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DISSERTATION

ON THE INDUCTIVE STRUCTURE OF WORKS OF ART

by

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Abstract

The purpose of the dissertation is to specify the meaning and consequences of the following proposition:

A great work of art is an inductive game.

I consider three axioms: the first states what an accurate induction is; the second asserts that the essential activity of a human organism is making accurate inductions; and the third asserts that a work of art is an inductive game which exercises the deepest habitual responses of the organism. Since the meaning of the latter two axioms depends upon the meaning of the first, I begin by constructing a formal logic of induction and illustrating its application to an inductive situation. I then specify the meaning of the latter two axioms. Lastly, I discuss and illustrate the structural consequences of the axioms with respect to the traditional formal canons of the arts, distinguishing between narrative and non-narrative arts.

The formal logic of induction firstly specifies the meaning of 'accurate induction' with respect to a set of precisely defined machines. Such machines are objects each of which is programmed to act as if each of a given set of propositions were true. Each machine is said

to believe each proposition of its program; and, if a believed proposition is indeed true, then the machine is said to know the proposition. Probabilities are then specified to be rational numbers which are assigned with respect to a given machine to propositions asserting that a proposition p implies a proposition q. The procedure involves assigning a code number to each element of an exclusive and exhaustive set P of propositions, each element of which the machine knows to imply p. The ratio of the sum of the code numbers of that subset Q of P, each element of which is known by the machine to imply q, to the sum of the code numbers of P is the probability with respect to the machine that p implies q. (A noteworthy feature of the logic is that Goodman's grue-paradox cannot be constructed within it, as detailed in an appendix to the dissertation.)

After proving the fundamental theorems of the probability calculus, and deriving a restricted version of Laplace's Law of Succession, I then specify the sense in which human organisms can be considered to be machines as discussed in the formalism.

The second and third axioms are then informally specified. A human organism at each moment of its existence encounters complex temporal events, only some of which are conducive to its well-being. To insure its self-preservation, the organism must seek the latter and avoid the others with maximum efficiency. To do so, it develops unconscious habits of inductive expectation, the thwarting of which gives rise to emotional reactions. Works of art are tools whereby a human organism is able to make habitual inductions as if its well-being depended upon their accuracy, without an actual, but rather an imaginary, threat being present.

By a detailed examination of the structural feature of a golf course (a less-detailed kind of inductive game), three structural conditions are derived which are necessary to a work of art being great. I argue lastly (a) that works of art constructed in accordance with the three conditions would in general conform to the traditional descriptive canons of the arts, including the tri-partite canons (Exposition - Development - Recapitulation or Climax) of the narrative and musical arts, (b) that the narrative genres of Tragedy and Comedy are thereby explicable, and finally (c) that paintings can be viewed as inductive games even though no traditional structural canons (eg. akin to the sonata form in music) exist against which to test the argument.

Preface

The purpose of this dissertation is to clarify the meaning and consequences of the following proposition:

A great work of art is an inductive game.

I do not offer an argument for the truth of the proposition, but rather, as a mathematician who specifies an axiom system and develops its consequences without arguing for the truth or falsity of his axioms, I specify its meaning, so that, if true, its structural consequences will be apparent to the working artist. It is not verities, hence, but utilities I seek, propositions useful to the working artist engaged in solving his daily compositional problems. For I consider it a scandal of aesthetics that, of the thousands of admittedly true propositions which have been uttered about works of art, the number which have proven useful to the working artist is negligible.

When the formal sections ahead get murky (and they will), may I suggest to the reader that he keep steadfastly before him the image of a dramatist who, having attempted to fit a few sketchy scenes for an unfinished play into rough structural order, senses that one of the scenes in that context 'doesn't work' (as the usual phrase nicely puts it). What ought he to do then? My task in this essay, simply put, is to place in focus a general conception of art which has structural implications for that dramatist in his situation, and for other artists in their's. (All references in the text of the dissertation are to the Bibliography.)

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Introduction

At the beginning of Chapter II, I state three axioms upon the assumed truth of which is based my subsequent discussion of the meaning and consequences of the proposition

A great work of art is an inductive game:

the first states what an accurate induction is; the second asserts that the essential activity of a human organism is making accurate inductions; and the third asserts that a work of art is an inductive game which exercises the deepest habitual responses of the organism. Since the meaning of the latter two axioms depends upon the meaning of the first, I begin in Chapter I by constructing a formal logic of induction and illustrating its application to an inductive situation. In the early part of Chapter II, I then specify the meaning of the latter two axioms. Lastly, I discuss and illustrate the structural consequences of the axioms with respect to the traditional formal canons of the arts, distinguishing between narrative and non-narrative arts.

The task of specifying the meaning of the first axiom involves a complex formalization which specifies firstly the meaning of 'accurate induction' with respect to a set of precisely defined machines, and then secondly specifies the sense in which human organisms can be considered to be such machines.

A proposition is defined to be anything which asserts that the number of elements in the intersection of two sets, a subject set and a predicate set, lies within an upper and lower numerical bound. If the cardinality of the intersection of the two sets indeed lies within the asserted limits, then the proposition is taken to be true (with suitable restrictions to avoid the Paradox of the Liar). Given the above, the truth-functions of the propositional calculus are defined in the customary way, with the assertion that p implies q being taken to be simply the assertion that the conditional of q given p is true.

A set of objects, called θ -machines, is then introduced. Such machines are objects each of which is programmed to act as if each of a given set of propositions were true. Each machine is said to *believe* each proposition of its program; and, if a believed proposition is indeed true, then the machine is said to *know* the proposition.

Probabilities are then specified to be rational numbers which are assigned unambiguously *with respect to a given θ -machine* to propositions asserting that a proposition p implies a proposition q. (I abbreviate 'the θ -probability of the truth of the conditional of q given p equals k' by the symbol

$$\dashv\theta\vdash(p \rightarrow q) = k \quad .)$$

The number assigned to a given implication proposition with respect to a given θ -machine depends upon the knowledge of the θ -machine; it is, therefore, subjective with respect to that machine. Since, however, the machine's knowledge depends upon the truth of the propositions it believes, it is objective with respect to the world.

The procedure for assigning the probabilities contains many formal subtleties designed to elude the various objectionable features of previous inductive logics. (Eg. The grue-paradox of Nelson Goodman is thereby avoided, as explained in Appendix II.) But, essentially, the procedure involves assigning a code number to each element of that set P of propositions which θ knows to contain only one true proposition (but doesn't know which) and which is such that θ knows that each element implies proposition p. The ratio of the sum of the code numbers of that subset Q of P, each element of which θ knows to imply q, to the sum of the code numbers of P is the probability with respect to θ that p implies q; i.e.

$$\dashv \theta \vdash (p \to q) .$$

The fundamental theorems of the probability calculus are then derived within the system at the end of Section I.

Applying the theorems of the probability calculus as specified in the formalism to an illustrative inductive situation in which I, as a θ -machine, know that each element of a set E is either white or black, and that some particular subset S of E having s elements contains exactly t black elements, I prove in Section II that the probability with respect to myself as θ that any particular other element of E is black is

$$\frac{t+1}{s+2} .$$

This is formally similar to Laplace's Law of Succession (1774); but whereas Laplace's law leads to inductive inconsistencies, the restrictions imposed by the formalism of my system avoid them. In Section III

I then remove an ambiguity in the Laplacean notion of 'sampled object' as applied to ordered sets by giving a formal specification of what a <u>theme</u> is.

Having formally specified and illustrated the procedure of assigning probabilities to assertions of implication with respect to Θ - machines, I immediately specify in Chapter II the sense in which human organisms can be considered to be such machines, and explicitly set forth the three axioms.

The second and third axioms are then informally specified. A human organism at each moment of its existence encounters complex temporal events, only some of which are conducive to its well-being. To insure its self-preservation, the organism must seek the latter and avoid the others with maximum efficiency. To do so, it develops unconscious habits of inductive expectation, the thwarting of which gives rise to emotional reactions. Works of art are tools whereby a human organism is able to make habitual inductions <u>as if</u> its well-being depended upon their accuracy, without an actual threat to its well-being being present. The organism consciously or unconsciously <u>imagines</u> such a threat, and exercises its habitual responses to avoid it.

The point of specifying the meaning of the three axioms, however, is to elicit more clearly the structural features of great works of art. Since sporting events have been shown to be inductive games also, though of a coarser structure and effect, Section II of Chapter II is devoted to a discussion of the basic structural conditions for great works of art as evidenced coarsely in the structural features of a well-designed golf course, culminating in the explicit statement of three structural conditions necessary for a work of art to be great.

I argue lastly, in the remainder of Chapter II, that works of art constructed in accordance with the above three conditions would in general conform to the traditional descriptive canons of the various arts (including, in particular, the tri-partite canon (Exposition - Development - Climax) of the narrative arts, and the structurally similar sonata form of the musical arts), that the narrative genres of Tragedy and Comedy are thereby explicable, and finally that paintings can be viewed as inductive games even though no traditional structural canons (eg. akin to the sonata form in music) exist against which to test the argument.

So much for the plans. Let us proceed to the foundations!

Chapter I, Section I:

The Logic of Probable Inference

1. Imagine a class A no two members of which are identical. I shall then say that <u>A is a set</u>.

1.1 Imagine a set A and an object c . I shall refer to the number of elements in A by the symbol

$N(A)$ (read 'the cardinality of A ');

to the set consisting of all and only those elements each of which is not an element of A by the symbol

$\breve{A}$ (read 'the complement of A ');

to the set consisting only of the object c by the symbol

$\{c\}$ (read 'the unit set of c ');

and to the set consisting of no elements by the symbol

Λ (read 'the null set').

1.11 Imagine two sets, A_1 and A_2 . I shall refer to the set consisting of all and only those elements each of which is an element of either A_1 or A_2 or both by the symbol

$A_1 + A_2$ (read 'the sum of A_1 and A_2 ').

