Second Order Regularity for the $p(x)$-Laplace Operator

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Abstract

In this paper, we extend some second order regularity results to the $p(x)$-Laplace operator, known when the function $p(.)$ is equal to some constant $p > 1$.

Key words : $p(x)$-Laplace operator, Variable Exponent Sobolev spaces, Second Derivatives.

Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$. For $\kappa \in [0,1]$ we consider the family of problems :

$$(P_\kappa)\begin{cases} u \in W^{1,p(\cdot)}(\Omega) \\ div\Big((\kappa + |\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u\Big) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^\infty(\Omega)$ and $\varphi \in W^{1,p(\cdot)}(\Omega)$.
p is a measurable real valued function defined on Ω and satisfying for some positive numbers p_- and p_+

$$1 < p_- = \inf_\Omega p(x) \leqslant p(x) \leqslant p_+ = \sup_\Omega p(x) \qquad \text{a.e. } x \in \Omega. \tag{0.1}$$

We recall some definitions of Lebesgue and Sobolev spaces with variable exponents (see for example [2], [3]) :

$$L^{p(\cdot)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \ \text{ measurable } / \ \rho(u) = \int_\Omega |u(x)|^{p(x)} < +\infty \ \Big\}$$

equipped with the Luxembourg norm

$$\|u\|_{p(.)} = \inf\left\{ \lambda > 0 / \rho\left(\left|\frac{u(x)}{\lambda}\right|\right) \leqslant 1 \right\}.$$

$\left(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(.)}\right)$ is a separable and reflexive Banach space. Moreover we have

- *i)* $\|u\|_{p(.)} \leqslant 1 \iff \rho(u) \leqslant 1.$
- *ii)* $\min\left(\|u\|_{p(.)}^{p_-}, \|u\|_{p(.)}^{p_+}\right) \leqslant \rho(u) \leqslant \max\left(\|u\|_{p(.)}^{p_-}, \|u\|_{p(.)}^{p_+}\right).$
- *iii)* If $q(x) = \dfrac{p(x)}{p(x)-1}$, we have the Hölder inequality

$$\forall u \in L^{p(\cdot)}(\Omega), \quad \forall v \in L^{q(\cdot)}(\Omega) \quad \left|\int_\Omega u(x)v(x)dx\right| \leqslant 2\|u\|_{p(.)}\|v\|_{q(.)}.$$

We consider now the variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \ / \ \nabla u \in \left(L^{p(\cdot)}(\Omega)\right)^n \right\}$$

equipped with the following norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}, \qquad \|\nabla u\|_{p(.)} = \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{p(.)}.$$

$\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.
The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Assume that p satisfies for some $L > 0$

$$-|p(x) - p(y)| \log|x-y| \leqslant L \qquad \forall x, y \in \overline{\Omega}. \tag{0.2}$$

Then we have (see [2])

- *iv)* $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$.
- *v)* $W_0^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.
- *vi)* $\forall u \in W_0^{1,p(\cdot)}(\Omega) \qquad \|u\|_{p(.)} \leqslant C\|\nabla u\|_{p(.)}$ (Poincaré's inequality).

By a solution of (P_κ) we shall mean any function $u \in W^{1,p(\cdot)}(\Omega)$ satisfying

$$\begin{cases} u - \varphi \in W_0^{1,p(\cdot)}(\Omega), \\ \displaystyle\int_\Omega (\kappa + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u.\nabla\zeta dx = \int_\Omega f\zeta dx \quad \forall \zeta \in W_0^{1,p(\cdot)}(\Omega). \end{cases}$$

In this paper we first prove $W^{2,2}_{loc}$ regularity for the solution of (P_κ) when $\kappa > 0$. We do this using $C^{1,\alpha}_{loc}$ estimate and the difference quotients Δ_h as in [4], Theorem 8.1 p267. To consider the singular or the degenerate operator corresponding to $\kappa = 0$, we establish suitable estimates independently on κ. Then we pass to the limit as κ goes to zero and show that $D(|\nabla u|^{\frac{p(.)-2}{2}}\nabla u) \in L^2_{loc}(\Omega)$.

1 L^∞ and $C^{1,\alpha}$ estimates

First we show that u is uniformly bounded independently of κ in Ω.

Lemma 1.1. *Let u be the solution of the problem (P_κ). Assume that $|\varphi|_{\infty,\overline{\Omega}} \leqslant M$, $(M > 0)$. There exists a positive constant C depending only on $|f|_\infty$, M, p_-, p_+ and $diam(\Omega)$ such that $|u|_{L^\infty(\Omega)} \leqslant C$.*

