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A three-pronged lesson in differential equations in a calculus course: analytical, numerical, experimental

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Physical experiments in classrooms have many benefits for student learning, including increased student interest, participation and knowledge retention. While experiments are common in engineering and physics classes, they are seldom used in first-year calculus, where the focus is on solving problems analytically, and occasionally numerically. In this paper, we detail a three-pronged lesson introducing differential equations using analytical, numerical and experimental approaches in a large first-year differential calculus course. Presenting the three approaches in succession allows students to evaluate advantages and disadvantages. The lesson incorporates software and programming, and provides opportunities for active, experiential, team-based learning.

Keywords: First-year calculus; Differential equations; Physical experiments; Numerical methods.

1. Introduction

University students often encounter differential equations (DEs) for the first time in their introductory calculus courses. The outline laid out in a textbook like Stewart (2012) is standard. The term *differential equation* is first used when the exponential function e^t is presented as an analytical solution to the DE $x'(t) = x(t)$. Apart from a passing mention in the section on antiderivatives, DEs are then withheld until integral calculus is well-established, at which point they are reintroduced and solved using analytical and numerical approaches.

Our aim in this paper is to present a different, three-pronged lesson introducing DEs in a differential calculus course at the start of students' university education. This lesson demonstrates the analytical and numerical approaches in a compressed sequence, and includes a third prong, the experimental approach, often consigned to engineering and physics courses. The lesson emphasizes the multitude of approaches and their different strengths, incorporates software and programming at low cost, and provides opportunities for active, experiential, team-based learning.

Though the approach is applicable to multiple physical models, the lesson plan here is centred on a classic one, the motion of an object of mass m, falling from rest. One standard model, which may be derived from Newton's Second Law of Motion, yields the DE

$$x''(t) = g - kx'(t) \tag{1.1}$$

along with the initial conditions $x(0) = 0$ and $x'(0) = 0$, where $x(t)$ is the positive distance (in metres) the object has fallen downwards at time t (in seconds), $g \approx 9.81\mathrm{m/s}^2$ is the acceleration due to gravity, and the parameter k describes air resistance. The unit for k is s^{-1}.

In Section 2, we survey similar classroom lessons recorded in the literature on teaching and learning in mathematics, physics and engineering. Our lesson is described in Section 3, and a follow-up homework assessment is described in Section 4, before we conclude with reflections in Section 5.

2. Background and Motivation

An *analytical* approach is one that produces an exact solution to a DE. A *numerical* approach is one that approximates a solution to a DE using an algorithm, often iterated by software. An *experimental* approach uses a physical procedure to produce data. All three approaches have been combined in dynamical systems and control research — see, for example, Campbell, Crawford, & Morris (2008) on the control and stability of a pendulum.

Analytical approaches predominate in the calculus classroom. Numerical approaches are often in a second, disconnected position, even though most DEs can only be solved numerically, not analytically. While there are entire courses and research areas dedicated to intricate numerical methods, the simplest approach, Euler's method, is accessible to a first-year calculus student.

Experimental approaches are even less common than numerical approaches in first-year calculus, but this is not always the case in other disciplines. For example, engineering design days, described in Rennick *et al.* (2018); Hurst *et al.* (2019); Li *et al.* (2017) and Rennick & McKay (2018), give teams of students multiple days to solve a design problem whose challenges span multiple courses. Students gain experience with teamwork, communication, experiential learning and reflection, which emulates the professional practice of engineering. Indeed, prior to its incorporation into the academy, engineering education was essentially supervised experimentation — that is, apprenticeship (Hurst *et al.*, 2019). The benefits of learning using experimentation and real-life applications are increased interest, participation and retention (Campbell *et al.*, 2008). Freeman *et al.* (2013) conducted a meta-analysis of 225 university-level studies that showed learning through activities and discussions in the classroom (commonly referred to as active learning) improved student performance on assessments and failure rates in science, engineering and mathematics disciplines. A few years later, this was followed by a similar study (Theobald *et al.*, 2019) noting active learning strengthens the course performance of underrepresented students in science, technology, engineering and mathematics.

