A barrier function for the regularity of the free boundary in $div(a(x)\nabla u) = -(h(x)\gamma)_x$ with $h_x < 0$

S. Challal

Department of mathematics, Glendon college, York university
2275 Bayview Ave. Toronto ON M4N 3M6 Canada

schallal@glendon.yorku.ca

Abstract

A barrier function $w = \dfrac{\lambda}{2}[(y - f(x))^+]^2$ is compared to the solution u near a free boundary point. The properties $div(a(x)\nabla u) \geqslant -(h)_x\chi([u > 0])$ and $\nabla w = 0$ on $[y = f(x)]$ avoided the comparison of the gradients of u and v as in the case $h_x \geqslant 0$. A regularity of the free boundary is established.

Key words : Variational methods, positive solutions, Linear elliptic equations.

AMS subject classification : 35A15, 35B09, 35J60.

1 Introduction

In the domain $\Omega = (0, 1) \times (0, 1)$, we consider the free boundary problem:

$$
(P)\begin{cases}
\text{Find } (u,\gamma)\in H^1(\Omega)\times L^\infty(\Omega) \text{ such that } :\\[2ex]
(i)\quad u\geqslant 0,\quad 0\leqslant\gamma\leqslant 1,\quad u(\gamma-1)=0 \ \text{ a.e. in } \Omega\\[2ex]
(ii)\quad u=\varphi \ \text{ on } \partial\Omega\\[2ex]
(iii)\quad \displaystyle\int_\Omega \big(a(x)\nabla u+\gamma h(x)e_x\big).\nabla\xi dxdy \;=\; 0 \qquad\qquad \forall\xi\in H_0^1(\Omega)
\end{cases}
$$

where $e_x=(1,0)$, $\varphi\in C^{0,1}(\overline{\Omega})$,

$$
\varphi(x,y)=\left\|\begin{array}{lll} 0 & \text{on} & \Gamma_0=(0,1)\times\{0\},\\[2ex] u_a & \text{on} & \Gamma_1=(0,1)\times\{1\},\end{array}\right. \quad \text{and} \quad \begin{array}{lll}\theta_0(y) & \text{on} & \{0\}\times[0,1]\\[2ex] \theta_1(y) & \text{on} & \{1\}\times[0,1]\end{array}
$$

with θ_i being regular and nondecreasing functions satisfying $0\leqslant\theta_i(y)\leqslant u_a$, $i=1,2$, and u_a is a positive constant.
The function h is $C^1([0,1])$ and satisfies for some positive constants $\overline{h}$ and h^*:

$$
0<h(x)\leqslant\overline{h},\qquad\qquad h'(x)\leqslant -h^*<0 \qquad \text{for} \quad x\in[0,1]. \tag{1.1}
$$

The matrix a depends only on the x-variable and satisfies:

$$
a\in W^{1,\infty}(0,1)\cap C^{0,1}[0,1] \tag{1.2}
$$
$$
m|\xi|^2\leqslant a_{ij}\xi_i\xi_j\leqslant M|\xi|^2 \qquad \forall\xi\in\mathbb{R}^2,\qquad m>0,\quad M>0. \tag{1.3}
$$

This problem has been studied by the author in [3] under higher regularity assumptions on the functions $h(x)$ and $a(x)$. The method used involved C^2 regularity of the solution u far from the free boundary.
In this paper, we will establish the regularity using comparison methods in the weak formulation of the problem. The motivation of this approach came from its application to similar problem with $h_x\geqslant 0$ in [6], [7] and for the study in a heterogeneous coastal aquifer in [8]. Another motivation (see [4], [10]), is the fact that the solution behaves like the solution of an obstacle problem because of the property

$$
div(a(x)\nabla u)\geqslant -(h)_x\chi([u>0]) \qquad \text{in} \quad \mathcal{D}'(\Omega), \tag{1.4}
$$

where $\chi=1$ if $x\in[u>0]$ and $\chi=0$ if not. The property (1.4) is established by taking $\pm(H_\epsilon(u)\xi)$, $\xi\in\mathcal{D}(\Omega)$, $\xi\geqslant 0$ as a test function in (P), with

$$
H_\epsilon(t)=\left\|\begin{array}{lll} 0 & \text{if} & t<0\\ t/\epsilon & \text{if} & 0\leqslant t\leqslant\epsilon\\ 1 & \text{if} & t>\epsilon\end{array}\right.
$$

