RESULTS ON STAR SELECTION PRINCIPLES AND WEAKENINGS OF NORMALITY IN Ψ-SPACES

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Abstract

The first part of this dissertation is related with the theory of star selection principles. In particular, with the star and the strongly star versions of Menger, Hurewicz and Rothberger. We provide an equivalence between the Lindelöf property and its star versions in the classes of metaLindelöf and paraLindelöf spaces. Because of this result and the characterization of paracompactness in terms of stars, we obtain a single proof of the equivalence between the properties Menger, Hurewicz, Rothberger and compactness with their respective star versions in the classes of metaLindelöf and paraLindelöf spaces. Then, we present a class of spaces that contains both the Ψ-spaces and the Niemytzki plane and show that the characterizations given by Bonanzinga and Matveev for Ψ-spaces, are preserved in this broader class of spaces. A characterization in this class of spaces of the strongly star-Menger property in terms of games is also provided. Furthermore, some results are obtained for the absolute versions of these star selections principles. For small spaces, there is an equivalence between the absolute version of the strongly star-Lindelöf property and the selective versions of both the strongly star-Menger property and the strongly star-Hurewicz property. We mention and review some of the examples that make a distinction between the Menger, Hurewicz and Rothberger properties and its star versions. We provide an example of a normal star-Menger not strongly star-Menger space. Regarding unions of spaces, we prove that Lindelöf spaces that can be written as a union of less than $\mathfrak{d}$ ($\mathfrak{b}$) many star-Hurewicz spaces are Menger (Hurewicz) and Lindelöf spaces that can be written as a union of less than $\mathfrak{b}$ many star-Menger spaces are Menger. Analogous results for the star versions of Lindelöf are obtained.

The second part of this dissertation deals with weakenings of normality in Mrówka-Isbell Ψ-spaces. We present an equivalence between π-normal and almost-normal spaces. Then we provide three relevant counterexamples: a mildly-normal not partly-normal Ψ-space, a quasi-normal not almost-normal Ψ-space (both in ZFC), and a consistent example of a Luzin mad family such that its associated Ψ-space is quasi-normal.

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Chapter 1

Introduction

The objective of this chapter is twofold. First we will introduce the basic notation, main concepts and definitions that will be used throughout this work (Section 1.1). Second, we will provide some background, history and motivation in the theory of selection principles (Section 1.2) and the study of Ψ-spaces (Section 1.3).

1.1 Basic Notation and Definitions

We will use standard topological and set-theoretic notation such as in [28] and [56]. A space X will always denote a regular topological space unless otherwise stated (that is, points in X are closed and can be separated from closed sets that do not contain them). Given a set D, $|D|$ and $\mathcal{P}(D)$ denote, respectively, the cardinality and the power set of D and, $[D]^\kappa = \{B \subseteq D : |B| = \kappa\}$, $[D]^{<\kappa} = \{B \subseteq D : |B| < \kappa\}$, $[D]^{\leq\kappa} = \{B \subseteq D : |B| \leq \kappa\}$ for some cardinal κ. Sometimes we write $[D]^\omega$ or $[D]^{\leq\omega_1}$ instead of $[D]^{\aleph_0}$ or $[D]^{\leq\aleph_1}$, respectively.

Recall that a family $\mathcal{A}$ of infinite subsets of ω is called an **almost disjoint family** if and only if any two distinct members meet in a finite set (for each $a, b \in \mathcal{A}$, $a \neq b \to |a \cap b| < \omega$). An almost disjoint family is **mad** (maximal almost disjoint), if it is not properly included in any larger almost disjoint family. All almost disjoint families considered here will be infinite.

The following two important classes of spaces will be considered many times in this work.

Definition 1.1.1 ([3], [63])**.** *Given an almost disjoint family $\mathcal{A}$, the* **Mrówka-Isbell Ψ-space** $\Psi(\mathcal{A})$ *is defined as follows: the underlying set is $\omega \cup \mathcal{A}$; if $n \in \omega$, $\{n\}$ is open and if $a \in \mathcal{A}$, then for any finite set $F \subset \omega$, $\{a\} \cup a \setminus F$ is a basic open set of a.*

Ψ-spaces are separable, first countable, zero dimensional regular spaces. We will discuss more about them in Section 1.3.

Definition 1.1.2. *The* **Niemytzki plane** *on a set $X \subseteq \mathbb{R}$, denoted by $N(X)$, has as underlying set $X \times \{0\} \cup \mathbb{R} \times (0, \infty)$. The open upper half-plane $\mathbb{R} \times (0, \infty)$ has the Euclidean topology and the set $X \times \{0\}$ has the topology generated by all sets of the form $\{(x, 0)\} \cup B$ where $x \in X$ and B is an open disc in $\mathbb{R} \times (0, \infty)$ which is tangent to $X \times \{0\}$ at the point $(x, 0)$.*

The Niemytzki plane is also called Niemytzki's tangent disk topology, bubble space or Moore plane (since it is a classic example of a separable, nonmetrizable Moore space). It is important to mention that $X = \mathbb{R}$ is the way that it was originally defined by Niemytzki (see [65]).

For a pair of functions $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means that $f(n) \leq g(n)$ for all n). A subset B of ω^ω is **unbounded** if there is no $g \in \omega^\omega$ such that $f \leq^* g$ for each $f \in B$. A subset D of ω^ω is **dominating** if for each $g \in \omega^\omega$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an un$\mathfrak{b}$ounded subset of ω^ω is denoted by $\mathfrak{b}$, and the minimal cardinality of a $\mathfrak{d}$ominating subset of ω^ω is denoted by $\mathfrak{d}$.

Recall that a **cover $\mathcal{U}$ of a space** X is a subset of the power set $\mathcal{P}(X)$ of X such that $\bigcup \mathcal{U} = X$. In addition, we call $\mathcal{U}$ an **open cover of** X, if each of its elements is an open set in X. Open covers $\mathcal{V}$ of X which satisfy that for each $x \in X$, x belongs to all but finitely many elements of $\mathcal{V}$ are called **γ-covers**.

Notation 1.1.3. *Given a space X, $\mathcal{O}(X)$ denotes the set of open covers of X and $\Gamma(X)$ denotes the set of all γ-covers of X. We will simply write $\mathcal{O}$ and Γ when there's no confusion.*

1.2 Selection Principles

In [74] it is stated that "The study of selection principles in mathematics is the study of diagonalization processes". That is, "a selection principle is a rule asserting the possibility of obtaining mathematically significant objects by selecting elements from given sequences of sets" [93]. Researchers agree (see for instance [73], [55]), that the beginnings of selection principles in Topology took place in articles by E. Borel [16], K. Menger [62], W. Hurewicz

[43] and F. Rothberger [68]. But, since the appearance of Scheepers's paper "Combinatorics of open covers I: Ramsey theory" in 1996 [72], the field has enjoyed much attention. In the present day, the theory of selection principles in Topology has many connections with other areas of mathematics such as Game Theory, Set Theory, Function spaces and hyperspaces, Ramsey Theory, etc.

All the star selection principles we study in this work derive from the three classical selection principles Menger, Hurewicz and Rothberger. In [72] Scheepers provided convenient notation for a family of selection principles that have become standard in the literature:

Let X be a topological space let $\mathscr{A}$ and $\mathscr{B}$ be families of covers of X. We consider the following selection hypotheses:

- $S_{fin}(\mathscr{A},\mathscr{B})$: for each sequence $\{\mathcal{A}_n : n \in \omega\}$ of elements of $\mathscr{A}$ there is a sequence $\{\mathcal{F}_n : n \in \omega\}$ such that for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{A}_n$ and $\bigcup_{n\in\omega} \mathcal{F}_n \in \mathscr{B}$.
- $S_1(\mathscr{A},\mathscr{B})$: for each sequence $\{\mathcal{A}_n : n \in \omega\}$ of elements of $\mathscr{A}$ there is a sequence $\{A_n : n \in \omega\}$ such that for each n, $A_n \in \mathcal{A}_n$ and $\{A_n : n \in \omega\}$ is an element of $\mathscr{B}$.
- $U_{fin}(\mathscr{A},\mathscr{B})$: for each sequence $\{\mathcal{A}_n : n \in \omega\}$ of elements of $\mathscr{A}$ there is a sequence $\{\mathcal{F}_n : n \in \omega\}$ such that for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{A}_n$ and $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathscr{B}$.

Recall (Notation 1.1.3) that $\mathcal{O}$ and Γ denote, respectively, the set of open covers and the set of γ-covers of a space X.

Definition 1.2.1. *A space X is:*

- **Menger** *(M) if $S_{fin}(\mathcal{O}, \mathcal{O})$ holds.*
- **Rothberger** *(R) if $S_1(\mathcal{O}, \mathcal{O})$ holds.*
- **Hurewicz** *(H) if $U_{fin}(\mathcal{O}, \Gamma)$ holds.*

Observation 1.2.2. *The following diagram holds:*

$$\begin{tikzcd} H \arrow[r] & M \arrow[dr] & & R \arrow[ll] \\ & & L & \end{tikzcd}$$

Diagram 1.1: The Classical Selection Principles.

Proof. Indeed, Assume a space X is Hurewicz (Rothberger, respectively), and let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers of X. There is a sequence $\{\mathcal{V}_n : n \in \omega\}$ ($\{U_n : n \in \omega\}$ resp.) such that for each n, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ ($U_n \in \mathcal{U}_n$ resp.) and $\{\bigcup \mathcal{V}_n : n \in \omega\} \in \Gamma$ ($\{U_n : n \in \omega\} \in \mathcal{O}$). Then, in particular, $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{O}$ (the sequence $\{\{U_n\} : n \in \omega\}$ satisfies that for each n, $\{U_n\} \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n\in\omega}\{U_n\} \in \mathcal{O}$). Thus, X is Menger. Now assume that X is Menger and let $\mathcal{U} \in \mathcal{O}(X)$. For each $n \in \omega$ let $\mathcal{U}_n = \mathcal{U}$. Then for the constant sequence $\{\mathcal{U}_n : n \in \omega\}$ there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each n, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{O}$. Observe that $\bigcup_{n\in\omega} \mathcal{V}_n \in [\mathcal{U}]^{\omega}$. Hence, X is Lindelöf. $\square$

These arrows do not reverse. Examples can be found in Section 2.6. In 1924 K. Menger introduced the following property for metric spaces:

Definition 1.2.3 ([62]). *A metric space* (X, d) *has the* **basis property** *if for every basis* $\mathcal{B}$ *of* X *there is a sequence* $\{B_n : n \in \omega\}$ *of elements of* $\mathcal{B}$ *such that*

$$\lim_{n\to\infty} diam(B_n) = 0 \qquad \text{and} \qquad X = \bigcup_{n\in\omega} B_n \,.$$

Menger conjectured that a metric space has the basis property if and only if it is σ-compact (a space is called *σ-compact* if it can be written as a countable union of compact spaces). Hurewicz proved in [43] that a metric space (X, d) has the basis property if and only if $S_{fin}(\mathcal{O}, \mathcal{O})$ holds and showed that for analytic spaces, Menger's conjecture is true. In addition, in [44] he attributed to Sierpiński that Luzin sets are Menger and not σ-compact (an uncountable subset of the reals that has countable intersection with every meager set it's called a *Luzin* set). Thus, assuming **CH**, Menger's conjecture is false (Luzin sets can be constructed using **CH**). It wasn't until 1988 that Fremlin and Miller proved in **ZFC** that Menger's conjecture is false (see [31]).

In 1925 Hurewicz [43] introduced the principle $U_{fin}(\mathcal{O}, \Gamma)$ and conjectured that a metric space satisfies $U_{fin}(\mathcal{O}, \Gamma)$ if and only if it is σ-compact. A Sierpiński set (a subset of the reals S is called a *Sierpiński set* if it is uncountable, and for each Lebesgue measure zero set N, $S \cap N$ is countable) satisfies $U_{fin}(\mathcal{O}, \Gamma)$ and it is not σ-compact. Since a Sierpiński set can be constructed using **CH**, Hurewicz's conjecture is consistently false. It was until 1996, that Miller proved in [48] that Hurewicz conjecture is false in **ZFC**.

In 1938 F. Rothberger [68] introduced the principle $S_1(\mathcal{O}, \mathcal{O})$ and showed that if a metric space satisfies $S_1(\mathcal{O}, \mathcal{O})$, then it has strong measure zero (X has *strong measure zero* if for each sequence of positive reals $\{\epsilon_n : n \in \omega\}$ there is a sequence of intervals $\{I_n : n \in \omega\}$ such that for each n, $diam(I_n) \leq \epsilon_n$ and $X \subseteq \bigcup_{n\in\omega} I_n$). In 1942 he proved [69] that the converse does not hold.

For a detailed study on the beginnings and evolution of the study of selection principles, see [73] (see also [74]). For a more recent work that contaings slightly simplified solutions to Menger's and Hurewicz's problems and conjectures see [90] (see also [92]).

Star selection principles were first introduced and studied by Kočinac in [54] as natural generalizations of the selection principles Menger, Rothberger and Hurewicz. They will be the objects of study in the following chapter. Some of the results in Chapter 2 appeared in my joint paper [18] with Javier Casas de la Rosa and Paul Szeptycki. Chapter 2 starts with the definition of *star* (Definition 2.0.1), and some particular refinements of open covers (Definition 2.0.3) that are used to define the properties metaLindelöf, metacompact, paraLindelöf and paracompact. In Section 2.1 a proof of the characterization of paracompactness in terms of stars (due to A. H. Stone), is presented (Theorem 2.1.3). Section 2.2 is devoted to introduce the star versions of the Lindelöf property and to show that in the class of metaLindelöf spaces the properties Lindelöf and strongly star-Lindelöf are equivalent and that in the class of paraLindelöf spaces the properties Lindelöf and star-Lindelöf are equivalent (Proposition 2.2.7). In Section 2.3 we define the star versions of the Menger, Rothberger, Hurewicz and compactness properties as well as the basic relationships between them. Furthermore, using the characterization of paracompactness in terms of stars and the equivalence between the Lindelöf property and its star versions we show that if $\mathcal{P} \in \{$*compact, Menger, Rothberger, Hurewicz*$\}$ then a space X is $\mathcal{P}$ if and only if X is strongly star-$\mathcal{P}$ and metaLindelöf if and only if X is star-$\mathcal{P}$ and paraLindelöf (Theorem 2.3.8).

In Section 2.4 we present a class of spaces that contains the class of Ψ-spaces and the Niemytzki plane and analize under which conditions they have certain star selection principles. That is, strongly star-Lindelöf spaces which consist of the disjoint union of a closed discrete set with a σ-compact subspace. In turns out that the characterizations given by Bonanzinga and Matveev for the strongy star-Menger and strongy star-Hurewicz properties in Ψ-spaces, still hold in this class of spaces (Theorems 2.4.5 and 2.4.11).

Absolute versions of star selection principles are presented in Section 2.5 and some results are given for spaces of size smaller than $\mathfrak{b}$, or $\mathfrak{d}$. For instance, there is an equivalence between the absolute version of the strongly star Lindelöf property and the selective versions of both the strongly star-Menger property and the strongly star-Hurewicz property in spaces of size smaller than $\mathfrak{d}$ and $\mathfrak{b}$, respectively (Theorem 2.5.18).

In Section 2.6 examples of spaces that distinguish some star selection principles to the other are provided. In particular, a consistent example of a normal star-Menger not strongly star-Menger space is given (Proposition 2.6.20). In Section 2.7 we discuss how selection principles naturally relate to games and give a partial characterization of the strongly star-Menger property in terms of games for the class of spaces studied in Section 2.4.

Finally, in Section 2.8, we provide some results about spaces that can be written as a "small" union of spaces satisfying some selection principle. The results contained in this section are part of a work in progress with Javier Casas de la Rosa and William Chen-Mertens. In particular, we prove that a Lindelöf space that can be written as a union of less than $\mathfrak{d}$ ($\mathfrak{b}$) many star-Hurewicz spaces is Menger (Hurewicz) and a Lindelöf space that can be written as a union of less than $\mathfrak{b}$ many star-Menger spaces is Menger (Theorem 2.8.2). This improves a result by Tall stating that a Lindelöf space that can be written as a union of less than $\mathfrak{d}$ many compact spaces is Menger. Analogous results for the star versions of Lindelöf are obtained. For instance, a strongly star- Lindelöf space that can be written as a union of less than $\mathfrak{d}$ many Hurewicz spaces is strongly star-Menger (Theorem 2.8.8).

1.3 Ψ-Spaces

Mrówka-Isbell Ψ-spaces or simply Ψ-spaces (see definition 1.1.1), give a number of interesting counterexamples in many areas of topology including normality and related covering properties. They were introduced by Mrówka in 1954. He built a Ψ-space using a mad family to provide an example of a completely regular pseudocompact not countably compact space. In [34] Gillman and Jerison call such a space a Ψ-space and attribute it to Isbell[1].

[1] That's the reason why the name "Mrówka-Isbell Ψ-space" is widely spread, even though (apparently) there's no published work of Isbell where Ψ-spaces are defined.

It is important to point out that in [3], Alexandroff and Urysohn basically built a Ψ-space: for each $r \in \mathbb{R}$ fix a sequence $Q_r \subset \mathbb{Q}$ converging to r, then the family $\{Q_r : r \in \mathbb{R}\}$ is almost disjoint and we can identify $\mathbb{Q}$ with ω to obtain a Ψ-space.

Ψ-spaces are part of the normal Moore space conjecture (all normal Moore spaces are metrizable) first stated by F. B. Jones. Recall that a **development** for a space X is a sequence $\{G_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{St(x, G_n) : n \in \omega\}$ (see Definition 2.0.1) is a local base for x. A regular space with a development is called a **Moore space**. In 1937 Jones proved that assuming $2^{\aleph_0} < 2^{\aleph_1}$ every separable normal Moore space is metrizable. Then in 1951 Bing showed that if there is a **Q-set** (an uncontable set of reals in which every subset is relatively G_δ) then there is a separable non-metrizable normal Moore space. On the other hand, Heath showed in 1964 that the existence of a separable non-metrizable normal Moore space implies the existence of a Q-set. Then Silver showed that it is consistent that Q-sets exist. Hence, the metrizability of separable normal Moore spaces is independent from **ZFC**.

Ψ-spaces are examples of Moore spaces, they are regular and given any almost disjoint family $\mathcal{A}$ such that $\bigcup \mathcal{A} = \omega$, if for each $n \in \omega$ and each $a \in \mathcal{A}$ we define $U_n(a) = \{a\} \cup (a \setminus n)$ and $\mathcal{U}_n = \{U_n(a) : a \in \mathcal{A}\} \cup \{\{i\} : i < n\}$ then $\{\mathcal{U}_n : n \in \omega\}$ is a development of $\Psi(\mathcal{A})$.

Furthermore, since $\mathcal{A}$ is a closed discrete subset of $\Psi(\mathcal{A})$ and $\Psi(\mathcal{A})$ is separable (ω is a dense subspace), given that separable metrizable spaces are Lindelöf, if $\mathcal{A}$ is uncountable, then $\Psi(\mathcal{A})$ is non-metrizable. Therefore, Ψ-spaces (where $\mathcal{A}$ is uncountable) are examples of separable non-metrizable Moore spaces. Because of this, finding out when a Ψ-space is normal becomes relevant. Tall provided the following equivalence:

Theorem 1.3.1 ([87]). *The following are equivalent:*

1. *There is a non-metrizable separable normal Moore space*
2. *There is an uncountable almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is normal*
3. *There is a Q-set.*

Recall that assuming $\mathcal{A}$ is infinite, $\Psi(\mathcal{A})$ is not countably compact. If in addition, $\mathcal{A}$ is mad, then $\Psi(\mathcal{A})$ is pseudocompact. Given that normal pseudocompact spaces are countably compact we get that every time $\mathcal{A}$ is mad, $\Psi(\mathcal{A})$ is not normal. By Jones Lemma (if X is normal and separable then for each closed and discrete $Y \subseteq X$, $2^{|Y|} \leq 2^{\aleph_0}$), if $|\mathcal{A}| = \mathfrak{c}$, then $\Psi(\mathcal{A})$ is not normal. In other words, to construct a normal Ψ-space, the almost disjoint family shouldn't be mad and it has to have size smaller than $\mathfrak{c}$. Furthermore, the following has to hold:

Proposition 1.3.2 (Folklore). *Given any almost disjoint family $\mathcal{A}$, $\Psi(\mathcal{A})$ is normal if for every $\mathcal{C} \subseteq \mathcal{A}$, there is $X \subset \omega$ such that $\mathcal{C} = \{a \in \mathcal{A} : a \subseteq^* X\}$ and $\mathcal{A} \setminus \mathcal{C} = \{a \in \mathcal{A} : a \cap X =^* \emptyset\}$.*

The X in the previous proposition is called a separation of $\mathcal{C}$ and $\mathcal{A} \setminus \mathcal{C}$. More in general it will be said that two subfamiles $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$, have a **separation** if there is $X \subseteq \omega$ such that for each $b \in \mathcal{B}$, $b \subseteq^* X$ and for each

$c \in \mathcal{C}$, $c \cap X =^* \emptyset$. An important kind of almost disjoint families that will be used in Chapter 3 is the following:

Definition 1.3.3 ([57]). *An almost disjoint family $\mathcal{A}$ is called* **Luzin** *if it can be enumerated as $\{A_\alpha : \alpha < \omega_1\}$ so that for each $\alpha < \omega_1$ and each $n \in \omega$, $\{\beta < \alpha : A_\alpha \cap A_\beta \subseteq n\}$ is finite.*

In addition to the structural properties that Luzin families have, they provide an example of an almost disjoint family $\mathcal{A}$ such that every pair of uncountable subfamilies of $\mathcal{A}$ have no separation. In [41] it is stated that Luzin's construction was probably influenced by the Hausdorff gap [37] (see also [49]). Therefore, Ψ-spaces built from Luzin families are not normal:

Theorem 1.3.4 ([57]). *In a Luzin family no pair of uncountable subfamilies have a separation.*

Proof. Assume $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is a Luzin family and $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{\omega_1}$ have a separation. Let $X \subseteq \omega$ such that for each $B \in \mathcal{B}$, $B \subseteq^* X$ and for each $C \in \mathcal{C}$, $C \cap X =^* \emptyset$. Define a function $f : \mathcal{B} \cup \mathcal{C} \to \omega$ such that

$$\Big(\bigcup_{B \in \mathcal{B}} B \setminus f(B)\Big) \subseteq X \qquad \textit{and} \qquad \Big(\bigcup_{C \in \mathcal{C}} (C \setminus f(C)\Big) \cap X = \emptyset.$$

By the Pigeon hole principle, there are $n_0, n_1 \in \omega$ and $\mathcal{B}' \in [\mathcal{B}]^{\omega_1}$, $\mathcal{C}' \in [\mathcal{C}]^{\omega_1}$ such that for each $B \in \mathcal{B}'$, $f(B) = n_0$ and for each $C \in \mathcal{C}'$, $f(C) = n_1$. Let $m = max\{n_0, n_1\}$. Observe $\bigcup\{B \setminus m : B \in \mathcal{B}'\} \cap \bigcup\{C \setminus m : C \in \mathcal{C}'\} = \emptyset$. Thus, $\bigcup \mathcal{B}' \cap \bigcup \mathcal{C}' \subseteq m$. Now, take any countable subset $D \in [\mathcal{C}']^\omega$, since $\mathcal{B}'$ is uncontable there exists $A_\alpha \in \mathcal{B}'$ such that for all $A_\beta \in D$: $\beta < \alpha$. Since $\mathcal{A}$ is Luzin, there is $\beta_0 < \alpha$ such that $A_{\beta_0} \in D$ and $A_\alpha \cap A_{\beta_0} \nsubseteq m$, which is a contradiction. □

For a detailed survey on open problems and recent work on almost disjoint families and Ψ-spaces see [40] (see also [41] and [42]).

Weakenings of normality have been considered in the literature since the late 60's and early 70's. A (regular) space X is normal if for every pair (A, B) of disjoint closed subsets of X there is a pair of disjoint open sets (U_A, U_B) so that $A \subseteq U_A$, and $B \subseteq U_B$. In weakenings of normality (see Definition 3.1.1) we don't just consider closed sets, but also regular closed sets and π-closed sets (a finite intersetion of regular closed sets). Since every regular closed set is π-closed and, every π-closed set is closed, normality is the strongest of all these properties. In Chapter 3 we will study some of these weakenings of normality in the context of Ψ-spaces. Some of the results in this chapter appear in my submitted paper [33] with Paul Szeptycki.

In Section 3.1, these weak normality properties are introduced and some motivation and basic facts are provided. We prove that the properties π-normality and almost-normality are equivalent. The most important results of the chapter are contained in Section 3.2. First we construct a quasi-normal not almost-normal Ψ-space (Example 3.2.3), then we build a mildly-normal not partly-normal Ψ-space (Example 3.2.5). Both of these examples are in **ZFC** and have size $\mathfrak{c}$. Furthermore, assuming the existence of a mad family of true cardinality $\mathfrak{c}$ (Definition 3.1.3), these constructions give examples of a quasi-normal not almost-normal Ψ-space (Corollary 3.2.4) and a mildly-normal not partly-normal Ψ-space (Corollary 3.2.6) whose associated almost disjoint family is mad. That is, mildly-normality and quasi-normality in Ψ-spaces are not rescricted by the size or the maximality of the almost disjoint family, opposed to normality (as previously discussed). Finally, Example 3.2.10 is a consistent example (assuming **CH**) of a Luzin mad family such that its associated Ψ-space is quasi-normal. In Section 3.3 we define *strongly* $\aleph_0$*-separated* almost disjoint families (Definition 3.3.1), prove that almost-normal almost disjoint families have this property (Lemma 3.3.2) and assuming **CH**, we construct a strongly $\aleph_0$-separated mad family (Proposition 3.3.3).

Chapter 2

Star Selection Principles

In this chapter we will study some star selection principles that derive from the selection principles Menger, Rothberger and Hurewicz. As an introduction, we will define the notion of *star*, which is fundamental in this work. Then we will define some important refinements of open covers of a space (Definition 2.0.3) and the topological properties that relate to them.

Definition 2.0.1. *Given any space X, if $\mathcal{U} \subseteq \mathcal{P}(X)$, and $A \subseteq X$, the set*

$$St(A,\mathcal{U}) := \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$$

is called the **star of the set A with respect to $\mathcal{U}$**. *If $x \in X$, $St(x,\mathcal{U}) := St(\{x\},\mathcal{U})$.*

Even though $\mathcal{U}$ is any subset of $\mathcal{P}(X)$, the suggestive notation indicates that we will restrict to the cases where $\mathcal{U}$ is a cover of X with certain properties.

Definition 2.0.2. *If $\mathcal{U}, \mathcal{V}$ are covers of a topological space X, $\mathcal{V}$ will be called a* **refinement** *of $\mathcal{U}$, and it will be denoted by $\mathcal{V} \prec \mathcal{U}$, if for each $V \in \mathcal{V}$,* *there exists $U \in \mathcal{U}$ such that $V \subseteq U$. If in addition, each element of $\mathcal{V}$ is open, it will be called an* **open refinement**.

Refinements of open covers play an important role in selection principles. The following ones define important properties in Topology:

Definition 2.0.3. *If $\mathcal{U}, \mathcal{V}$ are covers of a topological space X, such that $\mathcal{V} \prec \mathcal{U}$, we say that $\mathcal{V}$ is a:*

- **point finite refinement** *if for every $x \in X$ the set $\{V \in \mathcal{V} : x \in V\}$ is finite.*
- **point countable refinement** *if for every $x \in X$ the set $\{V \in \mathcal{V} : x \in V\}$ is countable.*
- **locally finite refinement** *if for every $x \in X$ there is a neighbourhood U of x such that the set $\{V \in \mathcal{V} : U \cap V \neq \emptyset\}$ is finite.*
- **locally countable refinement** *if for every $x \in X$ there is a neighbourhood U of x such that the set $\{V \in \mathcal{V} : U \cap V \neq \emptyset\}$ is countable.*
- **star refinement** *of $\mathcal{U}$, denoted by $\mathcal{V} \prec_S \mathcal{U}$, if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $St(V, \mathcal{V}) \subseteq U$.*
- **barycentric refinement** *of $\mathcal{U}$, denoted by $\mathcal{V} \prec_b \mathcal{U}$, if for each $x \in X$, there exists $U \in \mathcal{U}$ such that $St(x, \mathcal{V}) \subseteq U$.*

If in addition, each element of $\mathcal{V}$ is open, the word "open" is added, for instance, "locally finite open refinement" instead of just "locally finite refinement" or "open star refinement" instead of "star refinement".

Observe that for any space X, and every cover $\mathcal{U}$ of X, each star refinement of $\mathcal{U}$ is a barycentric refinement of $\mathcal{U}$.

Definition 2.0.4. *A space X is called* **metacompact** *(***metaLindelöf***) if and only if it is Hausdorff and every open cover has a point finite open refinement (point countable open refinement).*

Definition 2.0.5. *A space X is called* **paracompact** *(***paraLindelöf***) if and only if it is Hausdorff and every open cover has a locally finite open refinement (locally countable open refinement).*

The following diagram shows the relationship between these properties. None of the arrows reverse.

$$\begin{tikzcd}
Paracompact \arrow[r, Rightarrow] \arrow[d, Rightarrow] & Metacompact \arrow[d, Rightarrow] \\
ParaLindel\ddot{o}f \arrow[r, Rightarrow] & MetaLindel\ddot{o}f
\end{tikzcd}$$

Diagram 2.1: Paracompact, ParaLindelöf, Metacompact and MetaLindelöf.

2.1 Paracompactness in terms of stars

Paracompactness has played an important role in Topology. In 1940 Tukey defined the *fully normal spaces* (a space X is fully normal if each open cover has an open star refinement). Then Dieudonné defined paracompactness as a generalization of compactness. In [47], Junnila writes "*In 1948, the period of 'modern general topology' was started by A.H. Stone's landmark paper in which full normality and paracompactness where shown to be equivalent properties*". As we'll see in Section 2.3, paracompactness plays an important role in star selection principles as well. But, as it might be expected in this realm, it is the "fully normal" interpretation that comes in handy. Therefore, we present a proof of this equivalence here that follows the one presented in [28].

In general, for any $A, B \subseteq X$, $\overline{A} \cup \overline{B} = \overline{A \cup B}$. It might be the case that $\bigcup_{\alpha \in \kappa} \overline{A}_\alpha \neq \overline{\bigcup_{\alpha \in \kappa} A_\alpha}$ for some infinite κ. But, if $\{A_\alpha\}_{\alpha \in \kappa}$ is a locally finite family, this equality hold:

Proposition 2.1.1. *For each ordinal κ and every locally finite family $\{A_\alpha\}_{\alpha \in \kappa}$ in some space X: $\bigcup_{\alpha \in \kappa} \overline{A}_\alpha = \overline{\bigcup_{\alpha \in \kappa} A_\alpha}$.*

Proof. Let $x \in \overline{\bigcup_{\alpha \in \kappa} A_\alpha}$. By local finiteness, there is a neighbourhood U of x such that $F = \{\alpha \in \kappa : U \cap A_\alpha \neq \emptyset\}$ is finite. Then, $x \notin \overline{\bigcup_{\beta \in \kappa \smallsetminus F} A_\beta}$. Since

$$x \in \overline{\bigcup_{\alpha \in \kappa} A_\alpha} = \overline{\bigcup_{\alpha \in F} A_\alpha} \cup \overline{\bigcup_{\beta \in \kappa \smallsetminus F} A_\beta}\,,$$

it holds that $x \in \overline{\bigcup_{\alpha \in F} A_\alpha} = \bigcup_{\alpha \in F} \overline{A_\alpha} \subseteq \bigcup_{\alpha \in \kappa} \overline{A_\alpha}$. ☐

This is a key property of paracompact spaces. In particular, it is important in the proof that paracompact spaces are normal and it provides the following corollary:

Corollary 2.1.2. *For every locally finite family $\{A_\alpha\}_{\alpha \in \kappa}$, $\{\overline{A}_\alpha\}_{\alpha \in \kappa}$ is also locally finite.*

Theorem 2.1.3 (Michael, Stone, Tukey)**.** *The following are equivalent:*

1. *X is paracompact.*
2. *For each $\mathcal{U} \in \mathcal{O}(X)$ there is $\mathcal{V} \prec \mathcal{U}$ open σ-locally finite refinement.*
3. *For each $\mathcal{U} \in \mathcal{O}(X)$ there is $\mathcal{V} \prec \mathcal{U}$ locally finite refinement.*

4. *For each $\mathcal{U} \in \mathcal{O}(X)$ there is $\mathcal{V} \prec \mathcal{U}$ closed locally finite refinement.*
5. *For each $\mathcal{U} \in \mathcal{O}(X)$ there is $\mathcal{V} \prec \mathcal{U}$ open barycentric refinement.*
6. *For each $\mathcal{U} \in \mathcal{O}(X)$ there is $\mathcal{V} \prec \mathcal{U}$ open star refinement.*
7. *For each $\mathcal{U} \in \mathcal{O}(X)$ there is $\mathcal{V} \prec \mathcal{U}$ open σ-discrete refinement.*

Proof. To show the equivalences we will procced as follows:
$1 \to 2 \to 3 \to 4 \to 1$ and $4 \to 5 \to 6 \to 7 \to 2$.

1 $\to$ 2: This is immediate since each open locally finite refinement is an open σ-locally finite refinement.

2 $\to$ 3: Let $\mathcal{U} \in \mathcal{O}(X)$ and let $\mathcal{V}$ be an open σ-locally finite refinement of $\mathcal{U}$. Hence, $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$, where for each $n \in \omega$, $\mathcal{V}_n$ is a locally finite family of open sets. For each $n \in \omega$ and each $V \in \mathcal{V}_n$ define

$$W_n^V = V \setminus \Big(\bigcup_{k<n} \bigcup \mathcal{V}_k\Big).$$

Clearly the family $\mathcal{I} = \{W_n^V : n \in \omega \wedge V \in \mathcal{V}_n\}$ covers X and is a refinement of $\mathcal{U}$ (since it refines $\mathcal{V}$). To show that $\mathcal{I}$ is locally finite, fix $x \in X$, let $k = min\{n \in \omega : x \in \bigcup \mathcal{V}_n\}$ and pick $V_x \in \mathcal{V}_k$ such that $x \in V_x$. Observe that for each $n > k$, if $V \in \mathcal{V}_n$, then $V_x \cap W_n^V = \emptyset$. Since for each $n \in \omega$, $\mathcal{V}_n$ is locally finite, for each $i \leq k$ there is U_i open such that $x \in U_i$ and $\{V \in \mathcal{V}_i : V \cap U_i \neq \emptyset\}$ is finite. If we define $U = \left(\bigcap_{i\leq k} U_i\right) \cap V_x$, then $x \in U$ and $\{A \in \mathcal{I} : U \cap A \neq \emptyset\}$ is finite.

Proof: 3 $\to$ 4: Let $\mathcal{U} \in \mathcal{O}(X)$, since X is regular, there is $\mathcal{W} \in \mathcal{O}(X)$ such that $\{\overline{W} : W \in \mathcal{W}\} \prec \mathcal{U}$. Let $\mathcal{V} \prec \mathcal{W}$ be a locally finite refinement. For each

$V \in \mathcal{V}$, let $U_V \in \mathcal{U}$ so that $\overline{V} \subset U_V$. For each $U \in \mathcal{U}$, let $F_U = \bigcup_{U_V = U} \overline{V}$. Claim: $\{F_U : U \in \mathcal{U}\}$ is a locally finite closed refinement. First, for each $U \in \mathcal{U}$, define $E_U = \bigcup_{U_V = U} V$. Since $\mathcal{V}$ is locally finite, for each $x \in X$, let W open so that $\{V \in \mathcal{V} : V \cap W \neq \emptyset\}$ is finite. Hence, $\{E_U : U \in \mathcal{U} \wedge E_U \cap W \neq \emptyset\}$ is finite. That is, $\{E_U : U \in \mathcal{U}\}$ is locally finite. By Proposition 2.1.1, for $U \in \mathcal{U}$, $F_U = \overline{E_U}$ and by Corollary 2.1.2, $\{F_U : U \in \mathcal{U}\}$ is locally finite as well.

