# How to Pair the Real Numbers with the Integers: a Game of Kitchen Mathematics

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## How to Pair the Real Numbers with the Integers: a Game of Kitchen Mathematics[1]

> But concerning your proof, I protest above all against the use of an infinite quantity [Grösse] as a *completed* one, which in mathematics is never allowed. The infinite is only a *façon de parler*, in which one properly speak of limits.
>
> Johann Karl Friedrich Gauss[2]

> Mathematics is a game played according to certain simple rules with meaningless marks on paper.
>
> David Hilbert[3]

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[1] 'Kitchen mathematics' is an echo of the Elizabethan phrase 'kitchen criticism' commended for reuse by Clive James. As quoted by Dwight Garner in his "An Appraisal: Clive James, a Tireless Polymath Who Led With His Wit" published on 30 November 2019 in Section C, Page 1, of the New York edition of the New York Times, James once remarked that "It was the term that the Elizabethans once used for the analysis of poetic technique: when to invert the foot, how to get a spondee by dropping a trochee into an iamb's slot, and things like that. Kitchen criticism is a term that should be revived, because its unlovely first word might have the merit of persuading the fastidious to make themselves scarce until they can accept that there is an initial level of manufacture at which the *potatoes have to be peeled*." [italics EWC]. See in turn the 'Autobiographical Conclusion' to this essay, pages 25f below.]

[2] From a letter of Gauss to H. C. Schumacher (Göttingen, 12 July 1831), as quoted by Joseph Warren Dauben on page 120 of his *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton, New Jersey: Princeton University Press, 1990 [First Princeton Paperback printing; original copyright, Harvard University, 1979], hereafter referred to as 'Dauben [Cantor]'. The letter can be found within Vol. II of the *Briefwechsel zwishen C. F. Gauss und H. C. Schumacher*, edited by C. A. F. Peters (Altona, Germany: G. Esch). I am indebted to Dauben as well for his later account of the life and work of Abraham Robinson entitled *Abraham Robinson: The Creation of Nonstandard Analysis – A Personal and Mathematical Odyssey* (Princeton, New Jersey: Princeton University Press, 1995), hereafter referred to as 'Dauben [Robinson]'.

[3] A summary by Eric Temple Bell of a conviction of David Hilbert, represented as a quotation but without citation of source on page 21 of Bell's *Mathematics, Queen and Servant of Science* (1951, 1961). Bell often wrote incautiously on other matters (see Dauben [Cantor], pages 278 and 280, for example, on the 'Freudian analysis' by which he blamed Cantor's father, by all accounts a compassionate and devoted parent, for having deeply implanted within him the "anxieties" that would later blossom into the sequence of "severe mental breakdowns", often requiring long periods of isolation within hospital, that would eventually kill him). The sentence reproduced above, however, captures succinctly the sense of Hilbert's belief.

Can one pair the real numbers with the integers? I believe so, believing as well that the proof follows so simply from the work of Abraham Robinson a half-century ago that neither he nor readers thereafter, minds on other matters, noticed.[4]

> Had they done so, they would have realised that the pairing obliterates the conclusion and methods of Georg Cantor's 'proofs' to the contrary and therewith his insistence that sets are required for comprehending mathematics.
>
> Were there 'no more real numbers than integers', we should have no reason to suppose that there are uncountably many things of any kind within our world, numbers included, and hence no need for 'sets', much less sets of 'infinite size', with which to calculate how to push and shove them about.

Using the standard axioms of set theory, Robinson, an admirer of Leibniz, constructed a mirror image of the methods, constructions and conclusions of 'standard analysis' (the 'calculus' as commonly construed) that he called 'non-standard analysis', and rightly so, for it sanctioned the use of differences so small that no real numbers could represent them. By doing so, Robinson confirmed that one can speak as coherently of real numbers differing *infinitesimally* from one another as of the real numbers themselves.

Robinson's account encompassed Cantor's transfinite numbers, and he introduced infinitesimals 'from the top down', as it were, as reciprocals of them.[5] I, on the contrary, avoiding sets of any kind, shall work 'from the bottom up' with real numbers differing infinitesimally from one another, having long suspected that Cantor's 'proofs' that the

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[4] Abraham Robinson, *Non-Standard Analysis* (Amsterdam: North-Holland Publishing Company, 1966). Robinson first presented his ideas to a seminar at Princeton University in 1960.

[5] Provoked by Robinson's work, John Horton Conway soon after encompassed both transfinite and infinitesimal numbers among his 'surreal' numbers'. See D. E. Knuth's representation of Conway's work within his *Surreal Numbers: How Two Ex-Students Turned On to Pure Mathematics and Found Total Happiness a Mathematical Novelette* (Reading, Massachusetts: Addison-Wesley Publishing Company, 1974), the popular book from which readers, Conway among them, took the term 'surreal'.

Robinson's readers may have assumed, and perhaps correctly, that he believed at the time that one had to acknowledge sets of uncountably infinite size to play mathematical games of merit. Shortly before his death in 1974, however, he affirmed that he had now come to suspect that there was no need for them, though he neither published nor amplified his suspicion, remarking upon it only when conversing privately with students, colleagues and friends. If their reports are trustworthy, however, he would yet again have glimpsed with insight how the logic of mathematics could be reconceived for the better. See Dauben [Robinson], pages 354 and 355.

real numbers could never be paired with the integers were faulty, but sensing no way, before reading Robinson, of proving it.[6]

> I shall indeed take little more from Robinson's work than the assurance that one can play games with real numbers differing infinitesimally from one another – among them a game pairing the integers, *neighbours* alongside, with the real numbers, *neighbours* alongside.[7]

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[6] I shall within this essay, footnotes aside, write as 'naively' as I can in homage to Paul Halmos, presuming that readers competent to care can supply unmentioned premises as needed, prompted by hint and context. (See Paul R. Halmos, *Naïve Set Theory* (New York: Springer-Verlag, 1974 [1960]). I shall speak *almost always*, for example, as did Cantor and Richard Dedekind, as if the terms 'real number' and 'decimal' were interchangeable, and that each real number can be represented unambiguously by a decimal in standard notation. See pages 13f below, however, for the crucial counter to this practice (!). Note as well, and in particular, that I shall nowhere pretend that a decimal is an *infinite* number. A decimal (3.141591 . . . , the number $\pi$, for example) is rather the sum of an integer and a *finite* sequence of rational numbers having 0 to 9 as numerators and, from first to last, a power of 10 in increasing order of size as denominators, *accompanied by a rule* by which to calculate further terms of the sequence from those *already in hand*.