This function involves in the penalization problem:

$$(P_\epsilon)\begin{cases} \text{Find } u_\epsilon \in H^1(\Omega) \text{ such that :} \\ \\ (i) \qquad u_\epsilon = \varphi \text{ on } \partial\Omega \\ \\ (ii) \quad \displaystyle\int_\Omega \big(a(x)\nabla u_\epsilon + h(x)H_\epsilon(u_\epsilon)e_x\big)\nabla\xi dxdy = 0 \qquad \forall \xi \in H_0^1(\Omega) \end{cases}$$

where, for $\epsilon \in (0, \min(1, u_a))$, we establish, as in [5], that there exists a unique solution for (P_ϵ) satisfying:

$$u_\epsilon \rightharpoonup u \quad \text{in } H^1(\Omega), \qquad\qquad H_\epsilon(u_\epsilon) \rightharpoonup \gamma \quad \text{in } L^2(\Omega)$$

and that (u, γ) is a solution of (P).

Taking u_ϵ^- (resp. $(u_\epsilon - u_a)^+$) as a test function in (P_ϵ), shows that $u_\epsilon \geqslant 0$ (resp. $u_\epsilon \leqslant u_a$). Then, comparing $u_\epsilon^\eta = u_\epsilon(x, y + \eta)$ with u_ϵ as in [8], we obtain $(u_\epsilon)_y \geqslant 0$ and finally get

$$0 \leqslant u \leqslant u_a, \qquad\qquad \frac{\partial u}{\partial y} \geqslant 0 \qquad \text{a.e in } \Omega. \tag{1.5}$$

This work brings answers to the situation where the function h is not increasing. Applications of such a model appear, for example, in the Lubrication problem [2] and the dam problem [1].

In all what follows, we consider only monotone solutions of (P).
As a consequence, we deduce that [5]:

- $\forall (x_0, y_0) \in [u > 0] = [u(x, y) > 0] \cap \Omega, \quad \exists \delta > 0$ such that

$$u(x, y) > 0 \qquad\qquad \text{for } (x, y) \in B_\delta(x_0, y_0) \cup (x_0 - \delta, x_0 + \delta) \times [y_0, 1]$$

- $\Phi : (0, 1) \longrightarrow [0, 1)$ is well defined by

$$\Phi(x) = \inf\{y \in (0, 1) \,/ \quad u(x, y) > 0\}$$

  and is upper semi-continuous (u.s.c) on $(0, 1)$.

- $[u > 0] = [y > \Phi(x)]$.

We recall also some properties:

- $div(a(x)\nabla u) = -(h\gamma)_x$ in $\mathcal{D}'(\Omega)$.

- $u \in C^{0,\alpha}_{loc}(\Omega \cup \Gamma_0 \cup \Gamma_1)$ ([9] Theorem 8.24 p 202).
- $[u > 0]$ is an open set.
- $div(a(x)\nabla u) \geqslant 0$ and $(h\gamma)_x \leqslant 0$ in $\mathcal{D}'(\Omega)$.

2 A Barrier Function and Comparison

Let $x_0 \in (0,1)$ and $\epsilon_0 = \min(x_0, 1-x_0)/6$. Then

$$u \in C^{0,\alpha}\big([x_0-\epsilon_0, x_0+\epsilon_0] \times [0,1]\big) \qquad \text{for some } \alpha \in (0,1). \tag{2.1}$$

Assume that there exists $\epsilon_1 > 0$ such that

$$\forall \epsilon \in (0, \epsilon_1), \qquad \underline{y} - 3\epsilon > 0,$$

where $\underline{y} \in (0,1)$ and $\underline{y}$ may depend on ϵ.
Let $x_1 \in (0,1)$ satisfying

$$x_1 < x_0 \quad \text{and} \quad |x_1 - x_0| < \epsilon^{3/\alpha} \quad \text{with} \quad \epsilon \in \big(0, \min(\epsilon_0, \epsilon_1)\big).$$

