Polarization operators in superspace

by

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Abstract

The classical coinvariant rings and its variants are quotient rings with rich connections to combinatorics, symmetric function theory and geometry. Studies of a generalization of the classical coinvariant rings known as the diagonal harmonics have fruitfully produced many interesting discoveries in combinatorics including the q,t-Catalan numbers and the Shuffle Theorem.

The super coinvariant rings are a direct generalization of the classical coinvariant rings to one set of commuting variables and one set of anticommuting variables. N. Bergeron, Li, Machacek, Sulzgruber, and Zabrocki conjectured in 2018 that the super coinvariant rings are representation theoretic models for the Delta Conjecture at $t = 0$.

In this dissertation, we explore the super coinvariant rings using algebraic and combinatorial methods. In particular, we study the alternating component of the super harmonics and discover a novel basis using polarization operators. We use polarization equivalence to establish a triangularity relation between the new basis and a known basis due to two groups of researchers Bergeron, Li, Machacek, Sulzgruber, and Zabrocki and Swanson and Wallach. Furthermore, we prove a folklore result on the cocharge statistics of standard Young tableaux and propose a basis for every irreducible representation appearing in the super harmonics.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Introduction

This dissertation is motivated by the classical coinvariant theory of the symmetric group and its generalizations.

An invariant is something that is preserved when acted upon. A child spinning a basketball on their fingertip would quickly observe its centre does not move much. If one spends more time staring at a spinning basketball, then one naturally discovers that it rotates about a fixed line – an invariant commonly known as the axis of rotation! This is useful knowledge. To perfectly balance a basketball, we just have to keep our fingertips on one end of its axis of rotation so that the axis is perpendicular to the ground.

Invariants are everywhere in nature and they are useful. Physical conservation laws are such examples. Mathematicians have explored and exploited invariants extensively throughout history. Broadly and historically speaking, *invariant theory* refers to a subfield of abstract algebra where one studies how algebraic objects act on geometric objects with the goal of describing their invariants in simpler terms. *Coinvariant theory* is historically closely related to invariant theory. Thanks to the rich connection between invariant theory and combinatorics [67, 58], coinvariant theory has blossomed into a field all on its own.

Coinvariant theory dates back to Chevalley [14] and Shephard and Todd [64] in the 1950s. It started with a classification result in invariant theory known as the Chevalley-Shephard-Todd Theorem. The core idea has a clean presentation when one strips its

mathematical generality and considers the simplest case first worked out by Newton [69]. Consider multivariate polynomials in n variables. One can permute a polynomial by rearranging some of its arguments. If a polynomial stays the same no matter how its arguments are permuted, then it is called a polynomial invariant (of permutations on n symbols). Newton [69] showed that one can pick out n – the same number as the number of variables – special polynomials. All invariant polynomials can be expressed in terms of these n special polynomials using *only* addition and multiplication. Chevalley showed that Newton's result was just the tip of an iceberg. A much larger class of algebraic objects, called finite reflection groups, share this property. There are still n such special polynomials, called *basic (polynomial) invariants*. All other invariants decompose, through addition and multiplication only, into these n basic ones. Shephard and Todd [64] showed[1] the converse is true. If one can get polynomial invariants from a subgroup of the complex general linear group, then one must have started with a finite reflection group. More precisely, Shephard-Todd Theorem states that if G is a finite subgroup of $GL_n(\mathbb{C})$ such that there exists homogeneous algebraically independent $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]^G$ of positive degree which generate $\mathbb{C}[x_1, \ldots, x_n]^G$, then G is generated by complex reflections. The results of Chevalley, and Shephard and Todd yield a method to recognize a complex finite reflection group by its invariants, and vice versa.

Coinvariants are objects complementary to invariants. Springer [66] placed coinvariant theory on equal footing with invariant theory by showing that one can get Chevalley's Theorem by studying coinvariants. Moreover, Hiller [41] found coinvariants encode information about geometric objects studied by Schubert dating back to the late nineteenth century. Around the same time, Stanley published a seminal paper demonstrating invariant theory's many applications to combinatorics [67]. Coinvariant theory became a hub linking algebra, geometry, and combinatorics.

In this dissertation, we choose to focus on the algebraic and combinatorial aspects of a direct generalization of coinvariants of permutations acting on multivariate polynomials. The classical coinvariant rings, denoted by R_n/I_n, are the quotient rings of polynomial rings in *one* set of commuting variables modded out by their symmetric polynomials with no constant term. The classical coinvariant rings are naturally symmetric group representations under the action of permutation of variables.

[1] Historically, Shephard-Todd Theorem came first by about a year.

The dimensions of R_n/I_n are the n factorial numbers $n! = 1 \cdots n$. The Hilbert series, a polynomial that encodes dimension information of an algebraic object, of R_n/I_n is the q-analog of the n factorial numbers. The graded Frobenius image of R_n/I_n, a symmetric function that encodes the decomposition of a representation into irreducible components, is a refined enumeration formula for combinatorial objects known as standard Young tableaux of size n. The classical coinvariant rings have bases built from various combinatorial instructions encoded by permutations [3, 31] and standard Young tableaux [2]. Variations on the theme (of R_n/I_n) known as the Garsia-Procesi modules [30] are representation theoretic models for the Hall-Littlewood polynomials as well as geometric objects known as Springer fibres in type A [18, 75, 30].

From the author's point of view, the classical coinvariant rings are important because they connect objects in algebra, combinatorics and geometry. Generalizations of R_n/I_n share similar connections.

Around the early 1990s, Garsia and Haiman [27] generalized R_n/I_n to diagonal coinvariants and its isomorphic space diagonal harmonics. The diagonal coinvariants, denoted as DR_n/DI_n, are quotient rings of polynomial rings in *two* sets of commuting variables (in $x_1, \ldots, x_n, y_1, \ldots, y_n$) modded out by their ideal of diagonally symmetric polynomials with no constant term. The diagonal harmonics, denoted as DH_n is a certain orthogonal complement of DI_n. The diagonal coinvariants also affords a natural symmetric group representation under the diagonal action of permuting variables. It is known that DR_n/DI_n and DH_n are isomorphic as symmetric group representations. We choose to describe only the combinatorial connections of DR_n/DI_n here because its geometric connections are not central to this dissertation. Readers interested in the geometric connections of DR_n/DI_n please see [38].

The *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ are another fundamentally important family of numbers [68, 46]. In particular, C_n counts certain objects called *Dyck paths* defined as lattice walks from $(0,0)$ to (n,n) in the first quadrant of $\mathbb{Z}^2$ using only north steps, i.e., $(1,0)$, and east steps, i.e., $(0,1)$. Coming from a combinatorics background, the author is always very excited whenever the Catalan numbers show up.

In 1996, Garsia and Haiman [29, Equation (1)] defined $C_n(q,t)$ to be a rational function in q,t and conjectured that $C_n(q,t)$ is a polynomial in q,t. Garsia and Haiman [29, Equation (3)] showed that $\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix} = q^{\binom{n}{2}}C_n(q,q^{-1})$ and $C_n(q,1) = C_n(1,q)$. Moreover, the

expression $C_n(q,1)$ is the generating series of Dyck words with respect to the well-known area statistic. Furthermore, Garsia and Haiman [29, Equation (15)] showed that $C_n(q,t)$ is the coefficient of the single-column Schur function s_{1^n} in certain expression involving the modified Macdonald polynomials. Garsia and Haiman further conjectured [29, Equation (129)] that the symmetric function expression in [29, Equation (15)] is the bigraded Frobenius image of DH_n. It follows from [29, Equation (129)] that $C_n(q,t)$ is the bigraded Hilbert series of the alternating component of DH_n.

In 2000, Haglund [32] defined a statistic on Dyck words called *bounce* and conjectured that the weighted enumeration of Dyck words with respect to area and bounce is $C_n(q,t)$. Garsia and Haglund proved this conjecture in [26]. The polynomials $C_n(q,t)$ are known as the *q,t-Catalan numbers*.

One very interesting bijective problem on the q,t-Catalan numbers remains open. From an algebraic point of view, DR_n/DI_n is invariant under swapping the x's and the y's. It follows that the q,t-Catalan numbers possess a natural symmetry $C_n(q,t) = C_n(t,q)$. However, there is no bijective proof to this date. The author is partly motivated by this symmetry problem to study coinvariant theory.

We now describe the development that led to the topic in the dissertation, super coinvariant theory, starting from diagonal harmonics.

Let $\mathbb{Q}(q,t)$ the field of rational functions in formal variables q and t. Let $\Lambda_{\mathbb{Q}(q,t)}$ denote the ring of symmetric functions with coefficients in $\mathbb{Q}(q,t)$. Macdonald polynomials are certain generalization of the Hall-Littlewood polynomials mentioned earlier and form an interesting basis of $\Lambda_{\mathbb{Q}(q,t)}$.

Diagonal harmonics motivated significant development in symmetric function theory because their bigraded Frobenius images are interesting symmetric functions in $\Lambda_{\mathbb{Q}(q,t)}$. Using sophisticated algebraic geometry, Haiman [38] proved a "master" theorem that says the bigraded Frobenius image of DH_n is a certain symmetric function expression ∇e_n. The nabla operator ∇ is defined [5] to be a certain operation on $\Lambda_{\mathbb{Q}(q,t)}$ such that Macdonald polynomials are eigenfunctions.

Haglund, Haiman, Loehr, Remmel, and Ulyanov [34] conjectured a certain combinatorial expansion of the diagonal harmonics known as the Shuffle Conjecture at the time. About a decade later, Carlsson and Mellit [12] proved this in their Shuffle Theorem.

The Shuffle Theorem was generalized to the Delta Conjecture first announced in 2015 [36]. The Delta Conjecture has two versions: Rise and Valley. The Rise Version has now been resolved by two groups of researchers [17, 11] using independent methods. The Valley Version remains open. Representation theoretic models, up to a minor twist, for the Delta Theorem at $t = 0$ have been found [37, 47]. In 2020, Rhoades and Wilson [55] found a representation theoretic model, without any twist, for the Delta Conjecture at $t = 0$. But none of these models are direct generalizations of the classical coinvariants.

The development of the super coinvariant theory started just before the author started his PhD. The algebraic combinatorics research group at the Fields Institute defined the super coinvariant rings as a direct generalization of the classical coinvariants. The Fields Conjecture described in Section 1.2 says the super coinvariant rings are representation theoretic models for the Delta Theorem at $t = 0$.

1.2 The Fields conjecture

In the winter term of 2018, Mike Zabrocki presented joint work with Laura Colmenarejo in the Algebraic Combinatorics Seminar at the Fields Institute [15] and kick started the development of super coinvariant theory and super harmonics. The superspace is a polynomial ring $SR_n = \mathbb{Q}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$ where x's commute and θ's anticommute. Define an ideal

$$SI_n = \left\langle \theta_1^{\varepsilon} x_1^k + \cdots + \theta_n^{\varepsilon} x_n^k : \varepsilon \in \{0, 1\}, k \geq 0, \varepsilon + k \geq 1 \right\rangle.$$

The super coinvariant rings are SR_n/SI_n and are direct generalizations of the classical coinvariant rings.

Over the course of the 2018 to 2019 academic year, N. Bergeron, Li, Machacek, Sulzgruber and Zabrocki studied SR_n/SI_n in the Algebraic Combinatorics Seminar at the Fields Institute while the author was learning from the back of the seminar room. By comparing computational data with the Delta Conjecture, [10] conjectured the following connection between SR_n/SI_n and the Delta Theorem at $t = 0$.

Conjecture 1.1 (Fields [10])**.** *The graded Frobenius image of SR_n/SI_n is the symmetric function in the Delta Theorem at $t = 0$.*

Conjecture 1.1 led Zabrocki to generalize super coinvariant space yet again and to conjecture a full representation theoretic model for the Delta Conjecture [80]. The resolution of Conjecture 1.1 is a crucial step towards the full resolution of the Delta Conjecture.

Like other coinvariant spaces, the quotient SR_n/SI_n has a twin, as isomorphic modules, known as *super harmonics*. Our starting point is the alternating component of super harmonics since its historical variants are instrumental in their respective theories. Bergeron, Li, Machacek, Sulzgruber, and Zabrocki [10], and Swanson and Wallach [73] found the same basis for its alternating components. It is interesting to note that Swanson and Wallach generalized super coinvariants to the setting of pseudo-reflection groups [73, 74].

Most recently, as this dissertation was being written, Rhoades and Wilson published a preprint [56] confirming the quotient SR_n/SI_n in Conjecture 1.1 has the desired dimension using an indirect method and sophisticated theorems from commutative algebra. Sagan and Swanson [60] showed that the conjectured bigraded Hilbert series of SR_n/SI_n can be expressed as

$$\sum_{J\subseteq[n]} \left(\prod_{i=1}^{n} [\mathrm{st}(J)_i]_q \right) z^{|J|} \tag{1.1}$$

In a nutshell, [56] essentially looks at SR_n/SI_n one θ_J component at a time where J is a subset of $[n]$. At the crux of their argument are regular sequences[2] $p_{J,1}, \ldots, p_{J,n}$, one for each $J \subseteq [n]$ the Hilbert series of $\mathbb{C}[x_1, \ldots, x_n]/\langle p_{J,1}, \ldots, p_{J,n}\rangle$ is the product $[\mathrm{st}(J)_1]_q \cdots [\mathrm{st}(J)_n]_q$ appearing in the summands of (1.1). Rhoades and Wilson give an abstract straightening argument to show certain elements in SR_n closely related to $p_{J,1}, \ldots, p_{J_n}$, one for each $J \subseteq [n]$, descends to a spanning set of SR_n/SI_n. Rhoades and Wilson also prove an analog of Haiman's Operator Theorem for the harmonic side of the super coinvariant SR_n/SR_n and show that SR_n/SI_n has the desired dimension.

It is still an open problem to find a basis for the quotient SR_n/SI_n. The results in this dissertation approach Conjecture 1.1 from a perspective different than that of [56].

1.3 Outline of results

The dissertation studies some algebraic and combinatorial aspects of Conjecture 1.1. In this section, we highlight the main results of the thesis with comments on their connections

[2]By convention, regular sequences generate proper ideals. It can be shown that $1 \notin \langle p_{J,1}, \ldots, p_{J,n}\rangle$.

and importance. Precision definitions are left to later chapters.

A natural way to study a quotient ring is to try to find a Gröbner basis. This method has been applied successfully to many coinvariant spaces. The ones directly relevant to this dissertation are the classical coinvariant rings [3, 31] and the generalized coinvariant rings by Haglund, Rhoades, and Shimozono [37].

In most cases, techniques for finding a Groöbner are only developed for commutating variables [16]. Therefore, the situation for SR_n/SI_n is different. In an unpublished work, Bergeron, Li, Machacek, Sulzgruber, and Zabrocki [10] conjectured that a certain collection of monomials, called *super Artin monomials*, is a monomial basis for SR_n/SI_n. The precise definition of super Artin monomials is documented in [73, 56]. The group [10] searched exhaustively for a monomial ordering such that the standard monomials of SI_n are the super Artin monomials, and found that even for small values of n, there is no such monomial ordering. Furthermore, [10] studied data for each monomial ordering and found no natural way to describe their standard monomials. The findings of [10] *does not* suggest a *nice* monomial basis does not exist. The author decided to steer away from the Gröbner basis approach.

We look to other coinvariant spaces such as the classical harmonics and the diagonal harmonics for ideas. In both spaces, the entire harmonic space can be generated by taking the linear span of partial derivatives of their corresponding alternating components. The author was inspired by this classical story to seek a new basis of the super harmonics. One would hope that a nice basis of the alternating component would yield some insight to find a basis of the entire super harmonics.

We take advantage of the fact that the alternating component of superspace already has a basis. Our main result is a new basis for the alternating component of super harmonics.

Corollary 3.13. *A basis of the alternating component of SH_n is*

$$\{M_\alpha \diamond \Delta_n : \alpha \in \mathcal{A}_n\}.$$

The results leading up to Corollary 3.13 on page 49 are inspired by the study of (classical) diagonal harmonics. Haiman [38] used a certain set of *symmetric* operator called

polarization operators to describe the alternating components of diagonal harmonics. We study a set of analogous operators in superspace.

We define the polarization action, denoted by a diamond $\diamond$, on the superspace and study basic properties of polynomials as polarization operators. In particular, we define a concept called *polarization equivalence* to weed out cosmetic difference in polarization operators. Theorem 3.10 on page 49 is our main result. It states that certain symmetric polynomials M_α are triangularly related to a certain symmetric polynomials P_α under polarization equivalence. As polarization operators, the M_α's are new and the P_α's are already known. In particular, $\{P_\alpha \diamond \Delta_n : \alpha \in \mathcal{A}_n\}$ is the basis of the alternating component of super harmonics discovered by [10] and [73].

Theorem 3.10. *Let $<$ denote the lexicographical order on $\mathbb{N}^n$. For every $\alpha \in \mathcal{A}_n$, we have*

$$P_\alpha \equiv M_\alpha + \sum_{\phi > \alpha} c_\phi M_\phi \pmod{\mathfrak{I}_n},$$

for some integer c_ϕ where ϕ's are all weak staircases.

Remark 1.2. *Although unaware at the time of the discovery of the notion of polarization equivalence, the author recently found that the idea of polarization equivalence was implicitly used by Allen [1] to conjecture a basis for the alternating component of diagonal harmonics. But Allen's conjecture remains open.*

The new discovery of the polarization operators M_α encouraged the author to search for a basis of the entire super harmonics. By analyzing the combinatorics associated to Conjecture 1.1, the author guessed and proved a folklore result about the cocharge statistic of standard Young tableaux. The statement of Theorem 4.7 on page 73 seems believable among experts. To the author's best knowledge, the statement and the proof of Theorem 4.7 have not appeared in the literature.

Theorem 4.7. *Let x, y be counting variables. For every partition $\mu \vdash n$, we have*

$$\sum_{T \in \mathrm{SYT}(\mu)} x^{\mathtt{maj}(T)} y^{\mathtt{des}(T)} = \sum_{T \in \mathrm{SYT}(\mu)} x^{\mathtt{cc}(T)} y^{\mathtt{des}(T)}.$$

By analyzing experimental data of the intersection of cones under the alternating component of super harmonics, we conjecture an $\mathfrak{S}_n$ basis for each irreducible representation appearing in super harmonics.

Conjecture 4.17. *The set*

$$\mathcal{B}_n = \left\{ (M_\alpha G^{S'}_{T'}) \diamond \Delta_n : S \in \mathrm{SYT}(n), T \in \mathrm{SYT}(\mathrm{sh}(S)), \alpha \in \mathcal{A}_n, \max(\mathtt{Area}(\alpha)) \leq \mathtt{des}(S) \right\}$$

is a basis of the super harmonics SH_n.

At the beginning of 2019, [10] conjectured that the dimension of super harmonics is counted by ordered set partitions, also known as set compositions. It follows that a necessary condition for Conjecture 4.17 is that the set $\mathcal{B}_n$ is also counted by ordered set partitions. We achieve this in Theorem 4.21 on page 86 by modifying the well-known Robinson-Schensted-Knuth algorithm to establish a bijection Φ_k between ordered set partitions and certain pairs of descent coloured standard Young tableaux of the same shape.

Theorem 4.21. *The map Φ_k is well-defined and is a bijection.*

At the beginning of 2023, Rhoades and Wilson published a preprint in which they proved that the dimension of super harmonics is indeed counted by ordered set partitions [56].

Towards proving Conjecture 4.17 we generalize a bilinear form due to Ariki, Terasoma, and Yamada [2] from the classical harmonics space to super harmonics. We prove that this bilinear form is well-defined and non-degenerate. We suggest a natural pairing for the conjectured basis through an operator we call *complement*.

Conjecture 4.30. *The following matrix is invertible*

$$M_n = \left[\left\langle M_\alpha G^{S'}_{T'}, ((M_\beta G^P_Q) \diamond \Delta_n)^c \right\rangle \right]$$

where the indices run over all triples $((\mathtt{Area}(\alpha), S), T)$ such that $\alpha \in \mathcal{A}_n$ and $(\mathtt{Area}(\alpha), S)$ is a coloured standard Young tableau and $T \in \mathrm{SYT}(\mathrm{sh}(S))$, and similarly for $((\mathtt{Area}(\beta), P), Q)$.

Corollary 4.31. *Conjecture 4.30 implies Conjecture 4.17.*

Lastly, Conjecture 4.17 might be very difficult to prove because computational evidence shows that M_n is not a block diagonal matrix even for small values of n. For the special cases of hooks, we suggests a different pairing whose associated Gram matrix is block triangular.

Conjecture 4.33. *For each hook shaped partition $\mu \vdash n$ and $k \in [n-1]$, define a matrix*

$$M_\mu = \left[\langle M_\alpha G^{S'}_{T'}, G^{(P,\beta)}_Q \rangle \right]_{(S,T),(P,Q)},$$

where $(S,T),(P,Q) \in \mathrm{SYT}(\mu)^2$ and $(\mathtt{Area}(\alpha), S)$ and $(\mathtt{Area}(\beta), P)$ are both descent coloured tableaux in $\mathrm{SYT}_{\mathtt{des}}(\mu,k)$. Then M_μ is block triangular with invertible blocks down the diagonal.

Conjecture 4.33 does not imply Conjecture 4.30 for two reasons. Conjecture 4.30 concerns the entire SH_n while Conjecture 4.33 only deals with a subset of irreducible representations in SH_n. The pairing in Conjecture 4.30 are different from the pairing in Conjecture 4.33.

1.4 Organization

This dissertation is organized as follows.

In Chapter 2, we recall the necessary background in combinatorics, the symmetric group and its representations, and the coinvariant theory and its generalizations. When appropriate, we discuss techniques to be described later in relation to the well-known techniques.

In Chapter 3, we discuss a new basis for the alternating components of super harmonics and give an explicit basis change formula.

In Chapter 4, we prove a folklore result on cocharge, describe a conjectural basis for super harmonics and present supporting combinatorial results and computational evidence.

Chapter 2

Background

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of *non-negative integers*.

2.1 Combinatorics

We recall some common terminology in combinatorics. Standard references can be found in [70, 69]. A large part of the dissertation work comes from computations using SageMath. We sometimes choose terminologies to be consistent with the documentation of SageMath [77].

Let $A = \{a_1, \dots, a_k\}$ be a set of k elements. We often use the notation $A = \{a_1 < \cdots < a_k\}$ to denote the set A whose elements $a_1, \dots, a_k$ are canonically named such that $a_1 < \cdots < a_k$. If n is a positive integer, then we denote $[n] = \{1, \dots, n\}$. We write $(i < j)$ to denote a tuple (i, j) such that $i < j$.

The *q-analog* of an integer n is a polynomial in an indeterminate, typically q, defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}.$$

The *q-factorial* $[n]!$ and *q-binomial* $\begin{bmatrix} n \\ k \end{bmatrix}$ are

$$[n]! = \prod_{k=1}^{n} [k]_q \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}.$$

Notice we use the subscript to distinguish $[n]_q$ from the set $[n] = \{1, \ldots, n\}$, but we drop the subscript in q-factorials and q-binomials for the sake of readability.

We will frequently use integer vectors. To compare two integer vectors of the same length, we use the *lexicographical ordering*. Let $\alpha, \beta \in \mathbb{N}^n$, then $\alpha < \beta$ if $\alpha \neq \beta$ and there exists an index $k \in [n]$ such that $\alpha_i = \beta_i$ for all $i \in [k-1]$ but $\alpha_i < \beta_i$.

A *lattice word* w is a finite length word on the alphabet $\{E, N\}$ where E is short for east and N for north. The set of lattice words with m number of E's and n number N's is denoted by $L_{m,n}$. It is well-known that $|L_{m,n}| = \binom{m+n}{n}$ [70]. We draw $w \in L_{m,n}$ as a path in $\mathbb{N}^2$ starting at the $(0,0)$. Suppose the prefix $w_1 \cdots w_{i-1}$ has been drawn. If $w_i = N$, then we extend the path by a unit step in the direction $(0,1)$. Similarly, if $w_i = E$, then we extend the path by a unit step in the direction $(1,0)$. The resulting path is called the *lattice path* of w.

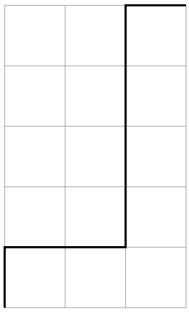


Figure 2.1: The path corresponding to the lattice word $w = NEENNNNE$ in $L_{3,5}$.

An (integer) *partition* is a tuple of positive integers with weakly decreasing entries $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$. An entry in λ is called a *part*. The *length of the partition* is the number of parts, denoted as $\ell(\lambda)$. The *weight of a partition* is the sum of its parts, denoted as $|\lambda| = \lambda_1 + \cdots + \lambda_k$. We write $\lambda \vdash n$ if λ is a partition of weight n and say "λ is a partition of n."

The *Ferrers diagram* of a partition λ is a visual representation of λ, drawn on a plane in matrix notation (also known as the English notation) by placing λ_1 left-justified unit squares in a row, then λ_2 left-justified unit squares in a row below, then λ_3 left-justified in the row below, and so on until all $|\lambda|$ unit squares are placed. Each unit square is called a

cell of λ. More formally, the Ferrers diagram of λ is the set

$$\text{Diagram}(\lambda) = \{(i,j) : 1 \leq i \leq \ell(\lambda) \text{ and } 1 \leq j \leq \lambda_i\}\,.$$

The number of rows in the diagram of a partition λ is $\ell(\lambda)$ and the number of columns is λ_1.

Sometimes we draw a little square $\square$ to denote a cell and say "$\square$ is a cell in λ" or "$\square \in \lambda$" as a shorthand for "$\square \in \text{Diagram}(\lambda)$." If a cell $\square$ appears in row i and column j of Diagram(λ), then we write $\square = (i,j)$.

We will need to describe the relative positions of cells. Let $\square^{(1)} = (i_1, j_1)$ and $\square^{(2)} = (i_2, j_2)$ be two cells in the same partition. If $i_1 < i_2$ then we say $\square^{(1)}$ *appears above* $\square^{(2)}$ and $\square^{(2)}$ *appears below* $\square^{(1)}$. If $j_1 < j_2$, then we say $\square^{(1)}$ *appears to the left* of $\square^{(2)}$ and $\square^{(2)}$ *appears to the right* of $\square^{(1)}$. The *first cell* in row i is the leftmost cell. The *last cell* in row i is the rightmost cell.

A fundamental involution on partitions is conjugation. The *conjugate* of a partition λ is also a partition, denoted by λ', whose entries are given by

$$\lambda'_i = \#\{k \in \lambda : k \geq i\}, \quad \text{for } 1 \leq i \leq \lambda_1.$$

Pictorially, the entry λ'_i counts the number of cells in column i of Diagram(λ). So the Ferrers diagram of λ' is obtained from that of λ by reflecting about the diagonal.

For example, let $\lambda = (5,3,3)$ be a partition. Then $\ell(\lambda) = 3$ and $|\lambda| = 5+3+3 = 11$. The conjugate of $\lambda = (5,3,3)$ is $\lambda' = (3,3,3,1,1)$. See Figure 2.1.

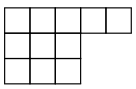
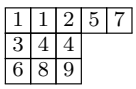
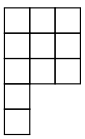
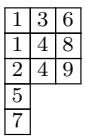
$\lambda =$  $T =$  $\lambda' =$  $T' =$ 

Figure 2.2: The partition $\lambda = 3311$, a semistandard Young tableau T of shape λ, and their conjugates.

A *tableau* is formally defined as a map from (the diagram of) a partition λ to some alphabet, typically an totally ordered set. The *shape* of T is the underlying partition λ,

denoted by $\mathrm{sh}(T)$. A cell in T refers to a cell in $\mathrm{sh}(T)$. If $\square$ is a cell in T, then we call $T(\square)$ the *value of the cell* $\square$. More intuitively, we visualize a tableau T by drawing the Ferrers diagram of $\mathrm{sh}(T)$ and place $T(\square)$ in $\square$ for each cell $\square \in \lambda$.

If the values of T weakly increase along each row read from left to right, i.e., $T(i,1) \leq T(i,2) \leq \cdots$ for each $i = 1, \ldots, \ell(\lambda)$, then we say T is *row weak*. If the values of T (strictly) increase along each row, then we say T is *row strict*. If the values of T weakly increase along each column when read from top to bottom, i.e., $T(1,j) \leq T(2,j) \leq \cdots$ for each $j = 1, \ldots, \lambda_1$, then we say T is *column weak*. If the values of T (strictly) increase along each column, then we say T is *column strict*.

A *semistandard Young tableau* is a row weak but column strict tableau. A *standard Young tableau* is a row and column strict tableau such that each $1, \ldots, n$ appears exactly once. The set of semistandard Young tableaux of shape λ and on alphabet $[n]$ is denoted by $\mathrm{SSYT}(\lambda)$. We define $\mathrm{SYT}(\lambda)$ and $\mathrm{SYT}(n)$ analogously for standard Young tableaux.

$$\mathrm{SSYT}(n) = \bigcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda) \quad \text{and} \quad \mathrm{SYT}(n) = \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda).$$

Remark 2.1. *In the combinatorics literature,* $\mathrm{SSYT}(\lambda)$ *also refers to the set of semistandard tableaux of shape* λ *but on alphabet* $\mathbb{N}$.