My suspicion that sets were unnecessary arose from my readings as an undergraduate student of Willard Quine's essays on ontological commitment and the scope and nature of logic, and a bit later on 'set theory'. Quine, with Nelson Goodman, had tried in 1947 to show how sets might be avoided within mathematics, but soon after gave up the quest. (See Nelson Goodman and W. V. Quine, "Steps toward a constructive nominalism", *Journal of Symbolic Logic 12* (1947), pages 105-122.) Goodman, however, continued to insist to the end of his life that sets could never be viably countenanced, despite the consensus within mathematics to the contrary. As summarised succinctly by Daniel Cohnitz and Marcus Rossberg, "Goodman's mature nominalism, from *The Structure of Appearance* onwards, is a rejection of the use of *sets* (and objects constructed from them) in constructional systems." (Entry on 'Nelson Goodman' within the *Stanford Encyclopedia of Philosophy*, copyright 2019 by Cohnitz and Rossberg.) Goodman's resistance to sets was and remains for me a model of philosophical rectitude, for, as this essay will confirm, he was right.

My suspicion that an *indirect* proof begs whatever question it purports to answer began even earlier when, as a junior in high school, I first encountered Euclid's 'proof' that the natural numbers could have no greatest member, since, having supposed one to be so, a greater could always be constructed by adding 1 to it. But if the supposed natural number were indeed the greatest of them, no natural number could be added to it. (More exactly, Euclid's rule of addition, applicable to lesser numbers, could never be applied to 'the greatest of them' without presupposing it to be other than it is.) The resonance of this suspicion was to amplify ever after for me. (See pages 9f. and pages 11f. below.)

[7] I shall presume quietly as well, however, that the burden of proof with respect to suggestions of incoherence brought against anyone speaking of infinitesimal differences, my

What are *neighbours*? and why are they important to the pairing of kinds of numbers with one another?

**Prolegomenon to the Proof:**
**Neighbours and Pairings**

Consider the real numbers 1.000000 . . . and 0.9999999 . . . , a paradigm of pairs of numbers often misconstrued as if identical. Differing infinitesimally, they are *neighbours*, other real numbers being bigger or smaller than either of them .[8]

1.000000 . . . is the *larger neighbour* of 0.999999 . . .
0.999999 . . . is the *smaller neighbour* of 1.000000 . . ..

Let 'ε' be the infinitesimal difference between them.

$$\varepsilon = (1.000000\ldots - 0.999999\ldots)$$

Were the equation for ε misconstrued, the difference as defined might seem unique to the decimals named within it. Being *infinitesimal*, however, the difference is *identical* to the infinitesimal difference between every other pair of neighbouring real numbers.

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own included, has – after Robinson's work – shifted. Claimants must now prove it so, rather than compelling those who know better to waste time tilting at fantasies.

[8] The two decimals are distinct, for no matter how far we may extend them, their expansions differ at every term (though only one would suffice). Their expansions converge to the same 'limit', but that is a tool of convenience for calculation having nothing to do with their identities. (See pages 10f. below for an account in historical context of why the notion of 'limit', if misconstrued otherwise, begs rather than addresses the question.)

Fun-loving readers may enjoy the accounts, available on the internet, of the frustrations shared by teachers of mathematics unable to convince their students of the supposed identity of the two numbers. (See, for example, Katz, K., & Katz, "When is .999... less than 1?" (*The Montana Mathematics Enthusiast, 7*(1), 2010), pages 3–30). Though competition is fierce, my prize for pretence goes to the 'educators' who, having written on the blackboard that '1/3 = .333333 . . . ', instructed their students to multiply each side of the equation by 3, writing below '3/3 = 1 = .999999 . . . , only to discover that a good many of the best of the students remain unconvinced. Few of the latter, of course, would have had the wit or nerve to insist that the teacher show them firstly how to multiply the decimal .333333 . . . by '4', '5', or '6' before ordering them to multiple it by '3', sensing correctly that if one doesn't know how to multiple a number by '4', '5' or '6', one doesn't know how to do it by '3' either. Few teachers of mathematics of my acquaintance, even at university level, would have known how even to begin to multiple a decimal arbitrarily chosen by an integer, much less by itself or by another decimal.

For any real number 'm', therefore, we may distinguish its smaller neighbour by subtracting ε from it, and its larger neighbour by adding ε to it.

the smaller neighbour of m = (m – ε)
the larger neighbour of m = (m + ε ).

Were m = π, for example (π = 3.14159 . . .), its smaller neighbour would be (π – (π – k)) and its larger neighbour (π + (π – k), each computable as accurately as the limited expansions of the real numbers with which we are working permit.

Every real number, consequently, has both a smaller and a larger neighbour that can be specified as fully and exactly as the real numbers brought to bring to bear upon it.

> When ordered by size, in other words, the real numbers are an unbroken and unbounded sequence of *neighbours*.

Of what importance could this be?

**The Pattern of Pairings**

Recall two occasions within the history of mathematics when numbers of diverse kinds were acknowledged to be pairable with the natural numbers, the first of which occurred half-a-millennia ago.

In 1638 Galileo noted an oddity with respect to the natural numbers. Some among them (the even, odd or square numbers, for example) could be paired with the natural numbers themselves.[9]

| | | | | | |
|---|---|---|---|---|---|
| Natural Numbers: | 1 | 2 | 3 | 4 | . . . |
| Even Numbers: | 2 | 4 | 6 | 8 | . . . |
| Odd Numbers: | 1 | 3 | 5 | 7 | . . . |
| Square Numbers: | 1 | 4 | 9 | 16 | . . . |

[9] See the discussion between Salviati, Sagredo and Simplicio on pages 40f. of Stillman Drakes' translation with introduction and notes of Galileo's *Two New Sciences* (Madison,

But how could this be? How could the natural numbers be paired with only *some* of them? Though perplexing, the pairings, having no use, were bypassed as curiosities, no one having asked, much less answered, a question of importance.

> What do the natural numbers of a kind have in common with those of another kind, and with the natural numbers themselves, that enables them to be paired?

Had the question been asked, an answer might have registered. Except for the number with which each sequence begins,

> Each of the natural numbers of its kind, and each of the natural numbers themselves, has a *neighbour* to its left and another to its right – a greatest number smaller and a smallest number greater than it.[10]
>
> To have paired a natural number of its kind with one of another kind, or with one of the natural numbers themselves, is to have paired their *neighbours* as well, and therewith every natural number to the left or right of them.

In 1873, two and half centuries later, Georg Cantor broadened the scope of the answer by proving that one could pair the natural numbers with the *rational* numbers as well. But how could this be? for, unlike the natural numbers, when the rational numbers are ordered by size, many lie between any two of them, none having a smaller or a greater neighbour.

> With unprecedented ingenuity, Cantor realised that he could *reorder* the rational numbers *other than by size* so that each of them would have a neighbour to either side.

The positive rational numbers can be arranged, for example, within a two-dimensional array with those within the columns from left to right ordered by the size of their numerators and those within the rows from top to bottom by the size of their denominators. Each number within the array can then be construed as part of a sequence within which each has a neighbour to either side. [Follow the arrows below.]

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Wisconsin: University of Wisconsin Press, 1974 [1638]. Here and elsewhere I shall speak of 'integers' rather than the 'natural numbers', even when discussing authors, like Galileo, who spoke only of the latter.

