RESULTS ON R-DIAGONAL OPERATORS IN BI-FREE PROBABILITY THEORY AND APPLICATIONS OF SET THEORY TO OPERATOR ALGEBRAS

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Abstract

The contents of this dissertation lie in the branch of pure mathematics known as functional analysis and are focused on the theory of bi-free probability and on the interplay between set theory and the field of operator algebras. The material is comprised of two main parts.

The first part of this dissertation investigates applications of set theory to operator algebras and is further divided into two chapters. The first chapter is focused on the Calkin algebra $\mathcal{Q}(H)$ and explores the class of C*-algebras which embed into it. We prove that under Martin's axiom every C*-algebra of density character less than $2^{\aleph_0}$ embeds into the Calkin algebra and, moreover, we show that the assertion "*every* C**-algebra of density character less than* $2^{\aleph_0}$ *embeds into* $\mathcal{Q}(H)$" is independent from ZFC. In the second chapter we investigate separably representable AF operator algebras from a descriptive set-theoretic viewpoint. Contrary to the case of separable AF C*-algebras which are classified up to isomorphism by the ordered K_0 groups, we show that the canonical isomorphism relations for separable, non-self-adjoint AF operator algebras are not classifiable by countable structures.

The second part of this dissertation focuses on the theory of bi-free probability. This part is further divided into four chapters, the first of which concerns the development of the theory of R-diagonal operators in the setting of bi-free probability theory. We define bi-R-diagonal pairs based on certain alternating cumulant conditions and give a complete description of their joint distributions in terms of their invariance under multiplication by bi-Haar unitary pairs. The final three chapters of this manuscript concern the development of non-microstate bi-free Fisher information and entropy with respect to completely positive maps. By extending the operator-valued bi-free structures and allowing the implementation of completely positive maps into bi-free conjugate variable expressions, we define notions of Fisher information and entropy which generalize the corresponding notions of entropy in the bi-free setting. As an application we show that minimal values of the bi-free Fisher information and maximal values of the non-microstates bi-free entropy are attained at bi-R-diagonal pairs of operators.

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Introduction

The study of operator algebras was initiated by Murray and von Neumann in the 1930's, who were aiming to provide the rigorous mathematical foundations to the developing theory of quantum mechanics. A key idea in their considerations renders algebras of operators as the canonical analogues of well-studied mathematical structures in the non-commutative world, such as topological and measure spaces. Since the pioneering work of Murray and von Neumann, the theory of operator algebras has evolved into a rich, independent research area in pure mathematics, with significant connections with multiple other mathematical areas, such as algebraic topology, group theory, probability theory and set theory. The key structures in this mathematical field consist of C*-algebras and von Neumann algebras. On the one hand, C*-algebras are given by algebras of bounded, linear operators on Hilbert spaces, closed in the norm topology and under the adjoint operation. In the abelian case, C*-algebras are classified by their spectra. Notably, the Gelfand-Naimark duality provides an equivalence between the category of unital, abelian C*-algebras and the category of compact, Hausdorff topological spaces and, under this lens, the theory of C*-algebras constitutes the non-commutative analogue of the theory of topological spaces. On the other hand, von Neumann algebras are defined similarly to C*-algebras as self-adjoint algebras of operators on Hilbert spaces and are required to be closed in the topology of pointwise convergence. Abelian von Neumann algebras acting on separable Hilbert spaces arise in the form $L_\infty(X, \mu)$ where X is a second countable, compact Hausdorff space and μ is a positive Borel measure on X, and, as a result, one views the theory of von Neumann algebras as the non-commutative analogue of measure theory.

Free probability theory was developed by Voiculescu in [77] as an extension of classical probability theory to the non-commutative setting, in order to tackle the (still open) problem of whether the free group factors $\mathcal{L}(\mathbb{F}_n)$ (which are the von Neumann algebras generated via the left regular representations of free groups on Hilbert spaces) are isomorphic. In this context, random variables consist of operators in C*-algebras or von Neumann algebras and their distribution is computed via the expectation given by states (i.e. positive linear functionals) on these algebras, parallel to the notion of expectation given by integration in classical probability. The advancement of free probability theory, particularly in conjunction with the developments of Fisher information and entropy in this non-commutative context, has offered a plethora of applications to the field of operator algebras. The key notion of independence here is that of free independence, which is modelled by the free product of algebras, replacing the model of tensor product corresponding to the commutative, classical setting. Operators whose distribution carry significant importance in free probability include the Haar unitary operators, which are unitary operators distributed according to the Haar measure on the circle and correspond to the canonical generators of the free group factors, and the semicircular operators, which are self-adjoint operators viewed as the analogue of the Gaussian random variables in the context of free probability. Significant depth was added to the theory of free probability with the development of the free cumulants by Speicher in [70], which consist of functionals defined with the aid of the combinatorial structure of the lattice of non-crossing partitions. The free cumulants express the distributions of non-commutative random variables and characterize the notion of free independence.

The recently developed theory of bi-free probability theory originated by Voiculescu in [82] as an extension of the free setting and involves the simultaneous study of left and right actions of algebras on reduced free product spaces. Here, the main objects of study are pairs of operators or algebras and their joint distributions. Bi-free probability is a rapidly evolving research area that provides the ground for extending techniques from free probability in order to solve problems pertaining to pairs of von Neumann algebras, such as von Neumann algebras and their commutants, or tensor products of von Neumann algebras. While free independence can be viewed as the distributional condition that follows via the left actions of algebras on reduced free product spaces, the corresponding notion of bi-free independence is regarded as the condition imposed on the joint distribution of pairs of algebras via their left and right actions on reduced free product spaces. Therefore, this concept generalizes free independence, but also facilitates an environment that captures classical independence. Bi-free independence found its combinatorial characterization via the bi-free cumulants in [11] in the scalar setting and in [10] in the context of bi-free probability with amalgamation.

Set theory was originated by Cantor in 1874 and deals with the investigation of cardinalities of subsets of the real line. The development of set theory had significant mathematical and metamathematical implications, especially in conjuction with Godel's incompleteness theorems. One of the most important results in this setting concerns the proof of the independence of the continuum hypothesis CH from the standard axiomatization of set theory ZFC, which was achieved via the method of forcing initiated by Cohen in 1963. Over the last 20 years, the application of set-theoretic methods to operator algebras has proved to be extremely fruitful, with many longstanding open problems in operator algebras being settled via the set-theoretic approach. The interplay between these two mathematical fields includes the investigation of the rich structure theory of coronas and ultraproducts of separable C*-algebras, the application of descriptive set-theoretic methods in operator-algebraic classification problems and the development of pathological examples of non-separable C*-algebras. Notably, with the set-theoretic approach, major breakthroughs were achieved in important open problems in the theory of operator algebras, such as Naimark's problem (which asks whether the C*-algebra $\mathcal{K}(H)$ of compact operators on a separable Hilbert space is the unique, up to isomorphism, C*-algebra with a unique irreducible representation up to unitary equivalence; [1]), Anderson's conjecture (which states that every pure state on the C*-algebra $\mathcal{B}(H)$ of all bounded, linear operators on a separable Hilbert space is diagonalizable; [46]) and also completely settling the question of whether the Calkin algebra $\mathcal{Q}(H)$, obtained as the quotient of $\mathcal{B}(H)$ by the compact operators $\mathcal{K}(H)$, has outer automorphisms ([64], [23]). We refer the reader to [25] for an exposition of the applications of logic, set theory and model theory to operator algebras.

This dissertation is divided into two main parts. The two chapters of Part I along with the first chapter of Part II are autonomous, while the last three chapters of this dissertation can be thought of as consisting of a single body of work. The first part focuses on set-theoretic applications in the field of operator algebras and concerns the embedding of non-separable C*-algebras into the Calkin algebra and the Borel classification of separable, non-self-adjoint AF operator algebras. In the second part we shift our attention to the theory of bi-free probability and investigate the analogue of R-diagonal operators and study notions of Fisher information and entropy within the bi-free context.

Chapter 1 revolves around the Calkin algebra $\mathcal{Q}(H)$ which has been the object of intensive investigations by researchers in operator algebras (with its importance being signified after the seminal work in [7] and the subsequent development of the theory of extensions of C*-algebras). The Calkin algebra constitutes an object that is especially amenable to the study from a set-theoretic perspective, due to its structural similarities with the Boolean algebra $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, of which it is considered the non-commutative analogue. Notably, the question asked in [7] of whether $\mathcal{Q}(H)$ has outer automorphisms remained open for 30 years, until it was settled using set-theoretic methods in [64] and [23], where it was shown that the existence of outer automorphisms of $\mathcal{Q}(H)$ is independent of the standard axiomatization of set theory ZFC.

Recently, significant attention was turned to the study of the class of C*-algebras which embed in the Calkin algebra and the investigation of the level of influence various set-theoretic axioms have on it (see [28], [74]). While a standard amplification argument shows that every separable C*-algebra embeds into $\mathcal{Q}(H)$, the situation beyond this is non-trivial. In [28], the authors showed that every C*-algebra of density character $\aleph_1$ (the least uncountable cardinal) embeds into the Calkin algebra and this shows that under the continuum hypothesis CH (which asserts that $\aleph_1$ equals the cardinality of the continuum $2^{\aleph_0} := |\mathbb{R}|$), the Calkin algebra is (injectively) $2^{\aleph_0}$-universal, in the sense that all C*-algebras with density character $2^{\aleph_0}$ embed into the Calkin algebra. Thus, in this instance the description of the class of C*-algebras that embed into $\mathcal{Q}(H)$ is the simplest possible: a C*-algebra embeds into $\mathcal{Q}(H)$ if and only if its density character is at most equal to $2^{\aleph_0}$. However, in the absence of CH such a satisfactory characterization does not hold, since if one assumes the proper forcing axiom PFA (which implies the negation of CH), there exist even abelian C*-algebras of density character $2^{\aleph_0}$ that do not embed into $\mathcal{Q}(H)$ ([28]).

In the first chapter we continue the aforementioned line of research (parts of the content of this chapter can be found in the joint work [29]). We prove that under Martin's axiom (a set-theoretic axiom independent from ZFC and consistent with the negation of CH) every C*-algebra of density character strictly less that $2^{\aleph_0}$ embeds into the Calkin algebra. This is obtained using the method of forcing, by constructing for each C*-algebra A, regardless of its density character, a forcing partial order that satisfies the countable chain condition and that forces the embedding of A into $\mathcal{Q}(H)$. Moreover, we show that the assertion "*every* C*-*algebra of density character less than* $2^{\aleph_0}$ *embeds into the Calkin algebra*" is independent from ZFC.

In chapter 2 we investigate the Borel complexity of separable, non-self-adjoint AF operator algebras. In the setting of Borel complexity theory, one of the most notable benchmarks in the complexity hierarchy is given by the notion of classifiability by countable structures. Precisely, a classification problem (X, E) is said to be classifiable by countable structures if there is a category of countable structures $\mathcal{C}$ such that E is Borel reducible to the isomorphism relation $\cong_{\mathcal{C}}$ within the class of objects of $\mathcal{C}$. The interplay between descriptive set theory and functional analysis has been fruitful over the past years, however classification problems in the context of operator theory tend not to be classifiable by countable structures. For instance, the natural isomorphism relations for von Neumann factors other than type I ([65])

and for unital, simple, separable, nuclear C*-algebras ([33]) are not classifiable by countable structures. On the opposite side of the spectrum, Elliot's classical K-theoretic classification of separable AF C*-algebras ([20]), which are inductive limits of finite-dimensional C*-algebras, implies that the isomorphism relation for separable AF C*-algebras is classifiable by countable structures.

In connection to the classification of AF C*-algebras, the question of whether the class of non-self-adjoint AF operator algebras can be similarly classified by the ordered K_0 groups, or by any other class of countable structures, remained open since the early stages of the development of operator-algebraic classification. In chapter 2 we provide a negative answer to this question and show that the relations of isomorphism, isometry, complete isomorphism and complete isometric isomorphism for separable, non-self-adjoint AF operator algebras are not classifiable by countable structures (these results are part of a joint work in preparation with N. C. Phillips). This is obtained by constructing a functor from the category of operator spaces to the category of non-self ajoint AF operator algebras and making use of the fact that the canonical isomorphism relations for operators spaces are not classifiable by countable structures.