Set

$$Z = (x_1, x_0) \times (\underline{y} - \epsilon, \underline{y}), \qquad D = (x_1, x_0) \times (0, \underline{y}),$$

$$w(x,y) = \frac{\lambda}{2}[(y - f(x))^+]^2,$$

$$f(x) = \underline{y} - \epsilon + \int_{x_1}^{x} \Big[\frac{a_{12}(s) + C_0}{a_{11}(s)}\Big] ds,$$

where C_0 and λ are constants. We choose $C_0 = 2M$ so that

$$1 = \frac{-M+2M}{M} \leqslant \frac{a_{12}(x) + C_0}{a_{11}(x)} \leqslant \frac{3M}{m} \qquad \forall x \in [x_0 - \epsilon_0, x_0 + \epsilon_0].$$

Then f satisfies

$$
\begin{aligned}
&f'(x) = \frac{a_{12}(x)+C_0}{a_{11}(x)} \geqslant 1 > 0 \qquad \text{and} \qquad - a_{11}(x)f'(x) + a_{12}(x) = -C_0,\\
&f(x_1) = \underline{y} - \epsilon\\
&f(x_0) = \underline{y} - \epsilon + \int_{x_1}^{x_0} \Big[\frac{a_{12}(s)+C_0}{a_{11}(s)}\Big] ds \leqslant \underline{y} - \epsilon + \frac{3M}{m}(x_0 - x_1)\\
&\qquad\quad \leqslant \underline{y} - \epsilon + \frac{3M}{m}\epsilon^{3/\alpha} < \underline{y} \qquad \text{if} \qquad \epsilon \in \Big(0, \Big(\frac{m}{3M}\Big)^{\frac{\alpha}{3-\alpha}}\Big).
\end{aligned}
$$

Note that $[y > f(x)] \cap Z \neq \emptyset$ since we have for $\epsilon \in \Big(0, \Big(\frac{m}{6M}\Big)^{\frac{\alpha}{3-\alpha}}\Big) = (0, \epsilon_2)$

$$
\frac{\epsilon}{2} < \epsilon - \frac{3M}{m}\epsilon^{3/\alpha} \leqslant \underline{y} - f(x_0) \leqslant \underline{y} - f(x) \leqslant \underline{y} - f(x_1) = \epsilon \qquad \forall x \in [x_1, x_0].
$$

Next, we have $w \in H^1(D)$ and

$$
\begin{aligned}
&\nabla w(x,y) = \lambda(y - f(x)) \begin{bmatrix} -f'(x) \\ 1 \end{bmatrix} \chi([y > f(x)])\\
&a(x)\nabla w = \lambda(y - f(x))^+ \begin{bmatrix} -f'(x)a_{11} + a_{12} \\ -f'(x)a_{21} + a_{22} \end{bmatrix}\\
&div(a(x)\nabla w) = \lambda a(x) \begin{bmatrix} -f'(x) \\ 1 \end{bmatrix} . \begin{bmatrix} -f'(x) \\ 1 \end{bmatrix} \qquad \text{in } [y > f(x)]\\
&\lambda m[1 + (f'(x))^2] \leqslant div(a(x)\nabla w) \leqslant \lambda M[1 + (f'(x))^2] \qquad \text{in } [y > f(x)].
\end{aligned}
$$

Lemma 2.1. *Let $\epsilon \in \big(0, \min(\epsilon_0, \epsilon_1, \epsilon_2)\big)$ and $\theta > 0$. Then, there exists $\lambda_0 > 0$ independent of ϵ such that: $\forall \lambda \in (0, \lambda_0)$, we have*

$$
\int_D \big[a(x)\nabla w.\nabla\zeta + \theta\chi([w > 0])\zeta\big] dxdy \geqslant 0 \qquad \forall \zeta \in H_0^1(D), \quad \zeta \geqslant 0.
$$