The *conjugate of a tableau* T is defined analogously to that of a partition. We conjugate the underlying partition together with their values. More formally, the conjugate of T is denoted as T' whose entries are given by $T'(i,j) = T(j,i)$ for every cell (i,j) in $\mathrm{sh}(T)'$.

A *(combinatorial) statistic* is simply a map $\mathtt{stat} : \mathcal{F} \to \mathbb{N}$ that assigns each object in a set of combinatorial objects $\mathcal{F}$ a non-negative integer. Combinatorial statistics are always typeset in monospaced fonts. We now recall some statistics central to this dissertation, mostly on tableaux.

Let T be a standard Young tableau of size n. An integer i in T is called a *descent* if $i+1$ appears in a row below i. The *set of descents* of T is denoted by $\mathtt{Des}(T)$ as a subset of $[n-1]$. The *number of descents* of T is the cardinality of $\mathtt{Des}(S)$ and is denoted by $\mathtt{des}(T)$. The *major index* of T is defined to be

$$\mathtt{maj}(T) = \sum_{i \in \mathtt{Des}(T)} i. \tag{2.1}$$

Remark 2.2. *There is also a notion of descents and the major index for the set of words. Interested readers please consult [69].*

Closely related to the major index is the cocharge statistic, due to Lascoux and Schützenberger [48] in the study of Hall-Littlewood polynomials. Let $T \in \mathrm{SYT}(\lambda)$. The *cocharge tableau* of T is a tableau of the same shape λ, denoted by $\mathtt{CC}_T$, and defined as follows. For each $i \in [n]$, let $\square^{(i)}$ be the cell in T with value i. The cells $\square^{(1)}, \ldots, \square^{(n)}$ belong to the common partition λ. We assign values of $\square^{(i)}$ in $\mathtt{CC}_T$ depending on their values in T. Initialize $\mathtt{CC}_T$ by placing 0 in the cell $\square^{(1)}$ in $\mathtt{CC}_T$, i.e., we set $\mathtt{CC}_T(\square^{(1)}) = 0$. Recursively for each $i \geq 2$, set

$$\mathtt{CC}_T(\square^{(i)}) = \begin{cases} \mathtt{CC}_T(\square^{(i-1)}) + 1 & \text{if } \square^{(i)} \text{ appears below } \square^{(i-1)} \\ \mathtt{CC}_T(\square^{(i-1)}) & \text{otherwise.} \end{cases} \tag{2.2}$$

The *cocharge word* of T is defined by

$$(\mathtt{CC}_T(\square^{(1)}), \ldots, \mathtt{CC}_T(\square^{(1)})). \tag{2.3}$$

The cocharge word, say α, is always a weakly increasing sequence such that $\alpha_1 = 0$ and $\alpha_{i+1} - \alpha_i \leq 1$ for all $i \geq 1$. Such type of sequences, called *weak staircases*, appears frequently in this dissertation and will be defined in Section 3.3.

The *cocharge* of T is the sum of the values in T, or equivalently the sum of the entries of its cocharge word, i.e.,

$$\mathtt{cc}(T) = \sum_{i \in [n]} \mathtt{CC}_T(\square^{(i)}). \tag{2.4}$$

We mention a dual notion called *charge* on standard Young tableaux, denoted by $\mathtt{ch}(T)$, for completeness. Charge does not appear in the rest of the dissertation. We define a charge tableau $\mathtt{CH}_T$ by setting $\mathtt{CH}_T(\square^{(1)}) = 0$ and recursively setting

$$\mathtt{CH}_T(\square^{(i)}) = \begin{cases} \mathtt{CH}_T(\square^{(i-1)}) & \text{if } \square^{(i)} \text{ appears below } \square^{(i-1)} \\ \mathtt{CH}_T(\square^{(i-1)}) + 1 & \text{otherwise.} \end{cases} \tag{2.5}$$

The charge word is $(\mathtt{CH}_T(\square^{(1)}, \ldots, \mathtt{CH}_T(\square^{(n)}))$ and the charge statistic $\mathtt{ch}(T)$ is the sum of the entries in its charge tableau, or equivalently the sum of the entries in its charge word. The relation between charge and cocharge is well-known.

Theorem 2.3 ([52]). *For every $T \in \mathrm{SYT}(n)$, we have*

$$\mathtt{ch}(T) + \mathtt{cc}(T) = \binom{n}{2}.$$

Remark 2.4. *The naming convention of charge and cocharge is not always consistent in the literature. Sometimes the cocharge (and charge respectively) statistic defined in this dissertation is called charge (and cocharge respectively) in the literature, e.g., [52, page 242]. We have chosen the definition to be consistent with the ones in [4, Page 43] and SageMath [77].*

Remark 2.5. *There is also a pair of notions cocharge and charge statistics on words. Interested readers please consult [52, page 242].*

Example 2.6. *Consider the standard Young tableau*

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}.$$

Its cocharge tableau is constructed visually by a process depending the relative positions of each i and $i-1$. Start by drawing the Ferrers diagram of $\mathrm{sh}(T)$. Initialize a counter $i = 0$. In the empty Ferrers diagram, place i in the cell containing 1 in T. We have

$$\begin{array}{|c|c|c|}\hline 0 & & \\ \hline & & \\ \cline{1-2} & \\ \cline{1-1} \end{array}.$$

The counter i is 0. We increment i because 2 appears below 1 in T. Now $i = 1$. Let $\square^{(2)}$ be the cell in T containing the value 2. We have $\square^{(2)} = (2, 1)$. In the partilly filled Ferrers diagram, place i in the cell $\square^{(2)}$. We have

$$\begin{array}{|c|c|c|}\hline 0 & & \\ \hline 1 & & \\ \cline{1-2} & \\ \cline{1-1} \end{array}.$$

The counter i is 1. We keep i unchanged because 3 does not appear below 2 in T. Let $\square^{(3)}$ be the cell in T containing the value 3. We have $\square^{(2)} = (1, 2)$. In the partially filled Ferrers diagram, place i in the cell $\square^{(3)}$. We have

$$\begin{array}{|c|c|c|}\hline 0 & 1 & \\ \hline 1 & & \\ \cline{1-2} & \\ \cline{1-1} \end{array}.$$

The process continues as follows

$$\begin{array}{|c|c|c|}\hline 0&1&\\\hline 1&2\\\cline{1-2}\end{array}, \qquad \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 1&2\\\cline{1-2}\end{array}, \qquad \mathtt{CC}_T = \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 1&2\\\cline{1-2}3\\\cline{1-1}\end{array}.$$

The cocharge word of T is $(0,1,1,2,2,3)$ and the cocharge of T is

$$\mathtt{cc}(T) = 0+1+1+2+2+3 = 9.$$

Example 2.7. *If we know the descent set of a tableau T, then we can skip the step of computing the cocharge tableau of T to find the cocharge of T. Consider the tableau T in Example 2.6. Its descent set is $\mathtt{Des}(T) = \{1 < 3 < 5\}$ and $n = 6$. Let $d_1 = 1, d_2 = 3, d_3 = 5$.*

If $i = 1$, then we notice the cocharge word of T increases from $i-1 = 0$ to $i = 1$ between positions $d_i = d_1 = 1$ and $d_i + 1 = 2$. This is true for every $i = 1, 2, 3$. And we have

$$\mathtt{cc}(T) = 1 \cdot (n - d_1) + 1 \cdot (n - d_2) + 1 \cdot (n - d_3) = 5 + 3 + 1.$$

This phenomenon is true in general.

Lemma 2.8. *Let $T \in \mathrm{SYT}(n)$. If $\mathtt{Des}(T) = \{d_1 < \cdots < d_k\}$, then*

$$\mathtt{cc}(T) = \sum_{i=1}^{k} (n - d_i).$$

If some C is the cocharge tableau of an unknown standard Young tableau T, then we can recover T from C. However, it is *not possible* to recover a standard Young tableau from knowing only its cocharge word and shape. There exist standard Young tableaux of the same shape with the same cocharge word as shown in the next example.

Example 2.9. *The cocharge words of the following two standard Young tableaux are both equal to $(0,1,1,2,2,3)$.*

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}6\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2}4\\\cline{1-1}\end{array}.$$

It is well-known that major index and cocharge on standard tableaux of the same shape are equidistributed.

Theorem 2.10 ([4, Page 44]). *Let μ be a partition. Then*

$$\sum_{S \in \mathrm{SYT}(\mu)} q^{\mathtt{maj}(S)} = \sum_{S \in \mathrm{SYT}(\mu)} q^{\mathtt{cc}(S)},$$

where q is a counting variable.

We will prove a refinement of Theorem 2.10 in Section 4.1.

The Robinson-Schensted correspondence is an important bijection transforming permutations on $[n]$ to pairs of standard Young tableaux of the same shape on the alphabet $1, \ldots, n$. The correspondence was first described by Robinson [57] in 1938 in a different form than we present here. Schensted rediscovered the correspondence in [61] using certain insertion algorithms in the study of increasing subsequences of permutations. We use Schensted's description. Knuth later extended the correspondence to include biwords [45]. The extension is now known as the *Robinson-Schensted-Knuth correspondence (RSK)*. For a comprehensive treatment of RSK, see [59]. We recall the essential parts used in this dissertation.

A *biword* (or a generalized permutation) is a two-row array $\left[\begin{smallmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{smallmatrix}\right]$ such that $a_1 \leq \cdots \leq a_k$, and if $a_i = a_{i+1}$ then $b_i < b_{i+1}$. We will only apply RSK to words, i.e., biwords of the form $\left[\begin{smallmatrix} 1 & \cdots & k \\ b_1 & \cdots & b_k \end{smallmatrix}\right]$. But we describe the full algorithm for completeness.

First, we define *Schensted insertion*, typically denoted as $P \leftarrow k$, where P is a semistandard tableau and k is an positive integer. The symbol $P \leftarrow k$ also refers to the tableau produced by Schensted insertion.

Schensted insertion was originally only described for permutations. The algorithm presented below is the generalized version due to Knuth and applies to all words.

1. Initialize the row index $i = 1$.
2. Find the *rightmost* cell $\square$ in row i of P such that if $P(\square)$ is replaced by k, then the resulting tableau is still row weak.
   2.1. Suppose such a cell $\square$ cannot be found in row i of P. Add a cell to the end of row i of P and place k there. Go to 3.

2.2. Suppose such a cell $\square$ is found. Save $k' = P(\square)$ and set $P(\square) = k$.

- Update $k = k'$.
- Update i to be $i + 1$.
- Go back to 2 and repeat with the new values of k and i.

3. Call the current $\square$ the last cell of the insertion. Terminate.

Step 2.2 is typically called a *bump* as in "we bump k' in row i."

We describe RSK in two stages, first only for words then for biwords.

Let the input to RSK be a word $w = \left[\begin{smallmatrix} 1 & \cdots & n \\ b_1 & \cdots & b_n \end{smallmatrix}\right]$. Repeatedly apply Schensted insertions as follows. Start by setting P_0 and Q_0 to the empty tableau. Suppose P_{i-1}, Q_{i-1} are defined for some $i \geq 1$. Let $P_i = (P_{i-1} \leftarrow b_i)$ and let $\square$ be the last cell of the insertion. We construct Q_i by duplicating Q_{i-1}, adding $\square$ to Q_i and setting $Q_i(\square) = i$. The correspondence returns (P_n, Q_n).

Let the input to RSK be a biword $w = \left[\begin{smallmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{smallmatrix}\right]$. Apply RSK to the word $b_1 \cdots b_k$ to obtain (P_n, Q_n). Set $P = P_k$. Replace each i by a_i in Q_k to obtain Q. The correspondence returns (P, Q).

Typically, P is called the *insertion tableau* and Q is called the *recording tableau*.

In Section 4.2, we use RSK to establish a bijection between ordered set partitions and a certain set of descent-coloured tableaux to verify that the set of polynomials in Conjecture 4.17 has the right number of elements.

Example 2.11. *Let $w = 815394627$ be a permutation written in one-line notation. Running Robinson-Schensted on w results in the following sequences of tableaux.*

$$
\begin{array}{lll}
P_0 \leftarrow 8 & P_1 = \begin{array}{|c|}\hline 8 \\ \hline \end{array} & Q_1 = \begin{array}{|c|}\hline 1 \\ \hline \end{array} \\
P_1 \leftarrow 1 & P_2 = \begin{array}{|c|}\hline 1 \\ \hline 8 \\ \hline \end{array} & Q_2 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \\
P_2 \leftarrow 5 & P_3 = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 8 \\ \cline{1-1} \end{array} & Q_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \\
P_3 \leftarrow 3 & P_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} & Q_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}
\end{array}
$$

$P_4 \leftarrow 9$	$P_5 = \begin{array}{\|c\|c\|c\|}\hline 1&3&9\\\hline 5\\\cline{1-1} 8\\\cline{1-1}\end{array}$	$Q_5 = \begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$
$P_5 \leftarrow 4$	$P_6 = \begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 5&9\\\cline{1-2} 8\\\cline{1-1}\end{array}$	$Q_6 = \begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}$
$P_6 \leftarrow 6$	$P_7 = \begin{array}{\|c\|c\|c\|c\|}\hline 1&3&4&6\\\hline 5&9\\\cline{1-2} 8\\\cline{1-1}\end{array}$	$Q_7 = \begin{array}{\|c\|c\|c\|c\|}\hline 1&3&5&7\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}$
$P_7 \leftarrow 2$	$P_8 = \begin{array}{\|c\|c\|c\|c\|}\hline 1&2&4&6\\\hline 3&9\\\cline{1-2} 5\\\cline{1-1} 8\\\cline{1-1}\end{array}$	$Q_8 = \begin{array}{\|c\|c\|c\|c\|}\hline 1&3&5&7\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1} 8\\\cline{1-1}\end{array}$
$P_8 \leftarrow 7$	$P_9 = \begin{array}{\|c\|c\|c\|c\|c\|}\hline 1&2&4&6&7\\\hline 3&9\\\cline{1-2} 5\\\cline{1-1} 8\\\cline{1-1}\end{array}$	$Q_9 = \begin{array}{\|c\|c\|c\|c\|c\|}\hline 1&3&5&7&9\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1} 8\\\cline{1-1}\end{array}$

In particular, the insertion $P_7 \leftarrow 2$ is performed as follows. First insert 2 into the first row of P_7 and bump 3 to obtain $P_{7,1} = \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 5&9\\\cline{1-2} 8\\\cline{1-1}\end{array}$. Then insert 3 into the second row of $P_{7,1}$ and bump 5 to obtain $P_{7,2} = \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&9\\\cline{1-2} 8\\\cline{1-1}\end{array}$. Lastly, insert 5 into the third row of $P_{7,2}$ and bump 8 to obtain P_8. The last bump creates a new cell $\square = (4,1)$ which is in P_8 but not in P_7. We obtain Q_8 from Q_7 by adding $\square$ and set $Q_8(\square) = 8$.

Besides many wonderful properties of the Robinson-Schensted correspondence, it yields a combinatorial explanation to the following celebrated identity.

$$\sum_{\lambda \vdash n} |\mathrm{SYT}(\lambda)|^2 = n!. \tag{2.6}$$

In Section 4.2, we use RSK to establish a similar bijective result.

2.2 Representations of the symmetric groups

We summarize and recall notations of the representation theory of the symmetric group $\mathfrak{S}_n$. For detailed references, see [59, 25] and [69, Section 7.6].

In this paragraph, we describe the *idea* of a representation. Formal definitions are given in the paragraph following Remark 2.12. A representation of a finite group arises naturally

from group actions. A finite group G acts on a set of objects X, denoted by $g \cdot x$, if we require $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$. The operation $g_1 g_2$ takes place within the group G. If we turn X into a vector space, then we obtain an example of representation, call a permutation representation. We remark not all representations are permutation representations. The idea of an representation is that a finite group G acts on a vector space V, not necessarily just permuting elements within a basis.

For example, let $\mathbb{C}G$ be the vector space of all formal linear combinations of elements G with coefficients in $\mathbb{C}$. We obtain a fundamentally important representation by letting G act on the left of $\mathbb{C}G$ by multiplication. Furthermore, each group element $g \in G$ can be identified with a matrix M_g in the basis of X with the property that they multiply homomorphically as in G, i.e., $M_{g_1} M_{g_2} = M_{g_1 g_2}$. The structure of the representation is completely encoded by a function $\chi : g \mapsto \text{trace}(M_g)$ called its character.

Remark 2.12. *In particular, the trace of representing matrix of the identity of G, i.e., $\chi(M_{id})$ is exactly the dimension of the representation V. This fact is useful when we use symmetric functions to encode decomposition of representations.*

Let G be a group and let V be a vector space. A *representation* of G is a choice of group homomorphism $\phi : G \to \text{GL}(V)$ where $\text{GL}(V)$ is the group of invertible linear transformations on V. We call V a *G-module* and leave the choice ϕ to be understood from the context. Representation and G-modules are equivalent concepts. We adopt the slightly vague but popular phrase "*V is a representation of G*" leaving the definition of ϕ to be understood from the context. We write ϕ_g to denote the image some $g \in G$ under ϕ. If a subspace $U \subseteq V$ is closed under the same action of G, i.e., $\phi_g(U) \subseteq U$ for all $g \in G$, then we say U is a *submodule* of V. If the only submodules of V are $\{0\}$ and V and $V \neq \{0\}$, then we say V is *irreducible*. Maschke's Theorem states that as long as G *is finite* and $|G|$ is invertible in the base field of V, then every representation V of G can be written as a direct sum

$$V = V^{(1)} \oplus \cdots \oplus V^{(k)},$$

where $V^{(1)}, \ldots, V^{(k)}$ are irreducible representations of G [59, Theorem 1.5.3].

From now on, we restrict our discussions to *finite* groups and work over $\mathbb{C}$.

Sometimes representations differ only cosmetically. A linear map $f : V \to W$ where both V, W are representations of the same group G is a *homomorphism* (of representations) if the action of G commutes with f. If f is a bijection, then f is said to be an *isomorphism*.

Schur's Lemma is a useful tool for comparing irreducible representations up to isomorphism.

Theorem 2.13 (Schur's Lemma [59, Theorem 1.6.5])**.** *Let V, W be irreducible representations of G. If $f : V \to W$ is a homomorphism, then f is either an isomorphism or the zero map.*

It is well-known that, up to isomorphism, the number of distinct irreducible representations is counted by the number of conjugacy classes of G.

By choosing a basis of a representation V we can realize each linear transformation ϕ_g as a matrix M_g. Everything we described above has a parallel description in terms of matrices. The *character* of V is defined to be $\chi(g) = \text{trace}(M_g)$. A character of an irreducible representation is called an *irreducible character*. The decomposition of V into irreducible representations is completely determined by its character.

A function $f : G \to \mathbb{C}$ is called a *class function* of G if $f(g_1) = f(g_2)$ whenever $g_1, g_2 \in G$ belong to the same conjugacy class. Characters are examples of class functions. The set of class functions of G is denoted by $\mathcal{K}_G$. We can define an inner product on $\mathcal{K}_G$ by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

It can be shown [59] that irreducible characters form an orthogonal basis of $\mathcal{K}_G$ with respect to the inner product. The following theorem is stated slightly differently from its reference but in an equivalent form.

Theorem 2.14 ([59, Corollary 1.9.1])**.** *Suppose $V^{(1)}, \ldots, V^{(k)}$ is a complete list of irreducible representations up to isomorphism. Let $\chi^{(1)}, \ldots, \chi^{(k)}$ be their corresponding characters. Suppose V is a representation of the same group with character χ. If V decomposes as*

$$V \cong m_1 V^{(1)} \oplus \cdots \oplus m_k V^{(k)},$$

then $m_i = \langle \chi, \chi^{(i)} \rangle$ for all $i = 1, \ldots, k$.

Theorem 2.14 says that if m_i is the number of times an irreducible representation $V^{(i)}$ appears in a representation V, then m_i can be computed directly from the character of V without the need to exhibit a basis of V.

The *symmetric group* $\mathcal{S}_n$ is the group of bijections on the ground set $[n]$ under function composition. If A is a set, then $\mathcal{S}_A$ consists of bijections on A. The *group algebra of* $\mathcal{S}_n$ is the vector space $\mathbb{C}[\mathcal{S}_n] = \mathrm{Span}_{\mathbb{C}}\{\sigma : \sigma \in \mathcal{S}_n\}$ endowed with the multiplication in $\mathcal{S}_n$.

For every $A \subseteq [n]$, we treat $\mathcal{S}_A$ as a subgroup of $\mathcal{S}_n$ and define, as elements in $\mathbb{C}[\mathcal{S}_n]$,

$$\mathrm{sym}_A = \sum_{\sigma \in \mathcal{S}_A} \sigma \quad \text{and} \quad \mathrm{alt}_A = \sum_{\sigma \in \mathcal{S}_A} \mathrm{sgn}(\sigma)\sigma. \tag{2.7}$$

We call sym_A as the symmetrization operator on A and alt_A the antisymmetrization operator on A. Moreover, we define

$$\mathrm{sym}_n = \mathrm{sym}_{[n]} \quad \text{and} \quad \mathrm{alt}_n = \mathrm{alt}_{[n]}.$$

Young symmetrizers are essential in the representation theory of $\mathcal{S}_n$. We recall relevant notations. Let $T \in \mathrm{SYT}(\mu)$ be a standard Young tableau of shape μ. Define $R_i(T)$ to be the set of values in its i-th row, i.e., $R_i(T) = \{T(i,j) : j \geq 1\}$. Similarly, define $C_j(T)$ to be the set of values in its j-th column, i.e., $C_j(T) = \{T(i,j) : i \geq 1\}$. The row and column groups of T

$$R(T) = \mathcal{S}_{R_1(T)} \times \cdots \times \mathcal{S}_{R_{\ell(\mu)}(T)} \quad \text{and} \quad C(T) = \mathcal{S}_{C_1(T)} \times \cdots \times \mathcal{S}_{C_{\mu_1}(T)},$$

where $\mathcal{S}_X$ denotes the subgroup of $\mathcal{S}_n$ on the ground set $X \subseteq [n]$. The *row symmetrizer* and the *column anti-symmetrizer* of T are elements of the group algebra $\mathbb{C}[\mathcal{S}_n]$ defined by

$$P(T) = \sum_{\sigma \in R(T)} \sigma \quad \text{and} \quad N(T) = \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\sigma.$$

An equivalent way to express $P(T)$ and $N(T)$ is

$$P(T) = \mathrm{sym}_{R_1(T)} \times \cdots \times \mathrm{sym}_{R_{\ell(\mu)}(T)} \quad \text{and} \quad P(T) = \mathrm{alt}_{C_1(T)} \times \cdots \times \mathrm{alt}_{C_{\ell(\mu)}(T)}$$

The *Young symmetrizer* is an element of the group algebra $\mathbb{C}[\mathcal{S}_n]$ defined by

$$\varepsilon_T = N(T)P(T). \tag{2.8}$$

Young symmetrizers are used to describe, up to isomorphism, all irreducible representations of $\mathcal{S}_n$. Let $\lambda \vdash n$ be a partition and let $T \in \mathrm{SYT}(\lambda)$. Define a monomial

$$x^T = \prod_{\square \in \lambda} x_{T(\square)}^{\mathrm{row}(\square)-1}.$$

The polynomial Δ_T defined by

$$\Delta_T(x_1, \ldots, x_n) = N(T) \cdot x^T$$

is often known as the Garnir polynomial [28]. It can be shown that Δ_T are linearly independent and

$$\mathcal{S}^\lambda = \mathrm{Span}_{\mathbb{C}} \left\{ \Delta_T : T \in \mathrm{SYT}(\lambda) \right\},$$

is an irreducible representation of $\mathcal{S}_n$ of shape λ.

From general representation theory (of finite groups) [59, Equation (1.26)], irreducible representations of $\mathcal{S}_n$ are indexed by partitions of n, and each irreducible representation appears in $\mathbb{C}[\mathcal{S}_n]$ exactly f^λ times where f^λ is the dimension of the irreducible representation indexed by a partition λ. It follows that the group algebra decomposes as

$$\mathbb{C}[\mathcal{S}_n] \cong \bigoplus_{\lambda \vdash n} f^\lambda \mathcal{S}^\lambda, \tag{2.9}$$

where $f^\lambda = |\mathrm{SYT}(\lambda)|$.

In particular, when T is the unique one-column standard Young tableau, then $N(T) = \mathrm{alt}_n$ and $x_T = x_1^0 \cdots x_n^{n-1}$, and

$$\Delta_T = \mathrm{alt}_n \cdot x_1^0 \cdots x_n^{n-1}. \tag{2.10}$$

We will see later, c.f., Equation (2.19) on page 32, that Δ_T is the Vandermonde determinant and plays an important role in coinvariant theory.

One can also construct all irreducible representations purely combinatorially using polytabloids. Interested readers should consult [59]. We root our understanding in the Garnir polynomials Δ_T because of the generalization by Ariki, Terasoma, and Yamada [2], which we describe in Section 2.5.

2.3 Symmetric functions

In this section, we recall some well-known facts of symmetric functions. Our references are [52], [69, Chapter 7] and [59].

Symmetric functions play two distinct roles in this dissertation. Firstly, as polynomials, they are used to generate an $\mathcal{S}_n$ representation on a quotient ring as described in Section 2.4. Secondly, as formal power series, they are used to encode the decomposition of $\mathcal{S}_n$ representations into irreducible representations. We describe the latter in this section.

A *formal power series* $f(x_1, x_2, \dots)$ is a formal, potentially infinite, sum of monomials in alphabet $x_1, x_2, \dots$ if each monomial appearing in f with non-zero coefficient has a bounded degree. By formal, we mean the analytic properties of such expressions do not play a role. For example, $\prod_{i\geq 1} x_i$ is not a formal power series since it does not have a bounded degree. But $\frac{1}{1-x}$ is a formal power series. A *symmetric function* is a formal power series of bounded total degree that is invariant under any permutation of any *finite* subset of arguments. Symmetric functions form a subring of the formal power series, denoted as Λ. The subspace $\Lambda^{(n)}$ denotes the subspace of Λ consisting of homogeneous degree n symmetric functions. It is well-known that $\dim \Lambda^{(n)}$ is counted by partitions of n.

There are five well-known bases of Λ. The *monomial symmetric functions* m_λ form a basis. They are obtained by symmetrizing, without duplicates, a monomial $x^\lambda = \prod_{i=1}^{\ell(\lambda)} x_i^{\lambda_i}$ where λ is a partition. The next three are multiplicative bases. The *elementary symmetric functions* e_k are defined to be the sum of square-free monomials of degree k. The *homogeneous symmetric functions* h_k are defined to be the sum of all monomials of degree k. The *power-sum symmetric functions* p_k are defined to be $p_k = \sum_{i\geq 1} x_i^k$. Given a partition λ, we define

$$e_\lambda = \prod_{k\in\lambda} e_k, \quad h_\lambda = \prod_{k\in\lambda} h_k, \quad \text{and} \quad p_\lambda = \prod_{k\in\lambda} p_k.$$

The fifth basis is the Schur basis, to be defined combinatorially. See [52] for equivalent definitions. Let T be a tableau. We associate a monomial $x^T = \prod_{i\geq 1} x_i^{w_i(T)}$ where $w_i(T)$ is the number of times the value i appears in T. Since T has finitely many values, all but finitely many $w_i(T)$ are 0. It follows that x^T has bounded degree and, consequently, a well-defined formal power series. The *Schur function* indexed by a partition λ is defined to

be

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

There is a fundamental involution which we denote $\omega : \Lambda \to \Lambda$ that sends e_k to h_k and is extended multiplicatively and linearly to all of Λ. Its effect on Schur function is conjugation [52, (3.8) on page 42], i.e.,

$$\omega(s_\lambda) = s_{\lambda'}.$$

Schur functions have many nice properties. The one most relevant to this dissertation is about the character table of the symmetric group.

Theorem 2.15 ([59, 52, 69])**.** *Let λ be a partition of n. Then*

$$s_\lambda = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi^\lambda(\sigma) p_{\mathrm{cyc}(\sigma)}, \tag{2.11}$$

where $\mathrm{cyc}(\sigma)$ *is the partition indexing the conjugacy class of* σ*.*

Recall from Section 2.2 that partitions of n are in bijective correspondence with irreducible representations of $\mathcal{S}_n$ up to isomorphism. Let $\mathcal{K}_n$ be the set of class functions on $\mathcal{S}_n$. There is a vector space isomorphism called the *characteristic (Frobenius) map* [52, 69, 59] $\mathrm{ch}^n : \mathcal{K}_n \to \Lambda^{(n)}$ by

$$\mathrm{ch}^n(\chi) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) p_{\mathrm{cyc}(\sigma)}, \tag{2.12}$$

where χ is a class function of $\mathcal{S}_n$. Comparing Equation (2.11) and (2.12), we can immediately conclude that if χ^λ is an irreducible character of $\mathcal{S}_n$, then

$$\mathrm{ch}^n(\chi^\lambda) = s_\lambda. \tag{2.13}$$

Equation (2.13) says the characteristic map associates, up to isomorphism, each irreducible representation to the Schur function indexed by the same partition. It also gives a concrete way to decompose an $\mathcal{S}_n$ representation V into irreducible representations without having to exhibit an $\mathcal{S}_n$-invariant basis if the character χ is known. Simply rewrite $\mathrm{ch}^n(\chi)$

in terms of the Schur functions. The resulting coefficient $[s_\lambda]\mathrm{ch}^n(\chi)$ is the multiplicity of the Specht module $\mathcal{S}^\lambda$ in V.