In the setting of free probability theory, in [60] the authors introduced the class of R-diagonal operators as a generalization of the Haar unitary operators and the circular operators (the non-normal version of the semicirculars). Specifically, an operator is called R-diagonal if it is of the form $u \cdot p$, where u is a Haar unitary which is freely independent from p. This class consists of in general non-normal operators that are particularly well behaved and are combinatorialy characterized by certain vanishing conditions on their free cumulants.

In particular, in [60] R-diagonal operators were found to satisfy a "free absorption" property, in the sense that if X is an R-diagonal operator which is freely independent from an operator Y, then the product $X \cdot Y$ remains R-diagonal. In [51], powers of R-diagonal operators were shown to be R-diagonal, while in [69] the authors showed that R-diagonal operators admit continuous families of invariant subspaces relative to the von Neumann algebras they generate. Moreover, in [59], a number of equivalent characterizations of R-diagonality were formulated, including conditions on moments, free cumulants and the freeness of certain self-adjoint matrices from the scalar matrices, while in [58] distributions of R-diagonal operators found applications in the non-microstate approach to free entropy, answering questions regarding the minimization of the free Fisher information in the tracial framework.

In chapter 3, the theory of R-diagonal operators in the context of bi-free probability is developed (this chapter can also be partly found in [42]). Adopting the combinatorial approach, the notion of a bi-R-diagonal pair of operators is defined by means of certain vanishing conditions on their joint bi-free cumulants. Canonical examples of bi-R-diagonal pairs include the bi-Haar unitary pairs of operators, which constitute the analogue of the Haar unitaries in bi-free probability, with their joint distribution modelled by the left and right regular representations of groups on Hilbert spaces. In this generalized setting, an analogous "bi-free absorption" property is shown to hold. More precisely, if (X, Y) is a bi-R-diagonal pair that is bi-freely independent from a pair (Z, W), then the pair obtained by the coordinate-wise product (XZ, WY) (with the opposite multiplication considered on the right operators) is also bi-R-diagonal. Moreover, bi-R-diagonal pairs are shown to be closed under the taking of arbitrary powers and a characterization of the condition of bi-R-diagonality in terms of bi-freeness with amalgamation is developed. Lastly, a complete description of the joint distributions of bi-R-diagonal pairs of operators is given, in the sense that they always arise in the form $(u_l X, Y u_r)$, where the pairs (u_l, u_r) and (X, Y) are bi-freely independent and (u_l, u_r) is a bi-Haar unitary pair.

In a series of groundbreaking papers originating with [78], Voiculescu developed analogues of the notions of entropy and Fisher information to the free probability setting, introducing the concepts of microstates and non-microstates free entropy. One the one hand, microstates free entropy measures the volumes of tuples of self-adjoint scalar matrices that approximate the distribution of tuples of self-adjoint operators in tracial von Neumann algebras, motivated by the connection between free probability and random matrix theory. The development of this notion led to several important results, giving answers to longstanding open problems regarding the structure of the free group factors, such as the absence of Cartan subalgebras ([79]) and the primeness of the free group factors ([35]).

On the other hand, the non-microstates approach to free entropy is motivated by the concept of Fisher information in classical probability and is based on conjugate variable systems defined with respect to non-commutative partial derivatives. The techniques developed throughout the advancement of this theory led to important applications in von Neumann algebras, as they were used to show that specific type II_1 factors do not have property Γ ([15]), as well as to show the absence of atoms and zero divisors from free product distributions (see [12] and [53]). Non-microstates free entropy was generalized in [66] in the operator-valued setting by allowing the implementation of completely positive maps into the conjugate variable expressions. This concept of free entropy with respect to completely positive maps was key in [58], where its was proved that, under distributional constraints, minimal values of the free Fisher information and maximal values of the non-microstates free entropy existed and were reached by R-diagonal operators.

Recently, in [13] and [14] the notions of microstate and non-microstate entropy were extended to the setting of bi-free probability. In the last three chapters of this dissertation we continue the study of entropy within the bi-free environment and extend the concept of non-microstate bi-free entropy to the operator-valued setting so as to involve the existence of completely positive maps in the conjugate variable formulae, and we examine its applications (parts of the content of these chapters can be found in the joint work [43]). In chapter 4, in order to accommodate for the typical absence of positivity in the expectations occurring in operator-valued bi-free probability, we extend the operator-valued structures in order to facilitate an environment amenable to analytical computations. These structures are modeled based on the canonical left and right actions of a tracial von Neumann algebra on its L_2- Hilbert space and with the addition of a compatible tracial state on the algebra of amalgamation, we develop notions of bi-free moment and cumulant maps with values in the L_2-Hilbert spaces of algebras. Based on this machinery, in chapter 5 the systems of conjugate variables and Fisher information are defined via these L_2-valued moment and cumulant relations by incorporating completely positive maps in a manner analogous to [66] appropriated to the bi-free context, and the corresponding notion of bi-free entropy is given as an integral of the bi-free Fisher information with respect to completely positive maps of perturbations of left and right operators by bi-freely independent operator-valued bi-semicircular pairs. Most of the salient features of conjugate variables, Fisher information and entropy are shown to hold in this generalized context. As an application, in chapter 6 we extend the results from [58] by showing that, modulo distributional conditions imposed in order to accommodate for the potential lack of traciality, minimal values of the bi-free Fisher information and maximal values of the non-microstates bi-free entropy are attained at bi-R-diagonal pairs of operators.

Throughout this dissertation we will assume familiarity with the basic theory of C*-algebras, von Neumann algebras and free probability and will refer to the excellent standard texts [57], [8], [5], [61] and [56] whenever necessary. Although we will be explicitly stating the required set-theoretic definitions and related notions, the reader will be assumed to be familiar with cardinal arithmetic, forcing and the basics of descriptive set theory. Standard references for these topics are [50] and [44].

Part I : Set Theory and Operator Algebras

Chapter 1

Embedding C*-algebras into the Calkin algebra

The Calkin algebra $\mathcal{Q}(H)$ is the quotient of $\mathcal{B}(H)$, the algebra of bounded linear operators on a complex, separable, infinite-dimensional Hilbert space H, modulo the ideal of the compact operators $\mathcal{K}(H)$. It is considered to be the noncommutative analogue of the Boolean algebra $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, where Fin denotes the ideal of all finite subsets of $\mathbb{N}$ (see e.g., [24] and [84]) and, as a consequence, results about $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ often translate into questions (frequently nontrivial) about $\mathcal{Q}(H)$. In this chapter we study the analogue of the question "Which linear orderings embed into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$?". In order to put our study into the proper context, we start by reviewing some known results about the latter problem.

Note that $\mathcal{P}(\mathbb{N})$ embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. To define an embedding, send $A \subseteq \mathbb{N}$ to the equivalence class of the set $\{(2n+1)2^m : n \in \mathbb{N},\ m \in A\}$. Every countable linear ordering $\mathbb{L}$ embeds into $\mathcal{P}(\mathbb{N})$, and therefore into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. One way to see this is to enumerate the elements of $\mathbb{L}$ as a_n, for $n \in \mathbb{N}$, and define $\Phi\colon \mathbb{L} \to \mathcal{P}(\mathbb{N})$ by $\Phi(a_m) = \{n : a_n \leq a_m\}$.

There is a simple characterization of linear orderings $\mathbb{L}$ that embed into $\mathcal{P}(\mathbb{N})$. A linear ordering $\mathbb{L}$ embeds into $\mathcal{P}(\mathbb{N})$ if and only if it has a countable subset $\{a_n : n \in \mathbb{N}\}$ which is separating in the sense that for all $x < y$ in $\mathbb{L}$ there exists n such that $x \leq a_n < y$ or $x < a_n \leq y$. To prove the direct implication, given $\{a_n : n \in \mathbb{N}\}$, one can define Φ as above. The converse implication is straightforward. No such characterization exists for the class of linear orderings that embed into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

Since $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a countably saturated atomless Boolean algebra, all linear orderings of cardinality $\aleph_1$ embed into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Thus the Continuum Hypothesis, CH, implies that a linear order embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ if and only if its cardinality is at most $2^{\aleph_0}$. By [52], if ZFC is consistent then the assertion that all linear orderings of cardinality at most $2^{\aleph_0}$ embed into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is relatively consistent with ZFC plus the negation of CH. Laver's model is however an exception, and in some models of ZFC (if there are any!) the class of linear orderings which embed into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ can be downright bizarre. This class is also very important. For example, Woodin's condition for the automatic continuity of Banach algebra homomorphisms from $C([0,1])$ asserts that if there exists a discontinuous homomorphism from $C([0,1])$ into a Banach algebra then a nontrivial initial segment of an ultrapower $\mathbb{N}^{\mathbb{N}}/\mathcal{U}$ embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ ([16]).[1] Every $*$-homomorphism between C*-algebras is automatically continuous, and all homomorphisms between C*-algebras are continuous in Woodin's model. It is not known whether it is provable in ZFC that every homomorphism between C*-algebras with dense range is continuous (see the introduction to [62]).

The question of what linear orderings embed into the poset of projections of the Calkin algebra or into the poset of self-adjoint elements of the Calkin algebra may be of an independent interest. However, the question that we consider here is strictly operator-algebraic: Which C*-algebras embed into the Calkin algebra? This is also a non-commutative analogue of the question of which abelian C*-algebras embed into ℓ_∞/c_0. By the Gelfand–Naimark duality, this corresponds to asking which compact Hausdorff spaces are continuous images of $\beta\mathbb{N}\setminus\mathbb{N}$, the Čech–Stone remainder of $\mathbb{N}$. By Parovičenko's Theorem, having weight not greater than $\aleph_1$ is a sufficient condition (alternatively, this can be proved by elementary model theory; see the discussion in [19, p. 1820]). However, the situation in ZFC is quite nontrivial ([18], [17]).

The analogue of the cardinality of a C*-algebra (or a topological space) A is the density character. It is defined as the least cardinality of a dense subset of A. Thus the C*-algebras of density character $\aleph_0$ are exactly the separable C*-algebras. The density character of a nonseparable C*-algebra is equal to the minimal cardinality of a generating subset and also to the minimal cardinality of a dense $(\mathbb{Q}+i\mathbb{Q})$-subalgebra. Every separable C*-algebra embeds into $\mathcal{B}(H)$ and therefore into $\mathcal{Q}(H)$ by a standard amplification argument, analogous to the argument for embedding $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ that was presented in the second paragraph of this chapter. In addition, all C*-algebras of density character $\aleph_1$ embed into $\mathcal{Q}(H)$, but the proof is surprisingly nontrivial ([28]) due to the failure of countable saturation in the Calkin algebra ([26, §4]). Since the density character of $\mathcal{Q}(H)$ is $2^{\aleph_0}$, C*-algebras of larger density character do not embed into $\mathcal{Q}(H)$ and once again CH gives the simplest possible characterization of the class of C*-algebras that embed into $\mathcal{Q}(H)$. In this chapter we

[1] This is usually stated in terms of embedding into the directed set $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, but a linear order embeds into $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ if and only if it embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$; see e.g., [22, Proposition 0.1] or [87, Lemma 3.2].

investigate what happens when CH fails, focusing on C*-algebras of density character strictly less than $2^{\aleph_0}$.

Theorem 1.0.1. *The assertion 'Every* C**-algebra of density character strictly less than* $2^{\aleph_0}$ *embeds into the Calkin algebra' is independent from ZFC. It is moreover independent from ZFC+*$2^{\aleph_0} = \aleph_3$*, and* $\aleph_3$ *is the minimal cardinal with this property.*

The most involved part in the proof of Theorem 1.0.1 is showing that the statement 'All C*-algebras of density character strictly less than $2^{\aleph_0}$ embed into $\mathcal{Q}(H)$' is consistent with ZFC+$2^{\aleph_0} > \aleph_2$. This will be achieved via Theorem 1.0.2 (which is proved in §1.3) using forcing.