[10] The initial number of each sequence, of course, has a neighbour only to the right.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1/1 | 2/1 | 3/1 | 4/1 | 5/1 | . . . | etc. |
| 1/2 | 2/2 | 3/2 | 4/2 | 5/2 | . . . | |
| 1/3 | 2/3 | 3/3 | 4/3 | 5/3 | . . . | |
| 1/4 | 2/4 | 3/4 | 4/4 | 5/4 | . . . | |
| 1/5 | 2/5 | 3/5 | 4/5 | 5/5 | . . . | |

. . . etc.

Neighbours in place along the sequence, the rational numbers can be paired with the natural numbers.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Natural Numbers: | 1 | 2 | 3 | 4 | 5 | 6 | . . . etc. |
| Rational Numbers: | 1/1 | 2/1 | 1/2 | 1/3 | 2/2 | 3/1 . . . | |

Cantor's proof was accepted immediately by his peers, for, however wondrous, it was by common consent 'constructive': one could calculate the natural number paired with any rational number, and conversely.

As everyone realised as well (accustomed, unlike Galileo, to playing with both negative and positive numbers, 0 in between), one could easily expand the pairings in hand to encompass those between the positive and negative integers of their kind with the integers themselves, the negative numbers extending to the left of 0, the positive numbers to its right.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Integers: | . . . | -3 | -2 | -1 | 0 | 1 | 2 | 3 | . . . etc. |
| | | | | | | | | | |
| Even Integers: | . . . | -6 | -4 | -2 | 0 | 2 | 4 | 6 | . . . |
| Odd Integers: | . . . | -5 | -3 | -1 | 0 | 1 | 3 | 5 | . . . |
| Square Integers | . . . | -9 | -4 | -1 | 0 | 1 | 4 | 9 | . . . |
| | | | | | | | | | |
| Rational Numbers | . . . | -1/2 | -2/1 | -1/1 | 0 | 1/1 | 2/1 | 1/2 | . . . |

The five pairings exemplify a pattern confirming a judgment crucial to the design of the proof that follows, for I am unable to conceive of a counterexample.

> An unbounded sequence of numbers can be paired with another *if and only if* each of the numbers within them has a *neighbour* to either side.
>
> If and only if, that is, each of the sequences consists of a succession of *neighbours*.

Since each of the real numbers has a neighbour to either side, as do the integers, the real numbers can be paired with the integers, and the *proof* is so simple that the word seems almost inappropriate.

## The Proof

Each of the real numbers has a larger and a smaller *neighbour* – a smallest real number greater than it and a greatest real number smaller than it – differing from it *infinitesimally*.[11] The real numbers are a succession of neighbours.

Each of the integers has a larger and a smaller *neighbour* – a smallest integer greater than it and a greatest integer smaller than it. The integers are a succession of neighbours as well.

> One can therefore pair each of the real numbers, neighbours alongside, with an integer, neighbours alongside, and conversely.

The real numbers can consequently be paired with the integers. **Q.E.D.**

[11] See pages 4f. above.

**What has been proven?**

Lest the simplicity of the proof mask its importance, let me amplify what has been proven.

Unlike Cantor's demonstration of 1873 that one can pair the rational numbers with the natural numbers, the proof is in a sense less 'constructive', for, having paired the real numbers with the integers, the real number paired with an integer arbitrarily chosen, or conversely, is indeterminable. In a more crucial sense, however, it is far more 'constructive' than the 3rd and most celebrated of Cantor's supposed 'proofs' to the contrary, for it is in no way 'indirect'.[12] No theses, presumed true at its beginning, are rejected as false at its end, having supposedly been shown to contradict its conclusion. Rather, it builds upon premises free of *negation*, relying solely upon rules of inference also free of *negation*.[13] The proof is thus as logically secure as the real numbers and integers paired within it.

**Epilegomenon to the Proof:**
**How and Why Cantor Failed to Prove Otherwise**

By the time I first encountered Cantor's 3rd 'proof' that the real numbers could never be paired with the integers, and that sets of uncountably infinite size had therefore to be accommodated within mathematics, I had long suspected that both its method and consequences were faulty, and I was in good company.

By the middle of the 19th century, mathematicians, acquainted with logic only by habit but determined nonetheless to purge the calculus of its 'illogical' reliance on infinitesimals (Leibniz and Newton having used them when creating it only to discard them when no longer useful), had contrived a clever way of playing the game of [standard] 'analysis' to comparable ends by pruning the terms within the supposed unending expansion of decimals of those too small to be of further use, pretending that they could thereby identify without impediment the decimals whose expansions converged upon identical 'limits', despite the terms and values of the decimals differing at every step of the way.

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[12] See the following section of this essay for a dissection of Cantor's 3rd 'proof'.

[13] Readers of other essays within the Evan Wm. Cameron Collection will already have registered my impatience with those unable to acknowledge that only games of logic pruned of 'negation' can be played coherently. See in particular the relevant passages of "Film, Logic and the Historical Reconstruction of the World" [1995].

The decimals 0.999999 . . . and 1.000000 . . ., for example, though differing everywhere in appearance and value, were deemed to be indistinguishable, for they converged upon the same 'limit' – the decimal 1.000000 . . . . Needing only the larger decimal to 'analyze' the events of the world, the smaller was banished from the cast of actors (decimals) to be used within the play of [standard] 'analysis' being performed.

> To do so, however, was to destroy the *raison d'être* of the very system of notation for decimals upon which Cantor would rely within the 3rd of his supposed 'proofs' that the real numbers between 0 and 1 could never be paired with the integers.

As noted above, Cantor had contrived wondrously in 1873 to pair the rational numbers with the integers by reordering the former other than by size.[14] Convinced, however, that no such reordering of the real numbers could be devised, he set out to prove that the real numbers could never be paired with the integers.

A year later, in 1874, he published the first of three supposed 'proofs' of the conjecture, revising it in 1879 and then, a dozen years later in 1891, advancing a third of a notably different kind relying upon what would later become known as the 'diagonal method', having insisted all along, but with increasing vehemence toward those who thought otherwise, that he had thereby shown that games of mathematics had to encompass transfinite numbers and therewith sets, uncountable and otherwise.[15]

> Why did Cantor put forward more than one 'proof' of the conjecture? Because, as he realised with increasing dismay, a goodly number of mathematicians found none of his arguments conclusive – among them (then or thereafter) luminaries such as Kronecker, Poincaré, Brouwer and Skolem.

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[14] See pages 6f above.

[15] Cantor put forward his 3rd 'proof' within an essay entitled "Uber eine elemantare Frage der Mannigfaltigkeitslehre", published on pages of 75-78 of *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. I (1891). The work was reprinted in 1932 as pages 278-280 of *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, edited by E. Zermelo (Berlin: J. Springer [reprinted Hildesheim: Olms (1966)].

No one of the dissenters managed to show to the satisfaction of others, however, how Cantor's 'proof's had failed, much less to prove to the contrary that the decimals could indeed be paired with the integers. Cantor and fellow travellers could therefore dismiss their discontents either as misunderstandings of the games now playable with transfinite numbers, or as misguided attempts to play different games by different rules of little consequence.