Proof. Without loss of generality, one can assume that $\Omega \subset (0,d)\times\mathbb{R}^{n-1}$, $(d>0)$. We consider, for $\alpha > 0$, the function $w(x) = M + (e^{\alpha d} - e^{\alpha x_1})$. We have $w_{x_1} = -\alpha e^{\alpha x_1}$, $w_{x_1x_1} = -\alpha^2 e^{\alpha x_1} = \alpha w_{x_1}$ and

$$\begin{aligned} L_\kappa w &= div\Big((\kappa + |\nabla w|^2)^{\frac{p(x)-2}{2}}\nabla w\Big)\\ &= (\kappa + w_{x_1}^2)^{\frac{p(x)-2}{2}}\Big[w_{x_1x_1}\Big(1 + (p(x)-2)\frac{w_{x_1}^2}{\kappa + w_{x_1}^2}\Big) + p_{x_1}.w_{x_1} ln(\kappa + w_{x_1}^2)^{1/2}\Big]\\ &\leqslant (\kappa + w_{x_1}^2)^{\frac{p(x)-2}{2}}\Big[-\alpha\min(1,p_-) + L|ln(\kappa + w_{x_1}^2)^{1/2}|\Big]|w_{x_1}|. \end{aligned}$$

Note that $\alpha \leqslant |w_{x_1}| = \alpha e^{\alpha x_1} \leqslant \alpha e^{\alpha d}$ for all $x_1 \in [0,d]$. Hence choosing $\alpha > 1$, we obtain

$$L_\kappa w \leqslant (1 + (\alpha e^{\alpha d})^2)^{\frac{p_+-1}{2}} ln(1 + (\alpha e^{\alpha d})^2)^{1/2}\Big[L - \frac{\alpha\min(1,p_-)}{ln(1+(\alpha e^{\alpha d})^2)^{1/2}}\Big] = \psi(\alpha)$$

with $\lim\limits_{\alpha\to+\infty}\psi(\alpha) = -\infty$. So there exists $\alpha_0 = \alpha_0(p_-, p_+, diam(\Omega), |f|_\infty)$ such that $\psi(\alpha_0) \leqslant -|f|_\infty$. We deduce that for $\alpha = \alpha_0$, the corresponding function w satisfies $L_\kappa w \leqslant -|f|_\infty$ in Ω. It follows that $L_\kappa(\pm u) = \pm f \geqslant -|f|_\infty \geqslant L_\kappa w$ in Ω. Since we have moreover $w \geqslant M \geqslant \pm u = \pm\varphi$ on $\partial\Omega$, we obtain by the weak maximum principle that $\pm u \leqslant w$ in Ω. Hence we have $|u| \leqslant w \leqslant M + e^{\alpha_0 d}$ in Ω. □

Next we show $C^{1,\alpha}$ estimates for u_κ independently of κ.

Lemma 1.2. *Let u be the solution of the problem (P_κ). Assume that $|\varphi|_{\infty,\overline{\Omega}} \leqslant M$, $(M > 0)$, $p \in C^{0,\beta}(\overline{\Omega})$, $\beta \in (0,1)$ and there is a positive constant L such that*

$$|p(x_1) - p(x_2)| \leqslant L|x_1 - x_2|^\beta \qquad \textit{for } x_1, x_2 \in \overline{\Omega}.$$

Then there exists a positive constant $\alpha(n, p_-, p_+, L, \beta, M, |f|_\infty) \in (0,1)$ *such that for each* $\Omega' \subset\subset \Omega$, *we have* $|u|_{1,\alpha,\Omega'} \leqslant C$ *where* $C(n, p_-, p_+, L, \beta, M, |f|_\infty, d(\Omega', \Omega))$ *is a positive constant independent of* κ.

Proof. We introduce the auxiliary variable t, the set $Q = (-1,1) \times \Omega$ and the function $U(t,x) = \sqrt{\kappa} t + u(x)$. Then we have

$$\nabla U(t,x) = (\sqrt{\kappa}, \nabla u(x)) \qquad \text{and} \qquad |\nabla U(t,x)| = (\kappa + |\nabla u|^2)^{1/2}.$$

For any $\zeta \in \mathcal{D}(Q)$, the function $\zeta(t,.)$ is a test function for (P_κ) for each t. It follows that

$$\int_Q |\nabla U|^{p(x)-2} \nabla U \nabla \zeta dt dx = \int_\Omega \Big(\int_{-1}^1 (\kappa + |\nabla u|^2)^{\frac{p(x)-2}{2}} \sqrt{\kappa}.\zeta_t dx \Big) dt$$
$$+ \int_{-1}^1 \Big(\int_\Omega (\kappa + |\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u.\nabla \zeta dx \Big) dt = \int_{-1}^1 \Big(\int_\Omega f \zeta dx \Big) dt = \int_Q f \zeta dx dt.$$

We deduce that U satisfies

$$U \in W^{1,p(.)}(Q), \qquad \Delta_{p(.)} U = f \quad \text{in } Q.$$

Moreover we have $|U|_{L^\infty(\partial Q)} \leqslant 1 + |u|_{L^\infty(\overline{\Omega})}$ which, by the previous lemma, leads to U bounded in Q by a constant independent of κ. Then, the lemma holds as a consequence of Theorem 1.1 in [1]. □

In the rest of this paper we assume $\beta = 1$ i.e, p Lipschitz.