I shall refer to the set consisting of all and only those elements each of which is an element of both A_1 and A_2 by the symbol

$$A_1 \times A_2 \qquad \text{(read 'the product of } A_1 \text{ and } A_2 \text{')}.$$

And I shall refer to the set $A_1 \times \breve{A}_2$ by the symbol

$$A_1 - A_2 \qquad \text{(read 'the difference of } A_1 \text{ and [redacted] } A_2 \text{')}.$$

1.12 Imagine a set A and an object c such that c is an element of A (i.e. such that, by 1.1 and 1.11, $N(\{c\} \times A) = 1$). I shall then write

$$c \in A \quad .$$

(If c were not an element of A (i.e. if, by 1.1 and 1.11, $N(\{c\} \times A) = 0$), I should then write

$$c \notin A \, .$$

1.13 Imagine two sets, A_1 and A_2, such that each element of A_1 is an element of A_2. I shall then say that A_1 is included in A_2, abbreviated

$$A_1 \subset A_2 \quad .$$

(if A_1 were not included in A_2, I should then write

$$A_1 \not\subset A_2 \quad . \quad)$$

1.14 Imagine two sets, A_1 and A_2, such that

(a) $A_1 \subset A_2$; and

(b) $A_2 \subset A_1$.

I shall then say that <u>A_1 and A_2 are e-identical</u>, abbreviated

$$A_1 \overset{e}{=} A_2 \ .$$

(if A_1 and A_2 were <u>not</u> e-identical, I should then write

$$A_1 \overset{e}{\neq} A_2 \ .)$$

1.15 Imagine two objects, a_1 and a_2, and a set A, such that neither object is an element of A unless the other is also an element of A. I shall then say that <u>a_1 is A-identical to a_2</u>, written

$$a_1 \overset{A}{=} a_2 \ .$$

(If a_1 were <u>not</u> A-identical to a_2, I should then write

$$a_1 \overset{A}{\neq} a_2 \ .)$$

1.2 Imagine a set A and a set $B = \{B_1, B_2, \ldots\}$ such that

(a) for each $B_i \in B$ $(0 < i)$, $B_i \overset{e}{\neq} \Lambda$;

(b) for each $B_i, B_j \in B$ $(0 < i, j$ and $i \neq j)$,

$$B_i \times B_j \stackrel{e}{=} \Lambda$$; and

(c) $$B_1 + B_2 + \cdots \stackrel{e}{=} A .$$

I shall then say that <u>B is a partition of A</u>.

1.3 Imagine two objects, a_1 and a_2, and the set $A = \{A_1, A_2\}$ such that $A_1 = \{a_1\}$ and $A_2 = \{a_1, a_2\}$. I shall refer to set A by the symbol

$\langle a_1, a_2 \rangle$ (read 'the 2-gram of a_1, a_2').

Imagine a third object a_3, and the set $A' = \langle\langle a_1, a_2\rangle, a_3\rangle$. I shall refer to set A' by the symbol

$\langle a_1, a_2, a_3 \rangle$ (read 'the 3-gram of a_1, a_2, a_3').

By extension, I shall refer to any set $\langle\langle a_1, a_2, \ldots, a_{k-1}\rangle, a_k\rangle$, where k is finite, by the symbol

$\langle a_1, a_2, \ldots, a_k \rangle$ (read 'the k-gram of a_1, a_2, ..., a_k').

1.4 Imagine a set $A = \{a_1, a_2, \ldots\}$ such that each $a_i \in A$ $(0 < i)$ is a 2-gram. I shall then say that <u>A is a relation.</u>

1.5 Imagine a set $A = \{a_1, a_2, \ldots\}$ and a relation B such that, for each $a_i, a_j, a_k \in A$ $(0 < i, j, k)$,

(a) if $a_i \neq a_j$, then either $\langle a_i, a_j \rangle \in B$ or $\langle a_j, a_i \rangle \in B$;

(b) if $\langle a_i, a_j \rangle \in B$, then $a_i \neq a_j$; and

(c) if $\langle a_i, a_j \rangle, \langle a_j, a_k \rangle \in B$, then $\langle a_i, a_k \rangle \in B$.

I shall then say that <u>A is ordered with respect to B</u> .

1.51 Imagine a set A such that A is ordered with respect to some relation B . I shall then say that <u>A is ordered</u>.

2. Imagine a 3-gram $P = \langle\langle A, B \rangle, m_1, m_2 \rangle$ such that

(a) A and B are sets;

(b) m_1 and m_2 are integers such that $m_1 \leq m_2$; and

(c) $P \notin A$.

I shall then say that <u>P is a proposition</u> (and that <u>A is the subject set of P</u> , and that <u>B is the predicate set of P</u>).

2.1 Imagine a proposition $P = \langle\langle A, B \rangle, m_1, m_2 \rangle$ such that

$$m_1 \leq N(A \times B) \leq m_2 \quad .$$

I shall then say that <u>P is true.</u> (If a proposition P were <u>not</u> true, I should then say that <u>P is false.</u>)

I shall refer to the set of true propositions by the symbol

$$T \quad .$$

(Since condition (c) of 2 prohibits propositions from referring to themselves, the Paradox of the Liar is not forthcoming. See Appendix I.)

2.2 Imagine a proposition P and a proposition $Q = \langle\langle\{P\},T\rangle, 1, 1\rangle$ (i.e. $Q = P \in T$). I shall refer to the proposition Q , which proposes that the proposition P is true, by the symbol

$$\vdash P$$

2.3 Imagine firstly a set $A = \{A_1, A_2, \ldots\}$ such that each $A_i \in A$ ($0 < i$) is itself a set consisting of exactly two propositions, p_{ia} and p_{ib} . Imagine secondly the set of propositions $S = \{S_1, S_2, \ldots\}$ such that each

$$S_i \in S \ (0 < i) \ \overset{e}{=} \ \langle\langle A_i, T\rangle, 0, 1\rangle .$$

(i.e. Imagine the set S consisting of all and only those propositions S_i ($0 < i$) each of which proposes that the product of the set of true propositions T with a particular A_i contains either p_{ia} or p_{ib} or neither, but not both.) For each $S_i \in S$ ($0 < i$), I shall say that <u>S_i is a truth-function of p_{ia} and p_{ib}</u> , abbreviated

$p_{ia} \mid p_{ib}$ (read 'the alternate-denial of p_{ia} and p_{ib} ');

and shall also say that <u>p_{ia} and p_{ib} are the components of S_i</u> . Furthermore, I shall abbreviate

$p_{ia} \mid p_{ia}$ by $\overline{p_{ia}}$ (or $- p_{ia}$).

(read 'the negation of p_{ia} ');

$(p_{ia} | p_{ib}) | (p_{ia} | p_{ib})$ by $p_{ia} \wedge p_{ib}$

(read 'the conjunction of p_{ia} and p_{ib} ');

(I shall omit the symbol ' $\wedge$ ' wherever clarity will not suffer thereby.)

$(p_{ia} | p_{ia}) | (p_{ib} | p_{ib})$ by $p_{ia} \vee p_{ib}$

(read 'the disjunction of p_{ia} and p_{ib} '); and

$p_{ia} | (p_{ib} | p_{ib})$ by $p_{ia} \longrightarrow p_{ib}$

(read 'the conditional of p_{ib} given p_{ia} ').

2.31 Imagine a proposition P such that

$$P: \; ([p_{ia} | (p_{ib} | p_{ib})] | [p_{ib} | (p_{ia} | p_{ia})]) | ([p_{ia} | (p_{ib} | p_{ib})] | [p_{ib} | (p_{ia} | p_{ia})]).$$

P proposes that $p_{ia} \in T$ if and only if $p_{ib} \in T$.
Hence, by 1.15,

$$P: \qquad p_{ia} \overset{T}{=} p_{ib} \; .$$

2.4 Imagine two propositions, p_1 and p_2 , such that

$$N(\{p_1 \rightarrow p_2\} \times T) = 1 \; .$$

I shall then say that <u>p_1 implies p_2</u> .

2.5 Imagine a truth-function S and a finite set of propositions $P = \{S, p_1, p_2, \ldots, p_n\}$ such that

(a) p_1 is a component of S ;

(b) for each $p_i, p_j \in P$ $(0 < i, j \leq n$ and $i + 1 = j)$, p_j is a component of p_i ; and

(c) p_n is not a truth-function.

I shall then say that

(1) P is an S-functional chain; and that

(2) p_n is the end of P .

2.51 Imagine a truth-function S and a finite set of propositions $P = \{S, p_1, p_2, \ldots, p_n\}$ such that

(a) S has both of its components as elements of P ;

(b) each $p_i \in P$ $(0 < i \leq n)$ is an element of an S-functional chain C_i included in P such that either

(1) p_i is the end of C_i ; or

(2) p_i has both of its components as elements of P .

I shall then say that P is the S-functional net.

2.52 Imagine a truth-function S, the S-functional net P, and a finite set of propositions Q, such that Q contains all and only those elements of P each of which is the end of an S-functional chain included in P . I shall then say that Q is

<u>the genetic set of S.</u>

2.53 Imagine two sets of propositions, $P = \{p_1, p_2, ---\}$ and Q, such that Q contains all and only those propositions such that, for each $p_i \in P$ $(0 < i)$,

(a) if p_i is not a truth-function, then $p_i \in Q$; and

(b) if p_i is a truth-function, then the genetic set of p_i is included in Q.

I shall then say that <u>Q is the reduction set of P</u>.

3. Imagine two finite sets, A and A', such that, if A is taken as the given set, Procedure I specifies that A' is 'the stop set of A'.

<u>Procedure I:</u> (1) Imagine a finite given set $B = \{B_1, B_2, \ldots, B_n\}$;

(2) Imagine some element $B_i \in B$ $(0 < i \leq n)$ as KEYED;

(3) Imagine the partition of B into two subsets, I and II, such that

(a) I contains all and only those $B_j \in B$ $(0 < j \leq n)$ such that $B_i \subset B_j$; and

(b) II contains all and only those $B_k \in B$ $(0 < k \leq n)$ such that $B_i \not\subset B_k$;

(4) Imagine the set B' which contains only

(a) B_i;

(b) for each $B_j \in I$, $B_j - B_i$; and

(c) each $B_k \in II$.

