Next, we will show that for the pre-retirement phase $J^{(0)}(w) = \frac{(w+\beta)^{1-\gamma}}{1-\gamma}h_w$ and $J(w) = \frac{(w+\beta)^{1-\gamma}}{1-\gamma}k_w$, where $k_w = k$ and $h_w = w$ for $\beta = \frac{y_0}{r}$. The subscript of h_w and k_w should not be confused with the partial derivative notation. The solution $h_w = h$ has been derived earlier and will not be repeated here. We will concentrate on the methodology for finding k_w and the value function $J(w)$. We know that $\frac{\partial J}{\partial w} = \hat{w}^{-\gamma}k_w$ and $\frac{\partial^2 J}{\partial w^2} = -\gamma\hat{w}^{-\gamma-1}k_w$ for $\hat{w} = w + \beta$. The optimal control variables are:

$$c_t^* = \hat{w}k_w^{-\frac{1}{\gamma}} \tag{4.3.55}$$

and

$$\hat{\pi}_t^* = \pi_t^* w = -\frac{\mu - r}{\sigma^2}\frac{\frac{\partial J}{\partial w}}{\frac{\partial^2 J}{\partial w^2}} = \frac{\mu - r}{\gamma\sigma^2}\hat{w}. \tag{4.3.56}$$

It is easy to see that

$$\begin{aligned} \pi_t^* w(\mu - r)\frac{\partial J}{\partial w} + \frac{(w\pi_t^*\sigma)^2}{2}\frac{\partial^2 J}{\partial w^2} &= \frac{\hat{\pi}_t^*(\mu - r)}{2}\frac{\partial J}{\partial w} \\ &= \frac{(\mu - r)^2}{2\gamma\sigma^2}\hat{w}^{1-\gamma}k_w. \end{aligned} \tag{4.3.57}$$

Next, we present the derivation of the health insurance control variable. We let $I_t^* = p^* \hat{w}$. We assume that the health-shock-cost amplitude, v, is uniformly distributed on $[0, \hat{w}]$ with density $\frac{1}{\hat{w}}$ similar to the case analyzed in the previous chapter. As before, $\frac{\partial J^{(0)}}{\partial w} = \hat{w}^{-\gamma} h_k$ and we evaluate the jump-term as follows:

$$\eta \int_v \frac{\partial J^{(0)}}{\partial w}(w - v + p^* \alpha \hat{w}) \phi_v dv = \frac{\eta}{\hat{w}}[J^{(0)}(w + \alpha p^* \hat{w}) - J^{(0)}(\alpha p^* \hat{w} - \beta)] \tag{4.3.58}$$

$$= \frac{\eta h_w}{\hat{w}(1-\gamma)}[(1 + \alpha p^* \hat{w})^{1-\gamma} - (\alpha p^* \hat{w})^{1-\gamma}]. \tag{4.3.59}$$

The optimal insurance control variable will satisfy the first-order condition

$$\eta \int_v \frac{\partial J^{(0)}}{\partial w}(w - v + \alpha p^* \hat{w}) \phi_v dv = \frac{\partial J}{\partial w}. \tag{4.3.60}$$

This is further reduced to:

$$\frac{\eta}{1-\gamma}[(1 + \alpha p^*)^{1-\gamma} - (\alpha p^*)^{1-\gamma}] = \frac{k_w}{h_w}. \tag{4.3.61}$$

The HJB for the healthy individual in the pre-retirement phase can be reduced to:

$$(k_w)^{-\frac{1}{\gamma}} = \frac{1}{\gamma}\{\lambda^{(1)} + \rho - (1-\gamma)[\frac{(\mu - r)^2}{2\gamma\sigma^2} + r - \alpha p^*] - \frac{\eta h_w}{2-\gamma}[(1 + \alpha p^*)^{2-\gamma} - (\alpha p^*)^{2-\gamma}]\}. \tag{4.3.62}$$

Note that equations (4.3.61) and (4.3.62) are identical to equation (4.3.54). Since $h = h_w$, we conclude that p^* and k_w are identical to p^* and k in the post-retirement case. Therefore, we have:

$$\Phi = \frac{w^{1-\gamma}}{1-\gamma} k \ell, \tag{4.3.63}$$

$$J = \frac{(w + \beta)^{1-\gamma}}{1-\gamma} k. \tag{4.3.64}$$

The critical retirement-triggering wealth is given by $\bar{w}^{1-\gamma}\ell = (\bar{w} + \beta)^{1-\gamma}$. We can see that we obtain the same solution as in the previous setups, $\bar{w} = \frac{\beta}{\ell^{\frac{1}{1-\gamma}} - 1}$.

