



Solutions to Module 7 Practice Problems: Summation Notation, Series and Sequences

1. Suppose A_i denotes the number of adults and C_i the number of children who visit a pizza store on day i .

(a) Use summation notation to denote (i) the total number of adults, and (ii) the total number of customers, who visit the pizza store in the month of January.

$$(i) \sum_{i=1}^{31} A_i, (ii) \sum_{i=1}^{31} (A_i + C_i)$$

(b) Use summation notation to denote the mean and variance of the number of adult customers per day in January.

$$\text{Mean} = \mu_A = \frac{1}{31} \sum_{i=1}^{31} A_i, \text{ Variance} = \frac{1}{31} \sum_{i=1}^{31} (A_i - \mu_A)^2$$

(c) If each adult customer orders 3 slices of pizza and each child orders 1 slice, how many slices were sold in total in the month of January?

$$\sum_{i=1}^{31} (3A_i + C_i)$$

2. There are n families in a wealthy village. b_i denotes the number of BITCOINs, h_i the number of houses and c_i the number of cars owned by family i .

(a) Use summation notation to denote (i) the total number of BITCOINs, and (ii) the total number of cars, in the village.

$$(i) \sum_{i=1}^n b_i, (ii) \sum_{i=1}^n c_i$$

(b) Use summation notation to denote the mean number of (i) BITCOINs, and (ii) cars, per family.

$$(i) \frac{1}{n} \sum_{i=1}^n b_i, (ii) \frac{1}{n} \sum_{i=1}^n c_i$$

(c) If each BITCOIN is valued at \$60,000, each house at \$1 million and each car at \$40,000, what is the total wealth possessed by all the families in the village? What is the average wealth per family?

$$\text{Total wealth} = \sum_{i=1}^n (60000b_i + 1000000h_i + 40000c_i)$$

$$\text{Average wealth} = \frac{1}{n} \sum_{i=1}^n (60000b_i + 1000000h_i + 40000c_i)$$

3. Write out what is meant by the following:

$$(a) \sum_{i=1}^5 4i = 4 + 8 + 12 + 16 + 20$$

$$(b) \sum_{i=1}^5 \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$(c) \sum_{i=1}^n i^3 = 1 + 2^3 + 3^3 + \dots + n^3$$

$$(d) \sum_{i=0}^4 i3^i = 3 + 18 + 81 + 324$$

$$(e) \sum_{i=1}^5 a_i^2 = a_1^2 + a_2^2 + \dots + a_5^2$$

$$(f) \sum_{i=1}^6 (-1)^i a_i^2 = -a_1^2 + a_2^2 - a_3^2 + a_4^2 - a_5^2 + a_6^2$$

$$(g) \sum_{i=1}^4 \left(-\frac{1}{i}\right)^i = -1 + \frac{1}{4} - \frac{1}{27} + \frac{1}{256}$$

4. Write the following using summation notation:

$$(a) 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \sum_{i=1}^6 i^2$$

$$(b) 4 + 5 + 6 + 7 + \dots + 100 = \sum_{i=4}^{100} i$$

$$(c) 1 + 3^2 + 3^3 + 3^4 + \dots + 3^{30} = \sum_{i=0}^{30} 3^i$$

$$(d) 1 \times 2 + 2 \times 3 + \dots + 19 \times 20 = \sum_{i=1}^{19} i(i+1)$$

$$(e) -1 + 2 - 3 + 4 - 5 + 6 - \dots + 20 = \sum_{i=1}^{20} (-1)^i i$$

$$(f) \frac{1}{n}[(x_1 - \mu_x)(y_1 - \mu_y) + (x_2 - \mu_x)(y_2 - \mu_y) + \dots + (x_n - \mu_x)(y_n - \mu_y)] = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

5. Find the following sums (*hint*: make use of Gauss' formula).

$$(a) 1 + 2 + 3 + 4 + \dots + 800 = 320,400$$

$$(b) 41 + 42 + 43 + 44 + \dots + 800 = 319,580$$

$$(c) 8 + 16 + 24 + 32 + \dots + 800 = 40,400$$

$$(d) 100 + \sum_{i=21}^{80} 4i = 12,220$$

$$(e) \sum_{i=21}^{80} (100 + 4i) = 18,120$$

$$(f) \sum_{i=1}^{80} 4i - \sum_{i=21}^{80} i = 9930$$

$$(g) 1 - 2 + 3 - 4 + \dots - 100 = -50$$

Activity: Use Excel / Google sheets to verify your answers above.

6. Find the following sums (*hint*: make use of the formula for the sum of a geometric series).

$$(a) 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = 2\left(1 - \frac{1}{2^5}\right) = 1.94$$

$$(b) 1 + 2 + 4 + 8 + \dots + 512 = 2^{10} - 1 = 1023$$

$$(c) 2^5 + 2^6 + 2^7 + \dots + 2^{10} = 2^5(2^6 - 1) = 2016$$

$$(d) 1 + 3^2 + 3^4 + 3^6 + \dots + 3^{10} = \frac{3^{12}-1}{8} = 66430$$

$$(e) 1 - 2 + 4 - 8 + 16 - \dots - 512 = \frac{(-2)^{10}-1}{-3} = -341$$

Activity: Use Excel/Google sheets to verify your answers above.

7. Excel/Google sheets activity

(a) In the first sheet of the Excel (or .ods) file “GAMES_Summation_PracticeQuestion.xlsx” are the revenues and costs of a fictional service company, Fictional Inc., over 15 days. Column B gives the company's day by day sales figures. The company employs two types of workers — Grade 1 and Grade 2. The number of workers of each type is given in Columns C and D. It pays \$100 a

day to each Grade 1 worker and \$160 a day to each Grade 2 worker. These are the only costs for the company.