2 $W^{2,2}_{loc}$ regularity when $\kappa > 0$

In this section we assume that $\kappa > 0$. Our main result is the following theorem.

Theorem 2.1. *Let* u *be a solution of the problem* (P_κ). *Then we have* $u \in W^{2,2}_{loc}(\Omega)$.

First let us define for each $h \neq 0$ and each vector e_s $(s = 1,..,n)$ of the canonical basis of $\mathbb{R}^2$ the difference quotient of a function g defined on Ω by

$$\Delta_{s,h} g(x) = \frac{g(x + he_s) - g(x)}{h}.$$

The function $\Delta_{s,h} g$ is well defined on the set $\triangle_{s,h}\Omega = \{x \in \Omega /\ x + he_s \in \Omega\}$ which contains the set $\Omega_{|h|} = \{x \in \Omega /\ d(x, \partial\Omega) > |h|\}$.

Since $W^{1,p(.)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow W^{1,1}(\Omega)$, some properties in [4] (p263) on difference quotient are still valid. In particular we have

. If $g \in W^{1,1}(\Omega)$, then $\Delta_{s,h} g \in W^{1,1}(\Omega)$ and we have $\nabla(\Delta_{s,h} g) = \Delta_{s,h}(\nabla g)$.

. $\Delta_{s,h}(g_1 g_2)(x) = g_1(x + he_s)\Delta_{s,h} g_2(x) + g_2(x)\Delta_{s,h} g_1(x)$ for functions g_1 and g_2 defined in Ω.

. If at least one of the functions g_1 or g_2 has support contained in $\Omega_{|h|}$, then

$$\int_\Omega g_1 \Delta_{s,h} g_2 = -\int_\Omega g_2 \Delta_{s,h} g_1.$$

. If $w \in W^{1,p}(B_{4R})$ $(p \geqslant 1)$ and $\zeta^2 \triangle_{s,h} w \in W^{1,1}(B_{3R})$, we have (see Lemma 8.1 [4]) for $|h| < R$ and some constant $c(n)$,

$$\begin{aligned} &|\triangle_{s,h} w|_{p,B_{2R}} \leqslant c(n)|D_s w|_{p,B_{3R}} \\ &|\triangle_{s,-h}(\zeta^2 \triangle_{s,h} w)|_{1,B_{2R}} \leqslant c(n)|D_s(\zeta^2 \triangle_{s,h} w)|_{1,B_{3R}}. \end{aligned}$$

Proof of Theorem 2.1. Let $R > 0$ such that the open ball B_{2R} satisfies $\overline{B_{2R}} \subset \Omega$. We consider a function $\zeta \in \mathcal{D}(B_{2R})$ such that

$$\begin{cases} 0 \leqslant \zeta \leqslant 1 \quad \text{in } B_{2R}, \qquad \zeta = 1 \quad \text{in } B_R \\ |\nabla\zeta|^2 + |D^2\zeta| \leqslant \dfrac{c}{R^2} \quad \text{in } B_{2R}. \end{cases}$$

Then $\Delta_{s,-h}(\zeta^2 \Delta_{s,h} u)$ is a test function for the problem (P_κ). For simplicity, we drop the symbol s. We have

$$\int_\Omega \big(\kappa + |\nabla u|^2\big)^{\frac{p(x)-2}{2}} \nabla u.\nabla\big(\Delta_{-h}(\zeta^2 \Delta_h u)\big)dx = \int_\Omega f \triangle_{-h}(\zeta^2 \triangle_h u)dx.$$

This leads to

$$\begin{aligned} &\int_\Omega \Delta_h\Big(\big(\kappa + |\nabla u|^2\big)^{\frac{p(x)-2}{2}} \nabla u\Big).\big(\zeta^2 \nabla(\Delta_h u) + 2\zeta \Delta_h u \nabla\zeta\big)dx \\ &\qquad = \int_\Omega f \triangle_{-h}(\zeta^2 \triangle_h u)dx. \end{aligned} \tag{2.1}$$

Set $x_h = x + he_s$. Then we have

$$\begin{aligned} &\Delta_h\Big(\big(\kappa + |\nabla u|^2\big)^{\frac{p(x)-2}{2}} \nabla u\Big) \\ &= \frac{1}{h}\Big(\big(\kappa + |\nabla u(x_h)|^2\big)^{\frac{p(x_h)-2}{2}} \nabla u(x_h) - \big(\kappa + |\nabla u(x)|^2\big)^{\frac{p(x)-2}{2}} \nabla u(x)\Big) \\ &= U + V \end{aligned} \tag{2.2}$$

where

$$U = \frac{1}{h}\Big(\big(\kappa + |\nabla u(x_h)|^2\big)^{\frac{p(x_h)-2}{2}} \nabla u(x_h) - \big(\kappa + |\nabla u(x_h)|^2\big)^{\frac{p(x)-2}{2}} \nabla u(x_h)\Big)$$

$$V = \frac{1}{h}\Big(\big(\kappa + |\nabla u(x_h)|^2\big)^{\frac{p(x)-2}{2}} \nabla u(x_h) - \big(\kappa + |\nabla u(x)|^2\big)^{\frac{p(x)-2}{2}} \nabla u(x)\Big).$$