Proof. Let $\zeta \in H_0^1(D)$, $\zeta \geqslant 0$. We have

$$
\begin{aligned}
&\int_D \big[a(x)\nabla w.\nabla\zeta + \theta\chi([w > 0])\zeta\big] dxdy\\
&= \int_{[y>f(x)]} \big[-div(a(x)\nabla w) + \theta\big]\zeta dxdy + \int_{[y=f(x)]} \big[a(x)\nabla w.\nu\big]\zeta
\end{aligned}
$$

where $\nu = \langle \nu_x, \nu_y \rangle = \dfrac{1}{\sqrt{1+f'^2(x)}} \langle f'(x), -1 \rangle$ is the unit normal to the curve $[y = f(x)]$ pointing towards the set $[y < f(x)]$.
Since $\nabla w = 0$ on $[y = f(x)]$, we have

$$\int_D \big[a(x)\nabla w.\nabla \zeta + \theta \chi([w>0])\zeta\big] dxdy = \int_{[y>f(x)]} \big[-div(a(x)\nabla w) + \theta\big]\zeta dxdy$$
$$\geqslant \int_{[y>f(x)]} \big[-\lambda M(1+\Big(\frac{3M}{m}\Big)^2) + \theta\big]\zeta dxdy$$

We conclude the lemma by choosing $\lambda_0 \in \Bigg(0, \dfrac{\theta}{M(1+\big(\frac{3M}{m}\big)^2)}\Bigg)$. $\square$

The following lemma compares u and w near a free boundary point, along the x direction. First, from (2.1), there exists a positive constant K^* such that for $(x,y),(x',y') \in [x_0 - \epsilon_0, x_0 + \epsilon_0] \times [0,1]$,

$$|u(x,y) - u(x',y')| \leqslant K^*\big(|x-x'|^2 + |y-y'|^2\big)^{\frac{\alpha}{2}}. \tag{2.2}$$

Lemma 2.2. *Assume that $u(x_0, \underline{y}) = 0$. Then for $\epsilon \in \big(0, \min(\frac{\lambda}{8K^*}, \epsilon_0, \epsilon_1, \epsilon_2)\big)$, we have*

$$u(x,\underline{y}) \leqslant w(x,\underline{y}) \qquad\qquad \forall x \in [x_1, x_0].$$

Proof. Since $u(x_0,\underline{y}) = 0$, we have by (2.2) in particular

$$u(x,\underline{y}) \leqslant K^*|x-x_0|^\alpha < K^*(\epsilon^{3/\alpha})^\alpha \qquad \text{for all } x \in [x_0 - \epsilon^{3/\alpha}, x_0 + \epsilon^{3/\alpha}].$$

On another hand, we have for $x \in [x_1, x_0]$,

$$w(x,\underline{y}) = \frac{\lambda}{2}[(\underline{y} - f(x))^+]^2 \geqslant \frac{\lambda}{2}[(\underline{y} - f(x_2))]^2 \geqslant \frac{\lambda}{2}\Big(\frac{\epsilon}{2}\Big)^2.$$

Hence,

$$u(x,\underline{y}) \leqslant K^*(\epsilon^{3/\alpha})^\alpha \leqslant \lambda\frac{\epsilon^2}{8} \leqslant w(x,\underline{y}) \qquad \Longleftrightarrow \qquad \epsilon \leqslant \frac{\lambda}{8K^*}.\square$$

3 Non Oscillation Lemma

The following Lemma shows that we cannot have a vertical segment where $u = 0$ without having $u = 0$ at some point to the left of the segment. When, we have a point to the right of the segment, where $u = 0$, we have necessarily another point to the left. This property is needed when proving the continuity of the free boundary.

Lemma 3.1. *Let u be a solution of (P). Let $(x_0, y_0) \in \Omega$ and $r > 0$ such that $B_r = B_r(x_0, y_0) \subset\subset \Omega$. Then we cannot have the following situations:*

$$\begin{array}{llllllll} i) & u > 0 & in & B_r \setminus S_r & and & u = 0 & in & S_r \\ ii) & u > 0 & in & B_r^- & and & u = 0 & in & B_r^+ \cup S_r \end{array}$$

with $S_r = \{x_0\} \times (y_0 - r, y_0 + r)$, $\quad B_r^- = B_r \cap [x < x_0]$ *and* $B_r^+ = B_r \cap [x > x_0]$.