If all we want to know about a representation is its decomposition into irreducible representations, the notation of the characteristic map is somewhat cumbersome because we have to mention characters. It is customary in the coinvariant community to use the term *Frobenius image*. Formally, let V be an $\mathcal{S}_n$ representation with character χ, the *Frobenius image* of V is defined to be

$$\mathrm{Frob}(V) = \mathrm{ch}(\chi)$$

where we extend ch^n to $\mathrm{ch} : \oplus_{n\geq 0}\mathcal{K}_n \to \Lambda$ in the natural way.

If V is a multi-graded representation $V = \oplus V^{(d_1,\ldots,d_k)}$, then we can define the *graded Frobenius image* of V by

$$\mathrm{grFrob}(V; q_1, \ldots, q_k) = \sum q_1^{d_1} \cdots q_k^{d_k} \mathrm{Frob}(V^{(d_1,\ldots,d_k)}),$$

where $q_1, \ldots, q_k$ are formal counting variables. If we are only concerned with the dimension information, we can use the *Hilbert series*

$$\mathrm{Hilb}(V; q_1, \ldots, q_k) = \sum q_1^{d_1} \cdots q_k^{d_k} \dim(V^{(d_1,\ldots,d_k)}).$$

We can obtain the Hilbert series of V from its Frobenius series in two different ways. Let U be a graded component of V and let χ be the character of U. By Remark 2.12, $\dim(U)$ is χ evaluated at (the matrix representing) the identity permutation. Then by Equation (2.12), the coefficient of p_{1^n} in $\mathrm{Frob}(U)$ is $\dim(U)$. Since the identity permutation is the only permutation whose cycle type is 1^n, we get

$$\dim(U) = \mathrm{Frob}(U)\Big|_{p_1=1,\, p_2=\cdots p_n=0},$$

where $\mathrm{Frob}(U)$ is expressed in the power-sum basis. It follows that $\mathrm{Hilb}(V)$ can be obtained from $\mathrm{grFrob}(V)$ by setting $p_1 = 1$ and $p_k = 0$ for all $k \geq 2$. Alternatively, Equation (2.13) says the coefficient of s_λ in $\mathrm{grFrob}(V)$ is the number of copies of $\mathcal{S}^\lambda$ in V. So we can obtain $\mathrm{Hilb}(V)$ by from $\mathrm{grFrob}(V)$ by replacing each s_λ in $\mathrm{grFrob}(V)$ by f^λ which is the dimension of $\mathcal{S}^\lambda$.

Lastly, we recall two formulae to be used in Section 3.3.

The elementary symmetric functions satisfy the following recursion.

Lemma 2.16 ([52]). *For every $k \geq 1$ and every $i \in [n]$, we have*

$$e_k(x_1, \ldots, x_n) = x_i e_{k-1}(\widehat{x_i}) + e_k(\widehat{x_i}). \tag{2.14}$$

The *dominance order on partitions*, also known as the natural order, is a partial order defined by comparing all partial weights. More precisely, for partitions λ, μ, we define $\lambda \succ \mu$ in dominance order if $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$ where we set $\lambda_i = 0$ for all $i > \ell(\lambda)$.

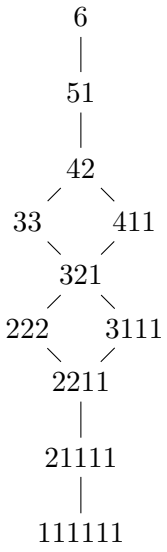


Figure 2.3: The Hasse diagram of $\{\lambda \vdash 6\}$ in the dominance order.

Theorem 2.17 ([52]). *Let $\lambda \vdash n$. Then*

$$e_{\lambda'} = m_\lambda + \sum_{\mu \prec \lambda} a_{\lambda\mu} m_\mu \tag{2.15}$$

for some non-negative integers $a_{\lambda\mu}$ where $\prec$ is the dominance order.

There is a map from symmetric functions to symmetric polynomials [52, 59]. This map sends a symmetric function f expressed in the power sum basis by sending each

p_k to $x_1^k + \cdots + x_n^k$ and we denote the image by $f(x_1, \ldots, x_n)$. To obtain a symmetric polynomial $f(x_1, \ldots, x_n)$ in $\mathbb{Q}[x_1, \ldots, x_n]$ from a symmetric function f, we specialize all irrelevant indeterminates $x_{n+1}, x_{n+2}, \ldots$ in the formal power series f to 0, i.e., $f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 0, 0, \ldots)$.

2.4 Coinvariant theory and diagonal harmonics

Let $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ be the ambient space. The symmetric group $\mathcal{S}_n$ acts on R_n by permuting variables, i.e., $\sigma(f)(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$ for every permutation $\sigma \in \mathcal{S}_n$ and every polynomial $f(x_1, \ldots, x_n) \in R_n$. This is a well-defined action turning R_n into an $\mathcal{S}_n$ representation. The ring of symmetric polynomials is precisely the subring of R_n that is invariant under this action. For convenience, we will drop the argument list from the notation and denote a polynomial $f(x_1, \ldots, x_n)$ by f, unless the argument list becomes necessary.

The ideal in R_n generated by symmetric polynomials with no constant terms is

$$I_n = \langle e_1, \ldots, e_n \rangle \tag{2.16}$$

where $e_1, \ldots, e_n$ are the elementary symmetric polynomials in R_n. The quotient R_n/I_n is well studied and is known as the *classical coinvariant ring* among the algebraic combinatorics community. Later in the dissertation, we will use the same set of generators to create ideals in different ambient spaces.

Various bases of R_n/I_n have been found. In 1942, Artin [3, page 38] gave a reduction argument to show that the set of partial homogeneous symmetric functions

$$h_i(x_i, \ldots, x_n) = \sum_{\substack{0 \le j_i \le \cdots \le j_n \le i \\ j_i + \cdots + j_n = i}} x_i^{j_i} \cdots x_n^{j_n} = x_i^i + (\text{other terms of degree } i)$$

for $i = 1, \ldots, n$ generate the ideal I_n. It follows that the so-called substaircase monomials

$$\{x^\alpha : \alpha_i < i \text{ for all } i \in [n]\}$$

descend to a basis of R_n/I_n, and is called the *Artin basis* of R_n/I_n.

These substaircase exponent vectors are in bijective correspondences to the so-called Lehmer codes of permutations. Let $\sigma \in \mathcal{S}_n$ be written in one-line notation. An *inversion*

pair starting at position i is a tuple $(i < j)$ such that $\sigma_i > \sigma_j$. The Lehmer code of σ is a vector of length n, denoted as $\text{code}(\sigma)$, whose i-th entry counts the number of inversion pairs starting at position i. The *inversion* statistic is $\texttt{inv}(\sigma) = |\text{code}(\sigma)|$. Artin's argument then shows R_n/I_n has dimension $n!$ and

$$\text{Hilb}(R_n/I_n; q) = \sum_{\sigma \in \mathcal{S}_n} q^{\texttt{inv}(\sigma)} = [n]_q!.$$

It is well-known that the q-factorial is also the generating series for $\mathcal{S}_n$ with respect to the major index [33]. A *descent* of a permutation σ written in one-line notation is a value i such that $\sigma_i > \sigma_{i+1}$. The *major index*, denoted by $\texttt{maj}(\sigma)$, of a permutation is the sum of its descents. In 1975, Steinberg [71] discovered a basis, commonly referred to as the Garsia-Stanton descent basis in the algebraic combinatorics community, for R_n/I_n in the general context of Weyl groups. Nine years later, Garsia and Stanton [31] developed combinatorial methods to obtain Steinberg's basis. Garsia and Stanton expressed their basis as

$$x_\sigma = \sum_{i \in \texttt{Des}(\sigma)} \prod_{j=1}^{i} x_{\sigma_j},$$

one for each permutation $\sigma \in \mathcal{S}_n$.

Chevalley studied coinvariant theory in the broader class of groups known as finite complex reflection groups [14] in 1955. But we will restrict ourselves to just the symmetric groups. A special case of Chevalley's result is that R_n/I_n is a graded version of the left-regular representation of $\mathcal{S}_n$. Furthermore, one can deduce from Chevalley's result that

$$\text{grFrob}(R_n/I_n; q) = \sum_{T \in \text{SYT}(n)} q^{\texttt{cc}(T)} s_{shape(T)},$$

where $\texttt{cc}$ is the cocharge statistic due to Lascoux and Schützenberger [48].

In 1992, Garsia and Procesi [30] studied a broader family of quotient rings R_n/I_μ from a geometric point of view where μ is a partition and I_μ is defined by Tanisaki [75]. Garsia and Procesi connected the Frobenius image of R_n/I_μ to the Hall-Littlewood symmetric functions indexed by the partition μ'. When μ is the partition (n), then $R_n/I_{(n)} = R_n/I_n$.

In 1993, Terasoma and Yamada announced an explicit basis for every irreducible representation appearing in R_n/I_n along with a proof sketch [76]. The full proof was published later in 1997 [2]. The basis elements due to [2] are known as the higher Specht polynomials.

Recall the cocharge tableau is defined in Equation (2.2). Let $S \in \mathrm{SYT}(\mu)$ be a standard Young tableau. For each $T \in \mathrm{SYT}(\mu)$, define a monomial, called the *cocharge monomial*,

$$x_T^S = \prod_{\square \in \mu} x_{T(\square)}^{\mathtt{cc}_S(\square)}.$$

The *higher Specht polynomials* are

$$G_T^S = \varepsilon_T x_T^S \tag{2.17}$$

where ε_T is the Young symmetrizer defined in Equation (2.8).

For each fixed $S \in \mathrm{SYT}(\mu)$, the space $V^S = \mathbb{Q}\{G_T^S : T \in \mathrm{SYT}(\mu)\}$ is an irreducible representation. Furthermore, these representations canonically describe all irreducible representations in R_n/I_n. More precisely, Ariki, Terasoma, and Yamada [2] proved that elements of $\cup_{S \in \mathrm{SYT}(n)} V^S$ descend to a basis of R_n/I_n, i.e.,

$$R_n/I_n = \bigoplus_{S \in \mathrm{SYT}(n)} \left\{ G_T^S + I_n : T \in \mathrm{SYT}(\mu) \right\}. \tag{2.18}$$

The decomposition in Equation (2.18) refines the one exhibited in Equation (2.9) and plays an important role in this dissertation as we will see in Section 4.2.

Example 2.18. *Suppose we have the following standard Young tableaux S, T together with the cocharge tableaux $\mathtt{cc}_S$*

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}, \quad \mathtt{cc}_S = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 3 & 4 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}, \quad \text{and} \quad T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 7 & 9 \\ \hline 3 & 6 & 8 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}.$$

We obtain the associated cocharge monomial $x_T^S = x_1^0 x_2^1 x_3^1 x_4^2 x_5^3 x_6^2 x_7^3 x_8^4 x_9^4$ by reading each cell $\square \in \mathrm{sh}(T)$, then $T(\square)$ is the subscript of x and $\mathtt{cc}_S(\square)$ is the exponent.

The two monomial bases and the higher Specht basis are good descriptions of the quotient side of the classical story. But there is a harmonic side. Define a bilinear form on R_n by

$$\langle f, g \rangle = \left(f(\partial x_1, \ldots, \partial x_n) \cdot g \right) \Big|_{x_1 = \cdots = x_n = 0},$$

where $f(\partial x_1, \ldots, \partial x_n)$ turns the polynomial f into a (formal) partial differential operator by replacing each x_i by ∂x_i. The *classical harmonics* is $H_n = I_n^{\perp}$ the orthogonal complement of I_n. One can deduce that polynomials $f \in H_n$, called harmonic polynomials, are exactly the solutions to the system of differential equations $p_k(\partial x_1, \ldots, \partial x_n) f = 0$ for all $k \geq 1$. In other words, we have

$$H_n = \{f \in R_n : p_k(\partial x_1, \ldots, \partial x_n) f = 0 \text{ for all } k \geq 1\}.$$

For a comprehensive treatment, see [30].

The classical harmonics H_n possesses several interesting properties that are useful for studying more general coinvariant spaces. First, H_n consists of polynomials not cosets. Computing Gröbner bases in general is a difficult task. Working with H_n has the advantage of avoiding computing Gröbner bases altogether. This also makes H_n more machine computation friendly. Secondly, the top degree component of H_n is the 1-dimensional alternating space generated by the *Vandermonde* (determinant) defined by

$$\Delta_n = \prod_{1 \leq i < j \leq n} (x_j - x_i). \tag{2.19}$$

We note that $\Delta_n = \text{alt}_n \cdot x_1^0 \cdots x_n^{n-1} = x_T$ where T is the unique one-column standard Young tableaux by Equation (2.10).

Define the *cone* of a polynomial $f \in R_n$ to be the linear span of all partial derivatives of f. We denote the cone of f by

$$\mathcal{L}_\partial(f) = \mathbb{Q}\{\partial x^\alpha \cdot f : \alpha \in \mathbb{N}^n\}. \tag{2.20}$$

Steinberg [71] showed that the harmonics are precisely the cone of the Vandermonde Δ_n. Thirdly, Bergeron and Garsia showed [9] that bases of R_n/I_n and H_n are *dual* to each other in the following sense. Define the flip map $\text{flip} : R_n \to H_n$ by $\text{flip}(f) = f(\partial)\Delta_n$.

Theorem 2.19 (Proposition 3.2 (b) in [9])**.** *For any $f \in R_n$, $\text{flip}(f) = 0$ if and only if $f \in I_n$. In other words, $\ker(\text{flip}) = I_n$.*

Theorem 2.19 immediately gives us an instrumental tool to transfer properties between the coinvariant side R_n/I_n and the harmonic side H_n.

Theorem 2.20 (Theorem 3.3 in [9]). *A set of polynomials B_n descends to a basis for R_n/I_n if and only if* $\text{flip}(B_n) = \{\text{flip}(b) : b \in B_n\}$ *is a basis of H_n.*

We now move on to a generalization of R_n/I_n known as diagonal harmonics. Our reference is [33]. Let $DR_n = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the ambient ring. Let $\mathcal{S}_n$ act on DR_n by permuting both sets of variables together. More precisely, define an action by

$$\sigma(f)(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}, y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}).$$

Let $DSym_n$ denote the $\mathcal{S}_n$ invariant subring in DR_n. The *diagonal coinvariant ring* is DR_n/DI_n where DI_n is generated by polynomials in $DSym_n$ with no constant terms. An explicit description of generators of DI_n is given by the mixed power sums [33]

$$DI_n = \left\langle \sum_{k=1}^{n} x_k^i y_k^j : i, j \in \mathbb{N} \text{ and } i + j \geq 1 \right\rangle.$$

Define a bilinear form on DR_n by

$$\langle f, g \rangle = (f(\partial x_1, \ldots, \partial x_n, \partial y_1, \ldots, \partial y_n) \cdot g)\Big|_{x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0}.$$

The *diagonal harmonics* is the orthogonal complement $DH_n = DI_n^{\perp}$.

An important theorem in the study of diagonal harmonics is Haiman's Operator Theorem. Define operators $E_k = y_1 \partial x_1^k + \cdots + y_n \partial x_n^k$ for each $k \geq 1$.

Theorem 2.21 (The Operator Theorem [38, Theorem 4.2]). *The diagonal harmonics DH_n is the smallest subspace of DR_n satisfying all of the following.*

1. *It contains $\Delta_n(x_1, \ldots, x_n)$.*
2. *It is closed under partial differentiation operators $\partial x_1, \ldots, \partial x_n$.*
3. *It is closed under the operators $E_1, \ldots, E_n$.*

2.5 Superspace and super harmonics

The *superspace*, also known as the super polynomial ring, is $SR_n = \mathbb{Q}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]$ where $x_1, \ldots, x_n$ are commuting variables, $\theta_1, \ldots, \theta_n$ are anticommuting variables, and the two sets of variables commute [22]. More precisely, for all indices $i, j \in [n]$, we have relations

$$x_i x_j = x_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i, \quad \text{and} \quad \theta_i x_j = x_j \theta_i.$$

Notice the middle relation implies $\theta_i^2 = 0$ for all $i \in [n]$. Hence, the subring $\mathbb{Q}[\theta_1, \ldots, \theta_n]$ is a copy of the exterior algebra. We index monomials in $\mathbb{Q}[\theta_1, \ldots, \theta_n]$ by a set and *canonically* write $\theta_A = \theta_{a_1} \cdots \theta_{a_k}$ if $A = \{a_1 < \cdots < a_k\}$.

It is interesting to note that the study of super symmetric polynomials originates in mathematical physics [19, 21] and has developed into an interesting topic in symmetric function theory [20, 24, 23].

Superspace carries natural notions of weight, degree and grading. The *weight* of a monomial $x^\alpha \theta_A \in SR_n$ is defined to be

$$\texttt{weight}(x^\alpha \theta_A) = (|\alpha|, |A|).$$

The two pairs of vertical bars carry different meanings. Because α is an exponent vector, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is its weight (as a vector). Because A is a set, $|A|$ is the cardinality of A. For example, $\texttt{weight}(x_1 x_2^3 \theta_1) = (4, 1)$. A polynomial is *bi-homogeneous* if every monomial on its support has the same weight. For each pair $i \in \mathbb{N}$ and $j \in \{0, 1, \ldots, n\}$, the (i, j)-bi-homogeneous component of SR_n is the vector subspace $SR_n^{(i,j)}$ consisting of 0 and all bi-homogenous polynomials whose weight is (i, j). The weight function yields a natural grading of SR_n, namely

$$SR_n = \bigoplus_{\substack{i \in \mathbb{N} \\ j \in \{0,1,\ldots,n\}}} SR_n^{(i,j)}.$$

The *(total) degree* of a monomial $x^\alpha \theta_A$ is defined to be

$$\deg(x^\alpha \theta_A) = |\alpha| + |A|.$$

A polynomial is *homogenous* if every monomial on its support has the same degree. The weight function is a natural grading for superspace.

The symmetric group acts on the superspace by a diagonal action. More precisely, for each $\sigma \in \mathcal{S}_n$ and $f \in SR_n$, we define an action by

$$\sigma(f)(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}, \theta_{\sigma^{-1}(1)}, \ldots, \theta_{\sigma^{-1}(n)}).$$

For example, if σ is the transposition (13) and $f = \theta_1\theta_2(x_2 - x_1)$, then

$$\sigma(f) = \theta_3\theta_2(x_2 - x_3) = -\theta_2\theta_3(x_2 - x_3).$$

If a polynomial $f \in SR_n$ is invariant under the symmetric group action, i.e., $\sigma(f) = f$ for all $\sigma \in \mathcal{S}_n$, then it is called a *symmetric polynomial in superspace*. The set of symmetric polynomials in SR_n is denoted by SSym_n. They form an algebra and have been studied by Desrosiers, Lapointe, and Mathieu [22]. Their motivation is connected to particle physics.

We note that in 1963 Solomon [65] studied invariants of S_n in superspace in the general context of finite complex reflection groups.

Motivated by diagonal harmonics and the development of symmetric functions in superspace, part of the Algebraic Combinatorics Workshop Group at Fields Institute [10], consisting of Nantel Bergeron, Shu Xiao Li, John Machacek, Robin Sulzgruber, and Mike Zabrocki, considered the quotient ring known as the super coinvariant ring defined as follows. Define an ideal

$$SI_n = \langle \text{bi-homogeneous polynomials in } \mathrm{SSym}_n^+ \rangle,$$

where SSym_n^+ consists of symmetric functions in superspace with no constant term. The ideal SI_n is known as the *super invariant ideal* and the *bi-graded* quotient ring

$$SR_n/SI_n$$

is the *super coinvariant ring*.

Analogous to the study of diagonal harmonics, super coinvariants have a natural harmonic side. Mike Zabrocki suggested the idea and Nantel Bergeron proved a harmonic side exists using formal differentiation and a bilinear form [6]. For each $i \in [n]$, define

formal partial differential operators ∂x_i and $\partial\theta_i$ by the following group of relations. For each $i, j \in [n]$, define

$$\partial x_i \cdot x_i^k = \begin{cases} k x_i^{k-1} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}, \tag{2.21}$$

$$\partial x_i \cdot x^\alpha = (\partial x_i \cdot x_i^{\alpha_i}) \prod_{\substack{j \in [n] \\ j \neq i}} x_j^{\alpha_j}, \tag{2.22}$$

$$\partial\theta_i \cdot \theta_A = \begin{cases} (-1)^{\#\{a \in A : a < i\}} \theta_{A \setminus \{i\}} & \text{if } i \in A \\ 0, & \text{if } i \notin A \end{cases}, \tag{2.23}$$

$$\partial x_i \cdot x_i^\alpha \theta_A = (\partial x_i \cdot x_i^\alpha)\theta_A, \tag{2.24}$$

$$\partial\theta_i \cdot x^\alpha \theta_A = x^\alpha (\partial\theta_i \cdot \theta_A), \tag{2.25}$$

$$\partial x_i \partial\theta_j = \partial\theta_j \partial x_i. \tag{2.26}$$

The relation in Equation (2.22) implies $\partial x_i \partial x_j = \partial x_j \partial x_i$. Similarly, Equation (2.23) implies $\partial\theta_i \partial\theta_j = -\partial\theta_j \partial\theta_i$ and $(\partial\theta_i)^2 = 0$. If $\alpha \in \mathbb{N}^n$ and $A = \{a_1 < \cdots < a_k\} \subseteq [n]$, then define

$$\partial\theta_A = \partial\theta_{a_1} \cdots \partial\theta_{a_k} \quad \text{and} \quad \partial x^\alpha = (\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}.$$

Extend this linearly to the entire superspace SR_n. If $f \in SR_n$, then let $f(\partial)$ be the differential operator obtained by replacing each x_i by ∂x_i and θ_i by $\partial\theta_i$, namely,

$$f(\partial) = \sum_{\alpha, A} c_{\alpha, A} \partial x^\alpha \partial\theta_A, \quad \text{if } f = \sum_{\alpha, A} c_{\alpha, A} x^\alpha \theta_A.$$

The presentation here follows closely the notions that were presented in [28]. Define *super harmonics* as the orthogonal complement of the super invariant ideal SI_n with respect to the bilinear form on SR_n defined by

$$\langle f, g \rangle = (f(\partial) \cdot g)(0, \cdots, 0, 0, \cdots, 0).$$

More explicitly, the super harmonics is defined to be

$$SH_n = \{g \in SR_n : \langle f, g \rangle = 0 \text{ for all } f \in SI_n\}.$$

Similar to diagonal harmonics, the super harmonics have a simpler description.

Theorem 2.22 ([6, 10, 55])**.** *For $n \geq 1$, we have*

$$SH_n = \{g \in SR_n : f(\partial) \cdot g = 0 \text{ for all } f \in SI_n\}.$$

Chapter 3

Polarization Operators

3.1 General definitions

The classical harmonic spaces H_n has the alternating representation occurring with multiplicity 1 in degree $\binom{n}{2}$. This space is spanned by the Vandermonde determinant Δ_n in Equation (2.19). Therefore H_n can be written as the linear span of partial derivatives of their alternating components.

We briefly describe polarization operators from a general point of view. F. Bergeron [4] considered the following generalization of the diagonal coinvariant spaces to the ambient polynomial ring $R_{k\times n}$ with k sets of n variables $x_{11},\dots,x_{1k},\dots,x_{k1},\dots,x_{kn}$. The symmetric group acts on each set of variables diagonally, namely, if $\sigma \in \mathcal{S}_n$ and $f \in R_{k\times n}$, then

$$\sigma(f) = f(x_{1\sigma^{-1}(1)},\dots,x_{1\sigma^{-1}(n)},\dots,x_{k\sigma^{-1}(1)},\dots,x_{k\sigma^{-1}(n)}).$$

We can construct an ideal $I_{k\times n}$ by taking as generators all $\mathcal{S}_n$-invariant polynomials with no constant term. Then $R_{k\times n}/I_{k\times n}$ is a well-defined $\mathcal{S}_n$-module, sometimes referred to as the higher diagonally coinvariant space.

In the study of higher diagonally coinvariant spaces $R_{k\times n}/I_{k\times n}$, it can be challenging to even get started due to the complex structure of their invariant ideals. However, we can *hope* the analog of Theorem 2.21 is true in the more general setting. One would study instead a space $\mathcal{E}_{k\times n}$ defined so that it is the *smallest* space such that

(i) it contains the Vandermonde determinant in any one set of variables,

(ii) it is closed under all partial derivatives, and

(iii) it is closed under all higher polarization operators of the form

$$\sum_{i=1}^{n} u_i \partial v_i^k,$$

where $k \geq 1$ and $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ can by any two sets of variables in the ambient space.

The alternating component of $\mathcal{E}_{k\times n}$ is the smallest space satisfying (i) and (iii). One then hopes there is a way to compute the Hilbert series of $\mathcal{E}_{k\times n}$ by picking out special polynomials in the cones of the alternating component. This strategy depends heavily on the properties of $E_{u \leftarrow v^k}$. Even in the case of the classical diagonal harmonics, we still do not have a basis of the alternating component.

We are motivated by this strategy to find a different set of polarization operators to describe the alternating component of SH_n *and* reveal combinatorial heuristics to build an invariant basis for every irreducible representation in SH_n. We now recall relevant known results.

In the study of super harmonics, the polarization operators are

$$E_k = \sum_{i=1}^{n} \theta_i \partial x_i^k \tag{3.1}$$

for $k = 1, \ldots, n$. In 2019, the algebraic combinatorics group at Fields Institute [10], and independently, Swanson and Wallach [73] used these operators to compute the Hibert series of the alternating component of SH_n. However, progress stalled until 2023 when Rhoades and Wilson [56] used tools from commutative algebra to successfully compute the Hilbert series of SH_n.

The E-operators in Equation (3.1) offer a key approach to studying diagonal harmonics of various kinds. The domain of each E_k is the classical harmonic space H_n and the *kernel of* E_k is $\ker E_k = \{f \in H_n : E_k f = 0\}$. Each E_k commutes with the $\mathcal{S}_n$-action. Hence,

they map $\mathcal{S}_n$-modules to $\mathcal{S}_n$-modules possibly killing some irreducible representations. If one can describe the kernels of E_k and the intersection of the images of E_k, then one can potentially understand the entire harmonic module.

However, this was shown to be a difficult task in the study of diagonal harmonics. Haiman's Operator Theorem [38] gives a spanning set for the alternating component in terms of $\sum_{i=1}^n y_i \partial x_i^k$ and $\sum_{i=1}^n x_i \partial y_i^k$ for $k \geq 1$. Its proof uses deep tools from algebraic geometry.

In superspace, the algebraic combinatorics group at the Fields Institute [10], and independently Swanson and Wallach [73], accomplished the task of describing the alternating components using E-operators in a few pages of leading term arguments.

Theorem 3.1 ([10, 73])**.** *A basis of the alternating component of SH_n is*

$$\{E_{k_m} \cdots E_{k_1} \cdot \Delta_n : 1 \leq k_1 < \cdots < k_m \leq n-1, m \geq 0\}. \tag{3.2}$$

It is interesting to note that this theorem has been generalized to finite reflection groups in a series of papers by Swanson and Wallach [73, 72, 74].

We refer to the set of polynomials in (3.2) as *the E-alternants*. This is not the only way to describe the alternating component of SH_n. We will describe another set of alternants in Theorem 3.10. For comparison, we compute the following examples.

Example 3.2. *For $n = 3$, the E-alternants are the following polynomials.*

$$\begin{aligned}
E_\emptyset \Delta_3 &= \Delta_3 \\
&= x_2x_3^2 - x_2^2x_3 - x_1x_3^2 + x_1^2x_3 + x_1x_2^2 - x_1^2x_2 \\
E_1\Delta_3 &= \left(x_2^2 - x_1^2 + 2x_1x_3 - 2x_2x_3\right)\theta_3 \\
&\quad + \left(x_1^2 - x_3^2 + 2x_2x_3 - 2x_1x_2\right)\theta_2 \\
&\quad + \left(x_3^2 - x_2^2 + 2x_1x_2 - 2x_1x_3\right)\theta_1 \\
E_2\Delta_3 &= (2x_1 - 2x_2)\,\theta_3 + (-2x_1 + 2x_3)\,\theta_2 + (2x_2 - 2x_3)\,\theta_1 \\
E_1E_2\Delta_3 &= -4\theta_2\theta_3 + 4\theta_1\theta_3 - 4\theta_1\theta_2.
\end{aligned}$$

If we start with a polynomial $\theta_1 x_1^k + \cdots + \theta_n x_n^k$ and replace each x_i with ∂x_i, then of course we obtain E_k.

We distill a general notion of polarization action by asking the super polynomial ring SR_n to act on itself by differentiation in x variables and multiplication in θ variables. More precisely, the *polarization action* is a bilinear map $\diamond : SR_n \times SR_n \to SR_n$ defined by

$$g \diamond h = g(\theta_1, \ldots, \theta_n, \partial x_1, \ldots, \partial x_n)h. \tag{3.3}$$

We now verify that polarization is indeed an action.