It follows then from (2.1) and (2.2) that

$$\begin{aligned} \int_\Omega \zeta^2 V.\nabla(\Delta_h u) = & -\int_\Omega \zeta^2 U.\nabla(\Delta_h u) - \int_\Omega 2\zeta(\Delta_h u) U.\nabla\zeta dx \\ & - \int_\Omega 2\zeta V.\nabla\zeta dx + \int_\Omega f \triangle_{-h}(\zeta^2 \triangle_h u) dx. \end{aligned} \tag{2.3}$$

Note that $\nabla u(x_h) = (\nabla u + h\Delta_h(\nabla u))(x)$ and then we have

$$V = \frac{1}{h}\int_0^1 \frac{d}{dt}\Big[\big(\kappa + |(\nabla u + th\Delta_h(\nabla u))(x)|^2\big)^{\frac{p(x)-2}{2}} (\nabla u + th\Delta_h(\nabla u))(x)\Big] dt.$$

Denoting $(\nabla u + th\Delta_h(\nabla u))(x)$ by θ_t, we deduce easily that

$$\begin{aligned} V = & \int_0^1 \big(\kappa + |\theta_t|^2\big)^{\frac{p(x)-2}{2}} \Delta_h(\nabla u) dt \\ & +(p(x)-2)\int_0^1 \big(\kappa + |\theta_t|^2\big)^{\frac{p(x)-4}{2}} < \theta_t, \Delta_h(\nabla u) > \theta_t dt. \end{aligned}$$

It follows that

$$\begin{aligned} & V.\nabla(\Delta_h u) = V.\Delta_h(\nabla u) = \int_0^1 \big(\kappa + |\theta_t|^2\big)^{\frac{p(x)-2}{2}} |\nabla(\Delta_h u)|^2 dt \\ & +(p(x)-2)\int_0^1 \big(\kappa + |\theta_t|^2\big)^{\frac{p(x)-4}{2}} < \theta_t, \Delta_h(\nabla u) >^2 dt. \end{aligned}$$

Then we have

$$\begin{aligned} \int_\Omega \zeta^2 V.\nabla(\Delta_h u) = & \int_\Omega \zeta^2 |\nabla(\Delta_h u)|^2 \left(\int_0^1 \big(\kappa + |\theta_t|^2\big)^{\frac{p(x)-2}{2}} dt \right) dx \\ & + \int_\Omega (p(x)-2)\zeta^2 \left(\int_0^1 \big(\kappa + |\theta_t|^2\big)^{\frac{p(x)-4}{2}} < \theta_t, \Delta_h(\nabla u) >^2 dt \right) dx \\ & = I_1 + I_2. \end{aligned} \tag{2.4}$$

Remark that

$$I_2 \geqslant \int_{\Omega\cap[p(x)<2]} (p(x)-2)\zeta^2 \left(\int_0^1 \big(\kappa + |\theta_t|^2\big)^{\frac{p(x)-4}{2}} < \theta_t, \Delta_h(\nabla u) >^2 dt \right) dx \tag{2.5}$$

and

$$\Big|\int_0^1 \big(\kappa+|\theta_t|^2\big)^{\frac{p(x)-4}{2}} < \theta_t, \Delta_h(\nabla u) >^2 dt\Big| \leqslant |\nabla(\Delta_h u)|^2 \int_0^1 \big(\kappa+|\theta_t|^2\big)^{\frac{p(x)-2}{2}} dt. \quad (2.6)$$

Setting $W(x) = \displaystyle\int_0^1 \big(\kappa+|\theta_t|^2\big)^{\frac{p(x)-2}{2}} dt$, we deduce from (2.5)-(2.6) that

$$I_2 \geqslant \int_{\Omega\cap[p(x)<2]} (p(x)-2)\zeta^2|\nabla(\Delta_h u)|^2 W(x)dx. \quad (2.7)$$

Then we get from (2.4) and (2.7)

$$\begin{aligned}
&\int_\Omega \zeta^2 V.\nabla(\Delta_h u) \geqslant \int_\Omega \zeta^2|\nabla(\Delta_h u)|^2 W(x) + \int_{\Omega\cap[p(x)<2]} (p(x)-2)\zeta^2|\nabla(\Delta_h u)|^2 W(x)\\
&\geqslant \int_{\Omega\cap[p(x)\geqslant 2]} \zeta^2|\nabla(\Delta_h u)|^2 W(x) + \int_{\Omega\cap[p(x)<2]} (p(x)-1)\zeta^2|\nabla(\Delta_h u)|^2 W(x)\\
&\geqslant \min(1, p_- -1)\int_\Omega \zeta^2|\nabla(\Delta_h u)|^2 W(x)dx. \quad (2.8)
\end{aligned}$$