Proof. **Case i).** Since $u > 0$ in $B_r \setminus S$ and $|S| = 0$, then $\gamma = 1$ a.e in B_r. We have

$$\int_\Omega \big(a(x)\nabla u + h(x)e_x\big).\nabla\xi dxdy \;=\; 0 \qquad\qquad \forall \xi \in H_0^1(\Omega)$$

Let $\eta \in (0, r/4)$ and $u_\eta(x, y) = u(x, y - \eta)$. Then, u_η satisfies

$$\int_{B_{r/4}} a(x)\nabla u_\eta.\nabla\xi(x, y)dxdy \;=\; \int_{B_{r/4}-\eta e_y} a(x')\nabla u(x', y').\nabla\xi(x', y' + \eta)dx'dy'$$

$$= -\int_{B_{r/4}-\eta e_y} h(x')\xi_{x'}(x', y' + \eta)dx'dy' = -\int_{B_{r/4}} h(x)\xi_x(x, y)dxdy$$

$$= \int_{B_{r/4}} a(x)\nabla u.\nabla\xi(x, y)dxdy \qquad\qquad \forall \xi \in \mathcal{D}(B_{r/4}).$$

We deduce that

$$\left\|\begin{array}{lll} div(a(x)\nabla(u - u_\eta)) = 0 & \text{in} & B_{r/4} \\ u - u_\eta \geqslant 0 & \text{in} & B_{r/4} \\ u - u_\eta = 0 & \text{on} & S_{r/4} \end{array}\right.$$

By the strong maximum principle, we deduce that $u - u_\eta = 0$ in $B_{r/4}$.
Consequently, $\dfrac{\partial u}{\partial y} = 0$ in $\mathcal{D}'(B_{r/4})$ which leads to $u(x, y) = \tau(x)$ in $B_{r/4}$.

Now, we have

$$div(a(x)\nabla u) = div(a(x)\nabla \tau) = -h'(x) \qquad \text{in} \quad B_{r/4}.$$

Using the monotony of u, we have, in particular

$$\left\| \begin{array}{c} div(a(x)\nabla(u-\tau)) = 0 \qquad \text{in} \quad D_{r/4} = (x_0 - \frac{r}{4}, x_0 + \frac{r}{4}) \times (y_0, 1) \\ u - \tau \geqslant 0 \qquad \text{in} \quad D_{r/4} \\ u - \tau = 0 \qquad \text{in} \quad B_{r/4} \cap D_{r/4}. \end{array} \right.$$

Applying the strong maximum principle, we deduce that: $u - \tau = 0$ in $D_{r/4}$.
The continuity of u up to Γ_1 implies that $\tau = 1$ for $x \in (x_0 - \frac{r}{4}, x_0 + \frac{r}{4})$. This contradicts $div(a(x)\nabla\tau) = -h'(x) > 0$.

Case ii). Because $\quad u = 0 \quad$ in $B_r \cap [x \geqslant x_0]$, we will have

$$(h(x)\gamma)_x = -div(a(x)\nabla u) = 0 \quad \Longrightarrow \quad h(x)\gamma = L(y) \quad \text{a.e. in } B_r^+.$$

Therefore, for $\xi \in \mathcal{D}(B_r)$,

$$\int_{B_r} a(x)\nabla u.\nabla\xi(x,y)dxdy \ = \ -\int_{B_r^-} h(x)\xi_x(x,y) + \int_{y_0-r}^{y_0+r} L(y)\xi(x_0,y)dy.$$

For u_η, as defined in case i), we have for $\xi \in \mathcal{D}(B_{r/4})$ and $\xi \geqslant 0$,

$$\int_{B_{r/4}} a(x)\nabla u_\eta.\nabla\xi(x,y)dxdy \ = \ -\int_{B_{r/4}^-} h(x)\xi_x(x,y) + \int_{y_0-r}^{y_0+r} L(y-\eta)\xi(x_0,y)dy.$$

Then, we obtain

$$\int_{B_{r/4}} a(x)\nabla(u-u_\eta).\nabla\xi(x,y)dxdy \ = \ \int_{y_0-r}^{y_0+r} (L(y) - L(y-\eta))\xi(x_0,y)dy.$$