This effect has been observed in mathematics classrooms. In Banks & Tran (2009), the authors detail experiments centred on partial differential equations describing three phenomena: heat transfer in copper and aluminum, vibrations in a beam, and acoustic wave propagation in a PVC pipe. They note:

> Our experience with this approach to teaching advanced mathematics with a strong laboratory experience has been, not surprisingly, overwhelmingly positive. It is one thing to hear lectures on natural modes and frequencies (eigenfunctions and eigenvalues) or even to compute them, but quite another to go to the laboratory; *excite* the structure, *see* the modes, and *take* data to verify your theoretical and computational models.

Experimental approaches have been used in first-year calculus courses too. Gruszka (1994) describes students using analytical and experimental approaches to model the motion of a buoyant balloon. The authors in Farmer & Gass (1992) design an experiment using a plastic bottle with a hole drilled in it to demonstrate Torricelli's law.

The simplicity and low cost of the experiments is a feature in both Gruszka (1994) and Farmer & Gass (1992). Decades later, the low cost of software and ubiquity of smartphones make experimental and numerical approaches even more attractive. In Mamolo *et al.* (2011), The Geometer's Sketchpad is used in an experiment on proportionality. MATLAB is used in Robinson & Jovanoski (2010) to model the projectile motion of an ejecting pilot. The Geometer's Sketchpad and MATLAB are commercial software programs with viable free versions. More recently, Wohak & Frank (2021) describe using the Julia programming language and Jupyter notebooks — both completely free and open-source — to teach linear algebra and signal compression.

The lesson described in this paper makes use of two widely available pieces of software: phyphox and Microsoft Excel. Phyphox is a free and open-source app that was developed to enhance physics teaching (Gianino, 2021; Nichols, 2022; Staacks *et al.*, 2018, 2022) and Microsoft Excel, while not free, is part of many computers' standard software suite. It is also interchangeable with other spreadsheet programs that are free.

3. Classroom Lesson

This lesson was taught in 2022 during the ninth week of a twelve week differential calculus course at the University of British Columbia in Canada. It took place within one lecture period of 110 minutes, with a 5-10 minute break halfway through. Total enrollment in the course was around 4000 students. Each student was assigned to one of 12 distinct sections, each taught by a single instructor. This paper focuses on two of these sections, of around 200 students and 400 students respectively.

Prior to this lecture, students had learned: how to verify if a function solves a differential equation; how to adjust parameters such that a given ansatz solves a differential equation; the solutions to certain linear first-order differential equations; and how to use phase line analysis to understand the qualitative behaviour of differential equations. This course has no physics or calculus prerequisites, and integration is covered in the course that follows it. The discussion of equation (1.1) was carried out under the assumption that students had no prior knowledge in physics or integral calculus. In particular, the form of the term $kx'(t)$ was not explicitly justified.

With one exception, the supplies needed for the experiment portion of the lesson are cheap and readily available household items: a ruler, pen, and measuring tape. The exception is not cheap, but it is certainly readily available: a smartphone with the free app phyphox (Staacks *et al.*, 2018) downloaded. Phyphox uses smartphone sensors to measure quantities such as acceleration, pitch, pressure and duration. In this lesson, it is used as a stopwatch to measure the time elapsed between two audio triggers.

3.1. *The analytical approach*

In the classroom lesson, the parameter k in (1.1) is set to be equal to 0; $k > 0$ is explored in the follow-up homework assessment described in Section 4.