4 $\rightarrow$ 1: Let $\mathcal{U} \in \mathcal{O}(X)$. First let $\mathcal{A}$ be a (closed) locally finite refinement of $\mathcal{U}$ and for each $x \in X$, fix V_x such that $x \in V_x$ and $\{A \in \mathcal{A} : A \cap V_x \neq \emptyset\}$ is finite. Since the family $\mathcal{V} = \{V_x : x \in X\}$ is an open cover of X, pick $\mathcal{F}$ closed locally finite refinement of $\mathcal{V}$. Now for each $A \in \mathcal{A}$ let $W_A = X \setminus \bigcup\{F \in \mathcal{F} : F \cap A = \emptyset\}$. Since $\mathcal{F}$ is locally finite, by Proposition 2.1.1, for each $A \in \mathcal{A}$, W_A is open and contains A. Furthermore, for each $A \in \mathcal{A}$ and $F \in \mathcal{F}$:

$$W_A \cap F \neq \emptyset \quad \textit{if and only if} \quad A \cap F \neq \emptyset \qquad (*)$$

For each $A \in \mathcal{A}$, let $U_A \in \mathcal{U}$ such that $A \subseteq U_A$ and define $B_A = W_A \cap U_A$. Observe that $\mathcal{B} = \{B_A : A \in \mathcal{A}\}$ is an open refinement of $\mathcal{U}$. Now we show that $\mathcal{B}$ is locally finite. Fix $x \in X$, there is U^x open such that $x \in U^x$ and $H = \{F \in \mathcal{F} : U^x \cap F \neq \emptyset\}$ is finite. In addition, for each $F \in H$, since $\mathcal{F}$ is a closed locally finite refinement of $\mathcal{V}$, there is some $y \in X$ such that $F \subseteq V_y$ and $\{A \in \mathcal{A} : A \cap V_y \neq \emptyset\}$ is finite. Hence, $\{A \in \mathcal{A} : \exists F \in H(F \cap A \neq \emptyset)\}$ is finite. By $(*)$, $\{A \in \mathcal{A} : \exists F \in H(F \cap W_A \neq \emptyset)\}$ is finite. Observe that since $\mathcal{F}$ is a cover, in particular $U^x \subseteq \bigcup H$. Since $\{A \in \mathcal{A} : \exists F \in H(F \cap W_A \neq \emptyset)\}$ is finite, and H is finite, it holds true that $\{A \in \mathcal{A} : W_A \cap U^x \neq \emptyset\}$ is finite. Thus, $\mathcal{B}$ is locally finite.

4 $\to$ **5:** Let $\mathcal{U} \in \mathcal{O}(X)$ and fix $\mathcal{H} \prec \mathcal{U}$ locally finite closed refinement. For each $H \in \mathcal{H}$, fix $U_H \in \mathcal{U}$ such that $H \subset U_H$. Since $\mathcal{H}$ is locally finite, for each $x \in X$, the set $T(x) = \{H \in \mathcal{H} : x \in H\}$ is finite. For $x \in X$, let

$$V_x = \bigcap_{H \in T(x)} U_H \cap \Big(X \setminus \bigcup_{H \in \mathcal{H} \setminus T(x)} H\Big)$$

By Proposition 2.1.1, $(\bigcup_{H \in \mathcal{H} \setminus T(x)} H)$ is closed and, therefore V_x is open. Hence, $\mathcal{V} = \{V_x : x \in X\} \in \mathcal{O}(X)$. Fix $x \in X$ and $H \in T(x)$, we show that $St(x, \mathcal{V}) \subseteq U_H$: first observe that $St(x, \mathcal{V}) = \bigcup\{V_y \in \mathcal{V} : x \in V_y\}$. Hence, if for some $y \in X$, $x \in V_y$, given that $x \in H$, it is the case that $H \in T(y)$ (otherwise $x \notin V_y$). Hence, $V_y \subseteq U_H$. Thus, $\mathcal{V} \prec_b \mathcal{U}$.

5 $\to$ **6:** Let $\mathcal{U} \in \mathcal{O}(X)$ and fix $\mathcal{V}, \mathcal{W} \in \mathcal{O}(X)$ such that $\mathcal{W} \prec_b \mathcal{V} \prec_b \mathcal{U}$. We show that $\mathcal{W} \prec_S \mathcal{U}$. Fix $W \in \mathcal{W}$. For each $x \in W$, let $V_x \in \mathcal{V}$ such that $St(x, \mathcal{W}) \subseteq V_x$. Then $St(W, \mathcal{W}) = \bigcup_{x \in W} St(x, \mathcal{W}) \subseteq \bigcup_{x \in W} V_x$. Fix $y \in W$ and observe that for each $x \in W$: $y \in W \subseteq St(x, \mathcal{W}) \subseteq V_x$. That is, if $x \in W$, then $y \in V_x$. Hence, $\bigcup_{x \in W} V_x \subseteq St(y, \mathcal{V})$. Fix $U \in \mathcal{U}$ so that $St(y, \mathcal{V}) \subseteq U$. Whence, $St(W, \mathcal{W}) \subseteq U$. That is, $\mathcal{W} \prec_S \mathcal{U}$.

6 $\to$ **7:** Let $\mathcal{U} \in \mathcal{O}(X)$. Define $\mathcal{U}_0 = \mathcal{U}$ and for each $n \in \omega$, let $\mathcal{U}_{n+1} \prec_S \mathcal{U}_n$. For each $U \in \mathcal{U}$ and $n > 0$ let

$$U^n = \{x \in X : \textit{ there is } W \textit{ open } (x \in W \wedge St(W, \mathcal{U}) \subseteq U)\}$$

Claim 1: for each $n > 0$, $\{U^n : U \in \mathcal{U}\}$ is an open refinemente of $\mathcal{U}$.

Fix $n > 0$,

- $\{U^n : U \in \mathcal{U}\}$ covers X: for $x \in X$, since $\mathcal{U}_n \prec_S \mathcal{U}$, for any $W \in \mathcal{U}_n$ with $x \in W$ there is $U \in \mathcal{U}$ such that $St(W,\mathcal{U}_n) \subseteq U$. Hence, $x \in U^n$.
- For each $U \in \mathcal{U}$, U^n is open: let $U \in \mathcal{U}$ and $x \in U^n$, then there is W open such that $x \in W$ and $St(W,\mathcal{U}_n) \subseteq U$. Since $\mathcal{U}_n$ is a cover of X, $W \subseteq St(W,\mathcal{U}_n) \subseteq U$. If $y \in W$, since $St(W,\mathcal{U}_n) \subseteq U$, then $y \in U^n$. That is, W is open, $x \in W$ and $W \subseteq U^n$.
- $\{U^n : U \in \mathcal{U}\}$ refines $\mathcal{U}$ since for each $U \in \mathcal{U}$, $U^n \subseteq U$.

Claim 2: for each $n > 0$ and each $U \in \mathcal{U}$, if $x \in U^n$ and $y \notin U^{n+1}$, then there is no $V \in \mathcal{U}_{n+1}$ such that $x, y \in V$.

Indeed, fix $n > 0$ and $U \in \mathcal{U}$. Since $\mathcal{U}_{n+1} \prec_S \mathcal{U}_n$, for each $V \in \mathcal{U}_{n+1}$ there is $W \in \mathcal{U}_n$ such that $St(V,\mathcal{U}_{n+1}) \subseteq W$. In particular, $V \subseteq W$. Therefore, if $x \in V \cap U^n$, since $x \in W \in \mathcal{U}_n$, $W \subseteq St(x,\mathcal{U}_n) \subseteq U$. This implies that $St(V,\mathcal{U}_{n+1}) \subseteq W \subseteq U$ and $V \subseteq U^{n+1}$.

Now list $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ and define, for each $\alpha < \kappa$ and $n > 0$

$$V_\alpha^n = U_\alpha^n \setminus \overline{\bigcup_{\beta<\alpha} U_\beta^{n+1}}$$

Let $\alpha, \beta < \kappa$ with $\beta < \alpha$, then $V_\alpha^n \subseteq X \setminus U_\beta^{n+1}$. Thus, by Claim 2, if $\alpha, \beta < \kappa$ with $\alpha \neq \beta$ and $x \in V_\alpha^n$, $y \in V_\beta^n$, then there is no $V \in \mathcal{U}_{n+1}$ such that $x, y \in V$. Hence, for each $n > 0$, the family of open sets $\{V_\alpha^n : \alpha < \kappa\}$ is discrete.

It only remains to show that $\bigcup_{n=1}^{\infty}\{V_\alpha^n : \alpha \in \kappa\} \in \mathcal{O}(X)$. Let $x \in X$ and let $\alpha = min\{\gamma < \kappa : x \in U_\gamma^i \textit{ for some } i > 0\}$. Observe that if $\beta < \alpha$, in particular, $x \notin U_\beta^{i+2}$.

Claim 3: $x \in V_\alpha^i$.

Assume on the contrary that $x \in \overline{\bigcup_{\beta<\alpha} U_\beta^{i+1}}$, then since $\mathcal{U}_{n+2}$ is a cover of X, pick $W_x \in \mathcal{U}_{n+2}$ such that $x \in W_x$. Thus, there is $\beta < \alpha$ and $y \in X$ such that $y \in W_x \cap U_\beta^{i+1}$. In addition, $x \notin U_\beta^{i+2}$ and this contradicts Claim 2. Whence, $\bigcup_{n=1}^{\infty}\{V_\alpha^n : \alpha \in \kappa\}$ is an open σ-discrete refinement of $\mathcal{U}$.

7 $\rightarrow$ 2: Immediate since every open σ-discrete refinement is open σ-locally finite refinement. $\square$

2.2 Lindelöfness in terms of stars

The goal of this section is to introduce the properties strongly star-Lindelöf, star-Lindelöf and, to present a characterization of Lindelöfness in terms of these properties together with the metaLindelöf and paraLindelöf properties (see Definitions 2.0.4 and 2.0.5). We will see in Section 2.3 how these star versions of the Lindelöf property relate with the star selection principles in a similar way as Lindelöf relates with the Menger, Hurewicz and Rothberger properties. First, let us recall the definition of the Lindelöf property and the proof that (regular) Lindelöf spaces are paracompact.

Definition 2.2.1. *A space X is called* **Lindelöf** *if and only if it is regular and every open cover has a countable subcover.*

The following result relies heavily on regularity.

Theorem 2.2.2 (Morita)**.** *Every open cover of a Lindelöf space has a locally finite open refinement.*

Proof: Assume X is Lindelöf and let $\mathcal{U} \in \mathcal{O}(X)$. Since X is, in particular, regular, for $x \in X$ fix $U_x, V_x \subseteq X$ open, such that $x \in U_x \subseteq \overline{U}_x \subseteq V_x \in \mathcal{U}$.

Given that X is Lindelöf, pick $E = \{e_n : n \in \omega\} \in [X]^\omega$ such that $\{U_{e_n} : n \in \omega\} \in \mathcal{O}(X)$. For $i \in \omega$ let

$$W_i = V_{e_i} \setminus \bigcup_{j<i} \overline{U}_{e_j}.$$

Clearly, W_i is open. We show that $\{W_i : i \in \omega\}$ is a locally finite refinement. First, it's a cover: let $x \in X$, and define $i(x) = min\{n \in \omega : x \in V_{e_n}\}$, thus $x \in W_{i(x)}$. Now, if $x \in X$, there is $n \in \omega$ so that $x \in U_{e_n}$. Then, for each $i > n$, $x \notin W_i$. ■

Corollary 2.2.3. *Every Lindelöf space is paracompact.*

Definition 2.2.4. *[[45], [58], [26]] A space X is called* **strongly star-Lindelöf** *(SSL) if, and only if, for each open cover $\mathcal{U} \in \mathcal{O}(X)$ there is a countable subset $C \in [X]^\omega$ so that $St(C,\mathcal{U}) = X$.*

Definition 2.2.5. *[[26]] A space X is called* **star-Lindelöf** *(SL) if, and only if, for each open cover $\mathcal{U} \in \mathcal{O}(X)$ there is a countable subset $\mathcal{V} \in [\mathcal{U}]^\omega$ so that $St(\bigcup\mathcal{V},\mathcal{U}) = X$.*

If a space X is Lindelöf then for any open cover $\mathcal{U}$, there is a countable subcover $\{U_n : n \in \omega\} \subseteq \mathcal{U}$ of X. If we select, for each $n \in \omega$, $x_n \in U_n$, and let $C = \{x_n : n \in \omega\}$, then $St(C,\mathcal{U}) = X$. Furthermore, if X is strongly star-Lindelöf then for any open cover $\mathcal{U}$, there is a countable subset $C = \{x_n : n \in \omega\} \in [X]^\omega$ so that $St(C,\mathcal{U}) = X$. If we select, for each $n \in \omega$, $U_n \in \mathcal{U}$ such that $x_n \in U_n$, and let $\mathcal{V} = \{U_n : n \in \omega\}$, then $St(\bigcup\mathcal{V},\mathcal{U}) = X$. In other words,

Observation 2.2.6. *Lindelöf $\to$ strongly star-Lindelöf $\to$ star-Lindelöf.*

The strongly star-Lindelöf property (as mentioned in [60]), is a joint generalization of the three well known topological properties: separability, the Lindelöf property and countable compactness. There are at least three places where it was studied independently: in [45] Ikenaga calls it *ω-1-star*, in [58] Matveev calls it *star-Lindelöf* and in [26], van Douwen, Reed, Roscoe and Tree call it *strongly 1-star-Lindelöf* (actually in [45] and [26] they define and study, for $n \in \mathbb{N}$, ω-n-star and strongly n-star-Lindelöf, respectively).

The following characterization of the Lindelöf property in terms of its star versions hadn't been noticed before (though, Song proved independently in [85] [Theorem 2.24], that paraLindelöf star-Lindelöf spaces are Lindelöf). In Section 2.3 this result will help us to obtain analogous characterizations for the class of compact, Menger, Hurewicz and Rothberger spaces (Theorem 2.3.8).

Proposition 2.2.7 ([18]). *For a topological space X the following are equivalent:*

1. *X is Lindelöf.*
2. *X is strongly star-Lindelöf and metaLindelöf.*
3. *X is star-Lindelöf and paraLindelöf.*

Proof. Trivially, (1) implies (2) and (3). Hence, it only remains to show (2) $\rightarrow$ (1) and (3) $\rightarrow$ (1).

(2) $\rightarrow$ (1): Assume X is a strongly star-Lindelöf, metaLindelöf space and fix $\mathcal{U} \in \mathcal{O}(X)$. Let $\mathcal{V}$ be a point-countable open refinement of $\mathcal{U}$. Fix $E \in [X]^\omega$ such that $St(E, \mathcal{V}) = X$. For each $e \in E$ let $\mathcal{V}_e = \{V \in \mathcal{V} : e \in V\}$. Since

$\mathcal{V}$ is point-countable, for each $e \in E$, $|V_e| \leq \omega$. Let $\mathcal{W} = \bigcup_{e \in E} V_e$. Thus, $\mathcal{W}$ is a countable subfamily of $\mathcal{V}$ and it is a cover of X. For each $W \in \mathcal{W}$, fix $U_W \in \mathcal{U}$ such that $W \subseteq U_W$. Then $\{U_W : W \in \mathcal{W}\} \in [\mathcal{U}]^\omega$. That is, X is Lindelöf.

(3) $\rightarrow$ (1): Assume X star-Lindelöf and paraLindelöf and fix $\mathcal{U} \in \mathcal{O}(X)$. Let $\mathcal{V}$ be a locally countable open refinement of $\mathcal{U}$. For each $x \in X$, take W_x an open neighbourhood of x such that $|\{W \in \mathcal{V} : W \cap W_x \neq \emptyset\}| \leq \omega$. For each $x \in X$, fix $V_x \in \mathcal{V}$ such that $x \in V_x$ and define $\mathcal{W} := \{W_x \cap V_x : x \in X\}$. Since X is star-Lindelöf, there exists $\mathcal{N} \in [\mathcal{W}]^{\leq\omega}$ such that $St(\bigcup \mathcal{N}, \mathcal{W}) = X$. The set $\mathcal{V}' = \{V \in \mathcal{V} : \exists N \in \mathcal{N}$ such that $N \cap V \neq \emptyset\}$ is countable since $\mathcal{V}' = \bigcup_{N \in \mathcal{N}} \mathcal{V}'_N$, where each $\mathcal{V}'_N = \{V \in \mathcal{V} : N \cap V \neq \emptyset\}$ is countable for each $N \in \mathcal{N}$. Moreover, $\bigcup \mathcal{V}' = X$. Indeed, let $x \in X$, then, there exists $y \in X$ such that $x \in V_y \cap W_y$ and $(V_y \cap W_y) \cap (\bigcup \mathcal{N}) \neq \emptyset$. Hence, there exists $N_y \in \mathcal{N}$ such that $(V_y \cap W_y) \cap N_y \neq \emptyset$. Thus $x \in V_y \in \mathcal{V}'$. For each $V \in \mathcal{V}'$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. The set $\{U_V : V \in \mathcal{V}'\} \in [\mathcal{U}]^\omega$ covers X. $\square$

2.3 Star versions of the Menger, Hurewicz and Rothberger principles

In this section we introduce the star versions of the Menger, Rothberger, Hurewicz and compactness properties as well as the basic relationships between them. Furthermore, we show that the characterization obtained in Section 2.2 for Lindelöf spaces can be obtained as well for the class of Compact, Menger, Rothberger and Hurewicz spaces.

In [54] Kočinac introduced the following general star selection principles (for a topological space X and $\mathscr{A}$, $\mathscr{B}$ families of covers of X):

- $S^*_{fin}(\mathscr{A},\mathscr{B})$: for each sequence $\{\mathcal{A}_n : n \in \omega\}$ of elements of $\mathscr{A}$ there is a sequence $\{\mathcal{F}_n : n \in \omega\}$ such that for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{A}_n$ and $\{St(\bigcup \mathcal{F}_n, \mathcal{A}_n) : n \in \omega\} \in \mathscr{B}$.
- $S^*_1(\mathscr{A},\mathscr{B})$: for each sequence $\{\mathcal{A}_n : n \in \omega\}$ of elements of $\mathscr{A}$ there is a sequence $\{A_n : n \in \omega\}$ such that for each n, $A_n \in \mathcal{A}_n$ and $\{St(A_n, \mathcal{A}_n) : n \in \omega\} \in \mathscr{B}$.
- $SS^*_{fin}(\mathscr{A},\mathscr{B})$: for each sequence $\{\mathcal{A}_n : n \in \omega\}$ of elements of $\mathscr{A}$ there is a sequence $\{F_n : n \in \omega\}$ such that for each n, F_n is a finite subset of X and $\{St(F_n, \mathcal{A}_n) : n \in \omega\} \in \mathscr{B}$.
- $SS^*_1(\mathscr{A},\mathscr{B})$: for each sequence $\{\mathcal{A}_n : n \in \omega\}$ of elements of $\mathscr{A}$ there is a sequence $\{x_n : n \in \omega\}$ such that for each n, $x_n \in X$ and $\{St(x, \mathcal{A}_n) : n \in \omega\} \in \mathscr{B}$.

Definition 2.3.1 ([54],[11]). *A space X is:*

- **star-Menger** *(SM) if $S^*_{fin}(\mathcal{O},\mathcal{O})$ holds.*
- **star-Rothberger** *(SR) if $S^*_1(\mathcal{O},\mathcal{O})$ holds.*
- **star-Hurewicz** *(SH) if $S^*_{fin}(\mathcal{O},\Gamma)$ holds.*
- **strongly star-Menger** *(SSM) if $SS^*_{fin}(\mathcal{O},\mathcal{O})$ holds.*
- **strongly star-Rothberger** *(SSR) if $SS^*_1(\mathcal{O},\mathcal{O})$ holds.*
- **strongly star-Hurewicz** *(SSH) if $SS^*_{fin}(\mathcal{O},\Gamma)$ holds.*

There is no harm in writting down the definitions explicitly:

Definition 2.3.2. *A space X is:*

- *star-Menger (SM) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.*
- *strongly star-Menger (SSM) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of X such that $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.*
- *star-Hurewicz (SH) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n.*
- *strongly star-Hurewicz (SSH) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of X such that for each $x \in X$, $x \in St(F_n, \mathcal{U}_n)$ for all but finitely many n.*
- *star-Rothberger (SR) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $U_n : n \in \omega\}$ such that for each $n \in \omega$, $U_n \in \mathcal{U}_n$ and $\{St(U_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.*
- *strongly star-Rothberger (SSR) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{x_n : n \in \omega\}$ of elements of X such that $\{St(x_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.*

The relationship between these properties is given in the next proposition.

Proposition 2.3.3 ([54],[11]). *The following diagram holds:*

$$\begin{tikzcd}
H \arrow[d] & M \arrow[d] & & R \arrow[d] \\
SSH \arrow[r] \arrow[dd] & SSM \arrow[dd] \arrow[dr] & & SSR \arrow[ll] \arrow[dd] \\
 & & SSL & \\
SH \arrow[r] & SM \arrow[dr] & & SR \arrow[ll] \\
 & & SL &
\end{tikzcd}$$

Diagram 2.2: Star Selection Principles.

Proof. The horizontal arrows follow from the definition. Let us prove the diagonal arrows. Assume a space X is strongly star-Menger (SM, respectively) and let $\mathcal{U} \in \mathcal{O}(X)$. For the constant sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$, where for each n, $\mathcal{U}_n = \mathcal{U}$, there is a sequence $\{F_n : n \in \omega\}$ ($\{\mathcal{V}_n : n \in \omega\}$) such that for n, $F_n \in [X]^{<\omega}$ ($\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$) and $\{St(F_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{O}(X)$ ($\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{O}(X)$). That is, $\bigcup_{n \in \omega} F_n$ is a countable subset of X such that $St(\bigcup_{n \in \omega} F_n, \mathcal{U}) = X$ ($\bigcup_{n \in \omega} \mathcal{V}_n$ is a countable subset of $\mathcal{U}$ such that $St(\bigcup_{n \in \omega} \mathcal{V}_n, \mathcal{U}) = X$). Hence, X is strongly star-Lindelöf (X is SL).

Now, for the vertical arrows, we will only show $M \to SSM \to SM$ since the remaining vertical arrows are proved similarly.

$M \to SSM$**:** Assume X is Menger and $\{\mathcal{U}_n : n \in \omega\}$ is a sequence of open covers of X. Fix a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that for each n, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n \in \omega} \mathcal{V}_n \in \mathcal{O}(X)$. Now, for each n and each $U \in \mathcal{V}_n$, pick $x_n^U \in U$ and let $F_n = \{x_n^U : U \in \mathcal{V}_n\}$. Since for each n, $\mathcal{V}_n$ is finite, F_n is finite as well.

Claim: $\{St(F_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{O}(X)$.

Indeed, if $x \in X$, there is some $n \in \omega$ and some $U \in \mathcal{V}_n$ such that $x \in U$. Thus, $x \in St(x_n^U,\mathcal{U}_n) \subseteq St(F_n,\mathcal{U}_n) \in \{St(F_n,\mathcal{U}_n) : n \in \omega\}$.

$SSM \to SM$**:** Assume X is SSM and $\{\mathcal{U}_n : n \in \omega\}$ is a sequence of open covers of X. Fix a sequence $\{F_n : n \in \omega\}$ such that for each n, $F_n \in [X]^{<\omega}$ and $\{St(F_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{O}(X)$. For each n and each $x \in F_n$, pick $U_n^x \in \mathcal{U}_n$ such that $x \in U_n^x$ and let $\mathcal{V}_n = \{U_n^x : x \in F_n\}$. For each n, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$.

Claim: $\{St(\bigcup \mathcal{V}_n,\mathcal{U}_n) : n \in \omega\} \in \mathcal{O}(X)$.

Indeed, if $x \in X$, there is some $n \in \omega$ such that $x \in St(F_n,\mathcal{U}_n)$. That is, there is $y \in F_n$ such that $x \in St(y,\mathcal{U}_n)$. Then, $x \in St(U_n^y,\mathcal{U}_n) \subseteq St(\bigcup \mathcal{V}_n,\mathcal{U}_n) \in \{St(\bigcup \mathcal{V}_n,\mathcal{U}_n) : n \in \omega\}$. □

None of the arrows in Proposition 2.3.3 reverse. In Section 2.6 we present examples that differentiate these properties.

One of the first things that Kočinac proved about star selection principles is that paracompact star-Menger (star-Rothberger) spaces are Menger (Rothberger) and that MetaLindeöf strongly star-Menger spaces spaces are Menger. In addition, Bonanzinga, Cammaroto and Kočinac proved in [11] that it is also true that paracompact star-Hurewicz spaces are Hurewicz. That is:

Theorem 2.3.4 ([54], [11])**.** *In the class of paracompact spaces the following holds :*

1. *the properties M, SM, and SSM are equivalent.*
2. *the properties R, SR, and SSR are equivalent.*
3. *the properties H, SH, and SSH are equivalent.*

As pointed out in Section 2.1 the tool used in these proofs is the equivalence between paracompactness and fully normality. Furthermore, the same idea can be applied to the class of compacts spaces. In [30] Fleishman proved that a Hausdorff space X is **countably compact** (every countable open cover of X has a finite subcover) if and only if for every open cover $\mathcal{U}$ of X there exists a finite subset $F \subseteq X$ such that $St(F,\mathcal{U}) = X$. This led to the study of the following star versions of compactness:

Definition 2.3.5 ([26])**.** *A space X is*

- **star-compact** *(SC) if for every open cover $\mathcal{U}$ of X there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\bigcup\mathcal{V},\mathcal{U}) = X$.*
- **strongly star-compact** *(SSC) if for every open cover $\mathcal{U}$ of X there exists a finite subset $F \subseteq X$ such that $St(F,\mathcal{U}) = X$.*

The following proposition shows the basic relationships of these star versions of compactness with the star selection principles previously defined.

Proposition 2.3.6 (Folklore)**.** *The following diagram holds:*

$$\begin{array}{ccccc} C & \longrightarrow & SSC & \longrightarrow & SC \\ \downarrow & & \downarrow & & \downarrow \\ H & & SSH & & SH \end{array}$$

Diagram 2.3: Compactness, Hurewicz and their Star Versions.

Proof. Assume a space X is compact, and let $\mathcal{U}$ be any open cover of X. Find $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $\bigcup\mathcal{V} = X$. For each $V \in \mathcal{V}$, pick $x_V \in V$, then $F = \{x_V : V \in \mathcal{V}\}$ is finite and $St(F,\mathcal{U}) = X$. That is, X is SSC. Now assume X is SSC, and let $\mathcal{U}$ be any open cover of X. Find a finite set $F \in [X]^{<\omega}$ such that $St(F,\mathcal{U}) = X$. For each $x \in F$, pick $V_x \in \mathcal{U}$ such that

$x \in V_x$. Then $\mathcal{V} = \{V_x : x \in F\} \in [\mathcal{U}]^{<\omega}$ and $St(\bigcup \mathcal{V}, \mathcal{U}) = X$. Hence, X is SC.

To prove the vertical arrows assume X is compact (SSC, SC, respectively) and let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Let $\{\mathcal{V}_n : n \in \omega\}$ ($\{F_n : n \in \omega\}$, $\{\mathcal{W}_n : n \in \omega\}$, respectively) such that for each n, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ ($F_n \in [X]^{<\omega}$, $\mathcal{W}_n \in [\mathcal{U}_n]^{<\omega}$, respectively) and $\bigcup \mathcal{V}_n = X$ ($St(F_n, \mathcal{U}_n) = X$, $St(\bigcup \mathcal{W}_n, \mathcal{U}_n) = X$, respectively). Hence, $\{\bigcup \mathcal{V}_n : n \in \omega\}$ ($\{St(F_n, \mathcal{U}_n) : n \in \omega\}$, $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$) is a γ-cover of X. Thus, X is Hurewicz (SSH, SH, respectively). $\square$

Putting together the results from Observation 2.2.6, Observation 1.2.2, Proposition 2.3.3 and Proposition 2.3.6 we get the following diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[dd] & H \arrow[r] \arrow[dd] & M \arrow[dr] \arrow[dd] & & R \arrow[ll] \arrow[dd] \\
& & & L \arrow[dd] & \\
SSC \arrow[r] \arrow[dd] & SSH \arrow[r] \arrow[dd] & SSM \arrow[dr] \arrow[dd] & & SSR \arrow[ll] \arrow[dd] \\
& & & SSL \arrow[dd] & \\
SC \arrow[r] & SH \arrow[r] & SM \arrow[dr] & & SR \arrow[ll] \\
& & & SL &
\end{tikzcd}$$

Diagram 2.4: The Complete Diagram.

In [26], van Douwen, Reed, Roscoe and Tree showed that metacompact strongly star-compact spaces are compact. It is also true that paracompact star-compact spaces are compact. Putting this result together with Theorem 2.3.4, we have:

Proposition 2.3.7. *Assume $\mathcal{P} \in \{$Compact, Menger, Rothberger, Hurewicz$\}$. Then, in the class of paracompact spaces: $\mathcal{P} \leftrightarrow$ star-$\mathcal{P}$.*

Proof. Since $\mathcal{P} \to$ star-$\mathcal{P}$, it only remains to show that $\mathcal{P} \leftarrow$ star-$\mathcal{P}$.
Assume X is paracompact and star-compact. Fix $\mathcal{U} \in \mathcal{O}(X)$. Since X is paracompact, let $\mathcal{V} \prec \mathcal{U}$ be an open star refinement. Using star-compactness on $\mathcal{V}$, fix $W \in [\mathcal{V}]^{<\omega}$ such that $St(\bigcup \mathcal{W}, \mathcal{V}) = X$. For each $W \in \mathcal{W}$, find $U_W \in \mathcal{U}$ such that $St(W, \mathcal{V}) \subseteq U_W$. Thus, $\{U_W : W \in \mathcal{W}\}$ is a finite subcover of X. Hence, X is compact.

Now, assume X is paracompact and star-Menger (star-Hurewicz). Let $(\mathcal{U}_n : n \in \omega)$ be any sequence of open covers of X. Since X is paracompact, for each $n \in \omega$ let $\mathcal{V}_n \prec \mathcal{U}_n$ be an open star refinement. For $n \in \omega$, it is possible to find $W_n \in [\mathcal{V}_n]^{<\omega}$ such that $\{St(\bigcup \mathcal{W}_n, \mathcal{V}_n) : n \in \omega\}$ is an open cover of X (γ-cover of X). For each $W \in \mathcal{W}_n$, find $U_n^W \in \mathcal{U}_n$ such that $St(W, \mathcal{V}_n) \subseteq U_n^W$. For $n \in \omega$, let $S_n = \{U_n^W : W \in \mathcal{W}_n\}$. Observe $S_n \in [\mathcal{U}]^{<\omega}$ and $\{\bigcup S_n : n \in \omega\}$ covers X (is a γ-cover of X). Thus, X is Menger (Hurewicz).
Similarly, it is possible to show that paracompact and star-Rothberger implies Rothberger. □

Going a step further (using the characterization of Lindelöfness given by Proposition 2.2.7), the following holds:

Theorem 2.3.8. *If $\mathcal{P} \in \{$Compact, Menger, Rothberger, Hurewicz$\}$, then the following are equivalent:*

1. *X is $\mathcal{P}$.*
2. *X is strongly star-$\mathcal{P}$ and metaLindelöf.*

3. X is star-$\mathcal{P}$ and paraLindelöf.

Proof. Let $\mathcal{P} \in \{$*Compact, Menger, Rothberger, Hurewicz*$\}$. Observe that:

- $\mathcal{P} \to$ strongly star-$\mathcal{P} \to$ star-$\mathcal{P}$.
- $\mathcal{P} \to$ Lindelöf.
- strongly star-$\mathcal{P} \to$ strongly star-Lindelöf.
- star-$\mathcal{P} \to$ star-Lindelöf.

(1) $\to$ (2): Assume X is $\mathcal{P}$. Then X is, in particular, strongly star-$\mathcal{P}$ and Lindelöf. Thus, X is strongly star-$\mathcal{P}$ and metaLindelöf.

(2) $\to$ (3): Assume X is strongly star-$\mathcal{P}$ and metaLindelöf. Then X is, in particular, star-$\mathcal{P}$, strongly star-Lindelöf and metaLindelöf. By Proposition 2.2.7, X is also paraLindelöf.

(3) $\to$ (1): Assume X is star-$\mathcal{P}$ and paraLindelöf. Then X is, in particular, star-Lindelöf and paraLindelöf. By Proposition 2.2.7, X is Lindelöf and, therefore, paracompact. Thus, X is star-$\mathcal{P}$ and paracompact, by Proposition 2.3.7, X is $\mathcal{P}$. □

2.4 A class of spaces that contains both the Ψ-spaces and the Niemytzki plane

In this section we study strongly star-Lindelöf spaces X which consist of the disjoint union of a closed discrete set with a σ-compact subspace. That is X is a topological space of the form $Y \cup Z$, where $Y \cap Z = \emptyset$, Z is a σ-compact subspace and Y is a closed discrete set. The motivation is that both Ψ-spaces and the Niemytzki plane (see Definition 1.1.1 and Definition 1.1.2, respectively) fall in this class of spaces. It turns out that the hypotheses needed for these spaces to satisfy some of the star selection principles are the same that the ones needed for Ψ-spaces (see Theorems 2.4.5 and 2.4.11 below). Things are not completely settled as the existence of a star-Menger not strongly star-Menger Ψ-space is still unknown. Bonanzinga and Matveev asked the previous question in [15] and were the first ones to study star selection principles on Ψ-spaces. They obtained the following important characterization:

Proposition 2.4.1 ([15])**.** *Given any almost disjoint family $\mathcal{A}$, the following assertions hold.*

1. *$\Psi(\mathcal{A})$ is strongly star-Menger if and only if $|\mathcal{A}| < \mathfrak{d}$.*
2. *$\Psi(\mathcal{A})$ is strongly star-Hurewicz if and only if $|\mathcal{A}| < \mathfrak{b}$.*

Proposition 2.4.1 motivated our study of these star versions of the Menger and Hurewicz properties in the Niemytzki plane. First, we observed that similar results do hold in the Niemytzki plane, and then we were able to generalize them to the class of spaces $X = Y \cup Z$ described at the beginning of this section.

Since Ψ-spaces are separable (and in particular SSM), Sakai pointed out in [71] that the "if" part of item 1, in Proposition 2.4.1, follows immediately from the more general result:

Proposition 2.4.2. *[71] Every (strongly) star-Lindelöf space of cardinality less than $\mathfrak{d}$ is (strongly) star-Menger.*

This can be seen as the selective version of the folklore result that Lindelöf spaces of size less than $\mathfrak{d}$ are Menger. To use these results in our characterization for the space $X = Y \cup Z$ (see Theorem 2.4.5 below), we slightly modify the proof by saying that a subset Y of a space X is **relatively strongly star-Menger in** X if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there is a sequence $\{F_n : n \in \omega\}$ such that for each $n \in \omega$, F_n is a finite subset of X and $Y \subseteq \bigcup\{St(F_n,\mathcal{U}_n) : n \in \omega\}$. Similarly, we will undertand the same when we say "a subset Y of a space X is **relatively** $\mathcal{P}$ **in** X" and $\mathcal{P}$ is some other selection principle.

Lemma 2.4.3. *If X is a (strongly) star-Lindelöf space, then every subset of X of size less than $\mathfrak{d}$ is relatively (strongly) star-Menger in X.*

Proof. We only show the case of a strongly star-Lindelöf space. The case of a star-Lindelöf space can be proved similarly. Let $Y \subseteq X$ such that $|Y| < \mathfrak{d}$ and let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is strongly star-Lindelöf, for each $n \in \omega$ there exists $A_n = \{x^n_m : m \in \omega\} \in [X]^\omega$ such that $St(A_n,\mathcal{U}_n) = X$. For each $y \in Y$ let $f_y \in \omega^\omega$ such that for each $n \in \omega$, $f_y(n) = min\{m \in \omega : y \in St(x^n_m,\mathcal{U}_n)\}$. Since the collection $\{f_y : y \in Y\}$ has size less than $\mathfrak{d}$, there exists $f^* \in \omega^\omega$ such that for each $y \in Y$, $f^* \not\leq^* f_y$. For each $n \in \omega$, let $F_n = \{x^n_m : m \leq f^*(n)\}$. If follows that $Y \subseteq \bigcup\{St(Fn,\mathcal{U}_n) : n \in \omega\}$. Indeed, let $y \in Y$, since $f^* \not\leq^* f_y$, there is $m \in \omega$ such that $f^*(m) > f_y(m)$. Hence, $y \in St(x^m_{f_y(m)},\mathcal{U}_m) \subseteq St(F_m,\mathcal{U}_m)$. Therefore Y is relatively strongly star-Menger in X. $\square$

Lemma 2.4.4. *Let X be a topological space of the form $B \cup V$ with $B \cap V = \emptyset$,*

B is a closed discrete set and V is a σ-compact subspace of X. If X is strongly star-Menger, then $|B| < \mathfrak{d}$.