(5)(a) If B' contains at least one element which has not been KEYED, then repeat steps (1) - (4), replacing B by B' and B_i by some element of B' which has not been KEYED;

(b) If B' contains no elements which have not been KEYED, then say that B' is 'the stop set of B ', and stop.

I shall then say that <u>A' is the stop set of A</u> .

3.1 Imagine a set of propositions $P = \{p_1, p_2, ---\}$ and a set A such that A contains all and only those sets each of which is either a subject or predicate set of some $p_i \in P \; (0 < i)$.
I shall then say that <u>A is the vocabulary set of P</u> .

3.11 Imagine two sets of propositions, P and P' , and two finite sets, A and A' , such that

(a) P' is the reduction set of P ;

(b) A is the vocabulary set of P' ; and

(c) A' is the stop set of A .

I shall then say that <u>A' is the definition set of P</u> .

4. Imagine a set $\Theta = \{\theta_1, \theta_2, ---\}$ such that, for each $\theta_i \in \Theta \; (0 < i)$, there exists a set of propositions $P_i =$

$\{p_{i1}, p_{i2}, \dots\}$ such that P_i contains all and only those propositions p_{ik} $(0 < k)$ such that Θ_i is programmed to act upon $\vdash p_{ik}$ (i.e. such that Θ_i is programmed to act as if p_{ik} were true). I shall then say that

(a) <u>Θ is the set of machines</u>;

(b) for each $\Theta_i \in \Theta$ $(0 < i)$, <u>Θ_i is a machine</u>; and

(c) for each P_i $(0 < i)$, <u>P_i is the program of Θ_i</u>.

4.1 Imagine a machine Θ, a set of propositions P, and a proposition p, such that

(a) P is the program of Θ; and

(b) $p \in P$.

I shall then say that <u>p is believed by Θ</u>.

4.11 Imagine a machine Θ, a set of propositions P, and a propositions p, such that

(a) P is the program of Θ;

(b) p is believed by Θ; and

(c) $p \in T$.

(i.e. such that p is both true and believed by Θ). I shall they say that <u>p is known by Θ</u>, abbreviated

$$\Theta(p).$$

4.2 Imagine a machine Θ and a set of propositions $P = \{p_1, p_2, \ldots\}$ such that, for each $p_i, p_j \in P$ ($0 < i, j$ and $i \neq j$), $\sim\Theta(p_i \rightarrow p_j)$. I shall then say that <u>P is a Θ-free set.</u>

4.21 Imagine a machine Θ and a set of propositions $P = \{p_1, p_2, \ldots\}$ such that $\Theta(N(P \times T) \geq 1)$. I shall then say that <u>P is a Θ-live set.</u> Imagine, rather, that $\Theta(N(P \times T) \leq 1)$. I shall then say that <u>P is a Θ-exclusive set.</u> Imagine, rather, that $\Theta(N(P \times T) = 1$. I shall then say that <u>P is a Θ-live-and-exclusive set.</u>

4.211 Imagine a machine Θ, a proposition p, and a set of propositions P, such that $\Theta(p \rightarrow [N(P \times T) = 1])$. I shall then say that <u>P is a $\langle\Theta, p\rangle$-live-and-exclusive set.</u>

4.22 Imagine a machine Θ, a proposition p, and a set of Θ-free propositions $P = \{p_1, p_2, \ldots\}$, such that

(a) for each $p_i \in P$ ($0 < i$),

(1) Θ (neither p nor any element of any p-functional chain is an element of any p_i-functional chain);

(2) were it the case that $\sim\Theta(p)$ and $\sim\Theta(\bar{p})$, it would be the case that $\Theta(p_i \rightarrow p)$; and

(b) it is <u>not</u> the case, for some $p_j \in P$ ($0 < j$) and Θ-free set $Q = \{q_1, q_2, \ldots\}$, that

(1) Q is $\langle\Theta, p_j\rangle$-live-and-exclusive; and

(2) for each $q_k \in Q\ (o < k)$,

(a') Θ (neither p nor p_j, nor any element of any p-functional or p_j-functional chain, is an element of any q_k-functional chain); and

(b') were it the case that $\sim\Theta(p)$ and $\sim\Theta(\bar{p})$, it would be the case that $\Theta(q_k \rightarrow p)$.

I shall then say that <u>P is a $\langle\Theta, p\rangle$-largest set</u>.

4.23 Imagine a machine Θ, a proposition p, and a set of propositions P, such that P is both $\langle\Theta, p\rangle$-live-and-exclusive and $\langle\Theta, p\rangle$-largest. I shall then say that <u>P is $\langle\Theta, p\rangle$-prime</u>.

4.3 Imagine a machine Θ, a proposition $P = \langle\langle A, B\rangle, m_1, m_2\rangle$, and a set C, such that

(a) Θ (C is ordered);

(b) for some $b_i \in B$ and $c_j \in C$, $\Theta(b_i = c_j)$; and

(c) for some $a_k \in A$ and $c_m, c_n \in C$ such that

(1) $\Theta(a_k = c_m)$; and

(2) $\Theta(c_m \neq c_n)$;

were it instead the case that

(3) $\Theta(a_k = c_n)$,

it would instead be the case that

(4) $\Theta(a_k \notin A)$.

I shall then say that <u>P is a Θ-non-random proposition</u>. (If P were <u>not</u> a Θ-non-random proposition, I should then say that <u>P is a Θ-random proposition</u>.)

(The reader will note later that only Θ-random propositions are to be included in probability calculations, thereby avoiding Goodman's grue-paradox (Goodman [1], pp. 72-81). See 6. below and Appendix II.)

5. Imagine a set A and a set of propositions $R = \{r_{00}, r_{01}, \ldots, r_{ij}, \ldots\}$ $(0 < i, j$ and $i \leq j)$ such that, for each $r_{ij} \in R$,

$$i \leq N(A) \leq j \quad .$$

I shall then say that <u>R is the field set of A</u>.

5.1 Imagine two propositions, p and q, and two sets, A and $R = \{r_1, r_2, \ldots\}$, such that

(a) R is the field set of A; and

(b) there is no $r_i \in R$ $(0 < i)$ which p would imply, were it true, that q would not imply, were it true, and conversely.

I shall then say that <u>p and q are A-similar</u>.

5.2 Imagine a set of propositions $P = \{p_1, p_2, \ldots\}$, a set A, and a partition $P' = \{P_1, P_2, \ldots\}$ of P, such that each $P_i \in P'$ $(0 < i)$ contains all and only those $p_k \in P$, $(0 < k)$

which are A -similar to each other. I shall then say that <u>P' is the A -partition of P</u>.

5.21 Imagine a finite set of propositions P, a subset $Q = \{q_1, q_2, ---, q_n\}$ of P, a set A, and a finite set of sets $P' = \{P_1, P_2, ---, P_z\}$, such that P' is the A -partition of P. By 5.2, each $q_i \in Q$ $(o < i \leq n)$ is an element of one and only one $P_k \in P'$; eg. P_j. Imagine a number m_i such that

$$m_i = \frac{1}{N(P') \times N(P_j)} .$$

I shall then say that <u>m_i is the $\langle P, A \rangle$ -subcode of q_i</u>.

5.22 Imagine two finite sets of propositions, P and $Q = \{q_1, q_2, ---, q_n\}$, a finite set $A = \{A_1, A_2, ---, A_z\}$, and, for each $q_i \in Q$ $(o < i \leq n)$, a number m_i, such that

$$m_i = \sum_{k=1}^{z} \text{the } \langle P, A_k \rangle \text{- subcode of } q_i .$$

I shall then say that <u>m_i is the $\langle P, A \rangle$ -code of q_i</u>.

5.23 Imagine two finite sets of propositions, P and $Q = \{q_1, q_2, ---, q_n\}$, a set A, and a number m, such that

$$m = \sum_{i=1}^{n} \text{the } \langle P, A \rangle \text{ - code of } q_i .$$

I shall then say that <u>m is the $\langle P, A \rangle$ -sum of Q</u>, written

$$\sum_{P}^{A} [Q] .$$

6. Imagine a machine Θ, two Θ-random propositions, p and q, a set of propositions A, a finite set A', and two finite sets of propositions, $P = \{p_1, p_2, ---, p_n\}$ and Q, such that

(a) $A = \{p, q\}$;

(b) A' is the definition set of A;

(c) P is $\langle \Theta, p \rangle$-prime; and

(d) Q contains all and only those $p_i \in P$ $(0 < i \leq n)$ such that $\Theta(p_i \rightarrow q)$.

By <u>the Θ-probability of the truth of the conditional of q given p</u> (i.e. the probability for Θ that p implies q), abbreviated

$$\dashv\Theta\vdash(p \rightarrow q),$$

I shall refer to the quotient of

$$\frac{\sum_{P}^{A'}\left[\text{the set of propositions of } P \text{ each of which implies } p \text{ and } q\right]}{\sum_{P}^{A'}\left[\text{the set of propositions of } P \text{ each of which implies } p\right]};$$

i.e.,

$$(1) \qquad \dashv\Theta\vdash(p \rightarrow q) = \frac{\sum_{P}^{A'}[P \times Q]}{\sum_{P}^{A'}[P]}.$$

(The reader ought to note that, on my account, implications may be assigned probabilities only with respect to a given machine Θ . For purposes of deciding consistent action, therefore, two implications assigned probabilities with respect to different such machines would be strictly incomparable.)

6.1 (In the pursuit of typographical sanity, I shall assume for the remainder of this Chapter that Θ and set A' remain constant, and that both summations given in any ratio of two summations are taken with respect to the set summed in the denominator (as in (1) above), and hence whall omit the respective symbols from all succeeding formulae whenever the meaning is clear. But note 6.2 and 6.3 .)