The nonbelievers had sensed correctly, however, that Cantor 'proofs' were faulty, though their instincts lacked focus, beclouded by the suspicion that one could never play coherently with real numbers differing *infinitesimally* from one another. Both their sense and suspicion deserve attention, however, for by bringing them to bear upon the question-begging opening of Cantor's 3rd 'proof', much can be learned about the dysfunctional methods that many players bring to their playing of mathematical games.

Cantor began his 3rd 'proof' by asking readers to suppose that the real numbers between 0 and 1 could be paired with the positive integers, giving as an example a sequence of positive integers to the left, ordered by size, paralleling one of decimals to the right, ordered otherwise.[16]

| | |
|---|---|
| 1. | 0.000000 . . . |
| 2. | 0.**2**78252 . . . |
| 3. | 0.4**5**8867 . . . |
| 4. | 0.15**2**210 . . . |
| 5. | 0.009**3**95 . . . |
| 6. | 0.0019**9**3 . . . |
| 7. | 0.63234**8** . . . |

Etc.

He then instructed his readers to imagine a decimal differing at each of its terms from the one running downward *diagonally* from 0.000000 . . . [bold-faced above]. The new decimal would then differ from each of the decimals within the pairing in at least one of its terms (the term replacing the one bold-faced above).

[16] The sequence that follows differs, but only typographically, from Cantor's own.

Consequently, or so Cantor insisted, the real numbers could never be paired with the integers, for one could always, as above, 'construct' one missing from any imagined pairing of them.[17]

> What's wrong with Cantor's 'proof'? It *begs the question* that it purports to answer.

**Decimals Missing from Cantor's 'Exemplary' Sequence**

Cantor relies within his 'proof' upon a rule of inference, the 'diagonal method' of constructing a decimal missing from a sequence of them, that could only be applied to a sequence of decimals from which one or more were missing – such as the one with which he begins the 'proof', thereby presupposing what it pretends to prove!

> Were the 'exemplary' sequence of decimals with which Cantor begins his 'proof' *complete*, encompassing them *without exception*, no decimal could be missing from it (however 'constructed' by whatever method, 'diagonal' or otherwise).

Let me amplify that conclusion. In the 1870s, Cantor and Richard Dedekind, while exchanging letters on their work, advanced a series of conjectures that we now lump together under the rubric 'the Cantor-Dedekind Axiom' – the suggestion that one can pair the real numbers with the points on the geometric line, the distance of each of the points to the left or right of 0.000000 . . . (a 'point of origin' arbitrarily chosen) expressible by a decimal paired with its real number. Convinced that no points on the line remained unpaired with a real number, Cantor and Dedekind referred to it as the 'Continuum', and the name stuck, despite the discomfort of mathematicians who remained unconvinced (Weyl, for example).

As we now know from Robinson's work, however, the term was inexact, for the 'Continuum' of Cantor and Dedekind failed to encompass one of every two real numbers differing infinitesimally from each other (admitting 1.000000 . . . , for example, but disallowing 0.999999 . . . , its smaller neighbour). Unsurprisingly, therefore, they are missing from the sequence of decimals with which Cantor began his 3rd 'proof'.

> How would Cantor's sequence of decimals have had to appear were it to have encompassed them without exception?

[17] Within the 'proof', Cantor attended solely to the real numbers between 0 and 1. The restriction is inconsequential, however, for had the 'proof' been valid, it could easily be extended to encompass the others.

The numerals after the decimal point within the name of a decimal, designating in order the numerators of its terms, can range from '0' to '9'. A complete representation of the decimals between 0.000000 . . . and 1.000000 . . . would therefore have had to encompass without exception every sequence of numerals capable of naming a decimal lying between them. Had it done so, no one could have 'constructed' by any method ('diagonal' or otherwise) a purported name for a decimal supposedly missing from it, for the name of each decimal between 0.000000 . . . and 1.000000 . . . would already have appeared within the sequence, naming a decimal.

But there is a depth yet to be plumbed to the inadequacy of the sequence of 'the real numbers' with which Cantor began his 'proof', for it was incomplete with respect to real numbers having no representation as a decimal as well. (Yes, Virginia, there are many such numbers!)

**Real Numbers having No Decimal Representation**

Until now I have spoken as if Cantor and Dedekind were correct in assuming that each real number could be expressed by a decimal. Their assumption requires renewed attention, however, for the sum of the initial terms of two kinds of real numbers can be used within arithmetical calculations of others, only the first of which can be represented unambiguously as decimals.

One can calculate both the smaller and the larger neighbours of any real number by respectively subtracting or adding to it the infinitesimal difference between 1.000000 . . . and 0.999999 . . . [18] When doing so, however, one encounters a variance between those that have and those that lack representation as decimals, among the latter a good many of the real numbers paired with another from which they differ infinitesimally – real numbers that consequently could never have appeared within the sequence of decimals with which Cantor began his 3rd 'proof'.[19]

When determining the smaller neighbour of 1.000000 . . ., for example, one can name in order the *completed* decimals comprising the sequence of sums as calculated step-by-step.

0.9, 0.99, 0.999, 0.9999, . . . etc.

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[18] See pages 4f. above.

[19] The importance of this to our understanding of the scope and nature of the real numbers has never, to my knowledge, been recognised.

Because each of the names reproduces the numeral(s) '9' appearing within the name preceding it, one can represent the smaller neighbour of 1.000000 . . . unambiguously as '0.999999 . . . '.

When determining the larger neighbour of 1.000000 . . . , however, one encounters an unsurmountable obstacle! One can, as before, name in order the completed decimals as calculated.

1.1, 1.01, 1.001, 1.0001, . . . etc..

Because, however, the first appearance of the numeral '1' after the decimal point moves relentlessly to the right within the names as the completed decimals are expanded, no *decimal* can represent unambiguously how the terms of the sequence would appear were one to expand it by rule from those already in hand!

> Within an intelligible decimal, the numerals that designate in order the numerators of the terms can only occur *before* the three-dots symbolising the unbounded continuation of the sequence [as in '1.01 . . .' , '1.001 . . '. or '1.0001 . . .' , etcetera].
>
> Were one to put forward instead a sequence of numerals before *and after* the three-dot symbol '. . .' within a supposed name of a decimal, (as, for example, within '1.010000 . . . 1234', or '3.141591 . . . 0026'), the numerals after the symbol '. . .' would be *meaningless* having (referring, but only purportedly, to numerators of pseudo ratios having no specifiable powers of 10 as denominators).

Though one may calculate term-after-term both the smaller and the larger neighbours of 1.000000 . . ., only its smaller neighbour can be represented unambiguously by a (0.999999 . . . ). No initial sequence of numerals within a decimal could represent its larger neighbour, for only a misconceived representation of a supposed 'decimal of infinite length' having a numeral 1 beyond reach at its *end* could impossibly do so.

The larger neighbour of 1.000000 . . . is only one among the many real numbers lacking unambiguous representation by a decimal, every one missing from Cantor's supposedly 'exemplary' sequence of them.