The method of forcing was introduced by Cohen to prove the independence of CH from ZFC, and later developed to deal with more general independence phenomena (see §1.1.2). The *countable chain condition* (or ccc) is a property of forcing notions that ensures no cardinals or cofinalities are collapsed, and all stationary sets are preserved, in the forcing extension (see Definition 1.1.5).

Theorem 1.0.2. *For every* C**-algebra* A *there exists a ccc forcing notion* $\mathbb{E}_A$ *which forces that* A *embeds into* $\mathcal{Q}(H)$*.*

Rephrasing the statement of Theorem 1.0.2, every C*-algebra, regardless of its density character, can be embedded into the Calkin algebra in a forcing extension of the universe obtained without collapsing any cardinals or cofinalities.

The following Corollary (proved as Corollary 1.3.8) is the consistency result needed to prove Theorem 1.0.1 and follows from the proof of Theorem 1.0.2.

Corollary 1.0.3. *Assume Martin's Axiom, MA. Then, every* C**-algebra with density character strictly less than* $2^{\aleph_0}$ *embeds into the Calkin algebra.*

In the case when the continuum is not greater than $\aleph_2$, the conclusion of Corollary 1.0.3 follows from [28]. A combination of this corollary with results from [73] yields the proof of Theorem 1.0.1.

Proof of Theorem 1.0.1. As pointed out above, if the cardinality of the continuum is not greater than $\aleph_2$ then all C*-algebras of density character strictly less than $2^{\aleph_0}$ embed into the Calkin algebra.

Martin's Axiom is relatively consistent with the continuum being equal to $\aleph_3$ ([50, Theorem V.4.1]), hence by Corollary 1.0.3 in this model all C*-algebras of density character not greater than $\aleph_2$ embed into the Calkin algebra.

On the other hand, in a model obtained by adding $\aleph_3$ Cohen reals to a model of CH we get $2^{\aleph_0} = \aleph_3$ and the Calkin algebra has no chains of projections of order type $\aleph_2$. This was proved in [73, Section 2.5] by adapting a well-known argument from Kunen's PhD thesis ([48, Section 12]). Therefore in this model the abelian C*-algebra $C(\aleph_2 + 1)$ (where the ordinal $\aleph_2 + 1$ is endowed with the order topology) does not embed into $\mathcal{Q}(H)$. $\square$

We remark that Theorem 1.0.2 was inspired by an analogous fact holding for partial orders and $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$: For every partial order $\mathbb{P}$ there is a ccc forcing notion which forces the existence of an embedding of $\mathbb{P}$ into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. While the proof of this latter fact is an elementary exercise, the proof of Theorem 1.0.2 is fairly sophisticated, and will take most of this chapter. At a critical place it makes use of some variations of Voiculescu's theorem ([8, Corollary 1.7.5]; see Theorem 1.1.2 and Corollary 1.1.3).

1.1 Preliminary Results

1.1.1 C*-algebras

In this chapter, H will always denote the complex, separable, infinite-dimensional Hilbert space $\ell_2(\mathbb{N})$ and $\mathcal{B}(H)$ will denote the space of linear, bounded operators on H. The space of all finite-rank operators on H is denoted $\mathcal{B}_{\mathrm{f}}(H)$. Its norm-closure, denoted $\mathcal{K}(H)$, is the ideal of compact operators. The notation $\mathcal{U}(H)$ is reserved for the group of unitary operators on H. The Calkin algebra $\mathcal{Q}(H)$ is the quotient of $\mathcal{B}(H)$ by the compact operators and for what follows $\pi : \mathcal{B}(H) \to \mathcal{Q}(H)$ will always denote the quotient map. For $h \in \mathcal{B}_{\mathrm{f}}(H)$, h^+ denotes the orthogonal projection onto its range and h^- is the projection onto the space of 1-eigenvectors of h (i.e. the space of all vectors ξ such that $h\xi = \xi$). We write $\mathcal{B}_{\mathrm{f}}(H)_+^{\leq 1}$ for the collection of all finite-rank positive contractions on H. An operator $T \in \mathcal{B}(H)$ is *way above* S, $T \gg S$ in symbols, if $TS = S$. For two projections P, Q we have $P \ll Q$ iff $P \leq Q$. We write $T \sim_{\mathcal{K}(H)} S$ and say that T and S *agree modulo the compacts* to indicate that $T - S \in \mathcal{K}(H)$. Similarly, given a C*-algebra A, two maps $\varphi_1 : A \to \mathcal{B}(H)$ and $\varphi_2 : A \to \mathcal{B}(H)$ are said to agree modulo the compacts if $\varphi_1(a) \sim_{\mathcal{K}(H)} \varphi_2(a)$ for every $a \in A$. A net of operators $\{T_i\}_{i \in I}$ *strongly converges* to an operator T if for each $\xi \in H$ the net $\{T_i\xi\}_{i \in I}$ converges to $T\xi$. We remark that to verify the strong convergence of a net it suffices to check it on a dense subset of H.

Given two vectors ξ and η of a normed vector space and $\epsilon > 0$, the notation $\xi \approx_\epsilon \eta$ stands for $\|\xi - \eta\| < \epsilon$. We abbreviate '$F$ is a finite subset of A' as $F \Subset A$. If F is a subset of a C*-algebra then $C^*(F)$ denotes the C*-algebra generated by F. If A is unital and $u \in A$ is a unitary element, then $\operatorname{Ad} u$ denotes the automorphism of A which sends a to uau^*. A representation $\Phi : A \to \mathcal{B}(H)$ is called *essential* if $\Phi(a) \in \mathcal{K}(H)$ implies $\Phi(a) = 0$ for all $a \in A$. Note that all (non-zero) representations of unital, simple, infinite-dimensional C*-algebras on H are faithful (i.e. injective) and essential. A unital, injective $*$-homomorphism $\Theta : A \to \mathcal{Q}(H)$ is *trivial* if there exists a unital (and necessarily essential) representation $\Phi : A \to \mathcal{B}(H)$ such that $\pi \circ \Phi = \Theta$ and, in this case, the map Φ is called a *lift* of Θ. Moreover, Θ is called *locally trivial* if its restriction to any unital separable C*-subalgebra of A is trivial.

Mainly for convenience, in the proof of Theorem 1.0.2 in section 1.3 we shall exclusively be concerned with embeddings of unital and simple C*-algebras into the Calkin algebra, as any unital $*$-homomorphism from a unital and simple C*-algebra into $\mathcal{Q}(H)$ is automatically injective. This causes no loss of generality, as a result of the next proposition.

Proposition 1.1.1 ([28, Lemma 2.1])**.** *Every* C*-*algebra A embeds into a unital and simple* C*-*algebra B of the same density character as A.* □

The following standard consequence of Voiculescu's theorem will be invoked frequently throughout the rest of this manuscript.

Theorem 1.1.2 ([8, Corollary 1.7.5])**.** *Let A be a unital, separable* C*-*algebra and let $\Phi : A \to \mathcal{B}(H)$ and $\Psi : A \to \mathcal{B}(H)$ be two faithful, essential, unital representations. Then, for every $F \Subset A$ and $\epsilon > 0$ there exists a unitary $u \in \mathcal{U}(H)$ such that:*

1. *The maps $\operatorname{Ad} u \circ \Phi$ and Ψ agree modulo the compacts.*
2. $\|\operatorname{Ad} u \circ \Phi(a) - \Psi(a)\| < \epsilon$ *for all* $a \in F$. □

See also [3] and [38, Section 3] for a detailed proof of the theorem above. We will also be using the next variant, which allows to find a unitary as in item 1 of the previous theorem which in addition is equal to the identity on a given finite-dimensional space:

Corollary 1.1.3. *Let A be a unital, separable* C*-*algebra and consider two faithful, essential, unital representations $\Phi : A \to \mathcal{B}(H)$ and $\Psi : A \to \mathcal{B}(H)$. Then, for every $F \Subset A$ and every finite-dimensional subspace $K \subseteq H$ there exists a unitary $w \in \mathcal{U}(H)$ such that:*

1. *The maps* $\operatorname{Ad} w \circ \Phi$ *and* Ψ *agree modulo the compacts.*

2. $\operatorname{Ad} w \circ \Phi(a)(\xi) = \Phi(a)(\xi)$ *for every* $a \in F$ *and* $\xi \in K$.

In particular, the set

$$Z = \{\operatorname{Ad} w \circ \Phi : w \in \mathcal{U}(H),\ \operatorname{Ad} w \circ \Phi(a) \sim_{K(H)} \Psi(a) \textit{ for all } a \in A\}$$

has Φ *in its closure with respect to strong convergence.*

Proof. Let $F \Subset A$, $K \subseteq H$ be a finite-dimensional subspace and we let $P \in \mathcal{B}(H)$ be the orthogonal projection onto K. By Theorem 1.1.2, we can find a unitary $v \in \mathcal{U}(H)$ such that $\operatorname{Ad} v \circ \Phi$ and Ψ agree modulo the compacts. Let Q be the finite-rank projection onto the subspace spanned by the set $K \cup \{\Phi(a)K : a \in F\}$ and let $w \in \mathcal{U}(H)$ be a finite-rank modification of v such that $wQ = Qw = Q$. Then $\operatorname{Ad} w \circ \Phi$ and $\operatorname{Ad} v \circ \Phi$ agree modulo the compacts and $(\operatorname{Ad} w \circ \Phi)(a)P = \Phi(a)P$ for all $a \in F$. □

The following lemma will be invoked for proving a density result (Proposition 1.3.4).

Lemma 1.1.4. *Let* $T \in \mathcal{B}(H)$ *be a finite-rank projection. For every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $S \in \mathcal{B}(H)$ *and* $\|T - S\| < \delta$, *then there is a unitary* $u \in \mathcal{U}(H)$ *satisfying the following:*

1. $uT[H] \subseteq S[H]$, *namely the image space of* uT *is contained in the image space of* S,

2. $\|(u - \mathrm{Id}_H)T\| < \epsilon$,

3. $u - \mathrm{Id}_H \in \mathcal{B}_{\mathrm{f}}(H)$,

4. *for every orthogonal projection* P *onto a subspace of* $T[H]$ *such that* $SP = P$, *we have that* $uP = P$ *holds.*

Proof. Let $\{\xi_1, \ldots, \xi_k\}$ be an orthonormal basis of the space of all eigenvectors of S whose eigenvalue is 1 and which are moreover contained in $T[H]$. Fix $\{\xi_1, \ldots, \xi_n\}$ an orthonormal basis of $T[H]$ extending $\{\xi_1, \ldots, \xi_k\}$. If $\|T - S\| < \delta < 1$, the set $\{S\xi_1, \ldots, S\xi_n\}$ (which linearly spans $ST[H]$) is linearly independent. In fact, if $\xi \in T[H]$ has norm one and is such that $S\xi = 0$, then $\|T\xi\| = \|\xi\| < \delta$, which is a contradiction. Applying the Gram-Schmidt process to $\{S\xi_1, \dots, S\xi_n\}$ we obtain an orthonormal basis $\{\eta_1, \dots, \eta_n\}$ for $ST[H]$, which for sufficiently small choice of δ (which depends on the dimension of $T[H]$) is such that

$$\|\xi_i - \eta_i\| < \frac{\epsilon}{n},\ i = 1, \dots, n.$$

Denote by V the finite-dimensional space spanned by $T[H]$ and $ST[H]$. Let $\{\xi_1, \dots, \xi_m\}$ be an orthonormal basis of V that extends $\{\xi_1, \dots, \xi_n\}$ and, similarly, $\{\eta_1, \dots, \eta_m\}$ an orthonormal basis of V extending $\{\eta_1, \dots, \eta_n\}$. This naturally defines a unitary $w : V \to V$ by sending the vector ξ_i to η_i for every $i = 1, \dots, m$. Finally, define $u \in \mathcal{U}(H)$ to be equal to w on V and equal to the identity on the orthogonal complement of V. The unitary u satisfies the desired properties, in particular item 4 of the statement holds since $\eta_i = \xi_i$ for $i \leq k$ by our initial choice of $\{\xi_1, \dots, \xi_k\}$, orthonormal basis of the space of all eigenvectors of S of eigenvalue 1 in $T[H]$. $\square$

1.1.2 Set Theory and Forcing

As stated in the introduction, Theorem 1.0.2 is an application of the method of forcing. For a standard introduction to this topic see [50]; see also [16] and [85].