4.3.6 Extension 3: Retirement-triggering Wealth for Time-dependent Force of Mortality

For the general case with time-varying mortality rate λ_{x+t}, we need to solve for J and $\bar{w}$ numerically. In this case, we briefly propose a numerical methodology for solving the problem considering a time-dependent force of mortality. In what follows, we present a semi-discrete numerical scheme where time is discretized by intervals t_i. Within the time interval $[t_i, t_{i+1}]$, we can solve the HJB as follows:

$$\frac{J^{i+1} - J^i}{t_{i+1} - t_i} + u_1(c_t^*) + \left[(\pi(\mu - r) + r)w + y_0 - c_t^*\right] \frac{\partial J^{i+1}}{\partial w} + \frac{(\pi_t^* w\sigma)^2}{2} \frac{\partial^2 J^{i+1}}{\partial w^2} = (\rho + \lambda_{x+t})J^i, \quad (4.3.65)$$

for the intermediate value of $J^i = J(t_i, w)$, with $J^{i+1} = J(t_{i+1}, w)$ given. When we carry out the computation, we also need to discretize in w and approximate the derivatives in w numerically. We also need to compute the optimal consumption rate and asset allocation c_t^* and π_t^* by approximating the following expressions numerically:

$$c_t^* = (J^{i+1})_w^{-\frac{1}{\gamma}}; \quad \pi_t^* = -\frac{\mu - r}{\sigma^2} \frac{J_w^{i+1}}{J_{ww}^{i+1}}. \quad (4.3.66)$$

Note that we have used a mixed implicit-explicit formula. In the second step, we pick the final value of J^i by:

$$J^i = \max\{J^i, \Phi^i\}. \quad (4.3.67)$$

The implementation of the numerical scheme for the case of time-dependent force of mortality is left for future work.

4.4 Conclusion

Using analytical and numerical techniques developed in Chapter 3 for the post-retirement phase, we have developed a new optimal portfolio model which can provide a working investor with valuable investment and consumption strategies in the pre-retirement period. Our approach for finding the critical retirement-triggering wealth is an extension of the study provided in Chapter 3. In addition to the theory shown in the previous chapter, we aimed at setting up a model where the investor is still working. The solution will give this individual knowledge of the critical wealth required to enter retirement. Previously, Farhi and Panageas (2007) have developed a similar model, but using an option-valuation approach and convex duality theory. Our study differs from these authors' study by introducing health and mortality risk, as well as considering asset rates of returns which follow a jump-diffusion process. As was observed in the derivation, our critical wealth approximation for both constant mortality and constant rate of becoming sick, as well as for asset returns with jumps, all resulted in the same explicit critical wealth formula. Our formulation provides only an approximate value for the retirement-triggering wealth. To obtain an exact solution, one would have to apply the smooth pasting condition as used by Farhi and Panageas (2007) to our formulation, which is a matter for future work. Our approximate solution is the result of a direct and analytically tractable approach. It can provide a quick guide for an investor interested in the possibility of retiring, by using the approximate critical wealth level information.

An immediate future extension of our method would be to explore how the model behaves under the introduction of a time-dependent force of mortality and a rate of becoming sick. One could follow the methodology proposed in this chapter in order to arrive at the numerical solution for this case. In addition, we expect this extended formulation to produce an exact solution for the critical wealth when one applies the smooth pasting condition as in Farhi and Panageas (2007) and is likewise a matter for future work.

An additional extension to our model could be derived by relaxing the assumption of a constant income stream throught the introduction of the more realistic time-varying or stochastic cases. Further, the model can be calibrated to historical health and mortality rates in order to capture trends observed in real-world data.