Since the numerator of (1) cannot exceed the denominator, while neither can be negative,

(2) $$0 \leq \dashv\vdash (p \rightarrow q) \leq 1 \ .$$

6.11 By (1), as the reader would expect,

$$\dashv\vdash (p \rightarrow p) = \frac{\Sigma [P \times P]}{\Sigma [P]} = \frac{\Sigma [P]}{\Sigma [P]} = 1 \ ;$$

and

$$\dashv\vdash(p \rightarrow \bar{p}) = \frac{\Sigma\,[P \times \breve{P}]}{\Sigma\,[P]} = \frac{\Sigma\,[\Lambda]}{\Sigma\,[P]} = \frac{0}{\Sigma\,[P]} = 0.$$

6.12 By (1),

$$\dashv\vdash(p \rightarrow \bar{q}) = \frac{\Sigma\,[P \times (P - Q)]}{\Sigma\,[P]}$$

$$= \frac{\Sigma\,[P - Q]}{\Sigma\,[P]}\,.$$

But $\Sigma\,[P - Q] = \Sigma\,[P] - \Sigma\,[P \times Q]$; hence

$$= \frac{\Sigma\,[P] - \Sigma\,[P \times Q]}{\Sigma\,[P]}$$

$$= 1 - \frac{\Sigma\,[P \times Q]}{\Sigma\,[P]}\,,$$

which, by (1), is

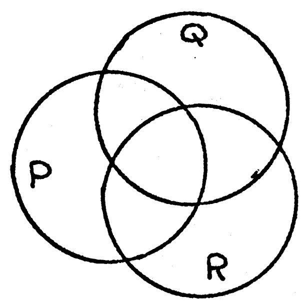
(3) $\dashv\vdash(p \rightarrow \bar{q}) = 1 - \dashv\vdash(p \rightarrow q)$.

6.2 Imagine now a third θ -random proposition $\curlyvee$ and a finite set of propositions R such that

(a) $r \in A$; and

(b) R contains all and only those $p_i \in P$ $(0 < i \leq n)$ such that $\theta(p_i \longrightarrow r)$.

(Reflection on the standard Venn diagram



may help to clarify the derivations below.) By (1),

$$\dashv\vdash (p \rightarrow qr) = \frac{\Sigma [P \times (Q \times R)]}{\Sigma [P]} .$$

But $\Sigma [P \times (Q \times R)] = \Sigma [P \times Q \times R]$; hence

(4) $$\dashv\vdash (p \rightarrow qr) = \frac{\Sigma [P \times Q \times R]}{\Sigma [P]} .$$

6.21 By 2.3 and 2.31,

$$(pq \rightarrow r) \stackrel{T}{=} [p \rightarrow (q \rightarrow r)] .$$

But, by (1),

$$\dashv\vdash(pq \rightarrow r) = \frac{\Sigma\,[(P \times Q) \times R]}{\Sigma\,[P \times Q]}$$

$$= \frac{\Sigma\,[P \times Q \times R]}{\Sigma\,[P \times Q]}.$$

Hence,

(5) $$\dashv\vdash(pq \rightarrow r) = \dashv\vdash[p \rightarrow (q \rightarrow r)]$$

$$= \frac{\Sigma\,[P \times Q \times R]}{\Sigma\,[P \times Q]}.$$

6.22 By (1),

(6) $$\dashv\vdash[p \rightarrow (q \vee r)] = \frac{\Sigma\,[P \times (Q + R)]}{\Sigma\,[P]}.$$

6.23 Since

$$\frac{\Sigma\,[P \times Q \times R]}{\Sigma\,[P \times Q]} = \frac{\dfrac{\Sigma\,[P \times Q \times R]}{\Sigma\,[P]}}{\dfrac{\Sigma\,[P \times Q]}{\Sigma\,[P]}},$$

by (1), (4), and (5),

(7) $$\dashv\vdash (pq \to r) = \frac{\dashv\vdash (p \to qr)}{\dashv\vdash (p \to q)}.$$

6.24 Since

$$\Sigma[P \times (Q+R)] = \Sigma[P \times Q] + \Sigma[P \times R] - \Sigma[P \times Q \times R],$$

by division,

$$\frac{\Sigma[P \times (Q+R)]}{\Sigma[P]} = \frac{\Sigma[P \times Q)}{\Sigma[P]} + \frac{\Sigma[P \times R]}{\Sigma[P]} - \frac{\Sigma[P \times Q \times R]}{\Sigma[P]};$$

which, by (1), (4), and (6), is

(8) $$\dashv\vdash [p \to (q \vee r)] = \dashv\vdash (p \to q) + \dashv\vdash (p \to r) - \dashv\vdash (p \to qr).$$

6.3 Imagine lastly two finite sets of cardinality m ,

$$\omega = \{\omega_1, \omega_2, \ldots, \omega_m\} \text{ ; and}$$

$$W = \{W_1, W_2, \ldots, W_m\};$$

such that

(a.) each $\omega_i \in \omega$ $(0 < i \leq m)$ is

(1) θ -random; and

(2) an element of A ;

(b) ω is

(1) θ -free; and

(2) θ -live-and-exclusive; and

(c) each $W_j \in W$ ($0 < j \leq m$) contains all and only those $p_i \in P$ ($0 < i \leq n$) such that $\theta(p_i \rightarrow \omega_i)$.

Imagine now a particular element of ω , written ω_a . By (7),

(9) $$\dashv\vdash(pq \rightarrow \omega_a) = \frac{\dashv\vdash(p \rightarrow q\omega_a)}{\dashv\vdash(p \rightarrow q)} .$$

Since ω is θ-free and θ -live-and-exclusive, however, by 2.3, 4.2, and 4.21,

(A') $q \overset{T}{=} [(q\omega_1) \vee (q\omega_2) \vee \ldots \vee (q\omega_m)]$; and

(B') $\dashv\vdash(p \rightarrow q\omega_i\omega_j) = 0$ (for all $0 < i, j \leq m$ and $i \neq j$).

Hence, by (A'),

$$\dashv\vdash(p \rightarrow q) = \dashv\vdash(p \rightarrow [(q\omega_1) \vee (q\omega_2) \vee \ldots \vee (q\omega_m)]),$$

which, by (8) extended and (B'),

$$= \dashv\vdash(p \rightarrow q\omega_1) + \dashv\vdash(p \rightarrow q\omega_2) + \ldots + \dashv\vdash(p \rightarrow q\omega_m),$$

which I shall abbreviate in customary fashion as .

$$(10) \qquad \dashv\vdash(p \to q) = \sum_{t=1}^{m} \dashv\vdash(p \to q\omega_t) \quad .$$

By substitution, (9) then becomes

$$\dashv\vdash(pq \to \omega_a) = \frac{\dashv\vdash(p \to q\omega_a)}{\sum_{t=1}^{m} \dashv\vdash(p \to q\omega_t)} ,$$

which, by (7) extended, is

$$(11) \quad \dashv\vdash(pq \to \omega_a) = \frac{\dashv\vdash(p \to \omega_a) \times \dashv\vdash(p\omega_a \to q)}{\sum_{t=1}^{m} \left[\dashv\vdash(p \to \omega_t) \times \dashv\vdash(p\omega_t \to q)\right]} \quad .$$

Equation (11) is the general form of Bayes's Theorem (1763). (Jeffreys, p. 30; and Skyrms, p. 134)

Chapter I, Section II:

The Logic Applied

To illustrate the results of Section I in application, I shall now apply them to a single illustrative inductive situation.[1]

Imagine a finite set of elements of cardinality n, $E = \{e_1, e_2, ---, e_n\}$, which I shall call the population. Imagine secondly two sets, S and S', which I shall call respectively the sample set and the new sample set, and two sets, B and W, which I shall call respectively the black set and the white set. Imagine thirdly that I (as θ) know, for each $e_i \in E\ (0 < i \leq n)$ that

$$e_i \in \breve{B} \overset{T}{=} e_i \in W \qquad \text{and} \qquad e_i \in B \overset{T}{=} e_i \in \breve{W} \quad ;$$

and hence that I can substitute $\breve{B}$ for W in all propositions.

I shall be concerned with the following propositions (or truth-functions of them):

$$\begin{aligned}
\omega &: \quad N(E) = n; \\
g &: \quad N(E \times S) = s; \\
h &: \quad N(E \times S \times B) = t; \\
y &: \quad N(S \times S') = 0; \\
g' &: \quad N((E - S) \times S') = 1; \\
h' &: \quad N((E - S) \times S' \times B) = 1;
\end{aligned}$$

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1. The general idea of the proof given in this Section is due to Jeffreys, and, in particular, the derivation from page 39 to page 41 is his alone. (Jeffreys, in turn, attributes the reduction on page to a suggestion from Dr. F.J. Whipple.) See Jeffreys, pp. 125-127.

and a set of propositions, $F = \{f_0, f_1, ---, f_n\}$, such that, for each $f_i \in F$ $(0 \leq i \leq n)$,

$$f_i : \quad N(E \times B) = b_i .$$

(For convenience sake, I shall abbreviate $wgg'y$ by Z.) The set A of these propositions has set $A' = \{E, S, S', B\}$ as its definition set. All probabilities calculated below will be with respect to myself as θ.