Thus

$$\left\| \begin{array}{c} div(a(x)\nabla(u-u_\eta)) \leqslant 0 \qquad \text{in} \quad B_{r/4} \\ u - u_\eta \geqslant 0 \qquad \text{in} \quad B_{r/4} \\ u - u_\eta = 0 \qquad \text{in} \quad B_{r/4}^+ \end{array} \right.$$

since we have $L_y \geqslant 0$. Indeed, we have $(u_\epsilon)_y \geqslant 0$, $(H_\epsilon)'(t) \geqslant 0$,

$$L_y = (h(x)\gamma)_y = h(x)\gamma_y \quad \text{in } \mathcal{D}'(B_r^+), \quad \text{and}$$

$$(H_\epsilon(u_\epsilon))_y = H'_\epsilon(u_\epsilon)).(u_\epsilon)_y \longrightarrow \gamma_y \quad \text{in } \mathcal{D}'(\Omega).$$

As in case i), we conclude that $u = \tau(x)$ and get a contradiction. $\square$

Arguing as in the previous proof, we establish the following property of the free boundary:

Lemma 3.2. *The set* $[y = \Phi(x)] \cap \Omega$ *has no horizontal segment.*

Proof. Assume that there exist $x_1, x_2 \in (0,1)$ such that

$$x_1 < x_2 \qquad \text{and} \qquad \Phi(x) = y_0 \qquad \forall x \in [x_1, x_2].$$

Using the monotony of u, we have

$$u > 0 \quad \text{on } (x_1, x_2) \times (y_0, 1] \qquad \text{and} \qquad u = 0 \quad \text{on } (x_1, x_2) \times (0, y_0]$$

from which we deduce that $h(x)\gamma = L(y)$ a.e in $(x_1, x_2) \times (0, y_0)$.
Let $x_0 \in (x_1, x_2)$ and let $r > 0$ such that $\Delta_r = (x_0 - r, x_0 + r) \times (y_0 - r, y_0 + r) \subset (x_1, x_2) \times (0, 1)$. For $\eta \in (0, r/4)$, define $u_\eta(x, y) = u(x, y - \eta)$. Then, for $\xi \in \mathcal{D}(\Delta_{r/2})$, we have

$$\int_{\Delta_{r/2}} a(x)\nabla(u - u_\eta).\nabla\xi(x,y)dxdy = \int_{\Delta_{r/2}\cap[y<y_0]} (L(y-\eta) - L(y))\xi_x dxdy$$

$$= \int_{y_0 - r/2}^{y_0} (L(y-\eta) - L(y))\Big[\xi(x_0 + r/2, y) - \xi(x_0 - r/2, y)\Big]dy = 0.$$

Thus

$$\left\| \begin{array}{c} div(a(x)\nabla(u - u_\eta)) = 0 \qquad \text{in} \quad \Delta_{r/2} \\ u - u_\eta \geqslant 0 \qquad \text{in} \quad \Delta_{r/2} \\ u - u_\eta = 0 \qquad \text{in} \quad \Delta_{r/2} \cap [y < y_0] \end{array} \right.$$

and we get a contradiction as in the proof of the previous lemma. $\square$

4 Regularity of the free boundary

The main result of this section is the following theorem on the regularity of the free boundary.

Theorem 4.1. Φ *is either continuous or left continuous on a point in* $(0,1) \cap [\Phi(x) > 0]$.

Proof. We will adopt the notations in section 2.
Let $(x_0, \Phi(x_0)) = (x_0, y_0) \in \Omega$ with $\Phi(x_0) > 0$. Set $\epsilon_0 = \min(x_0, 1 - x_0)/6$.
Using (2.2), for $\epsilon \in (0, \epsilon_0)$, there exists $\delta_1 \in (0, \epsilon^{3/\alpha})$ such that:

$$u(x, y) = |u(x, y) - u(x_0, y_0)| \leqslant K^* \epsilon^3 \qquad \forall (x, y) \in B_{\delta_1}(x_0, y_0).$$

1^{st} **situation.** In applying the oscillation Lemma 3.1 i), assume that

$$\exists (x_0^-, y_0^-) \in B_{\delta_1}(x_0, y_0), \qquad u(x_0^-, y_0^-) = 0, \qquad x_0^- < x_0, \qquad |x_0^- - x_0| < \delta_1 < \epsilon^{3/\alpha}.$$