Lemma 3.3. *For $f, g, h \in SR_n$, we have $(fg) \diamond h = f \diamond (g \diamond h)$.*

Proof. It suffices to check this for monomials. If $h \in SR_n$, then

$$\theta_A x^\alpha \diamond \Big(\theta_B x^\beta \diamond h\Big) = \theta_A x^\alpha \diamond \theta_B \partial x^\beta h = \theta_A \theta_B \partial x^\alpha \partial x^\beta h = \theta_A \theta_B x^\alpha x^\beta h = \Big(\theta_A x^\alpha \theta_B x^\beta\Big) \diamond h,$$

as desired. □

From the definition, polarization operators are described as polynomials in SR_n, it is possible that two distinct polynomials have the same effect as polarization operators. For example, the polynomials x_1 and $-x_2$ act on $H_2 = \mathrm{Span}_{\mathbb{Q}}\{1, x_1 - x_2\}$ in exactly the same way. We have $(x_1) \diamond (x_1 - x_2) = 1 = (-x_2) \diamond (x_1 - x_2)$ and $x_1 \diamond 1 = 0 = (-x_2) \diamond 1$.

Lemma 3.4 weeds out cosmetic difference in polarization operators. A small but straightforward change of perspective takes place and should be reflected in the notation. Define[1]

$$\mathcal{I}_n = \langle e_k(x_1, \ldots, x_n) : k \geq 1 \rangle \tag{3.4}$$

to be an ideal in SR_n. Recall I_n is defined in Equation (2.16). Note $\mathcal{I}_n$ contains I_n as a set and

$$\mathcal{I}_n \cong I_n \otimes \mathbb{Q}[\theta_1, \ldots, \theta_n].$$

That means every polynomial $f(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) \in SR$ descends to a coset

$$f + \mathcal{I}_n = \sum_{A \subseteq [n]} (f_A(x_1, \ldots, x_n) + I_n)\theta_A, \tag{3.5}$$

where f_A is the coefficient of θ_A in f and $f_A + I_n \in R_n/I_n$. Equation (3.5) allows us to understand symmetric functions in superspace as polarization operators from the perspective of R_n/I_n.

[1]The symbol $\mathcal{I}$ is a calligraphic I.

Lemma 3.4. *Let $f, g \in SR_n$. If*

$$f \equiv g \pmod{\mathcal{I}_n},$$

then $f \diamond h = g \diamond h$ for any $h \in H_n$.

Proof. Notice $f \equiv g \pmod{\mathcal{I}_n}$ implies $f - g \in \mathcal{I}_n$. Then we can write

$$f - g = \sum_{S \subseteq [n]} \phi_S(x_1, \ldots, x_n)\theta_S$$

and $\phi_S \in I_n$ for every $S \subseteq [n]$. Hence for each $S \subseteq [n]$, we have $\phi_S \diamond h = 0$ because h is harmonic in R_n. We conclude that $(f - g) \diamond h = 0$. □

In light of Lemma 3.4, we say two polynomials $f, g \in SR_n$ are *equivalent as polarization operators* if $f \equiv g \pmod{\mathcal{I}_n}$. This concept has been implicitly described by [1] in a conjectural statement describing the alternating component of the (classical) diagonal harmonics. The author rediscovered it independently.

Lastly, we relate the $\mathcal{S}_n$-action on SR_n to its action on the image of polarization.

Lemma 3.5. *For any $\sigma \in \mathcal{S}_n$ and any $f, h \in SR_n$, we have $\sigma(f) \diamond h = \sigma(f \diamond \sigma^{-1}h)$.*

Proof. By linearity of $\mathcal{S}_n$-action, it suffices to check this for monomials. From the definition, we have

$$\begin{aligned}
\sigma(\theta_A x^\alpha) \diamond h &= \sigma(\theta_A \partial x^\alpha)\sigma\sigma^{-1}h \\
&= \sigma\left(\theta_A \partial x^\alpha \sigma^{-1}(h)\right) \\
&= \sigma\left(\theta_A x^\alpha \diamond \sigma^{-1}h\right),
\end{aligned}$$

as desired. □

3.2 Finding symmetric polarization operators

In this section, we describe the context surrounding our main results in Section 3.3. We recall some development of the classical diagonal harmonics (in two sets of commuting variables $x_1, \ldots, x_n, y_1, \ldots, y_n$).

Let DA_n denote the alternating component of diagonal harmonics. Define $DHE_k = y_1\partial x_1^k + \cdots + y_n\partial x_n^k$ for $k \geq 1$ to be the well-known symmetric polarization operator in diagonal harmonics. The more popular notation for DHE_k in the literature is just E_k, but we have to avoid confusion with the E_k defined in Equation (3.1).

As mentioned in Section 2, DA_n has no known basis. Haiman [38] went through extensive algebraic geometry to describe a spanning set for DA_n using $\{DHE_k : k \geq 1\}$. In 2018, Carlsson and Oblomkov [13] described a monomial basis for the entire diagonal harmonics. However, to obtain a basis for DA_n, one would have to antisymmetrize the monomial basis of Carlsson and Oblomkov, resulting in a spanning set. The mysterious DA_n has attracted numerous attempts to find a basis. Partial progress includes [8, 49, 43, 78].

The combinatorics related to DA_n makes the problem of finding a basis of DA_n even more interesting because the dimension of DA_n is the famous n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Let $\mathcal{D}_n$ denote lattice paths from $(0,0)$ to (n,n) that never go below the diagonal line $y = x$. Elements of $\mathcal{D}_n$ are typically called Dyck paths or Catalan paths [70]. It is well-known that $C_n = |\mathcal{D}_n|$ [70]. The q,t-Catalan numbers are defined by

$$C_n(q,t) = \sum_{\pi \in \mathcal{D}_n} q^{\mathtt{area}(\pi)} t^{\mathtt{dinv}(\pi)}, \quad n \geq 1$$

where `area` and `dinv` are certain statistics on $\mathcal{D}_n$ [33]. We remark that there are other ways to define $C_n(q,t)$. Interested readers please consult [33]. It is clear from the definition that $C_n(1,1)$ is the n-th Catalan number. It is a celebrated result in the algebraic combinatorics community that the Hilbert series of DA_n is $C_n(q,t)$ [33]. The coefficient of $q^i t^j$ in $C_n(q,t)$ is the dimension of the bihomogeneous component of DA_n having x-degree i and y-degree j.

If we exchange x_i for y_i for every $i \in [n]$ in the definition of DA_n, it is clear that we obtain the same space. It is obvious, from an algebraic point of view, that $C_n(q,t)$ is

symmetric in q, t, namely, $C_n(q,t) = C_n(t,q)$. It follows that there exists a bijection on $\mathcal{D}_n$ that sends `area` to `dinv`.

Furthermore, there is not even a conjectured basis for the DA_n using only $\{DHE_k : k \geq 1\}$. Allen conjectured a basis for DA_n [1] which implicitly uses the idea of polarization equivalence but in the setting of two sets of commuting variables. This is known as the joint-symmetry problem and has been open for about 20 years. There exist numerous attempts and partial progress [50, 51, 39, 40, 78].

Both the problem of finding a basis of DA_n and the joint symmetry problem of the q,t-Catalan are fascinating and partly motivate this dissertation. One can think about $\{DHE_k : k \geq 1\}$ as "generators" of DA_n in the sense that they generate a space of operators, by taking all possible compositions, whose image of Δ_n is a spanning set of DA_n. One can ask whether there exist other, perhaps simpler, polarization operators that also generate DA_n in similar fashion.

We can take a broader perspective by looking at variations of the classical diagonal harmonics. Let's consider all symmetric polynomials in an ambient space which may be one of $R_{k\times n}$ or SR_n. A natural way to start looking for new polarization operators is to consider all symmetric polynomials in the ambient space. We would need to tackle two problems in order to find a basis for the alternating component.

(P1) We need to make sure we only find symmetric polarization operators that preserve the property of being harmonic. This is not a trivial task because there exist polarization operators that do not polarize Δ_n to a harmonic polynomial, even just in superspace.

For example, consider

$$f = \theta_1\theta_2(x_2^2 - x_1^2) + \theta_1\theta_3(x_3^2 - x_1^2) + \theta_2\theta_3(x_3^2 - x_2^2) \in SR_3.$$

Then f is symmetric but $f \diamond \Delta_3$ is not harmonic because $(\partial\theta_1 + \partial\theta_2 + \partial\theta_3)(f \diamond \Delta_3) \neq 0$. We remark that $f = (\theta_1 + \theta_2 + \theta_3)(\theta_1 x_1^2 + \theta_2 x_2^2 + \theta_3 x_3^3)$.

This example also shows that even if we have a basis for $\mathrm{SSym}_n/\mathcal{I}_n$ we would still need to characterize $f \in \mathrm{SSym}_n$ that polarize Δ_n to a harmonic polynomial. Conjecture 3.15 predicts, specifically for super harmonics, all symmetric polarization operators that do not preserve the property of being harmonic can be constructed in a way similar to the construction of f.

(P2) Suppose $\mathcal{B}$ is a set of symmetric polarization operators that polarize Δ_n to harmonic polynomials. We would still need to prove these images are linearly independent and spans the entire alternating component.

In the most general setting, both (P1) and (P2) are highly non-trivial. However, the situation is tamer in superspace. The E-alternants described in (3.2) provide a natural starting point in searching for a new basis of the alternating component, call it $\mathcal{B}$ for now. As long as we can triangularly relate $\mathcal{B}$ to the E-alternants, then we automatically have that $\mathcal{B}$ is harmonic and is a basis of the alternating components.

Our strategy to find a new basis for the alternating component of SH_n involves studying symmetric functions in superspace [22]. In the terminology of [22], the E_k's, as polynomials in superspace, are super symmetric power sum polynomials. Following the notations of [22], we define

$$\tilde{p}_k = \sum_{i=1}^{n} \theta_i x_i^k, \quad \text{for each } k \geq 1.$$

For convenience, we use the following notation for monomials due to Hivert [42]. Let $I = \{i_1 < \cdots < i_k\} \subseteq [n]$ be a set and let $v \in \mathbb{N}^k$ be an exponent vector. Define

$$x_I^v = \prod_{j=1}^{k} x_{i_j}^{v_j}. \tag{3.6}$$

Let $X \subseteq [n]$. Let $\mathcal{S}_X$ denotes the subgroup of $\mathcal{S}_n$ on the ground set X. Define operators

$$\mathrm{sym}_X = \sum_{\sigma \in \mathcal{S}_X} \sigma \quad \text{and} \quad \mathrm{alt}_X = \sum_{\sigma \in \mathcal{S}_X} \mathrm{sgn}(\sigma)\sigma. \tag{3.7}$$

Furthermore, we define

$$\mathrm{sym}_n = \mathrm{sym}_{[n]} \quad \text{and} \quad \mathrm{alt}_n = \mathrm{alt}_{[n]}. \tag{3.8}$$

Lemma 3.6. *For each $\phi = (\phi_1, \cdots, \phi_k)$ with $\phi_j \in [n-1]$ and any $B \subseteq [n]$ with $|B| = k$, we have*

$$(n-k)!\, \tilde{p}_{\phi_1} \cdots \tilde{p}_{\phi_k} = \mathrm{sym}_n \left(\theta_B x_B^{\phi} \right).$$

Proof. For any $S = \{s_1 < \cdots < s_k\} \subseteq [n]$ with $|S| = k$, there exists some $\sigma \in \mathcal{S}_n$ such that $s_{\sigma^{-1}(i)} = b_i$ for all $i \in [k]$. Hence, we get an equality $\mathrm{sym}_S \theta_S x_S^\phi = \sigma \mathrm{sym}_B \sigma^{-1} \theta_S x_S^\phi = \sigma \mathrm{sym}_B \theta_B x_B^\phi$ without introducing any sign.

Starting from the definition, expand the product of power sums by collecting terms with respect to θ. It follows that

$$\tilde{p}_{\phi_1} \cdots \tilde{p}_{\phi_k} = \sum_{\substack{S \subseteq [n] \\ |S|=k}} \mathrm{sym}_S \left(\theta_S x_S^\phi \right) = \sum_{\sigma \in \mathcal{S}_n / Stab(B)} \sigma \cdot \mathrm{sym}_B \left(\theta_B x_B^\phi \right).$$

On the other hand, expand $\mathrm{sym}_n \left(\theta_B x_B^\phi \right)$ with respect to the subgroup $\mathcal{S}_B$. Let $Stab(B)$ denote the stabilizer subgroup of B under the permutation action of $\mathcal{S}_n$. Let $[n] = A \cup B$ be a disjoint union. The sum can be refined using $Stab(B) = \mathcal{S}_A \times \mathcal{S}_B$. We get

$$\begin{aligned}
\mathrm{sym}_n \left(\theta_B x_B^\phi \right) &= \sum_{\sigma \in \mathcal{S}_n / \mathcal{S}_B} \sigma \mathrm{sym}_B \left(\theta_B x_B^\phi \right) \\
&= \sum_{\sigma \in \mathcal{S}_n / Stab(B)} \sum_{\pi \in Stab(B)/\mathcal{S}_B} \pi\sigma \mathrm{sym}_B \left(\theta_B x_B^\phi \right) \\
&= \sum_{\sigma \in \mathcal{S}_n / Stab(B)} \left(\sum_{\pi \in \mathcal{S}_A} \pi \right) \sigma \mathrm{sym}_B \left(\theta_B x_B^\phi \right) \\
&= (n-k)! \sum_{\sigma \in \mathcal{S}_n / Stab(B)} \sigma \cdot \mathrm{sym}_B \left(\theta_B x_B^\phi \right),
\end{aligned}$$

as desired. We used the fact that $|\mathcal{S}_A| = (n-k)!$ to deduce the last equality. □

3.3 A new set of symmetric polarization operators

In this section, we describe a new basis for the alternating component of super harmonics by describing a new set of symmetric polarization operators denoted as $\{M_\alpha : \alpha \in \mathcal{A}_n\}$. The main result is Theorem 3.10. Proofs are given in Section 3.4. Other properties of the new basis are stated and proved in Section 3.5.

Lemma 3.6 suggests a new set of polarization operators should be another subset of monomial symmetric functions. To define a different proposed set of polarization operators,

we need some simple combinatorial objects. A *weak staircase* of length n is a weakly increasing sequence $\alpha \in \mathbb{N}^n$ whose entries satisfy

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_{i+1} \leq \alpha_i + 1 \quad \text{for each} \quad i \in [n-1]. \tag{3.9}$$

We denote the set of weak staircases of length n by $\mathcal{A}_n$.

The *ascent set* of a weak staircase α is

$$\mathtt{Asc}(\alpha) = \{i \geq 2 : \alpha_i > \alpha_{i-1}\}. \tag{3.10}$$

Please note this definition differs from the usual convention [70, page 31] by 1. Elements in $\mathtt{Asc}(\alpha)$ are called an *ascent* of α. The *number of ascents* of α is $\mathtt{asc}(\alpha) = |\mathtt{Asc}(\alpha)|$. We typically list ascents in increasing order as $\mathtt{Asc}(\alpha) = \{i_1 < \cdots < i_k\}$.

The *area word* of a weak staircase is $\mathtt{Area}(\alpha) = (a_1, \cdots, a_k)$ where $k = \mathtt{asc}(\alpha)$ and

$$a_j = \#\{i \in [n] : \alpha_i \geq j\}, \text{ for each } i = 1, \ldots, k. \tag{3.11}$$

We remark that the entries of $\mathtt{Area}(\alpha)$ are always strictly decreasing.

Weak staircases are visualized by their associated diagrams in the first quadrant of the Cartesian plane. We will be able to easily read off ascents and area sets.

Suppose $\alpha \in \mathcal{A}_n$ and $k = \max(\alpha)$. The *path of* α is a lattice walk from $(0,0)$ to (n,k) drawn as follows. For each $i \geq 1$, there is a horizontal step from $(i-1, \alpha_i)$ to (i, α_i), and if i is an ascent of α, then a vertical step (i, α_{i-1}) to (i, α_i). A *box* is a unit square in the $n \times k$ rectangle and has *coordinate* equal to the coordinate of its upper right corner. The j-th row consists of boxes whose second coordinate is j. The i-th column consists of boxes whose first coordinate is i. We *shade* the collection of boxes

$$\{(i,j) : i \in [n], j \in [k], j \leq \alpha_i\}$$

because they appear below the path of α in the diagram. Notice that the path of α is the unique lattice path from $(0,0)$ to (n,k), using steps $(1,0)$ and $(0,1)$, that forms the boundary between the shaded and the unshaded boxes.

Example 3.7. *This is the running example in this chapter.*

Consider a weak staircase $\alpha = 00111223333344$. *We have*

$$\mathtt{Asc}(\alpha) = \{3, 6, 8, 13\} \quad \textit{and} \quad \mathtt{Area}(\alpha) = (12, 9, 7, 2).$$

Its corresponding diagram is drawn in Figure 3.1. The box decorated with a bullet $\bullet$ *has coordinate* $(8, 2)$.

We can read off $\mathtt{Area}(\alpha)$ *from the diagram by counting shaded boxes in each row. The entries of* $\mathtt{Area}(\alpha)$ *are listed on the right of their corresponding row.*

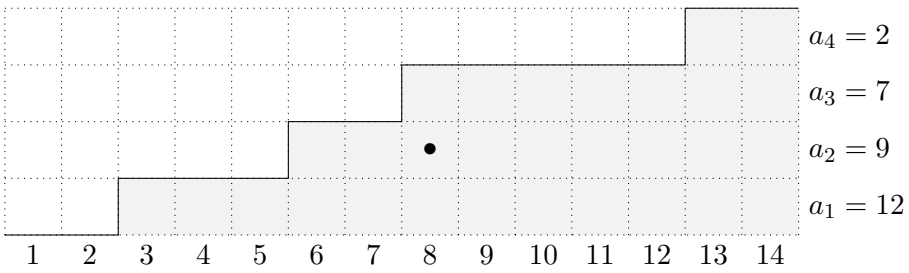


Figure 3.1: A weak staircase $\alpha = 00111223333344$.

The observations in Lemma 3.8 follow immediately from the geometric interpretation of weak staircases.

Lemma 3.8. *Let* n *be an arbitrary positive integer.*

(1) The map $\mathtt{Asc}$ *is a bijection between* $\mathcal{A}_n$ *and subsets of* $\{2, \ldots, n\}$ *and* $|\mathcal{A}_n| = 2^{n-1}$.

(2) Suppose $\alpha, \beta \in \mathcal{A}_n$. *In the lexicographical order, we have* $\alpha \leq \beta$ *if and only if* $\mathtt{Area}(\alpha) \leq \mathtt{Area}(\beta)$.

We now describe a subset M of super monomial symmetric functions as new polarization operators in superspace. Our notation differs from that of [22], but we will offer a translation. Let α be a weak staircase of length n. Define a symmetric polynomial

$$M_\alpha = \mathrm{sym}_n \left(\theta_{\mathtt{Asc}(\alpha)} x^\alpha \right). \tag{3.12}$$

Monomials in M_α appear with uniform multiplicity. For each $\alpha \in \mathcal{A}_n$, define a constant

$$z_\alpha = \prod_{k=1}^{n} (\text{number of } k\text{'s in } \alpha)!$$

Therefore, $z_\alpha^{-1} M_\alpha$ is a *monomial symmetric function* in the sense of [22]. We remark that $\{M_\alpha : \alpha \in \mathcal{A}_n\}$ is a *strict* subset of super monomial symmetric functions.

We will soon state and prove a new basis for the alternating component of the super harmonics. See Corollary 3.13. The proof will show the new basis is triangularly related to E-alternants described in Theorem 3.1. We prefer to work in the space of $SR_n/\mathcal{I}_n$ to take advantage of the theory of symmetric functions. In particular, we want to treat $E_k = \theta_1 \partial x_1^k + \cdots + \theta_n \partial x_n^k$ as a symmetric polynomial under polarization equivalence by "removing" the partial differential operators. Therefore, for each $\alpha \in \mathcal{A}_n$, define

$$P_\alpha = \tilde{p}_{a_1} \cdots \tilde{p}_{a_m}, \qquad \text{where } (a_1, \cdots, a_m) = \texttt{Area}(\alpha). \tag{3.13}$$

It follows that for any $h \in H_n$ and all $\alpha \in \mathcal{A}_n$,

$$P_\alpha \diamond h = E_{a_1} \cdots E_{a_m} \cdot h, \qquad \text{where } (a_1, \cdots, a_m) = \texttt{Area}(\alpha). \tag{3.14}$$

Moreover, Lemma 3.6 can be rephrased as, for any $\alpha \in \mathcal{A}_n$ and any $B \subseteq [n]$ with $|B| = \texttt{asc}(\alpha)$, we have

$$(n-k)! P_\alpha = \mathrm{sym}_n \left(\theta_B x_B^{\texttt{Area}(\alpha)} \right). \tag{3.15}$$

Example 3.9. *The monomial symmetric functions M_α and P_α are not necessarily the same as polynomials, although some are equivalent as polarization operators (up to scaling). For example, if $n = 3$, P_{001} is a constant multiple of M_{001}. However, P_{011} is not a constant multiple of M_{011} but it can be shown that P_{011} is equivalent to M_{011} as polarization operators (up to scaling).*

For $n = 5$, we note that $P_{00122} \in SR_5$ is not equivalent to any $M_\alpha \in \mathcal{A}_5$ as a polarization operator. In Example 3.19, we will rewrite P_{00122} as a linear combination of M_α's modulo $\mathcal{I}_n$.

The main result of this section is a triangularity relation between the M's and the P's.

Theorem 3.10. *Let $<$ denote the lexicographical order on $\mathbb{N}^n$. For every $\alpha \in \mathcal{A}_n$, we have*

$$P_\alpha \equiv M_\alpha + \sum_{\phi > \alpha} c_\phi M_\phi \pmod{\mathfrak{I}_n}, \tag{3.16}$$

for some integer c_ϕ where ϕ's are all weak staircases.

The proof of Theorem 3.10 is given on page 63.

Corollary 3.11. *For each $\alpha \in \mathcal{A}_n$, we have $M_\alpha \not\equiv 0 \pmod{\mathfrak{I}_n}$.*

Remark 3.12. *It will become clear in the proof of Theorem 3.10 that $M_\alpha \not\equiv 0 \pmod{\mathfrak{I}_n}$. Moreover, we will use a novel approach to show the non-vanishing (modulo $\mathfrak{I}_n$) of M_α in Lemma 3.31 in Section 3.5. The proof of Lemma 3.31 does not use Theorem 3.10.*

Corollary 3.13. *A basis of the alternating component of SH_n is*

$$\{M_\alpha \diamond \Delta_n : \alpha \in \mathcal{A}_n\}.$$

Proof. By the definition of P_α, Theorem 3.1 is equivalent to that $\{P_\alpha \diamond \Delta_n : \alpha \in \mathcal{A}_n\}$ is a basis of the alternating component of SH_n. Theorem 3.10 and Lemma 3.4 imply a desired triangularity

$$P_\alpha \diamond \Delta_n = (M_\alpha \diamond \Delta_n) + \sum_{\phi > \alpha} c_\phi (M_\phi \diamond \Delta_n), \tag{3.17}$$

where c_ϕ are coefficients in $\mathbb{Q}$. □

Recall the definition of $\mathfrak{I}_n$ in Equation 3.4 on page 40. Denote

$$\overline{\mathrm{SSym}_n} = \{F + \mathfrak{I}_n : F \in \mathrm{SSym}_n\}.$$

Notice $\overline{\mathrm{SSym}_n}$ is a subring of $SR_n/\mathfrak{I}_n$ because SSym_n is a ring. We have one corollary to Theorem 3.10 and one conjecture.

Corollary 3.14 (to Theorem 3.10)**.** *The vector subspace $\overline{\mathrm{SSym}_n}^{\mathrm{harmonic}}$ of $\overline{\mathrm{SSym}_n}$ defined by*

$$\overline{\mathrm{SSym}_n}^{\mathrm{harmonic}} = \{F \in \overline{\mathrm{SSym}_n} : F \diamond \Delta_n \in SH_n\}.$$

has dimension 2^{n-1}. Moreover,

$$\{P_\alpha : \alpha \in \mathcal{A}_n\} \quad \textit{and} \quad \{M_\alpha : \alpha \in \mathcal{A}_n\}.$$

are bases of $\overline{\mathrm{SSym}_n}^{\mathrm{harmonic}}$.

Proof. To shorten the notation, we let $B = \{P_\alpha : \alpha \in \mathcal{A}_n\}$.

By Theorem 3.1 and Equation (3.13), we see $P_\alpha \in \overline{\mathrm{SSym}_n}^{\text{harmonic}}$ for all $\alpha \in \mathcal{A}_n$. If we have a linear relation

$$\sum_{\alpha \in \mathcal{A}_n} c_\alpha P_\alpha \equiv 0 \pmod{\mathfrak{I}_n},$$

where c_α are coefficients, then

$$\sum_{\alpha \in \mathcal{A}_n} c_\alpha P_\alpha \diamond \Delta_n = 0$$

is a linear relation in the alternating component of SH_n. Theorem 3.1 and Equation (3.13) force $c_\alpha = 0$ for all $\alpha \in \mathcal{A}_n$. We see that B is linearly independent and the dimension of $\overline{\mathrm{SSym}_n}^{\text{harmonic}}$ is at least 2^{n-1}.

If B does not span $\overline{\mathrm{SSym}_n}^{\text{harmonic}}$, then there exists some non-zero (modulo $\mathfrak{I}_n$)

$$f \in \overline{\mathrm{SSym}_n} \setminus \mathrm{Span}\{P_\alpha : \alpha \in \mathcal{A}_n\}$$

such that $f \diamond \Delta_n$ is in the alternating component of SH_n but not in the span of $\{P_\alpha \diamond \Delta_n : \alpha \in \mathcal{A}_n\}$. However, because $f \not\equiv 0 \pmod{\mathfrak{I}_n}$, we must have $f \diamond \Delta_n \neq 0$. It follows that the dimension of the alternating component of SH_n is larger than 2^{n-1} contradicting Theorem 3.1. Therefore, B spans $\overline{\mathrm{SSym}_n}^{\text{harmonic}}$ and the dimension of $\overline{\mathrm{SSym}_n}^{\text{harmonic}}$ is 2^{n-1}.

Theorem 3.10 says M_α's and P_α's are triangularly related. Therefore $\{M_\alpha : \alpha \in \mathcal{A}_n\}$ is also a basis of $\overline{\mathrm{SSym}_n}^{\text{harmonic}}$. □

Conjecture 3.15. *The dimension of* $\overline{\mathrm{SSym}_n}$ *is* 2^n. *Moreover, let* B_n^{harmonic} *be a set. If* B_n^{harmonic} *is a basis for* $\overline{\mathrm{SSym}_n}^{\text{harmonic}}$, *then*

$$B_n^{\text{harmonic}} \cup \{(\theta_1 + \cdots + \theta_n)F : F \in B_n^{\text{harmonic}}\}$$

is a basis for $\overline{\mathrm{SSym}_n}$.

Corollary 3.16 (to Conjecture 3.15)**.** *If Conjecture 3.15 holds, then the vector subspace* $\overline{\mathrm{SSym}_n}^{\text{not harmonic}}$ *of* $\overline{\mathrm{SSym}_n}$ *defined by*

$$\overline{\mathrm{SSym}_n}^{\text{not harmonic}} = \{F \in \overline{\mathrm{SSym}_n} : F \diamond \Delta_n \notin SH_n\}.$$

has dimension 2^{n-1}*. Moreover, let* B_n^{harmonic} *be a set. If* B_n^{harmonic} *is a basis for* $\overline{\text{SSym}}_n^{\text{harmonic}}$*, then*

$$\{(\theta_1 + \cdots + \theta_n)F : F \in B_n^{\text{harmonic}}\}$$

is a basis for $\overline{\text{SSym}}_n^{\textit{not harmonic}}$*.*

Proof. It follows immediately from $\overline{\text{SSym}}_n = \overline{\text{SSym}}_n^{\text{harmonic}} \oplus \overline{\text{SSym}}_n^{\text{not harmonic}}$. □

3.4 Proofs

Toward the proof of Theorem 3.10, we expand each M_α in terms of θ-monomials to get

$$M_\alpha = \sum_{\substack{B \subseteq [n] \\ |B| = \mathtt{asc}(\alpha)}} \pi_B \cdot \theta_{\mathtt{Asc}(\alpha)} \text{sym}_{[n]\setminus \mathtt{Asc}(\alpha)} \left(x^\alpha \big|_{[n]\setminus \mathtt{Asc}(\alpha)} \right) \text{alt}_{\mathtt{Asc}(\alpha)} \left(x^\alpha \big|_{\mathtt{Asc}(\alpha)} \right), \tag{3.18}$$

where π_B is a permutation that sends $\theta_{\mathtt{Asc}(\alpha)}$ to θ_B *without* introducing a sign. Such a choice is always possible. If $\mathtt{Asc}(\alpha) = \{i_1 < \cdots < i_k\}$ and $B = \{b_1 < \cdots < b_k\}$, then $\pi_B = (i_1 b_1) \cdots (i_k b_k)$ in cycle notation. The long vertical bar notation $x^\phi|_B$ denotes the restriction to indices in B, i.e., if $\phi \in \mathbb{N}^n$ and $B \subseteq [n]$, then

$$x^\phi \big|_B = \prod_{i \in B} x_i^{\phi_i}.$$

The expression on the right-hand side of Equation (3.18) reveals that the coefficients of $\theta_{\mathtt{Asc}(\alpha)}$ in M_α is a product of a monomial symmetric function in variables indexed by $[n] \setminus \mathtt{Asc}(\alpha)$ and a monomial alternant in variables indexed by $\mathtt{Asc}(\alpha)$.

It turns out the Equation (2.15) is the right trick that allows us to rewrite P_α in terms of M_α. Hence, we should find a way to manipulate P_α such that elementary symmetric functions show up in the θ coefficients. The next elementary lemma tells us what to do.