Proof. Assume $|B| \geq \mathfrak{d}$. Write $V = \bigcup_{n<\omega} K_n$ an increasing union of compact sets (i.e. $n < m$ implies $K_n \subseteq K_m$). Let $\{f_\alpha : \alpha < \mathfrak{d}\} \subseteq \omega^\omega$ be a dominating family with respect to $\leq$ on every coordinate (i.e. for each $g \in \omega^\omega$ there is $\alpha < \mathfrak{d}$ such that for all $n \in \omega$, $g(n) \leq f_\alpha(n)$). Choose distinct points $p_{\alpha,\beta} \in B$ for $\alpha < \mathfrak{d}$ and $\beta < \omega_1$. Let $P = \{p_{\alpha,\beta} : \alpha < \mathfrak{d}, \beta < \omega_1\}$.

For each $\alpha < \mathfrak{d}$, $\beta < \omega_1$ and $n < \omega$, let $O_n(p_{\alpha,\beta})$ be an open neighborhood of $p_{\alpha,\beta}$ such that $O_n(p_{\alpha,\beta}) \cap B = \{p_{\alpha,\beta}\}$ and $O_n(p_{\alpha,\beta}) \cap K_{f_\alpha(n)} = \emptyset$. For each $n < \omega$ we define $\mathcal{U}_n = \{O_n(p_{\alpha,\beta}) : \alpha < \mathfrak{d}, \beta < \omega_1\} \cup \{X \setminus P\}$. Then $\mathcal{U}_n$ is an open cover of X.

Claim: the sequence $(\mathcal{U}_n : n < \omega)$ witnesses X is not strongly star-Menger.

Let $(F_n : n < \omega)$ be a sequence of finite subsets of X. For $n < \omega$, let $g(n) = min\{m : F_n \cap V \subseteq K_m\} + 1$ if $F_n \cap V \neq \emptyset$, otherwise, let $g(n) = 1$. Thus, there exists $\alpha < \mathfrak{d}$ such that $f_\alpha(n) \geq g(n)$ for each $n < \omega$. Further, there is $\beta < \omega_1$ such that $p_{\alpha,\beta} \notin \bigcup\{F_n : n < \omega\}$. It only remains to show that $p_{\alpha,\beta} \notin \bigcup\{St(F_n, \mathcal{U}_n) : n < \omega\}$. Indeed, suppose the opposite, then there exists $m \in \omega$ such that $p_{\alpha,\beta} \in St(F_m, \mathcal{U}_m)$. Since $O_m(p_{\alpha,\beta})$ is the only element of $\mathcal{U}_m$ that contains the point $p_{\alpha,\beta}$, then $O_m(p_{\alpha,\beta}) \cap F_m \neq \emptyset$. Moreover, $p_{\alpha,\beta} \notin F_m$ implies that there exists $x \in F_m \cap V$ such that $x \in O_m(p_{\alpha,\beta})$. Since $f_\alpha(m) \geq g(m)$, $x \in F_m \cap V \subseteq K_{g(m)} \subseteq K_{f_\alpha(m)}$. Therefore $x \in O_m(p_{\alpha,\beta}) \cap K_{f_\alpha(m)}$ which is a contradiction. Hence, X is not strongly star-Menger. □

Theorem 2.4.5. *Let X be a topological space of the form $Y \cup Z$, where $Y \cap Z = \emptyset$, Z is a σ-compact subspace and Y is a closed discrete set. If X is* *strongly star-Lindelöf, then $|Y| < \mathfrak{d}$ if and only if X is strongly star-Menger.*

Proof. If X is strongly star-Menger, then by Lemma 2.4.4, $|Y| < \mathfrak{d}$.
Now assume that $|Y| < \mathfrak{d}$. By Lemma 2.4.3 Y is relatively strongly star-Menger in X. Furthermore, since Z is σ-compact, it is relatively strongly star-Menger in X. Observe that it is always the case that if $T = \bigcup\{T_n : n \in \omega\}$, where each T_n is relatively strongly star-Menger in T, then T is strongly star-Menger. Thus, in particular, X is strongly star-Menger. □

Corollary 2.4.6. *The following assertions hold.*

1. *For any almost disjoint family $\mathcal{A}$, $\Psi(\mathcal{A})$ is strongly star-Menger if and only if $|\mathcal{A}| < \mathfrak{d}$.*
2. *For any $Y \subseteq \mathbb{R}$, $N(Y)$ is strongly star-Menger if and only if $|Y| < \mathfrak{d}$.*

Corollary 2.4.7. *$MA + \neg CH$ implies that for every $X \subseteq \mathbb{R}$ with $|X| < \mathfrak{c}$, $N(X)$ is strongly star-Menger.*

Proof. Assume $X \subseteq \mathbb{R}$ with $|X| < \mathfrak{c}$. Since MA implies $\mathfrak{d} = \mathfrak{c}$, by Corollary 2.4.6 it follows that $N(X)$ is strongly star-Menger. □

Now we turn to the cardinal $\mathfrak{b}$ and the strongly star-Hurewicz property. A similar characterization as the one in Theorem 2.4.5 is obtained for the space $X = Y \cup Z$.

Lemma 2.4.8. *If X is a (strongly) star-Lindelöf space, then every subset of X of size less than $\mathfrak{b}$ is relatively (strongly) star-Hurewicz in X.*

Proof. Assume X is star-Lindelöf. Let $Y \in [X]^{<\mathfrak{b}}$ and let $\langle \mathcal{U}_n : n < \omega \rangle$ be any sequence of open covers of X. Since X is star-Lindelöf, for each $n < \omega$ let

$\mathcal{W}_n = \{W_n^m : m < \omega\} \in [\mathcal{U}_n]^\omega$ such that $St(\bigcup \mathcal{W}_n, \mathcal{U}_n) = X$. For each $y \in Y$, define $f_y \in \omega^\omega$ as follows $f_y(n) = min\{m \in \omega : y \in St(W_n^m, \mathcal{U}_n)\}$. Since $|Y| < \mathfrak{b}$, the family $\{f_y : y \in Y\}$ is bounded, i.e., there exists $g \in \omega^\omega$ such that for each $y \in Y$, $f_y \leq^* g$. For each $n < \omega$, let $\mathcal{V}_n = \{W_n^m : m \leq g(n)\}$. We show $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n < \omega\}$ is a γ-cover of Y. Let $y \in Y$, there exists $m \in \omega$ such that for all $t \geq m$, $f_y(t) < g(t)$. Hence, for each $t \geq m$, $y \in St(W_t^{f_y(t)}, \mathcal{U}_t) \subseteq St(\bigcup \mathcal{V}_t, \mathcal{U}_t)$.

Now assume X is strongly star-Lindelöf. Let $Y \in [X]^{<\mathfrak{b}}$ and let $\langle \mathcal{U}_n : n < \omega \rangle$ be any sequence of open covers of X. Since X is strongly star-Lindelöf, for each $n < \omega$ let $A_n = \{x_n^m : m < \omega\} \in [X]^\omega$ such that $St(A_n, \mathcal{U}_n) = X$. For each $y \in Y$ define $f_y \in \omega^\omega$ as follows $f_y(n) = min\{m \in \omega : y \in St(x_n^m, \mathcal{U}_n)\}$. Since $|Y| < \mathfrak{b}$, the family $\{f_y : y \in Y\}$ is bounded, i.e., there exists $g \in \omega^\omega$ such that for each $y \in Y$, $f_y \leq^* g$. For each $n \in \omega$ define $F_n = \{x_n^m : m \leq g(n)\}$. We show $\{St(F_n, \mathcal{U}_n) : n < \omega\}$ is a γ-cover of Y. Let $y \in Y$, there exists $m \in \omega$ such that for all $t \geq m$, $f_y(t) < g(t)$. Hence, for each $t \geq m$, $y \in St(x_t^{f_y(t)}, \mathcal{U}_t) \subseteq St(F_t, \mathcal{U}_t)$. □

Corollary 2.4.9. *Every (strongly) star-Lindelöf space of cardinality less than $\mathfrak{b}$ is (strongly) star-Hurewicz.*

Lemma 2.4.10. *Let X be a topological space of the form $B \cup V$ with $B \cap V = \emptyset$, B is a closed discrete set and V is a σ-compact subspace of X. If X is strongly star-Hurewicz, then $|B| < \mathfrak{b}$.*

Proof. Assume $|B| \geq \mathfrak{b}$. Write $V = \bigcup_{n<\omega} K_n$ an increasing union of compact sets (i.e. $n < m$ implies $K_n \subseteq K_m$). Let $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family. Choose distinct points $p_{\alpha,\beta} \in B$ for $\alpha < \mathfrak{b}$ and $\beta < \omega_1$. Let $P = \{p_{\alpha,\beta} : \alpha < \mathfrak{b}, \beta < \omega_1\}$.

For each $\alpha < \mathfrak{b}$, $\beta < \omega_1$ and $n < \omega$, let $O_n(p_{\alpha,\beta})$ be an open neighborhood of $p_{\alpha,\beta}$ such that $O_n(p_{\alpha,\beta}) \cap B = \{p_{\alpha,\beta}\}$ and $O_n(p_{\alpha,\beta}) \cap K_{f_\alpha(n)} = \emptyset$. For each $n < \omega$ we define $\mathcal{U}_n = \{O_n(p_{\alpha,\beta}) : \alpha < \mathfrak{b}, \beta < \omega_1\} \cup \{X \setminus P\}$. Then $\mathcal{U}_n$ is an open cover of X.

Claim: the sequence $(\mathcal{U}_n : n < \omega)$ witnesses X is not strongly star-Hurewicz. Let $(F_n : n < \omega)$ be a sequence of finite subsets of X. We show that $\{St(Fn, \mathcal{U}_n) : n < \omega\}$ is not a γ-cover of X.

For $n < \omega$, let $g(n) = min\{m : F_n \cap V \subseteq K_m\} + 1$ if $F_n \cap V \neq \emptyset$, otherwise, let $g(n) = 1$. Thus, there exists $\alpha < \mathfrak{b}$ such that $f_\alpha \not\leq^* g$ for each $n < \omega$. Further, there is $\beta < \omega_1$ such that $p_{\alpha,\beta} \notin \bigcup\{F_n : n < \omega\}$. We show that for each $m \in \omega$, if $f_\alpha(m) > g(m)$, then $p_{\alpha,\beta} \notin St(F_m, \mathcal{U}_m)$. Assume there exists $m \in \omega$ such that $f_\alpha(m) > g(m)$ and $p_{\alpha,\beta} \in St(F_m, \mathcal{U}_m)$. Since $O_m(p_{\alpha,\beta})$ is the only element of $\mathcal{U}_m$ that contains the point $p_{\alpha,\beta}$, then $O_m(p_{\alpha,\beta}) \cap F_m \neq \emptyset$. Moreover, $p_{\alpha,\beta} \notin F_m$ implies that there exists $x \in F_m \cap V$ such that $x \in O_m(p_{\alpha,\beta})$. Since $f_\alpha(m) > g(m)$, $x \in F_m \cap V \subseteq K_{g(m)} \subseteq K_{f_\alpha(m)}$. Therefore $x \in O_m(p_{\alpha,\beta}) \cap K_{f_\alpha(m)}$ which is a contradiction. Hence, X is not strongly star-Hurewicz. □

Theorem 2.4.11. *Let X be a topological space of the form $Y \cup Z$, where $Y \cap Z = \emptyset$, Z is a σ-compact subspace and Y is a closed discrete set. If X is strongly star-Lindelöf, then $|Y| < \mathfrak{b}$ if and only if X is strongly star-Hurewicz.*

Proof. If X is strongly star-Hurewicz, then by Lemma 2.4.10, $|Y| < \mathfrak{b}$.

Now assume that $|Y| < \mathfrak{b}$. By Lemma 2.4.8 Y is relatively strongly star-

Hurewicz in X. Furthermore, since Z is σ-compact, it is relatively strongly star-Hurewicz in X. Thus, X is strongly star-Hurewicz. $\square$

Corollary 2.4.12. *The following assertions hold.*

1. *For any almost disjoint family $\mathcal{A}$, $\Psi(\mathcal{A})$ is strongly star-Hurewicz if and only if $|\mathcal{A}| < \mathfrak{b}$.*
2. *For any $Y \subseteq \mathbb{R}$, $N(Y)$ is strongly star-Hurewicz if and only if $|Y| < \mathfrak{b}$.*

As we have seen, the strongly star-Menger strongly star-Hurewicz properties on Ψ-spaces and the Niemytzki plane are completely characterized by the size of the closed and dicrete subset. The story is different for the star-Menger and the star-Hurewicz property. As mentioned at the beginning of this section, Bonanzinga and Matveev asked in [15] whether the properties star-Menger and strongly star-Menger are equivalent for Ψ-spaces. Since the star-Menger property deals with sequences of finite sets, Bonanzinga and Matveev introduced the following cardinal $\mathfrak{d}_\kappa$ (for an infinite cardinal κ) in [15]. This cardinal was studied also in [22], and denoted by $cof(Fin(\kappa)^{\mathbb{N}})$:

Definition 2.4.13 ([15])**.** *For an infinite set X, let $Fin(X)$ be the set of all finite subsets of X and let $Fin(X)^{\mathbb{N}}$ be the set of all sequences $(F_n : n \in \omega)$ of finite subsets of X. This set is partially ordered by defining the order $\leq$ as follows: given $\mathbb{F} = (F_n), \mathbb{G} = (G_n) \in Fin(X)^{\mathbb{N}}$, $\mathbb{F} \leq \mathbb{G}$ if $F_n \subseteq G_n$ for all $n \in \omega$. The cofinality of the partially ordered set $(Fin(X)^{\mathbb{N}}, \leq)$ is denoted by $cof(Fin(X)^{\mathbb{N}})$.*

They proved the following lemma and proposition that allowed them to show Corollary 2.4.16 and Corollary 2.4.17 below:

Lemma 2.4.14 ([15]). *The following hold.*

1. $cof(Fin(\omega)^{\mathbb{N}}) = \mathfrak{d}$,
2. *If* $\omega \leq \kappa \leq \mathfrak{c}$, *then* $max\{\mathfrak{d}, \kappa\} \leq cof(Fin(\kappa)^{\mathbb{N}}) \leq \mathfrak{c}$,
3. *If* $\omega \leq \kappa < \aleph_\omega$, *then* $cof(Fin(\kappa)^{\mathbb{N}}) = max\{\mathfrak{d}, \kappa\}$,
4. $cof(Fin(\mathfrak{c})^{\mathbb{N}}) = \mathfrak{c}$.

In the following results, $\mathcal{A}$ denotes an almost disjoint family.

Proposition 2.4.15 ([15]). *If $\mathcal{A}$ has cardinality κ and $cof(Fin(\kappa)^{\mathbb{N}}) = \kappa$, then $\Psi(\mathcal{A})$ is not star-Menger.*

Corollary 2.4.16 ([15]). *If $|\mathcal{A}| < \aleph_\omega$, then $\Psi(\mathcal{A})$ is star-Menger if and only if $\Psi(\mathcal{A})$ is strongly star-Menger.*

Corollary 2.4.17 ([15]). *If $|\mathcal{A}| = \mathfrak{c}$, then $\Psi(\mathcal{A})$ is not star-Menger.*

In order to answer the question whether in Ψ-spaces the properties star-Menger and strongly star-Menger are equivalent, Bonanzinga and Matveev asked in [15] whether (in **ZFC**) for each cardinal $\kappa \leq \mathfrak{c}$, $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa$. Observe that if the answer is yes, then the only candidates of star-Menger not strongly star-Menger Ψ-spaces, which are spaces $\Psi(\mathcal{A})$ with $|\mathcal{A}| \geq \mathfrak{d}$ (recall Corollary 2.4.6), fail to be star-Menger by Proposition 2.4.15 ($|\mathcal{A}| = \kappa \geq \mathfrak{d}$ implies $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa = \kappa$). Hence, star-Menger and strongly star-Menger would be equivalent in Ψ-spaces. Tsaban [91] answered this question in the negative establishing the special set theoretic hypothesis needed. As a consequence of this, it is still valid to ask whether there is, consistently, a star-Menger Ψ-space of size greater or equal than $\mathfrak{d}$. To have such an example, it is necessary that the size of $\mathcal{A}$ satisfies $\mathfrak{d} < \aleph_\omega \leq |\mathcal{A}| < \mathfrak{c}$. In

[17], Brendle proves it is consistent that the almost-disjointness number $\mathfrak{a}$ has countable cofinality. This construction might shed some light on the solution to this problem.

It is important to point out that in [22] it was shown that for any infinite cardinal κ, $cof(Fin(\kappa)^{\mathbb{N}}) = max\{\mathfrak{d}, cof[\kappa]^{\aleph_0}\}$. Given an infinite cardinal κ, finding the size of $cof[\kappa]^{\aleph_0}$ is a central question in Shelah's PCF theory, the theory of possible cofinalities. It is known that the function $\kappa \mapsto cof[\kappa]^{\aleph_0}$ has some fixed points. For instance, for each $n \geq 1$, $cof[\aleph_n]^{\aleph_0} = \aleph_n$. But for $\kappa = \aleph_\omega$ (and in general for uncountable cardinals with countable cofinality) $cof[\kappa]^{\aleph_0} > \kappa$. Under *Shelah's Strong Hypothesis* (the statement that for each uncountable cardinal κ with countable cofinality: $cof[\kappa]^{\aleph_0} = \kappa^+$) things simplify a bit, for each $\kappa > \aleph_0$: $cof[\kappa]^{\aleph_0} = \kappa$ if κ has uncountable cofinality and κ^+ otherwise. For more on PCF theory refer to [76] and [77] (see also [1]).

Sakai proved in [71] that Proposition 2.4.15 is not particular to Ψ-spaces:

Theorem 2.4.18 ([71])**.** *Let X be a star-Menger space. If Y is an infinite closed and discrete subspace of X, then $|Y| < cof(Fin(w(X))^{\mathbb{N}})$ holds.*

Lemma 2.4.14 and Theorem 2.4.18, let us get the same conclusions for the Niemytzki plane:

Corollary 2.4.19. *Let $Y \subseteq \mathbb{R}$ and $N(Y)$ denote the Niemytzki plane on Y (see Definition 1.1.2).*

- *If $cof(Fin(|Y|)^{\mathbb{N}}) = |Y|$, then $N(Y)$ is not a star-Menger space. In particular, if Y has cardinality $\mathfrak{c}$, $N(Y)$ is not star-Menger.*

- *If $|Y| < \aleph_\omega$, then $N(Y)$ is star-Menger if and only if it is strongly star-Menger.*

Proof. Assume $Y \subseteq \mathbb{R}$ such that $cof(Fin(|Y|)^{\mathbb{N}}) = |Y|$. Observe that $w(N(Y)) = |Y|$. Thus, $cof(Fin(w(N(Y)))^{\mathbb{N}}) = cof(Fin(|Y|)^{\mathbb{N}}) = |Y|$. Since Y is a closed discrete subset of $N(Y)$, by Theorem 2.4.18, $N(Y)$ is not star-Menger.
Now assume $|Y| < \aleph_\omega$. If $|Y| < \mathfrak{d}$, by Corollary 2.4.6, $N(Y)$ is strongly star-Menger, hence star-Menger. If $|Y| \geq \mathfrak{d}$, by Lemma 2.4.14 (3), $cof(Fin(|Y|)^{\mathbb{N}})$ $= max\{\mathfrak{d}, |Y|\} = |Y|$ and by Theorem 2.4.18, $N(Y)$ is not star-Menger and, therefore, it is not strongly star-Menger. □

The question whether there is a star-Menger not strongly star-Menger Ψ-space, remains open. In addition, Tsaban [91] asks whether there is consistently a star-Hurewicz Ψ-space of cardinality greater or equal than $\mathfrak{b}$. These questions (which apply to the Niemytzki plane as well), motivated our study of (normal) star-Menger not strongly star-Menger spaces and led to the examples discussed in Section 2.6.1.

2.5 Absolute versions of star selection principles

In this section we study some absolute versions of selection principles. First we present the "absolute version" of Theorem 2.3.8 and then we prove that for "small spaces" some absolute star selection principles are equivalent (Theorem 2.5.18).

Fleishman's equivalence between countably compactness and strongly star-compactness (see the paragraph before Definition 2.3.5), motivated Matveev to introduce in [59] the following interesting property:

Definition 2.5.1 ([59]). *A space X is* **absolutely countably compact** **(***acc***)** *if for any $\mathcal{U} \in \mathcal{O}(X)$ and any dense subset D of X, there is a finite subset $F \in [D]^{<\omega}$ such that $St(F,\mathcal{U}) = X$.*

Observe that this property can be called absolutely strongly star-compact as well. Clearly, compact $\rightarrow$ *acc* $\rightarrow$ countably compact. The space $[0, \omega_1)$ with the usual order topology is *acc* and is not compact (see Space **O** in Section 2.6). Matveev pointed out in [59] that Arhangel'skii asked him if every normal countably compact space is *acc*. Pavlov answered this question in the negative (around ten years later) in [67] where he presented a normal countably compact not absolutely countably compact space.

Bonanzinga defined and studied in [9] and [10] the absolute version of the strongly star-Lindelöf property.

Definition 2.5.2 ([9],[10]). *A space X is* **absolutely strongly star-Lindelöf** **(***aSSL***)** *if for any $\mathcal{U} \in \mathcal{O}(X)$ and any dense subset D of X, there is $C \in [D]^{\omega}$ such that $St(C,\mathcal{U}) = X$.*

Now, let us recall the following weakenings of metacompact and metaLindelöf.

Definition 2.5.3 ([38]). *X is called* **nearly metacompact** *(***nearly metaLindelöf***) provided that if $\mathcal{U} \in \mathcal{O}(X)$, then there is a dense set $D \subseteq X$*

and an open refinement $\mathcal{V} \prec \mathcal{U}$ such that $\mathcal{V}_d = \{V \in \mathcal{V} : d \in V\}$ is finite (countable) for all $d \in D$.

E. Grabner, G. Grabner and Vaughan proved in [36] that nearly metaLindelöf *acc* spaces are compact. Matveev realized that the following holds true as well:

Proposition 2.5.4 ([61]). *A space X is Lindelöf if and only if it is absolutely strongly star-Lindelöf and nearly metaLindelöf.*

Proof. Assume X is an absolutely strongly star-Lindelöf, nearly metaLindelöf space and fix $\mathcal{U} \in \mathcal{O}(X)$. Let $D \subseteq X$ be dense and $\mathcal{V} \prec \mathcal{U}$ be an open refinement such that $\mathcal{V}_x = \{V \in \mathcal{V} : x \in V\}$ is countable for all $d \in D$. Since X is aSSL, fix $E \in [D]^{\omega}$ such that $St(E, \mathcal{V}) = X$. For each $e \in E$ let $\mathcal{V}_e = \{V \in \mathcal{V} : e \in V\}$. Since $\mathcal{V}$ is point-countable, for each $e \in E$, $|V_e| \leq \omega$. Let $\mathcal{W} = \bigcup_{e \in E} V_e$. Thus, $\mathcal{W}$ is a countable subfamily of $\mathcal{V}$ and it is a cover of X. For each $W \in \mathcal{W}$, fix $U_W \in \mathcal{U}$ such that $W \subseteq U_W$. Then $\{U_W : W \in \mathcal{W}\} \in [\mathcal{U}]^{\omega}$. That is, X is Lindelöf. $\square$

Caserta, Di Maio and Kočinac were the first ones to define and study the absolute versions of the SSM and SSH properties:

Definition 2.5.5 ([20]). *A space X is:*

1. **absolutely strongly star-Menger** ($aSSM$) *if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, and each dense subset $D \subseteq X$, there exists a sequence $\{F_n : n \in \omega\}$ so that for each $n \in \omega$, $F_n \in [D]^{<\omega}$ and $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.*
2. **absolutely strongly star-Hurewicz** ($aSSH$) *if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, and each dense subset $D \subseteq X$, there*

exists a sequence $\{F_n : n \in \omega\}$ *so that for each* $n \in \omega$, $F_n \in [D]^{<\omega}$ *and* $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ *is a* Γ*-cover of* X*.*

The following diagram shows the relationships between these properties (C stands for "compact").

$$\begin{array}{ccccc}
C & \Longrightarrow & acc & \Longrightarrow & SSC \\
\Downarrow & & \Downarrow & & \Downarrow \\
H & \Longrightarrow & aSSH & \Longrightarrow & SSH \\
\Downarrow & & \Downarrow & & \Downarrow \\
M & \Longrightarrow & aSSM & \Longrightarrow & SSM \\
\Downarrow & & \Downarrow & & \Downarrow \\
L & \Longrightarrow & aSSL & \Longrightarrow & SSL
\end{array}$$

Diagram 2.5: Absolute Versions of Star Selection Principles.

As an application of Propositions 2.5.4 and 2.3.7, the following analogous to Theorem 2.3.8 is easily obtained.

Theorem 2.5.6. *If* $\mathcal{P} \in \{$*Compact, Menger, Rothberger, Hurewicz*$\}$*, then the following are equivalent:*

1. *X is $\mathcal{P}$.*
2. *X is absolutely strongly star-$\mathcal{P}$ and nearly metaLindelöf.*

Proof. If $\mathcal{P} \in \{$*Compact, Menger, Rothberger, Hurewicz*$\}$, then both properties absolutely strongly star-$\mathcal{P}$ and nearly metaLindelöf are weaker than $\mathcal{P}$. Hence, it is sufficient to show (2) $\rightarrow$ (1). Assume X is absolutely strongly star-$\mathcal{P}$ and nearly metaLindelöf. Since absolutely strongly star-$\mathcal{P}$ implies absolutely strongly star-Lindelöf, by Proposition 2.5.4 X is Lindelöf. Thus, in particular, X is strongly star-$\mathcal{P}$ and paracompact. By Proposition 2.3.7, X is $\mathcal{P}$. $\square$

To complete the picture, let us write down the following "selective" versions of the previous absolute properties. Instead of just taking a dense subset of the space, we now consider a sequence of dense sets:

Definition 2.5.7. *A space X is:*

1. **selectively strongly star-Lindelöf** (*selSSL*) [8] *if for each open cover $\mathcal{U}$ of X, and each sequence of dense subets of X $(D_n : n \in \omega)$, there exists a sequence $\{F_n : n \in \omega\}$ so that for each $n \in \omega$, $F_n \in [D_n]^{<\omega}$ and $\{St(F_n, \mathcal{U}) : n \in \omega\}$ is an open cover of X.*

2. **selectively strongly star-Menger** (*selSSM*) [7], [23] *if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, and each sequence of dense subsets of X $(D_n : n \in \omega)$, there exists a sequence $\{F_n : n \in \omega\}$ so that for each $n \in \omega$, $F_n \in [D_n]^{<\omega}$ and $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.*

3. **selectively strongly star-Hurewicz** (*selSSH*) *if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, and each sequence of dense subets of X $(D_n : n \in \omega)$, there exists a sequence $\{F_n : n \in \omega\}$ so that for each $n \in \omega$, $F_n \in [D_n]^{<\omega}$ and $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is a Γ-cover of X.*

With these new properties the following diagram holds:

$$
\begin{tikzcd}
H \arrow[r, Rightarrow] \arrow[d, Rightarrow] & selSSH \arrow[r, Rightarrow] \arrow[d, Rightarrow] & aSSH \arrow[r, Rightarrow] \arrow[d, Rightarrow] & SSH \arrow[d, Rightarrow] \\
M \arrow[r, Rightarrow] \arrow[d, Rightarrow] & selSSM \arrow[r, Rightarrow] \arrow[d, Rightarrow] & aSSM \arrow[r, Rightarrow] \arrow[d, Rightarrow] & SSM \arrow[d, Rightarrow] \\
L \arrow[r, Rightarrow] & selSSL \arrow[r, Rightarrow] & aSSL \arrow[r, Rightarrow] & SSL
\end{tikzcd}
$$

Diagram 2.6: Selective Versions of Star Selection Principles.

Recall that by Theorem 2.3.8 these properties are interesting when the spaces are not MetaLindelöf.

In [14] Bonanzinga, Cuzzupé and Sakai presented an example of a Tychonoff $aSSL$ space of cardinality $\mathfrak{d}$ which is not $selSSL$ and proved that every $aSSL$ space of cardinality less than $\mathfrak{d}$ is $selSSL$.

Basile, Bonanzinga and Cuzzupè, made us interested in the study of selectively strongly star-Menger spaces with their work [7], (see also [23]) and motivated the rest of this section. At first, we were interested in the study of the properties $selSSM$ and $selSSH$ in the space $X = Y \cup Z$ inspected in section 2.4. This led to the following definition:

Definition 2.5.8. *Given any space X, $B(X)$ stands for: for each open cover $\mathcal{U}$ of X and each dense $D \subseteq X$, there are $C \in [X]^\omega$ and $E \in [D]^\omega$ such that*
(1) $St(C,\mathcal{U}) = X$,
(2) $C \subseteq cl(E)$.

Definition 2.5.9 (Folklore)**.** *For a space X, let $CD(X)$ stand for "For each $D \subseteq X$ infinite dense, there exists $C \in [D]^\omega$ dense".*

Observation 2.5.10. *For a space X, the following holds:*

- *$B(X)$ implies X is SSL,*
- *If X is Lindelöf, then $B(X)$ holds,*
- *$CD(X) \to B(X)$*
- *if X is SSL and first countable (actually countably tight), then $B(X)$.*

Proposition 2.5.11. *Assume that for a space X, $B(X)$ holds. Then*

1. *$|X| < \mathfrak{d} \to X$ is selSSM.*
2. *$|X| < \mathfrak{b} \to X$ is selSSH.*

Proof. (1.) Write $X = \{x_\alpha : \alpha < \kappa\}$ with $|X| = \kappa < \mathfrak{d}$. Let $(\mathcal{U}_n : n \in \omega)$ be any sequence of open covers and let $(D_n : n \in \omega)$ be any sequence of dense subsets of X. We will find, for each $n \in \omega$, finite $F_n \subseteq D_n$ such that $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is a cover of X. Since $B(X)$ holds, for each $n \in \omega$ there exist $C_n \in [X]^\omega$ and $E_n \in [D_n]^\omega$ such that $St(C_n, \mathcal{U}_n) = X$ and $C_n \subseteq cl(E_n)$. Observe that for each $n \in \omega$, $St(E_n, \mathcal{U}_n) = X$: fix $n \in \omega$ and $x \in X$, there is $c_x \in C_n$ such that $x \in St(c_x, \mathcal{U}_n)$. Therefore, there is $U \in \mathcal{U}_n$ such that $x, c_x \in U$. Since $C_n \subseteq cl(E_n)$, pick $e \in E_n \cap U$, then $x \in St(e, \mathcal{U}_n) \subseteq St(E_n, \mathcal{U}_n)$.

Since for each $n \in \omega$, E_n is countable, write it as $E_n = \{e^n_s : s \in \omega\}$. Now, for each $n \in \omega$, the collection $\{St(e^n_s, \mathcal{U}_n) : s \in \omega\}$ is an open cover of X. For each $\alpha < \kappa$ define a fuction $f_\alpha : \omega \to \omega$ as follows $f_\alpha(n) = min\{s \in \omega : x_\alpha \in St(e^n_s, \mathcal{U}_n)\}$ for $n \in \omega$. Since $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{d}$, there is $g \in \omega^\omega$ such that for all $\alpha < \kappa$: $g \nleq^* f_\alpha$. For each $n \in \omega$, let

$F_n = \{e^n_s : s \leq g(n)\} \in [D_n]^{<\omega}$. It follows that $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X: let $x_\alpha \in X$, there is $m \in \omega$ such that $f_\alpha(m) < g(m)$. Hence $x_\alpha \in St(e^m_{f_\alpha(m)}, \mathcal{U}_m) \subseteq St(F_m, \mathcal{U}_m)$.

(2.) The same proof as before works with the respective modifications: Write $X = \{x_\alpha : \alpha < \kappa\}$ with $|X| = \kappa < \mathfrak{b}$. For $n \in \omega$, and $\alpha < \kappa$ find C_n, E_n as before and define f_α, similarly. Since the family $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there is $g \in \omega^\omega$ such that for all $\alpha < \kappa$: $f_\alpha \leq^* g$. For each $n \in \omega$, let $F_n = \{e^n_s : s \leq g(n)\} \in [D_n]^{<\omega}$. It follows that $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is a Γ-cover of X: let $x_\alpha \in X$, there is $m \in \omega$ such that for each $n \geq m$: $f_\alpha(n) < g(n)$. Hence for each $n \geq m$, $x_\alpha \in St(e^n_{f_\alpha(n)}, \mathcal{U}_n) \subseteq St(F_n, \mathcal{U}_n)$. □

Definition 2.5.12. *For a space X and a subset $Y \subseteq X$, Y will be called* **relatively** *$SelSSM$* **in** *X (respectively* **relatively** *$SelSSH$* **in** *X) if for every sequence of open covers $(\mathcal{U}_n : n \in \omega)$ and any sequence $(D_n : n \in \omega)$ of dense subsets of X, there is a sequence $(F_n : n \in \omega)$ such that for each $n \in \omega$, $F_n \in [D_n]^{<\omega}$ and $Y \subseteq \bigcup\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ (for each $y \in Y$, $\{n \in \omega : y \notin St(F_n, \mathcal{U}_n)\}$ is finite).*

Similarly as Lemmas 2.4.3 and 2.4.8 (using the proof of Proposition 2.5.11) the following holds:

Lemma 2.5.13. *If $B(X)$ holds for a space X, then every subset of size less than $\mathfrak{d}$ is relatively SelSSM in X and evey subset of size less than $\mathfrak{b}$ is relatively SelSSH in X.*

Lemmas 2.4.4, 2.4.10 and 2.5.13 allow us to prove that in the class of spaces $X = Y \cup Z$ that satisfy $B(X)$, the two properties SSM and $SelSSM$ are equivalent and the two properties SSH and $SelSSH$ are equivalent:

Theorem 2.5.14. *Let X be a space such that $X = Y \cup Z$ with $Y \cap Z = \emptyset$, Y closed discrete subspace and Z σ-compact. If $B(X)$ holds, then*

1. *X is $SSM \leftrightarrow X$ is $selSSM$.*
2. *X is $SSH \leftrightarrow X$ is $selSSH$.*

Proof. (1.) Since $selSSM$ always implies SSM, it is enough to show that if X is SSM, then X is $selSSM$. Assume that X is SSM, by Lemma 2.4.4, $|Y| < \mathfrak{d}$. By Lemma 2.5.13, Y is relatively $selSSM$ in X. In addition, Z is σ-compact and σ-compact spaces are Menger, that is, Z is relatively $selSSM$ in X. Hence, X is $selSSM$.

(2.) Similarly as (1.), since $selSSH$ always implies SSH, it is enough to show that if X is SSH, then X is $selSSH$. Assume that X is SSH, by Lemma 2.4.10, $|Y| < \mathfrak{b}$. By Lemma 2.5.13, Y is relatively $selSSH$ in X. In addition, Z is σ-compact and σ-compact spaces are Hurewicz, that is, Z is relatively $selSSH$ in X. Hence, X is $selSSH$.

□

By Observation 2.5.10 and Theorem 2.5.14 the following corollary holds.