Consider the question: What is the probability, given an s-membered sample of E of which t are black, that any particular one of the unsampled elements E is black? Or, formally, what is the value of

$$\dashv\vdash (Zh \longrightarrow h') \quad ?$$

By (10), since the set $F = \{f_0, f_1, ---, f_n\}$ is both θ-free and θ-live-and-exclusive,

$$(12) \qquad \dashv\vdash (Zh \rightarrow h') = \sum_{i=0}^{n} \dashv\vdash (Zh \rightarrow f_i h').$$

But, by (7),

$$\dashv\vdash (Zh \rightarrow f_i h') = \dashv\vdash (Zhf_i \rightarrow h') \times \dashv\vdash (Zh \rightarrow f_i);$$

and, by (11),

$$\dashv\vdash (Zh \rightarrow f_i) = \frac{\dashv\vdash (Z \rightarrow f_i) \times \dashv\vdash (Zf_i \rightarrow h)}{\sum_{k=0}^{n} [\dashv\vdash (Z \rightarrow f_k) \times \dashv\vdash (Zf_k \rightarrow h)]} .$$

Hence, by substitution in (12),

$$\dashv\vdash(zh \rightarrow h') = \sum_{i=0}^{n} \left[\dashv\vdash(zhf_i \rightarrow h') \times \frac{\dashv\vdash(z \rightarrow f_i) \times \dashv\vdash(zf_i \rightarrow h)}{\sum_{k=0}^{n} [\dashv\vdash(z \rightarrow f_k) \times \dashv\vdash(zf_k \rightarrow h)]} \right],$$

which reduces to

$$(13)\ \dashv\vdash(zh \rightarrow h') = \frac{\sum_{i=0}^{n} [\dashv\vdash(zhf_i \rightarrow h') \times \dashv\vdash(z \rightarrow f_i) \times (zf_i \rightarrow h)]}{\sum_{k=0}^{n} [\dashv\vdash(z \rightarrow f_k) \times \dashv\vdash(zf_k \rightarrow h)]}.$$

To assist in the calculations, imagine now three sets of propositions of cardinality 2^n :

$P = \{p_1, \bar{p}_1, p_2, \bar{p}_2, ---, p_n, \bar{p}_n\}$;

$Q = \{q_1, \bar{q}_1, q_2, \bar{q}_2, ---, q_n, \bar{q}_n\}$; and

$R = \{r_1, \bar{r}_1, r_2, \bar{r}_2, ---, r_n, \bar{r}_n\}$;

such that

each $p_i \in P$ $(0 < i \leq n)$: $N(\{e_i\} \times B) = 1$;

each $\bar{p}_i \in P$ $(0 < i \leq n)$: $N(\{e_i\} \times B) = 0$;

each $q_j \in Q$ $(0 < j \leq n)$: $N(\{e_j\} \times S) = 1$;

each $\bar{q}_j \in Q$ $(0 < j \leq n)$: $N(\{e_j\} \times S) = 0$;

each $r_k \in R$ $(0 < k \leq n)$: $N(\{e_k\} \times S') = 1$; and

each $\bar{r}_k \in R$ $(0 < k \leq n)$: $N(\{e_k\} \times S') = 0$.

Imagine secondly the conjunction of the disjunctions of each $p_i, \bar{p}_i \in P$ $(0 < i \leq n)$ (and similarly for sets Q and R):

$$(p_1 \vee \bar{p}_1) \wedge (p_2 \vee \bar{p}_2) \wedge \ldots \wedge (p_n \vee \bar{p}_n) \;;$$

$$(q_1 \vee \bar{q}_1) \wedge (q_2 \vee \bar{q}_2) \wedge \ldots \wedge (q_n \vee \bar{q}_n) \;; \text{ and}$$

$$(r_1 \vee \bar{r}_1) \wedge (r_2 \vee \bar{r}_2) \wedge \ldots \wedge (r_n \vee \bar{r}_n) \;.$$

By 2.3, each conjunction is respectively $\overset{T}{=}$ to

$$A: \quad p_1 p_2 \cdots p_n \vee \bar{p}_1 p_2 \cdots p_n \vee \cdots \vee \bar{p}_1 \bar{p}_2 \cdots \bar{p}_n \;;$$

$$B: \quad q_1 q_2 \cdots q_n \vee \bar{q}_1 q_2 \cdots q_n \vee \cdots \vee \bar{q}_1 \bar{q}_2 \cdots \bar{q}_n \;; \text{ and}$$

$$C: \quad r_1 r_2 \cdots r_n \vee \bar{r}_1 r_2 \cdots r_n \vee \cdots \vee \bar{r}_1 \bar{r}_2 \cdots \bar{r}_n \;.$$

Imagine thirdly conjoining proposition ω to each of the 2^n conjunctions disjoined in A (and similarly for B and C):

$$A^{\omega}: \quad \omega p_1 p_2 \cdots p_n \vee \omega \bar{p}_1 p_2 \cdots p_n \vee \cdots \vee \omega \bar{p}_1 \bar{p}_2 \cdots \bar{p}_n \;;$$

$$B^{\omega}: \quad \omega q_1 q_2 \cdots q_n \vee \omega \bar{q}_1 q_2 \cdots q_n \vee \cdots \vee \omega \bar{q}_1 \bar{q}_2 \cdots \bar{q}_n \;; \text{ and}$$

$$C^{\omega}: \quad \omega r_1 r_2 \cdots r_n \vee \omega \bar{r}_1 r_2 \cdots r_n \vee \cdots \vee \omega \bar{r}_1 \bar{r}_2 \cdots \bar{r}_n \;.$$

I shall refer to the set of conjunctions disjoined in A^{ω} as I ; to the set disjoined in B^{ω} as II ; and to the set disjoined in C^{ω} as III.

Imagine fourthly the conjunction of the disjunction of the elements of I, the disjunction of the elements of II , and the disjunction of the elements of III . By 2.3, this conjunction (with redundant ω's eliminated) is $\overset{T}{=}$ to

$$D^{\omega}: \quad \omega p_1 p_2 \cdots p_n q_1 q_2 \cdots q_n r_1 r_2 \cdots r_n \vee \omega \bar{p}_1 p_2 \cdots p_n q_1 \cdots r_n \vee \cdots \vee \omega \bar{p}_1 \bar{p}_2 \cdots \bar{q}_1 \bar{q}_2 \cdots \bar{r}_n \,.$$

I shall refer to the set of 2^{3n} conjunctions disjoined in D^{ω} as IV .

Consider firstly, now, the last term of the numerator of (13):

$$\dashv\vdash (Z f_i \longrightarrow h) .$$

There are $\binom{n}{b_i}$ elements of I which imply ω and f_i, and $\binom{n}{s}$ elements of II which imply ω and g. Conjoining each of the $\binom{n}{b_i}$ such elements of I with each of the $\binom{n}{s}$ such elements of II gives $\binom{n}{b_i}\binom{n}{s}$ conjunctions each of which implies ω, g, and f_i. And since, for each of the $\binom{n}{s}$ elements of II which imply ω and g, there are $\binom{n-s}{1} = n-s$ elements of III which, when conjoined respectively with it, imply $\omega g g' y$ ($\overset{T}{=} Z$), there are $\binom{n}{b_i}\binom{n}{s}(n-s)$ unique conjunctions of IV each of which imply $Z f_i$. The set of these $\binom{n}{b_i}\binom{n}{s}(n-s)$ conjunctions, which I shall call IV_1, is $\langle \theta, Z f_i \rangle$-prime. And since any two of these propositions have identical $\langle IV_1, A \rangle$-codes, which I shall call m_1, by 5.23

$$E_1: \qquad \sum_{IV_1}^{A'} [IV_1] = m_1 \binom{n}{b_i}\binom{n}{s}(n-s) .$$

Imagine a particular one of the $\binom{n}{b_i}$ elements of I which imply ω and f_i, which I shall call π_1. π_1 asserts that a particular b_i-membered subset of E is black and the remainder white. From this subset, I can select t elements in $\binom{b_i}{t}$ ways, and can then select $s-t$ elements from the remainder of E in $\binom{n-b_i}{s-t}$ ways. To each of these $\binom{b_i}{t}\binom{n-b_i}{s-t}$ selections, there corresponds a unique element of II which asserts the fact of that s-membered selection being the sample $E \times S$. And, as previously noted, there are $n - s$ elements of III which, when conjoined respectively with this element of II, jointly imply ω, g, g', and y. Hence, conjoined with Π_1, there are $\binom{b_i}{t}\binom{n-b_i}{s-t}(n-s)$ unique conjunctions implying $Zf_i h$. But Π_1 is only one of $\binom{n}{b_i}$ elements of I implying f_i. Hence, there are $\binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(n-s)$ elements of IV implying $Zf_i h$, the set of which I shall call IV_1'.

Since each element of IV which implies $Zf_i h$ implies Zf_i, IV_1' is a subset of IV_1'. Each element of IV_1', thus, has $\langle IV_1, A'\rangle$-code $= m_1$; therefore

$$E_2: \quad \sum_{IV_1}^{A'} [IV_1'] = m_1 \binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(n-s) .$$

Since IV_1 is $\langle \theta, Zf_i \rangle$-prime, by (1)

$$(13.1) \quad \dashv\vdash(Zf_i \longrightarrow h) = \frac{\sum_{IV_1}^{A'} [IV_1']}{\sum_{IV_1}^{A'} [IV_1]} .$$

By E_1 and E_2, then, (13.1) is

$$\dashv\vdash(Zf_i \rightarrow h) = \frac{m_1 \binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(n-s)}{m_1 \binom{n}{b_i}\binom{n}{s}(n-s)} ,$$

which reduces to

$$(13.11) \quad \dashv\vdash(Zf_i \rightarrow h) = \frac{\binom{b_i}{t}\binom{n-b_i}{s-t}}{\binom{n}{s}} .$$

Consider secondly the middle term of the numerator of (13):

$$\dashv\vdash (Z \rightarrow f_i)\ .$$

By conjoining respective elements of I, II, and III as in the previous discussion, there are

$\binom{n}{0}\binom{n}{s}(n-s)$ propositions of IV implying Zf_0 ;
$\binom{n}{1}\binom{n}{s}(n-s)$ propositions of IV implying Zf_1 ;

$\vdots$

$\binom{n}{n}\binom{n}{s}(n-s)$ propositions of IV implying Zf_n .