Now we have

$$\begin{aligned}
U &= \frac{1}{h}\Big[\big(\kappa+|\nabla u(x_h)|^2\big)^{\frac{p(x_h)-2}{2}}\nabla u(x_h) - \big(\kappa+|\nabla u(x_h)|^2\big)^{\frac{p(x)-2}{2}}\nabla u(x_h)\Big]\\
&= \frac{1}{h}\int_0^1 \frac{d}{dt}\Big(\big(\kappa+|\nabla u(x+he_s)|^2\big)^{\frac{p(x+the_s)-2}{2}}\nabla u(x+he_s)\Big)dt\\
&= \frac{1}{2}\Big(\int_0^1 \frac{\partial p}{\partial x_s}(x+the_s)\ln\big(\kappa+|\nabla u(x_h)|^2\big)\big(\kappa+|\nabla u(x_h)|^2\big)^{\frac{p(x+the_s)-2}{2}}dt\Big)\nabla u(x_h).
\end{aligned}$$

Using the fact that $u \in C^{1,\alpha}_{loc}(\Omega)$ (see [1]) and that p is uniformly Lipschitz continuous in Ω, we easily deduce from the above equality that for some positive constant C_κ, one has

$$|U| \leqslant C_\kappa.$$

Hence we get for any $\nu > 0$

$$\begin{aligned}
&\Big|\int_\Omega \zeta^2 U.\nabla(\Delta_h u)dx\Big| \leqslant \int_\Omega \zeta^2|U||\nabla(\Delta_h u)|dx\\
&\leqslant \nu\int_\Omega \zeta^2|\nabla(\Delta_h u)|^2dx + \frac{1}{4\nu}\int_\Omega \zeta^2||U|^2dx\\
&\leqslant \nu\int_\Omega \zeta^2|\nabla(\Delta_h u|^2dx + \frac{C_\kappa^2}{4\nu}|B_{2R}|. \quad (2.9)
\end{aligned}$$

We can estimate the second term of the right side of (2.3) as follows

$$\Big|\int_\Omega 2\zeta(\Delta_h u)U.\nabla\zeta dx\Big| \leqslant \frac{2C_\kappa c^{1/2}}{R}\int_{B_{2R}}|\Delta_h u|dx \leqslant \frac{2C_\kappa^2 c^{1/2}}{R}\int_{B_{2R}}|\nabla u|dx. \qquad (2.10)$$

In order to estimate the third term of the right hand side of (2.3), we need to estimate V, for which we have

$$\begin{aligned}
|V| &\leqslant \int_0^1 \left(\kappa+|\theta_t|^2\right)^{\frac{p(x)-2}{2}}|\Delta_h(\nabla u)|dt \\
&+|p(x)-2|\int_0^1 \left(\kappa+|\theta_t|^2\right)^{\frac{p(x)-4}{2}}|\theta_t|^2|\Delta_h(\nabla u))|dt \\
&\leqslant (1+|p(x)-2|)\int_0^1 \left(\kappa+|\theta_t|^2\right)^{\frac{p(x)-2}{2}}|\Delta_h(\nabla u)|dt \\
&= (1+|p(x)-2|)W(x)|\Delta_h(\nabla u)|.
\end{aligned}$$

It is easy to check that there exists two positive constants l_κ and L_κ, depending on κ, such that $l_\kappa \leqslant W(x) \leqslant L_\kappa$.
Since $|\Delta_h u| \leqslant |\nabla u|_{\infty,B_{3R}}$, it follows by Young's inequality that

$$\begin{aligned}
&\Big|\int_\Omega 2\zeta V\nabla\zeta\Delta_h u dx| \leqslant 2(p_+ +3)L_\kappa \int_\Omega \zeta|\Delta_h\nabla u||\nabla\zeta||\Delta_h u|dx \\
&\leqslant \mu\int_\Omega \zeta^2|\Delta_h\nabla u|^2dx + \frac{((p_+ +3)L_\kappa)^2}{\mu}\int_\Omega |\nabla\zeta|^2|\Delta_h u|^2dx. \qquad (2.11)
\end{aligned}$$

For the last term of (2.3), we have, since $f\in L^\infty(\Omega)$

$$\begin{aligned}
&\int_\Omega f\triangle_{-h}(\zeta^2\triangle_h u)dx \leqslant |f|_\infty\int_\Omega |\triangle_{-h}(\zeta^2\triangle_h u)|dx \leqslant c(n)|f|_\infty\int_\Omega |\nabla(\zeta^2\triangle_h u)|dx \\
&\leqslant c(n)|f|_\infty\int_\Omega \zeta^2|\nabla(\triangle_h u| + 2\zeta|\nabla\zeta||\triangle_h u|dx \\
&\leqslant \eta\int_\Omega \zeta^2|\nabla(\triangle_h u)|^2 + c^2(n)|f|_\infty^2\frac{|B_{2R}|}{4\eta} + \frac{2c^{1/2}}{R}c^2(n)|f|_\infty\int_{B_{2R}}|\nabla u|dx \qquad (2.12)
\end{aligned}$$