We start with some technical definitions. Two elements p, q of a partial order (or poset) $(\mathbb{P}, \leq)$ are *compatible* if there exists $s \in P$ such that $s \leq p$ and $s \leq q$. Otherwise, p and q are *incompatible*. A subset $A \subseteq \mathbb{P}$ is an *antichain* if its elements are pairwise incompatible. A subset $D \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$. A subset D of $\mathbb{P}$ is *open* if it is closed downwards, i.e. $p \in D$ and $q \leq p$ implies $q \in D$. A non-empty subset G of $\mathbb{P}$ is a *filter* if $q \in G$ and $q \leq p$ implies $p \in G$, and if for any $p, q \in G$ there exists $r \in G$ such that $r \leq p$, $r \leq q$. Given a family $\mathcal{D}$ of dense open subsets of $\mathbb{P}$, a filter G is *$\mathcal{D}$-generic* if it has non-empty intersection with each element of $\mathcal{D}$.

A *forcing notion* (or *forcing*) is a partially ordered set (poset), whose elements are called *conditions*. Naively, the forcing method produces, starting from a poset $\mathbb{P}$, an extension of von Neumann's universe V. The extension is obtained by adding to V a filter G of $\mathbb{P}$ which intersects *all* dense open subsets of $\mathbb{P}$. This generic extension, usually denoted by $V[G]$, is a model of ZFC, and its theory depends on combinatorial properties of $\mathbb{P}$ and (to some extent) on the choice of G. A condition $p \in \mathbb{P}$ *forces* a sentence φ in the language of ZFC if φ is true in $V[G]$ whenever G is a generic filter containing p. If φ is true in every generic extension $V[G]$, we say that $\mathbb{P}$ *forces* φ.

Unless $\mathbb{P}$ is trivial, no filter intersects every dense open subset of $\mathbb{P}$. For this reason, the forcing method is combined with a Löwenheim–Skolem reflection argument and applied to countable models of ZFC. If M is a countable model of ZFC and $\mathbb{P} \in M$, then the existence of an M-generic filter G (i.e. intersecting every open dense subset of $\mathbb{P}$ in M) of $\mathbb{P}$ is guaranteed by the Baire Category Theorem ([50, Lemma III.3.14])[2].

An obvious method for embedding a given C*-algebra A into the Calkin algebra is to generically add a bijection between a dense subset of A and $\aleph_0$ (i.e. to 'collapse' the density character of A to $\aleph_0$). The completion of A in the forcing extension (routinely identified with A) is then separable and therefore embeds into the Calkin algebra of the extension. However, if the density character of A is collapsed, then this results in a C*-algebra that has little to do with the original algebra A. We shall give two examples.

Fix an uncountable cardinal κ. If A is $C^*_{\mathrm{red}}(F_\kappa)$, the reduced group C*-algebra of the free group with κ generators, then collapsing κ to $\aleph_0$ makes $\mathbb{F}_\kappa$ isomorphic to $\mathbb{F}_{\aleph_0}$ and hence A becomes isomorphic to $C^*_{\mathrm{red}}(F_\infty)$. If the cardinal κ is not collapsed, then the completion of $C^*_{\mathrm{red}}(F_\kappa)$ in the extension is isomorphic to $C^*_{\mathrm{red}}(F_\kappa)$ as computed in the extension. This is not automatic as, for example, the completion of the ground model Calkin algebra in a forcing extension will be isomorphic to the Calkin algebra in the extension if and only if no new reals are added.

A more drastic example is provided by the 2^κ nonisomorphic C*-algebras each of which is an inductive limit of full matrix algebras of the form $M_{2^n}(\mathbb{C})$ for $n \in \mathbb{N}$ constructed in [30]. After collapsing κ to $\aleph_0$, all of these C*-algebras become isomorphic to the CAR algebra M_{2^∞}. This is because it can be proved that the K-groups of A are invariant under forcing and, by Glimm's classification result, unital and separable inductive limits of full matrix algebras are isomorphic (e.g. [5]). An analogous result holds for any UHF C*-algebra of infinite type by [31].

Instead of 'collapsing' the cardinality of A, our approach is to 'inflate' the Calkin algebra. More precisely, we prove that Martin's Axiom implies that the Calkin algebra has already been 'inflated'.

Forcing axioms are far-reaching extensions of the Baire Category Theorem that enable one to apply forcing without worrying about metamathematical issues. Corollary 1.0.3 will be proved by applying Martin's axiom, the simplest (and most popular) forcing axiom.

Definition 1.1.5. A poset $(\mathbb{P}, \leq)$ satisfies the *countable chain condition* (or *ccc*) if every

[2] For metamathematical reasons related to Gödel's Incompleteness Theorem, one usually considers models of a large enough finite fragment of ZFC. By other metamathematical considerations, for all practical purposes this issue can be safely ignored; see [50, Section IV.5.1].

antichain in $\mathbb{P}$ is at most countable.

Martin's Axiom, *MA*, asserts that for every ccc poset $\mathbb{P}$ and every family $\mathcal{D}$ of fewer than $2^{\aleph_0}$ dense open subsets of $\mathbb{P}$, there exists a $\mathcal{D}$-generic filter in $\mathbb{P}$.

It is relatively consistent with ZFC that Martin's axiom holds and the continuum is larger than any prescribed cardinal κ ([50, Theorem V.4.1]). The countable chain condition is the single most flexible property of forcing notions that enables one to iterate forcing and obtain forcing extensions with various prescribed properties (see e.g. [50, Theorem IV.3.4]). Our posets will have the following strong form of ccc. A poset $(\mathbb{P}, \leq)$ has *property K* if every uncountable subset of $\mathbb{P}$ contains a further uncountable subset in which any two elements are compatible.

The proof strategy in section 1.3 is as follows. Given a C*-algebra A, we start by defining a forcing notion $\mathbb{E}_A$ (Definition 1.3.2) whose sufficiently generic filters allow to build an embedding of A into $\mathcal{Q}(H)$ (Proposition 1.3.5). We then proceed to show that $\mathbb{E}_A$ is ccc (Proposition 1.3.7), and that the existence of sufficiently generic filters inducing the existence of an embedding of A into $\mathcal{Q}(H)$ is guaranteed in models of ZFC + MA (Corollary 1.3.8).

The following lemma will be used when proving that a given forcing notion is ccc. A family $\mathcal{C}$ of sets forms a Δ-*system* with *root* R if $X \cap Y = R$ for any two distinct sets X and Y in $\mathcal{C}$. When the sets in $\mathcal{C}$ are pairwise disjoint, one obtains the special case with $R = \emptyset$.

Lemma 1.1.6 (Δ-System Lemma, [50, Lemma III.2.6])**.** *Every uncountable family of finite sets contains an uncountable Δ-system.* □

1.2 The Cases of Abelian and Quasidiagonal C*-algebras

In this section, we discuss two special cases of Theorem 1.0.2, those corresponding to the classes of abelian and quasidiagonal C*-algebras. Their proofs (the first of which is standard) are intended to provide intuition and demonstrate the increase in complexity regarding the corresponding forcing notions that are implemented. It also displays the natural progression behind Theorem 1.0.2. We will omit most of the technical details in this section, as the results discussed here can be easily inferred by the proofs of the subsequent parts of the chapter. The reader eager to transition right away to the proof of Theorem 1.0.2 can safely skip ahead to section 1.3.

1.2.1 Embedding Abelian C*-algebras into ℓ_∞/c_0

The main focus in this part will be on obtaining the abelian version of Theorem 1.0.2:

Proposition 1.2.1. *For every abelian* C*-*algebra A there exists a ccc forcing notion which forces that A embeds into ℓ_∞/c_0.*

Exploiting the fact that the categories of Boolean algebras, Stone spaces (i.e. zero-dimensional, compact, Hausdorff spaces) and C*-algebras of continuous functions on Stone spaces are all equivalent (by a combination of the Stone duality [40, section II.4] and the Gelfand–Naimark duality [40, section IV.4]), one can translate the statement of the proposition above to a statement regarding Boolean algebras. In particular, it is enough to show that for any Boolean algebra B there exists a ccc forcing notion which forces that B embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. If B is a Boolean algebra, we denote by $\operatorname{St}(B)$ its *Stone space*, the space of all ultrafilters on B equipped with the Stone topology.

To see the aforementioned translation, first of all note that it suffices to prove the assertion of Proposition 1.2.1 for C*-algebras of the form $C(Y)$ with Y being a Stone space, as every abelian C*-algebra embeds into such an algebra. Indeed, any abelian C*-algebra $C(X)$ naturally embeds into the von Neumann algebra $L^\infty(X)$ which, being a real rank zero unital C*-algebra, is of the form $C(Y)$ with Y zero-dimensional, compact and Hausdorff. We provide an alternative proof for the reader who is not familiar with the theory of von Neumann algebras. Every non-unital, abelian C*-algebra embeds into its unitization, which is a C*-algebra of continuous functions on a compact, Hausdorff space X. For any compact, Hausdorff space X, let X_d consist of the underlying set of X equipped with the discrete topology. Then, the identity map from X_d to X uniquely extends to a continuous map from βX_d onto X and this, in turn, implies the existence of an embedding of $C(X)$ into $C(\beta X_d)$. The Čech–Stone compactification of a discrete space is always zero-dimensional and this establishes the previous claim.

Now, if X is a Stone space, consider the Boolean algebra B of all clopen subsets of X. Due to the Stone duality, the existence of a ccc forcing notion that forces the embedding of B into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ yields (in any generic extension of the universe) a continuous surjection from $\operatorname{St}(\mathcal{P}(\mathbb{N})/\mathrm{Fin}) \cong \beta\mathbb{N} \setminus \mathbb{N}$ onto $\operatorname{St}(B) \cong X$. By contravariance due to the Gelfand–Naimark duality, one obtains an injective $*$-homomorphism from $C(X)$ into $C(\beta\mathbb{N} \setminus \mathbb{N})$, with the latter being isomorphic to ℓ_∞/c_0.

Thus, we turn our attention to providing the forcing notion guaranteed by the following folklore proposition:

Proposition 1.2.2. *For every Boolean algebra B there exists a ccc forcing notion $\mathbb{P}_B$ which forces that B embeds into $\mathcal{P}(\mathbb{N})/Fin$.*

We identify the subsets of $\mathbb{N}$ with their characteristic functions, and we think them as elements of $2^{\mathbb{N}}$. With this in mind, we view the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$ as the space of all binary sequences $2^{\mathbb{N}}$ modulo the equivalence relation

$$x \sim y \text{ if and only if } |\{n \in \mathbb{N} : x(n) \neq y(n)\}| < \aleph_0$$

for all $x, y \in 2^{\mathbb{N}}$.

Definition 1.2.3. Fix a Boolean algebra B and let $\mathbb{P}_B$ be the set of all triples

$$p = (B_p, n_p, \psi_p)$$

where:

1. B_p is a finite Boolean subalgebra of B,
2. $n_p \in \mathbb{N}$,
3. $\psi_p : B_p \to 2^{n_p}$ is an arbitrary map.

For $p, q \in \mathbb{P}_B$, we say that p *extends* q and write $p < q$ if the following hold:

4. $B_q \subseteq B_p$,
5. $n_q < n_p$,
6. $\psi_q \subset \psi_p$ (i.e. $\psi_p(a)(i) = \psi_q(a)(i)$ for all $a \in B_q$ and $i \leq n_q$),
7. the map from B_q into $2^{n_p - n_q}$ given by

$$a \mapsto \psi_p(a)_{\restriction [n_q, n_p)}$$

is an injective homomorphism of Boolean algebras.