Calling the set of these propositions IV_2, any two of the above propositions have identical $\langle IV_2, \{S, S', E\}\rangle$-subcodes, the sum of which I shall call k . But the $\langle IV_2, B\rangle$-subcode of each is dependent upon the value of i of the f_i it implies. Specifically, for each proposition implying f_i (for some $0 < i \leq n$), its $\langle IV_2, B\rangle$-subcode is $\dfrac{1}{(n+1)\binom{n}{b_i}}$; hence, its $\langle IV_2, A'\rangle$-code $= \left[k + \dfrac{1}{(n+1)\binom{n}{b_i}}\right]$. But since it is only one of $\binom{n}{b_i}$ such propositions comprising a subset of IV_2, which I shall call IV_2', the $\langle IV_2, A'\rangle$-sum of the propositions of IV_2 implying Zf_i ($0 < i \leq n$) is $\binom{n}{b_i}\left[k + \dfrac{1}{(n+1)\binom{n}{b_i}}\right]$;

hence,

$$E_3:\quad \sum_{IV_2}^{A'} [IV_2'] = k\binom{n}{b_i} + \frac{1}{(n+1)}$$

There are $n+1$ such subsets of IV_2, however, each implying Z and a different one of the $n+1$ values of f_i. The $\langle IV_2, A' \rangle$-sum of IV_2, therefore, $= (n+1)\left[k\binom{n}{b_i} + \frac{1}{(n+1)}\right]$; hence,

$$E_4: \quad \sum_{IV_2}^{A'} [IV_2] = (n+1)\left[k\binom{n}{b_i} + \frac{1}{(n+1)}\right].$$

IV_2 is $\langle \theta, Z \rangle$ -prime. Since $IV_2' \subset IV_2$, therefore, by (1)

$$(13.2) \qquad \dashv\vdash(Z \rightarrow f_i) = \frac{\sum_{IV_2}^{A'} [IV_2']}{\sum_{IV_2}^{A'} [IV_2]}.$$

By E_3 and E_4, then, (13.2) is

$$\dashv\vdash(Z \rightarrow f_i) = \frac{k\binom{n}{b_i} + \frac{1}{(n+1)}}{(n+1)\left[k\binom{n}{b_i} + \frac{1}{(n+1)}\right]},$$

which reduces to

$$(13.21) \qquad \dashv\vdash(Z \rightarrow f_i) = \frac{1}{n+1}.$$

Consider lastly the first term of the numerator of (13):

$$\dashv\vdash(Z h f_i \rightarrow h').$$

As indicated on page 34 above, there are $\binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(n-s)$ propositions in IV implying Zhf_i . I shall here call this set of propositions IV_3. Each of these propositions has $\langle IV_3, A' \rangle$-code $= m_1$; hence,

$$E_5: \quad \sum_{IV_3}^{A'} [IV_3] = m_1 \binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(n-s) .$$

Imagine now one of the $\binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}$ propositions formed by conjoining an element of I and an element of II into a truth-function which implies $\omega g h f_i$, which I shall call Π_3 . Π_3 asserts that a particular $b_i - t$ -membered subset of $E - S$ is black and the remainder white. From this subset, I can select a single element in $\binom{b_i - t}{1} = b_i - t$ ways. To each of these $b_i - t$ selections, there corresponds a unique element of III which asserts the fact of that one-membered selection being the sole element of S' . Hence, conjoined with Π_3 , there are $b_i - t$ unique conjunctions implying $Zhf_i h'$. But Π_3 is only one of $\binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(b_i - t)$ such propositions implying $\omega g h f_i$. Hence, there are $\binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(b_i - t)$ elements of IV implying $Zhf_i h'$, the set of which I shall call IV_3'.

Since each element of IV which implies $Zhf_i h'$, implies Zhf_i , $IV_3' \subset IV_3$. Thus, each element of IV_3' has $\langle IV_3, A' \rangle$-code $= m_1$; hence

$$E_6: \quad \sum_{IV_3}^{A'} [IV_3'] = m_1 \binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(b_i - t) .$$

IV_3 is $\langle \theta, zhf_i \rangle$-prime. By (1), therefore,

$$(13.3) \qquad ⫣\!\vdash (zhf_i \rightarrow h') = \frac{\sum_{IV_3}^{A'} [IV_3{}']}{\sum_{IV_3}^{A'} [IV_3]} .$$

By E_5 and E_6, then, (13.3) is

$$⫣\!\vdash (zhf_i \rightarrow h') = \frac{m_1 \binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(b_i - t)}{m_1 \binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}(n-s)} ,$$

which reduces to

$$(13.31) \qquad ⫣\!\vdash (zhf_i \rightarrow h') = \frac{b_i - t}{n - s} .$$

By (13.11), (13.21), and (13.31), therefore, (13) becomes

$$⫣\!\vdash (zh \rightarrow h') = \frac{\sum_{i=0}^{n} \left[\left(\frac{b_i - t}{n - s} \right) \left(\frac{1}{n+1} \right) \left(\frac{\binom{b_i}{t}\binom{n-b_i}{s-t}}{\binom{n}{s}} \right) \right]}{\sum_{k=0}^{n} \left[\left(\frac{1}{n+1} \right) \left(\frac{\binom{b_k}{t}\binom{n-b_k}{s-t}}{\binom{n}{s}} \right) \right]} ,$$

which reduces to

$$(14) \quad ⫣\!\vdash (zh \rightarrow h') = \frac{\sum_{i=0}^{n} (b_i - t) \binom{b_i}{t}\binom{n-b_i}{s-t}}{(n-s) \sum_{k=0}^{n} \binom{b_k}{t}\binom{n-b_k}{s-t}} .$$

Consider now the summation in the denominator of (14):

$$\sum_{k=0}^{n} \binom{b_k}{t}\binom{n-b_k}{s-t}$$

Imagine a set of $n+1$ objects arranged in a given order, from which one wishes to select $s+1$ of them. There are $\binom{n+1}{s+1}$ unique selections. But one may proceed as follows:

(a) One may select an arbitrary member of the set. Let it be the $(b_k+1)^{th}$ in the order.

(b) From the remainder, one may then select t from those b_k objects before the $(b_k+1)^{th}$ in $\binom{b_k}{t}$ unique ways. Similarly, one may then select $s-t$ from those $n-b_k$ objects after the $(b_k+1)^{th}$ in $\binom{n-b_k}{s-t}$ unique ways. Hence, one may select t from those before, and $s-t$ from those after, the $(b_k+1)^{th}$ object in $\binom{b_k}{t}\binom{n-b_k}{s-t}$ unique ways.

(c) But one might have chosen any of $n+1$ values for b_k (and hence any one of the $n+1$ objects for the partition point). And all selections made for different values of b_k are unique, since the $(b_k+1)^{th}$ object of the set of $n+1$ objects must be the $(t+1)^{th}$ object of the selection made.

Hence,

$$\sum_{k=0}^{n} \binom{b_k}{t}\binom{n-b_k}{s-t} = \binom{n+1}{s+1} .$$

Equation (14), thus, reduces to

$$(14.1) \quad \dashv\vdash (zh \rightarrow h') = \frac{\sum_{i=0}^{n} (b_i - t)\binom{b_i}{t}\binom{n-b_i}{s-t}}{(n-s)\binom{n+1}{s+1}} .$$

But consider $(b_i - t)\binom{b_i}{t}$. By expansion,

$$(b_i - t)\binom{b_i}{t} = \frac{(b_i - t)\, b_i!}{t!\,(b_i - t)!} = \frac{t+1}{(t+1)!} \times \frac{b_i!}{(b_i - t - 1)!}$$

$$= (t+1) \times \frac{b_i!}{(t+1)!\,[b_i - (t+1)]!}$$

$$= (t+1)\binom{b_i}{t+1} .$$

Hence, (14.1) becomes

$$(14.2) \quad \dashv\vdash (zh \rightarrow h') = \frac{t+1}{(n-s)\binom{n+1}{s+1}} \sum_{i=0}^{n} \binom{b_i}{t+1}\binom{n-b_i}{s-t} .$$

By an argument strictly analogous to that given on page 34, however, the summation in (14.2) reduces to

$$\sum_{i=0}^{n} \binom{b_i}{t+1}\binom{n-b_i}{s-t} = \binom{n+1}{s+2} .$$

The right side of (14.2), thus, becomes

$$= \frac{t+1}{(n-s)\binom{n+1}{s+1}} \binom{n+1}{s+2}$$

$$= (t+1)\,\frac{(s+1)!\,(n-s)!}{(n-s)\,(n+1)!} \times \frac{(n+1)!}{(s+2)!\,(n-s-1)!},$$

which yields

$$(15) \qquad \dashv\vdash (Zh \rightarrow h') = \frac{t+1}{s+2} \,.$$

Equation (15) is formally similar to Laplace's Law of Succession (1774) (Jeffreys, p. 127). It asserts, under the restrictions assumed on page 29, that given a sample of s objects of which t are black, the probability of any particular unsampled object being black is $\frac{t+1}{s+2}$. Notice: since the right side of (15) is independent of ω, one needn't know the cardinality of the population E to assess the probability! (If it seems, dear reader, that I've taken a disproportionate amount of space to derive such an innocuous equation, may I suggest that you ponder the last remark again and then see Appendix III. Innocuity evanesces quickly.)

Chapter I, Section III:

Objects and Samples

Near the end of Section II, I said that equation (15) was 'similar' to Laplace's Law of Succession. I did not say 'identical', for

(A) the set of conditions placed upon the propositions involved in the derivation of (15) severly restrict its range of application in a manner foreign to the conception of Laplace; and

(B) Laplace's result can be put thusly:

> Given a sample of s objects of which t are black, the probability that the next object sampled is black is
>
> $$\frac{t+1}{s+2} \quad ;$$

but none of the propositions involved in the derivation of (15), whether singly or conjoined, contain information sufficient to give meaning to the work 'next' in Laplace's result.

I shall say of (A) only that Laplace's result leads to inductive inconsistency[1], whereas equation (15) apparently does not. But of (B) I

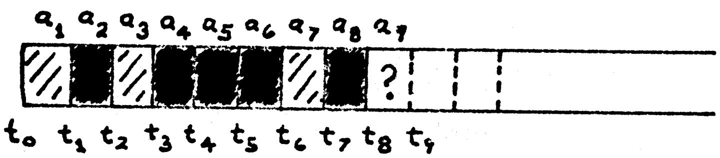
---

1. Imagine, for example, a sample of two objects, one black and one white, drawn from a population. By an unrestricted application of Laplace's Law of Succession, the probability of the next object being not black is 1/2. But since, by a similar unrestricted application, the probability of it being white is 1/2 and of it being red is 1/4, by the disjunctive rule the probability of the next object being either white

must say more, for in it inheres an ambiguity in the meaning of the words 'object' and 'sample' which must be clarified.