**Succinctly Summarised**

Cantor began his 3rd 'proof' with a sequence of decimals purporting to encompass 'the real numbers'. The sequence, supposedly exemplary, would have had to have been *complete* were the steps to follow to have been of even promissory consequence, for to 'construct' a decimal supposedly missing from it would otherwise be of no importance.

Cantor's sequence of 'the real numbers', however, was *incomplete*, lacking not only decimals differing infinitesimally from others but many unexpressible as decimals. The following steps of his 'proof' were consequently little more than the manipulation of numerals masquerading as the 'construction' of a number, the exercise begging the question that the 'proof' was pretending to prove.

> Cantor had misconstrued the 'mentioning' of numbers with the 'using' of them, presupposing that by replacing numerals within the name of a decimal (the 'diagonal'), he was thereby 'constructing' a decimal with terms were named by the numerals, despite having given no *mathematical* rule by which to calculate any of its terms beyond those supposedly named by the numerals *already in hand*.

So where and how did Cantor's train go off the tracks? It never left the tracks, having never left the station. Cantor's journey ended before it began.

## Cantor's Religious Game

The real numbers can be paired with the integers. One need never, therefore, contra Cantor, speak of sets save as chatter of convenience, for with no sets having uncountably many members, we need never speak of sets at all. How, then, could so many for so long have believed otherwise? The metaphoric answer is simple.

> Players within the upper reaches of the tree-houses of mathematics have been well-trained to avoid interrupting their games by glancing downward into the shadows cast by the tree, much less to do so by descending to ground level to ponder dark matters of logical import that might prove disturbing if too closely attended, especially in historical context. 'Playing by the rules' reinforces habits that preclude one from attending, as one plays, to the rules themselves and therewith to unintended consequences of them.[20]

But that, of course, is true of players of games of other kinds as well, one among them of especial significance.

For more than a millennium before Cantor, students of theology within the dominant cultures of the west spoke decisively of God, other gods, the Devil and angels (arch and otherwise, and whether on or off heads of pins), despite no one among them, or anyone else for that matter, having ever encountered any one of the things of which they spoke. As I write a century half later after Cantor, students of mathematics and its logic speak decisively of numbers, points, groups, fields, rings, surfaces, derivatives, categories – of how they *are* and of how they *relate* to others – despite no one among them, or anyone else for that matter, having ever encountered any of the things of which they speak.

Cautious thinkers gave up playing games of the earlier kind long ago. Players of games of the latter kind, however, are ubiquitous. Am I joking, then, when comparing the two? Perhaps, but that depends on the depth of one's sense of humour.

Playing a game may be fun. Playing with pretence is no joke.

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[20] For example, though Cantor's 'diagonal method' was inapplicable to the premises of his 'proof' without begging the question, mathematical logicians were nevertheless to rely thereafter upon adaptations of it as if warranted when fashioning the most celebrated conjectures of the following century, among them the 'proofs' by Gödel that arithmetic is incomplete, by Turing that some numbers are incomputable and by Cohen that the Continuum Hypothesis is undecidable. Are their arguments, as 'indirect' as Cantor's, mistaken? Formally, 'No', but, like his, substantially 'Yes', resting at root upon disguised confusions of use and mention (the very kind of possibility that, as Gödel admitted, he would have had to skirt within his argument for it to succeed by 'walking a line' without falling into a paradox akin to that 'the liar') – or so I have long suspected, though having only lately returned to the fun of reconstruing resolutely the completeness of their premises and the coherence of their rules of inference with the nonsense of negation pruned from them. But that, as they say, is 'the subject for another essay' (slightly larger?) that I have no intention of writing.

Consider a pair of puzzles with respect to how Cantor behaved as he tried recurringly to prove that the real numbers could never be paired with the integers while dismissing concurrently the attempts of others to work with infinitesimals.

**First Puzzle**

The flaws of Cantor's 3rd 'proof' are so evident that one can hardly avoid wondering how he could have failed to be aware of them. Or did he rather, while sensing or at least suspecting its defects, persist nonetheless in advancing the 'proof' in pursuit of an end of such encompassing importance as to render inadequacies of means unworthy of more than superficial attention.

> Was he unaware of the impotence of what he was doing, or ignoring it for 'higher reasons' left unadvertised?

Upon stepping back to register the historical record more amply, one notes that Cantor *often* begged the question of 'proofs' that he put forward by relying upon premises and rules of inference that presupposed the conclusions that he would thereafter claim to have proven by means of them. Why did he do it?

Consider, for example, Cantor's "ironical" failure (as Dauben puts it) to realise that the core of his criticism of others for playing with infinitesimals "could have been turned as effectively against the transfinite numbers as against infinitesimals".[21]

In the third century BCE, Archimedes suggested (in effect) that the smaller of a pair of real numbers could always be multiplied by another, the product exceeding the larger of them. Cantor, refusing to countenance 'numbers' that he deemed to be 'non-linear' (any, that is, that he was unwilling to acknowledge to be "bounded, continuous lengths of a straight line", among them any "smaller than any arbitrarily small finite number"), claimed in a letter of 1887 to Weierstrass that he could therefore 'prove' the conjecture of Archimedes rather than construing it as 'axiom', confirming, or so he insisted, that, since such numbers [infinitesimals] would "contradict the concept of linear numbers", they "do not exist".[22]

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[21] Dauben [Cantor], op. cit. (footnote 2 above), page 131.
[22] Ibid., page 130.

Dauben notes, however, that Cantor's 'proof' was not only circular, but obviously so.

> [Cantor's] . . . rejection of infinitesimals was certainly fallacious in *its* reliance upon a *petitio principii*. Having assumed that all numbers *must* be linear, this was equivalent to the Archimedean property, and it is thus no wonder that Cantor could 'prove' the axiom. The infinitesimals were excluded by his original assumptions, and his proof of their impossibility was consequently flawed by its own circularity.[23]

As the Swedish mathematician Mittag-Leffler, founding editor of *Acta Mathematica*, confirmed in 1883 by querying Cantor "about the possible existence of infinitesimals as numbers interpolated between the rational and the irrational numbers", talk of infinitesimals was common among Cantor's peers.[24] To many of them, indeed, it seemed obvious that if one could play games with transfinite numbers, as Cantor insisted, then one could do so with infinitesimals as well.

In a review of 1885 of Cantor's *Grundlagen …* , for example, Benno Kerry not only affirmed it, but showed how infinitesimals could be construed as reciprocals of transfinite numbers, foreshadowing the definition of Abraham Robinson a century later.[25,26]

> In my opinion a *formal definition* of definite, infinitely small numbers is indeed given in fixing the greatest of such numbers as one which produces the sum 1 by adding itself to itself ω times; the next smaller is then the one which produces 1 by adding itself to itself ω – 1 times, etc. The definite, infinitely small numbers would accordingly be denoted as:
>
> $$\frac{1}{\omega}, \quad \frac{1}{\omega+1}, \quad \ldots, \quad \frac{1}{2\omega}, \quad \ldots, \quad \frac{1}{\omega^2}$$
>
> etc. Of course whether numbers so defined have any *empirical applicability* is not decided here.