Corollary 2.5.15. *Let X be a space such that $X = Y \cup Z$ with $Y \cap Z = \emptyset$, Y closed discrete subspace and Z σ-compact, then:*

1. *If X is SSL and countably tight, then [X is $SSM \leftrightarrow X$ is $selSSM$] and [X is $SSH \leftrightarrow X$ is $selSSH$].*
2. *If $CD(X)$ holds, then [X is $SSM \leftrightarrow X$ is $selSSM$] and [X is SSH $\leftrightarrow X$ is $selSSH$].*

In particular, Ψ-spaces and Niemytzki planes are separable (hence SSL), countably tight, and satisfy the hypotheses of of Corollary 2.5.15. Therefore:

Corollary 2.5.16. *For Ψ-spaces and for the Niemytzki plane, the three properties SSM, $aSSM$ and $selSSM$ are equivalent, and the three properties SSH, $aSSH$ and $selSSH$ are equivalent.*

In oral communication with Javier Casas de la Rosa, he realized that for any space X, $B(X)$ holds if and only if X is absolutely SSL:

Proposition 2.5.17. *For any space X, $B(X)$ holds if and only if X is $aSSL$.*

Proof. Assume X is $aSSL$, let $\mathcal{U} \in \mathcal{O}(X)$ be any open cover and let $D \subseteq X$ be any dense set. Then, there is $C \in [D]^\omega$ such that $St(C,\mathcal{U}) = X$. Observe that $C \subseteq cl(C)$, that is, $B(X)$ holds.
Now assume that $B(X)$ holds let $\mathcal{U} \in \mathcal{O}(X)$ be any open cover and let $D \subseteq X$ be any dense set. Then, there is $C \in [X]^\omega$ and $E \in [D]^\omega$ such that $St(C,\mathcal{U}) = X$ and $C \subseteq cl(E)$. It is enough to show $St(E,\mathcal{U}) = X$. Let $x \in X$, there is $c_x \in C$ such that $x \in St(c_x,\mathcal{U})$. Hence, there is $U \in \mathcal{U}$ with $x, c_x \in U$. Then, since $C \subseteq cl(E)$, there is $e \in E$ such that $e \in U$. Thus, $x \in St(e,\mathcal{U}) \subseteq St(E,\mathcal{U})$. Whence, X is $aSSL$. $\square$

With the previous equivalence, Proposition 2.5.11 actually becomes the more interesting:

Theorem 2.5.18. *For any space X,*

- *If $|X| < \mathfrak{d}$, then X is $aSSL$ if and only if X is $selSSM$.*
- *If $|X| < \mathfrak{b}$, then X is $aSSL$ if and only if X is $selSSH$*

2.6 Counterexamples

In this section we mention the spaces that differentiate the star selection principles. In the following diagram the letter next to each arrow represents the space (described below) showing the converse does not hold.

$$
\begin{tikzcd}
C \arrow[r, "\sigma"] \arrow[dd, "\mathbf{O}"'] & H \arrow[r, "\mathbf{S_L}"] \arrow[dd, "\mathbf{O}"'] & M \arrow[dd, "\mathbf{O}"'] \arrow[dr, "\mathbf{S}"'] & & R \arrow[ll, "\mathbf{I}"'] \arrow[dd, "\mathbf{O}"'] \\
 & & & L \arrow[dd, "\mathbf{O}"'] & \\
SSC \arrow[r, "\mathbf{Y}"] \arrow[dd, "\mathbf{T}"'] & SSH \arrow[r, "\mathbf{B}"] \arrow[dd, "\mathbf{T}"'] & SSM \arrow[dd, "\mathbf{T}"'] \arrow[dr, "\mathbf{S}"'] & & SSR \arrow[ll, "\mathbf{M}"'] \arrow[dd, "\mathbf{T}"'] \\
 & & & SSL \arrow[dd, "\mathbf{T}"'] & \\
SC \arrow[r, "\mathbf{Y}"] & SH \arrow[r, "\mathbf{B}"] & SM \arrow[dr, "\mathbf{S}"'] & & SR \arrow[ll, "\mathbf{M}"'] \\
 & & & SL &
\end{tikzcd}
$$

Diagram 2.7: Counterexamples in the Complete Diagram.

Space σ: Any σ-compact (a space which can be written as a countable union of compact spaces), non-compact space is Hurewicz and is not compact. ■

Space $\mathbf{S_L}$: A *Luzin* set (an uncountable subset of the reals that has countable intersection with every meager set) is an example of a Menger (in fact Rothberger) space that is not Hurewicz. Luzin sets exist assuming **CH**. Hurewicz attributed this result to Sierpiński in [44]. A solution in **ZFC** of a Menger not Hurewicz space was provided in 2002 by Chaber and Pol in [21] (see also [90], [92]). ■

Space I: The unit invertal $\mathbf{I} = [0, 1]$ is an example of a compact (hence Hurewicz and Menger) space that is not Rothberger. ■

For the class of compact spaces the properties Rothberger and scattered (every nonempy subset has an isolated point) are equivalent. A proof of this fact, stated as the next proposition can be found in [13].

Proposition 2.6.1. *A compact space X is Rothberger if and only if it is scattered.*

In [54], $\omega_1 = [0, \omega_1)$ with the usual order topology was used as an example of a SSM not M space. This space also shows $SSR \not\to R$, $SSH \not\to H$, $SSL \not\to L$ and $SSC \not\to C$. In general, the following holds true.

Space O: If κ is a regular uncountable cardinal, then the space $\mathbf{O} = [0, \kappa)$ with the usual order topology is *acc* (and therefore SCC, $aSSH$, $aSSM$, SSH and SSM), and SSR. But, it is not Lindelöf (therefore not Menger, not Hurewicz , not compact and not Rothberger).

To check that it is *acc* and SSR, we use the Fodor's Pressing Down Lemma which states that for every regular uncountable cardinal κ and each regressive function $f : Lim(\kappa) \to \kappa$ (that is $f(\alpha) < \alpha$ for each $\alpha \in Lim(\kappa)$, $\alpha \neq 0$), there is some $\alpha < \kappa$ such that $f^{-1}\{\alpha\}$ is stationary.

O is absolutely countably compact: Let $\mathcal{U} \in \mathcal{O}(X)$ and $D \subseteq [0, \kappa)$ dense. Observe that $\{0\} \cup \{\alpha + 1 : \alpha < \kappa\} \subseteq D$. For each $\alpha \in \kappa$, fix $U_\alpha \in \mathcal{U}$ such that $\alpha \in U_\alpha$ and fix $\beta_\alpha = min\{\beta \leq \alpha : \textit{ the interval } [\beta, \alpha] \subseteq U_\alpha\}$.

Define $f : Lim(\kappa) \to \kappa$ as $f(\alpha) = \beta_\alpha$. Hence, f is regressive and by Fodor's Lemma, there is some $\alpha < \kappa$ such that $f^{-1}\{\alpha\}$ is stationary (observe that $\alpha = \gamma + 1$ for some $\gamma < \kappa$, i.e. $\alpha \in D$). Thus, For each $\gamma \geq \alpha$, $\gamma \in St(\alpha, \mathcal{U})$. Now, $[0, \alpha]$ is compact, hence it is relatively *acc* in $[0, \kappa)$. Fix $F_0 \in [D]^{<\omega}$ such that $[0, \alpha] \subseteq St(F_0, \mathcal{U})$. Let $F = \{\alpha\} \cup F_0 \in [D]^{<\omega}$. Thus, $St(F, \mathcal{U}) = X$.

X **is SSR:** Let $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{O}(X)$. Similarly, obtain $\alpha < \kappa$ such that for all $\gamma \geq \alpha$, $\gamma \in St(\alpha, \mathcal{U}_0)$. Now, $[0, \alpha]$ is compact and scattered, by Proposition 2.6.1, it is relatively Rothberger in $[0, \kappa)$ and, in particular, relatively SSR in $[0, \kappa)$. Hence, there is a sequence $\{\alpha_n : n \in \mathbb{N}\}$ such that $\bigcup_{n \in \mathbb{N}} St(\alpha_n, \mathcal{U}_n) = [0, \alpha]$. Thus, $\{St(\alpha, \mathcal{U}_0)\} \cup \{St(\alpha_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{O}(X)$. That is, X is SSR.

Whence, X is SSC, SSM, SSH and SSR. Since ω_1 is not Lindelöf (the set of intervals $\{[\beta, \alpha] : \beta \leq \alpha \wedge \alpha < \omega_1\}$ is an open cover of ω_1 with no countable subcover), it is not Menger, not Hurewicz, not compact, and not Rothberger. $\blacksquare$

The following example appears in [12] and [81] and shows that $SL \not\to SSL$, $SM \not\to SSM$, $SH \not\to SSH$, $SC \not\to SSC$ and $SR \not\to SSR$ $SH \not\to SSH$ for Tychonoff spaces.

Space T: Let κ be a regular uncountable cardinal.

Let $\mathbf{T} = X = \Big((D(\kappa) \cup \{\infty\}) \times \kappa^+\Big) \cup \Big(D(\kappa) \times \{\kappa^+\}\Big)$ where $D(\kappa) \cup \{\infty\}$ is the one point compactification of the discrete topology on κ and X has the subspace topology inherited from the product $(D(\kappa) \cup \{\infty\}) \times (\kappa^+ + 1)$. Then X is Tychonoff, SC (therefore SH and SM), and SR. But, it is not

SSM (therefore not SSH and not SSR)

X **is SC:** Let $\mathcal{U} \in \mathcal{O}(X)$. For each $\alpha < \kappa$, fix $\beta_\alpha < \kappa^+$ so that there is $U_\alpha \in \mathcal{U}$ neighbourhood of $\langle \alpha, \kappa^+ \rangle$ such that $\{\alpha\} \times [\beta_\alpha, \kappa^+] \subseteq U_\alpha$. Since $|\{\beta_\alpha : \alpha < \kappa\}| \leq \kappa$, then $\{\beta_\alpha : \alpha < \kappa\}$ is not cofinal in κ^+. Fix $\gamma < \kappa^+$ such that for each $\alpha < \kappa$, $\beta_\alpha \leq \gamma$. Observe that $X_\gamma = (D(\kappa) \cup \{\infty\}) \times [0, \gamma + 1]$ is compact and for each $\alpha < \kappa$, $U_\alpha \cap X_\gamma \neq \emptyset$. Hence, there is $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $X_\gamma \subseteq \bigcup \mathcal{V}$ and, therefore, $X_\gamma \cup \bigcup_{\alpha<\kappa} U_\alpha \subseteq St(\bigcup \mathcal{V}, \mathcal{U})$. Thus, in particular, $D(\kappa) \times [0, \kappa^+ + 1] \subseteq St(\bigcup \mathcal{V}, \mathcal{U})$. By Example **O**, $\{\infty\} \times \kappa^+$ is SSC (in particular, it is SC). Fix $\mathcal{W} \in [\mathcal{U}]^{<\omega}$ such that $\{\infty\} \times \kappa^+ \subseteq St(\bigcup \mathcal{W}, \mathcal{U})$. It is obtained that $St(\bigcup(\mathcal{V} \cup \mathcal{W}), \mathcal{U}) = X$.

X **is SR:** Let $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{O}(X)$. Similarly, for $\mathcal{U}_0$ find $\gamma < \kappa^+$ such that for all $\alpha < \kappa$ there is $\beta_\alpha < \kappa^+$ and $U_\alpha \in \mathcal{U}_0$ such that $\{\alpha\} \times [\beta_\alpha, \kappa^+] \subseteq U_\alpha$ and $\beta_\alpha \leq \gamma$. Fix any $U^\gamma \in \mathcal{U}_0$ such that $\langle \infty, \gamma \rangle \in U^\gamma$. Then, there is some finite set $F \in [\kappa]^{<\omega}$ such that $(D(\kappa) \setminus F) \times \{\gamma\} \subseteq U^\gamma$. Observe that by the way γ was defined, if we let $P_0 = (D(\kappa) \setminus F) \times [\gamma, \kappa^+]$, $P_0 \subseteq St(U^\gamma, \mathcal{U}_0)$. Let $P_1 = \{\{\alpha\} \times [\gamma, \kappa^+] : \alpha \in F\}$; $P_2 = (D(\kappa) \cup \{\infty\}) \times [0, \gamma + 1]$; $P_3 = \{\infty\} \times [0, \kappa^+)$. Observe that $X = P_0 \cup (\bigcup P_1) \cup P_2 \cup P_3$. By Example **O**, P_3 and each element of P_1 are SSR and, in particular, SR. In addition, P_2 is scattered and compact, by Proposition 2.6.1 it is Rothberger, and thus, SR. Hence, X can be written as a finite union of SR subspaces and therefore, X is SR.

X **is not SSL:** fix $\mathcal{U} \in \mathcal{O}(X)$ of basic open sets, such that for each $\alpha < \kappa$ there is a unique $U \in \mathcal{U}$ with $\langle \alpha, \kappa^+ \rangle \in U$ and, in addition, $U \subseteq \{\alpha\} \times [0, \kappa^+]$.

We show that for any countable set $C \in [X]^{<\omega}$, $X \nsubseteq \{St(C,\mathcal{U}\}$. Since C is countable, there is $\alpha < \kappa$, such that $([\alpha,\kappa) \times [0,\kappa^+]) \cap C = \emptyset$. Observe that for each β such that $\alpha < \beta < \kappa$, $\langle \beta, \kappa^+ \rangle \notin St(C,\mathcal{U})$. Hence, X is not SSL, and therefore, is not SSM, not SSH and not SSR. ■

Space S: The Sorgenfrey line **S** is Lindelöf (therefore SSL and SL) and is not star-Menger (therefore not SSM and not Menger). Let us show that **S** is not star-Menger. Since **S** is paracompact, by Theorem 2.3.8, it is enough to show that **S** is not Menger. Hence, we will build a sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ such that for each sequence $\{\mathcal{V}_n : n \in \omega\}$ with $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ for each $n \in \omega$, $\bigcup_{n\in\omega} \mathcal{V}_n \notin \mathcal{O}(\mathbf{S})$. Hence, let $\mathcal{U}_0$ be any cover of pairwise disjoint clopen sets, for instance $\mathcal{U}_0 = \{[n, n+1) : n \in \mathbb{Z}\}$. Now, for each $n \in \mathbb{Z}$, partition $[n, n+1)$ into countably many clopen sets. For instance, let $\{a_n^m \in \mathbb{Q} : m \in \omega\}$ be a strictly increasing sequence such that $a_n^0 = n$ and that converges to $n+1$. Hence $[n, n+1) = \bigcup_{j\in\omega}[a_n^j, a_n^{j+1})$ and let $\mathcal{U}_1 = \{[a_n^j, a_n^{j+1}) : j \in \omega, n \in \mathbb{Z}\}$. Build $\mathcal{U}_2$ subdividing each element of $\mathcal{U}_1$ in a similar fashion, and do the same for each $n \geq 3$.

Now let $\{\mathcal{V}_n : n \in \omega\}$ such that for each $n \in \omega$, $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$. Fix $s_0 \in \mathbb{Z}$ such that for each $m \geq s_0$, $[m, m+1) \notin \mathcal{V}_0$. Now, there is a sequence $\{b_1^j : j \in \omega\}$ such that $[s_0, s_0+1) = \bigcup_{j\in\omega}[b_1^j, b_1^{j+1})$ and $\{[b_1^j, b_1^{j+1}) : j \in \omega\} \subset \mathcal{U}_1$. Fix $s_1 \in \omega$ such that for each $m \geq s_1$, $[b_1^m, b_1^{m+1}) \notin \mathcal{V}_1$. For $2 \leq t \leq n$ assume $\{b_t^j : j \in \omega\}$ and s_t have been defined such that

$$[b_{t-1}^{s_{t-1}}, b_{t-1}^{s_{t-1}+1}) = \bigcup_{j\in\omega}[b_t^j, b_t^{j+1}) \quad \{[b_t^j, b_t^{j+1}) : j \in \omega\} \subset \mathcal{U}_t$$

and for all $m \geq s_t$, $[b_t^m, b_t^{m+1}) \notin \mathcal{V}_t$. Let $\{b_{n+1}^j : j \in \omega\}$ such that $[b_n^{s_n}, b_n^{s_n+1}) =$

$\bigcup_{j\in\omega}[b^j_{n+1}, b^{j+1}_{n+1})$ and $\{[b^j_{n+1}, b^{j+1}_{n+1}) : j \in \omega\} \subset \mathcal{U}_{n+1}$. Find $s_{n+1} \in \omega$ such that for each $m \geq s_{n+1}$, $[b^m_{n+1}, b^{m+1}_{n+1}) \notin \mathcal{V}_{n+1}$. For each $n \in \omega$, the limit of the sequence $\{b^{s_n}_n : n \in \mathbb{N}\}$ is not in $\bigcup \mathcal{V}_n$. Hence, $\mathbf{S}$ is not Menger (equivalently, not star-Menger). $\blacksquare$

Space Y: Let $\mathbf{Y} = ([0, \omega] \times [0, \omega]) \setminus \{\langle\omega, \omega\rangle\}$ be considered as a subspace of the product space $[0, \omega] \times [0, \omega]$. Song presented this space in [81] to provide an example of a Tychonoff strongly star-Hurewicz, not star-compact space. $\blacksquare$

Space M: In [15] Bonanzinga and Matveev, using a subspace of ω^ω of size $cov(\mathcal{M})$, build a Ψ-space of size $cov(\mathcal{M})$ that is not star-Rothberger. Hence, by Proposition 2.4.1, if we assume $cov(\mathcal{M}) < \mathfrak{d}$, this is a consistent example of a strongly star-Menger space (therefore SM), that is not star-Rothberger (therefore not strongly star-Rothberger). $\blacksquare$

Space B: In [91] Tsaban shows that assuming $\mathfrak{b} = \aleph_1 < \mathfrak{d}$, any almost disjoint family $\mathcal{A}$ of size $\aleph_1$ satisfies that $\Psi(\mathcal{A})$ is strongly star-Menger (therefore SM) and it is not star-Hurewicz (therefore is not strongy star-Hurewicz). $\blacksquare$

2.6.1 Star-Menger not strongly star-Menger spaces

In the literature, there were only examples of regular star-Menger not strongly star-Menger spaces. This motivated the following question:

Question 2.6.2 ([18] Question 2.4)**.** *Is there a normal star-Menger space which is not strongly star-Menger?*

The first attempt to answer this question resulted in the space $\mathbf{F}$ described below. It is a consistent example of a regular, metacompact, star-Menger not strongly star-Menger D-space (see Definition 2.6.6 below). It is not normal though.

Space F: Let 2^{ω_1} be given the product topology, and let $\mathbf{F} = F_{\omega_1} = \{x \in 2^{\omega_1} : 0 < |supp(x)| < \omega\}$, the subspace of elements of 2^{ω_1} with finite non-empty support. Recall that $supp(x) = \{\alpha < \omega_1 : x(\alpha) = 1\}$. Also, let $F^\alpha = \{x \in 2^{\omega_1} : |supp(x)| = \alpha\}$ and $F_\alpha = \{x \in \mathbf{F} : supp(x) \subseteq \alpha\}$ for each $\alpha \in \omega_1$. A basis for $\mathbf{F}$ is the set $\{W(x, A) : x \in \mathbf{F} \wedge A \in [\omega_1]^{<\omega}\}$ where $W(x, A) = \Big\{y \in \mathbf{F} : \forall\, \alpha \in A\Big[x(\alpha) = y(\alpha)\Big]\Big\}$. For a basic open set $U = W(x, A)$, let's write $dom(U) := A$. ■

Remarks about F: It is easy to verify the following facts for each $n \geq 1$:

1. The set $F^{\leq n} = \{x \in \mathbf{F} : |supp(x)| \leq n\}$ is a closed subset of $\mathbf{F}$.
2. F^n is discrete.
3. If $C_n \subseteq F^n$ is so that for every $x, y \in C_n$, $supp(x) \cap supp(y) = \emptyset$, then C_n is closed and discrete.
4. $(F^{n+1})' \subset F^{\leq n}$.
5. $\mathbf{F} = \bigcup_{n \geq 1} F^n$.

Proposition 2.6.3. $\mathbf{F}$ *is metacompact.*

Proof. Given an open cover $\mathcal{U}$ of $\mathbf{F}$ we can refine it to an open cover $\{W(x, A_x) : x \in \mathbf{F}\}$ such that for all $x \in \mathbf{F}$, $supp(x) \subseteq A_x$. Indeed, let $\mathcal{U}$ be an open cover of $\mathbf{F}$ and fix $y \in \mathbf{F}$, then there exists $u_y \in \mathcal{U}$ such that $y \in u_y$. Hence, there exists $x \in \mathbf{F}$ and $A \in [\omega_1]^{<\omega}$ such that $y \in W(x, A) = u_y$. Observe that $W(y, A \cup supp(y)) \subseteq W(x, A)$ because if $z \in W(y, A \cup supp(y))$ then for all $\alpha \in A \cup supp(y)$, $z(\alpha) = y(\alpha)$ then, in particular, for all $\alpha \in A$, $z(\alpha) = y(\alpha) = x(\alpha)$. Hence, we can build such a refinement.

Now, assume that there exists $x_0 \in \mathbf{F}$ such that there exists $B \in [\mathbf{F}]^\omega$ such that for all $x \in B$, $x_0 \in W(x, A_x)$. Since for all $x \in B$, $supp(x) \subseteq A_x$, we get $C := \bigcup_{x \in B} supp(x) \subseteq supp(x_0)$. C cannot be finite, otherwise, for $x \in B$, $supp(x) \in \mathcal{P}(C)$ and this would imply that you can only build finitely many different $x \in B$, which is a contradiction. Hence, this refinement is point finite, i.e. $\mathbf{F}$ is metacompact. $\square$

It is easy to show that $\mathbf{F}$ is not a Lindelöf space. Therefore, $\mathbf{F}$ cannot be strongly star-Menger (not even strongly star-Lindelöf) by Proposition 2.2.7. In addition, since $\mathbf{F}$ has the CCC property, it is a star-Lindelöf space (see Theorem 3.1.6 in [26]). Thus, by Proposition 2.2.7, $\mathbf{F}$ is not a paraLindelöf space.

Theorem 2.6.4. *$\omega_1 < \mathfrak{d}$ if and only if $\mathbf{F}$ is star Menger.*

Proof. Assume that $\omega_1 < \mathfrak{d}$ and let $(\mathcal{U}'_n)$ be a sequence of open covers of $\mathbf{F}$ which consists of basic open sets. Take $\mathcal{U}_n$ point-countable refinements for each one of them, respectively.

Claim 1: $\exists \alpha < \omega_1 : \forall x \in \mathbf{F}\, \forall n \in \omega \left[supp(x) \subseteq \alpha \rightarrow [\forall U \in \mathcal{U}_n\, (x \in U \rightarrow dom(U) \subseteq \alpha)] \right]$.

Let us start with $F_\omega = \{x \in \mathbf{F} : supp(x) \subseteq \omega\}$. Consider $\mathcal{V}_i = \{U \in \mathcal{U}_i : U \cap F_\omega \neq \emptyset\}$. Note that each $\mathcal{V}_i$ is countable because each $\mathcal{U}_i$ is point-countable and $|F_\omega| = \omega$. Hence, the set $\mathcal{W}_0 = \bigcup_{i \in \omega} \mathcal{V}_i$ has size ω. Let $\alpha_0 = sup\{dom(V) : V \in \mathcal{W}_0\} < \omega_1$. Put $\beta_0 = \alpha_0 + 1$. Recursively, assume we have built $\beta_n < \omega_1$ such that

$$\forall x \in \mathbf{F} : \forall m \in \omega : \forall j < n : \Big[supp(x) \subseteq \beta_j \to [\forall U \in \mathcal{U}_m (x \in U \to dom(U) \subseteq \beta_n)]\Big].$$

Since the set $F_{\beta_n} = \{x \in \mathbf{F} : supp(x) \subseteq \beta_n\}$ is countable, we do the same as the previous paragraph to obtain $\alpha_{n+1} < \omega_1$. Let $\beta_{n+1} = \alpha_{n+1} + 1$. So, we take $\alpha = sup\{\beta_n : n \in \omega\}$. Since α is the supremum of an strictly increasing sequence, α is a limit ordinal.

Let us see that α has the property of the claim. Let $x \in \mathbf{F}$ and $n \in \omega$. Suppose that $supp(x) \subseteq \alpha$. Since $|supp(x)| < \omega$, then there exists $m < \omega$ such that $supp(x) \subseteq \beta_m$. By construction, we have $\forall n \in \omega \forall U \in \mathcal{U}_n : x \in U \to dom(U) \subseteq \beta_m$ with $\beta_m < \alpha$.

Claim 2: $\forall x \in \mathbf{F}\, \forall n \in \omega\, \exists U_n \in \mathcal{U}_n \Big[dom(U_n) \subseteq \alpha \wedge (x \in U_n \vee x \in St(U_n, \mathcal{U}_n))\Big]$.

Let $x \in \mathbf{F}$ and $n \in \omega$. We consider the following cases:

Case 1: If $supp(x) \cap \alpha \neq \emptyset$. Let $x' = x \restriction_\alpha \frown \overline{0}$. Hence, $x' \in \mathbf{F}$ and $supp(x') \subseteq \alpha$. Since $\mathcal{U}_n$ is a cover of $\mathbf{F}$, there exists $U_n \in \mathcal{U}_n$ such that $x' \in U_n$. By the property of α, $dom(U_n) \subseteq \alpha$. Therefore, $x \in U_n$.

Case 2: If $supp(x) \cap \alpha = \emptyset$. Let $V \in \mathcal{U}_n$ such that $x \in V$. Let $y \in \mathbf{F}$ with the following properties:

(i) $y \restriction dom(V) \cap \alpha \equiv \overline{0}$

(ii) $\exists \beta \in (\alpha \backslash dom(V)) : y(\beta) = 1$

(iii) $y \restriction dom(V) \cap (\omega_1 \backslash \alpha) \equiv x$

Since $y \restriction dom(V) \equiv x \restriction dom(V)$, $y \in V$. Moreover, using the *Case 1*, there exists $U_n \in \mathcal{U}_n$ such that $y \in U_n$ and $dom(U_n) \subseteq \alpha$. Hence, $x \in St(U_n, \mathcal{U}_n)$ because $y \in V \cap U_n$ and $x \in V$.

Observe that for each $n \in \omega$, $\mathcal{C}_n := \{U \in \mathcal{U}_n : dom(U) \subseteq \alpha\}$ is countable, hence enumerate it as $\mathcal{C}_n = \{U^n_m : m \in \omega\}$. Now, for each $x \in \mathbf{F}$, define $f_x \in \omega^\omega$ so that for each $n \in \omega$: $f_x(n) = m$ such that $x \in U^n_m \in \mathcal{U}_n$ or $x \in St(U^n_m, \mathcal{U}_n)$. By claim 2, these functions are well defined. Furthermore, since $\omega_1 < \mathfrak{d}$, the set $\mathcal{F} := \{f_x : x \in \mathbf{F}\}$ has cardinality less than $\mathfrak{d}$. Let $f_0 \in \omega^\omega$ such that for each $x \in \mathbf{F}$: $f_0 \not\leq^* f_x$. If we define for each $n \in \omega$, $\mathcal{V}_n := \{U^n_m : m < f_0(n)\}$, then clearly $\mathbf{F} = \bigcup_{n\in\omega} St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$. $\square$

This completes the proof that $\omega_1 < \mathfrak{d}$ implies $\mathbf{F}$ is star-Menger. For the converse, assume $\omega_1 = \mathfrak{d}$.

Consider $D := \left([\mathcal{B}(\mathbf{F})]^{<\omega}\right)^\omega$, i.e. D is the set of all sequences of finite families of basic open sets in $\mathbf{F}$. We order D by $f \prec g$ if $f(n) \subseteq g(n)$ for all n. By Lemma 2.4.14, since $\mathfrak{d} = \omega_1$, we have that the cofinality of $(D, \prec)$ is ω_1. So we may enumerate a cofinal subset E of D as $\{(y^\alpha_n)_{n\in\omega} : \alpha \in \omega_1\}$ such that each element of E appears ω_1 times.

We will build recursively a sequence $(\mathcal{U}_n)_{n\in\omega}$ of covers of $\mathbf{F}$ such that none of the sequences $(y^\alpha_n)_{n\in\omega}$ witness that $\mathbf{F}$ is star Menger. Since these sequences are cofinal in $(D \prec)$ this implies our constructed sequence will witness that $\mathbf{F}$ is not star-Menger.

Basic step: For all $n \in \omega$ let $\mathcal{U}^\omega_n = \{W(\chi_{\{m\}}, \{m\}) : m \in \omega\}$, where $\chi_{\{m\}}$ is the characteristic function of $\{m\}$ ($\chi_{\{m\}}(t) = 1$ if and only if $t = m$). Then, for all $n \in \omega$:

1. $\mathcal{U}^\omega_n$ covers F_ω,
2. for each $U \in \mathcal{U}^\omega_n$, $U = W(x, A)$ for some $x \in F_\omega$ and $A \subseteq \omega$ such that $supp(x) \subseteq A$.

Successor step: Fix $\alpha < \omega_1$ and assume we have built $(\mathcal{U}^\alpha_n)_{n\in\omega}$ such that each $\mathcal{U}^\alpha_n$ covers F_α and for all $U \in \mathcal{U}^\alpha_n$, $U = W(x, A)$ for some $x \in F_\alpha$ and $A \subseteq \alpha$

with $supp(x) \subseteq A$.

Case (1) $\forall n \in \omega\, \forall V \in y_n^\alpha : supp(V) \subseteq \alpha$.

Let $x_\alpha = \chi_{\{\alpha\}}$ and for every $n \in \omega$ let

$$W_n^\alpha = W\Big(x_\alpha, \{\alpha\} \cup \bigcup\{A : \exists y : W(y, A) \in y_n^\alpha\}\Big).$$

We now define for all $n \in \omega$, $\mathcal{U}_n^{\alpha+1} = \mathcal{U}_n^\alpha \cup \{W_n^\alpha\}$. We need to show that $\mathcal{U}_n^{\alpha+1}$ covers $F_{\alpha+1}$: Let $n \in \omega$ and $z \in F_{\alpha+1}$. Then we have the following cases: (i) If $supp(z) = \{\alpha\}$, then $z = x_\alpha \in W_n^\alpha$. (ii) If $supp(z) \cap \alpha \neq \emptyset$, then the function $z' \in F_\alpha$ such that $z' \restriction_\alpha = z \restriction_\alpha$, belongs to some $W(x, A) \in \mathcal{U}_n^\alpha$ and therefore $z \in W(x, A)$ because $A \subseteq \alpha$.

Case (2) $\exists n \in \omega\, \exists V \in y_n^\alpha : supp(V) \nsubseteq \alpha$.

Then for all $n \in \omega$ let $\mathcal{U}_n^{\alpha+1} = \mathcal{U}_n^\alpha \cup \{W(\chi_{\{\alpha\}}, \{\alpha\})\}$. Observe that for all $n \in \omega$, $\mathcal{U}_n^{\alpha+1}$ also covers $F_{\alpha+1}$.

Furthermore, in either case we have that for all $U \in \mathcal{U}_n^{\alpha+1}$, $U = W(x, A)$ for some $x \in F_{\alpha+1}$ and $A \subseteq \alpha + 1$ such that $supp(x) \subseteq A$. Thus, the recursion is complete for the successor step.

Limit step: If $\gamma < \omega_1$ is a limit ordinal, for all $n \in \omega$, let $\mathcal{U}_n^\gamma = \bigcup_{\alpha<\gamma} \mathcal{U}_n^\alpha$. Clearly, $\mathcal{U}_n^\gamma$ covers F_γ and satisfies all recursive hypothesis.

For all $n \in \omega$ let $\mathcal{U}_n = \bigcup_{\alpha<\omega_1} \mathcal{U}_n^\alpha$. For each $n \in \omega$, $\mathcal{U}_n$ covers $\mathbf{F}$. We show now that $(\mathcal{U}_n)_{n\in\omega}$ witnesses that $\mathbf{F}$ is not star Menger.

For all $n \in \omega$ let $B_n \in [\mathcal{U}_n]^{<\omega}$. We show $\bigcup\{St(\bigcup B_n, \mathcal{U}_n) : n \in \omega\}$ doesn't cover $\mathbf{F}$.

Since for each $n \in \omega :\ domB_n := \bigcup_{A\in B_n} domA$ is finite, then $\bigcup_{n\in\omega} domB_n$ is countable. Thus, we can get $\beta < \omega_1$ such that $\bigcup_{n\in\omega} domB_n \subseteq \beta$. Hence, for all $n \in \omega$, $B_n \subseteq \mathcal{U}_n^\beta$. Using the property of the sequence $\{(y_n^\alpha)_{n\in\omega} : \alpha \in \omega_1\}$, there exists $\alpha > \beta$ such that for all $n \in \omega$: $y_n^\alpha \supseteq B_n$ and for each $V \in y_n^\alpha, supp(V) \subseteq \alpha$. Therefore, $\mathcal{U}_n^{\alpha+1}$ is defined as in Case (1) above.

Claim: $x_\alpha = \chi_{\{\alpha\}}$ is not covered by $\bigcup\{St(\bigcup y_n^\alpha, \mathcal{U}_n) : n \in \omega\}$. Indeed:

1. For each $n \in \omega$ and for each $U \in \mathcal{U}_n^\alpha$, $x_\alpha \notin U$: This is true because $x_\alpha = \chi_{\{\alpha\}}$ and the construction of $\mathcal{U}_n$.
2. For each $n \in \omega$, $x_\alpha \notin St(\bigcup y_n^\alpha, \mathcal{U}_n^{\alpha+1})$: Fix $n \in \omega$, from 1., we get that the unique element in $\mathcal{U}_n^{\alpha+1}$ that contains x_α is W_n^α, but such W_n^α was built so that for each $V \in y_n^\alpha$, $W_n^\alpha \cap V = \emptyset$.
3. For each $n \in \omega$ and for each $\beta > \alpha+1$, $x_\alpha \notin St(\bigcup y_n^\alpha, \mathcal{U}_n^\beta)$: Assume the opposite, hence $\exists n \in \omega\, \exists \beta > \alpha+1\, \exists U_{x_\alpha} \in \mathcal{U}_n^\beta$ such that $x_\alpha \in U_{x_\alpha}$ and $U_{x_\alpha} \cap (\bigcup y_n^\alpha) \neq \emptyset$. Fix β minimal with this property. By the recursive construction of $\mathcal{U}_n$, $U_{x_\alpha} = W(x, A)$ with $x = \chi_{\{\beta\}}$ and $\beta \in A$, this implies that $x_\alpha(\beta) = \chi_{\{\alpha\}}(\beta) = 1$, which is a contradiction.

From 1., 2., and 3. the claim is true. Since for each $n \in \omega, B_n \subseteq y_n^\alpha$, it follows that $\{St(\bigcup B_n, \mathcal{U}_n) : n \in \omega\}$ does not cover $\mathbf{F}$. Thus, $\mathbf{F}$ is not star Menger.

Proposition 2.6.5. $\mathbf{F}$ *is not normal.*

Proof. For each $\alpha \in \omega_1$, let us define $x_\alpha = \chi_{\{\alpha\}} \in 2^{\omega_1}$ as the characteristic function of $\{\alpha\}$ ($x_\alpha(\beta) = 1$ if and only if $\beta = \alpha$). It is easy to verify that the set $F^1 = \{x_\alpha : \alpha \in \omega_1\}$ is a closed subset of $\mathbf{F}$. Let us choose subsets C_1, C_2 of ω_1 such that $|C_1| = \omega_1$, $|C_2| = \omega$, $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = \omega_1$. We put $E_1 = \{x_\alpha : \alpha \in C_1\}$ and $E_2 = \{x_\alpha : \alpha \in C_2\}$. Note that E_1 and E_2 are disjoint subsets of F, $|E_1| = \omega_1$ and $|E_2| = \omega$. Furthermore, both sets are closed (and discrete) subsets in $\mathbf{F}$.