Hence, choosing $\nu=\mu=\eta=\min(1,p_1-1)l_\kappa/3$, we obtain from (2.8), (2.9), (2.10), (2.11) and (2.12)

$$\int_{B_R}|\Delta_h\nabla u|^2dx \leqslant C(\kappa,p_-,p_+,L,R).$$

Letting $h\longrightarrow 0$ in the last inequality, we obtain the desired result (see Lemma 8.2 [4]). □

3 Second order regularity when $\kappa = 0$

In this section, we assume that $\kappa = 0$ and consider the problem

$$(P)\begin{cases} div\Big(|\nabla u|^{p(x)-2}\nabla u\Big) = f & \text{in } \Omega, \\ u \in W^{1,p(.)}_{loc}(\Omega), \qquad |u|_\infty \leqslant M & \text{in } \Omega. \end{cases}$$

Our main result here is the following regularity for a solution u of (P).

Theorem 3.1. *Let u be a solution of the problem (P). If $f \in W^{1,q(.)}_{loc}(\Omega) \cap L^\infty(\Omega)$ then we have*

$$D\Big(|\nabla u|^{\frac{p(x)-2}{2}}\nabla u\Big) \in L^2_{loc}(\Omega).$$

In particular, we have $D^2u \in L^2_{loc}([p(.) \leqslant 2])$.

Proof. We introduce for $\epsilon \in (0,1)$ the following approximated problem

$$(P_\epsilon)\begin{cases} u_\epsilon \in W^{1,p(.)}(B_{4R}) \\ div\Big((\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\nabla u_\epsilon\Big) = f & \text{in } B_{4R} \\ u_\epsilon = u \qquad \text{on } \partial B_{4R} \end{cases}$$

where $R > 0$ and $B_{4R} \subset\subset \Omega$.
Note that from Lemma 1.1, u_ϵ is bounded uniformly and independently of ϵ in B_{4R}. To prove Theorem 3.1, we need the following uniform estimate :

Theorem 3.2. *Let u be a solution of the problem (P_ϵ). Assume $f \in W^{1,q(.)}_{loc}(\Omega) \cap L^\infty(\Omega)$. Then there exists a constant C independent of ϵ such that*

$$\int_{B_R} [(\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}|D^2(u_\epsilon)|]^2 \leqslant C.$$

Proof. Let $\xi \in \mathcal{D}(\Omega)$. Since for each $i = 1, 2, .., n$, ξ_{x_i} is a test function for (P_ϵ), we have

$$\int_\Omega (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\nabla u_\epsilon.\nabla(\xi_{x_i})dx = -\int_\Omega f\xi_{x_i}dx.$$

Since $\nabla(\xi_{x_i}) = (\nabla\xi)_{x_i}$ and $D^2u_\epsilon \in L^2_{loc}(\Omega)$, we get

$$\int_\Omega \Big((\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\nabla u_\epsilon\Big)_{x_i}.\nabla\xi dx = \int_\Omega f\xi_{x_i}dx. \tag{3.1}$$

Now, let $\zeta \in \mathcal{D}(B_{2R})$, $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in B_R and $|\nabla\zeta|^2 + |D^2\zeta| \leqslant C/R^2$. Taking $\xi = u_{x_i}\zeta^2$ as a test function in (3.1), we obtain

$$\int_\Omega (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\Big[\nabla u_{\epsilon x_i}+(p(x)-2)\frac{\nabla u_{\epsilon x_i}.\nabla u_\epsilon}{\epsilon+|\nabla u_\epsilon|^2}\nabla u_\epsilon+p_{x_i}\nabla u_\epsilon ln(\epsilon+|\nabla u_\epsilon|^2)^{1/2}\Big]$$
$$.[\zeta^2\nabla u_{\epsilon x_i}+2\zeta\nabla\zeta.u_{\epsilon x_i}]=\int_\Omega f(u_{\epsilon x_i}\zeta^2)_{x_i}dx$$

which can be written as

$$\begin{aligned}
&\int_\Omega \zeta^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\Big[|\nabla u_{\epsilon x_i}|^2+(p(x)-2)\frac{|\nabla u_{\epsilon x_i}.\nabla u_\epsilon|^2}{\epsilon+|\nabla u_\epsilon|^2}\Big]dx\\
&=-\int_\Omega \zeta^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\Big[\nabla u_{\epsilon x_i}.\nabla u_\epsilon p_{x_i} ln(\epsilon+|\nabla u_\epsilon|^2)^{1/2}\Big]dx\\
&-\int_\Omega (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\Big[\nabla u_{\epsilon x_i}+(p(x)-2)\frac{\nabla u_{\epsilon x_i}.\nabla u_\epsilon}{\epsilon+|\nabla u_\epsilon|^2}\nabla u_\epsilon\Big].(2\zeta\nabla\zeta.u_{\epsilon x_i})dx\\
&-\int_\Omega (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\Big[p_{x_i} ln(\epsilon+|\nabla u_\epsilon|^2)^{1/2}\nabla u_\epsilon\Big].(2\zeta\nabla\zeta.u_{\epsilon x_i})dx+\int_\Omega f(u_{\epsilon x_i}\zeta^2)_{x_i}dx
\end{aligned}$$