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[23] Ibid., page 131.

[24] Ibid., page 131. Quotation from a letter of Mittag-Leffler to Cantor, dated 07 February 1883, reproduced on page 234 of H. Meschkowski, *Probleme des Unendlichen: Werk und Leben Georg Cantors* (Braunschweig, Germany: Vieweg, 1967).

[25] Georg Cantor, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre: Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen* (Leipzig: B. G. Teubner, 1883).

[26] Dauben [Cantor], pages 129 and 130.

## Second Puzzle

Convinced of the need for transfinite numbers while deeming infinitesimals a delusion, Cantor could have commended the former while discounting the latter without much shouting. Instead, however, he denigrated both the playing of games with infinitesimals and the players themselves with a fury that bewildered his peers. As Dauben recounts,

> In 1893 [Giulio] Vivanti, the Italian mathematician, wrote to Cantor suggesting that his rejection of infinitesimals was unjustified. As du Bois-Reymond had shown, 'the orders of infinity of functions constitute a class of one-dimensional magnitudes, which include infinitely many small and infinitely large elements. Thus there is no doubt that your assertions cannot apply to the most general concept of number.'[27]
>
> In defending his point of view, Cantor wrote back to Vivanti with the vitriolic remark that to the best of his knowledge, Johannes Thomae was the first to 'infect mathematics with the Cholera-Bacillus of infinitesimals'. Paul de Bois-Reymond, however, was soon to follow. Cantor claimed that in systematically extending Thomae's ideas, de Bois-Reymond found 'excellent nourishment for the satisfaction of his own burning ambition and conceit.' Cantor went on to discredit du Bois-Reymond's infinitesimals because they were self-contradictory, since he rejected without compromise the existence of linear numbers which were non-zero yet smaller than any arbitrarily small real number. Infinitesimals, Cantor replied, were complete nonsense. He reserved his strongest words for du Bois-Reymond's *Infinitäre Pantarchie* and his orders of infinitesimals: 'Can one still call things numbers? You see, therefore, that the 'Infinitäre Pantarchie', belongs in the waste-basket *as nothing but paper numbers!* Cantor placed the theory of actual infinitesimals on a par with attempts to square the circle, as impossible, sheer folly, belonging to the scrap heap rather than in print.

[27] Dauben [Cantor], page 131. Vivanti's comment and Cantor's denigration of Johannes Thomae in the following paragraph can be found on page 505 of H. Meschkowski's "Aus den Briefbüchern Georg Cantors", pages 503-519 of the *Archive for History of Exact Sciences 1* (1965) [Dauben's footnotes: 33 and 34]. Cantor's vilifications of du Bois-Reymond and his work are respectively from pages 505 and 506 of the same text [Dauben's footnotes: 35 and 36].

**The Puzzles Resolved**

Why did Cantor, a mathematician of surpassing originality, persist in advancing 'proofs' of obvious circularity when attempting to show that one could never pair the real numbers with the integers while condemning with vituperation both the games that others were trying to play with infinitesimals and the players themselves?

> Because he suspected, or so I surmise, that were such games to be played, players of talent and industry might prove that the real numbers could indeed be paired with the integers, rendering his 'proofs' to the contrary a pretense and collapsing the house of cards that he was trying so hard to erect within the upper reaches of the mathematical tree to accommodate transfinite numbers.

But that only scratches the surface of a compulsion of far greater depth and historical resonance.

> Cantor was driven to do as he was doing by a passion of *religious* dimensions, convinced that he had been 'chosen' to open a window for others onto a universe of infinities exemplifying the vast and encompassing greatness of God himself.

To mirror God's greatness, the infinities had to be of grandiose scale. Infinitesimals had consequently to be condemned with fervour as sinful examples of the 'Cholera-Bacillus' rather than being simply shunned as a waste of time. Commissioned by God to enlighten the mathematical world while preserving it from corruption, Cantor was doing as he had been commanded to do.[28]

Consider a short-list of occasions upon which Cantor confirmed the conviction with increasing intensity as he corresponded with others.

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[28] Cantor, unlike others, may have deemed his earlier proof that the rational numbers could be paired with the integers as a caution rather than cause for celebration, for it failed to confirm that sets of uncountable size were needed. He may indeed have stumbled upon the proof with consternation while seeking to prove the contrary, provoking him to try with renewed vigor to show that no comparable pairing of the real numbers could be made.

To Georg Woldemar (Cantor's father) (25 May 1862) upon learning at age seventeen of his father's acceptance of his wish to pursue the study of mathematics:

> . . . I hope that you will still be proud of me one day, dear Father, for my soul, my entire being lives in my calling; whatever one wants and is able to do, whatever it is toward which an unknown, secret voice calls him, *that* he will carry through to success.[29]

To Mittag-Leffler (31 January 1884):

> [Cantor affirms that his awareness of transfinite numbers had come to him from] "a more powerful energy".[30]

To Ignatius Jeiler (Whitsuntide 1888):

> I entertain no doubts as to the truth of the transfinites, which I have recognised with God's help and which, in their diversity, I have studied for more than twenty years; every year, and almost every day brings me further in this science.[31]

To K. F. Heman (1888):

> My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all, because I have followed its roots, so to speak, to the first infallible cause of all created things.[32]

In May of 1884, shortly after his thirty-ninth birthday, Cantor, prone since his late teens to 'periods of depression', suffered the first of the sequence of 'serious mental breakdowns' that would require him to endure prolonged periods of isolation in

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[29] Dauben [Cantor], pages 277 (and 284). Dauben's translation of a part of the letter as cited in A. Fraenkel's "George Cantor", *Jahresbericht der Deutschen Mathematiker-Vereinigung* 39 (1930), page 193, and later on page 5 of Meschkowski (1967), op. cit. (footnote 24).

[30] Ibid., page 290. Dauben here amplifies the remark succinctly: "he was only the means by which set theory might be made known". See page 23 below, however, for Dauben's more ample summary (from page 146 of his book) of the content of the phrase within the context of the letter, and another that preceded it on 23 December 1883.

[31] Ibid., page 147.

[32] Ibid., page 298.

hospital.[33] By 1894, he had decided that these episodes were gifts of time from God to enable him to deepen both his religious and mathematical thinking – a construal that he would reaffirm in 1908 when the religious impetus behind his devotion to propagating his 'tranfinites' had become so obvious that, following the seeming desecration of it by Poincaré in his condemnation of 'Cantorism' at an International Congress in Rome, he responded by insisting they could only be acknowledged appropriately as confirming the all-embracing nature of God himself.