The lesson begins with the analytical approach. Students consider the simplified DE $x''(t) = g$ and solve it analytically by trying to come up with a function that, when twice differentiated, yields g. They are given time to discuss it with their neighbours. Afterwards, the instructor facilitates a discussion with the whole class, drawing out students to propose solutions. First one solution is proposed; then, with prompting, more solutions; and finally, the general solution $x(t) = \frac{1}{2}gt^2 + At + B$ is proposed and checked. The initial conditions $x(0) = 0$ and $x'(0) = 0$ are invoked to come up with the specific solution

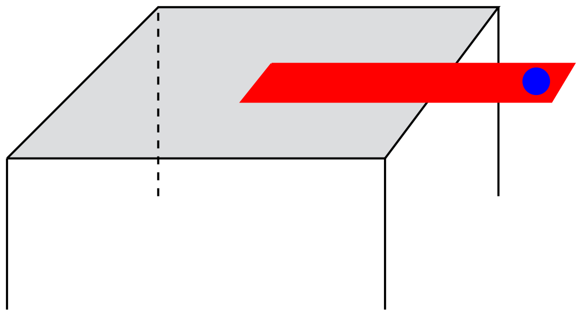

FIG. 1. A ruler (red) at the edge of a table with a small circular object (blue) placed at the end of the ruler. The size of the ruler has been exaggerated in the diagram for emphasis.

$x(t) = \frac{1}{2}gt^2$. This is graphed for $t \in [0,1]$ (see Figure 2). A classroom discussion follows in which students are asked to speculate about the accuracy of the graph in depicting the actual motion of the falling object. One of the aims of this discussion is to remind students that the DE has been simplified by setting $k = 0$. Another is to prompt students to wonder how the accuracy of the analytical solution in describing the real world phenomenon of a falling object can be confirmed.

3.2. *The experimental approach*

The experimental approach is a means of validating the mathematical model behind the analytical and numerical approaches. Experimental approaches have the additional benefit of uncovering behaviours in the absence of any mathematical model.

In this lesson, students perform trials in which an object is dropped from a known height by an action that generates an audio trigger. The object makes another sound when it hits the floor, and the acoustic stopwatch program in phyphox measures the elapsed time.

While the general experimental setup is predetermined, students are involved in the experimental design. Their contributions are solicited in a classroom discussion. *How should the known height be set? What object should be dropped? How should the initial audio trigger be generated?*

In the end, it is decided to drop the object from an initial position on a ruler at the edge of a table, as illustrated in Figure 1. Striking the ruler with a pen simultaneously generates an audio trigger and moves the ruler, dropping the object. Students suggest various objects to be dropped: a rock (too large), a phone adaptor (too expensive), a banana (too soft, too absurd). Eventually, a coin is chosen as a small, inexpensive object that will make a sufficient sound when dropped.

The distance between the ruler and the ground is measured and denoted x_1. The time the object takes to hit the ground is denoted t_1. Multiple trials are attempted for various heights to produce points (t_i, x_i), which are then plotted on the same axes as the graph of $x(t) = \frac{1}{2}gt^2$ which was generated in the analytical approach (Figure 2).

There are slight variations in how the experiment is run in the two observed sections. In the first section, the course instructor brings all the materials, including a phyphox-enabled phone. In the second section, the course instructor asks students to bring materials. In the second section, during the demonstration of the experiment at the front of the class, the remote access function of phyphox is used to project the user interface and acoustic stopwatch onto the classroom screen. Groups of students are then sent to repeat the experiment in groups.

i	t_i	$v(t_i) = x'(t_i)$	$x(t_i)$
0	0	0	0
1	0.1	0.981	0
2	0.2	1.962	0.0981
3	0.3	2.943	0.2943
4	0.4	3.924	0.5886
5	0.5	4.905	0.981
6	0.6	5.886	1.4715
7	0.7	6.867	2.0601
8	0.8	7.848	2.7468
9	0.9	8.829	3.5316
10	1	9.81	4.4145

TABLE 1 *Values generated using Euler's method for* $n = 10$ *subintervals on* $x''(t) = g$.