Lemma 3.17. *In the ambient ring* R_n*, for any* $i \in [n]$ *and any* $k \geq 1$*, we have*

$$x_i^k \equiv (-1)^k e_k(\widehat{x_i}) \pmod{I_n}.$$

where $e_k(\widehat{x_i}) = e_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Proof. Fix an $i \in [n]$. Proceed by induction on k. It is clear that the base case is

$$x_i \equiv (-1)(x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n) \pmod{I_n}.$$

Inductively, we apply Equation (2.14) to get

$$\begin{aligned} 0 &\equiv -e_k(x_1, \ldots, x_n) \pmod{I_n} \\ &= -x_i e_{k-1}(\widehat{x_i}) - e_k(\widehat{x_i}) \\ &\equiv -x_i(-1)^{k-1}x_i^{k-1} - e_k(\widehat{x_i}) \pmod{I_n}. \end{aligned}$$

Reorganize to get the desired relation. □

We can visualize Lemma 3.17 to get some intuition for the upcoming proofs. We can visualize x_{n-k}^k as a single column of k boxes in column $n-k$. Apply Lemma 3.17 to x_{n-k}^{k-1}. Notice the lexicographically largest monomial of $e_{k-1}(\widehat{x_{n-k}})$ is $x_{n-k+1}\cdots x_n$. Therefore, we have

$$\begin{aligned} x_{n-k}^k &\equiv (-1)^{n-k-1}x_{n-k}e_k(\widehat{x_{n-k}}) \pmod{I_n} \\ &= x_{n-k}x_{n-k+1}\cdots x_n + (\text{lexicographically smaller monomials}). \end{aligned}$$

The key to the proof of our main result is to apply this procedure repeatedly to a well-chosen set B in the context of Lemma 3.6.

Example 3.18. *Figure 3.18 visualizes the application of Lemma 3.17 to the monomial $x_4^6x_7^3$ in R_9 modulo I_9. The left figure represents the monomial $x_4^6x_7^3$ and the right figure represents* the leading monomial *after applying Lemma 3.17 to the lighter coloured boxes. The set $\{3,6\}$ is the set of exponents of $x_4^6x_7^3$ and is exactly the area set of the weak staircase* 000111222*.*

The following example illustrates the idea behind the proof of Theorem 3.10.

Example 3.19. *Let $n = 5$. We expand P_{00122} in terms of M_α. By Lemma 3.6, we can write $P_{00122} = -3!\mathrm{sym}_5(\theta_3\theta_4x_3^3x_4^2)$. Now repeatedly apply Lemma 3.17 to the monomial $x_3^2x_4$.*

$$\begin{aligned} x_3^3x_4^2 &= x_3x_4 \cdot x_3^2x_4 \\ &\equiv x_3x_4 \cdot e_2(\widehat{x_3})(-1)e_1(\widehat{x_4}) \pmod{I_5} \end{aligned} \tag{3.19}$$

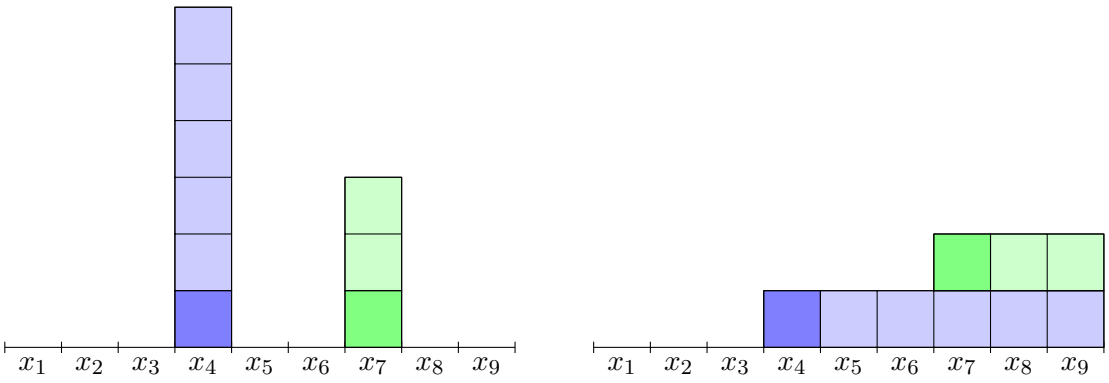


Figure 3.2: Application of Lemma 3.17 to $x_3^6 x_6^3$.

Notice $\mathtt{Asc}(\alpha) = \{2, 3\}$. *We would like to separate* x_i *for each* $i \in \mathtt{Asc}(\alpha)$, *i.e.,* x_2, x_3, *from the elementary symmetric functions. Denote* $\hat{e}_\mu = e_\mu(x_1, x_2, x_5)$. *Apply Equation* (2.14) *to isolate* x_2, x_3. *The right-hand side of Equation* (3.19) *is equal to*

$$x_3x_4 \cdot e_2(\widehat{x_3})(-1)e_1(\widehat{x_4}) = -x_3x_4 \cdot (x_4\hat{e}_1 + \hat{e}_2)(x_3 + \hat{e}_1) \tag{3.20}$$

$$= -x_3x_4 \cdot (x_3x_4\hat{e}_1 + x_4\hat{e}_{11} + x_3\hat{e}_2 + \hat{e}_{21}). \tag{3.21}$$

The computation up to Line (3.21) *will be generalized in Lemma 3.23.*

Using Equation (3.5), *Line* (3.21) *implies*

$$P_{00122} \equiv \mathrm{sym}_5\left((-1)\theta_3\theta_4 x_3x_4 \cdot (x_3x_4\hat{e}_1 + x_4\hat{e}_{11} + x_3\hat{e}_2 + \hat{e}_{21})\right) \pmod{\mathfrak{I}_5}. \tag{3.22}$$

Notice the expression $x_3x_4\hat{e}_1 + \hat{e}_{21}$ *is symmetric in* x_3, x_4. *Therefore* $\mathrm{sym}_5(\theta_3\theta_4(x_3x_4\hat{e}_1 + \hat{e}_{21})) = 0$.

We permute the remaining terms at the expense of introducing signs and collect terms.

$$\textit{Line } (3.22) = \mathrm{sym}_5\left((-1)\theta_3\theta_4 x_3x_4 \cdot (x_3^0x_4^1\hat{e}_{11} + x_3^1x_4^0\hat{e}_2)\right) \tag{3.23}$$

$$= \mathrm{sym}_5(\theta_3\theta_4 x_3x_4 \cdot x_3^0x_4^1(\hat{e}_2 - \hat{e}_{11})) \tag{3.24}$$

$$= \mathrm{sym}_5(\theta_3\theta_4 x_3x_4 \cdot x_3^0x_4^1\hat{e}_2) - \mathrm{sym}_5(\theta_3\theta_4 x_3x_4 \cdot x_3^0x_4^1\hat{e}_{11}). \tag{3.25}$$

The computation from lines (3.22) *to* (3.25) *will be generalized in Lemma 3.29.*

Finally we rewrite elementary symmetric functions in terms of monomial symmetric functions using Equation (2.15).

$$\begin{aligned} \textit{Line (3.25)} &= \mathrm{sym}_5(\theta_3\theta_4 x_3 x_4^2 \hat{m}_{11}) - \mathrm{sym}_5(\theta_3\theta_4 x_3 x_4^2 (2\hat{m}_{11} + \hat{m}_2)) && (3.26)\\ &= -\mathrm{sym}_5(\theta_3\theta_4 x_3 x_4^2 \hat{m}_{11}) - \mathrm{sym}_5(\theta_3\theta_4 x_3 x_4^2 \hat{m}_2)). && (3.27) \end{aligned}$$

All monomials in $x_3x_4^2\hat{m}_{11}$ are permutations applied to $x_2x_3x_4x_5^2$. Similarly, all monomials in $x_3x_4^2\hat{m}_2$ are permutations applied to $x_3x_4^2x_5^2$. Therefore, we have

$$P_{00122} \equiv c_{01112} M_{01112} + c_{00122} M_{00122} \pmod{\mathfrak{I}_n} \tag{3.28}$$

for some non-zero coefficients $c_{01112}, c_{00122} \in \mathbb{Q}$.

Example 3.20. *We repeat an analogous calculation for P_{01112} with some details omitted. Write $\hat{e}_2 = e_2(\widehat{x_2}, \widehat{x_5})$. Because $\mathtt{Asc}(01112) = \{2, 5\}$, we isolate x_2, x_5 from the elementary symmetric functions. We obtain*

$$\begin{aligned} x_2^4 x_5 &\equiv -x_2 x_5^2 \hat{e}_2 \pmod{I_5}\\ -x_2 x_5^2 \hat{e}_2 &= -x_2 x_5^2 \hat{m}_{11}. \end{aligned}$$

This implies

$$P_{01112} \equiv c_{01112} M_{01112} \pmod{\mathfrak{I}_5}. \tag{3.29}$$

We generalize the computation in Example 3.19 from Equation (3.19) to (3.20) in Lemma 3.21.

Lemma 3.21. *Let $I \subseteq [n]$, and $k \geq 1$. Then*

$$e_k(x_1, \ldots, x_n) = \sum_{J \subseteq I} x_J \hat{e}_{k-|J|}, \tag{3.30}$$

where $x_J = \prod_{j \in J} x_j$ and $\hat{e}_m = e_m(\widehat{x_i} : i \in I)$ for all $m \geq 1$. Moreover, if $k - |J| < 0$ or $k - |J| > n - |I|$, then $\hat{e}_{k-|J|} = 0$.

Proof. It is well-known [52, Example 25, Section I.5 on pages 91 to 92] that elementary symmetric functions satisfy

$$e_k(x_1, \ldots, x_n) = \sum_{j=0}^{k} e_{k-j}(x_i : i \in I) e_j(\widehat{x_i} : i \in I).$$

Equation (3.30) follows because

$$e_{k-j}(x_i : i \in I) = \sum_{\substack{J \subseteq I \\ |J|=k-j}} x_J$$

by definition. The vanishing conditions follow by degree consideration. □

Example 3.22. *Let $n = 5$ and $k = 4$. Fix $I = \{4, 5\}$. Note $\hat{e}_m = e_m(\widehat{x_i} : i \in I) = e_m(\widehat{x_4}, \widehat{x_5}) = e_m(x_1, x_2, x_3)$ for all $m \geq 1$. Note also that $\hat{e}_4 = 0$, $\hat{e}_3 = x_1x_2x_3$ and $\hat{e}_2 = x_1x_2 + x_1x_3 + x_2x_3$. We have*

$$\begin{aligned} e_4(x_1, \ldots, x_5) &= 0 + (x_1x_2x_3x_4) + (x_1x_2x_3x_5) + (x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5) \\ &= x_\emptyset \hat{e}_4 + x_4\hat{e}_3 + x_5\hat{e}_3 + x_{\{4,5\}}\hat{e}_2 \end{aligned}$$

The next lemma generalizes the computation in Example 3.19 up to Line (3.21).

Lemma 3.23. *The following expressions are elements in R_n modulo I_n.*

Let $\alpha \in \mathcal{A}_n$ be a weak staircase. Let i_j be elements of $\mathtt{Asc}(\alpha)$ such that $\mathtt{Asc}(\alpha) = \{i_1 < \cdots < i_k\}$, and $\mathtt{Area}(\alpha) = (a_1, \cdots, a_k)$. Denote $\hat{e}_m = e_m(\widehat{x_i} : i \in \mathtt{Asc}(\alpha)) = e_m(x_i : i \notin \mathtt{Asc}(\alpha))$ for all $m \geq 0$. We have

$$x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \equiv (-1)^{|\alpha|-k} x_{\mathtt{Asc}(\alpha)} \sum_J x_{J_1} \cdots x_{J_k} \hat{e}_{\lambda(J)'} \pmod{I_n}, \tag{3.31}$$

where the sum runs over sequences of sets $J = (J_1, \ldots, J_k)$ satisfying

(i) for each $j \in [k]$,

$$J_j \subseteq \mathtt{Asc}(\alpha) \setminus \{i_j\}, \tag{3.32}$$

(ii) $\lambda(J)'$ is a partition of length at most k defined by

$$\lambda(J)' = \mathrm{sort}_{\mathrm{desc}}(a_1 - 1 - |J_1|, \ldots, a_k - 1 - |J_k|). \tag{3.33}$$

Remark 3.24. *The conjugation in Lemma 3.23 is deliberate since we will rewrite elementary symmetric functions to monomial symmetric functions by applying Equation (2.15) in the proof of the Theorem 3.10.*

Proof. We apply Lemma 3.17 to each $x_{i_j}^{a_j-1}$ to obtain

$$x_{i_1}^{a_1}\cdots x_{i_k}^{a_k} = x_{i_1}\cdots x_{i_k}\cdot x_{i_1}^{a_1-1}\cdots x_{i_k}^{a_k-1} \tag{3.34}$$

$$\equiv x_{\mathtt{Asc}(\alpha)}\cdot(-1)^{|\alpha|-k}\prod_{j=1}^{k} e_{a_j-1}(\widehat{x_{i_j}}) \pmod{I_n}. \tag{3.35}$$

Apply Lemma 3.21 to each $e_{a_j-1}(\widehat{x_{i_j}})$ in the right-hand side of Equation (3.35). We get

$$x_{\mathtt{Asc}(\alpha)}\cdot(-1)^{|\alpha|-k}\prod_{j=1}^{k} e_{a_j-1}(\widehat{x_{i_j}}) = (-1)^{|\alpha|-k}x_{\mathtt{Asc}(\alpha)}\prod_{j=1}^{k}\sum_{J_j\subseteq \mathtt{Asc}(\alpha)\setminus\{i_j\}} x_{J_j}\hat{e}_{a_j-1-|J_j|}. \tag{3.36}$$

We naively expand the product into expressions partly supported on $\mathtt{Asc}(\alpha)$ and partly on its complement. Equation (3.36) continues as

$$\begin{aligned}&(-1)^{|\alpha|-k}x_{\mathtt{Asc}(\alpha)}\prod_{j=1}^{k}\sum_{J_j\subseteq \mathtt{Asc}(\alpha)\setminus\{i_j\}} x_{J_j}\hat{e}_{a_j-1-|J_j|}\\ &= (-1)^{|\alpha|-k}x_{\mathtt{Asc}(\alpha)}\sum_{J} x_{J_1}\cdots x_{J_k}\cdot\hat{e}_{a_1-1-|J_1|}\cdots\hat{e}_{a_k-1-|J_k|},\end{aligned} \tag{3.37}$$

where $J=(J_1,\ldots,J_k)$ satisfies (3.32). We can write $\hat{e}_{a_1-1-|J_1|}\cdots\hat{e}_{a_k-1-|J_k|} = \hat{e}_{\lambda(J)'}$ for some partition $\lambda(J)'$ with at most k parts and $\lambda(J)'$ satisfies Equation (3.33). There is a subtlety that $\lambda(J)'$ may have negative parts. But in this case, we have $\hat{e}_{\lambda(J)'}=0$. We can safely assume without the loss of generality that all $\lambda(J)'$ are well-defined partitions. □

We will show in Lemma 3.29 that if we apply Equation (3.31) to P_α, then the resulting expression is a sum over a certain[2] set $\mathcal{J}_\alpha^+$ defined as follows.

Let $\alpha\in\mathcal{A}_n$ be a weak staircase. Let i_j be elements of $\mathtt{Asc}(\alpha)$ such that $\mathtt{Asc}(\alpha)=\{i_1<\cdots<i_k\}$, and $\mathtt{Area}(\alpha)=(a_1,\cdots,a_k)$. Define

$$\mathcal{J}_\alpha = \{(\alpha,J) : J=(J_1,\ldots,J_k) \text{ satisfying Equation (3.32)}\}. \tag{3.38}$$

The *weight* of an $(\alpha,J)\in\mathcal{J}_\alpha$ is

$$\mathtt{weight}(\alpha,J) = \mathrm{sort}_{\mathrm{decr}}(a_1-1-|J_1|,\ldots,a_k-1-|J_k|). \tag{3.39}$$

[2]The symbol $\mathcal{J}$ is a calligraphic J.

Also define a vector $v(\alpha, J) = (v_1, \ldots, v_k)$ by

$$v_m = \#\{J_j : i_m \in J_j\} \quad \text{for each } m \in [k]. \tag{3.40}$$

A pair $(\alpha, J) \in \mathcal{J}_\alpha$ is called *admissible* if $\mathtt{weight}(\alpha, J)$ is a well-defined partition whose largest part is at most $n - \mathtt{asc}(\alpha) - 1$ and $v(\alpha, J)$ is a permutation of $\delta_k = (0, 1, \ldots, k-1)$. The subset of $\mathcal{J}_\alpha$ containing admissible elements is denoted by $\mathcal{J}_\alpha^+$.

Lemma 3.25. *Let $\alpha \in \mathcal{A}_n$ and $(\alpha, J) \in \mathcal{J}_\alpha$. We have $x_{J_1} \cdots x_{J_k} = x_{\mathtt{Asc}(\alpha)}^{v(\alpha,J)}$. Moreover, if $v(\alpha, J)$ is a permutation of δ_k, then $\mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{J_1} \cdots x_{J_k}) = \pm \mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{\mathtt{Asc}(\alpha)}^{\delta_k})$; otherwise $\mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{J_1} \cdots x_{J_k}) = 0$.*

Proof. The first claim $x_{J_1} \cdots x_{J_k} = x_{\mathtt{Asc}(\alpha)}^{v(\alpha,J)}$ holds by definition.

Let $j \in [k]$. The set J_j does not contain i_j, so the variable x_{i_j} cannot appear in x_{J_j}. Therefore, the exponent of x_{i_j} in $x_{J_1} \cdots x_{J_k}$ is at most $k-1$ because each $x_{J_1}, \ldots, x_{J_k}$ is a square-free monomial.

If $v(\alpha, J)$ is a permutation of δ_k, then clearly

$$\mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{J_1} \cdots x_{J_k}) = \mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{\mathtt{Asc}(\alpha)}^{v(\alpha,J)}) = \pm \mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{\mathtt{Asc}(\alpha)}^{\delta_k}).$$

Otherwise, suppose $v(\alpha, J)$ is not a permutation of δ_k. By the preceding paragraph, the largest entry in $v(\alpha, J)$ is $k-1$. By the definition, all entries in $v(\alpha, j)$ are non-negative integers. It follows that $v(\alpha, J)$ must contain repeated entries and, consequently, $\mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{J_1} \cdots x_{J_k}) = 0$. □

For each $(\alpha, j) \in \mathcal{J}_\alpha$, we define a decorated diagram as follows. Start with the diagram of α as described on page 46. Decorate each box (i_j, j) with a wedge $\wedge$ for each $j \in [k]$ to indicate that i_j is an ascent. On the diagram of α, decorate the boxes (i', j) with a bullet $\bullet$ for every $j \in [k]$ and every $i' \in J_j$. Because each J_j is a subset of $\mathtt{Asc}(\alpha)$, bullets only appear in a column decorated with a wedge. Moreover, because J_j does not contain i_j, a box cannot be decorated with both a bullet and a wedge. The j-th entry of $v(\alpha, J)$ counts the number of bullets in column i_j.

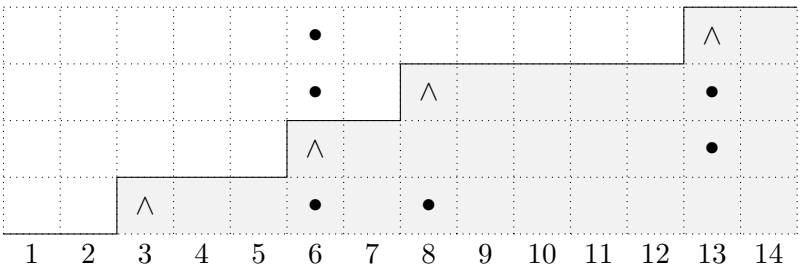


Figure 3.3: The diagram of an admissible (α, J).

Example 3.26. *Let α be the running example in Figure 3.1.*

Figure 3.3 shows an element $(\alpha, J) \in \mathcal{J}_\alpha^+$. The choice of J is $J_1 = \{6, 8\}, J_2 = \{13\}, J_3 = \{6, 13\}, J_4 = \{6\}$. The weight is $(9, 7, 4, 0)$ and $v(\alpha, J) = (0, 3, 1, 2)$.

Figure 3.4 shows an element $(\alpha, J) \in \mathcal{J}_\alpha$ such that (α, J) is not admissible. The choice of J is $J_1 = \emptyset, J_2 = \{3, 8, 13\}, J_3 = \{3, 13\}, J_4 = \{1\}$. The weight is $(11, 5, 4, 0)$. This (α, J) is not *admissible because $a_1 - 1 - |J_1| = 11$ is greater than $n - \mathtt{asc}(\alpha) - 1 = 14 - 4 - 1 = 9$. For completeness, we note $v(\alpha, J) = (3, 0, 1, 2)$.*

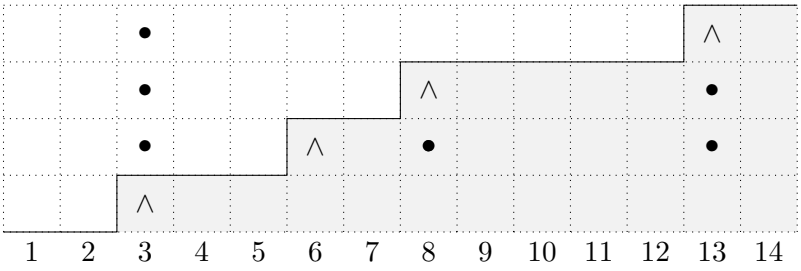


Figure 3.4: The diagram of a not admissible (α, J).

We can rephrase Lemma 3.23 in terms of $\mathcal{J}_\alpha$.

Corollary 3.27 (of Lemma 3.23)**.** *The following expressions are elements in R_n modulo I_n.*

Let $\alpha \in \mathcal{A}_n$ be a weak staircase. Let i_j be elements of $\mathtt{Asc}(\alpha)$ such that $\mathtt{Asc}(\alpha) = \{i_1 < \cdots < i_k\}$, and $\mathtt{Area}(\alpha) = (a_1, \cdots, a_k)$. We need a notation for omitting variables from the

arguments of a function. Denote

$$e_m(\widehat{x_i} : i \in \mathtt{Asc}(\alpha)) = e_m(x_j : j \notin \mathtt{Asc}(\alpha)) \tag{3.41}$$

for all $m \geq 0$. *We also use the shorthand* $\hat{e}_m = e_m(\widehat{x_i} : i \in \mathtt{Asc}(\alpha))$. *We have*

$$x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \equiv (-1)^{|\alpha|-k} x_{\mathtt{Asc}(\alpha)} \sum_{(\alpha,J)\in\mathcal{J}_\alpha} x_{J_1} \cdots x_{J_k} \hat{e}_{\mathtt{weight}(\alpha,J)} \pmod{I_n}. \tag{3.42}$$

Remark 3.28. *The* (α, J)*'s appearing in Equation* (3.42) *need not be admissible.*

The next lemma generalizes the computations in Example 3.19 from Line (3.22) to (3.25).

Lemma 3.29. *The following expressions are elements of* SR_n *modulo* $\mathcal{I}_n$. *Define* $\delta_k + 1 = (1, 2, \ldots, k)$.

Let $\alpha \in \mathcal{A}_n$ *be a weak staircase. Let* i_j *be elements of* $\mathtt{Asc}(\alpha)$ *such that* $\mathtt{Asc}(\alpha) = \{i_1 < \cdots < i_k\}$, *and* $\mathtt{Area}(\alpha) = (a_1, \cdots, a_k)$. *Let* $\hat{e}_m = e_m(x_i : i \notin \mathtt{Asc}(\alpha))$ *for all* $m \geq 0$. *We have*

$$P_\alpha \equiv \sum_{(\alpha,J)\in\mathcal{J}_\alpha^+} c_J \operatorname{sym}_n \left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)}^{\delta_k+1} \hat{e}_{\mathtt{weight}(\alpha,J)}\right) \pmod{\mathcal{I}_n}, \tag{3.43}$$

where $c_J = \pm(n-k)!$.

Proof. By Lemma 3.6, we have

$$P_\alpha = (n-k)! \cdot \operatorname{sym}_n \left(\theta_{\mathtt{Asc}(\alpha)} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}\right). \tag{3.44}$$

Equation (3.5) says to work with P_α in $SR_n/\mathcal{I}_n$ is to work with its θ coefficients in R_n/I_n. We apply Corollary 3.27 to the coefficient of $\theta_{\mathtt{Asc}(\alpha)}$ to obtain

$$x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \equiv (-1)^{|\alpha|-k} x_{\mathtt{Asc}(\alpha)} \sum_{(\alpha,J)\in\mathcal{J}_\alpha} x_{J_1} \cdots x_{J_k} \hat{e}_{\mathtt{weight}(\alpha,J)} \pmod{I_n}.$$

It follows that

$$P_\alpha \equiv (-1)^{|\alpha|-k}(n-k)! \sum_{(\alpha,J)\in\mathcal{J}_\alpha} \operatorname{sym}_n \left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)} \cdot x_{J_1} \cdots x_{J_k} \hat{e}_{\mathtt{weight}(\alpha,J)}\right) \pmod{\mathcal{I}_n}. \tag{3.45}$$

We need to check that the summation in Equation (3.45) can be reduced to just the admissible ones. Take an $(\alpha, J) \in \mathcal{J}_\alpha$ indexing the summands of Equation (3.45).

Let m be the largest entry of $\mathtt{weight}(\alpha, J)$. Suppose $m \geq n - \mathtt{asc}(\alpha)$. Because $x_{\mathtt{Asc}(\alpha)}\hat{e}_m$ has degree at least $n - \mathtt{asc}(\alpha) + \mathtt{asc}(\alpha) = n$ and $\hat{e}_m$ is a sum of square free monomials supported on $[n] \setminus \mathtt{Asc}(\alpha)$, we see every term in $x_{\mathtt{Asc}(\alpha)}\hat{e}_m$ is supported on $[n]$. Such terms are congruent to 0 modulo I_n because $x_1 \cdots x_n = e_n \in I_n$. So the entire expression $\mathrm{sym}_n\left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)} \cdot x_{J_1} \cdots x_{J_k} \hat{e}_{\mathtt{weight}(\alpha,J)}\right) \equiv 0 \pmod{\mathcal{I}_n}$.

Regroup sym_n by cosets of $\mathcal{S}_{\mathtt{Asc}(\alpha)} \times \mathcal{S}_{[n]\setminus\mathtt{Asc}(\alpha)}$ to get

$$\begin{aligned}
&\mathrm{sym}_n\left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)} \cdot x_{J_1} \cdots x_{J_k} \hat{e}_{\mathtt{weight}(\alpha,J)}\right) \\
&= \sum_{\substack{A \subseteq [n] \\ |A|=k}} \pi_A \cdot \mathrm{sym}_{\mathtt{Asc}(\alpha)}\left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)} \cdot x_{J_1} \cdots x_{J_k}\right) \mathrm{sym}_{[n]\setminus\mathtt{Asc}(\alpha)}(\hat{e}_{\mathtt{weight}(\alpha,J)}) \qquad (3.46) \\
&= \sum_{\substack{A \subseteq [n] \\ |A|=k}} \pi_A \cdot \theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)} \mathrm{alt}_{\mathtt{Asc}(\alpha)}(x_{J_1} \cdots x_{J_k}) \mathrm{sym}_{[n]\setminus\mathtt{Asc}(\alpha)}(\hat{e}_{\mathtt{weight}(\alpha,J)}). \qquad (3.47)
\end{aligned}$$

It follows from Lemma 3.25 that if (α, J) is not admissible, then the right-hand side of Equation (3.47) is 0; otherwise, if (α, J) is admissible, then Equation (3.47) becomes

$$\begin{aligned}
&\mathrm{sym}_n\left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)} \cdot x_{J_1} \cdots x_{J_k} \hat{e}_{\mathtt{weight}(\alpha,J)}\right) \\
&= \pm \mathrm{sym}_n\left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)} \cdot x_{\mathtt{Asc}(\alpha)}^{\delta_k} \hat{e}_{\mathtt{weight}(\alpha,J)}\right) \qquad (3.48) \\
&= \pm \mathrm{sym}_n\left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)}^{\delta_k+1} \hat{e}_{\mathtt{weight}(\alpha,J)}\right). \qquad (3.49)
\end{aligned}$$

The lemma follows. □

For each $\alpha \in \mathcal{A}_n$, we define $J^\alpha = (J_1, \ldots, J_k)$ by $J_j = \{i_{i'} \in \mathtt{Asc}(\alpha) : i' > j\}$ for each $j \in [k]$. See Figure 3.5 for the decorated diagram of (α, J^α) of the running example α. We note

$$\mathtt{weight}(\alpha, J^\alpha) = (\mathtt{Area}(\alpha)_1 - k, \ldots, \mathtt{Area}(\alpha)_k - 1).$$

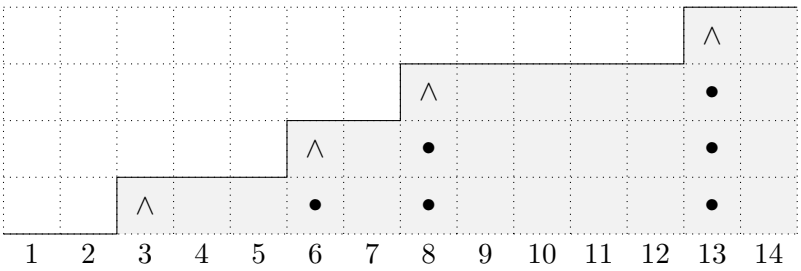


Figure 3.5: The decorated diagram corresponding to (α, J^α) in $\mathcal{J}_\alpha$.