To illustrate the ambiguity, imagine firstly a set A , each element of which is either black or grey, and which is temporally ordered in accordance with the following diagram:

Diagram I:



i.e. the elements a_1 , a_3 , and a_7 , which correspond respectively to temporal intervals $t_1 - t_0$, $t_3 - t_2$, and $t_7 - t_6$, are grey, while elements a_2 , a_4 thru a_6 , and a_8 , which correspond respectively to intervals $t_2 - t_1$, $t_4 - t_3$, $t_5 - t_4$, $t_6 - t_5$, and $t_8 - t_7$, are black.

Imagine secondly that we are at moment t_8 , have experienced the previous elements of A , and wish to determine the probability that the element a_9 which corresponds to temporal interval $t_9 - t_8$ will be black by equation (15). Of what objects ought our population, and our sample set, to be taken to consist? Laplace would have assumed implicitly that, since we are concerned to establish the probability

---

or red is 3/4. Hence, the probability of it being not black must be at least 3/4, which is inconsistent with the original assessment.

Note also a further inconsistency. By Laplacean standards (as by our own), the probability of any inference cannot exceed 1. But since, by an unrestricted application of Laplace's Law of Succession, the probability of the next object being white is 1/2, being red is 1/4, being blue is 1/4, and being green is 1/4, the probability that the next object is either white or red or blue or green exceeds 1, contrary to the above standard.

of the blackness of the next element a_9 of set A, our population ought to consist of the elements of A and our sample set ought to consist of elements a_1 thru a_8 of A. But since set A is temporally <u>ordered</u>, many other kinds of objects (eg. ordered pairs of elements of A, ordered triads of elements of A, etc.) have been experienced which could be taken to comprise a sample set relevant to determining the probability that element is black. Indeed, our sample set

(a) could consist of eight objects, each of temporal length $t_n - t_{n-1}$ $(0 < n \leq 8)$, five of which are black (Laplace's assumption); or rather

(b) could consist of seven objects, each of temporal length $t_n - t_{n-2}$ $(1 < n \leq 8)$, three of which have their first temporal half grey and their second temporal half black, two of which have their first temporal half black and their second temporal half grey, and two of which have both their temporal halves black; or rather

(c) could consist of six objects, each of temporal length $t_n - t_{n-3}$ $(2 < n \leq 8)$, one of which has its first temporal third grey, its second temporal third black, and its last temporal third grey, two of which have ...

.
.
.

(h) could consist of one object of temporal length $t_n - t_{n-8}$ $(7 < n \leq 8)$ which has its first temporal eighth grey, its second temporal eight black, its third temporal eighth grey, ...

Our choice is important, because the calculated value of the probability that element a_9 is black is a function of our choice. For example, were we to choose (a), the probability would be

$$\frac{t+1}{s+2} = \frac{5+1}{8+2} = \frac{4}{5} .$$

Were we to choose (b), on the other hand, and hence were we to be calculating the probability, in effect, that the second temporal half of the object which corresponds to interval $t_9 - t_7$ is black (knowing that its first temporal half, i.e. a_8, is black), our relevant sample set would consist of those objects of length $t_n - t_{n-2}$ ($1 < n \leq 8$) whose first temporal halves were black, of which two have their second temporal halves grey and two have their second temporal halves black. Hence, the probability would be

$$\frac{t+1}{s+2} = \frac{2+1}{4+2} = \frac{1}{2} \quad .$$

Without having specified firstly how to derive a unique sample set from an ordered set of elements in hand, therefore, equation (15) cannot be unambiguously applied to an ordered set of objects in hand.

I turn, consequently, to the task of specifying a rule by which to derive a unique sample set from an ordered set of objects in hand.

Firstly: Imagine a set $A = \{a_1, a_2, \ldots\}$ ordered with respect to a relation ϕ, and a set $A' = \{a'_1, a'_2, \ldots\}$, such that

(a) $A' \subset A$; and

(b) for each $a'_i, a'_{i+1} \in A'$, there does not exist three elements $a_j, a_k, a_L \in A$ such that

(1) $a_j = a'_i$;
(2) $a_L = a'_{i+1}$; and
(3) $j < k < L$.

I shall then say that <u>A' is a ϕ-object of A</u>.

Secondly: Imagine a set A ordered with respect to a relation ϕ, and two ϕ-objects of A, A_1 and A_2, such that

$$N(A_1) = N(A_2) = k .$$

I shall then say that <u>A_1 and A_2 have ϕ-length $= k$</u>.

Thirdly: Imagine a ϕ-ordered set $A = \{a_1, a_2, \dots, a_n, a_{n+1}, \dots\}$, two sets F_1 and F_2, and a set A', such that, for each $a_i \in A$ ($0 < i$),

$$a_i \in \breve{F}_1 \overset{\top}{=} a_i \in F_2 \qquad ; \text{ and}$$
$$a_i \in F_1 \overset{\top}{=} a_i \in \breve{F}_2 \qquad ; \text{ and}$$

if a_{n+1} is taken to be the <u>next</u> ϕ-object of A, then A' contains all and only those ϕ-themes of A to a_n with respect to F_1 and F_2, as specified by Procedure R:

<u>Procedure R:</u> Imagine an α-ordered set $B = \{b_1, b_2, \dots, b_n\}$, and two sets, H_1 and H_2, such that, for each $b_i \in B$ ($0 < i \leq n+1$),

$$b_i \in \breve{H}_1 \overset{\top}{=} b_i \in H_2 \qquad ; \text{ and}$$
$$b_i \in H_1 \overset{\top}{=} b_i \in \breve{H}_2 .$$

I(1)(A) Consider the α-object of B whose α-length $= 2$, whose second α-half is b_n, and whose first α-half is b_{n-1}.

(B) Does there exist, for some m $(1 < m \leq n-1)$, an α -object of B whose second α -half is b_m and whose first α -half is b_{m-1}, and which is such that

(a) either $b_m \overset{H_1}{=} b_n$ or $b_m \overset{H_2}{=} b_n$; and

(b) either $b_{m-1} \overset{H_1}{=} b_{n-1}$ or $b_{m-1} \overset{H_2}{=} b_{n-1}$?

(C) If the answer to (B) is 'yes', proceed to (2); if the answer to (B) is 'no', consider each α -object of B whose α -length $= 1$ to be an ' α -motif of B to b_n with respect to H_1 and H_2 ', and proceed to II.

(2)(A) Consider the α -object of B whose α -length $= 3$, whose third α -third is b_n, whose second α -third is b_{n-1}, and whose first α -third is b_{n-2}.

(B) Does there exist, for some m $(2 < m \leq n-1)$, an α -object of B whose third α -third is b_m, whose second α -third is b_{m-1}, and whose first α -third is b_{m-2}, and which is such that

(a) either $b_m \overset{H_1}{=} b_n$ or $b_m \overset{H_2}{=} b_n$;

(b) either $b_{m-1} \overset{H_1}{=} b_{n-1}$ or $b_{m-1} \overset{H_2}{=} b_{n-1}$; and

(c) either $b_{m-2} \overset{H_1}{=} b_{n-2}$ or $b_{m-2} \overset{H_2}{=} b_{n-2}$?

(C) If the answer to (B) is 'yes', proceed to (3); if the answer to (B) is 'no', consider each of the α -objects of B whose α-length = 2 , and whose second α -half is either $\overset{H_1}{=} b_n$ or $\overset{H_2}{=} b_n$, and whose first α -half is either $\overset{H_1}{=} b_{n-1}$ or $\overset{H_2}{=} b_{n-1}$, but whose last α -half is $\neq b_n$, to be an ' α -motif of B to b_n with respect to H_1 and H_2 ', and proceed to II.

$$\vdots$$

II(1) Consider any α -motif of B to b_n with respect to H_1 and H_2 , written $\{b_k, b_{k+1}, \ldots, b_{k+p}\}$.

(2) Does there exist, for some m ($k \leq m \leq k+q$), an α -shortest α -object of B ,

$$b' = \{b_m, b_{m+1}, \ldots, b_{m+q}\},$$

such that

either $b_k \overset{H_1}{=} b_m$ or $b_k \overset{H_2}{=} b_m$;

either $b_{k+1} \overset{H_1}{=} b_{m+1}$ or $b_{k+1} \overset{H_2}{=} b_{m+1}$;

$$\vdots$$

either $b_{k+x} \overset{H_1}{=} b_{m+q}$ or $b_{k+x} \overset{H_2}{=} b_{m+q}$;

either $b_{k+x+1} \overset{H_1}{=} b_m$ or $b_{k+x+1} \overset{H_2}{=} b_m$;

either $b_{k+x+2} \overset{H_1}{=} b_{m+1}$ or $b_{k+x+2} \overset{H_2}{=} b_{m+1}$;

$\vdots$

either $b_{k+2x} \overset{H_1}{=} b_{m+q}$ or $b_{k+2x} \overset{H_2}{=} b_{m+q}$;

either $b_{k+2x+1} \overset{H_1}{=} b_m$ or $b_{k+2x+1} \overset{H_2}{=} b_m$;

either $b_{k+2x+2} \overset{H_1}{=} b_{m+1}$ or $b_{k+2x+2} \overset{H_2}{=} b_{m+1}$;

$\vdots$

either $b_{k+p-1} \overset{H_1}{=} b_{m+q-1}$ or $b_{k+p-1} \overset{H_2}{=} b_{m+q-1}$; and

either $b_{k+p} \overset{H_1}{=} b_{m+q}$ or $b_{k+p} \overset{H_2}{=} b_{m+q}$?