Let us show that E_1 and E_2 cannot be separated by open sets in $\mathbf{F}$. Let U be an open set in $\mathbf{F}$ so that $E_1 \subseteq U$. By the topology on $\mathbf{F}$, for each $\alpha \in C_1$, there exists a finite subset A_α of ω_1 such that $\alpha \in A_\alpha$ and $W(x_\alpha, A_\alpha) \subseteq U$. Using the Δ-system Lemma on the collection $\{A_\alpha : \alpha \in C_1\}$, there are an uncountable set $C \subseteq C_1$ and a finite set $A \subset \omega_1$ such that for all $\alpha, \beta \in C$, $A_\alpha \cap A_\beta = A$. Hence, the family $\{A_\alpha \backslash A : \alpha \in C\}$ is disjoint. Since C_2 is infinite and A is finite, $C_2 \backslash A \neq \emptyset$. Let $\alpha^* \in C_2 \backslash A$. Then $x_{\alpha^*} \in E_2$ and $x_{\alpha^*} \in \overline{U}$. Indeed, let V be an open set in $\mathbf{F}$ such that $x_{\alpha^*} \in V$. Then, there is a finite subset A^* of ω_1 such that $\alpha^* \in A^*$ and $W(x_{\alpha^*}, A^*) \subseteq V$. As the set C is uncountable and the sets A and A^* are finite, there is an element $\beta^* \in C$ such that $A^* \cap (A_{\beta^*} \backslash A) = \emptyset$ and $\beta^* \in A_{\beta^*} \backslash A$. Note that $\alpha^* \neq \beta^*$ because $\alpha^* \in A^*$ and $\beta^* \in A_{\beta^*} \backslash A$. So we define the function $y \in 2^{\omega_1}$ so that $y(\alpha) = 1$ if $\alpha \in \{\alpha^*, \beta^*\}$ and $y(\alpha) = 0$ otherwise. Let us show that $y \in W(x_{\beta^*}, A_{\beta^*}) \cap W(x_{\alpha^*}, A^*)$. Let $\beta \in A_{\beta^*} \backslash \{\beta^*\}$. If $\beta \in A$, then $\beta \neq \alpha^*$ since $\alpha^* \in C_2 \backslash A$. Hence $y(\beta) = 0$. If $\beta \notin A$, then $\beta \in A_{\beta^*} \backslash A$. Since $A^* \cap (A_{\beta^*} \backslash A) = \emptyset$ and $\alpha^* \in A^*$, $\beta \neq \alpha^*$. Hence $y(\beta) = 0$. Thus $y \in W(x_{\beta^*}, A_{\beta^*})$. Now, let $\beta \in A^* \backslash \{\alpha^*\}$. Since $\beta^* \in A_{\beta^*} \backslash A$ and $A^* \cap (A_{\beta^*} \backslash A) = \emptyset$, $\beta \neq \beta^*$. Hence $y(\beta) = 0$. Thus $y \in W(x_{\alpha^*}, A^*)$. Therefore, we have

$$y \in W(x_{\beta^*}, A_{\beta^*}) \cap W(x_{\alpha^*}, A^*) \subseteq U \cap V.$$

This show that $x_{\alpha^*} \in \overline{U}$. Hence the sets E_1 and E_2 cannot be separated. $\square$

To show that $\mathbf{F}$ is a D-space, let us recall the definition:

Definition 2.6.6 ([25]). *Given a space (X, τ):*

- *A function $N : X \to \tau$ such that for each $x \in X$, $x \in N(x)$ is called an* **open neighbourhood assignment** (ONA) **of** (X, τ) *(τ is omitted when there's no confusion).*
- *Given an ONA N, a subset $D \subseteq X$ is called a* **kernel of** X **with respect to** N *if $N(D) = \{N(x) : x \in D\}$ is a cover of X.*
- *We say X is a* **D-space** *if and only if every ONA N of X has a closed and discrete kernel, i.e., for each ONA N of X there is some closed and discrete subset $D \subseteq X$ such that $N(D) \in \mathcal{O}(X)$.*

D-spaces were introduced by van Douwen in the 1970's and the most important open problem about them asks whether every regular Lindelöf space is a D-space. For more information on D-spaces refer to [25], [27], [35]).

In [43] Hurewicz provided a non-trivial characterization of the Menger property in terms of a two-player game. Two-player games can be naturally associated to most selection principles (in the next section we discuss this in more detail). L. Aurichi [5], uses the game characterization of the Menger property to prove that every Menger space is a D-space. Aurichi's result opened a window of opportunity to wonder what kind of (star) selection principles, other than Menger, turn out to be D-spaces. For instance, the following is not known:

Question 2.6.7 ([18]). *Is it true that every metaLindelöf (metacompact) star-Menger space is a D-space?*

Since **F** is a D-space, it gives some hope that the previous question might be answered in the positive. The following lemma, easy to prove, will be used to show that $\mathbf{F}$ is a D-space.

Lemma 2.6.8. *Let X be a topological space and $A \subseteq X$ be a set with $A \cap A' = \emptyset$. If $\mathcal{U}$ is a collection of open sets so that $A' \subseteq \bigcup \mathcal{U}$, then $A \setminus \bigcup \mathcal{U}$ is a closed and discrete subset of X.*

Proposition 2.6.9. $\mathbf{F}$ *is a D-space.*

Proof. Let N be an ONA of $\mathbf{F}$. We know that F^1 is a closed discrete subset of $\mathbf{F}$. Put $D_1 = F^1$ and define

$$n_2 = min\{n \in \omega : F^n \setminus \bigcup_{x \in D_1} N(x) \neq \emptyset\}.$$

By Lemma 2.6.8, $F^{n_2} \setminus \bigcup_{x \in D_1} N(x)$ is closed and discrete. Put $D_2 = F^{n_2} \setminus \bigcup_{x \in D_1} N(x)$. In general, we define

$$n_k = min\{n \in \omega : F^n \setminus \bigcup_{x \in \bigcup_{i<k} D_i} N(x) \neq \emptyset\}$$

and

$$D_k = F^{n_k} \setminus \bigcup_{x \in \bigcup_{i<k} D_i} N(x).$$

Let $D = \bigcup_{k \geq 1} D_k$. Since each D_k is a closed and discrete subset of $\mathbf{F}$, D is also closed and discrete in $\mathbf{F}$. Furthermore, $\mathbf{F} = \bigcup_{x \in D} N(x)$. Indeed, note that by definition we have $k \leq n_k$ for each $k \geq 1$. Hence, for each $k \geq 1$, $F^k \subseteq \bigcup_{x \in \bigcup_{i \geq k} D_i} N(x)$ and thus $\mathbf{F} = \bigcup_{k \geq 1} F^k \subseteq \bigcup_{k \geq 1} \bigcup_{x \in \bigcup_{i \leq k} D_i} N(x) = \bigcup_{x \in D} N(x)$. □

In the pursuit of answering Question 2.6.2, it is natural to consider some important pathologies in the theory of normal spaces, so we can ask if perhaps one of the Dowker spaces in the literature could be an example of a star-Menger not strongly star-Menger normal space. Or perhaps there is a theorem:

Question 2.6.10 ([18] Question 2.21)**.** *Are normal, countably paracompact star-Menger spaces strongly star-Menger? I. e., if X is normal, star-Menger, not strongly star-Menger, is X a Dowker space?*

Recall that Dowker proved in [24] that a normal space X is countably paracompact if, and only if, the product of X with the closed unit interval is normal. He asked whether there exists a normal space which is not countably paracompact. This became the well known Dowker space problem which was solved by M. E. Rudin in 1971 (see [70]), providing a ZFC example of a normal not countably paracompact space which is a subspace of the box product $\square_{1<n<\omega}\omega_n$. Because of this, normal spaces which are not countably paracompact are called **Dowker spaces**. In [18] we showed that the hereditarily separable, first countable De Caux - type Dowker space constructed from $\clubsuit$ by M.E. Rudin in [70] is strongly star-Rothberger and therefore strongly star-Menger. We recall the definition of $\clubsuit$-sequence:

Definition 2.6.11. *A sequence $\langle S_\alpha : \alpha \in LIM(\omega_1)\rangle$ is a* **$\clubsuit$-sequence** *if and only if for all α:*

(i) $S_\alpha \subseteq \alpha$;

(ii) $ordertype(S_\alpha) = \omega$;

(ii) $Sup\, S_\alpha = \alpha$; *and*

(iv) Each uncountable subset of ω_1 contains an S_α.

Example 2.6.12. *In [70, Dowker Spaces 3.1] M. E. Rudin defines a de Caux type space, as follows: let $\langle S_\alpha : \alpha \in LIM(\omega_1)\rangle$ be a $\clubsuit$-sequence. For* *each α, partition S_α into infinite, disjoint subsets $\{S_{\alpha ijn} : i, j, n \in \omega\}$. Let $X = \omega_1 \times \omega$ and $U \subseteq X$ is open if and only if for every $\langle \alpha + j, n\rangle \in U$, with $\alpha \in LIM(\omega_1)$, $n, j \in \omega$ and for every $i \leq n$, there is a cofinite subset S^i of $S_{\alpha ijn}$ such that $\{\langle \beta, i\rangle : \beta \in S^i\} \subseteq U$. Defined in this fashion, X is a Dowker space.*

Proposition 2.6.13. *If $U \subseteq X$ is an open set such that for some $n \in \omega$ and some stationary set $E \subseteq \omega_1$, $E \times \{n\} \subseteq U$, then $(\omega_1 \times \{n\}) \smallsetminus U$ is countable.*

Proof. Let us assume on the contrary that $(\omega_1 \times \{n\}) \smallsetminus U$ is uncountable. Since $\langle S_\alpha : \alpha \in LIM(\omega_1)\rangle$ is a $\clubsuit$-sequence, there exists $\alpha_0 \in LIM(\omega_1)$ such that $S_{\alpha_0} \times \{n\} \subseteq (\omega_1 \times \{n\}) \smallsetminus U$. Now, let $\beta > \alpha_0$, let us show that $\langle \beta, n\rangle \notin U$. If $\langle \beta, n\rangle \in U$, by the definition of an open set in X, we can find, in a finite number of steps, $j \in \omega$ such that $\langle \alpha_0 + j, n\rangle \in U$. Again, using that U is open, there exists a finite set $F \subseteq S_{\alpha_0 njn} \subseteq S_{\alpha_0}$ such that $(S_{\alpha_0 njn} \setminus F) \times \{n\} \subseteq U$, which contradicts that $S_{\alpha_0} \times \{n\} \subseteq (\omega_1 \times \{n\}) \smallsetminus U$. Therefore, for all $\beta > \alpha_0$ $\langle \beta, n\rangle \notin U$, which is a contradiction. Therefore, $(\omega_1 \times \{n\}) \smallsetminus U$ is countable. □

Proposition 2.6.14. *Let $\mathcal{U}$ be an open cover of X, then for all $n \in \omega$ there exists $\beta < \omega_1$ such that $St(\langle \beta, n\rangle, \mathcal{U})$ contains a stationary subset of $\omega_1 \times \{n\}$.*

Proof. Let $\mathcal{U}$ be an open cover of X and fix $n \in \omega$. For each $\alpha \in LIM(\omega_1)$, pick $U_\alpha \in \mathcal{U}$ such that $\langle \alpha, n\rangle \in U_\alpha$. Define $f_n : LIM(\omega_1) \to \omega_1$ as follows: for each $\alpha \in LIM(\omega_1)$, $f_n(\alpha)$ is so that $\langle f_n(\alpha), n\rangle \in [S_\alpha \times \{n\}]$. Since f_n is regressive, by the Pressing Down Lemma, there exists $\beta < \omega_1$ such that$f_n^{-1}(\beta)$ is stationary. Hence, $St(\langle \beta, n\rangle, \mathcal{U})$ contains the stationary $\{\langle \gamma, n\rangle : \gamma \in f_n^{-1}(\beta)\} \subseteq \omega_1 \times \{n\}$. □

Theorem 2.6.15. *The de Caux space X is strongly star-Rothberger.*

Proof. Let $(\mathcal{U}_n)_{n<\omega}$ be a sequence of covers of X. By Propositions 2.6.13 and 2.6.14, for all $n \in \omega$, there exists $\beta_n < \omega_1$ such that $(\omega_1 \times \{n\}) \setminus St(\langle\beta_n, n\rangle, \mathcal{U}_{2n})$ is countable. Hence, $X \setminus \left[\bigcup_{n<\omega} St(\langle\beta_n, n\rangle, \mathcal{U}_{2n})\right]$ is countable. Let us enumerate this set as $\{\gamma_j : j < \omega\}$. Then,

$$X = \bigcup_{n<\omega} St(\langle\beta_n, n\rangle, \mathcal{U}_{2n}) \cup \bigcup_{n<\omega} St(\gamma_n, \mathcal{U}_{2n+1}).$$

Therefore, X is strongly star-Rothberger. $\square$

During a work in progress with William Chen-Mertens and Javier Casas-de la Rosa, William brought to our attention that Example 2.6.20 below can be used to answer consistently in the afirmative Question 2.6.2 and in the negative Question 2.6.10, i.e. there is a normal star-Menger not strongly star-Menger space that, in addition, it is not a Dowker space.

Example 2.6.19 (below) was presented By Song in [82] to provide a *feebly Lindelöf space* X (every locally finite family of non-empty open sets in X is countable) which is not Lindelöf star kernel (X is *Lindelöf star kernel* if for every $\mathcal{U} \in \mathcal{O}(X)$ there is $L \subseteq X$ Lindelöf subspace of X such that $St(L, \mathcal{U}) = X$. This notion is called "Star-Lindelöf" in that article). Song provided this space to partially answer, in the negative, a question of Alas, Junqueira and Wilson [2] on whether T_4 feebly Lindelöf spaces are Lindelöf star kernel. Then he used this space again in [84] to present a normal star-Lindelöf space which is not neighbourhood star-Lindelöf (X is said to be *neighborhood star-Lindelöf* if for every $\mathcal{U} \in \mathcal{O}(X)$ there exists a countable subset A of X such that for every open $O \supseteq A$, $X = St(O, \mathcal{U})$). This space is a modification of a space given by Tall in [88] (see Example 2.6.18 below) to provide an example of a separable normal space with an uncountable discrete subspace. Let us present both spaces. Since the spaces require independent families, first, we provide the definition of (strongly) independent families and the proof of the existence of such families of size 2^κ for any infinite cardinal κ.

Definition 2.6.16. *Let κ be an infinite cardinal. A family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is called* **independent** *if for all pairs of disjoint $F, G \in [\mathcal{F}]^{<\omega}$ we have:*

$$C_{F,G} = \bigcap_{A \in F} A \cap \bigcap_{A \in G} (\kappa \setminus A) \neq \emptyset$$

(Assume $\bigcap \emptyset = \kappa$). If in addition, for each pair (F, G) as above, $|C_{F,G}| = \kappa$, $\mathcal{F}$ is called **strongly independent***.*

Theorem 2.6.17 (Fichtenholz-Kantorovitch-Hausdorff)**.** *Let κ be an infinite cardinal. Then there exists a strongly independent family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{F}| = 2^\kappa$.*

The following proof is taken from [49].

Proof. Let $K = [\kappa]^{<\omega} \times \left[[\kappa]^{<\omega}\right]^{<\omega}$. We build a strongly independent family using K that codes a strongly independent family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of size 2^κ. For $A \in \mathcal{P}(\kappa)$ let

$$B_A = \{(s, T) \in K : s \cap A \in T\}.$$

Let $\mathcal{B} = \{B_A : A \in \mathcal{P}(\kappa)\}$. First observe that if $A_0, A_1 \in \mathcal{P}(\kappa)$ are different, WLOG pick $x \in A_0 \setminus A_1$, then $(\{x\}, \{\{x\}\}) \in B_{A_0} \setminus B_{A_1}$. Thus $|\mathcal{B}| = 2^\kappa$. Now to show that $\mathcal{B}$ is strongly independent let (F, G) be a pair of finite disjoint subsets of $\mathcal{B}$. Write $F \cup G = \{B_{A_i} : i \leq n\}$. For $i < j < n$ fix $\xi_{i,j} \in A_i \triangle A_j$. Let $s = \{\xi_{i,j} : i < j < n\}$ and $T = \{s \cap A_i : B_{A_i} \in F\}$.

Claim: If $R \in \left[[\kappa]^{<\omega}\right]^{<\omega}$ is such that $T \subseteq R$ and $s \cap A_j \notin R$ whenever $B_{A_j} \in G$, then

$$(s, R) \in \Big(\bigcap_{B_{A_i} \in F} B_{A_i}\Big) \cap \Big(\bigcap_{B_{A_j} \in G} (K \smallsetminus B_{A_j})\Big).$$

Indeed, if $B_{A_i} \in F$, then $s \cap A_i \in T \subseteq R$ and therefore, $(s, R) \in B_{A_i}$. Now, if $B_{A_j} \in G$, then $s \cap A_j \notin R$, which implies that $(s, R) \notin B_{A_j}$. Hence $(s, R) \in K \smallsetminus B_{A_j}$. Observe that there are κ many R that satisfy the assumptions of the claim. Thus, $\mathcal{B}$ is strongly independent.

Now let $f : K \to \kappa$ be a bijection and let $\mathcal{F} = \{f''B : B \in \mathcal{B}\}$. If $B_0, B_1 \in \mathcal{B}$ and $B_0 \neq B_1$, then $f''B_0 \neq f''B_1$. Hence $\mathcal{F}$ has size 2^κ. To show that $\mathcal{F}$ is strongly independent let $F, G \in [\mathcal{B}]^{<\omega}$, it should hold that the set $I = \left(\bigcap_{B \in F} f''B\right) \cap \left(\bigcap_{B \in G}(\kappa \smallsetminus f''B)\right)$ has size κ. But we know that the set $J = \left(\bigcap_{B \in F} B\right) \cap \left(\bigcap_{B \in G}(K \smallsetminus B)\right)$ has size κ and for each $(s, T) \in J$, $f((s, T)) \in I$. Since f is a bijection, I has size κ. $\square$

Example 2.6.18 ([88] Example E)**.** *Assuming $2^{\aleph_0} = 2^{\aleph_1}$ there exists a separable normal T_1 space with an uncountable closed discrete subspace.*

Construction: Let L be a set of cardinality $\aleph_1$ disjoint from ω. By Theorem 2.6.17, there exists $\mathcal{F}$ strongly independent family of subsets of $\aleph_0 = \omega$ of size $2^{\aleph_0} = \mathfrak{c}$. Write $\mathcal{F} = \{A_\alpha : \alpha < \mathfrak{c}\}$. Since $|L| = \aleph_1$, $|\mathcal{P}(L)| = 2^{\aleph_1}$. Assuming $2^{\aleph_0} = 2^{\aleph_1}$ it is possible to build a function $f : \mathcal{P}(L) \to \{A_\alpha : \alpha < \mathfrak{c}\} \cup \{\omega \smallsetminus A_\alpha : \alpha < \mathfrak{c}\}$ which is bijective and complement-preserving (for each $B \subseteq L$, $f(L \smallsetminus B) = \omega \smallsetminus f(B)$).

Now let $X = L \cup \omega$ with a subbase φ for a topology defined by

1. if $M \subseteq L$, then $M \cup f(M) \in \varphi$,
2. if $n \in \omega$, then $\{n\} \in \varphi$,
3. if $p \in X$, then $X \smallsetminus \{p\} \in \varphi$.

Observe that by condition (3) X is T_1. By (2) ω is open, therefore $L = X \smallsetminus \omega$ is closed and, by (1) for any $x \in L$, $\{x\} \cup f(\{x\})$ is open such that $[\{x\} \cup f(\{x\})] \cap L = \{x\}$, that is L is discrete. X is separable since ω is dense in X: let U be any nonempty basic open set, then

$$U = \bigcap_{U \in F} U \cap \bigcap_{U \in G} U \cap \bigcap_{U \in H} U$$

where F, G, H are finite (possibly empty), each $U \in F$ is a subbasic open set defined as in (1), each $U \in G$ is a subbasic open set defined as in (2), and each $U \in H$ is a subbasic open set defined as in (3). To show $U \cap \omega \neq \emptyset$ it is enough to observe that $|\left(\bigcap_{U \in F} U\right) \cap \omega| = \omega$. This is always the case since $\mathcal{F}$ is a strongly independent family. Now let Y, Z be disjoint closed subsets of X and observe:

$$\begin{aligned} U_Y &= \Big((Y \smallsetminus L) \cup [(Y \cap L) \cup f(Y \cap L)]\Big) \cap (X \smallsetminus Z) \\ &= \Big(Y \cup f(Y \cap L)\Big) \cap (X \smallsetminus Z) \\ U_Z &= \Big((Z \smallsetminus L) \cup [(L \smallsetminus Y) \cup f(L \smallsetminus Y)]\Big) \cap (X \smallsetminus Y) \end{aligned}$$

are open sets and $Y \subseteq U_Y$, $Z \subseteq U_Z$. Assume $x \in U_Y \cap U_Z$, then $x \in X \smallsetminus (Y \cup Z)$ and $x \in f(Y \cap L) \cap f(L \smallsetminus Y)$. But this is a contradiction since f is complement preserving: $f(L \smallsetminus Y) = f(L \smallsetminus (Y \cap L)) = \omega \smallsetminus (Y \cap L)$. Hence, X is normal. ■

Example 2.6.19 ([82], [84])**.** *Assuming $2^{\aleph_0} = 2^{\aleph_1}$ there exists a normal T_1 space which is star-Lindelöf and not strongly star Lindelöf.*

Construction: Let $X_0 = L \cup \omega$ denote the space built in Example 2.6.18. Let $X = L \cup (\omega_1 \times \omega)$ and topologize it as follows, a basic open set of

(i) $x \in L$ is a set of the form $V^U_\alpha(x) = (U \cap L) \cup \big((\alpha, \omega_1) \times (U \cap \omega)\big)$ where U is a neighbourhood of $x \in X_0$ and $\alpha < \omega_1$.

(ii) $\langle \alpha, n\rangle \in (\omega_1 \times \omega)$ is a set of the form $V_W(\langle \alpha, n\rangle) = W \times \{n\}$ where W is a neighbourhood of α in ω_1 with the usual topology.

Condition (i) guarantees that X is T_1. Furthermore, $\omega_1 \times \omega$ is open in X and for $x \in L$, if we let $U = \{x\} \cup f(\{x\})$. then for any $\alpha < \omega_1$, $V^U_\alpha(x) \cap L = \{x\}$. That is, L is closed and discrete in X.

X **is normal:** Let $Y, Z \subseteq X$ closed and disjoint. Define $Y_L = Y \cap L$ and $Z_L = Z \cap L$ and for each $n \in \omega$, $Y_n = Y \cap (\omega_1 \times \{n\})$, $Z_n = Z \cap (\omega_1 \times \{n\})$. Since $Y \cap Z = \emptyset$ and $\omega_1 \times \{n\}$ is a copy of ω_1 with the usual topology (for each $n \in \omega$), then we can find clopen sets $Y'_n, Z'_n \subseteq \omega_1 \times \{n\}$ such that $Y'_n \cap Z'_n = \emptyset$, $Y_n \subseteq Y'_n$, $Z_n \subseteq Z'_n$ and so that for each $n \in \omega$, Y'_n is cofinal in $\omega_1 \times \{n\}$ if and only if Y_n is cofinal in $\omega_1 \times \{n\}$ and Z'_n is cofinal in $\omega_1 \times \{n\}$ if and only if Z_n is cofinal in $\omega_1 \times \{n\}$. This is possible since for each $n \in \omega$, Y_n and Z_n cannot be both cofinal (otherwise $Y_n \cap Z_n \neq \emptyset$). Let

$$\mathcal{Y} = Y_L \cup \bigcup_{n \in \omega} Y'_n, \qquad \mathcal{Z} = Z_L \cup \bigcup_{n \in \omega} Z'_n$$

Observe $Y \subseteq \mathcal{Y}$, $Z \subseteq \mathcal{Z}$ and $\mathcal{Y} \cap \mathcal{Z} = \emptyset$.

Claim: $\mathcal{Y}$ and $\mathcal{Z}$ are closed in X.

Indeed, if $\langle \alpha, m\rangle \in (\omega_1 \times \omega) \setminus \mathcal{Y}$, since Y'_m is clopen in $\omega_1 \times \{m\}$, then there is U open neighbourhood of $\langle \alpha, m\rangle$ in $\omega_1 \times \{m\}$ (and therefore open neighbourhood in X), such that $U \cap Y'_m = \emptyset$. Now, let $x \in L \setminus \mathcal{Y}$ and assume that for each U open neighbourhood of x in X_0 and each $\alpha < \omega_1$,

$V_\alpha^U(x) \cap \mathcal{Y} \neq \emptyset$. This implies that for each U open neighbourhood of x in X_0 and each $\alpha < \omega_1$ there is some $n \in \omega$ such that $V_\alpha^U(x) Y_n' \neq \emptyset$ and Y_n' is cofinal in $\omega_1 \times \{n\}$. Then Y_n is cofinal in $\omega_1 \times \{n\}$ and $V_\alpha^U(x) Y_n \neq \emptyset$. Hence, $x \in \overline{Y} = Y$ which is a contradition. Thus, $\mathcal{Y}$ is closed. A similar argument shows that $\mathcal{Z}$ is closed.

Since Y_L and Z_L are disjoint closed subsets of X_0 and X_0 is normal (recall X_0 is the space constructed in Example 2.6.18), then there exist disjoint open sets U_Y, U_Z in X_0 such that $Y_L \subseteq U_Y$, $Z_L \subseteq U_Z$. Let

$$V_Y = (U_Y \cap Y) \cup \bigcup_{n \in U_Y \cap \omega} (\omega_1 \times \{n\}, \qquad V_Z = (U_Z \cap Z) \cup \bigcup_{n \in U_Z \cap \omega} (\omega_1 \times \{n\}.$$

Observe that V_Y and V_Z are disjoint open subsets in X and $Y_L \subseteq V_Y$, $Z_L \subseteq V_Z$. Let $W_Y = \mathcal{Y} \cup (V_Y \setminus \mathcal{Z})$, $W_Z = \mathcal{Z} \cup (V_Z \setminus \mathcal{Y})$. Hence, W_Y and W_Z are open sets in X, $W_Y \cap W_Z = \emptyset$, and $y \subseteq W_Y$, $Z \subseteq W_Z$.

X **is not strongly star-Lindelöf:** List $L = \{x_\alpha : \alpha < \omega_1\}$. Since L is a closed discrete subset of X_0, for $\alpha < \omega_1$ let D_α be an open neighbourhood of x_α in X_0 such that $D_\alpha \cap L = \{x_\alpha\}$. Hence,

$$\mathcal{U} = \{V_\alpha^{D_\alpha}(x_\alpha) : \alpha < \omega_1\} \cup \{\omega_1 \times \omega\} \in \mathcal{O}(X).$$

Assume $E \in [X]^\omega$, we show $St(E,\mathcal{U}) \neq X$. Since E is countable, fix $\beta_0, \beta_1 < \omega_1$ such that $sup\{\alpha : x_\alpha \in E \cap L\} < \beta_0$ and $sup\{\gamma : \langle \gamma, n\rangle \in E \textit{ for some } n \in \omega\} < \beta_1$. Let $\alpha = max\{\beta_0, \beta_1\}$ and observe $E \cap V_\alpha^{D_\alpha}(x_\alpha) = \emptyset$ Since $V_\alpha^{D_\alpha}(x_\alpha)$ is the only element of $\mathcal{U}$ that contains x_α, then $x_\alpha \notin St(E,\mathcal{U})$. Thus, X is not strongly star-Lindelöf.

X **is star-Lindelöf:** Let $\mathcal{U} \in \mathcal{O}(X)$ and define

$$M = \{n \in \omega : (\exists U \in \mathcal{U})(\exists \beta < \omega_1)[(\beta, \omega_1) \times \{n\} \subseteq U]\}.$$

For each $n \in M$ fix $U_n \in \mathcal{U}$ and $\beta_n < \omega_1$ such that $(\beta, \omega_1) \times \{n\} \subseteq U_n$. Put $\mathcal{V}' = \{U_n : n \in M\}$.

Claim: $L \subseteq St(\bigcup \mathcal{V}', \mathcal{U})$.

Indeed, let $x \in L$, there is $U^x \in \mathcal{U}_n$ such that $x \in U^x$ and therefore, there is U open neighbourhood of x in X_0 and $\alpha < \omega_1$ such that $V^U_\alpha(x) \subseteq U^x$. Since $V^U_\alpha(x) \cap (\omega_1 \times \omega) = (\alpha, \omega_1) \times (U \cap \omega)$ and $U = N \cup f(N)$ for some $N \subseteq L$, with $x \in N$, it holds true that $n \in f(N) \to n \in M$. Then, for $n \in f(N)$, $V^U_\alpha(x) \cap (\omega_1 \times \{n\}) \cap [(\beta_n, \omega_1) \times \{n\}] \neq \emptyset$. Thus, $V^U_\alpha(x) \cap U_n \neq \emptyset$. Hence, $U^x \cap U_n \neq \emptyset$. Therefore $x \in St(U_n, \mathcal{U}) \subseteq St(\bigcup \mathcal{V}', \mathcal{U})$. Now, $\omega_1 \times \omega$ is a countable union of strongly star compact spaces (see Example 2.6 above), then there is a countable $\mathcal{V}'' \subseteq \mathcal{U}$ such that $\omega_1 \times \omega \subseteq St(\bigcup \mathcal{V}'', \mathcal{U})$. If we let $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$, then $St(\bigcup \mathcal{V}, \mathcal{U}) = X$. ■

Proposition 2.6.20. *Assuming $2^{\aleph_0} = 2^{\aleph_1}$ and $\aleph_1 < \mathfrak{d}$ the space X built in Example 2.6.19 is normal, star-Menger, and is not either strongly star-Menger nor Dowker.*

Proof. It has been shown that X is normal and not strongly star-Lindelöf (in particular, X is not strongly star-Menger). It remains to show that it is star-Menger and is not a Dowker space.

X **is star-Menger**: let $(\mathcal{U}_n : n \in \omega)$ be any sequence of open covers of X. Write $L = \{x_\alpha : \alpha < \omega_1\}$ and for each $\alpha < \omega_1$ and each $n \in \omega$, let $f_\alpha(n) =$

$min\{i \in \omega : (\exists U \in \mathcal{U}_n)(\exists \beta < \omega_1)[x_\alpha \in U \wedge (\beta, \omega_1) \times \{i\} \subseteq U]\}$. Observe that for each $\alpha < \omega_1$, $f_\alpha : \omega \to \omega$ is well defined. Since $\{f_\alpha : \alpha < \omega_1\}$ has size less than $\mathfrak{d}$, there is a funtion $g \in \omega^\omega$ such that for all $\alpha < \omega_1 : g \not\leq^* f_\alpha$. For $n \in \omega$ let

$$M_n = \{i \in \omega : (\exists U \in \mathcal{U}_n)(\exists \beta < \omega_1)[(\beta, \omega_1) \times \{i\} \subseteq U]\}.$$

Now, for each $n \in \omega$ and each $i \in M_n$, fix $U_n^i \in \mathcal{U}_n$ and $\beta_n^i < \omega_1$ such that $(\beta_n^i, \omega_1) \times \{i\} \subseteq U_n^i$ and let $\mathcal{V}_n = \{U_n^i : i \in M_n \cap g(n)\}$.
Claim: $L \subseteq \bigcup\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$.
Indeed, fix $x_\alpha \in L$. There is $n \in \omega$ such that $f_\alpha(n) < g(n)$. Hence, there are $U \in \mathcal{U}_n$ and $\beta < \omega_1$ such that $x_\alpha \in U$ and $(\beta, \omega_1) \times \{f_\alpha(n)\} \subseteq U$. Thus, $f_\alpha(n) \in M_n$ and $U_n^{f_\alpha(n)} \in \mathcal{V}_n$. In addition, $U_n^{f_\alpha(n)} \cap U \neq \emptyset$. Hence, $x \in St(\bigcup \mathcal{V}_n, \mathcal{U}_n) \subseteq \bigcup\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$.

X **it is not a Dowker space**: Let us recall the following characterization: A normal space D is a Dowker space (see [70]) if, and only if, D has a countable increasing open cover $\{U_n : n \in \omega\}$ such that there is no closed cover $\{F_n : n \in \omega\}$ of D with $F_n \subseteq U_n$ for each $n \in \omega$. Hence, let $\{U_n : n \in \omega\}$ be any countable increasing open cover ($U_0 \subseteq U_1 \subseteq \cdots$) of X, we must find a countable cover of closed sets $\{F_n : n \in \omega\}$, such that for each $n \in \omega$, $F_n \subseteq U_n$.

For each $i \in \omega$ define $n_i = min\{n \in \omega : i \leq n \wedge (\exists \gamma < \omega_1)\Big[[\gamma, \omega_1) \times \{i\} \subseteq U_n\Big]\}$. Observe that since $\{U_n : n \in \omega\}$ is a countable cover of X, n_i is well defined for each $i \in \omega$. In addition, for each $n \in \omega$ and $i \in \omega$ with $i \leq n_i \leq n$

let

$$\gamma_i^n = min\{\gamma < \omega_1 : [\gamma, \omega_1) \times \{i\} \subseteq U_n\} \qquad (*)$$

Since for each $n \in \omega$, $U_n \subseteq U_{n+1}$, then γ_i^n is well defined. Now, for $n \in \omega$ let

$$F_n = \Big(\bigcup_{i \leq n}\{[\gamma_i^n, \omega_1) \times \{i\} : i \leq n_i \leq n\}\Big) \cup (U_n \cap L).$$

Claim:

(1) For each $n \in \omega$, F_n is closed,

(2) For each $n \in \omega$, $F_n \subseteq U_n$,

(3) $\bigcup_{n\in\omega} F_n = X$.

Indeed, to show (1), fix $n \in \omega$. First assume $x \in (X \setminus F_n) \cap (\omega_1 \times \omega)$. Hence $x = \langle \alpha, m\rangle$ for some $\alpha < \omega_1$ and $m \in \omega$. If $F_n \cap (\omega_1 \times \{m\}) = \emptyset$, any $U \subseteq \omega_1 \times \{m\}$ open neighbourhood of x is disjoint from F_n. If $F_n \cap (\omega_1 \times \{m\}) \neq \emptyset$, then $\alpha < \gamma_m^n$ and for each $\beta < \alpha$, $(\beta, \alpha] \times \{m\}$ is an open neighbourhood of x disjoint from F_n. Now, assume $x \in (X \setminus F_n) \cap L$, let $N \subseteq L$ such that $N \cap F_n = \emptyset$ and $x \in N$. Observe that $U = N \cup f(N) \setminus (n + 1) = \Big(N \cup f(N)\Big) \cap \Big(\bigcap_{j \leq n+1}(X_0 \setminus \{j\})\Big)$ is an open neighbourhood of x in X_0 (see condition (1) and (3) of Example 2.6.18). Hence, for any $\alpha < \omega_1$, $V_\alpha^U(x)$ ($= [U \cap L] \cup [(\alpha, \omega_1) \times (U \cap \omega)]$) is an open neighbourhood of x in X such that $V_\alpha^U(x) \cap F_n = \emptyset$ since $F_n \subseteq \omega_1 \times [0, n]$ and $V_\alpha^U(x) \cap (\omega_1 \times [0, n]) = \emptyset$. Thus, F_n is closed.

To show (2), fix $n \in \omega$. If $x \in F_n \cap L$, then $x \in U_n$. If $x = \langle \alpha, m\rangle \in F_n \cap (\omega_1 \times \omega)$, then there is some $i \leq n_i \leq n$ such that $\langle \alpha, m\rangle \in [\gamma_i^n, \omega_1) \times \{i\}$. Thus , $m = i$ and $[\gamma_i^n, \omega_1) \times \{i\} \subseteq U_n$. Hence $F_n \subseteq U_n$.

Let us show (3). If $x \in X \cap L$, then there is some $n \in \omega$ such that $x \in U_n$.

Hence, $x \in U_n \cap L \subseteq F_n$. If $x \in X \setminus L$, there is some $i \in \omega$ such that $x \in \omega_1 \times \{i\}$. By $(*)$ and the fact that $U_n \subseteq U_{n+1}$, $\{\gamma_i^n : n \in \omega\}$ is a decreasing sequence of ordinals. Since $U_n : n \in \omega)$ covers X, there is some $m \in \omega$ such that $\gamma_i^m = 0$. Thus, $x \in F_m$. $\square$

It is still unknown whether there is an example in **ZFC** of a normal star-Menger not strongly star-Menger space. It is also worth considering under which other hypothesis it is possible to build such examples.

2.7 A result on Games

Associated with the Menger property we have the **Menger game** played (on a space X) as follows: Two players, Alice and Bob, play an inning per positive integer. In the n-th inning Alice chooses an open cover $\mathcal{U}_n$, and Bob responds by choosing a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. The play $(\mathcal{U}_1, \mathcal{V}_1, \mathcal{U}_2, \mathcal{V}_2, \ldots, \mathcal{U}_n, \mathcal{V}_n, \ldots)$ is won by Bob if $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is an open cover of X; otherwise, Alice wins.