Set

$$x_1 = x_0^-, \qquad\qquad \underline{y} = \min(\Phi(x_0), y_0^-)$$

$$Z = (x_1, x_0) \times (\underline{y} - \epsilon, \underline{y}), \qquad D = (x_1, x_0) \times (0, \underline{y}),$$

$$w(x, y) = \frac{\lambda}{2}[(y - f(x))^+]^2, \qquad f(x) = \underline{y} - \epsilon + \int_{x_1}^{x} \Big[\frac{a_{12}(s) + C_0}{a_{11}(s)}\Big] ds.$$

We deduce from Lemma 2.2 that $u(x, \underline{y}) \leqslant w(x, \underline{y})$ and that $(u - w)^+ \in H_0^1(D)$. We can write then

$$\int_D a(x)\nabla u.\nabla(u - w)^+ dxdy \;\leqslant\; \int_D h'(x)\chi([u > 0])(u - w)^+ dxdy$$

$$\int_D a(x)\nabla w.\nabla(u - w)^+ dxdy \;\geqslant\; \int_D -\theta\chi([w > 0])(u - w)^+ dxdy$$

Subtracting the two inequalities, we obtain

$$\begin{aligned}
&\int_D a(x)\nabla(u - w)^+.\nabla(u - w)^+ dxdy \\
&\leqslant \int_{D\cap[w=0]} h'(x)\chi([u > 0])(u - w)^+ dxdy \;+\; \int_{D\cap[w>0]} (h'(x) + \theta)(u - w)^+ dxdy \\
&\leqslant \int_{D\cap[w>0]} (-h^* + \theta)(u - w)^+ dxdy
\end{aligned}$$

since $h'(x) \leqslant -h^* < 0$ on $(0, 1)$.
First, we choose $\theta \leqslant h^*$. Then, we use the ellipticity of the matrix $a(x)$ and apply Poincaré's inequality to conclude that $(u - w)^+ = 0$ in D. In particular, we have $u(x, \underline{y} -$

$\epsilon) = 0$; that is the free boundary remains above the horizontal line segment $(x_0^-, x_0) \times \{\underline{y} - \epsilon\}$ which we can express by writing that:

$$\Phi(x) \geqslant \underline{y} - \epsilon \geqslant \Phi(x_0) - \epsilon^{3/\alpha} - \epsilon \qquad \forall x \in (x_0^-, x_0). \tag{4.1}$$

We conclude from (4.1) that Φ is lower semi continuous at x_0 at the left. Therefore Φ is continuous to the left at this point since it is u.s.c at x_0.

2^{nd} **situation.** Using oscillation Lemma 3.1 i), we assume this time

$$\exists (x_0^+, y_0^+) \in B_{\delta_1}(x_0, y_0), \qquad u(x_0^+, y_0^+) = 0, \qquad x_0^+ > x_0, \qquad |x_0^+ - x_0| < \delta_1 < \epsilon^{3/\alpha}.$$

Set

$$x_1 = x_0, \qquad x_0' = x_0^+, \qquad \underline{y} = \min(\Phi(x_0), y_0^+)$$

$$Z = (x_1, x_0') \times (\underline{y} - \epsilon, \underline{y}), \qquad D = (x_1, x_0') \times (0, \underline{y}),$$

$$w(x,y) = \frac{\lambda}{2}[(y - f(x))^+]^2, \qquad f(x) = \underline{y} - \epsilon + \int_{x_1}^{x} \Big[\frac{a_{12}(s) + C_0}{a_{11}(s)}\Big] ds.$$

Arguing as in the first situation, we deduce that $u(x, \underline{y} - \epsilon) = 0$ and

$$\Phi(x) \geqslant \underline{y} - \epsilon \geqslant \Phi(x_0) - \epsilon^{3/\alpha} - \epsilon \qquad \forall x \in (x_1, x_0') = (x_0, x_0^+). \tag{4.2}$$

We conclude from (4.2) that Φ is lower semi continuous at x_0 to the right. So Φ is continuous at x_0 to the right.