(3) If the answer to (2) is 'yes', consider each of the α-objects of B which is such that

(a) it has the same α-length as b' ; and

(b) its first α-part is either $\overset{H_1}{=} b_m$ or $\overset{H_2}{=} b_m$;

its second α-part is either $\overset{H_1}{=} b_{m+1}$ or $\overset{H_2}{=} b_{m+1}$;

$\vdots$

its last α-part is either $\overset{H_1}{=} b_{m+q}$ or $\overset{H_2}{=} b_{m+q}$;

to be an 'α-theme of B to b_n with respect to H_1 and H_2 '; if the answer to (2) is 'no', consider each of the α-motifs of B to b_n with respect to H_1 and H_2 to be an 'α-theme of B to b_n with respect to H_1 and H_2 '.

I shall then say that <u>A' is the ϕ-theme set of A to a_n</u>

<u>with respect to F_1 and F_2</u>.

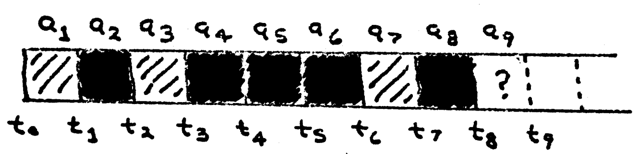
The required rule, then, which I shall call <u>the Thematic Rule</u>, is as follows:

<u>Thematic Rule:</u> Given a ϕ-ordered set $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ and two sets, F_1 and F_2, such that, for each $a_i \in A$ $(0 < i)$,

$$a_i \in \overset{\cup}{F_1} \overset{\top}{=} a_i \in F_2 \quad ; \text{ and}$$
$$a_i \in F_1 \overset{\top}{=} a_i \in \overset{\cup}{F_2} \quad ,$$

consider the ϕ-theme set of A to a_n with respect to F_1 and F_2 to be the sample set when applying equation (15) to determine the probability that a_{n+1} is F_1.

To illustrate the Thematic Rule in application, consider again the temporally-ordered set A, discussed on page 38 above, a sample of which conforms to Diagram I:

Diagram I: 

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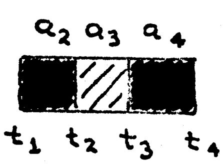
The set of t-motifs of A to a_8 with respect to the set of black things and the set of grey things consists of a single object [redacted],



which does <u>not</u> include an object of shorter temporal length satisfying condition II(2) of Procedure R. Hence, the set of t-themes of

A to a_8 with respect to the set of black things and the set of grey things consists of the same single object, which I shall call 'π'.

π is such that

(a) its first temporal third is black;

(b) its second temporal third is grey; and

(c) its third temporal third is black.

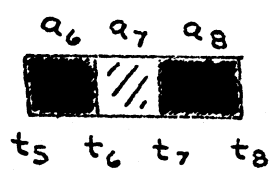
It is also such that

(d) the following element (i.e. a_5) is black.

Since it is the sole object in the sample set, the probability by (15) that the <u>next</u> object satisfying conditions (a) - (c) will also satisfy condition (d) is

$$\frac{t+1}{s+2} = \frac{1+1}{1+2} = \frac{2}{3} .$$

But the next object satisfying conditions (a) - (c) is object



Hence, the probability by (15) is $2/3$ that this object will be followed by a black element (as a_9) - which solves uniquely the problem initially set before us, and indicates how the Thematic Rule eliminates in practice the Laplacean ambiguity of applying equation (5) to ordered samples.

I shall assume the application of the Thematic Rule (or an extension of it) in all succeeding discussions whenever ordered samples are to serve as the basis for inductive inferences.

Appendix I

Imagine that condition (c) of 2 were absent, and that P were the proposition $\langle\langle\{P\}, T\rangle, 0, 0\rangle$; i.e. that P were

(M): $$P \notin T$$

Imagine that P were false. Then (M) would be true. But since (M) is identical to P , then P would be true. Imagine, on the contrary, that P were true. Then $P \in T$. But since (M) is identical to P , (M) would then be true. But (M) says that $P \notin T$. Hence, P would be false.

In either case, therefore, P would be both true and false: a contradiction.

Without condition (c), therefore, 2 and 2.1 would imply the above version of the Paradox of the Liar.

Appendix II

What if I had not excluded non-random propositions from probability calculations in 6.? Consider the consequences as exemplified in the grue-paradox of Goodman (Goodman [1], pp. 72-81):

Imagine five sets, $A_1 - A_5$, defined thusly:

A_1	$=$	the set of rubies;
A_2	$=$	the set of observed things;
A_3	$=$	the set of green things;
A_4	$=$	the set of temporal events prior to the advent of the year 2000 A.D.;
A_5	$=$	the set of temporal events including and posterior to the advent of 2000 A.D.;

and a sixth set A_6 defined as

$$A_6 = (A_3 \times A_4) + (\breve{A}_3 \times A_5)$$

(i.e. the set of grue things).

Imagine further that it is a few moments before New Year's Eve, December 31, 1999, and that proposition P_1 is true:

$$P_1: \quad \langle\langle A_1 \times A_2 \times A_4 , \breve{A}_3 \rangle , 0, 0 \rangle$$

(i.e. every observed ruby has been green). But then, by the definition of A_6 , proposition P_2 is also true:

$$P_2: \quad \langle\langle A_1 \times A_2 \times A_4 , \breve{A}_6 \rangle , 0, 0 \rangle$$

(i.e. every observed ruby has been grue).

By (15), consequently (with suitable assumptions), the probability of the next examined ruby being green, and the probability of it being grue, would be equal and greater than $\frac{1}{2}$. But by the definition of A_6, being grue after the advent of 2000 A. D. implies being not green. Hence, on the same evidence, the probability of the next examined ruby being green, and the probability of it being not green, would be equal and greater than $\frac{1}{2}$ – a paradox noticeably fatal to inductive consistency.

The paradox is avoided, however, by excluding non-random propositions from probability calculations, for P_2 is a non-random proposition:

(A) The set $A_7 = A_4 + A_5$ (i.e. the set of temporal events) is ordered, thereby satisfying 4.3(a);

(B) any non-green object existing before the advent of 2000 A.D. is a temporal event, thereby satisfying 4.3(b); and

(C) any one of the rubies observed before the advent of 2000 A.D. satisfies 4.3(c)(1) for some temporal event c_m, and thus is such that, if it were instead identical to some c_n, for any c_n identical to a temporal event after the advent of 2000 A.D., it could not be a ruby observed before the advent of 2000 A.D., hence satisfying 4.3(c)(1) – (4).

And thus the paradox, and its fellows, are avoided.

Appendix III

The reader may have wondered why I did not offer

$$(1') \qquad \dashv\vdash(p \rightarrow q) = \frac{N\,(P \times Q)}{N\,(P)},$$

instead of the more complicated summation formula given as (1). Consider the consequences: Equations (13.11) and (13.31) would remain as derived:

$$(13.11) \qquad \dashv\vdash(Zf_i \rightarrow h) = \frac{\binom{b_i}{t}\binom{n-b_i}{s-t}}{\binom{n}{s}}; \text{ and}$$

$$(13.31) \qquad \dashv\vdash(Zhf_i \rightarrow h') = \frac{b_i - t}{n - s}$$

But equation (13.21) would fail. Since the cardinality of the set of elements of IV which imply Z (i.e. IV_2) is $2^n \binom{n}{s}(n-s)$, while the cardinality of the set of elements of IV which imply Zf_i (i.e. IV_2') is $\binom{n}{b_i}\binom{n}{s}(n-s)$, equation (13.21) would become instead, by (1'),

$$(13.21') \qquad \dashv\vdash(Z \rightarrow f_i) = \frac{\binom{n}{b_i}\binom{n}{s}(n-s)}{2^n\binom{n}{s}(n-s)} = \frac{\binom{n}{b_i}}{2^n}.$$

Equation (14), therefore, would become

$$(14') \quad \dashv\vdash (Zh \rightarrow h') = \frac{\sum_{i=0}^{n} (b_i - t)\binom{n}{b_i}\binom{b_i}{t}\binom{n-b_i}{s-t}}{(n-s)\sum_{k=0}^{n}\binom{n}{b_k}\binom{b_k}{t}\binom{n-b_k}{s-t}},$$

which reduces algebraically to

$$= \frac{\sum_{i=0}^{n} \frac{1}{(b_i-t-1)!\,[(n-b_i)-(s-t)]!}}{(n-s)\sum_{k=0}^{n}\frac{1}{(b_k-t)!\,[(n-b_k)-(s-t)]!}}.$$

Since, for any integer k, $(-k)! = \infty$, one may eliminate the zero terms of the summations by rewriting them thusly:

$$= \frac{\sum_{i=t+1}^{n-(s-t)} \frac{1}{(b_i-t-1)!\,[(n-b_i)-(s-t)]!}}{(n-s)\sum_{k=t}^{n-(s-t)}\frac{1}{(b_k-t)!\,[(n-b_k)-(s-t)]!}}.$$

This, as Isaac Newton might have put it, "amounts to the same thing"[1] as

$$= \frac{\dfrac{\sum_{r=0}^{n-s-1}\binom{n-s-1}{r}}{(n-s-1)!}}{(n-s)\left(\dfrac{\sum_{r=0}^{n-s}\binom{n-s}{r}}{(n-s)!}\right)}$$

1. See, for example, Newton's "On Fluxions" as translated by Evelyn Walker and reprinted in part in Smith, Vol. II, p. 614.

(which phrase, freely translated, means 'I shall not give a proof I haven't got, but it's valid anyway, as the reader may attest by working out a few examples'), which then reduces algebraically to

$$= \frac{\sum_{r=0}^{n-s-1} \binom{n-s-1}{r}}{\sum_{r=0}^{n-s} \binom{n-s}{r}} .$$

But since Pascal proved that $\sum_{r=0}^{n} \binom{n}{r} = 2^{n}$, this becomes

$$= \frac{2^{n-s-1}}{2^{n-s}} ,$$

and finally

$$(15') \qquad \dashv\vdash(Zh \rightarrow h') = \frac{1}{2} .$$

Equation (15'), roughly and in general, implies that one ought never to let one's past experiences influence one's expectations for the future. The choice of (1') rather than (1), therefore, would lead to results which, to borrow Carnap's apt phrase, "would obviously be in striking contradiction to the basic principle of all inductive reasoning" (Carnap [3], p. 45 (reprint)).