The following theorem is a well-known characterization of the Menger property in terms of games proved by Hurewicz in [43].

Theorem 2.7.1. *A topological space X has the Menger property if, and only if, Alice does not have a winning strategy in the Menger game.*

The proof of his theorem is also known for being technical and difficult. Szewczak and Tsaban provide more intuition about this result in [86].

Kočinac mentions in [54] the game that naturally relates to each selection property but Hurewicz-like characterizations for the star versions of the

Menger and Hurewicz properties are not known. Though, here we give one for the strongly star-Menger property on a strongly star-Lindelöf space which consist of the disjoint union of a closed discrete set with a σ-compact subspace (i. e., the space $X = Y \cup Z$ studied in Section 2.4). We recall the definition of the game related to this property.

Definition 2.7.2 ([54])**.** *Given a non-empty topological space X, define the SSM-game as follows:*

$$\begin{array}{l|c|c|c|l} \text{Alice} & \mathcal{U}_0 & \mathcal{U}_1 & \mathcal{U}_2 & \\ \hline \text{Bob} & F_0 & F_1 & F_2 & \end{array} \cdots$$

Diagram 2.8: The Strongly Star-Menger Game.

In the n-th inning, Alice gives an open cover $\mathcal{U}_n$ of X. Bob responds by choosing a finite subset F_n of X. The play $(\mathcal{U}_0, F_0, \mathcal{U}_1, F_1, \ldots, \mathcal{U}_n, F_n, \ldots)$ is won by Bob if $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X; otherwise, Alice wins.

Observe that it is always the case that for any space X if Alice does not have a winning strategy in the SSM-game on X, then X is strongly star-Menger.

Theorem 2.7.3. *Let X be a strongly star-Lindelöf space of the form $Y \cup Z$, where $Y \cap Z = \emptyset$, Z is a σ-compact subspace and Y is a closed discrete set. If X is strongly star-Menger then Alice does not have a winning strategy in the SSM-game on X.*

Proof. Let σ be a strategy for Alice in the SSM-game on X. Let $K := \{K_n : n \in \omega\}$ be a $\subseteq$-increasing sequence of compact subsets of Z such that $\bigcup_{n\in\omega} K_n = Z$. Let θ be a large enough cardinal such that $H(\theta)$ contains everything relevant to the proof and take M a countable elementary submodel of $H(\theta)$ such that $K, X, \sigma \in M$. Observe that if $s \in dom(\sigma)$, then s is a finite sequence $\langle F_0, F_1, \dots, F_{n-1}\rangle$, for some $n \in \omega$, where for each $i < n$, F_i is the play of Bob at inning i. We let $\mathcal{U}_s := \sigma(s)$ be the response by Alice at inning $n+1$ following σ.

Claim: if $B := X \cap M$, then for all $s \in dom(\sigma) \cap M : St(B, \mathcal{U}_s) = X$.

Notice that if $s \in dom(\sigma) \cap M$, then $\sigma(s) \in M$. Since X is strongly star-Lindelöf, $M \vDash \exists D_s \subseteq X(|D_s| = \omega \wedge St(D_s, \mathcal{U}_s) = X)$. Given that $D_s \in M$ and $|D_s| = \omega$, we get $D_s \subseteq M$. Thus, $D_s \subseteq B$ and therefore, $St(B, \mathcal{U}_s) = X$. If we list B as $\{b_i : i < \omega\}$ and let $B_n = \{b_i \in B : i < n\}$ in M, by an elementarity argument, for each $n \in \omega$ and each $s \in dom(\sigma) \cap M$, there exists $m \in \omega$ such that $K_n \subseteq St(B_m, \mathcal{U}_s)$. For each $x \in Y$, we recursively define $f_x \in \omega^\omega$ as follows: $f_x(0) = min\{m : x \in St(B_m, \mathcal{U}_\emptyset)\}$, and having defined $f_x \restriction_n$, consider $s \in \omega^n$ such that $s < f_x \restriction_n$, let $\mathcal{U}_s := \sigma(\langle B_{s(0)}, B_{s(1)}, \dots, B_{s(n-1)}\rangle)$ the play by Alice at inning n. Since $\mathcal{U}_s \in M$, the following is well defined

$$f_x(n) = min\{m : \forall s \in \omega^n[s < f_x \restriction_n, \to x \in St(B_m, \mathcal{U}_s)]\}.$$

Similarly, for $t \in \omega$ we can recursively define $h_t \in \omega^\omega$ as follows $h_t(0) = min\{m : K_t \subseteq St(B_m, \mathcal{U}_\emptyset)\}$ and $h_t(n) = min\{m : \forall s \in \omega^n[s < h_t \restriction_n, \to K_t \subseteq St(B_m, \mathcal{U}_s)]\}$.

Since X is strongly star-Menger, by Theorem 2.4.5, $|\{f_x : x \in Y\} \cup \{h_t : t \in \omega\}| < \mathfrak{d}$. Thus, there exists $g \in \omega^\omega$ such that for every $x \in Y$, $g \not\leq^* f_x$ and for every $t \in \omega$ $g \not\leq^* h_t$. Let us define a way for Bob to defeat Alice using σ. For each $n \in \omega$, let Bob play $B_{g(n)}$. We show $X = \bigcup_{n\in\omega} St(B_{g(n)}, \mathcal{U}_{g\restriction_n})$.

Let $x \in Y$ and fix $t \in \omega$ such that $f_x(t) < g(t)$. If there exists $j < t$ such that $x \in St(B_{g(j)}, \mathcal{U}_{g\restriction_j})$, then x was already covered before inning t. Assume that for all $j < t$, $x \notin St(B_{g(j)}, \mathcal{U}_{g\restriction_j})$. By the definition of $f_x(0)$, we get that $g(0) < f_x(0)$, and, in general, for each $j < t$, we get that $g(j) < f_x(j)$. Thus, $g\restriction_j < f_x\restriction_j$. Hence, by the definition of $f_x(t)$,

$$x \in St(B_{f_x(t)}, \mathcal{U}_{g\restriction_t}) \subseteq St(B_{g(t)}, \mathcal{U}_{g\restriction_t}).$$

A similar argument show that for each $n \in \omega$, K_n is eventually covered. Hence, σ is not a winning strategy for Alice.

$\square$

As an immediate consequence, we have

Corollary 2.7.4. *If X is a Ψ-space or the Niemytzki plane, then X is strongly star-Menger if and only if Alice does not have a winning strategy in the SSM-game on X.*

A Full Hurewicz type characterization of strongly star-Menger spaces in terms of the SSM-game, is still unknown, i.e. Is it true that a space X is strongly star-Menger if and only if Alice does not have a winning strategy in the SSM-game on X? Characterizations for the other star selection principles in terms of games haven't been studied.

2.8 Small unions of star spaces

One the first things we notice when dealing with Menger spaces is the fact that a countable union of them is also Menger. Indeed, assume X_0, X_1, $\ldots$,

X_n, ... are Menger and let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of $X = \bigcup_{i\in\omega} X_i$. We can first partition $\{\mathcal{U}_n : n \in \omega\}$ into countably many pairwise disjoint infinite subsequences $\{\mathcal{U}_n : n \in \omega\} = \{\mathcal{U}_n^m : n, m \in \omega\}$. Then, since each X_n is Menger, for each $n, m \in \omega$ find $\mathcal{V}_n^m \in [\mathcal{U}_n^m]^{<\omega}$ such that $\bigcup_{m\in\omega} \mathcal{V}_n^m \in \mathcal{O}(X_n)$. Then $\bigcup_{n,m\in\omega} \mathcal{V}_n^m \in \mathcal{O}(X)$. That is, X is Menger. In [89] Tall improved the previous observation showing that if a space X is Lindelöf and it can be written as a union of less than $\mathfrak{d}$ (less than $\mathfrak{b}$) compact spaces is Menger (is Hurewicz). The idea is as follows: assume X is Lindelöf and it can be written as a union of κ many compact spaces with $\kappa < \mathfrak{d}$. Let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers of X. Enumerate as K_α with $\alpha < \kappa < \mathfrak{d}$ each compact space. Since X is Lindelöf, for each n, $\mathcal{U}_n = \{U_n^m : m \in \omega\}$ can be assumed to be countable. Now, for each $\alpha < \kappa$, define $f_\alpha(n) = min\{m \in \omega : K_\alpha \subseteq \bigcup_{i\leq m} U_n^i\}$. For each $\alpha < \kappa$, $f_\alpha \in \omega^\omega$ is well defined since K_α is compact. Given that $\kappa < \mathfrak{d}$, fix $g \in \omega^\omega$ such that for all $\alpha < \kappa : g \not\leq^* f_\alpha$. For each n, define $\mathcal{V}_n = \{U_n^m : m \leq g(n)\}$. Thus, for each $n : \mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{O}(X)$. That is, X is Menger.

Motivated by this result we wondered if this could be improved. It turns out that we can replace “compact” by “star-Hurewicz” in Tall’s results (see Proposition 2.8.3 below), or we can replace “$\mathfrak{d}$” and “compact” by “$\mathfrak{b}$” and “star-Menger” (Proposition 2.8.7). In addition, a Lindelöf space that can be written as a union of less than $\mathfrak{b}$ star-Hurewicz spaces, is Hurewicz (Proposition 2.8.4). These results are contained in Theorem 2.8.2 below.

Furthermore, we investigated what happens if instead of starting with a

Lindelöf space that can be written as some small union, we consider a star-Lindelöf space or a strongly star-Lindelöf space. Some other interesting relationships were obtained and they are described in Theorem 2.8.8 below. Let us first introduce some notation that allows to present these results in an organized manner.

Definition 2.8.1. *Let X be any space, A and B denote some properties and κ is some cardinal.*

$$\Big(A, (< \kappa, B)\Big)$$

stands for "X satisfies property A and it can be written as a union of less than κ spaces each of them satisfying property B".

For instance, if L, C and M denote Lindelöf, compact and Menger, respectively, then Tall's results can be written as "$\Big(L, (< \mathfrak{d}, C)\Big) \rightarrow M$" and "$\Big(L, (< \mathfrak{b}, C)\Big) \rightarrow H$". More in general, we have:

Theorem 2.8.2. *For any space X, the following diagram holds:*

$$\begin{tikzcd}
 & & \Big(L, (< \mathfrak{d}, SH)\Big) \arrow[drr, bend left, "\mathbf{A}"] & & \\
 & \Big(L, (< \mathfrak{b}, SH)\Big) \arrow[ur] \arrow[dl] \arrow[rr, "\mathbf{B}"] & & H \arrow[r] & M \\
\Big(L, (< \mathfrak{b}, SM)\Big) \arrow[urrrr, bend right, "\mathbf{C}"'] & & & &
\end{tikzcd}$$

Diagram 2.9: Smalls Unions: The Lindelöf Diagram.

Observe that unlabelled arrows are immediate. The proof is divided as Propositions 2.8.3, 2.8.4 and 2.8.7.

Proposition 2.8.3 (**A**). *If X is a Lindelöf space and X is the union of less than $\mathfrak{d}$ star-Hurewicz spaces, then X is Menger.*

Proof. Let κ be a cardinal smaller than $\mathfrak{d}$ and put $X = \bigcup_{\alpha<\kappa} Y_\alpha$ with each Y_α being a star-Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is Lindelöf, we can assume that for each $n \in \omega$, $\mathcal{U}_n$ is countable and put $\mathcal{U}_n = \{U_n^i : i \in \omega\}$. Since each Y_α is star-Hurewicz, for each $\alpha < \kappa$, there exists a finite subset $\mathcal{V}_n^\alpha$ of $\mathcal{U}_n$ such that $\{St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of Y_α. Define, for each $\alpha < \kappa$, a function f_α as follows: for each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : \mathcal{V}_n^\alpha \subseteq \{U_n^j : j \leq i\}\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{d}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $g \not\leq^* f_\alpha$. For each $n \in \omega$, let $\mathcal{W}_n = \{U_n^i : i \leq g(n)\}$. We show $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Let $x \in X$. Then, there exists $\alpha < \kappa$ such that $x \in Y_\alpha$. Hence, there is $n_0 \in \omega$ so that for every $n \geq n_0$, $x \in St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n)$. Since $g \not\leq^* f_\alpha$, we can take $n > n_0$ such that $g(n) > f_\alpha(n)$. Then $x \in St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n) \subseteq St(\bigcup_{j\leq f_\alpha(k)} U_n^j, \mathcal{U}_n) \subseteq St(\bigcup_{j\leq g(n)} U_n^j, \mathcal{U}_n) = St(\bigcup \mathcal{W}_n, \mathcal{U}_n)$. Therefore, the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is star-Menger. Since X is Lindelöf, by Theorem 2.3.8 X is Menger. □

Proposition 2.8.4 (**B**). *If X is a Lindelöf space and X is the union of less than $\mathfrak{b}$ star-Hurewicz spaces, then X is Hurewicz.*

Proof. Let κ be a cardinal smaller than $\mathfrak{b}$ and put $X = \bigcup_{\alpha<\kappa} Y_\alpha$ with each Y_α being a star-Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is Lindelöf, we can assume that for each $n \in \omega$, $\mathcal{U}_n$ is countable and put $\mathcal{U}_n = \{U_n^i : i \in \omega\}$. For each $\alpha < \kappa$, there exists a finite subset $\mathcal{V}_n^\alpha$ of $\mathcal{U}_n$ such that $\{St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of Y_α. Define, for each $\alpha < \kappa$, a function f_α as follows: for each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : \mathcal{V}_n^\alpha \subseteq \{U_n^j : j \leq i\}\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $\mathcal{W}_n = \{U_n^i : i \leq g(n)\}$. Let us show $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. Let $x \in X$. Then, there exists $\alpha < \kappa$ such that $x \in Y_\alpha$. Hence, there is $n_0 \in \omega$ so that for every $n \geq n_0$, $x \in St(\bigcup \mathcal{V}_n^\alpha, \mathcal{U}_n)$. Since $f_\alpha \leq^* g$, there is $n_1 \in \omega$ such that for every $n \geq n_1$, $f_\alpha(n) \leq g(n)$. Put $m = max\{n_0, n_1\}$. Hence, for each $k \geq m$, $x \in St(\bigcup \mathcal{W}_k, \mathcal{U}_k)$. Indeed, let $k \geq m$. Then $x \in St(\bigcup \mathcal{V}_k^\alpha, \mathcal{U}_k) \subseteq St(\bigcup_{j \leq f_\alpha(k)} U_k^j, \mathcal{U}_k) \subseteq St(\bigcup_{j \leq g(k)} U_k^j, \mathcal{U}_k) = St(\bigcup \mathcal{W}_k, \mathcal{U}_k)$. Therefore, the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. Thus, X is star-Hurewicz. Since X is Lindelöf, by Theorem 2.3.8 X is Hurewicz. □

Let us introduce the following class of covers:

Definition 2.8.5. *A cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of a space X it's called* **large** *if for every $\alpha < \kappa$, $\{U_\beta : \alpha \leq \beta < \kappa\}$ is a cover of X. We denote the class of large covers of X by $\mathcal{L}(X)$.*

The following lemma will be useful in the proof of Propositions 2.8.7, 2.8.11 and 2.8.14.

Lemma 2.8.6 (Folklore)**.** *For any space X:*

- $S_{fin}(\mathcal{O}, \mathcal{O}) \leftrightarrow S_{fin}(\mathcal{O}, \mathcal{L})$.
- $S^*_{fin}(\mathcal{O}, \mathcal{O}) \leftrightarrow S^*_{fin}(\mathcal{O}, \mathcal{L})$.

Proof. Let X be any space. Observe that $S_{fin}(\mathcal{O},\mathcal{L}) \to S_{fin}(\mathcal{O},\mathcal{O})$ and $S^*_{fin}(\mathcal{O},\mathcal{L}) \to S^*_{fin}(\mathcal{O},\mathcal{O})$ are immediate. Now, assume $S_{fin}(\mathcal{O},\mathcal{O})$ ($S^*_{fin}(\mathcal{O},\mathcal{O})$ respectively) holds. Let $\{\mathcal{U}_n : n \in \omega\}$ be any sequence of open covers of X and let $m \in \omega$. Since the collection $\{\mathcal{U}_n : m \leq n < \omega\}$ is a sequence of open covers of X, then for each $n \geq m$ there exists a finite subset $\mathcal{V}^m_n$ of $\mathcal{U}_n$ such that $\bigcup\{\mathcal{V}^m_n : m \leq n < \omega\}$ ($\{St(\bigcup \mathcal{V}^m_n, \mathcal{U}_n) : m \leq n < \omega\}$, resp.) is an open cover of X. So, for each $n \in \omega$ we define $\mathcal{W}_n = \bigcup_{m \leq n} \mathcal{V}^m_n$. Hence, for each $m \in \omega$ the collection $\bigcup\{\mathcal{W}_n : m \leq n < \omega\}$ ($\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : m \leq n < \omega\}$, resp.) is an open cover of X. That is, for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ and the collection $\bigcup\{\mathcal{W}_n : n < \omega\}$ ($\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n < \omega\}$, resp.) is a large cover of X. Hence, $S_{fin}(\mathcal{O},\mathcal{L})$ ($S^*_{fin}(\mathcal{O},\mathcal{L})$, resp.) holds. $\square$

Proposition 2.8.7 (C). *If X is a Lindelöf space and X is the union of less than $\mathfrak{b}$ star-Menger spaces, then X is Menger.*

Proof. Let κ be a cardinal smaller than $\mathfrak{b}$ and put $X = \bigcup_{\alpha<\kappa} Y_\alpha$ with each Y_α being a star-Menger space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is Lindelöf, we can assume that for each $n \in \omega$, $\mathcal{U}_n$ is countable and put $\mathcal{U}_n = \{U^i_n : i \in \omega\}$. Since or each $\alpha < \kappa$, Y_α is star-Menger, by Lemma 2.8.6, for each $\alpha < \kappa$ and each $n \in \omega$ there exists $l^\alpha_n \in \omega$ such that $\{St(\bigcup_{i \leq l^\alpha_n} U^i_n, \mathcal{U}_n) : n \in \omega\}$ is a large cover of M_α.
For each $\alpha < \kappa$, and each $n \in \omega$, let $f_\alpha(n) = l^\alpha_n$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $\mathcal{W}_n = \{U^j_n : j \leq g(n)\}$.
Claim: $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.
Let $x \in X$ and fix $\alpha < \kappa$ such that $x \in Y_\alpha$. Then, for the function f_α there is $n_0 \in \omega$ so that for every $n \geq n_0$, $f_\alpha(n) \leq g(n)$. Let $m \geq n_0$ such

that $x \in St(\bigcup_{i \leq l_m^\alpha} U_m^i, \mathcal{U}_m) \subseteq St(\bigcup_{i \leq g(m)} U_m^i, \mathcal{U}_m) = St(\bigcup \mathcal{W}_m, \mathcal{U}_m)$. Therefore, the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is star-Menger. Since X is Lindelöf, by Theorem 2.3.8 X is Menger. □

The following theorem is the analogous of Theorem 2.8.2 for the star and strongly star version of the Lindelöf property. Observe that instead of requiring that the "pieces" of the space are star-Hurewicz as in Propositions 2.8.3 and 2.8.4 (or star-Menger as in Proposition 2.8.7), we need them to be Hurewicz (Menger, respectively).

Theorem 2.8.8. *For any space X, the following diagram holds:*

$$\begin{tikzcd}
 & & \big(SSL,(<\mathfrak{d},H)\big) \arrow[drr, bend left, "\mathbf{D}"] \arrow[ddd] & & \\
 & \big(SSL,(<\mathfrak{b},H)\big) \arrow[ur] \arrow[rr, "\mathbf{E}"] \arrow[dl] \arrow[ddd] & & SSH \arrow[r] \arrow[ddd] & SSM \arrow[ddd] \\
\big(SSL,(<\mathfrak{b},M)\big) \arrow[urrrr, bend right, "\mathbf{F}"] \arrow[ddd] & & & & \\
 & & \big(SL,(<\mathfrak{d},H)\big) \arrow[drr, bend left, "\mathbf{G}"] & & \\
 & \big(SL,(<\mathfrak{b},H)\big) \arrow[ur] \arrow[rr, "\mathbf{I}"] \arrow[dl] & & SH \arrow[r] & SM \\
\big(SL,(<\mathfrak{b},M)\big) \arrow[urrrr, bend right, "\mathbf{J}"] & & & &
\end{tikzcd}$$

Diagram 2.10: Smalls Unions: The Star Version of the Lindelöf Diagram.

Unlabelled arrows are immediate. The proof is divided as Propositions 2.8.9, 2.8.10, 2.8.11, 2.8.12, 2.8.13 and 2.8.14.

Proposition 2.8.9 (**D**). *If X is a strongly star-Lindelöf space and X is the union of less than $\mathfrak{d}$ Hurewicz spaces, then X is strongly star-Menger.*

Proof. Let κ be any cardinal smaller than $\mathfrak{d}$ and put $X = \bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is strongly star-Lindelöf, for each $n \in \omega$ there exists $C_n \in [X]^{\leq\omega}$ such that $St(C_n, \mathcal{U}_n) = X$. For each $n \in \omega$, put $C_n = \{x_n^i : i \in \omega\}$. Note that $St(C_n, \mathcal{U}_n) = \bigcup_{i\in\omega} St(x_n^i, \mathcal{U}_n)$ for each $n \in \omega$. So, for each $n \in \omega$, the collection $\{St(x_n^i, \mathcal{U}_n) : i \in \omega\}$ is an open cover of X. Hence, for each $\alpha < \kappa$, there exists a finite subset $\mathcal{F}_n^\alpha$ of $\{St(x_n^i, \mathcal{U}_n) : i \in \omega\}$ such that $\{\bigcup \mathcal{F}_n^\alpha : n \in \omega\}$ is a γ-cover of H_α. Define, for each $\alpha < \kappa$, a function f_α as follows: for each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : \mathcal{F}_n^\alpha \subseteq \{St(x_n^j, \mathcal{U}_n) : j \leq i\}\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{d}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $g \not\leq^* f_\alpha$. For each $n \in \omega$, let $F_n = \{x_n^i : i \leq g(n)\}$.
Claim: $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.
Indeed, let $x \in X$ and fix $\alpha < \kappa$ such that $x \in H_\alpha$. Hence, there is $n_0 \in \omega$ so that for every $n \geq n_0$, $x \in \bigcup \mathcal{F}_n^\alpha$. Since $g \not\leq^* f_\alpha$, we can take $m \geq n_0$ such that $g(m) > f_\alpha(m)$. Then, $x \in \bigcup \mathcal{F}_m^\alpha \subseteq \bigcup_{j\leq f_\alpha(m)} St(x_m^j, \mathcal{U}_m) \subseteq \bigcup_{j\leq g(m)} St(x_m^j, \mathcal{U}_m) = St(F_m, \mathcal{U}_m)$. Therefore, the collection $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is strongly star-Menger. $\square$

Basically the same idea (now the fact that we have less than $\mathfrak{b}$ many pieces, let us get a γ-cover at the end) yields:

Proposition 2.8.10 (E). *If X is a strongly star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Hurewicz spaces, then X is strongly star-Hurewicz.*

Proof. Let κ be any cardinal smaller than $\mathfrak{b}$ and put $X = \bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is strongly star-Lindelöf, for each $n \in \omega$ there exists $C_n \in [X]^{\leq\omega}$ such that $St(C_n, \mathcal{U}_n) = X$. For each $n \in \omega$, put $C_n = \{x_n^i : i \in \omega\}$. Note that $St(C_n, \mathcal{U}_n) = \bigcup_{i\in\omega} St(x_n^i, \mathcal{U}_n)$ for each $n \in \omega$. So, for each $n \in \omega$, the collection $\{St(x_n^i, \mathcal{U}_n) : i \in \omega\}$ is an open cover of X. Hence, for each $\alpha < \kappa$, there exists a finite subset $\mathcal{V}_n^\alpha$ of $\{St(x_n^i, \mathcal{U}_n) : i \in \omega\}$ such that $\{\bigcup \mathcal{V}_n^\alpha : n \in \omega\}$ is a γ-cover of H_α. Define, for each $\alpha < \kappa$, a function f_α as follows: for each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : \mathcal{V}_n^\alpha \subseteq \{St(x_n^j, \mathcal{U}_n) : j \leq i\}\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $F_n = \{x_n^i : i \leq g(n)\}$.
Claim: $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X.
Indeed, let $x \in X$. Then, there exists $\alpha < \kappa$ such that $x \in H_\alpha$. Hence, there is $n_0 \in \omega$ so that for every $n \geq n_0$, $x \in \bigcup \mathcal{V}_n^\alpha$. Since $f_\alpha \leq^* g$, there is $n_1 \in \omega$ such that for every $n \geq n_1$, $f_\alpha(n) \leq g(n)$. Put $m = max\{n_0, n_1\}$. Let us show that for each $k \geq m$, $x \in St(F_k, \mathcal{U}_k)$. Fix $k \geq m$. Then $x \in \bigcup \mathcal{V}_k^\alpha \subseteq \bigcup_{j\leq f_\alpha(k)} St(x_k^j, \mathcal{U}_k) \subseteq \bigcup_{j\leq g(k)} St(x_k^j, \mathcal{U}_k) = St(F_k, \mathcal{U}_k)$. Therefore, the collection $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. Thus, X is strongly star-Hurewicz. □

In the next proposition, even though we have less than $\mathfrak{b}$ many pieces as well, the fact that each piece is Menger and not (necessarily) Hurewicz, allows to conclude only, that X is strongly star-Menger.

Proposition 2.8.11 (**F**). *If X is a strongly star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Menger spaces, then X is strongly star-Menger.*

Proof. Let κ be any cardinal smaller than $\mathfrak{b}$ and put $X = \bigcup_{\alpha<\kappa} M_\alpha$ with each M_α being a Menger space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is strongly star-Lindelöf, for each $n \in \omega$ there exists $C_n \in [X]^{\leq\omega}$ such that $St(C_n, \mathcal{U}_n) = X$. For each $n \in \omega$, put $C_n = \{x_n^i : i \in \omega\}$. Observe that for each $n \in \omega$, $St(C_n, \mathcal{U}_n) = \bigcup_{i\in\omega} St(x_n^i, \mathcal{U}_n)$. So, for each $n \in \omega$, the collection $\{St(x_n^i, \mathcal{U}_n) : i \in \omega\}$ is an open cover of X. Since or each $\alpha < \kappa$, M_α is Menger, by Lemma 2.8.6, for each $\alpha < \kappa$ and each $n \in \omega$ there exists $l_n^\alpha \in \omega$ such that $\{\bigcup_{i\leq l_n^\alpha} St(x_n^i, \mathcal{U}_n) : n \in \omega\}$ is a large cover of M_α. For each $\alpha < \kappa$, and each $n \in \omega$, let $f_\alpha(n) = l_n^\alpha$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $F_n = \{x_n^i : i \leq g(n)\}$.

Claim: $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

Indeed, let $x \in X$ and fix $\alpha < \kappa$ such that $x \in M_\alpha$. Then, for the function f_α there is $n_0 \in \omega$ so that for every $n \geq n_0$, $f_\alpha(n) \leq g(n)$. Let $m \geq n_0$ such that $x \in \bigcup_{i\leq l_m^\alpha} St(x_m^i, \mathcal{U}_m) \subseteq \bigcup_{i\leq g(m)} St(x_m^i, \mathcal{U}_m) = St(F_m, \mathcal{U}_m)$. Therefore, the collection $\{St(F_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is strongly star-Menger. $\square$

The remaining three propositions are the star-Lindelöf version of the previous ones. The ideas of their proofs are similar but we write them down for completeness.

Proposition 2.8.12 (**G**). *If X is a star-Lindelöf space and X is the union of less than $\mathfrak{d}$ Hurewicz spaces, then X is star-Menger.*

Proof. Let κ be any cardinal smaller than $\mathfrak{d}$ and put $X = \bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is star-Lindelöf, for each $n \in \omega$ there exists $\mathcal{V}_n \in [\mathcal{U}_n]^{\leq\omega}$ such that $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = X$. For each $n \in \omega$, put $\mathcal{V}_n = \{V_n^i : i \in \omega\}$. Note that $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = \bigcup_{i\in\omega} St(V_n^i, \mathcal{U}_n)$ for each $n \in \omega$. So, for each $n \in \omega$, the collection $\{St(V_n^i, \mathcal{U}_n) : i \in \omega\}$ is an open cover of X. Hence, for each $\alpha < \kappa$, there exists a finite subset $\mathcal{W}_n^\alpha$ of $\{St(V_n^i, \mathcal{U}_n) : i \in \omega\}$ such that $\{\bigcup \mathcal{W}_n^\alpha : n \in \omega\}$ is a γ-cover of H_α. Define, for each $\alpha < \kappa$, a function f_α as follows: for each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : \mathcal{W}_n^\alpha \subseteq \{St(V_n^j, \mathcal{U}_n) : j \leq i\}\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{d}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $g \not\leq^* f_\alpha$. For each $n \in \omega$, let $\mathcal{W}_n = \{V_n^j : j \leq g(n)\}$. It follows that the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X: let $x \in X$ and fix $\alpha < \kappa$ such that $x \in H_\alpha$. Hence, there is $n_0 \in \omega$ so that for every $n \geq n_0$, $x \in \bigcup \mathcal{W}_n^\alpha$. Further, since $g \not\leq^* f_\alpha$, there is $m \geq n_0$ such that $g(m) > f_\alpha(m)$. Hence, $x \in \bigcup \mathcal{W}_m^\alpha \subseteq \bigcup_{j\leq f_\alpha(m)} St(V_m^j, \mathcal{U}_m) \subseteq \bigcup_{j\leq g(m)} St(V_m^j, \mathcal{U}_m) = St(\bigcup \mathcal{W}_m, \mathcal{U}_m)$. Therefore, the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is star-Menger. $\square$

Proposition 2.8.13 (I). *If X is a star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Hurewicz spaces, then X is star-Hurewicz.*

Proof. Let κ be any cardinal smaller than $\mathfrak{b}$ and put $X = \bigcup_{\alpha<\kappa} H_\alpha$ with each H_α being a Hurewicz space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is star-Lindelöf, for each $n \in \omega$ there exists $\mathcal{V}_n \in [\mathcal{U}_n]^{\leq\omega}$ such that $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = X$. For each $n \in \omega$, put $\mathcal{V}_n = \{V_n^i : i \in \omega\}$. Note that $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = \bigcup_{i\in\omega} St(V_n^i, \mathcal{U}_n)$ for each $n \in \omega$. So, for each $n \in \omega$, the collection $\{St(V_n^i, \mathcal{U}_n) : i \in \omega\}$ is an open cover of X. Hence, for each $\alpha < \kappa$, there exists a finite subset $\mathcal{W}_n^\alpha$ of $\{St(V_n^i, \mathcal{U}_n) : i \in \omega\}$ such that $\{\bigcup \mathcal{W}_n^\alpha : n \in \omega\}$ is a γ-cover of H_α. Define, for each $\alpha < \kappa$, a function f_α as follows: for each $n \in \omega$, let $f_\alpha(n) = min\{i \in \omega : \mathcal{W}_n^\alpha \subseteq \{St(V_n^j, \mathcal{U}_n) : j \leq i\}\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $\mathcal{W}_n = \{V_n^j : j \leq g(n)\}$. It follows that $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X: let $x \in X$ and fix $\alpha < \kappa$ such that $x \in H_\alpha$. Hence, there is $n_0 \in \omega$ so that for every $n \geq n_0$, $x \in \bigcup \mathcal{W}_n^\alpha$. Further, since $f_\alpha \leq^* g$, there is $n_1 \in \omega$ such that for every $n \geq n_1$, $f_\alpha(n) \leq g(n)$. Put $m = max\{n_0, n_1\}$. Hence, for each $k \geq m$, $x \in St(\bigcup \mathcal{W}_k, \mathcal{U}_k)$. Indeed, let $k \geq m$. Since $k \geq n_0$ and $k \geq n_1$, then $x \in \bigcup \mathcal{W}_k^\alpha \subseteq \bigcup_{j \leq f_\alpha(k)} St(V_k^j, \mathcal{U}_k) \subseteq \bigcup_{j \leq g(k)} St(V_k^j, \mathcal{U}_k) = St(\bigcup \mathcal{W}_k, \mathcal{U}_k)$. Therefore, the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X. Thus, X is star-Hurewicz. □

Proposition 2.8.14 (J). *If X is a star-Lindelöf space and X is the union of less than $\mathfrak{b}$ Menger spaces, then X is star-Menger.*

Proof. Let κ be any cardinal smaller than $\mathfrak{b}$ and put $X = \bigcup_{\alpha < \kappa} M_\alpha$ with each M_α being a Menger space. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X. Since X is star-Lindelöf, for each $n \in \omega$ there exists $\mathcal{V}_n \in [\mathcal{U}_n]^{\leq \omega}$ such that $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = X$. For each $n \in \omega$, put $\mathcal{V}_n = \{V_n^i : i \in \omega\}$. Note that $St(\bigcup \mathcal{V}_n, \mathcal{U}_n) = \bigcup_{i \in \omega} St(V_n^i, \mathcal{U}_n)$ for each $n \in \omega$. So, for each $n \in \omega$, the collection $\{St(V_n^i, \mathcal{U}_n) : i \in \omega\}$ is an open cover of X. Since or each $\alpha < \kappa$, M_α is Menger, by Lemma 2.8.6, for each $\alpha < \kappa$ and each $n \in \omega$ there exists $l_n^\alpha \in \omega$ such that $\{\bigcup_{i \leq l_n^\alpha} St(V_n^i, \mathcal{U}_n) : n \in \omega\}$ is a large cover of M_α. For each $\alpha < \kappa$, and each $n \in \omega$, let $f_\alpha(n) = l_n^\alpha$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $f_\alpha \leq^* g$. For each $n \in \omega$, let $\mathcal{W}_n = \{V_n^j : j \leq g(n)\}$.

Claim:$\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

Let $x \in X$ and fix $\alpha < \kappa$ such that $x \in M_\alpha$. Then, for the function f_α there is $n_0 \in \omega$ so that for every $n \geq n_0$, $f_\alpha(n) \leq g(n)$. Let $m \geq n_0$ such that $x \in \bigcup_{i \leq l_m^\alpha} St(V_m^i, \mathcal{U}_m) \subseteq \bigcup_{i \leq g(m)} St(V_m^i, \mathcal{U}_m) = St(\bigcup \mathcal{W}_m, \mathcal{U}_m)$. Therefore, the collection $\{St(\bigcup \mathcal{W}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Thus, X is star-Menger. $\square$

Chapter 3

Weak Normality Properties in Ψ-spaces

Weakenings of normality have been considered in the literaure since the late 60's and early 70's. For instance, quasi-normal [94], almost-normal [4], mildly-normal [75], [78], and more recently π-normal [51] and partly-normal [52]. In [53] L. Kalantan and P. Szeptycki prove that any product of ordinals is mildly-normal. Kalantan builds a Ψ-space which is not mildly-normal in [50] and, in [52], using **CH** constructs a mad family so that the associated Ψ-space is quasi-normal. As we will see in Section 3.2, Example 3.2.3 constitutes an improvement of this result.

We will present some examples of Ψ-spaces that satisfy particular weakenings of normality. The main constructions are Example 3.2.3 which is a quasi-normal not almost-normal Ψ-space; Example 3.2.5 which is a mildly-normal not partly-normal Ψ-space and, Example 3.2.10, a consistent example (assuming **CH**) of a Luzin mad family such that its associated Ψ-space is quasi-normal. The tool that allows us to build Example 3.2.3 and Example 3.2.5 is an almost disjoint family of true cardinality $\mathfrak{c}$ (Definition 3.1.3 below). In Section 3.1 we introduce these weak normality properties, almost disjoint families of true cardinality $\mathfrak{c}$ and provide some motivation and basic facts. In addition, we prove that a space is π-normal if and only if it is almost-normal. Section 3.2 contains the construction of the spaces. Finally in Section 3.3 we define *strongly* $\aleph_0$*-separated* almost disjoint families (Definition 3.3.1), and present a couple of results.