Lemma 3.30. *Let $\alpha \in \mathcal{A}_n$. Every $(\alpha, J) \in \mathcal{J}_\alpha^+$ satisfies*

$$\texttt{weight}(\alpha, J) \geq \texttt{weight}(\alpha, J^\alpha) \tag{3.50}$$

in lexicographical order. Furthermore, the inequality in Equation (3.50) *is strict if $J \neq J^\alpha$.*

Proof. The integer n is fixed throughout the proof. The idea is that if J_1 is not the largest possible choice, then we can directly verify Equation (3.50); otherwise, on the decorated diagram of (α, J), we delete the first row to get a new weak staircase β and a new choice of K such that $(\beta, K) \in \mathcal{J}_\beta^+$ and apply induction.

We proceed by induction on k, which is the number of ascents. If $k = 1$, then $\alpha = (0^m, 1^{n-m})$ for some $m = 1, \ldots, n-1$ and $(\alpha, (\emptyset))$ is the only element of $\mathcal{J}_\alpha$ and the result holds trivially.

Suppose for some $k \geq 2$, Equation (3.50) holds for by every weak staircase in $\mathcal{A}_n$ whose number of ascents is less than k.

Let $\alpha \in \mathcal{A}_n$ with $\texttt{asc}(\alpha) = k$. Let i_j be elements of $\texttt{Asc}(\alpha)$ such that $\texttt{Asc}(\alpha) = \{i_1 < \cdots < i_k\}$, and $\texttt{Area}(\alpha) = (a_1, \cdots, a_k)$. Choose some $(\alpha, J) \in \mathcal{J}_\alpha^+$. We consider two cases.

Suppose J_1 is a strict subset of $\texttt{Asc}(\alpha) \setminus \{i_1\} = \{i_2, \ldots, i_k\}$. We have $|J_1| < k - 1$ and, by definition of $\texttt{weight}(\alpha, J)$ in Equation (3.39),

$$\texttt{weight}(\alpha, J)_1 \geq a_1 - 1 - |J_1| > a_1 - k = \texttt{weight}(\alpha, J^\alpha)_1. \tag{3.51}$$

It follows that $\texttt{weight}(\alpha, J) > \texttt{weight}(\alpha, J^\alpha)$ in lexicographical order.

Otherwise, suppose $J_1 = \{i_2, \ldots, i_k\}$. Define $\beta = \alpha - (0^{i_1-1}, 1^{n-i_1})$ by coordinate-wise subtraction and $K = (J_2, \ldots, J_{k-1})$. The decorated diagram of (β, K) is obtained by deleting the first row of the decorated diagram of (α, J). It follows that the weight of (β, J) is obtained from $\texttt{weight}(\alpha, J)$ by deleting the entry $\texttt{Area}(\alpha)_1 - 1 - |J_1| = a_1 - k$. We remark that $a_1 - 1$ is not necessarily the largest entry in $\texttt{weight}(\alpha, J)$.

We claim (β, K) is an admissible element in $\mathcal{J}_\beta$. The condition on $\texttt{weight}(\beta, K)$ is trivially satisfied. Since (α, J) is admissible, if i_1 appears in any one of $J_2, \ldots, J_k$, then 0 does not appear $v(\alpha, J)$, contradicting $v(\alpha, J)$ is a permutation of δ_k. We see that $v(\beta, K)$ is obtained from $v(\alpha, K)$ by removing the entry 0 then decreasing all remaining entries by 1. Therefore, $v(\beta, K)$ is a permutation of δ_{k-1}. The claim follows.

Because $\texttt{asc}(\beta) = k - 1$, we can apply the induction hypothesis to (β, K) to get $\texttt{weight}(\beta, K) \geq \texttt{weight}(\beta, J^\beta)$. To shorten notations for the remainder of the proof, we let $\texttt{weight}(\beta, K) = (u_1, \ldots, u_{k-1})$ and $\texttt{weight}(\alpha, J^\alpha) = (w_1, \ldots, w_k)$. The following relations follow immediately from the construction of (β, K). Recall by definition, we have $u_1 \geq \cdots \geq u_{k-1}$ and $w_1 \geq \cdots \geq w_k$.

If $\texttt{weight}(\beta, K) = \texttt{weight}(\beta, J^\beta)$, then we have $(\beta, K) = (\beta, J^\beta)$. The definition of K makes it clear that $(\alpha, J) = (\alpha, J^\alpha)$.

Assume $\texttt{weight}(\beta, K) \neq \texttt{weight}(\beta, J^\beta)$. Let j be the smallest index

$$u_1 = w_2, \ldots, u_{j-1} = w_j, \quad \text{but} \quad u_j > w_{j+1}.$$

We need to compare $\texttt{weight}(\alpha, J)$ to $\texttt{weight}(\alpha, J^\alpha)$. The following relations follow immediately from the construction of (β, K).

$$\begin{aligned} \texttt{weight}(\alpha, J) &= \mathrm{sort}_{\mathrm{decr}}(w_1, u_1, \ldots, u_{k-1}), \\ \texttt{weight}(\beta, J^\beta) &= (w_2, \ldots, w_{k-1}). \end{aligned}$$

We consider two sub cases. In the first sub case, we assume $w_1 \geq u_j$. Because $w_1 \geq w_2 = u_1$, we have

$$\texttt{weight}(\alpha, J) = (w_1 \geq u_1 \geq \cdots \geq u_{k-1}). \tag{3.52}$$

It follows that $\texttt{weight}(\alpha, J)_1 = \texttt{weight}(\alpha, J^\alpha)_1, \ldots, \texttt{weight}(\alpha, J)_j = \texttt{weight}(\alpha, J^\alpha)_j$ but $\texttt{weight}(\alpha, J)_{j+1} > \texttt{weight}(\alpha, J^\alpha)_{j+1}$ and $\texttt{weight}(\alpha, J) > \texttt{weight}(\alpha, J^\alpha)$. In the second sub case, we assume $w_1 < u_j$. If $j > 1$, then u_{j-1} is well-defined and $w_1 \geq w_j = u_{j-1} \geq u_j > w_1$ is a contradiction. It follows that $j = 1$ and w_1 appears in $\texttt{weight}(\alpha, J)$ at index at least 2. It follows that

$$\texttt{weight}(\alpha, J)_1 = u_1 > w_1 = \texttt{weight}(\alpha, J^\alpha)_1$$

and $\texttt{weight}(\alpha, J) > \texttt{weight}(\alpha, J^\alpha)$. □

We are ready to prove Theorem 3.10.

Proof of Theorem 3.10. Let i_j be elements of $\texttt{Asc}(\alpha)$ such that $\texttt{Asc}(\alpha) = \{i_1 < \cdots < i_k\}$, and $\texttt{Area}(\alpha) = (a_1, \cdots, a_k)$. Apply Lemma 3.29 to express P_α as

$$P_\alpha \equiv \sum_{(\alpha,J)\in\mathcal{J}_\alpha^+} c_J \operatorname{sym}_n \left(\theta_{\texttt{Asc}(\alpha)} x_{\texttt{Asc}(\alpha)}^{\delta_k+1} \hat{e}_{\texttt{weight}(\alpha,J)}\right) \pmod{\mathfrak{I}_n}. \tag{3.53}$$

For each partition μ, denote $\hat{m}_\mu = m_\mu(\widehat{x_i} : i \in \texttt{Asc}(\alpha)) = m_\mu(x_j : j \notin \texttt{Asc}(\alpha))$ analogous to that of $\hat{e}_m$ in Equation (3.41). We apply Equation (2.15) to rewrite each $\hat{e}$ in terms of $\hat{m}$. It follows that

$$P_\alpha \equiv \sum_{(\alpha,J)\in\mathcal{J}_\alpha^+} c_J \operatorname{sym}_n \left(\theta_{\texttt{Asc}(\alpha)} x_{\texttt{Asc}(\alpha)}^{\delta_k+1} \left(\sum_{\mu\preceq\texttt{weight}(\alpha,J)'} a_{J,\mu}\hat{m}_\mu\right)\right) \pmod{\mathfrak{I}_n}, \tag{3.54}$$

where $a_{J,\mu}$ is short for $a_{\texttt{weight}(\alpha,J)',\mu}$ and $a_{\texttt{weight}(\alpha,J)',\texttt{weight}(\alpha,J)'} = 1$.

Let $\lambda' = (a_1 - k, \ldots, a_k - 1)$. Notice $\texttt{weight}(\alpha, J^\alpha) = \lambda'$ and $\lambda_1 = \ell(\lambda) \leq k$ by definition.

If a partition μ appears in the inner summation of Equation (3.54), then by Lemma 3.30, we have $\texttt{weight}(\alpha, J) \geq \lambda'$. The inequality is reversed after conjugating both sides. We have $\mu \preceq \texttt{weight}(\alpha, J)' \leq \lambda$. The $\leq$ is in the latter inequality is strict if and only if $J = J^\alpha$. It follows that

$$\begin{aligned} P_\alpha \equiv & c_{J^\alpha} \operatorname{sym}_n \left(\theta_{\texttt{Asc}(\alpha)} x_{\texttt{Asc}(\alpha)}^{\delta_k+1} \hat{m}_\lambda\right) \\ & + \sum_{\substack{(\alpha,J)\in\mathcal{J}_\alpha^+ \\ J\neq J^\alpha}} \sum_{\mu\preceq\texttt{weight}(\alpha,J)'<\lambda} c_J a_{J,\mu} \operatorname{sym}_n \left(\theta_{\texttt{Asc}(\alpha)} x_{\texttt{Asc}(\alpha)}^{\delta_k+1} \hat{m}_\mu\right) \pmod{\mathfrak{I}_n}. \end{aligned} \tag{3.55}$$

For every partition ν with length at most $n-k-1$, define a vector

$$\beta(\nu) = \mathrm{sort}_{\mathrm{incr}}(0^{n-k-\ell(\nu)}, 1, \ldots, k, \nu_1, \ldots, \mu_{\ell(\nu)}). \tag{3.56}$$

It follows immediately that $\beta(\lambda) = \alpha$ because the diagram of λ is obtained from the diagram of α by deleting all columns indexed by elements of $\mathtt{Asc}(\alpha)$ and all zeros. Therefore,

$$\mathrm{sym}_n\left(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)}^{\delta_k+1} \hat{m}_\lambda\right) = M_\alpha.$$

We only use the lexicographical order to compare vectors for the remainder of the proof. Let μ be a partition indexing the inner summation in Equation (3.55). It remains to show that $\mathrm{sym}_n(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)}^{\delta_k+1}) = \pm M_{\beta(\mu)}$ and $\beta(\mu) > \alpha$.

Because $\mu < \lambda$, we have $\max(\mu) = \mu_1 \leq \lambda_1 \leq k$ hence $\beta(\mu) \in \mathcal{A}_n$. Every monomial in $\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)}^{\delta_k+1} \hat{m}_\mu$ is a permutation of the monomial $\theta_{\mathtt{Asc}(\beta(\mu))} x^{\beta(\mu)}$ by definition. Therefore $\mathrm{sym}_n(\theta_{\mathtt{Asc}(\alpha)} x_{\mathtt{Asc}(\alpha)}^{\delta_k+1} \hat{m}_\mu) = \pm M_{\beta(\mu)}$ by the definition of $M_{\beta(\mu)}$.

Lastly, we show $\beta(\mu) > \alpha$. By Lemma 3.8, it suffices to show $\mathtt{Area}(\beta(\mu)) > \mathtt{Area}(\alpha)$ as vectors. Notice

$$\mathtt{Area}(\beta(\mu)) = (\mu_1' + k, \ldots, \mu_k' + 1) \quad \text{and} \quad \mathtt{Area}(\alpha) = (\lambda_1' + k, \ldots, \lambda_k' + 1).$$

Because $\mu < \lambda$, we have $\mu' > \lambda'$. Then $\mathtt{Area}(\beta(\mu)) > \mathtt{Area}(\alpha)$ as desired. □

We remark that manipulating P_α using any relation modulo $\mathcal{I}_n$ results in polarization equivalent operators. However, computer experiments using SageMath [77] on different ways to apply Lemma 3.17 seem to suggest that M_α are the only nice ones. For example, instead of applying Lemma 3.17 to $x_{a_j}^{i_j - 1}$ as shown in Equation (3.35), we apply Lemma 3.17 to $x_{a_j}^{i_j}$ to replace Equation (3.35) by the following

$$x_{a_1}^{i_1} \cdots x_{a_k}^{i_k} \equiv (-1)^{|\alpha|} \prod_{j=1}^{k} e_{i_j}(\widehat{x_{a_j}}) \pmod{I_n}. \tag{3.57}$$

The expressions resulting from applying Equation (3.57) to the definition of P_α are not easy to understand. It is not clear whether there is any triangularity to either M_α or P_α.

3.5 Some properties of M_α

We first prove that $M_\alpha \diamond \Delta_n$ is non-zero. Although Theorems 3.1 and 3.10 imply this fact, the proof below demonstrates a novel feature of M_α's. Proofs of non-zeroness of alternants often involve picking out a well-chosen leading term. Our proof avoids such technical bookkeeping. Instead, we use the classical higher Specht polynomials.

Lemma 3.31. *For each weak staircase $\alpha \in \mathcal{A}_n$, we have*

$$M_\alpha \not\equiv 0 \pmod{\mathfrak{I}_n}.$$

Proof. For convenience, denote $A = \mathtt{Asc}(\alpha)$, $B = [n] \setminus \mathtt{Asc}(\alpha)$ and $g_\alpha = [\theta_{\mathtt{Asc}(\alpha)}]M_\alpha$. Then by Equation (3.18), we have

$$g_\alpha = \mathrm{sym}_B \left(x^\alpha\big|_B\right) \mathrm{alt}_A \left(x^\alpha\big|_A\right).$$

To understand g_α, we work in R_n/I_n. By Equation (3.5) and (3.18), it suffices to show that g_α is non-zero modulo I_n. To this end, we show that g_α "generates" a classical higher Specht polynomial of a hook shape.

Let $k = \mathtt{asc}(\alpha)$. Define a hook-shaped standard Young tableau $T \in \mathrm{SYT}(n-k, 1^k)$ whose first row is the set B and, consequently, whose first column is the set $\{1\} \cup \mathtt{Asc}(\alpha)$. The cocharge word of T is exactly α.

The row symmetrizer $P(T)$ is sym_B. The set of cycles $\{(a_0 a_1 \cdots a_k) : i = 0, 1, \ldots, k\}$ forms a (left) transversal of $\mathfrak{S}_A$ in $\mathfrak{S}_{\{1\}\cup A}$. It follows that

$$N(T) = \sum_{i=0}^{k} (-1)^i (a_0 a_1 \cdots a_i) \mathrm{alt}_A. \tag{3.58}$$

Recall that higher Specht polynomial is defined in Equation (2.17) on page 31. It follows Equation (3.58) that the classical higher Specht polynomial G_T^T can be written as

$$G_T^T = N(T)P(T)x^{cocharge(T)} = \sum_{i=0}^{k} (-1)^i (a_0 a_1 \cdots a_i) \cdot g_\alpha.$$

Because G_T^T is an element of a basis of the irreducible representation in R_n/I_n indexed by the standard Young tableau T, we have $G_T^T \not\equiv 0 \pmod{I_n}$. We must have $g_\alpha \not\equiv 0 \pmod{I_n}$ as desired. □

The space of alternating polynomials in SR_n has a spanning set consisting of monomial alternating polynomials, i.e., $\mathrm{alt}_n(\theta_{i_1} \cdots \theta_{i_n} x^\alpha)$ where we can canonically choose α to be a weakly increasing exponent vector and $(i_1, \ldots, i_k) \in \mathbb{N}^n$. The next straightforward lemma records a method to expand $M_\alpha \diamond \Delta_n$ into linear combinations of monomial alternants.

Lemma 3.32. *Fix a weak staircase $\alpha \in \mathcal{A}_n$. Then*

$$M_\alpha \diamond \Delta_n = \mathrm{alt}_n\left(\theta_{\mathtt{Asc}(\alpha)} x^\alpha \diamond \Delta_n\right) = \sum_{\sigma \in \mathcal{S}_n} \mathrm{alt}_n\left(\theta_{\mathtt{Asc}(\alpha)} \partial x^\alpha \cdot x^{\sigma(\delta_n)}\right).$$

Every surviving monomial in $x^\alpha \diamond \Delta_n$ appears in the expansion into monomial alternants.

Proof. Apply Lemma 3.5 to the definition of M_α. Then we have

$$\begin{aligned} M_\alpha \diamond \Delta_n &= \sum_{\sigma \in \mathcal{S}_n} \sigma\left(\theta_{\mathtt{Asc}(\alpha)} x^\alpha\right) \diamond \Delta_n \\ &= \sum_{\sigma \in \mathcal{S}_n} \sigma\left(\theta_{\mathtt{Asc}(\alpha)} x^\alpha \diamond \sigma^{-1}(\Delta_n)\right) \\ &= \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) \sigma\left(\theta_{\mathtt{Asc}(\alpha)} x^\alpha \diamond \Delta_n\right) \\ &= \sum_{\sigma \in \mathcal{S}_n} \mathrm{alt}_n\left(\theta_{\mathtt{Asc}(\alpha)} \partial x^\alpha \cdot x^{\sigma(\delta_n)}\right). \end{aligned}$$

□

Chapter 4

Super harmonics

4.1 The conjectured Frobenius series

The super harmonics SH_n is closely related to the generalized coinvariant space $R_{n,k}$ first studied by Haglund, Rhoades, and Shimozono [37] and the Delta Conjecture [35]. For each pair of integers $1 \leq k \leq n$, define a symmetric function due to Rhoades [54, Equation (3.25)]

$$C_{n,k}(q) = \sum_{\mu \vdash n} \sum_{S \in \mathrm{SYT}(\mu)} q^{\mathtt{maj}(S) + \binom{n-k}{2} - (n-k)\cdot \mathtt{des}(S)} \begin{bmatrix} \mathtt{des}(S) \\ n-k \end{bmatrix} s_\mu. \tag{4.1}$$

We briefly describe the history of $C_{n,k}$ and its connections.

The Delta Conjecture is due to Haglund, Remmel, and Wilson [35, Conjecture 1.1] and first appears as a preprint in 2015 [36]. The Delta Conjecture [35] predicts combinatorial expansions of certain symmetric function expression $\Delta'_{e_k} e_n$. The Delta operator Δ'_f is defined in Equation (2) of [35]. Let $\mathrm{Rise}_{n,k}(x;q,t)$ and $\mathrm{Val}(x;q,t)$ be defined by Equations (19) and (21) in [35]. The Delta Conjecture predicts

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(x;q,t) = \mathrm{Val}_{n,k}(x;q,t). \tag{4.2}$$

The x in Equation (4.2) is an infinite set of variables $x_1, x_2, \ldots$. The same set of variables appears implicitly in the Schur function s_μ in Equation (4.1). We remark that the Delta Conjecture has been resolved by [11] and [17].

At the specialization $t = 1$, the work of Remmel and Wilson [53], Wilson [79] and Rhoades [54] combined shows $C_{n,k}(q)$ is equal to all of the following expressions

$$\mathrm{Rise}_{n,k}(x; q, 0) = \mathrm{Rise}_{n,k}(x; 0, q) = \mathrm{Val}_{n,k}(x; q, 0) = \mathrm{Val}_{n,k}(x; 0, q). \tag{4.3}$$

For each pair of integers $1 \leq k \leq n$, Haglund, Rhoades, and Shimozono [37, Equation (1.1)] define a coinvariant space

$$R_{n,k} = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n \rangle. \tag{4.4}$$

Define $\mathrm{rev}_q(f) = q^{\deg(f)} f(q^{-1})$ for any polynomial $f \in \mathbb{Z}[q]$. For example, $\mathrm{rev}_q(q^3 + 2q^2 + 3q) = 3 + 2q + q^2$. More intuitively, $\mathrm{rev}_q(f)$ is obtained from f by reversing the coefficient sequence of f. Equation (6.54) of [37] shows that

$$\mathrm{grFrob}(R_{n,k}; q) = (\omega \circ \mathrm{rev}_q) C_{n,k}(q). \tag{4.5}$$

Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be commuting variables and $\theta_1, \ldots, \theta_n$ be anticommuting variables. The x's and y's commute with the θ's. Zabrocki [80] define another coinvariant space

$$SDC_n = \frac{\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n]}{\langle \text{diagonally } \mathcal{S}_n\text{-invariant polynomials with no constant terms} \rangle}$$

Zabrocki [80] conjectured that graded Frobenius image of the θ-degree d component of SDC_n is exactly the symmetric function expression $\Delta'_{e_{n-d}} e_n$ appearing in the Delta Conjecture. The specialization of $t = 1$ in $\Delta'_{e_{n-d}} e_n$ corresponds to setting all y variables to 0. It follows from Zabrocki's conjecture that $C_{n,n-d}(q)$ is the graded Frobenius series of the θ-degree d component of the super coinvariant ring SR_n/SI_n.

In this section, we prove a folklore result about the cocharge statistic on standard Young tableaux and use the result to give a different expression for $C_{n,n-d}(q)$.

The Vandermonde determinant Δ_n spans the 1-dimensional top-degree component of the classical harmonics H_n. The cone under Δ_n is the vector space spanned by all x partial derivatives of Δ_n. One way to think about the super harmonics is to see it as the union of cones under each alternating component $M_\alpha \diamond \Delta_n$. However, it can be checked by a

computer algebra system that there exist weak staircases α, β such that the intersection of cones

$$\mathcal{L}_{\partial x}\left(M_\alpha \diamond \Delta_n\right) \bigcap \mathcal{L}_{\partial x}\left(M_\beta \diamond \Delta_n\right)$$

contains more than just the 0 polynomial. Significant effort went into studying experimental data to choose polynomials from each cone to conjecture a basis. Hence, the results in Section 4.1 and 4.2 are *not* written in the order that they were discovered. In retrospect, a conjectured basis was hiding in plain sight in the conjectured Frobenius series $C_{n,n-d}(q)$.

We aim to identify the combinatorics associated with $C_{n,n-d}(q)$ towards conjecturing a basis. From a general point of view, suppose $F_1, \ldots, F_m$ are bihomogeneous polarization operators, not necessarily symmetric. If $F_1 \diamond \Delta_n, \ldots, F_m \diamond \Delta_n$ are linearly independent, then the Hilbert series of their linear span is

$$\sum_{i=1}^{n} q^{\binom{n}{2}-\deg_x F_i} z^{\deg_\theta F_i}. \tag{4.6}$$

We can interpret the subtraction $-\deg_x(F_i)$ in the Hilbert series as the effect of polarization. Hence, we wish to manipulate the expression of $C_{n,n-d}$ in Equation (4.1) so that a meaningful subtraction shows up.

To state Lemma 4.5, we need to describe a pair of well-known statistics on lattice paths that we call area and coarea. Our terminology differs from the reference [70] and we offer a translation in Remark 4.2. Recall from page 12, $L_{m,n}$ is the set of lattice paths in $\mathbb{N}^2$ starting at $(0,0)$ and ending at (m,n). The *coarea word* of a lattice path $w \in L_{m,n}$ is $\texttt{Coarea}(w) = (c_1, \ldots, c_n)$ where each c_i counts the total number of E steps appearing *before* the i-th N step. The coarea of w is $\texttt{coarea}(w) = |\texttt{Coarea}(w)|$. The *area word* of $w \in L_{m,n}$ is $\texttt{Area}(w) = (a_1, \ldots, a_n)$ where each a_i counts the total number of E steps appearing *after* the i-th N-step. The area of w is $\texttt{area}(w) = |\texttt{Area}(w)|$. Pictorially, each c_i (or a_i) counts the number of cells above (or below respectively) the path w in row i.

Remark 4.1. *The path of a weak staircase is a lattice path. If we write a weak staircase $\alpha \in \mathcal{A}_m$ with n ascents as a lattice word $w \in L_{m,n}$, then the definition of area on α is consistent with the definition of area on w, i.e.,* $\texttt{Area}(\alpha) = \texttt{Area}(w)$.

Remark 4.2. *The coarea of a lattice path w is also known as the weight of the partition λ associated to w in the literature [70]. The partition λ is obtained from w by taking the collection of cells above the path of w. See Example 4.4.*

The following statement is well-known and rephrased from our reference.

Theorem 4.3 ([70, Proposition 1.7.3 on page 59])**.** *For each integer $n \geq 1$,*

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \sum_{w \in L_{m,n}} q^{\texttt{coarea}(w)}.$$

Example 4.4. *The lattice path $w = EENENEENEEE \in L_{8,3}$ is shown in Figure 4.1. We have* $\texttt{Coarea}(w) = (2,3,5)$, $\texttt{coarea}(w) = 2{+}3{+}5$ *and* $\texttt{Area}(w) = (6,3,3)$ *and* $\texttt{area}(w) = 12$*. For completeness, we note the partition defined by w is* 532*.*

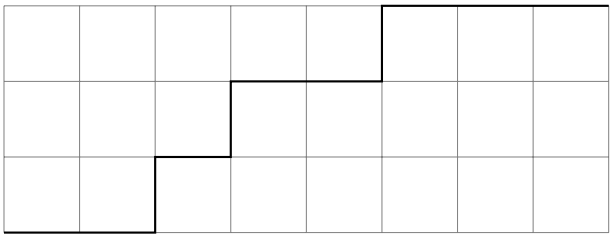


Figure 4.1: A lattice path $w = EENENEENEEE$ in $L_{8,3}$.

The next lemma allow us to manipulate the conjecture Frobenius series of SH_n. For a set D, define $\Sigma(D)$ to be the sum of elements in D.

Lemma 4.5. *For each pair $0 \leq d \leq d'$, we have*

$$q^{\binom{d}{2}} q^{-d \cdot d'} \begin{bmatrix} d' \\ d \end{bmatrix} = \sum_{\substack{D \subseteq [d'] \\ |D| = d}} q^{-\Sigma(D)}.$$

Proof. Apply Theorem 4.3 to rewrite

$$q^{\binom{d}{2}} \begin{bmatrix} d' \\ d \end{bmatrix} = \left(\prod_{i=1}^{d} q^{i-1} \right) \sum_{w \in L_{d'-d,d}} \prod_{i=1}^{d} q^{Coarea(w)_i} = \sum_{w \in L_{d'-d,d}} \prod_{i=1}^{d} q^{Coarea(w)_i + i - 1}.$$

Fix a path $w \in L_{d'-d,d}$. Its corresponding monomial $\prod_{i=1}^{d} q^{\texttt{Coarea}(w)_i + i - 1}$ appearing on the right-hand side can be interpreted combinatorially. Partition a d'-by-d grid into two sub-grids, one on the left having width d and one on the right having width $d' - d$.

In the left grid, draw the staircase $(NE)^d$. In the right grid, draw w. Shade the cells above each path within their sub-grid. We obtain a path $\pi \in L_{d',d}$ by left justifying all *shaded* cells of both paths. See Figure 4.2 for an example. The boundary between the shaded cells and non-shaded cells is a lattice path with no consecutive N steps and ends in an E step. Let $\texttt{Coarea}(\pi) = (c_1, \ldots, c_d)$ and $\texttt{Area}(\pi) = (a_1, \ldots, a_d)$. It follows that $0 \leq c_1 < \cdots < c_d \leq d'-1$. Because $c_i + a_i = d'$ for all $i \in [d]$, we have $d' \geq a_1 > \cdots > a_d \geq 1$. If we let $D = \{a_1, \ldots, a_d\}$, then D is a d-subset of $[d']$.

Conversely, if a d-subset $D = \{a_1, \ldots, a_d\} \subseteq [d']$ is given such that $a_1 > \cdots > a_d$, we can uniquely construct a lattice path $\pi \in L_{d',d}$ with no consecutive N steps and ends in an E steps that satisfies $\texttt{Area}(\pi) = (a_1, \ldots, a_d)$. It follows that

$$q^{-d \cdot d'} q^{\binom{d}{2}} \begin{bmatrix} d' \\ d \end{bmatrix} = \sum_w q^{-area(w)} = \sum_{D \subseteq [d']} q^{-\Sigma(D)},$$

where w runs over all lattice paths in $L_{d',d}$ such that w has no consecutive N steps and ends in an E step. □

Figure 4.2 shows a visual proof of Lemma 4.5.

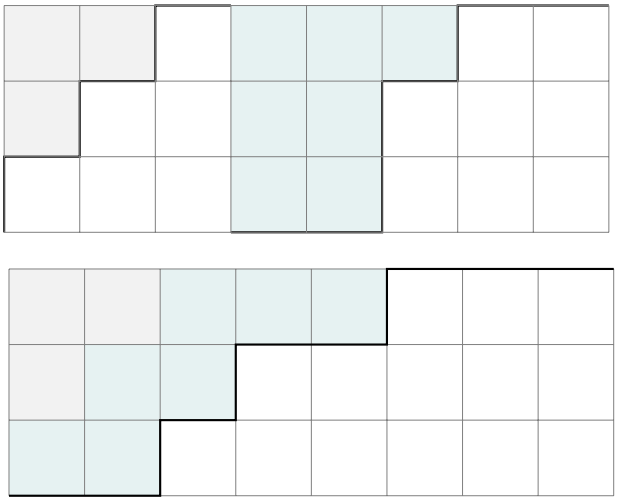


Figure 4.2: A visualization of the proof of Lemma 4.5.