This defines a strict partial order on $\mathbb{P}_B$. Conditions in $\mathbb{P}_B$ represent partial maps from a finite subset of B to an initial segment of a characteristic function corresponding to a subset of $\mathbb{N}$. Any finite Boolean subalgebra of B is isomorphic to the Boolean algebra given by the powerset of a finite set and hence can be embedded into 2^m for $m \in \mathbb{N}$ large enough. Therefore one can always extend a given condition $p \in \mathbb{P}_B$ to a $q < p$ such that B_q contains any arbitrary finite subset of B and $n_q > n_p$, while making sure that in the added segment the map is actually an injective homomorphism. For this reason, a generic filter G in $\mathbb{P}_B$ provides a pool of maps which can be 'glued' together in a coherent way, inducing thus a function Ψ_G which, by genericity, is defined everywhere on B:

$$\begin{aligned} \Psi_G : B &\to \mathcal{P}(\mathbb{N}) \\ b &\mapsto \bigcup_{\{p \in G : b \in B_p\}} \psi_p(b). \end{aligned}$$

Here we identify $\psi_p(b) \in 2^{n_p}$ with the corresponding subset of n_p. Moreover, by definition of the order relation on $\mathbb{P}_B$, the map Ψ_G is, modulo the ideal of finite sets, injective and preserves all Boolean operations.

By using a standard uniformization argument and an application of the Δ-System Lemma (Lemma 1.1.6), when given an uncountable set of conditions $U \subseteq \mathbb{P}_B$, it is possible to find an uncountable $W \subseteq U$, $n \in \mathbb{N}$ and $Z \Subset B$ such that $n_p = n$, $B_p \cap B_q = Z$ and $\psi_p(b) = \psi_q(b)$ for all $p, q \in W$ and $b \in Z$. Thus the problem of whether $\mathbb{P}_B$ is ccc is reduced to the following:

Lemma 1.2.4. *Let $p, q \in \mathbb{P}_B$ be two conditions such that $n_p = n_q$ and the maps ψ_p, ψ_q agree on $B_p \cap B_q$. Then, p and q are compatible.*

To see that this holds, define B_s to be the (finite) Boolean subalgebra of B that is generated by $B_p \cup B_q$ and choose a Boolean algebra isomorphism $f : B_s \to 2^m$ for some $m \in \mathbb{N}$. Set $n_s = n_p + m$ and define the map ψ_s to be equal to ψ_p concatenated with f on B_p, equal to ψ_q concatenated with f on $B_q \setminus B_p$ and equal to zero elsewhere. Then, the condition $s = (B_s, n_s, \psi_s)$ extends both p and q.

1.2.2 Embedding Quasidiagonal C*-algebras into $\mathcal{Q}(H)$

Quasidiagonal C*-algebras possess strong local properties and can be thought (at least in the separable case) as consisting of compact pertubations of simultaneously block-diagonalisable operators. A map $\varphi : A \to B$ between unital C*-algebras is called *unital completely positive* (abbreviated as *u.c.p.*) if it is unital, linear and the tensor product map $\varphi \otimes \mathrm{Id}_n : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$ defined on matrix algebras over A and B is positive for all $n \in \mathbb{N}$ ([5], section II.6.9). U.c.p. maps are always contractive and $*$-preserving. For a C*-algebra A, we will denote its unitization by $\tilde{A}$.

Definition 1.2.5. A C*-algebra A is *quasidiagonal* if for every finite set $F \Subset \tilde{A}$ and $\epsilon > 0$, there exist $n \in \mathbb{N}$ and a u.c.p. map $\varphi : \tilde{A} \to M_n(\mathbb{C})$ such that

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \epsilon \text{ for all } a, b \in F$$

and

$$\|\varphi(a)\| > \|a\| - \epsilon \text{ for all } a \in F.$$

This section is devoted to the following:

Proposition 1.2.6. *For every quasidiagonal* C*-*algebra A there exists a ccc poset $\mathbb{QD}_A$ which forces an embedding of A into $\mathcal{Q}(H)$.*

As opposed to the proof of Theorem 1.0.2 in section 1.3, where we can apply Proposition 1.1.1, we will not assume that A is simple in the proof of Proposition 1.2.6. Such assumption would have made Definition 1.2.7 slightly simpler, but, to our knowledge, it is not known whether it is possible to embed a given quasidiagonal C*-algebra into a simple quasidiagonal one (an application of the Downward Löwenheim–Skolem Theorem ([27, Theorem 2.6.2]) would then provide a quasidiagonal simple C*-algebra with the same density character as the one we started with). We may assume though that A is unital. Fix $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis of H and for every $n \in \mathbb{N}$ let R_n be the orthogonal projection onto the linear span of the set $\{e_k : k \leq n\}$. Since for every $n \in \mathbb{N}$ the space $R_n\mathcal{B}(H)R_n$ is finite-dimensional, choose D_n a countable dense subset that contains R_n. For $n < m \in \mathbb{N}$, we also require that $D_n \subseteq R_n D_m R_n$.

Similar to the case of Boolean algebras, we define a forcing notion for a quasidiagonal C*-algebra whose conditions represent partial maps from a finite subset of A to an "initial segment" in $\mathcal{B}(H)$, which in this case is a *corner* $R_n\mathcal{B}(H)R_n$ for some $n \in \mathbb{N}$. Extensions of conditions are defined as to yield better approximations, maps are defined on a bigger domain and take values on a larger corner in $\mathcal{B}(H)$. It is only on a sufficient part of the larger corner that we shall request that the new maps preserve the norm of elements and all algebraic operations, modulo a small error (which disappears once one passes to the Calkin algebra).

Definition 1.2.7. Let A be a unital, quasidiagonal C*-algebra and define $\mathbb{QD}_A$ to be the set of all tuples

$$p = (F_p, n_p, \epsilon_p, \psi_p)$$

such that:

1. $F_p \Subset A$ is such that $1 \in F_p$,
2. $n_p \in \mathbb{N}$,
3. $\epsilon_p \in \mathbb{Q}^+$,
4. $\psi_p : F_p \to D_{n_p}$ is a unital map such that $\|\psi_p(a)\| \leq \|a\|$ for all $a \in F_p$. This map is *not* required to be linear or self-adjoint.

For $p, q \in \mathbb{QD}_A$, we write $p < q$ if the following hold:

5. $F_q \subseteq F_p$,
6. $n_q < n_p$,
7. $\epsilon_p < \epsilon_q$,
8. $\psi_p(a)R_{n_q} = R_{n_q}\psi_p(a) = \psi_q(a)$ for all $a \in F_q$,
9. $\|\psi_p(a)(R_{n_p} - R_{n_q})\| > \|a\| - \epsilon_q$ for all $a \in F_q$,
10. for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$ define

$$\begin{aligned} \Delta^{p,+}_{a,b,\lambda,\mu} &:= \psi_p(\lambda a + \mu b) - \lambda\psi_p(a) - \mu\psi_p(b), \\ \Delta^{p,*}_{a} &:= \psi_p(a^*) - \psi_p(a)^*, \\ \Delta^{p,\cdot}_{a,b} &:= \psi_p(ab) - \psi_p(a)\psi_p(b). \end{aligned}$$

    Then we require

    (a) $\|\Delta^{p,+}_{a,b,\lambda,\mu}(R_{n_p} - R_{n_q})\| < \epsilon_q - \epsilon_p$ if $a, b, \lambda a + \mu b \in F_q$,
    (b) $\|\Delta^{p,*}_{a}(R_{n_p} - R_{n_q})\| < \epsilon_q - \epsilon_p$ if $a, a^* \in F_q$,
    (c) $\|\Delta^{p,\cdot}_{a,b}(R_{n_p} - R_{n_q})\| < \epsilon_q - \epsilon_p$ if $a, b, ab \in F_q$.

Item 8 above displays the block-diagonal fashion of the extension of conditions and plays a crucial role in ascertaining that the relation $<$ is transitive. To demonstrate it, by considering multiplication as an example, for conditions $p < q < s$ in $\mathbb{QD}_A$ we have that

$$\begin{aligned} \|\Delta^{p,\cdot}_{a,b}(R_{n_p} - R_{n_s})\| &\leq \|\Delta^{p,\cdot}_{a,b}(R_{n_p} - R_{n_q})\| + \|\Delta^{p,\cdot}_{a,b}(R_{n_q} - R_{n_s})\| \\ &< \epsilon_q - \epsilon_p + \|\Delta^{p,\cdot}_{a,b}(R_{n_q} - R_{n_s})\|. \end{aligned}$$

Item 8 implies that

$$\psi_p(c)(R_{n_q} - R_{n_s}) = \psi_q(c)(R_{n_q} - R_{n_s}) = (R_{n_q} - R_{n_s})\psi_q(c)(R_{n_q} - R_{n_s}),$$

for all $c \in F_s$. Thus

$$\begin{aligned}\psi_p(a)\psi_p(b)(R_{n_q} - R_{n_s}) = \psi_p(a)(R_{n_q} - R_{n_s})\psi_q(b)(R_{n_q} - R_{n_s}) \\ = \psi_q(a)\psi_q(b)(R_{n_q} - R_{n_s}),\end{aligned}$$

which in turn yields

$$\|\Delta^{p,\cdot}_{a,b}(R_{n_q} - R_{n_s})\| < \epsilon_s - \epsilon_q.$$

Note that for any finite set $F \Subset A$ and $n \in \mathbb{N}$ there are only countably many maps $\psi : F \to D_n$ as in condition 4. This, along with a standard uniformization argument and an application of the Δ-System Lemma (Lemma 1.1.6), reduces (similarly to the case of Boolean algebras) the problem of whether the poset $\mathbb{QD}_A$ is ccc to the following:

Lemma 1.2.8. *Let $p, q \in \mathbb{QD}_A$ be two conditions such that $n_p = n_q, \epsilon_p = \epsilon_q$ and the maps ψ_p, ψ_q agree on $F_p \cap F_q$. Then, p and q are compatible.*

To see this, for $\epsilon_s = \epsilon_p/8$ and $F_s = F_p \cup F_q$, let $m \in \mathbb{N}$ and $\varphi : F_s \to M_m(\mathbb{C})$ be given as in Definition 1.2.5. By setting $n_s = n_p + m$, identifying $M_m(\mathbb{C})$ with the corner $(R_{n_s} - R_{n_p})\mathcal{B}(H)(R_{n_s} - R_{n_p})$ and approximating φ via the dense sets up to ϵ_s, define a map ψ_s which block-diagonally extends both ψ_p and ψ_q via this approximation of φ. In this manner, the resulting condition $s = (F_s, n_s, \epsilon_s, \psi_s) \in \mathbb{QD}_A$ extends both p and q.

The previously described argument also gives the basic idea of how to extend a given condition (allowing also to enlarge the domain) by diagonally adjoining a finite-dimensional block in which, modulo a small error, all algebraic operations and the norm of all elements are preserved. This hints that a generic filter induces (analogously to the case of Boolean algebras in the previous subsection; see also Proposition 1.3.5) a map from A into $\mathcal{Q}(H)$ which is an isometric (and thus injective) $*$-homomorphism.

1.3 The General Case

In this section we proceed to define the forcing notion $\mathbb{E}_A$ and give the proof of Theorem 1.0.2.

1.3.1 The Definition of the Poset

For what follows let A be a simple, unital C*-algebra. We begin by fixing an increasing countable family of projections $\mathcal{P} \subseteq \mathcal{B}(H)$ converging strongly to the identity and a countable dense subset C of $\mathcal{B}_{\mathrm{f}}(H)_+^{\leq 1}$. For $R \in \mathcal{P}$ and $h \in C$ let $S_{R,h}$ be the orthogonal projection onto the span of $h^+[H] \cup R[H]$, where h^+ and h^- are as defined in the first paragraph of the section on preliminary results of this chapter. Fix a countable dense subset

$$D_{R,h} \subseteq \{S_{R,h} T h^+ : T \in \mathcal{B}(H)\}$$

that contains h^+. We need the dense sets $D_{R,h}$ and C to satisfy certain closure properties in order to carry out the arguments below. We explicit these properties in detail here, but the reader can safely ignore them for now and come back to them when reading the proof of Proposition 1.3.4.

Definition 1.3.1. The countable sets C and $D_{R,h}$ previously defined are required to have the following closure properties.