3.1 Weakenings of Normality

Recall that a subset A of a space X is called **regularly closed** (also called closed domain), if $A = \overline{int(A)}$ ($cl_X(A)$ or simply $cl(A)$ will denote the closure of A in the space X as well). A set A will be called π**-closed**, if A is a finite intersection of regularly closed sets. Two subsets A and B of a space X are said to be **separated** if there exist two disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Definition 3.1.1. *A regular space X is called:*

1. π**-normal** [51] *if any two disjoint sets A and B, where A is closed and B is π-closed, are separated.*
2. **almost-normal** [4] *if any two disjoint sets A and B, where A is closed and B is regularly closed, are separated.*
3. **quasi-normal** [94] *if any two disjoint π-closed sets A and B are sep-*

arated.

4. **partly-normal** [52] *if any two disjoint sets A and B, where A is regular closed and B is π-closed, are separated.*
5. **mildly-normal** *(also called κ-normal),* [75] [78] *if any two disjoint regular closed sets A and B are separated.*

Since "regular closed $\to$ π-closed $\to$ closed" holds, it follows that normal spaces are π-normal and the following diagram holds:

$$\begin{tikzcd}
 & \textit{quasi-normal} \arrow[dr] & & \\
\pi\textit{-normal} \arrow[ur] \arrow[dr] & & \textit{partly-normal} \arrow[r] & \textit{mildly-normal.} \\
 & \textit{almost-normal} \arrow[ur] & &
\end{tikzcd}$$

Diagram 3.1: Weakenings of Normality.

In the study of a possible construction of an almost-normal not π-normal Ψ-space, we were able to show that in scattered spaces (a space is *scattered* if every nonempty subset has an isolated point), of finite height, the properties π-normal and almost-normal are equivalent. Observe that Ψ-spaces are scattered of height 2. Afterwards, we dropped those hypotheses:

Proposition 3.1.2. *Almost-normal spaces are π-normal.*

Proof. Assume X is an almost-normal space. For a positive integer n, call a set n-π-closed, if it is the intersection of n many regular closed sets. We will show by induction on n, that in X every n-π-closed set can be separated from a closed set, provided they are disjoint. This is enough to show that X is π-normal.

Base case: $n = 1$. Since X is almost normal, every closed H and 1-π-closed set K in X such that $H \cap K = \emptyset$ can be separated (K is a regular closed set).
Inductive step: Assume that for all $1 \leq i \leq n$ if H is closed, K is i-π-closed in X and, $H \cap K = \emptyset$, then H and K can be separated. Let $H \subset X$ be a closed set and let K be an $(n+1)$-π-closed set such that $H \cap K = \emptyset$. Thus, $K = \bigcap_{0 \leq j \leq n} K_j$, where each K_j is a regular closed set in X. We show that H and K can be separated.

Case 1: $H \cap (\bigcap_{j<n} K_j) = \emptyset$ (or $H \cap K_n = \emptyset$).
Then, by the inductive hypothesis, we can find $U, V \subset X$ open such that $U \cap V = \emptyset$, $H \subseteq U$, $\bigcap_{j<n} K_j \subseteq V$ ($K_n \subseteq V$, respectively). Since $K \subseteq \bigcap_{j<n} K_j$ ($K \subseteq K_n$), H and K are separated by U and V.

Case 2: $H \cap (\bigcap_{j<n} K_j) \neq \emptyset \neq H \cap K_n$.
Given that $H \cap K = \emptyset$, $[H \cap (\bigcap_{j<n} K_j)] \cap K_n = \emptyset$. In addition, $H \cap (\bigcap_{j<n} K_j)$ is closed, non-empty and K_n is a regular closed set, since X is almost-normal, there are $U_n, V_n \subset X$ open such that $U_n \cap V_n = \emptyset$, $H \cap (\bigcap_{j<n} K_j) \subseteq U_n$, $K_n \subseteq V_n$.
Now, $H \setminus U_n = H \cap (X \setminus U_n)$ is closed, non-empty (since $H \cap K_n \subseteq H \setminus U_n$), and disjoint from $\bigcap_{j<n} K_j$, which is an n-π-closed set. Hence, by the inductive hypothesis, there are $U_K, V_K \subset X$ open such that $U_K \cap V_K = \emptyset$, $H \setminus U_n \subseteq U_K$, $\bigcap_{j<n} K_j \subseteq V_K$. Let $U = U_n \cup U_K$, $V = V_n \cap V_K$.

Claim: U and V are a separation of H and K.
Assume there is $x \in U \cap V$, then $x \in U_n \cap V_n$ or $x \in U_K \cap V_K$, which is a

contradiction. Thus, $U \cap V = \emptyset$. In addition, $H = (H \cap U_n) \cup (H \setminus U_n) \subseteq U_n \cup U_K = U$ and $K = (\bigcap_{j<n} K_j) \cap K_n \subseteq V_K \cap V_n = V$. Hence, H and K are separated.

Therefore, for any closed set H and for each n, if K is n-π-closed and $H \cap K = \emptyset$, then H and K can be separated. Whence, X is π-normal. $\square$

Hence, Diagram 3.1 is simplified as the following diagram:

almost-normal $\to$ *quasi-normal* $\to$ *partly normal* $\to$ *mildly-normal.*

Diagram 3.2: Weakenings of Normality Revisited.

The examples constructed in the next section show that in Ψ-spaces at least three of these four properties are distinct (Example 3.2.3 is a quasi-normal not almost normal Ψ-space and Example 3.2.5 is a mildly-normal not partly-normal Ψ-space). The next definition contains standard notation and terminology and, two important classes of almost disjoint families that will be used in the examples.

Definition 3.1.3. *Given an almost disjoint family $\mathcal{A}$,*

- *If $B \subseteq \omega$, let $\mathcal{A} \restriction_B = \{a \in \mathcal{A} : |a \cap B| = \omega\}$.*
- *$\mathcal{I}^+(\mathcal{A}) = \{B \subseteq \omega : |\mathcal{A} \restriction_B| \geq \omega\}$ is the family of big sets (the sets that have infinite intersection with infinite many members of the family).*
- *$\mathcal{I}(\mathcal{A}) = \{B \subseteq \omega : |\mathcal{A} \restriction_B| < \omega\}$, the family of small sets. This family forms an ideal.*
- *$\mathcal{A}$ will be called* **completely separable** *[39] if for each $B \in \mathcal{I}^+(\mathcal{A})$, there is some $a \in \mathcal{A}$ with $a \subseteq B$.*

- *$\mathcal{A}$ will be called* **of true cardinality $\mathfrak{c}$** *if for every $B \subseteq \omega$ either $\mathcal{A} \restriction_B$ is finite, or it has size $\mathfrak{c}$.*
- *If $\Psi(\mathcal{A})$ is a normal space (almost-normal, quasi-normal, partly-normal, mildly-normal), it will be said that $\mathcal{A}$ is normal (almost-normal, quasi-normal, partly-normal, mildly-normal, respectively).*

Hechler introduced the notion of a completely separable almost disjoint family in [39] and proved that such families exist assuming Martin's Axiom. His original definition implies maximality: $\mathcal{A}$ is *completely separable* if for every $M \subseteq \omega$ either $M \subseteq \bigcup \mathcal{B}$ for some finite $\mathcal{B} \subseteq \mathcal{A}$ or there is some $a \in \mathcal{A}$ with $a \subseteq M$. Erdős and Shelah asked in [29] whether completely separable mad families exist in ZFC. Since then, there has been a lot of interest in the construction of completely separable mad families and many interesting partial answers have been obtained (see [42]).

The following observation (known in Prague since the 70's[1]), implies that the existence of a completely separable almost disjoint familiy it is equivalent to the existence of an almost disjoint family of true cardinality $\mathfrak{c}$.

Observation 3.1.4. (Folklore)

1. *Completely separable almost disjoint families are of true cardinality $\mathfrak{c}$.*
2. *Given an almost disjoint family is of true cardinality $\mathfrak{c}$, it is possible to build a completely separable almost disjoint family.*

The way completely separable almost disjoint families are stated in Definition 3.1.3 does not imply maximality and it was shown in [6] that such families

[1] According to Michael Hrušák in private communication.

exist in ZFC. (see also [32]). Thus, by Observation 3.1.4, almost disjoint families of true cardinality $\mathfrak{c}$ exist in ZFC as well. Since (non maximal) almost disjoint families of true cardinality $\mathfrak{c}$ will be used in the constructions of the examples in the next section, let us prove part (1) of Observation 3.1.4. Let us present first, a general property of completely separable almost disjoint families:

Observation 3.1.5 (Folklore)**.** *If $\mathcal{A}$ is a completely separable almost disjoint family and $B \in \mathcal{I}^+(\mathcal{A})$, then $|\{a \in \mathcal{A} : a \subseteq B\}| = \mathfrak{c}$.*

Proof. For $n \in \omega$, choose distinct sets $a_n \in \mathcal{A}$ with $|B \cap a_n| = \omega$. For each n, choose an infinite set $M_n \subseteq (B \cap a_n) \setminus \bigcup_{i<n} a_i$ with $|a_n \setminus M_n| = \omega$. Choose an almost disjoint family $\mathcal{C}$ of size $\mathfrak{c}$. For each $c \in \mathcal{C}$, the set $\bigcup\{M_n : n \in c\}$ belongs to $\mathcal{I}^+(\mathcal{A})$ and by the complete separability of $\mathcal{A}$, it contains some $a_c \in \mathcal{A}$. Now we show that if $c \neq c'$, then $a_c \neq a_{c'}$. Indeed, let $l > max(c \cap c')$, then $\bigcup\{M_n : n \in c \setminus l\} \cap \bigcup\{M_n : n \in c' \setminus l\} = \emptyset$. Otherwise, there are $n, m \in \omega$ so that $m > n > l$, $n \in c$, $m \in c'$ and $x \in M_n \cap M_m$, but this implies that $x \in a_n$ and $x \in a_m \setminus a_n$ which is a contradiction.
Furthermore, for each $c \in \mathcal{C}$ and $n \in c$, $|a_c \cap M_n| < \omega$ (otherwise a_c would have infinite intersection with some $a_n \in \mathcal{A}$, and for all n, $a_c \neq a_n$). Thus, $c \neq c'$,implies $a_c \neq a_{c'}$. Now, since for each $c \in \mathcal{C}$, $a_c \subseteq B$, we get $|\{a \in \mathcal{A} : a \subseteq B\}| = \mathfrak{c}$. □

Hence, by Observation 3.1.5, if $\mathcal{A}$ is completely separable, then for any $B \subseteq \omega$, the set $\mathcal{A} \restriction_B$ is either finite or it has size $\mathfrak{c}$. That is, $\mathcal{A}$ is of true cardinality $\mathfrak{c}$. Furthermore, every infinite almost disjoint family $\mathcal{A}$ of true cardinality $\mathfrak{c}$, has size $\mathfrak{c}$ and thefore $\mathcal{A}$ is not normal (as a consequence of Jones' Lemma). Actually, something slightly stronger holds:

Observation 3.1.6. *If $\mathcal{A}$ is an almost disjoint family of true cardinality $\mathfrak{c}$, then for all $\mathcal{C} \in [\mathcal{A}]^{\aleph_0}$, $\mathcal{C}$ and $\mathcal{A} \smallsetminus \mathcal{C}$ cannot be separated in $\Psi(\mathcal{A})$.*

Proof. Let U, V be any open sets in $\Psi(\mathcal{A})$ so that $\mathcal{C} \subseteq U$, $\mathcal{A} \setminus \mathcal{C} \subseteq V$. Let $W = U \cap \omega$, then for all $c \in \mathcal{C}$, $c \subseteq^* W$. Hence, $|\mathcal{A} \restriction_W| \geq \omega$. Thus, $|\mathcal{A} \restriction_W| = \mathfrak{c}$. Pick $a \in \mathcal{A} \setminus \mathcal{C}$ such that $|W \cap a| = \omega$. Since $a \subseteq^* V \cap \omega$, $U \cap V \neq \emptyset$. □

The following observations are not hard to show and they will be used in various occasions in the next section.

Observation 3.1.7. *Given any almost disjoint family $\mathcal{A}$, if $W \subseteq \omega$, then $cl_{\Psi(\mathcal{A})}(W)$ is a regular closed subset of $\Psi(\mathcal{A})$.*

Observation 3.1.8. *Given any almost disjoint family $\mathcal{A}$, if $H \subset \Psi(\mathcal{A})$ is a regular closed set, then for each $a \in \mathcal{A}$, $a \in H$ if and only if $|a \cap H| = \omega$.*

Observation 3.1.9. *Given any almost disjoint family $\mathcal{A}$ and $H, K \subset \Psi(\mathcal{A})$ such that H and K are closed sets, $H \cap K = \emptyset$ and $|H \cap \mathcal{A}| < \omega$, then H and K can be separated. In particular, for each closed set $H \subset \Psi(\mathcal{A})$ that has finite intersection with $\mathcal{A}$, H and $\mathcal{A} \smallsetminus H$ can be separated.*

3.2 Examples of Ψ-spaces

Example 3.2.3 provides a quasi-normal not almost-normal almost disjoint family $\mathcal{F}$ which is constructed from a particular non almost-normal almost disjoint family $\mathcal{A}$ of true cardinality $\mathfrak{c}$. Each element of $\mathcal{F}$ will be a finite union of elements of $\mathcal{A}$. In order to make $\mathcal{F}$ quasi-normal, all pairs of disjoint π-closed sets in $\Psi(\mathcal{F})$ have to be separated. By Observation 3.1.9, the only pairs of π-closed sets (A, B) that might be difficult to separate are the ones where $A \cap \mathcal{F}$ and $B \cap \mathcal{F}$ are infinite. Using that $\mathcal{A}$ is of true cardinality $\mathfrak{c}$ it will be possible to build $\mathcal{F}$ so that all such pairs have a point in common. Thus, all pairs of disjoint π-closed sets in $\Psi(\mathcal{F})$ will be trivial, i.e. one of them will have finite intersection with $\mathcal{F}$. Hence, $\mathcal{F}$ will be quasi-normal. In addition, it won't be hard to carry this construction out so that the non almost-normality of $\mathcal{A}$ is preserved in $\mathcal{F}$. That is, a closed set $\mathcal{C}$ and a regular closed set E with empty intersection that cannot be separated in $\Psi(\mathcal{A})$ will be transformed into a pair of witnesses of non almost-normality in $\Psi(\mathcal{F})$. Now, let us obtain the required non almost-normal almost disjoint family of true cardinality $\mathfrak{c}$.

The following example is an instance of a machine for converting two almost disjoint families of the same cardinality, into a single almost disjoint family $\mathcal{A}$ with a countable set $\mathcal{C} \subset \mathcal{A}$ and a set $E \subset \Psi(\mathcal{A})$ such that $\mathcal{C}$ is closed and E is regular closed in $\Psi(\mathcal{A})$, $\mathcal{C} \cap E = \emptyset$ and $\mathcal{A} \subset \mathcal{C} \cup E$.

Example 3.2.1. *There is an almost disjoint family $\mathcal{A}$ of true cardinality $\mathfrak{c}$ on ω so that there is $\mathcal{C} \in [\mathcal{A}]^\omega$ and $W \in [\omega]^\omega$, such that $cl_{\Psi(\mathcal{A})}(W) \cap \mathcal{A} = \mathcal{A} \setminus \mathcal{C}$. In particular, there is a non almost-normal almost disjoint family of true cardinality $\mathfrak{c}$.*

Proof. Partition ω into two infinite disjoint sets V, W. Let $\mathcal{A}_0, \mathcal{A}_1$ be almost disjoint families of true cardinality $\mathfrak{c}$ on V and W, respectively, and let $\mathcal{C} \in [\mathcal{A}_0]^\omega$. Now, a new family is built as follows, let $\alpha : \mathcal{A}_0 \setminus \mathcal{C} \leftrightarrow \mathcal{A}_1$ be a bijective function. Let $\mathcal{A} = \{a \cup \alpha(a) : a \in \mathcal{A}_0 \setminus \mathcal{C}\} \cup \mathcal{C}$.
Let us check that $\mathcal{A}$ is the desired family. Clearly, it is almost disjoint. To see

that it has true cardinality $\mathfrak{c}$ let $M \subseteq \omega$ such that $|\mathcal{A} \restriction_M| \geq \omega$. Then, either $|\mathcal{C} \restriction_M| \geq \omega$ or $|(\mathcal{A} \setminus \mathcal{C}) \restriction_M| \geq \omega$. Hence, $|\mathcal{A}_0 \restriction_M| \geq \omega$ or $|\mathcal{A}_1 \restriction_M| \geq \omega$. Therefore, $|\mathcal{A}_0 \restriction_M| = \mathfrak{c}$ or $|\mathcal{A}_1 \restriction_M| = \mathfrak{c}$. In any case, $|\mathcal{A} \restriction_M| = \mathfrak{c}$. Thus, $\mathcal{A}$ is of true cardinality $\mathfrak{c}$.

Now, $a \in cl_{\Psi(\mathcal{A})}(W) \cap \mathcal{A} \leftrightarrow a \in \mathcal{A} \wedge |a \cap W| = \omega \leftrightarrow a \in \mathcal{A} \wedge \big(\exists a_0 \in \mathcal{A}_0[a = a_0 \cup \alpha(a_0)]\big) \leftrightarrow a \in \mathcal{A} \setminus \mathcal{C}$. By Observation 3.1.6, $\mathcal{A}$ is not almost-normal.

$\square$

If in the previous example we assume, in addition, that $\mathcal{A}_0, \mathcal{A}_1$ are mad families of the same cardinality, the resulting family $\mathcal{A}$ is mad as well: If $M \in [\omega]^\omega$, then M has infinite intersection either with V or with W, since $\mathcal{A}_0, \mathcal{A}_1$ are both mad, there is $a \in \mathcal{A}$ such $|a \cap M| = \omega$. Hence, the following holds:

Corollary 3.2.2. *The existence of a mad family of true cardinality $\mathfrak{c}$ implies the existence of a mad family $\mathcal{A}$ of true cardinality $\mathfrak{c}$ on ω so that there is $\mathcal{C} \in [\mathcal{A}]^\omega$ and $W \in [\omega]^\omega$, such that $cl_{\Psi(\mathcal{A})}(W) \cap \mathcal{A} = \mathcal{A} \setminus \mathcal{C}$. In particular, the existence of a mad family of true cardinality $\mathfrak{c}$ implies the existence of a non almost-normal mad family of true cardinality $\mathfrak{c}$.*

As pointed out at the beginning of this section, we use a particular non almost-normal almost disjoint family of true cardinality $\mathfrak{c}$ to build an example of a quasi-normal not almost normal Ψ-space. That is, we will use a family like the one given by Example 3.2.1. Observe we do not require that this family is maximal, i.e. the following example is a **ZFC** example.

Example 3.2.3. *There is a quasi-normal not almost-normal almost disjoint family of true cardinality $\mathfrak{c}$.*

Proof. Let $\mathcal{A}$ be a not almost-normal almost disjoint family of true cardinality $\mathfrak{c}$ as in Example 3.2.1. Hence, let $\mathcal{C} \in [\mathcal{A}]^\omega$ and $W \in [\omega]^\omega$, with $|\omega \setminus W| = \omega$, such that $cl_{\Psi(\mathcal{A})}(W) \cap \mathcal{A} = \mathcal{A} \setminus \mathcal{C}$. Consider the family of finite subsets of $[\omega]^\omega$, $\mathcal{E} = \left[[\omega]^\omega\right]^{<\omega}$ and let $\mathcal{B} = \{\{C, D\} \in [\mathcal{E}]^2 : (\bigcap C) \cap (\bigcap D) = \emptyset\}$. Since $|\mathcal{B}| = \mathfrak{c}$, we can list it as $\mathcal{B} = \{\{C_\alpha, D_\alpha\} : \alpha < \mathfrak{c}\}$. A sequence of finite sets $\mathcal{F}_\alpha \in [\mathcal{A}]^{<\omega}$ will be built recursively in $\mathfrak{c}$ many steps.

For $\alpha = 0$, consider $\{C_0, D_0\} \in \mathcal{B}$. If for each $C \in C_0$ and $D \in D_0$, $\mathcal{A} \restriction_C$ and $\mathcal{A} \restriction_D$ all have size $\mathfrak{c}$, then for each $C \in C_0$ and $D \in D_0$ pick $a_C, b_D \in \mathcal{A} \setminus \mathcal{C}$ such that $|a_C \cap C| = \omega = |b_D \cap D|$ and all the a_C's and b_D's are distinct ($|\{a_C, b_D : C \in C_0, D \in D_0\}| = |C_0| + |D_0|$). Let $\mathcal{F}_0 = \{a_C, b_D : C \in C_0, D \in D_0\}$. If there is $C \in C_0$ (or $D \in D_0$) such that $\mathcal{A} \restriction_C$ is finite ($\mathcal{A} \restriction_D$ is finite), let $\mathcal{F}_0 = \emptyset$. Observe that these are the only two possibilities as $\mathcal{A}$ is of true cardinality $\mathfrak{c}$.

Now assume $0 < \alpha < \mathfrak{c}$ and that for each $\beta < \alpha$, $\mathcal{F}_\beta$ is either empty of a finite subset of $\mathcal{A} \setminus (\mathcal{C} \cup \bigcup_{\gamma<\beta} \mathcal{F}_\gamma)$. Consider the pair $\{C_\alpha, D_\alpha\}$. If for each $C \in C_\alpha$ and $D \in D_\alpha$, $\mathcal{A} \restriction_C$ and $\mathcal{A} \restriction_D$ all have size $\mathfrak{c}$, then for each $C \in C_\alpha$ and $D \in D_\alpha$ pick $a_C, b_D \in \mathcal{A} \setminus (\mathcal{C} \cup \bigcup_{\beta<\alpha} \mathcal{F}_\beta)$ such that $|a_C \cap C| = \omega = |b_D \cap D|$ and all the a_C's and b_D's are distinct ($|\{a_C, b_D : C \in C_\alpha, D \in D_\alpha\}| = |C_\alpha| + |D_\alpha|$). Let $\mathcal{F}_\alpha = \{a_C, b_D : C \in C_\alpha, D \in D_\alpha\}$. If there is $C \in C_\alpha$ (or $D \in D_\alpha$) such that $\mathcal{A} \restriction_C$ is finite ($\mathcal{A} \restriction_D$ is finite), let $\mathcal{F}_\alpha = \emptyset$. Let

$$\mathcal{F} = \Big\{\bigcup \mathcal{F}_\alpha : \alpha < \mathfrak{c}\Big\} \cup \Big(\mathcal{A} \setminus \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha\Big).$$

Since each $a \in \mathcal{F}$ is either an element of $\mathcal{A}$ or a finite union of elements of $\mathcal{A}$, it is clear that $\mathcal{F}$ is an almost disjoint family of true cardinality $\mathfrak{c}$.

$\Psi(\mathcal{F})$ is quasi-normal:

Let $A \neq \emptyset \neq B$ be disjoint π-closed subsets of $\Psi(\mathcal{F})$. $A = \bigcap_{i=1}^{n} A_i$, $B = \bigcap_{j=1}^{m} B_j$, where each A_i and B_j are regular closed sets. It can be assumed that for each $i \leq n$ and for each $j \leq m$, $|A_i \cap \omega| = \omega = |B_j \cap \omega|$. Let $\alpha < \mathfrak{c}$ be minimal such that $C_\alpha = \{A_i \cap \omega : i \leq n\}$ and $D_\alpha = \{B_j \cap \omega : j \leq m\}$. At stage α, either $\mathcal{F}_\alpha = \emptyset$ or $\mathcal{F}_\alpha = \{a_C, b_D : C \in C_\alpha, D \in D_\alpha\}$. The latter is not possible since for each $C \in C_\alpha$ and each $D \in D_\alpha$ the a_C's and b_D's were chosen so that $|a_C \cap C| = \omega = |b_D \cap D|$ and this implies $\bigcup \mathcal{F}_\alpha$ is in the closure of each $C \in C_\alpha$ and each $D \in D_\alpha$ (see Observation 3.1.7 and Observation 3.1.8). Hence $\bigcup \mathcal{F}_\alpha \in A \cap B$, but it is assumed that A and B are disjoint.

Thus, $\mathcal{F}_\alpha = \emptyset$. This means that there exists $C \in C_\alpha$, such that $\mathcal{A} \restriction_C = H$ for some finite set H (or there exists $D \in D_\alpha$, such that $\mathcal{A} \restriction_D = H$ for some finite set H). Without loss of generality assume there exists such $C \in C_\alpha$. Hence, $\mathcal{A} \restriction_C = H_0$ for some finite set H_0. Observe that since for each $a \in \mathcal{F}$, either $a \in \mathcal{A}$ or a is a finite union of elements of $\mathcal{A}$, then $\mathcal{F} \restriction_C = H_1$ for some finite H_1 so that $|H_1| \leq |H_0|$. Now fix $i \leq n$ such that $A_i \cap \omega = C$. Since A_i is regular closed, by 3.1.8 $A_i \cap \mathcal{F} = H_0$. Thus, $A \cap \mathcal{F} \subseteq H_0$ and by Observation 3.1.9, A and B can be separated. Therefore $\Psi(\mathcal{F})$ is quasi-normal.

$\Psi(\mathcal{F})$ is not almost-normal:

Fix $a \in \mathcal{F} \setminus \mathcal{C}$, then $a \in \mathcal{A} \setminus \mathcal{C}$ or a is a finite union of elements of $\mathcal{A} \setminus \mathcal{C}$. Since $cl_{\Psi(\mathcal{A})}(W) \cap \mathcal{A} = \mathcal{A} \setminus \mathcal{C}$, $|W \cap a| = \omega$. Hence, $a \in cl_{\Psi(\mathcal{F})}(W)$, i.e., $\mathcal{F} \setminus \mathcal{C} \subseteq cl_{\Psi(\mathcal{F})}(W)$. On the other hand, if $c \in \mathcal{C}$, $c \notin cl_{\Psi(\mathcal{A})}(W)$, thus

$|c \cap W| < \omega$ and therefore $c \notin cl_{\Psi(\mathcal{F})}(W)$.

Hence, $\mathcal{C}$ is a closed set, $cl_{\Psi(\mathcal{F})}(W)$ is a regular closed set, they do not intersect and by Observation 3.1.6 they cannot be separated. □

If in the construction of Example 3.2.3, a mad family as in Corollary 3.2.2 is chosen, then the resulting family $\mathcal{F}$ is mad, quasi-normal and not almost-normal. Thus:

Corollary 3.2.4. *The existence of a mad family of true cardinality $\mathfrak{c}$ implies the existence of a quasi-normal, non almost-normal mad family of true cardinality $\mathfrak{c}$.*

The following (**ZFC**) example provides a mildly-normal not partly-normal almost disjoint family $\mathcal{F}$ of true cardinality $\mathfrak{c}$ which is constructed using three almost disjoint families of true cardinality $\mathfrak{c}$. In order to make $\mathcal{F}$ mildly-normal all pairs of disjoint regular closed sets in $\Psi(\mathcal{F})$ have to be separated. A similar approach as in Example 3.2.3 is followed. It will be possible to build $\mathcal{F}$ so that all pairs of disjoint regular closed sets in $\Psi(\mathcal{F})$ will be trivial, i.e., one of them will have finite intersection with $\mathcal{F}$ (Observation 3.1.9 guarantees they can be separated). To make $\mathcal{F}$ not quasi-normal, there will be a regular closed set A disjoint from a π-closed set B that cannot be separated. The basic idea is to partition ω into three infinite sets, W, V_0, V_1, take an almost disjoint family of true cardinality $\mathfrak{c}$ on each one of them (we use the property of true cardinality $\mathfrak{c}$ to make $\mathcal{F}$ mildly-normal), and build $\mathcal{F}$ so that in $\Psi(\mathcal{F})$, $A = cl_{\Psi(\mathcal{F})}(W)$ and $B = cl_{\Psi(\mathcal{F})}(V_0) \cap cl_{\Psi(\mathcal{F})}(V_1)$ are disjoint but cannot be separated.

Example 3.2.5. *There exists a mildly-normal not partly-normal almost disjoint family of true cardinality $\mathfrak{c}$.*

Proof. Partition ω into three disjoint infinite pieces, that is $W, V_0, V_1 \in [\omega]^\omega$ and $W \cup V_0 \cup V_1 = \omega$. If $Y \in \{W, V_0, V_1\}$ let $\mathcal{A}_Y$ be an almost disjoint family of true cardinality $\mathfrak{c}$ on Y. List all pairs of infinite subsets of ω with empty intersection as $\{\{C_\alpha, D_\alpha\} : \alpha < \mathfrak{c}\}$. A sequence of finite sets $\mathcal{F}_\alpha \subset \mathcal{A}_W \cup \mathcal{A}_{V_0} \cup \mathcal{A}_{V_1}$ will be built recursively in $\mathfrak{c}$ many steps.

Fix $\alpha < \mathfrak{c}$, assume that for each $\beta < \alpha$, $\mathcal{F}_\beta$ has been defined such that $\mathcal{F}_\beta$ is a possibly empty finite set $\mathcal{F}_\beta \subset (\mathcal{A}_W \cup \mathcal{A}_{V_0} \cup \mathcal{A}_{V_1}) \setminus \bigcup_{\gamma<\beta} \mathcal{F}_\gamma$ such that either $\mathcal{F}_\beta \subset \mathcal{A}_W$ or $\mathcal{F}_\beta$ has nonempty intersection with exactly two elements of $\{\mathcal{A}_W, \mathcal{A}_{V_0}, \mathcal{A}_{V_1}\}$. Consider $\{C_\alpha, D_\alpha\}$.

Case 1: Either all three sets $\mathcal{A}_W \restriction_{C_\alpha}$, $\mathcal{A}_{V_0} \restriction_{C_\alpha}$, $\mathcal{A}_{V_1} \restriction_{C_\alpha}$ are finite, or all three sets $\mathcal{A}_W \restriction_{D_\alpha}$, $\mathcal{A}_{V_0} \restriction_{D_\alpha}$, $\mathcal{A}_{V_1} \restriction_{D_\alpha}$ are finite. In this case, let $\mathcal{F}_\alpha = \emptyset$.

Case 2: Case 1 is false. That is (given that $\mathcal{A}_W$, $\mathcal{A}_{V_0}$, $\mathcal{A}_{V_1}$ are of true cardinality $\mathfrak{c}$): at least one of the three sets $\mathcal{A}_W \restriction_{C_\alpha}$, $\mathcal{A}_{V_0} \restriction_{C_\alpha}$, $\mathcal{A}_{V_1} \restriction_{C_\alpha}$ has size $\mathfrak{c}$ and at least one of the three sets $\mathcal{A}_W \restriction_{D_\alpha}$, $\mathcal{A}_{V_0} \restriction_{D_\alpha}$, $\mathcal{A}_{V_1} \restriction_{D_\alpha}$ has size $\mathfrak{c}$. Choose the smallest i such that Subcase 2.i (below) holds, define $\mathcal{F}_\alpha$ accordingly, and ignore the other subcases.

Subcase 2.1: $|\mathcal{A}_W \restriction_{C_\alpha}| = \mathfrak{c} = |\mathcal{A}_W \restriction_{D_\alpha}|$. Pick $c_\alpha, d_\alpha \in \mathcal{A}_W \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ such that $c_\alpha \neq d_\alpha$ and $|c_\alpha \cap C_\alpha| = \omega = |d_\alpha \cap D_\alpha|$. Let $\mathcal{F}_\alpha = \{c_\alpha, d_\alpha\}$.

Subcase 2.2: There exists $i \in \{0, 1\}$ so that $|\mathcal{A}_{V_i} \restriction_{C_\alpha}| = \mathfrak{c} = |\mathcal{A}_{V_i} \restriction_{D_\alpha}|$. Pick $c_\alpha, d_\alpha \in \mathcal{A}_{V_i} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, such that $c_\alpha \neq d_\alpha$ and $|c_\alpha \cap C_\alpha| = \omega = |d_\alpha \cap D_\alpha|$.

In addition, pick $e_\alpha \in \mathcal{A}_{V_{1-i}} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. Let $\mathcal{F}_\alpha = \{c_\alpha, d_\alpha, e_\alpha\}$.

Subcase 2.3: $|\mathcal{A}_{V_0} \restriction_{C_\alpha}| = \mathfrak{c} = |\mathcal{A}_{V_1} \restriction_{D_\alpha}|$. Pick $c_\alpha \in \mathcal{A}_{V_0} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ and $d_\alpha \in \mathcal{A}_{V_1} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ such that $|c_\alpha \cap C_\alpha| = \omega = |d_\alpha \cap D_\alpha|$ and let $\mathcal{F}_\alpha = \{c_\alpha, d_\alpha\}$.

Subcase 2.4: $|\mathcal{A}_W \restriction_{C_\alpha}| = \mathfrak{c}$ and there exists $i \in \{0,1\}$ so that $|\mathcal{A}_{V_i} \restriction_{D_\alpha}| = \mathfrak{c}$. Pick $c_\alpha \in \mathcal{A}_W \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ and $d_\alpha \in \mathcal{A}_{V_i} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ such that $|c_\alpha \cap C_\alpha| = \omega = |d_\alpha \cap D_\alpha|$ and let $\mathcal{F}_\alpha = \{c_\alpha, d_\alpha\}$.

This finishes Case 2 and the construction of $\mathcal{F}_\alpha$ for $\alpha < \mathfrak{c}$. Let

$$\mathcal{F} = \Big\{ \bigcup \mathcal{F}_\alpha : \alpha < \mathfrak{c} \Big\} \cup \Big((\mathcal{A}_W \cup \mathcal{A}_{V_0} \cup \mathcal{A}_{V_1}) \setminus \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha \Big).$$

It will be shown that $\mathcal{F}$ is the desired almost disjoint family. Given that each of $\mathcal{A}_W$, $\mathcal{A}_{V_0}$ and $\mathcal{A}_{V_1}$ is of true cardinality $\mathfrak{c}$ and if we let $a \in \mathcal{F}$, then either a is an element or a finite union of elements of $\mathcal{A}_W \cup \mathcal{A}_{V_0} \cup \mathcal{A}_{V_1}$, then $\mathcal{F}$ is an almost disjoint family of true cardinality $\mathfrak{c}$.

$\Psi(\mathcal{F})$ is not partly-normal:

Let $A = cl_{\Psi(\mathcal{F})}(W)$ and $B = cl_{\Psi(\mathcal{F})}(V_0) \cap cl_{\Psi(\mathcal{F})}(V_1)$. By Observation 3.1.7, A is regular closed and B is a π-closed set. Observe that since $\mathcal{A}_{V_0}$ and $\mathcal{A}_{V_1}$ are of true cardinality $\mathfrak{c}$, there are infinite many pairs $\{C_\alpha, D_\alpha\}$ such that $C_\alpha \subset V_0$, $D_\alpha \subset V_1$, and $|\mathcal{A}_{V_0} \restriction_{C_\alpha}| = \mathfrak{c} = |\mathcal{A}_{V_1} \restriction_{D_\alpha}|$. For such pairs Subcase 2.3 applies and therefore $|B \cap \mathcal{F}| \geq \omega$. In addition, $A \cap B = \emptyset$: assume there is $a \in A \cap B$. Since $V_0 \cap V_1 = \emptyset$, $B \cap \omega = \emptyset$, hence $a \in \mathcal{F} \cap A \cap B$. By Observation 3.1.8, $|a \cap W| = |a \cap V_0| = |a \cap V_1| = \omega$. This implies that $a \notin \mathcal{A}_W \cup \mathcal{A}_{V_0} \cup \mathcal{A}_{V_1}$. There is $\alpha < \mathfrak{c}$ such that $a = \bigcup \mathcal{F}_\alpha$, but by the construction, $\mathcal{F}_\alpha \subset \mathcal{A}_W$ or $\mathcal{F}_\alpha$ intersects exactly two elements of $\{\mathcal{A}_W, \mathcal{A}_{V_0}, \mathcal{A}_{V_1}\}$ which contradicts that a has infinite intersection with W, V_0 and V_1. Whence, $A \cap B = \emptyset$.

It remains to show that A and B cannot be separated. Assume, on the contrary, that there are $S, T \subseteq \Psi(\mathcal{F})$ open such that $A \subseteq S$, $B \subseteq T$ and $S \cap T = \emptyset$. Let $\alpha < \mathfrak{c}$ such that $C_\alpha = \omega \cap S$ and $D_\alpha = \omega \cap T$. For the pair $\{C_\alpha, D_\alpha\}$, either Case 1 or Case 2 of the construction holds.