Apply Lemma 4.5 to Equation (4.1) we obtain an interesting expression for $C_{n,n-d}(q)$.

Corollary 4.6. *For every $d = 0, 1, \ldots, n-1$, we have*

$$C_{n,n-d}(q) = \sum_{S \in \mathrm{SYT}(n)} \sum_{\substack{D \subseteq [\mathtt{des}(S)] \\ |D|=d}} q^{\mathtt{maj}(S)-\Sigma(D)} s_{\mathrm{sh}(S)}. \tag{4.7}$$

We refer back to the discussion surrounding Equation (4.6). The subtraction $-\Sigma(D)$ in Equation (4.7) is meaningful in that it can be interpreted as the effect of applying some *symmetric polarization operator* to some irreducible representation indexed by the standard Young tableau S in the classical harmonic H_n. We look in R_n/I_n instead of H_n because we wish to treat R_n/I_n as a set of polarization operators. We are a step closer to conjecturing a basis. There are no known bases of irreducible components of R_n/I_n associated to the major index of standard Young tableaux. However, we know bases of irreducible components of R_n/I_n are associated to the cocharge statistic. Apply wishful thinking, for now, to replace the major index by the cocharge statistic in Equation (4.7) to define a new symmetric function

$$D_{n,n-d}(q) = \sum_{S \in \mathrm{SYT}(n)} \sum_{\substack{D \subseteq [\mathtt{des}(S)] \\ |D|=d}} q^{\mathtt{cc}(S)-\Sigma(D)} s_{\mathrm{sh}(S)}, \tag{4.8}$$

where $d = 0, 1, \ldots, n-1$.

There was no evidence such wishful thinking would have worked. It was a pleasant surprise when a computer program using SageMath [77] confirmed that $C_{n,n-d} = D_{n,n-d}$ up to $n = 11$. To prove this equality, we need a refined enumeration of standard Young tableaux that has never appeared in the literature of coinvariant theory.

It is well-known that the major index and the cocharge statistics are equidistributed over a fixed shape [69, 7.21.5 on page 376] and [4, page 144], i.e., for every fixed partition $\mu \vdash n$,

$$\sum_{T \in \mathrm{SYT}(\mu)} q^{\mathtt{maj}(T)} = f^{\lambda}(q) = \sum_{T \in \mathrm{SYT}(\mu)} q^{\mathtt{cc}(T)}.$$

The polynomial $f^{\lambda}(q)$ is a special case of the Kostka-Foulkes polynomial. However, this result is not strong enough to replace `maj` with `cc` in Equation (4.7). We need the descent statistic to be preserved.

Theorem 4.7. *Let x, y be counting variables. For every partition $\mu \vdash n$, we have*

$$\sum_{T \in \mathrm{SYT}(\mu)} x^{\mathtt{maj}(T)} y^{\mathtt{des}(T)} = \sum_{T \in \mathrm{SYT}(\mu)} x^{\mathtt{cc}(T)} y^{\mathtt{des}(T)}.$$

Theorem 4.7 seems to be folklore. The author does not claim originality of its statement. Among experts, Theorem 4.7 seems believable but no one seems to know a direct proof. An argument for a similar problem was given in [44]. Killpatrick defines an injection $h : \mathrm{SYT}(\mu) \to W_\mu$ sending a standard Young tableau T to certain word $h(w)$ such that $\mathtt{maj}(T) = \mathtt{cc}(h(w))$. Notice $\mathtt{cc}(h(w))$ is the cocharge statistic on words. The naive approach of applying Robinson-Schensted to the image of Killpatrick's map does not work as the resulting recording tableaux are not unique. Nantel Bergeron suggested that this sequence of operations feels like Schützenberger's evacuation [7]. To the author's best knowledge, the proof of Theorem 4.7 using evacuation appears to be new.

Evacuation is a well-known involution on standard Young tableaux of a fixed shape [62]. It repeatedly uses a sliding rule called *jeu de Taquin* to remove the inner corner cell $(1, 1)$ to obtain a standard Young tableau with one fewer cell. The shapes of these tableaux form a path in Young's lattice from μ to the empty shape $\emptyset$ and, hence, define a standard Young tableau of shape μ. We now describe both procedures and prove Theorem 4.7.

Let μ be a partition. A *removable outer corner* of μ is a cell $(i, j) \in \mu$ such that both $(i + 1, j)$ and $(i, j + 1)$ are not in μ. An *addable inner corner* of a skew shape λ/μ is a removable outer corner of μ.

Jeu de Taquin is a procedure defined by Schützenberger that slides an addable inner corner of a skew tableau to its outer boundary [63]. Let T be a standard Young tableau of skew shape λ/μ and let $\square$ be an addable inner corner of T. Initialize $\square^{(0)} = \square$ and $T^{(0)} = T$. For $k \geq 0$, if $\square^{(k)}$ is a removable outer corner of $T^{(k)}$, then we terminate by removing $\square^{(k)}$ from $T^{(k)}$ to obtain $\mathrm{jdt}_{\square}(T)$. Otherwise, let $\square^{(k+1)}$ be the cell either to the right or below $\square^{(k)}$ depending on whichever one has a smaller value in $T^{(k)}$. We obtain $T^{(k+1)}$ by moving the value from $\square^{(k+1)}$ to $\square^{(k)}$. See Figure 4.3 for an example.

We need some terminology to describe certain non-crossing properties of evacuation revealed by the proof of Theorem 4.7. Let T be a standard Young tableau. Let $T^{(0)} = T$ and for $i \geq 1$, let $T^{(i)}$ be the tableau obtained by applying jeu de Taquin to the inner corner

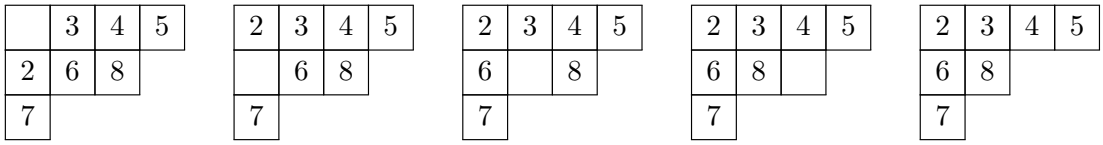


Figure 4.3: An example of applying jeu de Taquin.

of $T^{(i-1)}$. The *i-th evacuation path* of T, denoted as $\pi^{(i)}$ is a tuple of cells $(\square_{i,1}, \ldots, \square_{i,\ell_i})$ such that $\square_{i,1}$ is the inner corner of T, and if we apply jeu de Taquin to the inner corner of $T^{(i-1)}$, then $\square_{i,1}$ slides to $\square_{i,2}$, $\square_{i,2}$ slides to $\square_{i,3}$ and so on.

The evacuation path in Figure 4.3 is shown as the list of bullets in Figure 4.4

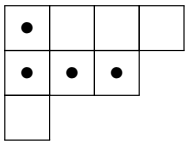


Figure 4.4: The evacuation path in Figure 4.3

The *evacuation* of T is a tableau of the same shape as T in which the value $n - i + 1$ is placed in the last cell of $\pi^{(i)}$.

Theorem 4.8 ([59, Theorem 3.9.4 on page 136])**.** *Evacuation is an involution on standard Young tableaux of a fixed shape.*

The next lemma is a straightforward property of jeu de Taquin.

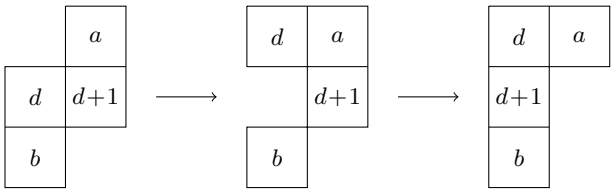
Lemma 4.9. *For every $i \geq 1$, the inner corner of $T^{(i-1)}$ is i and the fillings of $T^{(i-1)}$ are $i, \ldots, n$.*

Proof. For induction, assume the inner corner of $T^{(i-1)}$ is i for some $i \geq 1$. Then $i + 1$ must appear either immediately to the right or immediately below i. Either way, a jeu de Taquin slide starting at the inner corner of $T^{(i-1)}$ will place $i + 1$ in the inner corner of the $\mathrm{jdt}_{\square_{i,1}}(T^{(i-1)})$ which is just $T^{(i)}$. Because i is the only filling removed from $T^{(i-1)}$, the second conclusion holds. □

The next lemma says applying jeu de Taquin cannot create descents.

Lemma 4.10. *For each $i \geq 1$, the descent set of $T^{(i)}$ is $\mathtt{Des}(T) \setminus \{1, \ldots, i\}$.*

Proof. Let i be the smallest index such that the desired equality does not hold. Choose d to be the smallest descent in $T^{(i)}$ that is not in $\mathtt{Des}(T) \setminus \{1, \ldots, i\}$. Then d is not a descent in $T^{(i-1)}$. Notice $d+1$ must appear to the right or above d. If $d+1$ appears immediately to the right of d, then the sequence of slides affecting d and $d+1$ in $T^{(i-1)}$ must be



The first slide above forces $d < a$ contradicting the column strictness $a < d+1$. If $d+1$ appears above d, then it must also appear to the right of d. At least two applications of jeu de Taquin are required to slide $d+1$ below and weakly to the left of d. This is not possible. □

The next lemma describes the relative positions of two successive evacuation paths.

Lemma 4.11. *Let $\pi^{(k)}$ and $\pi^{(k+1)}$ be two evacuation paths of a standard tableau T.*

(a) Suppose there exist cells $x \in \pi^{(k)}, y \in \pi^{(k+1)}$ such that x appears immediately to the right of y. If x slides down, then y also slides down.

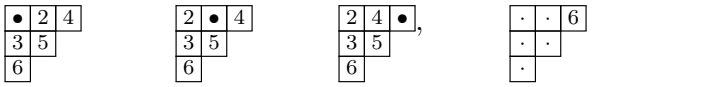
(b) Suppose there exist cells $x \in \pi^{(k)}, y \in \pi^{(k+1)}$ such that x appears immediately below y. If x slides to the right, then y also slides to the right.

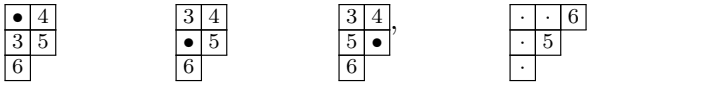
Proof. Consider the case (a). Suppose the filling below y is a, and the filling below x is b. Notice a appears to the left of b in $T^{(k-1)}$. By row strictness, we have $a < b$. Sliding x down does not affect the filling below y. Hence, to decide where y slides, we compare a and b. Then we see y slides down.

Consider the case (b). Suppose the filling to the right of y is a and the filling tot he right of x is b. Notice a appears below b in $T^{(k-1)}$. By column strictness, we have $a < b$. Sliding x to the right does not affect the filling to the right of y. Hence, to decide where y slides to, we compare a and b. Then we see y slides to the right. □

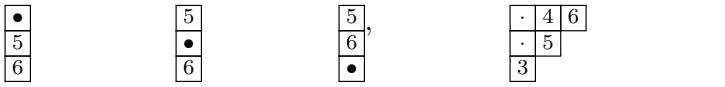
Example 4.12. *We demonstrate the consequence of Lemma 4.11. Consider the following standard tableau T and its evacuation paths. The development of* $\mathrm{evac}(T)$ *is shown in the rightmost column.*

$T =$  .

$\pi^{(1)} =$ 

$\pi^{(2)} =$ 

$\pi^{(3)} =$ 

$\pi^{(4)} =$ 

$\pi^{(5)} =$ 

$\pi^{(6)} =$  $= \mathrm{evac}(T)$

Case (a) of Lemma 4.11 is demonstrated by $\pi^{(1)}$ and $\pi^{(2)}$. Notice $i = 1$ is not a descent

in T and $n-i=5$ is not a descent in $\mathrm{evac}(T)$*. This is because the path $\pi^{(1)}$ always stays weakly above $\pi^{(2)}$, overlapping only in the first row. This is visualized in Figure 4.5.*

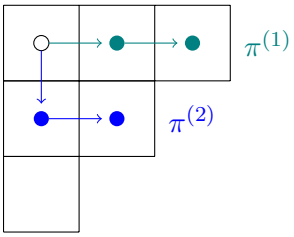


Figure 4.5: Case (a) of Lemma 4.11

Case (b) of Lemma 4.11 is demonstrated by $\pi^{(2)}$ and $\pi^{(3)}$. Notice $i=2$ is a descent in T and $n-i=4$ is not a descent in $\mathrm{evac}(T)$*. This is because the path $\pi^{(2)}$ always stays weakly above $\pi^{(1)}$, overlapping only in the first column. This is visualized in Figure 4.6.*

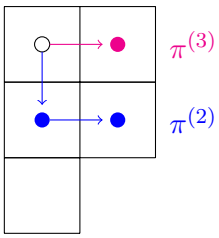


Figure 4.6: Case (b) of Lemma 4.11

Lemma 4.13. *For $i=1,\ldots,n-1$, the filling i is a descent of T if and only if $n-i$ is a descent of* $\mathrm{evac}(T)$*.*

Proof. Let i be a filling in T. The cells in which $n-i$ and $n-i+1$ appear in $\mathrm{evac}(T)$ are the last cells of $\pi^{(i+1)}$ and $\pi^{(i)}$ respectively. By Lemma 4.9, i appears in the inner corner of $T^{(i-1)}$.

Suppose i is not a descent in T. By Lemma 4.10, $i+1$ appears to the right of i in $T^{(i-1)}$. Let x be the second cell of $\pi^{(i)}$ and y be the first cell of $\pi^{(i+1)}$. If $\pi^{(i)}$ turns south at x, meaning x slides down, then we apply Lemma 4.11 (a) to force y to slide down. That means $\pi^{(i+1)}$ also turns south at y. If $\pi^{(i)}$ moves to the right after x, then $\pi^{(i+1)}$ is free to

move to the right or turn south. Either way, the cell after y in $\pi^{(i+1)}$ appears to the left, or below and to the left of the cell after x in $\pi^{(i)}$. It follows that for every cell $y' \in \pi^{(i+1)}$ either there exists a cell $x' \in \pi^{(i)}$ that appears in the same row and to the right of y', or y' is below the last cell of $\pi^{(i+1)}$. Hence, the last cell in $\pi^{(i+1)}$, which is $n-i$, appears to the left or below the last cell of $\pi^{(i)}$, which is $n-i+1$. This means $n-i$ is not a descent in T.

The case for when i is a descent has a similar argument using Lemma 4.11 (b).

Suppose i is a descent in T. By Lemma 4.10, $i+1$ appears below i in $T^{(i-1)}$. Let x be the second cell of $\pi^{(i)}$ and y be the first cell of $\pi^{(i+1)}$. If $\pi^{(i)}$ moves to the right at x, meaning x slides to the right, then we apply Lemma 4.11 (b) to force y to slide to the right. That means $\pi^{(i+1)}$ also turns right at y. If $\pi^{(i)}$ turns south at x, then $\pi^{(i+1)}$ is free to move to the right or turn south. Either way, the cell after y in $\pi^{(i+1)}$ appears above, or above and to the right of the cell after x in $\pi^{(i)}$. It follows that for every cell $y' \in \pi^{(i+1)}$ either there exists a cell $x' \in \pi^{(i)}$ that appears in the same column and above y', or y' is above the last cell of $\pi^{(i+1)}$. Hence, the last cell in $\pi^{(i+1)}$, which is $n-i$, appears above the last cell of $\pi^{(i)}$, which is $n-i+1$. This means $n-i$ is a descent in T. □

Theorem 4.7 is a straightforward consequence of Lemma 4.13.

Proof of Theorem 4.7. We have to check that for every $T \in \mathrm{SYT}(n)$, we have $\mathrm{sh}(\mathrm{evac}(T)) = \mathrm{sh}(T)$, $\mathtt{des}(T) = \mathtt{des}(\mathrm{evac}(T))$ and $\mathtt{maj}(T) = \mathtt{cc}(\mathrm{evac}(T))$.

Let $T \in \mathrm{SYT}(n)$. By Theorem 4.8, we have $\mathrm{sh}(\mathrm{evac}(T)) = \mathrm{sh}(T)$.

Suppose $\mathtt{Des}(T) = \{d_1, \ldots, d_k\}$ and assume $d_1 < \cdots < d_k$. By Lemma 4.13, we have $\mathtt{Des}(\mathrm{evac}(T)) = \{n-d_1, \ldots, n-d_k\}$ and, consequently, $\mathtt{des}(T) = \mathtt{des}(\mathrm{evac}(T))$.

We can compute α the cocharge word of $\mathrm{evac}(T)$ from $\mathtt{Des}(\mathrm{evac}(T))$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$, then by Equation (2.3) and (2.2) we have $\alpha_i = \#\{d \in \mathtt{Des}(\mathrm{evac}(T)) : d < i\}$ for each $i \in [n]$. It follows that for every $i \in [n]$,

$$\alpha_i = \#\{j \in [k] : n - d_j < i\}.$$

Because α is a weak staircase, α has an area word. By the definition of area words in Equation (3.11), we have $\mathtt{Area}(\alpha)_j = \#\{i \in [n] : \alpha_i \geq j\} = d_j$. Because $|\mathtt{Area}(\alpha)| = \alpha$, it follows that $\mathtt{maj}(T) = |\mathtt{Area}(\alpha)| = |\alpha| = \mathtt{cc}(\mathrm{evac}(T))$. □

Example 4.14. *We demonstrate the proof of Theorem 4.7. Consider the following standard Young tableau* $S \in \mathrm{SYT}(6)$ *and its image under evacuation.*

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \quad \textit{and} \quad \mathrm{evac}(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}.$$

Here we have $n = 6$. *We have* $\mathtt{Des}(T) = \{1, 3, 4\}$ *and* $d_1 = 1, d_2 = 3, d_3 = 4$.

We also see $\mathtt{Des}(\mathrm{evac}(T)) = \{2, 3, 5\} = \{n - d_3, n - d_2, n - d_1\}$. *The cocharge word of* $\mathrm{evac}(T)$ *is* $\alpha = 001223$ *and* $\mathtt{Area}(\alpha) = (4, 3, 1) = (d_3, d_2, d_1)$.

Example 4.15. *Table 4.1 demonstrates Theorem 4.7. Tableaux* $S \in \mathrm{SYT}(321)$ *such that* $\mathtt{des}(S) = 3$ *and* $\mathrm{evac}(S)$ *are listed along their cocharge and major index.*

$S \in \mathrm{SYT}(321)$ with $\mathtt{des}(S) = 3$	$\mathtt{cc}(S)$	$\mathtt{maj}(S)$	$\mathrm{evac}(S)$	$\mathtt{cc}(\mathrm{evac}(S))$	$\mathtt{maj}(\mathrm{evac}(S))$
$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$	11	7	$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$	7	11
$\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$	10	8	$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$	8	10
$\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$	10	8	$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$	8	10
$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$	9	9	$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$	9	9
$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$	8	10	$\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$	10	8
$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$	9	9	$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$	9	9
$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$	8	10	$\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$	10	8
$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$	7	11	$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$	11	7

Table 4.1: Evacuation on the subset of SYT(321) with exactly 3 descents.

Theorem 4.7 immediately allows us to replace the major index by the cocharge statistic in Equation (4.7) to obtain a new symmetric function expression for the conjecture Frobenius of SH_n.

Corollary 4.16. *We have*

$$C_{n,n-d}(q) = D_{n,n-d}(q)$$

for every $d = 0, 1, \ldots, n-1$. *In other words, the symmetric function* $D_{n,n-d}$ *is the conjectured Frobenius series of the* θ*-degree* d *component of the super harmonics* SH_n.

4.2 A conjectured basis

In Chapter 2, we recalled the construction of an $\mathcal{S}_n$ invariant basis of the classical harmonics. In this classical story, the dimension of the classical coinvariant space was already well-known before the discovery of the higher Specht basis. Robinson-Schensted correspondence provides crucial combinatorial insight into the discovery of the higher Specht basis. It relates permutations to pairs of Young tableaux of the same shape. Permutations are the underlying objects of $\mathrm{Hilb}(R_n/I_n; q)$ and the indexing objects of a monomial basis. Pairs of Young tableaux of the same shape are the underlying objects of $\mathrm{grFrob}(R_n/I_n; q)$ and the indexing objects of an $\mathcal{S}_n$ invariant basis. We prove a completely analogous result by modifying Robinson-Schensted to transform ordered set partitions, which describes the Hilbert series of SH_n, to pairs of certain descent-coloured standard Young tableaux of the same shape, which describe the conjectured Frobenius series of SH_n. The image of our new bijection provides combinatorial insight to conjecture a basis for SH_n.

In Equation (4.8) in Section 4.1, each summand $q^{\mathtt{cc}(S)-\Sigma(D)} s_{\mathrm{sh}(S)}$ can be interpreted as polarizing the classical *harmonic* irreducible component indexed by the tableau S by a super symmetric polynomial $M_D \in SR_n$ whose θ-degree is $|D|$ and x-degree is $\Sigma(D)$. Each super symmetric polynomial M_D must be harmonic as a polarization operator. However, this point of view would force us to work with harmonic polynomials. If we replace $\mathtt{cc}(S) = \binom{n}{2} - \mathtt{cc}(S')$ in Equation (4.8) and the set D by a set $A = \{a_1, \ldots, a_k\}$ whose elements are defined by $(a_1, \ldots, a_k) = \mathtt{Area}(\alpha)$, we obtain a more informative expression

$$D_{n,n-d}(q) = \sum_{S \in \mathrm{SYT}(n)} \sum_{\substack{\alpha \in \mathcal{A}_n \\ \max(\mathtt{Area}(\alpha)) \leq \mathtt{des}(S)}} q^{\binom{n}{2} - \mathtt{cc}(S') - |\alpha|} s_{\mathrm{sh}(S)}. \tag{4.9}$$

For each $\alpha \in \mathcal{A}_n$, we note $\max(\mathtt{Area}(\alpha)) \leq \mathtt{des}(S)$ if and only if the set $A = \{a_1, \ldots, a_k\}$ whose elements are defined by $(a_1, \ldots, a_k) = \mathtt{Area}(\alpha)$ satisfies $A \subseteq [\mathtt{des}(S)]$.

We can interpret the expression $\binom{n}{2} - \mathtt{cc}(S') - |\alpha|$ as follows. The degree of the Vandermonde determinant Δ_n is $\binom{n}{2}$. The degree of a higher Specht polynomial $G_{T'}^{S'}$ is $\mathtt{cc}(S')$. Then $G_{T'}^{S'} \diamond \Delta_n$ is the classical *harmonic* polynomial indexed by (S, T) and is a basis element of the irreducible component indexed by $\mathrm{sh}(S)$. If we polarize $G_{T'}^{S'} \diamond \Delta_n$ by M_α, then its degree is $\binom{n}{2} - \mathtt{cc}(S') - |\alpha|$.

Conjecture 4.17. *The set*

$$\mathcal{B}_n = \left\{ M_\alpha(G_{T'}^{S'}) \diamond \Delta_n : S \in \mathrm{SYT}(n), T \in \mathrm{SYT}(\mathrm{sh}(S)), \alpha \in \mathcal{A}_n, \max(\mathtt{Area}(\alpha)) \leq \mathtt{des}(S) \right\}$$

is a basis of the super harmonics SH_n.

See Example 4.19 for a pictorial explanation of Conjecture 4.17.

Remark 4.18. *We can express $\mathcal{B}_n$ in Conjecture 4.17 as a disjoint union*

$$\mathcal{B}_n = \bigcup_{\mu \vdash n} \bigcup_{\substack{\alpha \in \mathcal{A}_n \\ S \in \mathrm{SYT}(\mu) \\ \max(\mathtt{Area}(\alpha)) \leq \mathtt{des}(S)}} \left\{ M_\alpha \diamond (G_{T'}^{S'} \diamond \Delta_n) : T \in \mathrm{SYT}(\mu) \right\},$$

where the innermost set is a basis for a single irreducible component of shape μ.

The connection to between Conjecture (4.9) *and Equation* (4.9) *is reflected in the degree of each polynomial in $\mathcal{B}_n$ and the shape of each irreducible component. The degree of $M_\alpha \diamond (G_{T'}^{S'} \diamond \Delta_n)$ is exactly $\binom{n}{2} - cc(S') - |\alpha|$ and is an element of an irreducible component of shape* $\mathrm{sh}(S)$.

Unfortunately, the author is unable to predict an expression for the intersection of cones of the alternating polynomials $M_\alpha \diamond \Delta_n$.

Example 4.19. *The conditions on α and S in $\mathcal{B}_n$ of Conjecture 4.17 select, for each M_α, certain irreducible components of H_n for polarization by M_α. Figure 4.7 illustrates this selection using SH_3. For each pair α, S satisfying the conditions in $\mathcal{B}_3$, we draw the tableau S and an arrow with label M_α from the classical harmonics H_3.*

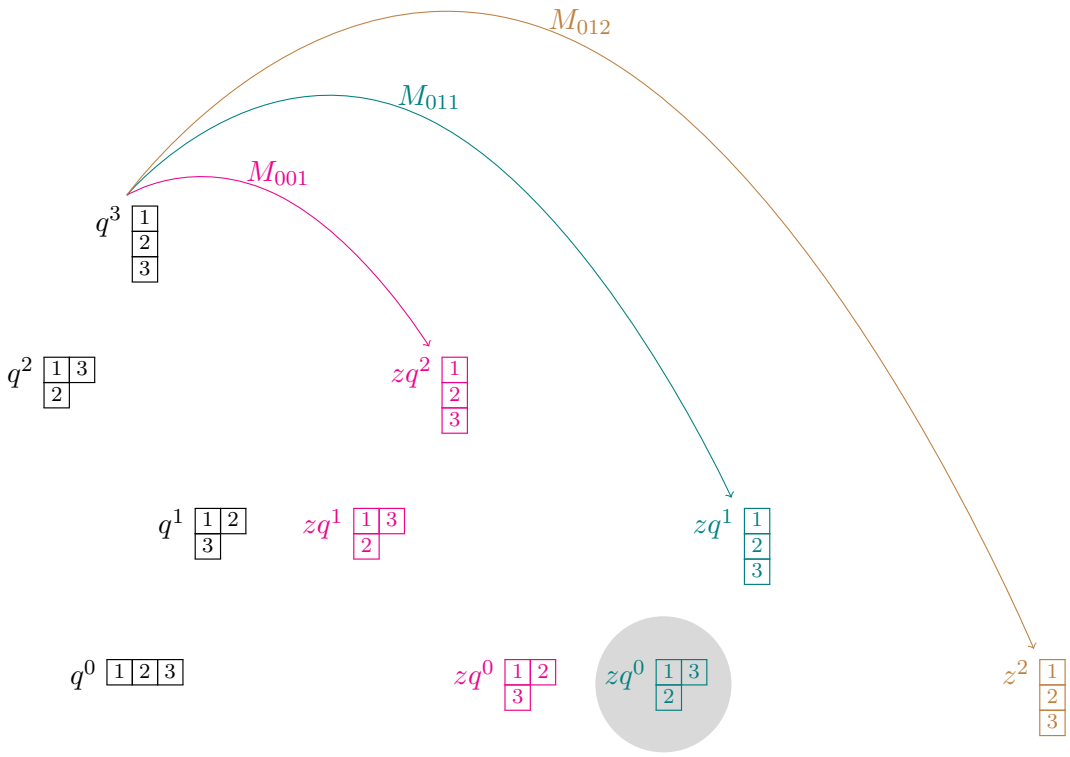


Figure 4.7: A pictorial description of $\mathcal{B}_3$.

Each tableau, except the one enclosed in the grey circle, represents a distinct irreducible component appearing in the decomposition of SH_3. The classical harmonic H_3 is represented by the leftmost four tableaux (drawn in black). Each arrow represents the polarization of H_3 by some M_α. The monomial that appears to the left of a tableau indicates the degrees at which it appears in SH_3.

For example, the magenta arrow is labelled by M_{001}. Therefore, the magenta coloured tableau zq^1 $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ represents the set of polynomials

$$\left\{(M_{001}G^{S'}_{T'}) \diamond \Delta_3 : T \in \mathrm{SYT}\left(\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}\right)\right\} \quad \textit{where} \quad S = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

This set of polynomials is a conjectured basis for an irreducible representation of shape $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$ *in SH_3 at θ-degree 1 and x-degree 1.*

If a tableau does not appear in Figure 4.7, then its corresponding polynomials are 0*. For example,* $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ *does not appear because, by degree consideration, its corresponding polynomials* $(M_{011}G_{T'}^{S'})\diamond\Delta_3$ *are* 0 *for all* $T\in\mathrm{SYT}(21)$ *where* $S=\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$.

Recall the definition of a cone in Equation (2.20) *on page 32. Tableaux of the same colour represent a basis of the cone under the (unique up to scaling) alternating polynomial of the same colour. For example, the two teal tableaux represent a basis of the cone under* $M_{011}\diamond\Delta_3$.