Summing up with respect to i and using (0.1), we get

$$\begin{aligned}
&\min(1,p_--1)\int_\Omega \zeta^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}|D^2u|^2dx\\
&\leqslant\sum_{i=1}^{i=n}\int_\Omega \zeta^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}|\nabla u_{\epsilon x_i}|.|\nabla u_\epsilon|L.|ln(\epsilon+|\nabla u_\epsilon|^2)^{1/2}|\\
&+2(p_++3)\frac{C^{1/2}}{R}\sum_{i=1}^{i=n}\int_\Omega (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}|\nabla u_{\epsilon x_i}||\nabla u_\epsilon|.\zeta dx\\
&+2L\frac{C^{1/2}}{R}\int_{B_{2R}}(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-1}{2}}|\nabla u_\epsilon||ln(\epsilon+|\nabla u_\epsilon|^2)^{1/2}dx\\
&+\int_\Omega f div(\zeta^2\nabla u)dx=I_1+I_2+I_3+I_4.
\end{aligned}\tag{3.2}$$

Now, by applying Young's inequality, we get

$$\begin{aligned}
I_1&\leqslant\sum_{i=1}^{i=n}\int_\Omega \zeta^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}(\nu_1|\nabla u_{\epsilon x_i}|^2+\frac{L^2}{\nu_1}|\nabla u_\epsilon|^2|ln(\epsilon+|\nabla u_\epsilon|^2)^{1/2}|^2dx\\
&\leqslant\nu_1\int_\Omega \zeta^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}|D^2u_\epsilon|^2dx+\frac{nL^2}{\nu_1}\int_{B_{2R}}(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)}{2}}|ln(\epsilon+|\nabla u_\epsilon|^2)^{1/2}|^2dx.
\end{aligned}$$

Moreover, since $|\nabla u|_{\infty,\overline{B_{3R}}}\leqslant K$, we have $(\epsilon+|\nabla u_\epsilon|^2)^{1/2}\leqslant 1+K$. The continuous function $t\mapsto t^{p(x)}|lnt|^2$ is bounded on $[0,1+K]$. We deduce that the second term in the right hand side of the above inequality is uniformly bounded.

By similar arguments as above we get the following estimates for I_2 and I_3

$$
\begin{aligned}
I_2 &\leqslant \sum_{i=1}^{i=n} \int_\Omega \zeta^2 (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}} (\nu_2 |\nabla u_{\epsilon x_i}|^2 + 4(p_+ + 3)^2 \frac{C}{\nu_2 R^2} |\nabla u_\epsilon|^2) dx \\
&\leqslant \nu_2 \int_\Omega \zeta^2 (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}} |D^2 u_\epsilon|^2 dx + 4n(p_+ + 3)^2 \frac{C}{\nu_2 R^2} \int_{B_{2R}} (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)}{2}} dx,
\end{aligned}
$$

$$
I_3 \leqslant 2L \frac{C^{1/2}}{R} \int_{B_{2R}} (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)}{2}} |ln(\epsilon + |\nabla u_\epsilon|^2)^{1/2} dx.
$$

For the last term, we have

$$
|I_4| = \Big| - \int_\Omega \zeta^2 \nabla f . \nabla u_\epsilon dx \Big| \leqslant \int_{B_{2R}} |\nabla f| . |\nabla u_\epsilon| dx \leqslant 2 |\nabla f|_{L^{q(\cdot)}(B_{2R})} . |\nabla u_\epsilon|_{L^{p(\cdot)}(B_{2R})}.
$$

Finally by choosing $\nu_1 = \nu_2 = \min(1, p_- - 1)/3$ we get the desired result since the following estimate holds for a constant $C_* = C_*(n, p_-, p_+, L, R)$

$$
\begin{aligned}
&\int_{B_R} (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}} |D^2 u|^2 dx \\
&\leqslant C_* \Big(\int_{B_{2R}} (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)}{2}} |ln(\epsilon + |\nabla u_\epsilon|^2)^{1/2}|^2 dx + \int_{B_{2R}} (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)}{2}} dx \\
&+ \int_{B_{2R}} (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)}{2}} |ln(\epsilon + |\nabla u_\epsilon|^2)^{1/2} dx + |\nabla f|_{L^{q(\cdot)}(B_{2R})} . |\nabla u_\epsilon|_{L^{p(\cdot)}(B_{2R})} \Big).
\end{aligned}
$$

□

Proof of Theorem 3.1. First, note that we have

$$
|(\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}} \nabla u_\epsilon|^2 = (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}} |\nabla u_\epsilon|^2 \leqslant (\epsilon + |\nabla u_\epsilon|^2)^{p(x)/2}.
$$