1. For all $c_1, \ldots, c_k \in C$ and $R \in \mathcal{P}$, the intersection of C with the set (recall that $h \gg c$ stands for $hc = c$)

$$\{h \in \mathcal{B}_{\mathrm{f}}(H)_+^{\leq 1} : h \gg c_1, \ldots, h \gg c_k, h \geq R\}$$

is dense in the latter.

2. Given $R \in \mathcal{P}$ and $h, k \in C$, the intersection of $D_{R,h}$ with the set

$$\{T \in S_{R,h}\mathcal{B}(H)h^+ : Tk^-[H] \subseteq h^-[H], Th^-[H] \subseteq h^+[H]\}$$

is dense in the latter.

3. Given $R, R' \in \mathcal{P}$, $h_1, h_2, k \in C$, and $T' \in D_{R',h_2}$, the intersection of D_{R,h_1} with the set

$$\begin{aligned}\{T \in S_{R,h_1}\mathcal{B}(H)h_1^+ :\, & Th_1^+ = T', h_2^- T = h_2^- T', \\ & Tk^-[H] \subseteq h_1^-[H], Th_1^-[H] \subseteq h_1^+[H]\}\end{aligned}$$

is dense in the latter.

It is straightforward to build countable dense sets with such properties by countable iteration.[3] This idea appears in [86], where ccc forcing was used to study the poset of projections in the Calkin algebra.

Before proceeding to the definition of the poset, we pause to give some insight and justify the considerably higher complexity it possesses when compared with the abelian or quasidiagonal case. The rough idea is, again, to define a poset where each condition represents a partial map from a finite subset of A into some finite-dimensional corner of $\mathcal{B}(H)$ and where the ordering guarantees that stronger conditions behave like $*$-homomorphisms on larger and larger subspaces of H up to an error which tends to zero. The countable, dense sets $D_{R,h}$ considered in the beginning of this section serve as the codomains of these partial maps and, as a result, for any finite subset of A there are only countably many possible maps into any given corner. The main difference with the quasidiagonal case is that we cannot expect conditions to look like block-diagonal matrices anymore. This has troublesome consequences, mostly caused by the multiplication (and to a minor extent by the adjoint operation). The main issue is that, given $p < q$, one cannot expect that a property similar to condition 8 of Definition 1.2.7, that is

$$R_{n_q}\psi_p(a)(1 - R_{n_q}) = (1 - R_{n_q})\psi_p(a)R_{n_q} = 0,$$

can hold in general. As a first consequence (and with the comments succeeding Definition 1.2.7 in mind), even defining a partial order that is transitive proves to be non-trivial. An even bigger issue that comes up is the extension of a condition to a stronger one with larger domain. While in the quasidiagonal case it is sufficient to add a finite-dimensional block with some prescribed properties, completely ignoring how ψ_p is defined, in the general case one has to explicitly require for ψ_p to allow at least one extension in order to avoid $\mathbb{E}_A$ having atomic conditions[4]. To this end, the poset $\mathbb{E}_A$ is defined as follows:

Definition 1.3.2. Let $\mathbb{E}_A$ be the set of the tuples

$$p = (F_p, \epsilon_p, h_p, R_p, \psi_p)$$

where

1. $F_p \Subset A$, $1 \in F_p$ and if $a \in F_p$ then $a^* \in F_p$,

[3] A logician can use a large enough countable elementary submodel of a sufficiently large hereditary set containing all the relevant objects as a parameter to outright define these sets.

[4] Given a poset $(P, <)$, $p \in P$ is *atomic* if $q \leq p$ implies $q = p$.

2. $\epsilon_p \in \mathbb{Q}^+$,

3. $h_p \in C$,

4. $R_p \in \mathcal{P}$,

5. $\psi_p : F_p \to D_{R_p,h_p}$ and there exist a faithful, essential, unital $*$-homomorphism $\Phi_p : C^*(F_p) \to \mathcal{B}(H)$ and a projection $k_p \leq h_p^-$ such that for all $a \in F_p$

   (a) $k_p = k^-$ for some $k \in C$,

   (b) $\psi_p(1) = h_p^+$,

   (c) $\|(\psi_p(a) - \Phi_p(a))(h_p^+ - k_p)\| < \frac{\epsilon_p}{3M_p}$, where

$$L(F_p) = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \exists \mu \in \mathbb{C},\ \exists a, b \in F_p \\ \text{s.t. } a \neq 0 \text{ and } \lambda a + \mu b \in F_p\}$$

   and

$$M_p = \max\{3\|a\|, 3\|\psi_p(a)\|, L(F_p) : a \in F_p\},$$

   (d) $\|\psi_p(a) + \Phi_p(a)(1 - h_p^+)\| < \frac{3}{2}\|a\|$,

   (e) $\psi_p(a)k_p[H] \subseteq h_p^-[H]$ and $\psi_p(a)h_p^-[H] \subseteq h_p^+[H]$,

   (f) $\Phi_p(a)k_p[H] \subseteq h_p^-[H]$ and $\Phi_p(a)h_p^-[H] \subseteq h_p^+[H]$.

Such pair (k_p, Φ_p) will henceforth be referred to as a *promise* for the condition p. Given $p, q \in \mathbb{E}_A$, we say that p *is stronger than* q and write $p < q$ if and only if

6. $F_p \supseteq F_q$,

7. $\epsilon_p < \epsilon_q$

8. $h_p \gg h_q$,

9. $R_p \geq R_q$,

10. $\psi_p(a)h_q^+ = \psi_q(a)$ for all $a \in F_q$,

11. $h_q^-\psi_p(a) = h_q^-\psi_q(a)$ for all $a \in F_q$,

12. (a) $\|\Delta^{p,+}_{a,b,\lambda,\mu}(h_p^- - h_q^-)\| < \epsilon_q - \epsilon_p$ for $a, b, \lambda a + \mu b \in F_q$,

(b) $\|\Delta_a^{p,*}(h_p^- - h_q^-)\| < \epsilon_q - \epsilon_p$ for $a \in F_q$,

(c) $\|\Delta_{a,b}^{p,\cdot}(h_p^- - h_q^-)\| < \epsilon_q - \epsilon_p$ for $a, b, ab \in F_q$,

where the quantities $\Delta_{a,b,\lambda,\mu}^{p,+}$, $\Delta_a^{p,*}$ and $\Delta_{a,b}^{p,\cdot}$ are defined as in Definition 1.2.7.

Item 5e above is an example of how the problem of transitivity is addressed and this becomes clear in Claim 1.3.3.3 of the next proposition. The promise in item 5 is witnessing that there is at least one way to extend p (via Φ_p) to conditions with arbitrarily large (finite-dimensional) domain. We will see later (see Propositions 1.3.4, 1.3.6 and 1.3.7) how Theorem 1.1.2 and Corollary 1.1.3 imply that the choice of a specific Φ_p is not a real constraint on what extensions of p are going to look like.

Proposition 1.3.3. *The relation $<$ defined on $\mathbb{E}_A$ is transitive.*

Proof. Let $p, q, s \in \mathbb{E}_A$ be such that $p < q < s$. It is straightforward to check that conditions 6–9 hold between p and s. Items 10 and 11 follow since $h_q \gg h_s$ implies $h_q^- \geq h_s^+$. We recall that for two projections p, q the relation $p \leq q$ is equivalent to $pq = qp = p$. We divide the proof of condition 12 in three claims, one for each item.

Claim 1.3.3.1. *If $a, b, \lambda a + \mu b \in F_s$ then $\|\Delta_{a,b,\lambda,\mu}^{p,+}(h_p^- - h_s^-)\| < \epsilon_s - \epsilon_p$.*

Proof. We have

$$\|\Delta_{a,b,\lambda,\mu}^{p,+}(h_p^- - h_s^-)\| \leq \|\Delta_{a,b,\lambda,\mu}^{p,+}(h_p^- - h_q^-)\| + \|\Delta_{a,b,\lambda,\mu}^{p,+}(h_q^- - h_s^-)\|.$$

Since $p < q < s$, we know that $\psi_p(c)h_q^+ = \psi_q(c)$ for all $c \in F_q$ (item 10) and thus $\|\Delta_{a,b,\lambda,\mu}^{p,+}(h_q^- - h_s^-)\| = \|\Delta_{a,b,\lambda,\mu}^{q,+}(h_q^- - h_s^-)\|$. Hence we can conclude

$$\|\Delta_{a,b,\lambda,\mu}^{p,+}(h_p^- - h_q^-)\| + \|\Delta_{a,b,\lambda,\mu}^{p,+}(h_q^- - h_s^-)\| < \epsilon_q - \epsilon_p + \epsilon_s - \epsilon_q = \epsilon_s - \epsilon_p,$$

as required. □

Claim 1.3.3.2. *If $a \in F_s$ then $\|\Delta_a^{p,*}(h_p^- - h_s^-)\| < \epsilon_s - \epsilon_p$.*

Proof. We have

$$\|\Delta_a^{p,*}(h_p^- - h_s^-)\| \leq \|\Delta_a^{p,*}(h_p^- - h_q^-)\| + \|\Delta_a^{p,*}(h_q^- - h_s^-)\|.$$

Since $p < q < s$, for all $c \in F_q$ we have that $\psi_p(c)h_q^+ = \psi_q(c)$ and that $h_q^-\psi_p(c) = h_q^-\psi_q(c)$ (items 10 and 11). The latter relation entails that $\psi_p(c)^*h_q^- = \psi_q(c)^*h_q^-$. Thus, we conclude

$$\begin{aligned}\|\Delta_a^{p,*}(h_p^- - h_q^-)\| + \|\Delta_a^{p,*}(h_q^- - h_s^-)\| &= \|\Delta_a^{p,*}(h_p^- - h_q^-)\| + \|\Delta_a^{q,*}(h_q^- - h_s^-)\| \\ &< \epsilon_s - \epsilon_p,\end{aligned}$$

as required. $\square$

Claim 1.3.3.3. *If $a, b, ab \in F_s$ then $\|\Delta_{a,b}^{p,\cdot}(h_p^- - h_s^-)\| < \epsilon_s - \epsilon_p$.*

Proof. We have

$$\begin{aligned}\|\Delta_{a,b}^{p,\cdot}(h_p^- - h_s^-)\| &\leq \|\Delta_{a,b}^{p,\cdot}(h_p^- - h_q^-)\| + \|\Delta_{a,b}^{p,\cdot}(h_q^- - h_s^-)\| \\ &< \epsilon_q - \epsilon_p + \|\Delta_{a,b}^{p,\cdot}(h_q^- - h_s^-)\|.\end{aligned}$$

Since $\psi_p(c)h_q^+ = \psi_q(c)$ for all $c \in F_q$ (item 10) we get

$$(\psi_p(ab) - \psi_p(a)\psi_p(b))(h_q^- - h_s^-) = (\psi_q(ab) - \psi_p(a)\psi_q(b))(h_q^- - h_s^-)$$

and therefore $(\psi_p(ab) - \psi_p(a)\psi_p(b))(h_q^- - h_s^-)$ is equal to

$$\Delta_{a,b}^{q,\cdot}(h_q^- - h_s^-) + (\psi_q(a) - \psi_p(a))\psi_q(b)(h_q^- - h_s^-).$$

The rightmost term is zero since $\psi_q(b)\xi \in h_q^+[H]$ for all $\xi \in h_q^-[H]$ (item 5e) and $\psi_p(a)h_q = \psi_q(a)h_q$ (this follows from item 10). This ultimately leads to the thesis since $\|\Delta_{a,b}^{q,\cdot}(h_q^- - h_s^-)\| < \epsilon_s - \epsilon_q$. $\square$

This completes the proof. $\square$

1.3.2 Density and the Countable Chain Condition

As in Definition 1.3.2, for $F \Subset A$, let

$$\begin{aligned}L(F) = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ and } \exists \mu \in \mathbb{C}, \exists a, b \in F \\ \text{s.t. } a \neq 0 \text{ and } \lambda a + \mu b \in F\}\end{aligned}$$

and

$$J(F) = \max\{\|a\| : a \in F\}.$$

For $p \in \mathbb{E}_A$, let

$$M_p = \max\{3\|a\|, 3\|\psi_p(a)\|, L(F_p) : a \in F_p\}.$$

For $F \Subset A$ and $p \in \mathbb{E}_A$ let

$$M(p, F) = 3\max\{3M_p + 1, L(F), 2J(F) + 1\}.$$

Finally, for $p \in \mathbb{E}_A$ and a fixed promise (k_p, Φ_p) for the condition p, define the constants

$$N(p, \Phi_p) = \max\{\|(\psi_p(a) - \Phi_p(a))(h_p^+ - k_p)\| : a \in F_p\}$$

and

$$D(p, \Phi_p) = \min\{3\|a\|/2 - \|\psi_p(a) + \Phi_p(a)(1 - h_p^+)\| : a \in F_p\}.$$

The main density result reads as follows:

Proposition 1.3.4. *Given $F \Subset A$, $\epsilon \in \mathbb{Q}^+$, $h \in C$ and $R \in \mathcal{P}$, the set*

$$\mathcal{D}_{F,\epsilon,h,R} = \{p \in \mathbb{E}_A : F_p \supseteq F, \epsilon_p \leq \epsilon, h_p \gg h, R_p \geq R\}$$

is open dense in $\mathbb{E}_A$.