Computation shows cones can overlap. In particular, the two components represented by zq^0 $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ *and* zq^0 $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ *overlap. Notice the two tableaux are different. More explicitly, we have*

$$\mathbb{Q}\left\{(M_{001}G_{T'}^{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,'})\diamond\Delta_3 : T\in\mathrm{SYT}\left(\begin{array}{|c|c|}\hline \\\hline \\\cline{1-1}\end{array}\right)\right\}=\mathbb{Q}\left\{(M_{011}G_{T'}^{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,'})\diamond\Delta_3 : T\in\mathrm{SYT}\left(\begin{array}{|c|c|}\hline \\\hline \\\cline{1-1}\end{array}\right)\right\}$$

Therefore, we only need to include one of zq^0 $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ *and* zq^0 $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ *in* $\mathcal{B}_3$*. The conditions on* α, S *in* $\mathcal{B}_3$ *avoid overlaps by selecting only one of the two descriptions. Because the number of descents of* $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ *is* 1 *and* $\mathtt{Area}(011)=(2)$*, the irreducible component* $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ *in* H_3 *is excluded from the polarization by* M_{011}*. In other words, the grey circle says we do not need to polarize the irreducible component* $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ *by* M_{011}.

Before introducing the next result, we need to recall the definition of ordered set partitions, also known as set composition. Let $n\in\mathbb{N}$. An ordered set partition of $[n]$ is a tuple $(B_1,\ldots,B_k)$ such that $B_1,\ldots,B_k$ are non-empty disjoint subsets of $[n]$ and $B_1\cup\cdots\cup B_k=[n]$. Each entry B_i is called a block. The set of ordered set partitions with k blocks is denoted by $\mathrm{OSP}(n,k)$. Ordered set partitions are closely related to *pack words* which we define now. A word w of length n on the alphabet $\mathbb{N}$ is a *packed word* if there exists a k such that $1,\ldots,k$ all appear in w. For example, if $n=4$, then $w=1123$ is a packed word, but $w=1433$ is not. There exists a natural bijection between $\mathrm{OSP}(n,k)$ and packed words of length n whose largest entry is k. Let $(B_1,\ldots,B_k)\in\mathrm{OSP}(n,k)$. List

Conjecture 4.17 predicts that triples $(\mathtt{Area}(\alpha),S,T)$ satisfying $\max(\mathtt{Area}(\alpha))\leq\mathtt{des}(S)$ where S,T are standard Young tableaux of the same shape and $\alpha\in\mathcal{A}_n$, count the dimension

of SH_n. The results by Rhoades and Wilson [56] imply that the dimension is counted by ordered set partitions. One would guess that there should be a natural bijection between these two sets of objects. Our next result is the description and the proof of this bijection.

The condition $\max(\mathtt{Area}(\alpha)) \leq \mathtt{des}(S)$ is one of the constraints in $\mathcal{B}_n$. Notice if $\alpha \in \mathcal{A}_n$ such that $\mathtt{Area}(\alpha) = (a_1, \ldots, a_k)$, then

$$\max(\mathtt{Area}(\alpha)) \leq \mathtt{des}(S) \quad \text{if and only if} \quad \{a_1, \ldots, a_k\} \subseteq [\mathtt{des}(S)]. \tag{4.10}$$

If there exists a bijection between ordered set partitions and the conjectured basis $\mathcal{B}_n$, then $\mathtt{Area}(\alpha)$ should be interpreted as decorations on descents of S.

For each partition $\mu \vdash n$, define $\mathrm{SYT}_{\mathtt{des}}(\mu, k)$ to be a set of tuples (I, P) such that $P \in \mathrm{SYT}(\mu)$ and $I = (i_1, \ldots, i_k)$ such that entries of I are all distinct and $\max(I) \leq \mathtt{des}(P)$. In other words, we can treat I as a k-subset of $[\mathtt{des}(P)]$. The reason for this rather strange description is that I's will be the area words, which is a partition, of a weak staircase in the place of I. Entries of I are *not* descents of P. Entries of I are chosen from $[\mathtt{des}(P)]$ and *index* the descents of P. We call $\mathrm{SYT}_{\mathtt{des}}(\mu, k)$ *descent-coloured standard Young tableaux*. Pictorially, we draw (I, P) by colouring the cells of P selected by I.

Notice $\mathrm{SYT}_{\mathtt{des}}(\mu, 0)$ is trivially in bijection with $\mathrm{SYT}(\mu)$ and $\mathrm{SYT}_{\mathtt{des}}(\mu, n)$ is empty.

We also define

$$\mathrm{SYT}_{\mathtt{des}}(n, k) = \bigoplus_{\mu \vdash n} \mathrm{SYT}_{\mathtt{des}}(\mu, k) \quad \text{and} \quad \mathrm{SYT}_{\mathtt{des}}(n) = \bigoplus_{k=0}^{n-1} \mathrm{SYT}_{\mathtt{des}}(n, k).$$

Example 4.20. *Let P be the standard tableaux drawn below on the left. Its descents set is $\mathtt{Des}(P) = \{1, 3, 7, 8\}$. When $I = (2, 4)$, the selected descents are 3 and 8. Then $((2, 4), P)$ is drawn below on the left.*

$$P = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 7 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 9 \\ \cline{1-1} \end{array} \qquad ((2,4), P) = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 7 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 9 \\ \cline{1-1} \end{array}$$

Recall $\mathrm{OSP}(n, k)$ denotes the set of ordered set partitions of $[n]$ with k parts. We remark that ordered set partitions are also called set compositions. We follow the terminology from [37] to call them ordered set partitions.

Let $1 \leq k \leq n$. The map

$$\Phi_k : \mathrm{OSP}(n,k) \to \bigcup_{\mu \vdash n} \mathrm{SYT}_{\mathtt{des}}(\mu, n-k) \times \mathrm{SYT}(\mu)$$

is defined by conjugating and standardizing a pair of Young tableaux obtained from an ordered set partition and colouring appropriate descents. We now describe the details.

Take an ordered set partition $\pi \in \mathrm{OSP}(n,k)$ and encode it as a packed word w. Apply Robinson-Schensted-Knuth algorithm to w to obtain a pair of tableaux (P_0, Q_0) of the same shape where P_0 is semistandard and Q_0 is standard. Conjugate both tableaux to set $P = P_0'$ and $Q = Q_0'$. To obtain the coloured cells, standardize P and record appropriate values. We describe this process in detail.

Because P_0 is semistandard, cells of P with the same value form vertical strips. We call such strips *constant-value vertical strips*. We will turn each such vertical strip into a contiguous block of values in a unique way and identify a subset of descents in the process. We leave Q unchanged.

List cells $\square_1, \ldots, \square_n$ of P by reading its values from the smallest to the largest, breaking the tie by reading each constant-value vertical strip from the lowest indexed row to the highest indexed row.

Start with the last cell in this order and set a pointer $i = n$. Initialize $D = \emptyset$.

- WHILE $i > 1$.
  - ($\bigstar$) If $P(\square_{i-1}) = P(\square_i)$, then place $\square_{i-1}$ into the set D and increment $P(\square_j)$ by 1 for every $j \geq i$.
    - Otherwise, do nothing.
- Decrement i by 1.
- REPEAT.

When the while-loop terminates, it is clear that P is standard. Let d_j be elements of $\mathtt{Des}(P)$ such that $\{d_1 < \cdots < d_m\}$. Define

$$I = (i : d_i = P(\square) \text{ for some } \square \in D).$$

Assign $\Phi_k(w) = ((I, P), Q)$.

Theorem 4.21. *The map Φ_k is well-defined and is a bijection.*

Proof. In step ($\bigstar$), $\square_{i-1}$ and $\square_i$ must live in the same constant-value vertical strip. Then by definition $\square_{i-1}$ appears in a lower indexed row than $\square_i$ and $P(\square_{i-1})+1 = P(\square_i)$. Hence $P(\square_{i-1})$ is a descent of P. It follows that I is a subset of $[\mathtt{des}(P)]$. If the largest symbol in w is k, then the largest value in P_0 is k and P_0' has k distinct constant-value vertical strips. For each such vertical strip, Step ($\bigstar$) is executed on all but the lowest indexed cell. Hence D has $n-k$ elements. Therefore, Φ_k is well-defined. It is clear from construction that Φ_k is injective.

We now describe Φ^{-1}. Let P, Q be standard tableaux of the same shape and choose $I = (i_1, \ldots, i_{n-k})$ such that $1 \leq i_1 < \cdots < i_{n-k} \leq \mathtt{des}(P)$. List cells $\square_1, \ldots, \square_n$ of P in increasing order of its values, i.e., $\square_i = P^{-1}(i)$. Conjugate both tableaux to set $P_0 = P'$ and $Q_0 = Q'$. We leave Q_0 unchanged.

Start with the first cell and set a pointer $i = 1$. Let d_j be elements of $\mathtt{Des}(P)$ such that $\mathtt{Des}(P) = \{d_1 < \cdots < d_m\}$ and set $D = \{d_j : j \in I\}$.

- WHILE $i < n$.

  ($\bigstar'$) If $i \in D$, then decrease $P(\square_j)$ by 1 for every $j > i$.

  - Otherwise, do nothing.
  - Increment i by 1.

- END WHILE.

Apply the inverse of Robinson-Schensted to (P_0, Q_0) to obtain a packed word which bijectively corresponds to an ordered set partition π.

Assign $\Phi_k^{-1}((I, P), Q) = \pi$. It is clear that Φ_k^{-1} is well-defined and injective. $\square$

Example 4.22. *The following example illustrates the bijection.*

Suppose $P = \begin{array}{|c|c|c|}\hline 1&3&8\\\hline 2&6\\\cline{1-2} 4&7\\\cline{1-2} 5\\\cline{1-1}\end{array}$ *and* $I = \{1,3,4\}$. *It follows that*

$$(I,P) = \begin{array}{|c|c|c|}\hline \cellcolor{green}1&3&8\\\hline 2&\cellcolor{green}6\\\cline{1-2} \cellcolor{green}4&7\\\cline{1-2} 5\\\cline{1-1}\end{array}\ .$$

Apply $\Phi_3^{-1}((I,P),Q)$ *to get*

$$(\ \begin{array}{|c|c|c|}\hline \cellcolor{green}1&3&8\\\hline 2&\cellcolor{green}6\\\cline{1-2} \cellcolor{green}4&7\\\cline{1-2} 5\\\cline{1-1}\end{array}\ ,Q) \mapsto (\ \begin{array}{|c|c|c|}\hline 1&2&7\\\hline 1&\cellcolor{green}5\\\cline{1-2} 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array}\ ,Q) \mapsto (\ \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 1&\cellcolor{green}4\\\cline{1-2} 3&5\\\cline{1-2} 3\\\cline{1-1}\end{array}\ ,Q) \mapsto (\ \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 1&4\\\cline{1-2} 3&4\\\cline{1-2} 3\\\cline{1-1}\end{array}\ ,Q) \mapsto (\ \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&4&4\\\cline{1-3} 5\\\cline{1-1}\end{array}\ ,Q').$$

Apply RSK inverse to the right-most pair of tableaux to get a packed word. We can then obtain an ordered set partition from the packed word.

Recall from Chapter 2 that the higher Specht polynomials, denoted by $G_T^S(x_1,\ldots,x_n)$, where S,T run overall standard tableaux of n of the same shape form a basis of the classical coinvariant R_n/I_n [3]. For every $S \in \mathrm{SYT}(n)$, denote the higher Specht basis of the irreducible representation indexed by S by

$$G^S = \left\{G_T^S : T \in \mathrm{SYT}(\mathrm{sh}(S))\right\}.$$

Using the flip map defined just before Theorem 2.20, we immediately get the corresponding irreducible representation on the harmonic side by differentiating against the Vandermonde determinant Δ_n. For every $S \in \mathrm{SYT}(n)$, denote the harmonic higher Specht basis of the irreducible representation indexed by S by

$$\mathrm{flip}(G^{S'}) = \left\{G_{T'}^{S'} \diamond \Delta_n : T \in \mathrm{SYT}(\mathrm{sh}(S))\right\}.$$

The conjugatation is necessary becasue $\deg(f \diamond \Delta_n) = \binom{n}{2} - \deg(f)$ for any homogeneous polynonial $f \in R_n$ and $f \diamond \Delta_n \neq 0$. Since $\mathtt{cc}(S') = \binom{n}{2} - \mathtt{cc}(S)$, we have that both G^S and $\mathrm{flip}(G^{S'})$ have degree $\mathtt{cc}(S)$. Moreover, the flip map conjugates the shape of the irreducible representation. So it is necessary to flip $G^{S'}$ to obtain the harmonic side of G^S.

Suppose $\alpha \in \mathcal{A}_n$ is a weak staircase such that $(\texttt{Area}(\alpha), S)$ is a descent-coloured standard Young tableau of shape μ. Define

$$M_\alpha G^{S'} = \left\{ M_\alpha G^{S'}_{T'} : T \in \mathrm{SYT}(\mu) \right\}$$

and

$$(M_\alpha G^{S'}) \diamond \Delta_n = \mathbb{Q}\{ f \diamond \Delta_n : f \in M_\alpha G^{S'} \}.$$

Another way to read $(M_\alpha G^{S'}) \diamond \Delta_n$ is that we polarize $\mathrm{flip}(G^{S'})$ by M_α.

We can read off from Conjecture 4.17 the full decomposition of SH_n into irreducible representations.

Corollary 4.23. *Suppose Conjecture 4.17 holds. Then*

$$SH_n = \bigoplus_{\substack{\alpha \in \mathcal{A}_n \\ (\texttt{Area}(\alpha), S) \in \mathrm{SYT}_{\texttt{des}}(n)}} (M_\alpha G^{S'}) \diamond \Delta_n.$$

Moreover, each $M_\alpha G^{S'} \diamond \Delta_n$ is an irreducible representation of $\mathcal{S}_n$.

Proof. If Conjecture 4.17 holds, then for each descent-coloured tableau $(\texttt{Area}(\alpha), S)$, the classical irreducible $G^{S'}$ is not in the kernel of M_α. By Schur's Lemma, $M_\alpha G^{S'}$ is an irreducible representation in superspace. □

Example 4.24. *We list the complete set of combinatorial objects indexing the irreducible representation of SH_3.*

θ-degree 0 and x-degree 0:

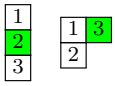
θ-degree 0 and x-degree 1:

θ-degree 0 and x-degree 2:

θ-degree 0 and x-degree 3:

θ-degree 1 and x-degree 0:

θ-degree 1 and x-degree 1:



θ-degree 1 and x-degree 2:

θ-degree 2 and x-degree 0:

There are no other irreducible representations in SH_3.

Lastly, we verify the following.

Lemma 4.25. *The polynomials in Conjecture 4.17 are all harmonic.*

Proof. Let $f \in SI_n$ and let $(M_\alpha G_{T'}^{S'}) \diamond \Delta_n$ be a conjectured basis element. Then $f G_{T'}^{S'} \in SI_n$. Because $M_\alpha \diamond \Delta_n$ is harmonic, we have

$$f(\partial) \cdot (M_\alpha G_{T'}^{S'}) \diamond = f(\partial) G_{T'}^{S'}(\partial) \cdot M_\alpha \diamond \Delta_n = 0.$$

□

4.3 Towards a proof

Conjecture 4.17 is supported by computational data up to SH_6. Thanks to the dimensional argument of Rhoades and Wilson [56], it suffices to check that the proposed basis is linearly independent. We used SageMath [77] in our experiments. In this section, we discuss strategies and partial results towards a complete proof of Conjecture 4.17.

The polynomials $(G_{T'}^{S'}M_\alpha)\diamond\Delta_n$ in Conjecture 4.17 are suitable for computer exploration because they can be represented by a coefficient matrix whose rank can be checked by a computer algebra system. Rank computations are done through Gaussian elimination. However, Gaussian elimination does not preserve the combinatorial properties of these polynomials. Computational data does not directly suggest a leading term argument.

We look for ways to boost results and generalize methods from the well-understood classical coinvariant spaces R_n/I_n. The flip map in the classical coinvariant theory transfers results between the coinvariant side and the harmonics side.

Observation 4.26. *Conjecture 4.17 holds if and only if*

$$\left\{M_\alpha G_{T'}^{S'} : \alpha \in \mathcal{A}_n, (\mathtt{Area}(\alpha), S) \in \mathrm{SYT}_{\mathtt{des}}(n), T \in \mathrm{SYT}(\mathrm{sh}(S))\right\}$$

is linearly independent in $SR_n/\mathcal{I}_n$.

We recall a basic linear algebra result. Let U, V be general vector spaces over $\mathbb{Q}$. Suppose $\langle\cdot,\cdot\rangle : U \times V \to \mathbb{Q}$ is a bilinear map. Let $u_1, \ldots, u_m \in U$ and $v_1, \ldots, v_m \in V$ vectors. The matrix $[\langle u_i, v_j\rangle]_{i,j\in[m]}$ is called the *Gram matrix* associated to the bilinear form for $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$. Its invertibility is related to the linear independence of both u, v as stated in the following lemma.

Lemma 4.27. *Let $\langle\cdot,\cdot\rangle$ be a non-degenerate pairing between vector spaces U and V. If the Gram matrix*

$$[\langle u_i, v_j\rangle]_{i,j\in[m]}$$

is invertible, then $u_1, \ldots, u_m$ are linearly independent in U and $v_1, \ldots, v_m$ are linear independent in V.

To prove that the higher Specht polynomials are linearly independent in the classical quotient R_n/I_n, Ariki, Terasoma, and Yamada developed a bilinear form

$$\langle f, g\rangle_0 = \frac{1}{\Delta_n}\mathrm{alt}_n\left(fg\right)\Big|_{x_1=\cdots=x_n=0},$$

in the ambient space R_n, where the evaluation $x_1 = \cdots = x_n = 0$ is the last operation. The bilinear form $\langle f, g\rangle_0$ computes the signed sum of coefficients of $x^{\sigma(\delta_n)}$ in the product fg summed over every permutation $\sigma \in \mathcal{S}_n$, i.e.,

$$\langle f, g\rangle_0 = \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)[x^{\sigma(\delta_n)}]fg.$$

Ariki, Terasoma, and Yamada showed that the matrix

$$\left[\langle G_{T_j'}^{S_i'}, G_{T_j}^{S_j}\rangle\right]_{i,j=1,\ldots,n!}$$

is upper triangular, where (S_i, T_i) are pairs of standard Young tableaux of the same shape and ordered by the dictionary order induced by the last letter order on standard Young tableaux.

We can interpret $\langle\cdot,\cdot\rangle_0$ as a way to compute a certain *complement* within the cone of the top-degree component. In the classical coinvariant theory, the top-degree component is the unique 1-dimensional alternating component. This is not the case in super harmonics. However, we can take the unique top-degree 1-dimensional component spanned by the super *symmetric* polynomial $\theta_1\cdots\theta_n\Delta_n$.

On the space $SR_n/\mathcal{I}_n$, define the *super Ariki-Terasoma-Yamada bilinear form*, analogous to that of $\langle -, -\rangle_0$, to be

$$\langle f, g\rangle = \frac{\partial\theta_n\cdots\partial\theta_1}{\Delta_n}\mathrm{sym}_n\left(fg\right)\Big|_{x_1=\cdots=x_n=0},$$

where the evaluation $x_1 = \cdots = x_n = 0$ is the last operation. The partial differentiation is just a fancy way of taking the coefficient of $\theta_1\cdots\theta_n$. This bilinear form can also be expressed as a signed sum

$$\langle f, g\rangle = \sum_{\sigma\in\mathcal{S}_n}\sum_{A\subseteq[n]} \mathrm{sgn}(\sigma)\mathrm{sgn}(A)[\theta_A x^{\sigma(\delta_n)}]fg,$$

where $\mathrm{sgn}(A) = \partial\theta_n \cdots \partial\theta_1 \cdot \theta_A \theta_{[n]\setminus A}$.

Because the bilinear form is defined over a quotient ring, we need to show that it does not depend on the choice of coset representatives.

Lemma 4.28. *The bilinear form $\langle -, - \rangle$ is well-defined on $SR_n/\mathcal{I}_n$.*

Proof. Elements in $\mathcal{I}_n$ are linear combinations of ge_λ for some $g \in SR_n$ and some non-empty partition λ. Hence, it suffices to check $\langle f, ge_\lambda \rangle = 0$ for all $f \in SR_n$. By definition, we have

$$\langle f, ge_\lambda \rangle = \frac{\partial\theta_n \cdots \partial\theta_1}{\Delta_n} \mathrm{sym}(fge_\lambda)\Big|_{x_1=\cdots=0} = \frac{\partial\theta_n \cdots \partial\theta_1}{\Delta_n} e_\lambda \mathrm{sym}(fg)\Big|_{x_1=\cdots=0}.$$

The last expression is 0 since $e_\lambda(0, \ldots, 0) = 0$ when λ is not empty. □

Lemma 4.29. *The bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate on $SR_n/\mathcal{I}_n$.*

Proof. It suffices to show that the Gram matrix of $\langle \cdot, \cdot \rangle$ with respect to a known basis of $SR_n/\mathcal{I}_n$ is invertible. Define a bilinear form on $\mathbb{Q}[\theta_1, \ldots, \theta_n]$ by $\langle \theta_A, \theta_B \rangle_1 = \partial\theta_n \cdots \partial\theta_1 \cdot \theta_A\theta_B$. It is clear that $\langle \theta_A, \theta_B \rangle_1 = \pm 1$ if and only if $A = [n] \setminus B$. It follows that for any monomials, we have a factorization

$$\langle x^\alpha \theta_A, x^\beta \theta_B \rangle = \langle x^\alpha, x^\beta \rangle_0 \langle \theta_A, \theta_B \rangle_1.$$

List substaircase monomials (in any order) as $x^{\alpha_1}, \ldots, x^{\alpha_{n!}}$ and list subsets of $[n]$ (in any order) as $A_1, \ldots, A_{2^n}$. Then $x^{\alpha_i}\theta_{A_s}$ where $i \in [n!]$ and $s \in [2^n]$ is a basis of $SR_n/\mathcal{I}_n$. Then the Gram matrix of $\langle \cdot, \cdot \rangle$ factors as

$$\left[\langle x^{\alpha_i}\theta_{A_s}, x^{\alpha_j}\theta_{A_t} \rangle\right]_{i,j\in[n!],s,t\in[2^n]} = \left[\langle x^{\alpha_i}, x^{\alpha_j} \rangle_0\right]_{i,j\in[n!]} \otimes \left[\langle \theta_{A_s}, \theta_{A_t} \rangle_1\right]_{s,t\in[2^n]}.$$

The Gram matrix on the left-hand side is invertible since both matrices on the right-hand side are invertible. □

We now suggest a pairing for the conjectured basis in $SR_n/\mathcal{I}_n$. From the definition $\langle \cdot, \cdot \rangle$, if $\langle f, g \rangle \neq 0$, then f, g must be degree complement to each other, i.e.,

$$\deg_x(f) + \deg_x(g) = \binom{n}{2} \quad \text{and} \quad \deg_\theta(f) + \deg_\theta(g) = n.$$

This condition suggests a natural complement for $M_\alpha G^{S'}_{T'}$.

Let $f \in SR_n/\mathcal{I}_n$ whose θ_A coefficient is written as $f_A = f_A(x_1, \ldots, x_n)$. The *complement* of f is defined to be

$$f^c = \sum_{A \subseteq [n]} \operatorname{sgn}(A) f_A \theta_{[n]\setminus A}.$$

Computational exploration suggests this remarkably naive idea is the right idea.

Conjecture 4.30. *The following matrix is invertible*

$$M_n = \left[\left\langle M_\alpha G^{S'}_{T'}, ((M_\beta G^P_Q) \diamond \Delta_n)^c \right\rangle \right]$$

where the indices run over all triples $((\mathtt{Area}(\alpha), S), T)$ *such that* $\alpha \in \mathcal{A}_n$ *and* $(\mathtt{Area}(\alpha), S)$ *is a coloured standard Young tableau and* $T \in \mathrm{SYT}(\mathrm{sh}(S))$*, and similarly for* $((\mathtt{Area}(\beta), P), Q)$*.*

Corollary 4.31. *If Conjecture 4.30 holds, then Conjecture 4.17 holds.*

In our attempt to understand the matrix M_n in Conjecture 4.30, we run into a roadblock. The Ariki-Terasoma-Yamada bilinear form satisfies a nice property that if $\sigma \in \mathfrak{S}_n$ and $f, g \in R_n$, then

$$\langle \sigma f, g \rangle_0 = \langle f, \operatorname{sgn}(\sigma) \sigma^{-1} g \rangle_0. \tag{4.11}$$

This property implies $\langle \varepsilon_T f, g \rangle = \langle f, \varepsilon_{T'} g \rangle$. The computation for higher Specht polynomial simplifies to considering only a row symmetrizer as follows

$$\langle \varepsilon_T x^\alpha, \varepsilon_{T'} x^\beta \rangle = \langle x^\alpha, \varepsilon_{T'} x^\beta \rangle = \langle P(T) x^\alpha, P(T) x^\beta \rangle.$$

However Equation 4.11 does not hold for $\langle \cdot, \cdot \rangle$. We have instead

$$\langle \sigma f, g \rangle = \langle f, \sigma g \rangle.$$

The matrix M_n in Conjecture 4.30 is not triangular in general. However, when restricted to hook shape, there seems to be another pairing for $M_\alpha G^{S'}_{T'}$ whose resulting matrix is block triangular with invertible blocks down the diagonal. We describe this conjectural construction in the remaining part of this section.

Let $(S, \mathtt{Area}(\alpha))$ be a descent coloured tableau where $\mathrm{sh}(S) = (1+a, 1^b)$ is a hook. Because $i \in \mathtt{Des}(S)$ if and only if $i+1$ is in the leg of S, we have $\mathtt{des}(S) = b$. Let $\mathtt{Area}(\alpha) = (a_1, \ldots, a_k)$. We have $\{a_1, \ldots, a_k\} \subseteq [\mathtt{des}(S)] = [b]$. So $\alpha_{n-b-1} = 0$ by the definition of conjugation of a weak staircase. Let $\alpha^* = (\alpha_{n-b-1}, \ldots, \alpha_n)$ be the last $b+1$ entries of α.

Let C be the cocharge tableau of S. The definition of a cocharge tableau together with $\mathtt{des}(S) = b$ imply that the first column of C is $(0, 1, \ldots, b)$. Define a new vector c with entries in $\{0, 1, \overline{1}, 2, \overline{2}, \ldots\}$ in two steps. First, let c be the coordinate-wise difference $c = (0, 1, \ldots, b) - \alpha^*$. By the preceding paragraph, entries of c are non-negative integers. Next, if i is an ascent in α^*, then replace c_i by $\overline{c_i}$. Finally, let the index tableau $i(S, \alpha)$ be obtained by replacing the first column of C by c.

Let $T \in \mathrm{SYT}(\mathrm{sh}(S))$. Define

$$x_T^{i(S,\alpha)} = \prod_{\square \in \mathrm{sh}(S)} z_{T(\square)}^{i(S,\alpha)(\square)},$$

where $z_i^j = x_i^j$ if j is unbarred and $x_i^j = \theta_i x_i^j$ if j is barred. The *super Higher Specht polynomial associated with a hook-shaped descent coloured tableau* is defined to be

$$G_T^{(S,\alpha)} = \varepsilon_T x_T^{i(S,\alpha)}.$$

Example 4.32. *Let S, T be the following tableaux and let $\mathtt{Area}(\alpha) = \{2, 3\}$. Taking the inverse of $\mathtt{Area}$, we get $\alpha = (0, 0, 0, 0, 1, 2, 2)$.*

$$S = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \quad \textit{and} \quad T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$$

Because $\mathtt{Des}(S) = \{2, 3, 4, 6, 7\}$, the tuple $(\mathtt{Area}(\alpha), S)$ is a descent coloured tableau. Then $\alpha^* = (0, 0, 0, 1, 2, 2)$. The cocharge tableau of S is

$$C = \begin{array}{|c|c|c|}\hline 0 & 0 & 3 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} .$$

The vector c is the coordinate-wise difference of the first column of C and α^* with certain barred entries. We get $c = (0, 1, 2, \bar{2}, \bar{2}, 3)$. Replace the first column of C by c to get the index tableau

$$i(S,\alpha) = \begin{array}{|c|c|c|}\hline 0 & 0 & 3 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \quad .$$

Lastly, we get $x_T^{i(S,\alpha)} = x_1^0 x_2^0 x_3^3 x_4^1 x_5^2 \theta_6 x_6^2 \theta_7 x_7^2 x_8^3$.

Computational evidence suggests the following conjecture.

Conjecture 4.33. *For each hook shaped partition $\mu \vdash n$ and $k \in [n-1]$, define a matrix*

$$M_\mu = \left[\langle M_\alpha G_{T'}^{S'}, G_Q^{\langle P,\beta \rangle} \rangle \right]_{(S,T),(P,Q)},$$

where $(S,T),(P,Q) \in \mathrm{SYT}(\mu)^2$ and $(\mathtt{Area}(\alpha), S)$ and $(\mathtt{Area}(\beta), P)$ are both descent coloured tableaux in $\mathrm{SYT}_{\mathtt{des}}(\mu, k)$. Then M_μ is block triangular with invertible blocks down the diagonal.

Corollary 4.34. *If Conjecture 4.33 is true, then Conjecture 4.17 is true for hook-shaped irreducible components. More precisely,*

$$\mathcal{B}_n^{hooks} = \{ (M_\alpha G_{T'}^{S'}) \diamond \Delta_n \in B_n : \mathrm{sh}(S) \text{ is a hook } \}$$

is a basis for the hook-shaped components of SH_n.

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