Since $|\nabla u_\epsilon|_{L^\infty(B_{3R})} \leqslant K$, then $(\epsilon + |\nabla u_\epsilon|^2)^{p(x)/2} \leqslant (1 + K^2)^{p^+/2}$ and

$$
\int_{B_R} |(\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}} \nabla u_\epsilon|^2 \leqslant C.
$$

Next, since u_ϵ has weak second derivatives, a simple calculation shows that

$$
\begin{aligned}
&D_i \big((\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}} \nabla u_\epsilon \big) = (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}} . \\
&\Big[\nabla u_{\epsilon x_i} + \frac{p(x) - 2}{2} \frac{\nabla u_{\epsilon x_i} . \nabla u_\epsilon}{\epsilon + |\nabla u_\epsilon|^2} \nabla u_\epsilon + \frac{p_{x_i}}{4} \nabla u_\epsilon ln(\epsilon + |\nabla u_\epsilon|^2) \Big].
\end{aligned}
$$

It follows that

$$|D_i\big((\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}\nabla u_\epsilon\big)| \leqslant \frac{p_+ + 4}{2}(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}.|\nabla u_{\epsilon x_i}|$$
$$+\frac{L}{4}(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}.(\epsilon+|\nabla u_\epsilon|^2)^{1/2}|ln(\epsilon+|\nabla u_\epsilon|^2)|.$$

From Theorem 3.2, we know that $(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}|\nabla u_{\epsilon x_i}|$ is bounded in $L^2(B_R)$ independently of ϵ. On the other hand, for given $\delta > 0$, we have $\lim\limits_{t\to+\infty}\frac{logt}{t^\delta}=0$ and $\lim\limits_{t\to 0^+} t^\delta logt = 0$. We deduce that there is a positive constant $c(\delta)$ depending only on δ such that

$$|ln(\epsilon+|\xi|^2)| \leqslant c(\delta) + (\epsilon+|\xi|^2)^{\delta/2} + (\epsilon+|\xi|^2)^{-\delta/2} \qquad \forall\epsilon>0, \quad \forall\xi\in\mathbb{R}^n.$$

Using this inequality with $\delta = 1/2$, we get

$$(\epsilon+|\nabla u_\epsilon|^2)^{1/4}|ln(\epsilon+|\nabla u_\epsilon|^2)| \leqslant c(1/2)(\epsilon+|\nabla u_\epsilon|^2)^{1/4} + (\epsilon+|\nabla u_\epsilon|^2)^{1/2} + 1.$$

Thus we have

$$\Big|(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}|\nabla u_\epsilon||ln(\epsilon+|\nabla u_\epsilon|^2)|\Big|^2 \leqslant (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-1}{2}}.\Big|(\epsilon+|\nabla u_\epsilon|^2)^{1/4}|ln(\epsilon+|\nabla u_\epsilon|^2)|\Big|^2$$

and deduce that $D((\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}\nabla u_\epsilon)$ is bounded in $L^2(B_R)$ independently of ϵ. Hence, up to a subsequence, still denoted by ϵ, there exists $v \in H^1(B_R)$ such that

$$(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}\nabla u_\epsilon \rightharpoonup v \quad \text{in } H^1(B_R)$$
$$(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}\nabla u_\epsilon \longrightarrow v \quad \text{in } L^2(B_R)$$

Since $|u_\epsilon|_{1,\alpha,\overline{B_R}} \leqslant C$, we get by Ascoli-Arzela's theorem that $u_\epsilon \longrightarrow u$ uniformly in $C^1(\overline{B_R})$. Hence we obtain $v = |\nabla u|^{\frac{p(x)-2}{4}}\nabla u$ on $[\nabla u \neq 0]$. Now since

$$|(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{4}}\nabla u_\epsilon| \leqslant (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)}{4}} \longrightarrow |\nabla u|^{\frac{p(x)}{2}} \qquad \text{a.e. in } B_R.$$

we deduce that $|v| \leqslant |\nabla u|^{\frac{p(x)}{2}}$ and then $v = 0$ a.e. on $[\nabla u = 0]$. Hence $v = |\nabla u|^{\frac{p(x)-2}{4}}\nabla u \in H^1(B_R)$ and $Dv = D(|\nabla u|^{\frac{p(x)-2}{4}}\nabla u) \in L^2(B_R)$.
When $B_R \subset Int([p(.) \leqslant 2])$, we obtain from Theorem 3.2

$$\int_{B_R}|D^2u_\epsilon|^2dx \leqslant (1+K^2)^{\frac{2-p_-}{2}}\int_{B_R}|(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p(x)-2}{2}}\nabla u_\epsilon|^2 \leqslant C$$

from which we deduce that $D^2u_\epsilon \rightharpoonup D^2u$ in $L^2(B_R)$. Hence $u \in W^{2,2}_{loc}(Int([p(.) \leqslant 2]))$. $\square$

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