- (i) Calculate the company's cost and profit for each day.
- (ii) What is the company's average sales and average cost per day?
- (iii) How much profit did the company make over the course of the 15 days?

See Excel (or .ods) Solutions file in the module.

(b) In the second sheet of the file are the student marks in a fictional course Econ 000. Column A contains the student id numbers. Columns B, C and D contain their marks on Assignments 1, 2 and 3 respectively. All three assignments were graded out of 40. Column E contains their marks in the midterm (which was out of 50) and column F contains their marks in the final exam (which was out of 100).

- (i) Determine the maximum, minimum and the average mark on each assignment and in the two exams.
- (ii) The overall mark in the course for a student is given by a weightage of 10% on each of the three assignments, 30% on the midterm score and 40% on the final. Determine the overall mark for each student.
- (iii) Determine the maximum, minimum and the average overall mark in the course. Identify the student(s) who have the lowest and the highest scores in the course.

See Excel (or .ods) Solutions file in the module.

Research: Suppose in part (b), two of the students did not take the final exam and so their scores are blank. How does this affect your calculations? In particular, can you get Excel to ignore these students when calculating the average for the class?

8. An annuity is a stream of n cash flows paid at regular intervals. For example, if one were to win a big lottery ticket (e.g. Powerball or Lottomax), one has the option of either receiving the whole amount all at once or as an annuity. In the latter case, one gets 30 equal payments over 30 years. Some retirement benefits also take the same form.

Consider an annuity that will pay \$200 each year over the next 10 years (starting from next year). The present value of receiving \$200 t years from now is given by $\frac{200}{(1+r)^t}$, where r is the yearly interest rate.

- (a) How much is the present value of receiving \$200 five years from now?

$$\frac{200}{(1+r)^5}$$

- (b) Write the total present value of this annuity as a sum of 10 terms, and also using the summation notation.

$$\frac{200}{(1+r)} + \frac{200}{(1+r)^2} + \dots + \frac{200}{(1+r)^{10}} = \sum_{t=1}^{10} \frac{200}{(1+r)^t}$$

- (c) Determine the total present value of this annuity if the rate of interest per year is 5% i.e. $r = 0.05$.

$$\frac{200}{0.05} \left(1 - \frac{1}{(1+0.05)^{10}}\right) = 1544.3$$

- (d) Suppose the lottery ticket that you win offers to give you \$1700 as a lump-sum right now or as an annuity of \$200 per year over the next 10 years. If the rate of interest per year is 5%, which option should you choose according to your present value calculation?

Take the lump-sum now.

- 9.** Consider a car purchase that requires you to pay \$6000 up-front and \$4000 per year over the next 5 years (starting from next year).

- (a) Write the present value of this annuity using the summation notation.

$$6000 + \sum_{t=1}^5 \frac{4000}{(1+r)^t}$$

- (b) Determine the total present value of this annuity if the rate of interest per year is 5% i.e. $r = 0.05$.

$$6000 + \frac{4000}{0.05} \left(1 - \frac{1}{(1+0.05)^5}\right) = 23,318.$$

- (c) Suppose you could buy the car outright for \$24,000 today. Which option should you choose according to your present value calculations?

Better to buy the car with the plan of \$6000 up-front and \$4000 per year for next 5 years.

- 10.** Jim's physiotherapist asks him to increase his weight-training regimen gradually over time to rebuild his strength. She asks Jim to start with 5 lbs on day 1, and then increase it by 2 lbs. every day.

- (a) If Jim were to follow this regimen, how much weight would he be lifting on day 10?

23 lbs

- (b) How much total weight would he have lifted over the 10 days?

140 lbs

(c) If Jim were to follow this regimen, how much weight would he be lifting on day n ? What would be the total weight he would have lifted by then?

$$5 + 2(n - 1), \frac{n(2n+8)}{2}$$

11. Jim's brother Kim goes to a different physiotherapist, who also asks him to increase his weight-training regimen gradually over time to rebuild his strength. He asks Kim to start with 5 lbs on day 1, and then increase it by 20% every day.

(a) If Kim were to follow this regimen, how much weight would he be lifting on day 10?

$$5(1.2)^9$$

(b) How much total weight would he have lifted over the 10 days?

$$129.79$$

(c) If Kim were to follow this regimen, how much weight would he be lifting on day n ? What would be the total weight he would have lifted by then?

$$5(1.2)^{n-1}, 25[(1.2)^n - 1]$$

12. Find the limits of the following sequences:

(a) $1, 4, 9, 16, \dots, n^2, \dots$ Limit = ∞

(b) $1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$ Limit = 0

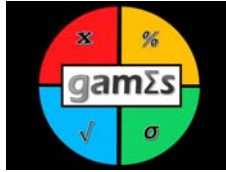
(c) $4, \frac{10}{2}, \frac{18}{3}, \frac{28}{4}, \dots, \frac{3n+n^2}{n}, \dots$ Limit = ∞

(d) $4, \frac{10}{8}, \frac{18}{27}, \frac{28}{64}, \dots, \frac{3n+n^2}{n^3}, \dots$ Limit = 0

(e) $36, \frac{49}{4}, \frac{64}{9}, \frac{81}{16}, \dots, \frac{(5+n)^2}{n^2}, \dots$ Limit = 1

(f) $1, a^4, a^9, a^{16}, \dots, a^{n^2}, \dots$ where $a < 1$. Limit = 0

(g) $1, a^4, a^6, a^8, \dots, a^{2n}, \dots$ where $a > 1$. Limit = ∞



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