Proof. Clearly $\mathcal{D}_{F,\epsilon,h,R}$ is open. Fix a condition $q = (F_q, \epsilon_q, h_q, R_q, \psi_q)$ and let (k_q, Φ_q) be a promise for the condition q. By item 5c of Definition 1.3.2 there is a δ such that

$$N(q, \Phi_q) < \delta < \frac{\epsilon_q}{3M_q}$$

Fix moreover a *small enough* γ, more precisely

$$\gamma \leq \min\{\epsilon, \epsilon_q - 3M_q\delta, D(q, \Phi_q)\}.$$

Let $F_p = F_q \cup F \cup F^*$. Applying Theorem 1.1.2, let Φ be a faithful, essential, unital representation of $C^*(F_p)$ such that

$$\|\Phi_{\restriction F_q} - \Phi_{q\restriction F_q}\| < \frac{\gamma}{36M}$$

with $M = M(q, F_p)$. Consider, by condition 1 of Definition 1.3.1, an operator $k \in C$ such that $k \gg h, k \gg h_q, k \gg R_q$ and denote k^- by k_p. Let T be the finite-rank projection onto the space spanned by the set $\{\Phi(a)k[H] : a \in F_p\}$. By item 1 of Definition 1.3.1, since $T \gg k$, we can choose $l \in C$ such that $l \gg k$ and $l \approx_{\frac{\gamma}{18M}} T$. Moreover, by Lemma 1.1.4, picking l closer to T if needed, there is a unitary $u \in \mathcal{U}(H)$ such that

1. u is a compact perturbation of the identity,
2. $uT[H] \subseteq l[H]$,
3. u is the identity on $k_p[H]$ (since $l \gg k_p$),
4. $\|(\operatorname{Ad} u(\Phi(a)) - \Phi(a))k_p\| < \frac{\gamma}{36M}$ for all $a \in F_p$.

This entails that $\Phi' = \operatorname{Ad} u \circ \Phi$ is such that $\Phi'(a)k_p[H] \subseteq l[H]$ and

$$\|(\Phi'(a) - \Phi_q(a))k_p\| < \frac{\gamma}{18M}$$

for all $a \in F_q$. Let Q be the finite-rank projection onto the space spanned by the set $\{\Phi'(a)l[H] : a \in F_p\}$ and let K be the finite-rank operator equal to the identity on $l[H]$, equal to $\frac{1}{2}\operatorname{Id}$ on $Q(H) \cap l[H]^\perp$ (remember that $Q \geq l^+$ since $1 \in F_p$) and equal to zero on $Q[H]^\perp$. By item 1 of Definition 1.3.1 there is $h_p \in C$ such that $h_p \gg l$ and $h_p \approx_{\frac{\gamma}{15M}} K$. Moreover, by picking h_p closer to K if necessary, we may assume that $\dim(h_p Q[H]) = \dim(Q[H])$ and that $h_p^- = l^+$. The first equality can be obtained with the argument exposed at the beginning of the proof of Lemma 1.1.4, while the second is as follows: Suppose $\xi \in l[H]^\perp$ is a norm one vector, then $\xi = \xi_1 + \xi_2$, where ξ_1 and ξ_2 are orthogonal vectors of norm smaller than 1 such that $K\xi_1 = \frac{1}{2}\xi_1$ and $K\xi_2 = 0$. Hence, if h_p is close enough to K it follows that $\|h_p\xi\| < 1$. The equality $\dim(h_p Q[H]) = \dim(Q[H])$ allows us to find a unitary v such that

5. v is a compact perturbation of the identity,
6. v sends $Q[H]$ in $h_p[H]$,
7. v is the identity on $l[H]$.

The representation $\Phi_p = (\operatorname{Ad} v) \circ \Phi'$ is such that

8. $\Phi_p(a)k_p[H] \subseteq h_p^-[H]$ for all $a \in F_p$,
9. $\Phi_p(a)h_p^-[H] \subseteq h_p^+[H]$ for all $a \in F_p$,

10. $\|(\Phi_p(a) - \Phi_q(a))k_p\| < \frac{\gamma}{18M}$ for all $a \in F_q$.

Let $R_p \in \mathcal{P}$ be such that $R_p \geq R, R_p \geq R_q$ and

$$\|(1 - R_p)\Phi_p(a)h_p^+\| < \frac{\gamma}{18M}$$

for all $a \in F_p$. Consider now, given $a \in F_q$, the operator

$$\varphi(a) = \psi_q(a) + (1 - h_q^-)\Phi_p(a)(h_p^- - h_q^+) + (1 - h_q^-)R_p\Phi_p(a)(h_p^+ - h_p^-)$$

and for $a \in F_p \setminus F_q$ the operator

$$\varphi(a) = \Phi_p(a)h_p^- + R_p\Phi_p(a)(h_p^+ - h_p^-).$$

For all $a \in F_p$ we have $\varphi(a)k_p[H] \subseteq h_p^-[H]$ and $\varphi(a)h_p^-[H] \subseteq h_p^+[H]$. Moreover, for $a \in F_q$ we also have $\varphi(a)h_q^+ = \psi_q(a)$ and $h_q^-\varphi(a) = h_q^-\psi_q(a)$. Let $\psi_p : F_p \to D_{R_p,h_p}$ be a function such that:

11. $\psi_p(1) = h_p^+$,
12. for all $a \in F_p$, $\psi_p(a) \approx_{\frac{\gamma}{18M}} \varphi(a)$ and we also require that
    (a) $\psi_p(a)k_p[H] \subseteq h_p^-[H]$ and $\psi_p(a)h_p^-[H] \subseteq h_p^+[H]$ for all $a \in F_p$,
    (b) $\psi_p(a)h_q^+ = \psi_q(a)$ and $h_q^-\psi_p(a) = h_q^-\psi_q(a)$ for all $a \in F_q$.

Such a function ψ_p exists because of the requirements on D_{R_p,h_p} we asked in items 2 and 3 of Definition 1.3.1.

Claim 1.3.4.1. *For all $a \in F_p$ we have $\|(\psi_p(a) - \Phi_p(a))(h_p^+ - k_p)\| < \frac{\gamma}{6M}$.*

Proof. The inequality is trivially true for $a = 1$. For $a \in F_p \setminus F_q$ we have

$$\psi_p(a)(h_p^+ - k_p) \approx_{\frac{\gamma}{18M}} \Phi_p(a)(h_p^- - k_p) + R_p\Phi_p(a)(h_p^+ - h_p^-) \approx_{\frac{\gamma}{18M}} \Phi_p(a)(h_p^+ - k_p)$$

since $h_q^+(h_p^+ - k_p) = 0$, $(h_p^+ - k_p) \geq (h_p^- - h_q^+)$, $(h_p^+ - k_p) \geq (h_p^+ - h_p^-)$ and where the last approximation is a consequence of how we defined R_p, in particular of

$$\|(1 - R_p)\Phi_p(a)h_p^+\| < \frac{\gamma}{18M}.$$

Now let $a \in F_q \setminus \{1\}$. Similarly to the previous case we get

$$\psi_p(a)(h_p^+ - k_p) \approx_{\frac{\gamma}{9M}} (1 - h_q^-)\Phi_p(a)(h_p^+ - k_p).$$

By the definition of the promise (item 5f of Definition 1.3.2), we have that $(h_p^+ - h_q^+)\Phi_q(a)h_q^- = 0$. Remember that by definition of Φ_p we have

$$\|(\Phi_p(a) - \Phi_q(a))k_p\| < \frac{\gamma}{18M}.$$

Use this inequality and $k_p \geq h_q^-$ to infer that $(h_p^+ - h_q^+)\Phi_p(a)h_q^- \approx_{\frac{\gamma}{18M}} 0$. Since F_q is self-adjoint, we also obtain that

$$h_q^-\Phi_p(a)(h_p^+ - h_q^+) \approx_{\frac{\gamma}{18M}} 0.$$

This allows us to conclude that $\psi_p(a)(h_p^+ - k_p) \approx_{\frac{\gamma}{6M}} \Phi_p(a)(h_p^+ - k_p)$. $\square$

Claim 1.3.4.2. *For all $a \in F_p$ we have $\|\psi_p(a) + \Phi_p(a)(1 - h_p^+)\| < \frac{3}{2}\|a\|$.*

Proof. Let $a \in F_p \setminus F_q$. Then we have

$$\begin{aligned}\psi_p(a) + \Phi_p(a)(1 - h_p^+) \approx_{\frac{\gamma}{18M}} \Phi_p(a)h_p^- + R_p\Phi_p(a)(h_p^+ - h_p^-) \\ +\Phi_p(a)(1 - h_p^+) \approx_{\frac{\gamma}{18M}} \Phi_p(a).\end{aligned}$$

hence the thesis follows since $\|\Phi_p(a)\| \leq \|a\|$ and we can assume $\gamma \leq \|a\|$. Consider now $a \in F_q$. Since in the previous claim we showed that

$$h_q^-\Phi_p(a)(h_p^+ - h_q^+) \approx_{\frac{\gamma}{18M}} 0,$$

we have

$$\psi_p(a) + \Phi_p(a)(1 - h_p^+) \approx_{\frac{\gamma}{18M}} \varphi(a) + \Phi_p(a)(1 - h_p^+) \approx_{\frac{\gamma}{9M}} \psi_q(a) + \Phi_p(a)(1 - h_q^+).$$

Recall that $\Phi_p = (\operatorname{Ad} w) \circ \Phi$, where w is a unitary which behaves like the identity on k_p (hence on h_q^+ and R_q as well), thus $w(1 - h_q^+) = (1 - h_q^+)w$ and $\psi_q(a) = \operatorname{Ad} w(\psi_q(a))$ for all $a \in F_q$. Moreover Φ was defined so that

$$\|\Phi_{\restriction F_q} - \Phi_{q\restriction F_q}\| < \frac{\gamma}{36M}.$$

Therefore the following holds

$$\begin{aligned}\|\psi_q(a) + \Phi_p(a)(1 - h_q^+)\| &= \|\psi_q(a) + \Phi(a)(1 - h_q^+)\| \\ &\approx_{\frac{\gamma}{36M}} \|\psi_q(a) + \Phi_q(a)(1 - h_q^+)\| < \frac{3}{2}\|a\|,\end{aligned}$$

which implies the thesis since $\gamma \leq \|a\|$. $\square$

This finally entails that, letting $\epsilon_p = \frac{\gamma}{6}$,

$$p = (F_p, \epsilon_p, h_p, R_p, \psi_p)$$

is an element of $\mathcal{D}_{F,\epsilon,h,R}$. It is in fact straightforward to check that if γ is small enough, then $M_p \leq M = M(q, F_p)$. We are left with checking that $p < q$. The conditions 6–10 in Definition 1.3.2 follow from the definition of p.