Both sections include an in-class reflection on experimental design. Students bring up issues of ambient noise; human error and the need for repeated trials; and when comparing experimentally generated points with the graph of the analytic solution to the simplified DE, the parameter k.

The four points generated in the first section — $(0.461, 0.74), (0.561, 1.54), (0.513, 1.05), (0.321, 0.47)$ — are plotted onto Figure 2. The points align well with the graph of $x(t) = \frac{1}{2}gt^2$.

3.3. *The numerical approach*

Finally, the class turns to the numerical approach. The course instructor sets up the basic idea: that the solution is constructed out of linear approximations. The time interval is divided into n subintervals, indexed by i, of equal length $\Delta t = t_{i+1} - t_i$. Then we approximate

$$x(t_{i+1}) \approx x(t_i) + x'(t_i)\Delta t$$

for $i = 0, 1, 2, \ldots$. But what is $x'(t_i)$? Again, a numerical approach is proposed:

$$x'(t_{i+1}) = v(t_{i+1}) \approx v(t_i) + v'(t_i)\Delta t = v(t_i) + g\Delta t,$$

where $v(t)$ is the downward velocity of the object at time t.

The technique is named as *Euler's method*, and thoroughly discussed before any values are computed. The technique for approximating $v(t_{i+1})$ is especially instructive: in the case that the analytical solution turns out to be a line, the linear approximations lie along the line itself, and there is no difference between the numerical solution and the analytical solution.

Finally, values are computed. Using an instructor-prepared Excel spreadsheet, the values of $x(t_0), x(t_1), \ldots, x(t_n)$ are calculated for various choices of n. The values for $n = 10$ are shown in Table 1, and also plotted onto Figure 2.

3.4. *The three prongs*

The plot of solutions reproduced in Figure 2 illustrates the three prongs — analytical, experimental and numerical — of the combined approach.

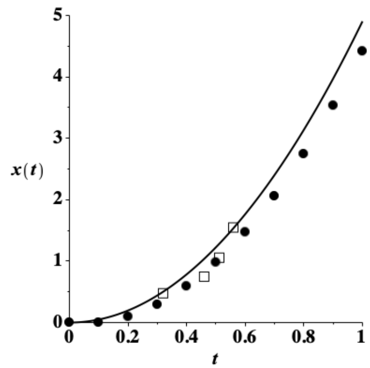

FIG. 2. Three solutions to $x''(t) = g$: analytic, $x(t) = \frac{1}{2}gt^2$, as denoted by the curve, experimental as depicted by the squares, and numerical (from Table 1) as shown by the dots.

In class, the figure is used to motivate a classroom discussion about the various approaches. A student asks "Is experimental the best way?", which generates conversations about what "best" means. Students conclude that the approaches are tools with strengths and weaknesses. Analytical approaches produce the most complete solutions, but require functions to be "well-behaved". Numerical approaches make fewer demands: for Euler's method, for example, the solution's derivative must be computable, even if the solution itself is not. The accuracy of both approaches depends entirely on the accuracy of the mathematical model. Experimental approaches have no such dependence, but are subject to errors in experimental design. None of the approaches is superfluous.

4. Assessments

Three types of graded assessments followed the classroom lesson to reinforce the material.

The first type, which we do not recount in detail here, asks students to use Euler's method to generate numerical solutions to straightforward problems on an online homework system. The goal of those problems is for students to gain technical familiarity with the method.

The second type of graded assessment is the written homework assignment. This is one of five assignments. Students are given two weeks to complete the assignment in teams of three to five students. In the assignment, the parameter k in (1.1) is not restricted to be equal to 0. In the first question, students are asked to make the substitution $v(t) = x'(t)$ to transform (1.1) into a first order DE, and then to solve the DE analytically using the ansatz $v(t) = Ae^{-kt} + B$. In the second question, students are asked to write an Excel program to solve (1.1) numerically for a particular nonzero value of k using Euler's method. The written assignment is an opportunity for students to extend their knowledge by solving more challenging material with the support of their other team members.