Now, by Lemma 3.1 ii), we cannot have $u > 0$ to the left of the vertical line $\{x_0\} \times (0, \underline{y} - \epsilon)$. Therefore,

$$\exists (x_0^-, y_0^-) \in B_{\delta_1}(x_0, \underline{y} - \epsilon) \cap [y < \underline{y} - \epsilon], \quad u(x_0^-, y_0^-) = 0, \quad x_0^- < x_0, \quad |x_0^- - x_0| < \delta_1 < \epsilon^{3/\alpha}.$$

Set

$$x_1 = x_0^-, \qquad \underline{y}' = y_0^-$$

$$Z = (x_1, x_0) \times (\underline{y}' - \epsilon, \underline{y}'), \qquad D = (x_1, x_0) \times (0, \underline{y}'),$$

$$w(x,y) = \frac{\lambda}{2}[(y - f(x))^+]^2, \qquad f(x) = \underline{y}' - \epsilon + \int_{x_1}^{x} \Big[\frac{a_{12}(s) + C_0}{a_{11}(s)}\Big] ds.$$

We then have $u(x, \underline{y}' - \epsilon) = 0$ and Φ is lower continuous to the left of x_0. Indeed, we have

$$\Phi(x) \geqslant \underline{y}' - \epsilon \geqslant \underline{y} - \epsilon - \epsilon^{3/\alpha} - \epsilon \geqslant \Phi(x_0) - 2\epsilon^{3/\alpha} - 2\epsilon \qquad \forall x \in (x_1, x_0') = (x_0^-, x_0). \quad (4.3)$$

Finally we conclude from (4.3) that Φ is lower semi continuous at x_0 to the left. Therefore Φ is continuous at this point.□

Remark 4.1. *We weren't able to establish the continuity of Φ completely at any point in $(0,1) \cap [\Phi(x) > 0]$. Indeed, if the situation*

$$iii) \qquad u > 0 \quad in \quad B_r^+ \qquad and \qquad u = 0 \quad in \quad B_r^- \cup S_r$$

couldn't occur, this would complete the proof in situation 1 and show that Φ is continuous at x_0. Our attempt to establish iii) by similar arguments as in the proof of i) and ii) in Lemma 3.1 led to

$$\left\| \begin{array}{ll} div(a(x)\nabla(u - u_\eta)) \geqslant 0 & in \quad B_{r/4} \\ u - u_\eta \geqslant 0 & in \quad B_{r/4} \\ u - u_\eta = 0 & in \quad B_{r/4}^- \end{array} \right.$$

which prevented us to apply the maximum principle.

References

[1] H.W. Alt : The fluid flow Through Porous Media. Regularity of the free Surface. Manuscripta Math. 21, (1977), 255-272.

[2] Guy Bayada & Michéle Chambat: Nonlinear variational formulation for a cavitation problem in lubrication. Journal of Mathematical Analysis and Applications Volume 90, Issue 2, 286-298, (1982).

[3] S. Challal: Regularity of the Free Boundary in $div(a(x)\nabla u(x,y)) = -(h(x)\gamma(u))_x$ with $h'(x) < 0$. http://hdl.handle.net/10315/38566

[4] S. Challal: Continuity of the free boundary in a non-degenerate p-obstacle problem type with monotone solution. Applicable Analysis vol. 92, no. 7, $1462 - 1473$ (2013).

[5] M. Chipot: Variational Inequalities and Flow in Porous Media. Springer-Verlag New York Inc (1984).

[6] M. Chipot: On the Continuity of the Free Boundary in some Class of Dimensional Problems. Interfaces Free Bound. Vol. 3, No. 1, 81-99, (2001).

[7] S. Challal & A. Lyaghfouri: Continuity of the Free Boundary in Problems of type $div(a(x)\nabla u) = -(\chi(u)h(x))_{x_1}$. Nonlinear Analysis : Theory, Methods & Applications, Vol. 62, No. 2, 283-300 (2005).

[8] S. Challal & A. Lyaghfouri: A stationary flow of fresh and salt groundwater in a heterogeneous coastal aquifer. Bollettino della Unione Matematica Italiana, (8) no 2, 505-533 (2000).

[9] D. Gilbarg & N.S. Trudinger: Elliptic Partial Differential Equations of Second Order. Springer-Verlag (1983).

[10] J. F. Rodrigues: Obstacle problems in Mathematical physics. North Holland (1987).