To Charles Hermite (1894):

> But now I thank God, the all-wise and all-good, that He always denied me the fulfillment of this wish [for a position at university in either Göttingen or Berlin], for he thereby constrained me, through a deeper penetration into theology, to serve Him and His Holy Roman Church better than I have been able with my exclusive preoccupation with mathematics.[34]

To Grace Chisholm Young [1908]:

> A peculiar fate, which thank goodness has in no way broken me, but in fact made me stronger inwardly, happier and more expectantly joyful than I have been in a couple of years, has [again] kept me far from home – I can also say far from the world . . . In my lengthy isolation neither mathematics nor in particular the theory of transfinite numbers has slept or lain fallow in me . . . .
>
> I have never proceeded from any "Genus supremum" of the actual infinite. Quite the contrary, I have rigorously proven that there is absolutely no "Genus supremum" of the actual infinite. What surpasses all that is finite and transfinite is no "Genus"; it is the single, completely individual unity in which everything is included, which includes the "Absolute", incomprehensible to the human understanding. This is the "Actus Purissimus" which by many is called "God".[35]

From the beginning to the end of his crusade, that is, Cantor never wavered from the conviction conveyed to Thomas Esser in February of 1896:

> From me, Christian Philosophy will be offered for the first time the true theory of the infinite.[36]

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[33] The phrases "periods of depression" and "serious mental breakdowns" are Dauben's. (Dauben [Cantor], pages 287 and 280). Cantor was to die from his malady in January, 1918.

[34] Dauben [Cantor], page 147.

[35] Ibid., page 290.

[36] Ibid., page 147.

Readers, finding this wondrous, would do well to ponder the epic of Cantor's life and the unfolding of his mathematical work within it as recounted with care by Joseph Dauben, registering the author's assessments of the religious fervour that determined the trajectory of it.

> From the very beginning, apparently, Cantor had felt some inner compulsion to study mathematics. [His father, Georg Woldemar's] original doubts and disappointment over his son's wish to study pure science were set aside, but this also left Cantor with a sense of having to prove that his father would not be disappointed. Characteristically, Cantor reinforced the validity of his decision by saying that it was not his alone – an unknown, secret voice was making its impression even then [1862].[37]
>
> By the early part of 1884, he could write to Mittag-Leffler that he was not the creator of his new work, but merely a reporter. *God* had provided the inspiration, leaving Cantor responsible only for the way in which his articles were written, for their style and organisation, but not for their content.[38] . . . Cantor believed in the absolute truth of his set theory because it had been *revealed* to him. Thus he may have seen himself not only as God's messenger, accurately recording, reporting, and transmitting the newly revealed theory of the transfinite numbers but as God's ambassador as well. If so, Cantor would not only have felt it appropriate, but more accurately, his duty, to use the knowledge which was his by the grace of God to prevent the Church from committing any grave errors with respect to doctrines concerning the nature of infinity.[39]
>
> During the long months of seclusion his mind was left free to ponder many things, and in the silence he could perceive the workings of a divine muse, he could hear a secret voice from above which brought him both inspiration and enlightenment. . . . [These convictions linked] Cantor's periods of introvertive contemplation, the long silences, and his mathematics. . . . Cantor's inner voice knew more than the details of Christian history; his Muse was also a mathematician.[40]

[37] Ibid., page 277.

[38] Dauben here refers within a footnote to Cantor's letters to Mittag-Leffler of 23 December 1883 and 31 January 1884 as they appeared in 1927 on pages 15 and 16 of A. Schoenflies's "Die Krisis in Cantor's mathematischem Schaffen", *Acta Mathematica 50* (pages 1 to 27).

[39] Ibid., pages 146 and 147.

[40] Ibid., pages 289 and 290.

> There can be no mistake about Cantor's identification of his mathematics with some greater absolute unity in God. This also paralleled his identification of transfinite set theory with divine inspiration. . . . Thus the periods of isolation in the hospital could be regarded as periods during which, as he told Mrs. Young [1908; see above], the transfinite numbers lay neither fallow nor forgotten, but might be further elucidated by the grace of God sent to inspire new lines of research. All this was very much in keeping with the principles Cantor had inherited from his father and from his religious upbringing.[41]
>
> Had he not been able to cast himself in the rôle of God's messenger, inspired from some higher source of inspiration, he might never have asserted the unquestionable, indubitable correctness of his research. The religious dimension which Cantor attributed to the *Transfinitum* should not be discounted as merely an aberration. Nor should it be forgotten or separated from his life as a mathematician. The theological side of Cantor's set theory, though perhaps irrelevant for understanding its mathematical content, is nevertheless essential for the full understanding of his theory and the development he gave it. Cantor believed that God endowed the transfinite numbers with a reality making them very special. Despite all the opposition and misgivings of mathematicians in Germany and elsewhere, he would never be persuaded that his results could be imperfect. This belief in the absolute and necessary truth of his theory was doubtless an asset, but it also constituted for Cantor an imperative of sorts. . . . He felt a duty to keep on, in the face of adversity, to bring the insights he had been given as God's messenger to mathematicians everywhere.[42]

When summarised historically, therefore,

> . . . underlying even [Cantor's] earliest decisions was a sense of destiny, a feeling that unknown forces were at work and that there propelling him toward a career that he felt he could not deny.[43]
>
> By the end of his life, in the spirit of *Aeterni Patris*, Cantor saw himself as the servant of God, a messenger or reporter who could use the mathematics that he had been given to serve the Church.[44]

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[41] Ibid., page 290.
[42] Ibid., page 291.
[43] Ibid., page 280.
[44] Ibid., page 147.

Were Dauben to have spoken with less scholarly sympathy, he might simply have said that Cantor, concerned oddly with mathematics but acting otherwise as religious enthusiasts before and after him have done, played his games by whatever means he could to 'transfinite' ends of his own devising, disregarding missteps while denigrating without remorse those who played others contrary to his own – driven by *faith* rather than fact – as fellow formalist David Hilbert would later confirm, albeit unwittingly, within a sermon of his own.

> Whenever there is any hope of salvage, we will carefully investigate fruitful definitions and deductive methods. We will nurse these, strengthen them, and make them useful. No one shall drive us out of the *paradise* which Cantor has created for us. [italics EWC][45]

Were fact rather than faith at issue with respect to Cantor's 'paradise', no preaching of reassurance to the converted would have been necessary.

## Autobiographical Conclusion: Peeling the Potatoes[46]

> As is frequently the case with work in architecture, work on philosophy is actually closer to working on oneself. On one's own understanding. On the way one sees things. (And on what one demands of them.)
>
> Ludwig Wittgenstein[47]

Understanding being historical, and, when philosophical, autobiographical, I conclude with two stories of how this essay came to be written.

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[45] From a lecture by David Hilbert of 04 June 1925 to a congress of the Westphalian Mathematical Society in honor of Karl Weierstrass. The initial portion of the text, as it appeared within the *Mathematische Annalen, No. 95* (Berlin, Germany, 1925), was then translated by Erna Putnam and Gerald J. Massey and reproduced on pages 134-151 under the title "On the Infinite" within *Philosophy of Mathematics: Selected Readings*, edited with an introduction Paul Benacerraf and Hilary Putnam (Oxford, England: Basil Blackwood, 1964). The quotation appears on page 141.

[46] See footnote 1.