The third type of graded assessment is the in-person, invigilated, final exam. The final exam is 2.5 hours, and students are required to work independently without course aids of any kind. Out of 15 exam questions, two refer to the classroom lesson and homework assignment described above. The first question, worth a small number of points, asks students to apply Euler's method in a straightforward way to a given DE. The second question, worth a larger number of points, asks students first to solve the DE $v'(t) = g - kv(t)$ analytically; and then to describe the asymptotic behaviour of its solutions, as well as of the solutions to the DE $v'(t) = g - kv(t)^2$, using steady state analysis of their phase lines.

5. Goals, Features and Reflections

The goals of the lesson described in Section 3 and the assessments described in Section 4 are for students to:

- encounter a deliberate and balanced presentation of three approaches to solving DEs: analytical, numerical and experimental,
- practice solving DEs using multiple approaches,
- articulate the advantages and disadvantages of the different approaches to understanding mathematical models, and
- use technology in support of mathematical tools.

These serve the overall goal in first year mathematics courses of understanding mathematical concepts and navigating back and forth between those abstract concepts and their concrete application. The lesson also contained the following features:

- teamwork,
- classroom discussion,
- active experimentation, and
- use of physical manipulatives.

These serve the overall goal of taking opportunities to make learning active, experiential and memorable.

By direct observation, students recognized the importance of the features and enjoyed the lesson. They were eager to volunteer, provide suggestions, watch the demonstration at the front of the class and try it out for themselves afterward. A number of students commented favourably on the lesson in their anonymous end-of-course surveys. In one section, students responded to the prompt asking what their instructor did well to support their learning by writing:

> "I think that the [...] physical demonstrations were helpful to my learning."

> "Did demonstrations/fun activities to get us engaged in the content."

> "[He] somehow made a math lecture interesting? I love how he integrated memorable activities like [...] calling people up for a [physics] demo, since it made learning the content way more engaging."

> "[He] was very energetic and it was clear he was putting in effort to keep us engaged ([...] like the physics experiment)."

When asked what the instructor could have done differently to further support their learning, one student wrote:

> "I think more in-class activities to solidify learning, such as the gravity experiment we did in one of the lectures, would be beneficial."

The instructors intend to repeat the lesson in the next iteration of this course. We propose three extensions to them, and to others considering using the lesson.

First, we propose that questions prompting students to examine or use the experimental approach should be included on one or more of the follow-up assessments. For example, on their homework assignment, students could adapt the classroom experiment to allow the parameter k in (1.1) to be nonzero, by attaching an air-filled balloon to the dropped object. They could collect data experimentally, perhaps providing video evidence as part of their homework submission, and then estimate the value of k analytically and numerically.

Second, we propose that, for engineering students and other students with prior knowledge of physics, equation (1.1) be explicitly derived in class from physical principles. This derivation could prompt a broader discussion around the appropriateness of the model, as well as the nuances of mathematical modelling in general.

Third, we propose that the three-pronged approach be replicated in the subsequent integral calculus course. For example, a classroom lesson could be constructed around analytically, numerically and experimentally determining the centre of mass of a plywood cutout of irregular area but uniform density per unit area. It is important, especially for first-year students in large classes, to have opportunities for active, experiential, team-based learning.

References

Banks, H. and Tran, H. (2009) *Mathematical and Experimental Modeling of Physical and Biological Processes*, CRC Press.

Campbell, P., Patterson, E., Busch-Vishniac, I. and Kibler, T. (2008) Integrating applications in the teaching of fundamental concepts, *Proceedings of the 2008 Annual Conference and Exposition of the American Society for Engineering Education.*

Campbell, S., Crawford, S. and Morris, K. (2008) Friction and the inverted pendulum stabilization problem. *ASME J. Dynamic Systems, Measurement and Control*, 130(5).