If Case 1 holds: since $W \subseteq C_\alpha$, $\mathcal{A}_W \restriction_{C_\alpha}$ is not finite. Hence, $\mathcal{A}_W \restriction_{D_\alpha}$, $\mathcal{A}_{V_0} \restriction_{D_\alpha}$, $\mathcal{A}_{V_1} \restriction_{D_\alpha}$ are finite. Thus, $\mathcal{F} \restriction_{D_\alpha}$ is finite. Since $cl_{\Psi(\mathcal{F})}(D_\alpha)$ is regular closed and $\mathcal{F} \restriction_{D_\alpha}$ is finite, by Observation 3.1.8, $\mathcal{F} \cap cl_{\Psi(\mathcal{F})}(D_\alpha)$ is finite. Now, T is open and $D_\alpha = \omega \cap T$, therefore $T \subseteq cl_{\Psi(\mathcal{F})}(D_\alpha)$. Hence, $\mathcal{F} \cap T$ is finite. Given that $|B \cap \mathcal{F}| \geq \omega$, $B \not\subseteq T$, which is a contradiction.
If Case 2 holds: Either $\mathcal{F}_\alpha \subset \mathcal{A}_W$ or $\mathcal{F}_\alpha$ intersects exactly two elements of $\{\mathcal{A}_W, \mathcal{A}_{V_0}, \mathcal{A}_{V_1}\}$. In any case $\bigcup \mathcal{F}_\alpha$ is an element of A or B. In addition, there exist $c_\alpha, d_\alpha \in \mathcal{F}_\alpha$ such that $|c_\alpha \cap C_\alpha| = \omega = |d_\alpha \cap D_\alpha|$. If $\bigcup \mathcal{F}_\alpha \in A$, then for each open neighbourhood U of $\bigcup \mathcal{F}_\alpha$, $U \cap T \neq \emptyset$ (which implies $U \not\subseteq S$), and this contradicts that S is open. We reach a similar contradiction if $\bigcup \mathcal{F}_\alpha \in B$. Hence, A and B cannot be separated.

$\Psi(\mathcal{F})$ is mildly-normal:

Let $C \neq \emptyset \neq D$ be disjoint regular closed subsets of $\Psi(\mathcal{F})$. It can be assumed that $|C \cap \omega| = \omega = |D \cap \omega|$. Fix $\alpha < \mathfrak{c}$ such that $C \cap \omega = C_\alpha$ and $D \cap \omega = D_\alpha$. For the pair $\{C_\alpha, D_\alpha\}$, either Case 1 or Case 2 holds. If Case 2 holds, there exist $c_\alpha, d_\alpha \in \mathcal{F}_\alpha$ such that $|c_\alpha \cap C_\alpha| = \omega = |d_\alpha \cap D_\alpha|$. Thus, $\bigcup \mathcal{F}_\alpha \in cl_{\Psi(\mathcal{F})}(C_\alpha) \cap cl_{\Psi(\mathcal{F})}(D_\alpha) \subseteq cl_{\Psi(\mathcal{F})}(C) \cap cl_{\Psi(\mathcal{F})}(D) = C \cap D$. This contradicts $C \cap D = \emptyset$.
Thus, Case 1 holds. This means that all three sets $\mathcal{A}_W \restriction_{C_\alpha}$, $\mathcal{A}_{V_0} \restriction_{C_\alpha}$, $\mathcal{A}_{V_1} \restriction_{C_\alpha}$

are finite, or all three sets $\mathcal{A}_W \restriction_{D_\alpha}$, $\mathcal{A}_{V_0} \restriction_{D_\alpha}$, $\mathcal{A}_{V_1} \restriction_{D_\alpha}$ are finite.

Without loss of generality, assume the former. This implies that $\mathcal{F} \restriction_{C_\alpha}$ is finite. Given that C is a regular closed set and $C_\alpha = C \cap \omega$, by Observation 3.1.8 $C \cap \mathcal{F}$ is finite and by Observation 3.1.9, C and D can be separated. Therefore $\Psi(\mathcal{F})$ is mildly-normal. □

Observe that if in the construction of Example 3.2.5, the families $\mathcal{A}_W$, $\mathcal{A}_{V_0}$ and $\mathcal{A}_{V_1}$ are mad of true cardinality $\mathfrak{c}$, then the family $\mathcal{F}$ is mad as well. Therefore:

Corollary 3.2.6. *If there exists a mad family of true cardinality $\mathfrak{c}$, then there is a mildly-normal, not partly-normal mad family of true cardinality $\mathfrak{c}$.*

Definition 3.2.7. *For a positive $n \in \omega$, a regular space will be called n-*partly-normal *if any two nonintersecting sets A and B, where A is regularly closed and B is the intersection of at most n regularly closed sets, are separated.*

Observe that 1-partly-normal coincides with mildly-normal, and for each positive $n \in \omega$, partly-normal $\to$ $(n+1)$-partly-normal $\to$ n-partly-normal $\to$ mildly-normal. It is possible to extend the idea in Example 3.2.5 (partition ω into $n+2$ pairwise disjoint infinite pieces, take an almost disjoint family of true cardinality $\mathfrak{c}$ on each piece and let $\{\mathbb{C}_\alpha : \alpha < \mathfrak{c}\}$ list all sets $\mathbb{C} \subset [\omega]^\omega$ such that $2 \leq |\mathbb{C}| \leq n+1$), to show the following:

Theorem 3.2.8. *For each positive $n \in \omega$, there exists a n-partly-normal not $(n+1)$-partly-normal almost disjoint family of true cardinality $\mathfrak{c}$.*

Proof. Fix a positive $n \in \omega$ and partition ω into $n+2$ disjoint infinite pieces, that is $W, V_0, \ldots, V_n \in [\omega]^\omega$ and $W \cup V_0 \cup \cdots \cup V_n = \omega$. If $Y \in$

$\{W, V_0, \ldots, V_n\}$ let $\mathcal{A}_Y$ be an almost disjoint family of true cardinality $\mathfrak{c}$ on Y. Let ${}^{2\leq}[[\omega]^\omega]^{\leq n+1}$ denote the family of all sets $\mathbb{C} \subset [\omega]^\omega$ such that $2 \leq |\mathbb{C}| \leq n+1$ and list it as ${}^{2\leq}[[\omega]^\omega]^{\leq n+1} = \{\mathbb{C}_\alpha : \alpha < \mathfrak{c}\}$. A sequence of finite sets $\mathcal{F}_\alpha \subset \mathcal{A}_W \cup \left(\bigcup_{i\leq n} \mathcal{A}_{V_i}\right)$ will be built recursively in $\mathfrak{c}$ many steps.

Fix $\alpha < \mathfrak{c}$, assume that for each $\beta < \alpha$, $\mathcal{F}_\beta$ has been defined such that $\mathcal{F}_\beta$ is a possibly empty finite set $\mathcal{F}_\beta \subset \left[\mathcal{A}_W \cup \left(\bigcup_{i\leq n} \mathcal{A}_{V_i}\right)\right] \setminus \bigcup_{\gamma<\beta} \mathcal{F}_\gamma$ and $\mathcal{F}_\beta$ has non-empty intersection with at most $n+1$ elements of $\{\mathcal{A}_W, \mathcal{A}_{V_0}, \ldots, \mathcal{A}_{V_n}\}$.

Consider $\mathbb{C}_\alpha$:

Case 1: There is $C \in \mathbb{C}_\alpha$ such that all the $n+2$ sets $\mathcal{A}_W \restriction_C$, $\mathcal{A}_{V_0} \restriction_C$, $\ldots$, $\mathcal{A}_{V_n} \restriction_C$ are finite. In this case, let $\mathcal{F}_\alpha = \emptyset$.

Case 2: Case 1 is false. That is (given that $\mathcal{A}_W$, $\mathcal{A}_{V_0}$, $\ldots$, $\mathcal{A}_{V_n}$ are of true cardinality $\mathfrak{c}$): for each $C \in \mathbb{C}_\alpha$ at least one of the $n+2$ sets $\mathcal{A}_W \restriction_C$, $\mathcal{A}_{V_0} \restriction_C$, $\ldots$, $\mathcal{A}_{V_n} \restriction_C$ has size $\mathfrak{c}$. Choose the smallest i such that Subcase 2.i (below) holds, define $\mathcal{F}_\alpha$ accordingly, and ignore the other subcases.
Subcase 2.1: For each $C \in \mathbb{C}_\alpha$, $|\mathcal{A}_W \restriction_C| = \mathfrak{c}$.
For each $C \in \mathbb{C}_\alpha$, pick distinct $a^C \in \mathcal{A}_W \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, such that $|a^C \cap C| = \omega$. That is, if we let $E = \{a^C : C \in \mathbb{C}_\alpha\}$, then $|E| = |\mathbb{C}_\alpha|$. Let $\mathcal{F}_\alpha = E$.

Subcase 2.2: For each $C \in \mathbb{C}_\alpha$, there is $i(C) \leq n$ such that $|\mathcal{A}_{V_{i(C)}} \restriction_C| = \mathfrak{c}$.
For each $C \in \mathbb{C}_\alpha$, pick distinct $a^C \in \mathcal{A}_{V_{i(C)}} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, such that $|a^C \cap C| = \omega$. That is, if we let $E = \{a^C : C \in \mathbb{C}_\alpha\}$, then $|E| = |\mathbb{C}_\alpha|$. In addition,

for each $i \leq n$ such that $E \cap \mathcal{A}_{V_i} = \emptyset$, pick $e^i \in \mathcal{A}_{V_i} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. Let $\mathcal{F}_\alpha = E \cup \{e^i : i \leq n \textit{ and } E \cap \mathcal{A}_{V_i} = \emptyset\}$.

Subcase 2.3: There are $C, D \in \mathbb{C}_\alpha$, such that $|\mathcal{A}_W \restriction_D| < \mathfrak{c}$ and for each $i \leq n$ $|\mathcal{A}_{V_i} \restriction_C| < \mathfrak{c}$.

Let $F^W = \{C \in \mathbb{C}_\alpha : |\mathcal{A}_W \restriction_C| = \mathfrak{c} \wedge \forall i \leq n(|\mathcal{A}_{V_i} \restriction_C| < \mathfrak{c})\}$. Observe $1 \leq |F^W| < |\mathbb{C}_\alpha|$. For each $C \in F^W$, pick distinct $a^C \in \mathcal{A}_W \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, such that $|a^C \cap C| = \omega$. That is, if we let $E = \{a^C : C \in F^W\}$, then $|E| = |F^W|$. Let $F^V = \mathbb{C}_\alpha \smallsetminus F^W$. Observe $1 \leq |F^V| < |\mathbb{C}_\alpha|$. For each $C \in F^V$, fix $i(C) \leq n$ such that $|\mathcal{A}_{V_{i(C)}} \restriction_C| = \mathfrak{c}$ and pick distinct $a^C \in \mathcal{A}_{V_{i(C)}} \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$, such that $|a^C \cap C| = \omega$. That is, if we let $G = \{a^C : C \in F^V\}$, then $|G| = |F^V|$. Note that $|F^V| < |\mathbb{C}_\alpha|$ implies that there is $i_0 \leq n$ such that $G \cap \mathcal{A}_{V_{i_0}} = \emptyset$. Let $\mathcal{F}_\alpha = E \cup G$.

This finishes Case 2 and the construction of $\mathcal{F}_\alpha$ for $\alpha < \mathfrak{c}$. Now, for each $\alpha < \mathfrak{c}$, $\mathcal{F}_\alpha \subset \left[\mathcal{A}_W \cup \left(\bigcup_{i\leq n} \mathcal{A}_{V_i}\right)\right] \setminus \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ is finite. Furthermore, in each subcase $\mathcal{F}_\beta$ has non-empty intersection with at most $n+1$ elements of $\{\mathcal{A}_W$, $\mathcal{A}_{V_0}, \ldots, \mathcal{A}_{V_n}\}$. Hence, the recursive hypothesis is satisfied. Let

$$\mathcal{F} = \left\{\bigcup \mathcal{F}_\alpha : \alpha < \mathfrak{c}\right\} \cup \left(\left[\mathcal{A}_W \cup \left(\bigcup_{i\leq n} \mathcal{A}_{V_i}\right)\right] \setminus \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha\right).$$

It will be shown that $\mathcal{F}$ is the desired almost disjoint family. Given that each of $\mathcal{A}_W, \mathcal{A}_{V_0}, \ldots, \mathcal{A}_{V_n}$ is of true cardinality $\mathfrak{c}$ and if we let $a \in \mathcal{F}$, then either a is an element or a finite union of elements of $\mathcal{A}_W \cup \mathcal{A}_{V_0} \cup \cdots \cup \mathcal{A}_{V_n}$, then $\mathcal{F}$ is an almost disjoint family of true cardinality $\mathfrak{c}$.

$\Psi(\mathcal{F})$ is not $(n+1)$-partly-normal:

Let $A = cl_{\Psi(\mathcal{F})}(W)$ and $B = \bigcap_{i \le n} cl_{\Psi(\mathcal{F})}(V_i))$. By Observation 3.1.7, A is regular closed and B is a π-closed set. Observe that since all $\mathcal{A}_{V_0}$, $\ldots$, $\mathcal{A}_{V_n}$ are of true cardinality $\mathfrak{c}$, there are infinite many $\alpha < \mathfrak{c}$ such that $\mathbb{C}_\alpha = \{C_0, C_1, \ldots, C_n\}$ and for each $i \le n$ $C_i \subset V_i$ and, $|\mathcal{A}_{V_i} \restriction_{C_i}| = \mathfrak{c}$. For such pairs Subcase 2.2 applies and therefore $|B \cap \mathcal{F}| \ge \omega$. In addition, $A \cap B = \emptyset$: assume there is $a \in A \cap B$. Since $V_0 \cap \cdots \cap V_n = \emptyset$, $B \cap \omega = \emptyset$, hence $a \in \mathcal{F} \cap A \cap B$. By Observation 3.1.8, $|a \cap W| = |a \cap V_0| = \cdots = |a \cap V_n| = \omega$. This implies that $a \notin \mathcal{A}_W \cup \mathcal{A}_{V_0} \cup \cdots \cup \mathcal{A}_{V_n}$. There is $\alpha < \mathfrak{c}$ such that $a = \bigcup \mathcal{F}_\alpha$, but by the construction, $\mathcal{F}_\alpha$ intersects at most $n+1$ elements of $\{\mathcal{A}_W, \mathcal{A}_{V_0}, \ldots, \mathcal{A}_{V_n}\}$ which contradicts that a has infinite intersection with all W, V_0, $\ldots$, V_n. Whence, $A \cap B = \emptyset$.

It remains to show that A and B cannot be separated. Assume, on the contrary, that there are $S, T \subseteq \Psi(\mathcal{F})$ open such that $A \subseteq S$, $B \subseteq T$ and $S \cap T = \emptyset$. Let $\alpha < \mathfrak{c}$ such that $\mathbb{C}_\alpha = \{\omega \cap S, \omega \cap T\}$. For the pair $\mathbb{C}_\alpha$, either Case 1 or Case 2 of the construction holds.

If Case 1 holds: Since $W \subseteq \omega \cap S$, $\mathcal{A}_W \restriction_{(\omega \cap S)}$ is not finite. Hence, all $\mathcal{A}_W \restriction_{(\omega \cap T)}$, $\mathcal{A}_{V_0} \restriction_{(\omega \cap T)}$, $\ldots$, $\mathcal{A}_{V_n} \restriction_{(\omega \cap T)}$ are finite. Thus, $\mathcal{F} \restriction_{(\omega \cap T)}$ is finite. Since $cl_{\Psi(\mathcal{F})}(\omega \cap T)$ is regularly closed and $\mathcal{F} \restriction_{(\omega \cap T)}$ is finite, by Observation 3.1.8, $\mathcal{F} \cap cl_{\Psi(\mathcal{F})}(\omega \cap T)$ is finite. Now, since T is open, $T \subseteq cl_{\Psi(\mathcal{F})}(\omega \cap T)$. Hence, $\mathcal{F} \cap T$ is finite. Given that $|B \cap \mathcal{F}| \ge \omega$, $B \not\subseteq T$, which is a contradiction.

If Case 2 holds: Since $\mathcal{F}_\beta$ has non-empty intersection with at most $n+1$ elements of $\{\mathcal{A}_W$, $\mathcal{A}_{V_0}$, $\ldots$, $\mathcal{A}_{V_n}\}$, then $\bigcup \mathcal{F}_\alpha \in cl_{\Psi(\mathcal{F})}(W) = A$ (if Subcases

2.1 or 2.3 hold) or $\bigcup\mathcal{F}_\alpha \in \left(\bigcap_{i\leq n} cl_{\Psi(\mathcal{F})}(V_i)\right) = B$ (if Subcase 2.2 holds). In addition, there exist $a^S, e^T \in \mathcal{F}_\alpha$ such that $|a^S \cap (\omega\cap S)| = \omega = |e^T \cap (\omega\cap T)|$. If $\bigcup\mathcal{F}_\alpha \in A$, then (because of a^S) for each open neighbourhood U of $\bigcup\mathcal{F}_\alpha$, $U\cap T \neq \emptyset$ (which implies $U \not\subseteq S$), and this contradicts that S is open. We reach a similar contradiction if $\bigcup\mathcal{F}_\alpha \in B$. Hence, A and B cannot be separated. Whence, $\Psi(\mathcal{F})$ is not $(n+1)$-partly-normal.

$\Psi(\mathcal{F})$ is n-partly-normal:

Let $A, B \subset \Psi(\mathcal{F})$ be nonempty such that $A\cap B = \emptyset$, A is regularly closed and, $B = \bigcap_{j\leq i} B_j$, where each B_j is regularly closed and $1 \leq i \leq n$. It can be assumed that $|A\cap\omega| = \omega$ and that for each $j \leq i$, $|B_j \cap\omega| = \omega$. Fix $\alpha < \mathfrak{c}$ such that $\mathbb{C}_\alpha = \{A\cap\omega\} \cup \{B_j\cap\omega : j \leq i\}$. For $\mathbb{C}_\alpha$ either Case 1 or Case 2 holds. Case 2 is not possible since otherwise, $\mathcal{F}_\alpha = \{a^A, e^0, \ldots, e^i\}$, satisfies $|a^A \cap (A\cap\omega)| = |e^0 \cap (B_0\cap\omega)| = \cdots = |e^i \cap (B_i \cap\omega)| = \omega$. Thus, $\bigcup\mathcal{F}_\alpha \in cl_{\Psi(\mathcal{F})}(A\cap\omega)\cap\left(\bigcap_{j\leq i} cl_{\Psi(\mathcal{F})}(B_j\cap\omega)\right) \subseteq cl_{\Psi(\mathcal{F})}(A)\cap\left(\bigcap_{j\leq i} cl_{\Psi(\mathcal{F})}(B_j)\right) = A\cap B$. This contradicts $A\cap B = \emptyset$.

Thus, Case 1 holds. This means that there is some $C \in \mathbb{C}_\alpha$ such that all $n+2$ sets $\mathcal{A}_W\restriction_C$, $\mathcal{A}_{V_0}\restriction_C$, $\ldots$, $\mathcal{A}_{V_n}\restriction_C$ are finite.

If $C = A\cap\omega$, then $\mathcal{F}\restriction_{(A\cap\omega)}$ is finite. Since A is regularly closed, By Observation 3.1.8, $A\cap\mathcal{F}$ is finite and by Observation 3.1.9, A and B can be separated. Similarly, if $C = B_j\cap\omega$ for some $j \leq i$, it is also true that $\mathcal{F}\restriction_{(B_j\cap\omega)}$ is finite. Since B_j is regularly closed, $B_j\cap\mathcal{F}$ is finite. Given that $B \subseteq B_j$, $B\cap\mathcal{F}$ is finite as well and, therefore, A and B can be separated. Thus, $\Psi(\mathcal{F})$ is n-partly-normal. □

Similarly as Corollary 3.2.6, it also holds true:

Corollary 3.2.9. *If there exists a mad family of true cardinality $\mathfrak{c}$, then for each positive $n \in \omega$, there is a n-partly-normal not $(n+1)$-partly-normal mad family of true cardinality $\mathfrak{c}$.*

Corollary 3.2.4 says, in particular, that there is a quasi-normal mad family, provided there is a completely separable mad family. Example 3.2.10 below shows that, assuming **CH**, not only a quasi-normal mad family exists, but one that it is also Luzin (Definition 1.3.3). This is interesting since, as discussed in Section 1.3, Luzin families are far from being normal. Hence, even though no mad family is normal and no Luzin family is normal, there is, consistently, a quasi-normal Luzin mad family:

Example 3.2.10 (**CH**)**.** *There is a Luzin mad family $\mathcal{A}$ which is quasi-normal.*

Proof. The standard construction of a Luzin family is modified to build a family $\mathcal{A}$ with the extra following property: for each $X \subseteq \omega$, either X is covered by finitely many elements of $\mathcal{A}$ or the set of elements of $\mathcal{A}$ that has finite intersection with X is countable.

The idea is to use **CH** to list all infinite subsets $X_\alpha \subseteq \omega$, with $\alpha < \omega_1$ and, at stage $\alpha < \omega_1$ of the construction of the family, X_α will be covered by the α-th element of the family, together with finitely many elements of the family previously constructed or, if X_α has infinite intersection with infinitely many elements of the family constructed so far, it will be guaranteed that, from that stage until the end, all elements of the family will have infinite intersection with X_α.

Partition ω into infinite pairwise disjoint subsets a_i, with $i \in \omega$, that is

$\omega = \bigcup_{i\in\omega} a_i$, and $i \neq j$ implies $a_i \cap a_j = \emptyset$. List all infinite subsets of ω as $[\omega]^\omega = \{X_\alpha : \alpha < \omega_1\}$ such that for each $n \in \omega$, $X_n = a_n$. If α is such that $\omega \leq \alpha < \omega_1$, recursively assume we have constructed a_β for $\beta < \alpha$ such that $\{a_\beta : \beta < \alpha\}$ is an almost disjoint family and for each $\beta < \alpha$, X_β is covered by finitely many elements of $\{a_\gamma : \gamma \leq \beta\}$ or for each $\beta \leq \gamma < \alpha$, $|X_\beta \cap a_\gamma| = \omega$.

The α-th element of the family will be constructed. Reenumerate the sets $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\}$ and $J_\alpha = \{X_\beta : \beta \leq \alpha\}$ as $\mathcal{A}_\alpha = \{a_n^\alpha : n \in \omega\}$ and $J_\alpha = \{X_n^\alpha : n \in \omega\}$. Let $I_\alpha = \{n \in \omega : X_n^\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)\}$.

There are two cases, either $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$ or $X_\alpha \notin \mathcal{I}^+(\mathcal{A}_\alpha)$. We will construct a_α depending on whether at this stage, I_α is still empty or not.

If $I_\alpha = \emptyset$ (observe that in particular $X_\alpha \notin \mathcal{I}^+(\mathcal{A}_\alpha)$), let $p_n^\alpha \subseteq a_n^\alpha \setminus \bigcup_{i<n} a_i^\alpha$, such that $|p_n^\alpha| = n$ and let $a_\alpha = \bigcup_{n\in\omega} p_n^\alpha \cup (X_\alpha \setminus \bigcup(\mathcal{A}_\alpha \restriction_{X_\alpha}))$.

If $I_\alpha \neq \emptyset$. Let $\{Y_n : n \in \omega\}$ list all X_n^α such that $n \in I_\alpha$ and so that not only each X_n^α appears infinitely often but for each $n \in I_\alpha$ and for each $m \in \omega$, there is some $s \geq m$ such that $Y_s = X_n^\alpha$ and $|a_s^\alpha \cap Y_s| = \omega$. For $n \in \omega$, if $|a_n^\alpha \cap Y_n| < \omega$, let $p_n^\alpha \subseteq a_n^\alpha \setminus \bigcup_{i<n} a_i^\alpha$, such that $|p_n^\alpha| = n$. If $|a_n^\alpha \cap Y_n| = \omega$, let $p_n^\alpha \subseteq (a_n^\alpha \setminus \bigcup_{i<n} a_i^\alpha) \cap Y_n$ such that $|p_n^\alpha| = n$. Let $a_\alpha = \bigcup_{n\in\omega} p_n^\alpha \cup (X_\alpha \setminus \bigcup(\mathcal{A}_\alpha \restriction_{X_\alpha}))$. Observe that if $X_\alpha \notin \mathcal{I}^+(\mathcal{A}_\alpha)$, then the construction of a_α guarantees that X_α is covered by finitely many elements of $\mathcal{A}_\alpha \cup \{a_\alpha\}$. On the other hand, if $X_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, then X_α appears infinitely often in $\{Y_n : n \in \omega\}$, thus, it has infinite intersection with a_α and it will have infinite intersection with each a_β for each $\beta > \alpha$.

Finally, let $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$. The construction guarantees that $\mathcal{A}$ is Luzin: let $\alpha \in \omega_1$ and $n \in \omega$. Recall that $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\} = \{a_m^\alpha : m \in \omega\}$ and for each $m \geq n$, $p_m^\alpha \subseteq a_\alpha \cap a_m^\alpha$ and $|p_m^\alpha| = m \geq n$. Hence, $\{\beta < \alpha : a_\beta \cap a_\alpha \subseteq n\}$ is finite. Let us verify that it is mad. Let $\alpha, \beta \in \omega_1$ such that $\beta < \alpha$. There is $n \in \omega$ with $a_\beta = a_n^\alpha$. Observe that for $i \leq n$, p_i^α is finite, for $i > n$, $p_i^\alpha \cap a_\beta = \emptyset$ and, $(X_\alpha \setminus \bigcup(\mathcal{A}_\alpha \restriction_{X_\alpha})) \cap a_\beta$ is finite. Hence, $a_\beta \cap a_\alpha$ is finite. Now, let $X \in [\omega]^\omega$ and $\alpha < \omega_1$ such that $X = X_\alpha$. Either $X \notin \mathcal{I}^+(\mathcal{A}_\alpha)$, in which case X is covered by finitely many elements of $\mathcal{A}_\alpha \cup \{a_\alpha\}$ (i.e. X has infinite intersection with some element of $\mathcal{A}$), or $X \in \mathcal{I}^+(\mathcal{A}_\alpha)$, in which case for each $\gamma > \alpha$, $|X \cap a_\gamma| = \omega$. Thus, $\mathcal{A}$ is mad and it has the desired property.

Let us show that $\mathcal{A}$ is quasi-normal. Let $A, B \subseteq \Psi(\mathcal{A})$ such that A and B are π-closed sets and $A \cap B = \emptyset$. Thus, $A = \bigcap_{i<n} A_i$, $B = \bigcap_{j<m} B_j$, where each A_i, B_j are regular closed subsets of $\Psi(\mathcal{A})$ for $i < n$ and $j < m$. Assume that for each $i < n$ and for each $j < m$, $|A_i \cap \mathcal{A}| \geq \omega$ and $|B_j \cap \mathcal{A}| \geq \omega$. Hence, for each $i < n$ and for each $j < m$, $A_i \cap \omega \in \mathcal{I}^+(\mathcal{A})$ and $B_j \cap \omega \in \mathcal{I}^+(\mathcal{A})$. By the construction of $\mathcal{A}$, for each $i < n$ and for each $j < m$ the sets $\{a \in \mathcal{A} : |a \cap (A_i \cap \omega)| < \omega\}$ and $\{a \in \mathcal{A} : |a \cap (B_j \cap \omega)| < \omega\}$ are countable. Since the A_i's, B_j's are regular closed sets, then for each $i < n$ and for each $j < m$, $|\mathcal{A} \setminus A_i| \leq \omega$ and $|\mathcal{A} \setminus B_j| \leq \omega$. Thus, $|\mathcal{A} \setminus \bigcap_{i<n} A_i| \leq \omega$ and $|\mathcal{A} \setminus \bigcap_{j<m} B_j| \leq \omega$. Therefore, $A \cap B \neq \emptyset$. Hence, there exists some $i < n$ (or $j < m$), such that $|A_i \cap \mathcal{A}| < \omega$ ($|B_j \cap \mathcal{A}| < \omega$). Then $|A \cap \mathcal{A}| < \omega$ ($|B \cap \mathcal{A}| < \omega$) and, by Observation 3.1.9, A can be separated from B. $\square$

3.3 Strongly $\aleph_0$ separated almost disjoint families

It is still open whether there could be (e.g., assuming **CH**) a mad family whose Ψ-space is almost normal, or one whose Ψ-space is almost normal but not normal. However, we can construct a mad family with a slightly weaker property:

Definition 3.3.1. *An almost disjoint family $\mathcal{A}$ will be called* **strongly $\aleph_0$-separated***, if and only if for each pair of disjoint countable subfamilies there is a clopen partition of $\mathcal{A}$ that separates them. That is, for each $A, B \in [\mathcal{A}]^\omega$, with $A \cap B = \emptyset$, there is $X \subset \omega$ such that*

1. *For each $a \in \mathcal{A}$, $a \subseteq^* X$ or $a \cap X =^* \emptyset$,*
2. *For each $a \in A$, $a \subseteq^* X$,*
3. *For each $a \in B$, $a \cap X =^* \emptyset$.*

Lemma 3.3.2. *Almost-normal almost disjoint families are strongly $\aleph_0$-separated.*

Proof. Let $\mathcal{A}$ be an almost-normal almost disjoint family. First, let us recall that each pair of disjoint countable closed subsets of a regular space can be separated. Hence, given that $\Psi(\mathcal{A})$ is regular and $\mathcal{A}$ is a closed discrete subset of $\Psi(\mathcal{A})$, if we consider $A, B \in [\mathcal{A}]^\omega$ so that $A \cap B = \emptyset$, then A and B can be separated. Thus, there exist U_A, U_B open subsets of $\mathcal{A}$ such that $U_A \cap U_B = \emptyset$ and $A \subseteq U_A$, $B \subseteq U_B$. Let $C = cl_{\Psi(\mathcal{A})}(U_A \cap \omega)$. By Observation 3.1.7, C is a regular closed set. Then C and $\mathcal{A} \setminus C$ is a pair of a regular closed set and a closed set with empty intersection. Since $\mathcal{A}$ is almost-normal, there exist V, W open subsets of $\Psi(\mathcal{A})$ such that $V \cap W = \emptyset$ and $C \subseteq V$, $\mathcal{A} \setminus C \subseteq W$.

Let us check that $X = V \cap \omega$ has the desired properties. Indeed, let $a \in \mathcal{A}$, if $a \in C$, then $a \subseteq^* V \cap \omega = X$. If $a \in \mathcal{A} \setminus C$, then $a \subseteq^* W \cap \omega$, thus $a \cap X =^* \emptyset$. Now, if $a \in A$, $a \subseteq^* U_A \cap \omega \subseteq C \cap \omega \subseteq V \cap \omega = X$. If $b \in B$, $|b \cap U_A| < \omega$ thus, $b \in \mathcal{A} \setminus C$. Hence, $b \subseteq^* W$, i.e. $b \cap X =^* \emptyset$. Hence, $\mathcal{A}$ is strongly $\aleph_0$-separated. $\square$

Proposition 3.3.3 (CH). *There is a strongly $\aleph_0$-separated mad family.*

Proof. Let $\{(A_\beta, B_\beta) \in [\omega_1]^\omega \times [\omega_1]^\omega : \omega \leq \beta < \omega_1\}$ list all disjoint pairs of countable subsets of ω_1 in such a way that for each $\omega \leq \beta < \omega_1$, $A_\beta \cup B_\beta \subseteq \beta$. In addition, list $[\omega]^\omega$ as $\{Y_\alpha : \omega \leq \alpha < \omega_1\}$.

Let $\omega \leq \alpha < \omega_1$ and assume that for each $\omega \leq \beta < \alpha$, the sets X_β, $a_\beta \subset \omega$ have been defined such that:

1. For each $\gamma \in A_\beta : a_\gamma \subseteq^* X_\beta$,
2. For each $\gamma \in B_\beta : a_\gamma \cap X_\beta =^* \emptyset$,
3. For each $\gamma < \alpha : a_\gamma \subseteq^* X_\beta$ or $a_\gamma \cap X_\beta =^* \emptyset$,
4. If there is $\gamma < \beta$ such that $|a_\gamma \cap Y_\beta| = \omega$, then $a_\beta = \emptyset$. Otherwise, $|a_\beta \cap Y_\beta| = \omega$,
5. For each $\eta, \gamma < \alpha$, $a_\eta \cap a_\gamma =^* \emptyset$,

Let us construct X_α. List $\alpha \setminus B_\alpha$ and B_α as $\alpha \setminus B_\alpha = \{\gamma_n : n \in \omega\}$, $B_\alpha = \{\beta_n : n \in \omega\}$. Since $A_\alpha \cup B_\alpha \subseteq \alpha$, then $A_\alpha \subseteq \alpha \setminus B_\alpha$ and for each $n \in \omega$, $\gamma_n, \beta_n < \alpha$. That is, $a_{\gamma_n}, a_{\beta_n}$ have been defined. In addition, for $n \in \omega$, $W_n = a_{\gamma_n} \setminus [\bigcup_{j \leq n} a_{\beta_j}]$ is either empty of infinite. Define $X_\alpha = \bigcup_{n \in \omega} W_n$. Observe that (A_α, B_α) and X_α satisfy properties 1. and 2. of the recursive construction.

Now let us build a_α. Reenumerate $\{X_\beta : \beta < \alpha\} \cup \{X_\alpha\}$ as $\{X^n : n \in \omega\}$. For

$n \in \omega$, let $X_1^n = X^n$, $X_0^n = \omega \setminus X^n$. If there is $\gamma < \alpha$ such that $|a_\gamma \cap Y_\alpha| = \omega$, then let $a_\alpha = \emptyset$. On the other hand, if for each $\gamma < \alpha$, $|a_\gamma \cap Y_\alpha| < \omega$, for $n \in \omega$, pick $i(n) \in \{0, 1\}$ so that $Y_\alpha \cap \bigcap_{j \leq n} X_{i(j)}^j$ is infinite. For each $n \in \omega$, pick $p_n \in \left[Y_\alpha \cap \bigcap_{j \leq n} X_{i(j)}^j\right] \setminus \{p_j : j < n\}$. In this case, let $a_\alpha = \{p_n : n \in \omega\}$. Since $a_\alpha \subseteq Y_\alpha$, then for ech $\beta < \alpha$, $a_\beta \cap a_\alpha$ is finite.

This finishes the recursive construction of X_α and a_α. Regardless of whether a_α is empty or not, it satisfies properties 4. and 5. In addition, it holds true that for each $\gamma, \beta \leq \alpha$: $a_\gamma \subseteq^* X_\beta$ or $a_\gamma \cap X_\beta =^* \emptyset$. Thus, property 3. is satisfied. Let $\mathcal{A} = \{a_\alpha : \omega \leq \alpha < \omega_1 \ \textit{and} \ a_\alpha \neq \emptyset\}$. Observe that properties 4. and 5. guarantee that $\mathcal{A}$ is a mad family. Properties 1., 2. and 3. guarantee that $\mathcal{A}$ is strongly $\aleph_0$-separated. Hence, $\mathcal{A}$ is the desired family. □

The following questions are currently being studied[2]:

Question 3.3.4. *Is there a partly-normal not quasi-normal almost disjoint family?*

Question 3.3.5. *Is there an almost-normal not normal almost disjoint family?*

Question 3.3.6. *Is there an almost-normal mad family?*

If $\mathcal{A}$ is mad, $\Psi(\mathcal{A})$ is a pseudocompact and not countably compact space. Recall that normal pseudocompact spaces are countably compact and so it is natural to ask the following more general question

Question 3.3.7. *Are almost-normal pseudocompact spaces countably compact?*

[2]Recently, Vinicius de Oliveira and Victor dos Santos have answered in [66], consistently in the positive Question 3.3.5 and negatively Question 3.3.7

Since Ψ-spaces are always Tychonoff and not countably compact, the existence of an almost-normal mad family would answer this question in the negative. Finally, we have not considered the relationship between these weakenings of normality and countable paracompactness:

Question 3.3.8. *Is there a relationship between countably paracompact and any of these weakenings of normality?*

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