[47] From *Ludwig Wittgenstein: Philosophical Occasions -1912-1951*, edited by James C. Klagge and Alfred Nordmann (Indianapolis, Indiana: Hackett Publishing, 1993), pages 161 and 162 [of a reproduction on facing pages of German and English of "Philosophie" [sections 86-93, pages 405-435 of the so-called *Big Typescript* (Catalogue Number 213) as drafted by Wittgenstein in 1933], edited by Heikki Nyman and published in the *Revue Internationale de*

**A Curbside Conversion**

In 1958, as I was stepping up onto a curb after crossing a street in Milwaukee, Wisconsin, I realised that there was no God – or, more exactly, that the presence or absence of one would never again make any difference to me. Pausing on the walkway to revel at the 'incredible lightness of being' that the awareness occasioned, I realised with astonishment that the heaviest weight that had been depressing me from childhood had evanesced, never to return.

I was sixteen years old at the time and unaware of the question-begging arguments to the contrary that wishful philosophers had preached in the past (ontological, by design, or others). Rather, as a young male obliged to write letters to young females far away, if one wished to entice them to attend to you, I had done so with two of them at once, each of whom, learning from the other of the duplicitous game that I was playing, having determined to never again have anything to do with me. Entrapped by deceit, I had been wondering whether, as trained from childhood, I ought perhaps to 'pray' for a miraculous release – only to realise that no God, however attentive, could save me without ripping asunder the causal net of the universe that he had 'with infinite wisdom put in place when creating it' (as my parents would have insisted).

My realisation was thus *pragmatic* rather than erudite, the first of many lessons from which I was to learn how to acknowledge the sense and import of a maxim that would with increasing efficiency guide me through life.

A difference that makes no difference is no difference at all.[48]

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*Philosophie*, vol. 43, no. 169, 1989, pages 175-203, alongside the translation of it by C. Grant Luckhardt and Maximilian A. E. Aue published as "Philosophy" in *Synthese*, vol. 87, April 1991, pages 3-22.

[48] A common reparsing, often attributed to William James, of 'the rule for attaining the third [highest] grade of clearness of apprehension' of Charles S. Peirce from his essay on "How to Make Our Ideas Clear", pages 286 to 302 of *Popular Science Monthly* (January, 1878), as reproduced on page 124 of Philip P. Wiener's *Values in a Universe of Chance: Selected Writings of Charles S. Peirce (1839-1914)* (Stanford, California: Stanford University Press, 1959). Peirce's admonition reads in its entirety: "Consider what effects, which might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object."

Fifteen years later in 1893, as Wiener notes (page 181), Peirce added a *religious* caution to the prescription within a series of footnotes: "Before we undertake to apply this rule, let us reflect a little upon what it implies. It has been said to be a skeptical and materialistic principle. But it is only an application of the sole principle of logic recommended by Jesus: 'Ye may know

**A Consequence**

In 1965 I withdrew from the systematic study of mathematical logic, and therewith mathematics itself, after listening to a lecture by Paul Cohen, a visiting professor at Harvard only eight years older that I, who had recently proven from within the axioms of set theory and to justifiable acclaim that Cantor's 'Continuum Hypothesis' was independent of them.

Cohen had published a clear account what he had done within a book that I purchased and pondered.[49] The conclusion to which I came upon doing so, however, differed diametrically from the one shared by its author and the larger community of those attempting to think logically about mathematics (the 'congregation' that I had been hoping soon to join).

The 'Continuum Hypothesis', though approachable only through the highest complexities of set theory, can be as easily understood, or almost so, as the notion of 'set' itself. If the real numbers comprise a set and the integers another, and Cantor was right when insisting that one can always construct members of the first to elude any purported pairing of them (the first being therefore 'larger' than the second), the question arises.

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them by their fruits', and it is very intimately related with the ideas of the Gospel. We must certainly guard ourselves against understanding this rule in a too individualistic sense." Wiener remarks, in turn (page 181), that "This was [William] James's error according to Peirce."

If so, then Peirce may well have been responding to James's well-intentioned but 'over-representation' of the dictum within Chapter II of his *What Pragmatism Means: a New Name for Some Old Ways of Thinking* [delivered as 'Popular Lectures on Philosophy' in November and December, 1906 (Boston) and in January, 1907 (New York): "There can *be* no difference anywhere that doesn't *make* a difference – no difference in abstract truth that doesn't express itself in a difference in concrete fact and in conduct consequence upon that fact, imposed on somebody, somehow, somewhere, and somewhen. The whole function of philosophy ought to be to find out what definite difference it will make to you and me, at definite instants of our life, if this world-formula or that world-formula be the true one." (See page 508 of *William James: Writings 1902-1910* (New York, New York: Literary Classics of America [The Library of America], 1987).

[49] Paul J. Cohen, *Set Theory and the Continuum Hypothesis* (New York, New York: W. A. Benjamin, Incorporated, 1966).

> Is there a set smaller than the first but larger than the second?
>
> The 'Continuum Hypothesis' says 'no'. The real numbers comprise the set next larger in size to that of the natural numbers with no others in between.

To many mathematical logicians, Gödel foremost among them, Cohen's work implied that we needed to supplement the axioms of set theory with others that would enable us to answer the question (or alternatively, to come up with a new set of them, differing from the old, that would do so).[50]

To me, however, Cohen's result pointed in another direction. Were the 'Continuum Hypothesis' independent of the axioms of set theory, we could suppose freely either that there are one or more sets intermediate in size to the sets of the natural and of the real numbers exists, or none of them.

> But if we knew 'of what we were speaking' when speaking of sets, how could the existence of one or more of them be a matter of *choice*?
>
> And to assume that we could avoid the ontological predicament by *postulating* ourselves into paradise (pretending, that is, that we could render it innocuous by playing a game of 'axiomatic method' *supposing* whatever objects would be presupposed within its axioms) was *to beg the question!*

Suddenly, but with conviction, I sensed that 'we' (the community of players of games of mathematics and its logic built upon the 'axiomatic method') knew neither 'of what we were talking about' when referring to 'sets' (or to 'numbers', for that matter) nor 'of what we were doing' when talking about them, despite the formal clarity of the magnificent edifice that 'we' had erected believing otherwise.

> Our situation was comparable – and exactly so! – to that of the Scholastic theologians striving industriously half-a-millennia ago to refine the finely-wrought nuances of a formally-coherent conceptual superstructure resting upon the core presupposition that 'we know what we are talking about' when using the word 'God', despite their being no evidence whatsoever that anything of the kind existed.

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[50] No one, however, then or now, has suggested any one or more of them that would do the trick.

Though I have never lost my fascination with the games of mathematics and its logic, and have persisted to this day in replaying one or another of them on occasion, I have never since had occasion to reverse a conviction.

> We can do no better, I believe, than to play games of 'kitchen mathematics' hoping thereby to lessen the temptation to misconstrue how better to play them, having beforehand done the same with the game of logic itself.[51]

A matter of fact rather than faith, eh?

[51] See footnote 15 above.