Farmer, T. and Gass, F. (1992) Physical demonstrations in the calculus classroom. *College Mathematics Journal*, 23, pp. 146148.

Freeman, S., Eddy, S., McDonough, M, Smith, M, Okoroafor, N, Jordt, H and Wenderoth, M. (2013) Active Learning Increases Student Performance in Science, Engineering and Mathematics. *Psychological and Cognitive Sciences*, 111(23), pp. 84108415.

Gianino, C. (2021) Uniform circular motion measurements employing a smartphone using the phyphox app and a turntable. *Phys. Educ.*, (UK) 56(1).

Gruszka, T. (1994) A balloon experiment in the classroom. *College Mathematics Journal,* 25(5), pp. 442 444.

Hurst, A., Rennick, C. and Bedi, S. (2019) A lattice approach to design education: Bringing real and integrated design experience to the classroom through engineering design days. *Proceedings of the 22nd International Conference on Engineering Design (ICED19)*, Delft, The Netherlands.

Li, E., Rennick, C., Hulls, C., Robinson, M., Cooper-Stachowsky, M., Boghaert, E., Melek, W. and Bedi, S. (2017) Tron days: Horizontal integration and authentic learning. *Proc. 2017 Canadian Engineering Education Association (CEEA17) Conf.*, University of Toronto.

Mamolo, A., Sinclair, M. and Whiteley, W. (2011) Proportional reasoning with a pyramid. *Mathematics Teaching in the Middle School*, 16(9), pp. 544549.

Nichols, D. (2022) Measuring mass with a rubber band and a smartphone. *The Physics Teacher,* 60(7), pp. 608609.

Rennick, C., Hulls, C., Wright, D., Milne, A. and Bedi, S. (2018) Engineering design days: Engaging students with authentic problem-solving in an academic hackathon. *ASEE Annual Conference*, Salt Lake City.

Rennick, C. and McKay, K. (2018). Componential theories of creativity: A case study of teaching creative problem solving. *Proc. 2018 Canadian Engineering Education Association (CEEA-ACEG18) Conf.*, University of British Columbia.

Robinson, G. and Jovanoski, Z. (2010). Fighter pilot ejection study as an educational tool. *Teaching Mathematics and Its Applications*, 29, pp. 197192.

Staacks, S., Dorsel, D., Hütz, S., Stallmach, F., Splith, T., Heinke, H. and Stampfer, C. (2022) Collaborative smartphone experiments for large audiences with phyphox. *Eur. J. Phys. (UK)* 43(5).

Staacks, S., Hütz, S., Heinke, H. and Stampfer, C. (2018). Advanced tools for smartphone-based experiments: phyphox. *Physics Education* 53(4).

Stewart, J. (2012) *Calculus: Early Transcendentals.* Brooks/Cole.

Theobald E., Hill, M., Tran, E., Agrawal, S., Arroyo, E., Behling, D., Chambwe, N., Cintrn, D., Cooper, J., Dunster, G., Grummer, J., Hennessey, K., Hsiao, J., Iranon, N., Jones, L., Jordt, H., Keller, M., Lacey, M., Littlefield, C., Lowe, A., Newman, S., Okolo, V., Olroyd, S., Peecook, B., Pickett, S., Slager, D., Caviedes-Solis, I., Stanchak, K., Sundaravardan, V., Valdebenito, C., Williams, C., Zinsli, K., and Freeman, S. (2019) Active Learning Narrows Achievement Gaps for Underrepresented Students in Undergraduate Science, Technology, Engineering and Math. *Psychological and Cognitive Sciences*, 117(12), pp. 64766483.

Wohak, K. and Frank, M. (2021) Compressing audio signals with interactive and cloud-based learning material–a workshop for high-school students. *Teaching Mathematics and Its Applications,* 41, pp. 240255.

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