5 Conclusion

The subject matter studied in this dissertation is related to the personal finance management of individuals, either before or after retirement. The questions we are trying to answer through this work play an important role for an individual wishing to make the optimal retirement and investment decisions, as the worldwide retirement system offers more freedom to the investors. The first topic is introduced in Chapter 2. Here, we developed a lifetime ruin probability (LRP) model, by assuming that a jump-diffusion process drives the investment return of the agent. Our objective was to investigate the impact that jumps in the portfolio-generating process have on this popular risk-measure. The value of the LRP is important to an investor who wants to find out the probability of running out of money, while maintaining a desired standard of living for the rest of his life. Today, many consumers are formulating their retirement income plans in light of the LRP calculation. The solution methodology adopted in this chapter is based on the theory developed by Huang, Milevsky, and Wang (2004). In our case, we derived an expression for the LRP under the jump-diffusion setup. Employing tools from stochastic control, we arrived at a partial-integro-differential equation for the LRP and related risk measures. We implemented the model with the help of an efficient and robust solution algorithm methodology. We tested our solutions with historical equity return data, by developing a calibration methodology through a moment-matching technique. Results were then compared against diffusion-related LRP values that do not assume jumps. Contrary to common intuition, our main result indicated that the LRP and some related probabilities are in fact lower when we introduced jumps. We speculated that this result was caused by the reduction in the volatility estimate obtained through our calibration method. In other word, we associate the cause of relatively low ruin probability values to the suitable reduction in the diffusion volatility triggered by jumps. Higher values of this parameter are more likely to lead to early losses, thereby increasing the diffusion process probabilities. Our main take-away here, is that practitioners interested in measuring the impact of asset return jumps on investments, can compute the probability risk measure under GBM dynamics and create the effect of jumps through an increase in the associated volatility. The result is not general, as under lower initial investment values, when ruin is more likely, the GBM-related probabilities are lower than those induced by jumps. One could further investigate this model by developing other calibration techniques. To add realism to this model this work could be extended to investigate the effect of more complicated stochastic processes for the volatility and consumption.

The second topic was developed in Chapter 3 and further utilized to develop Chapter 4. The results of that chapter provide optimal investment strategies for an investor in both the post- and pre-retirement phases under more realistic conditions than the ones in the existing literature. Retirees, being generally older investors, are often exposed to large and often unpredictable medical expenses due to health-related shocks. In order to hedge this risk, the retired agent was allowed to purchase out-of-pocket medical insurance. We modeled this health shock as jumps with a known distribution. This facilitated the derivation of solutions for the optimal consumption, investment and medical insurance proportions under some necessary conditions and assumptions. In addition, we considered that the investor also experiences mortality risk. The problem was formulated mathematically as an optimization question and solved through the application of the dynamic programming principle and stochastic calculus theory. We developed a solution methodology, which allowed us to arrive at analytical solutions for the optimal consumption-allocation to risky assets proportions in the case of a sick investor. For a healthy agent, results for the optimal medical insurance were retrieved by solving a system of differential-algebraic equations. The solutions were further confirmed through a comparison between the analytical result and a finite difference numerical scheme. Another objective was to arrive at an optimal investment strategy when the underlying risky asset return follows a jump-diffusion dynamics. For this case, we arrived at a differential-integro-algebraic system. Further implementation of this methodology is left for future work. The proposed methodology for asset returns with jumps ties in with the numerical and calibration methods developed in Chapter 2. We detailed our findings in the conclusion section of this chapter. The main take-away of our results is finding that it is optimal for more risk-averse consumers to allocate a larger proportion to health insurance, thereby hedging the health-decline risk. Moreover, we showed that risk-averse retirees are more likely to consume a smaller portion of their wealth. We also assessed the cases with and without mortality risk, and found that the increased probability of dying before the terminal time, increases the demand for medical insurance and also raises the optimal consumption levels . Meanwhile, the absence of force of mortality results in a lower demand for health insurance or motivation to consume wealth before the imposed terminal time.

A first extension to this model would be to include other distributions for the health cost and to find suitable calibration techniques to the existing health cost data. Other future work on this model, could involve relaxing the assumption that the health status is irreversible. Further, one could investigate the effects of other utility functions, such as Hyperbolic Absolute Risk Aversion (HARA).

These days, voluntary retirement plays an important role in making personal finance decisions. Individuals are often concerned with the best investment strategy that can ensure financial stability throughout their lives. In Chapter 4, we extended the second topic in order to arrive at the best investment choices that provide security both pre- and post-retirement. In considering the option of optimal voluntary retirement time, we formulated the problem by taking an optimal control approach in the life-cycle framework. We arrived at the optimal portfolio choice strategy in the pre-retirement phase and linked it to the answer of when it is optimal to enter retirement. Our model for finding the optimal retirement-triggering wealth opened up the possibility of rich interactions between different risk factors and the optimal consumption-allocation-retirement time strategy. This represents an important financial value for the investor who is interested in exiting the work-force while ensuring that he will be financially secure if he does so. We have observed that, by introducing both health and mortality risks, the trend of the retirement-triggering-wealth levels was maintained. In addition, the wealth level is sensitive to leisure. Small leisure values raise the critical wealth level. Many more interesting extensions are possible. An immediate extension of this model can be developed by considering a time-dependent or a stochastic force of mortality or rate of becoming sick. Moreover, cases for other utility functions more consistent with the real world can also be explored. Other future work, in the context of this model, could involve relaxing the assumption that the health shock is irreversible. Another immediate extension could consider deterministic time-dependent or stochastic income streams.

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