Claim 1.3.4.3. *For all $a, b, \lambda a + \mu b \in F_q$ we have that $\|(\Delta^{p,+}_{a,b,\lambda,\mu})(h_p^- - h_q^-)\| < \epsilon_q - \epsilon_p$.*

Proof. Given $c \in F_q$ we have, by definition of δ (see the beginning of the proof), $\|(\psi_q(c) - \Phi_q(c))(h_q^+ - k_q)\| < \delta$, and the same is true if we replace $(h_q^+ - k_q)$ with $(h_p^- - h_q^-)$, since $(h_q^+ - k_q) \geq (h_p^- - h_q^-)$. Moreover, by definition of Φ_p, $\|(\Phi_p(c) - \Phi_q(c))k_p\| < \frac{\gamma}{18M}$ holds. This, along with the fact that F_q is self-adjoint, $\Phi_q(c)h_q^-[H] \subseteq h_q^+[H]$ (item 5f of Definition 1.3.2) and $k_p \geq h_q^+$, entails that $\|h_q^- \Phi_p(c)(h_p^+ - k_p)\| < \frac{\gamma}{18M}$. Therefore

$$(\Delta^{p,+}_{a,b,\lambda,\mu})(h_p^- - h_q^-) \approx_{\frac{\gamma}{6}} (\varphi(\lambda a + \mu b) - \lambda\varphi(a) - \mu\varphi(b))(h_p^- - h_q^-) \approx_{3M_q\delta + \frac{\gamma}{3}} 0$$

as required. $\square$

Claim 1.3.4.4. *For all $a \in F_q$ we have $\|(\Delta^{p,*}_a)(h_p^- - h_q^-)\| < \epsilon_q - \epsilon_p$.*

Proof. Using approximations analogous to previous claim, we have that

$$\begin{aligned}(\Delta^{p,*}_a)(h_p^- - h_q^-) &\approx_{\frac{\gamma}{9}} (\varphi(a^*) - \varphi(a)^*)(h_p^- - h_q^-) \\ &\approx_{\delta + \frac{\gamma}{9}} (\Phi_p(a^*) - \psi_q(a)^* - (h_p^- - h_q^+)\Phi_p(a^*)(1 - h_q^-) \\ &\quad - (h_p^+ - h_p^-)\Phi_p(a^*)R_p(1 - h_q^-))(h_p^- - h_q^-).\end{aligned}$$

Since F_p is self-adjoint and by definition of R_p

$$\|h_p^+ \Phi_p(c)(1 - R_p)\| < \frac{\gamma}{18M}$$

for all $c \in F_q$, thus $(h_p^+ - h_p^-)\Phi_p(a^*)R_p(1-h_q^-) \approx_{\frac{\gamma}{18M}} (h_p^+ - h_p^-)\Phi_p(a^*)(1-h_q^-)$. Hence we obtain

$$(\Delta_a^{p,*})(h_p^- - h_q^-) \approx_{\delta + 5\frac{\gamma}{18}} (\Phi_p(a^*) - \psi_q(a)^* - (h_p^+ - h_q^+)\Phi_p(a^*)(1-h_q^-))(h_p^- - h_q^-).$$

Furthermore we have

$$\begin{aligned}\psi_q(a)^*(h_p^- - h_q^-) = ((h_p^- - h_q^-)\psi_q(a))^* &= ((h_p^- - h_q^-)\psi_q(a)h_q^+)^* \\ &= ((h_p^- - h_q^-)\psi_q(a)(h_q^+ - k_q))^*,\end{aligned}$$

where the last equality is a consequence of $\psi_q(c)k_qH \subseteq h_q^-H$ for all $c \in F_q$ (item 5e of Definition 1.3.2). Since

$$\|(\psi_q(c) - \Phi_q(c))(h_q^+ - k_q)\| < \delta, \ \|(\Phi_p(c) - \Phi_q(c))k_p\| < \frac{\gamma}{18M},$$

we get that

$$(\Delta_a^{p,*})(h_p^- - h_q^-) \approx_{2\delta + \frac{\gamma}{3}} \Phi_p(a^*)(h_p^- - h_q^-) - (h_p^+ - k_q)\Phi_p(a^*)(h_p^- - h_q^-).$$

Moreover, by how we defined Φ_p we have

$$\Phi_p(a^*)(h_p^- - h_q^-) = h_p^+\Phi_p(a^*)(h_p^- - h_q^-)$$

and

$$(1-h_q^-)\Phi_p(c)k_q \approx_{\frac{\gamma}{18M}} (1-h_q^-)\Phi_q(c)k_q = 0$$

for all $c \in F_q$. This last approximation entails, since F_q is self-adjoint, that

$$\|k_q\Phi_p(c)(1-h_q^-)\| < \frac{\gamma}{18M}$$

for all $c \in F_q$. □

Claim 1.3.4.5. *For all $a, b, ab \in F_q$ we have $\|(\Delta_{a,b}^{p,\cdot})(h_p^- - h_q^-)\| < \epsilon_q - \epsilon_p$.*

Proof. Similarly to the previous claims, we have the following approximations

$$\begin{aligned}(\Delta_{a,b}^{p,\cdot})(h_p^- - h_q^-) &\approx_{\frac{\gamma}{6}} (\varphi(ab) - \varphi(a)\varphi(b))(h_p^- - h_q^-) \\ &\approx_{2M_q\delta + \frac{2\gamma}{9}} \|(\Phi_p(ab) - \varphi(a)\Phi_p(b))(h_p^- - h_q^-)\|.\end{aligned}$$

As noted in the previous claim, for all $c \in F_q$ we have

$$\|k_q\Phi_p(c)(1 - h_q^-)\| < \frac{\gamma}{18M},$$

hence the same is true with $(h_p^- - h_q^-)$ in place of $(1 - h_q^-)$. Thus

$$\begin{aligned}\varphi(a)\Phi_p(b)(h_p^- - h_q^-) &\approx_{\frac{\gamma}{18M}} \varphi(a)(1 - k_q)\Phi_p(b)(h_p^- - h_q^-) \\ &\approx_{M_q\delta + \frac{\gamma}{6}} \Phi_p(a)(1 - k_q)\Phi_p(b)(h_p^- - h_q^-) \\ &\approx_{\frac{\gamma}{18M}} \Phi_p(a)\Phi_p(b)(h_p^- - h_q^-),\end{aligned}$$

as required. $\square$

This completes the proof. $\square$

Let B be the $(\mathbb{Q} + i\mathbb{Q})$-$*$-algebra generated by a dense subset of A with cardinality equal to the density character of A. We define the family $\mathcal{D}$ as follows (C and $\mathcal{P}$ were defined at the beginning of §1.3):

$$\mathcal{D} = \{\mathcal{D}_{F,\epsilon,h,R} : F \Subset B, \epsilon \in \mathbb{Q}^+, h \in C, R \in \mathcal{P}\}.$$

Proposition 1.3.5. *Suppose there exists a $\mathcal{D}$-generic filter G for $\mathbb{E}_A$. Then there exists a unital embedding of A into the Calkin algebra.*

Proof. Let G be a $\mathcal{D}$-generic filter and fix $a \in B$. The net $\{\psi_p(a)\}_{\{p\in G: a\in F_p\}}$ (indexed according to $(G, >)$, which is directed since G is a filter) is strongly convergent in $\mathcal{B}(H)$. Indeed, by Proposition 1.3.4 let

$$p = p_0 > p_1 > \cdots > p_n > \dots$$

be an infinite decreasing sequence of elements of G satisfying that $a \in F_p$, $\epsilon_{p_n} < 1/n$ and such that the sequence $\{h_{p_n}\}_{n\in\mathbb{N}}$ is an approximate unit for $\mathcal{K}(H)$ (which is possible by density of C and by genericity of G). The sequence $\{\psi_{p_n}(a)\}_{n\in\mathbb{N}}$ is strongly convergent to an operator in $\mathcal{B}(H)$ (since $\|\psi_{p_n}(a)\| < 3\|a\|/2$) which we denote by $\Psi(a)$. In order to show that the whole net $\{\psi_p(a)\}_{\{p\in G: a\in F_p\}}$ strongly converges to $\Psi(a)$, let $\xi_1, \ldots, \xi_k$ be norm one vectors belonging to $h_{p_n}[H]$ for some $n \in \mathbb{N}$. Then, for all $q \in G$ such that $q < p_n$ we have

$$\psi_q(a)\xi_j = \psi_q(a)h^+_{p_n}\xi_j = \psi_{p_n}(a)\xi_j$$

for all $j \leq k$. Since $\epsilon_n \to 0$ as $n \to \infty$, and $\{h_{p_n}[H] : n \in \mathbb{N}\}$ is dense in H (by genericity, $(h_{p_n})_{n\in\mathbb{N}}$ is an approximate unit of $\mathcal{K}(H)$), it follows that the net strongly converges to $\Psi(a)$ on H. Let $\Phi_G = \pi \circ \Psi$.

Claim 1.3.5.1. *The map $\Phi_G : B \to \mathcal{Q}(H)$ defined above is a unital, bounded $*$-homomorphism of $(\mathbb{Q} + i\mathbb{Q})$-algebras.*

Proof. For $a, b \in B$, we will prove that $\Psi(ab) - \Psi(a)\Psi(b)$ is compact. Let $\epsilon > 0$ and pick $p \in G$ such that $a, b, ab \in F_p$ and $\epsilon_p < \epsilon$. We claim that

$$\|(\Psi(ab) - \Psi(a)\Psi(b))(1 - h^-_p)\| < \epsilon.$$

Suppose this fails, and let $\xi \in (1 - h^-_p)H$ be a norm one vector such that

$$\|(\Psi(ab) - \Psi(a)\Psi(b))\xi\| > \epsilon.$$

By genericity of G we can find $q \in G$ such that $q < p$ and

$$\|(\Psi(ab) - \Psi(a)\Psi(b))\eta\| > \epsilon$$

where $\eta = h_q\xi$. Now let $s < q$ in G such that $\Psi(b)\eta$ is close enough to $h_s\Psi(b)\eta$ to obtain

$$\|(\psi_s(ab) - \psi_s(a)\psi_s(b))\eta\| > \epsilon.$$

But this is a contradiction since $s < p$ implies

$$\|(\psi_s(ab) - \psi_s(a)\psi_s(b))(h^-_s - h^-_p)\| < \epsilon_p < \epsilon.$$

Similarly it can be checked that Φ_G is $(\mathbb{Q} + i\mathbb{Q})$-linear and self-adjoint. Moreover, Φ_G is bounded since Ψ is. The claim follows since Ψ maps the unit of A to the identity on H. $\square$

By extending Φ_G to the complex linear span of B, we obtain a unital and bounded

$*$-homomorphism into the Calkin algebra. This is a dense (complex) $*$-subalgebra of A, hence we can uniquely extend to obtain a unital $*$-homomorphism from A into $\mathcal{Q}(H)$, which is injective, since A is simple. $\square$

Note that the fact that Φ_G above is bounded is crucial in allowing one to extend it and obtain a $*$-homomorphism defined on all of the algebra A. To see how this can fail, the identity map on the (algebraic) group algebra of any non-amenable discrete group cannot be extended to a $*$-homomorphism from the reduced group C*-algebra to the universal one (see [8, Theorem 2.6.8]).

With the only part of Theorem 1.0.2 remaining unproven being the fact that the poset is ccc, we begin with the following lemma yielding sufficient conditions for the compatibility of elements of $\mathbb{E}_A$.

Lemma 1.3.6. *Suppose that $p, q \in \mathbb{E}_A$ satisfy the following conditions.*

1. *$h_p = h_q$ and $R_p = R_q$.*
2. *$\psi_p(a) = \psi_q(a)$ for all $a \in F_p \cap F_q$.*
3. *There exist two unital $*$-homomorphisms $\Phi_p : C^*(F_p) \to \mathcal{B}(H)$ and $\Phi_q : C^*(F_q) \to \mathcal{B}(H)$ which are faithful and essential, and a projection k satisfying the following:*
   - *(a) The pairs (k, Φ_p) and (k, Φ_q) are promises for p and q, respectively.*
   - *(b) There are constants δ_p and δ_q such that $N(p, \Phi_p) < \delta_p < \frac{\epsilon_p}{3M_p}$ and $N(q, \Phi_q) < \delta_q < \frac{\epsilon_q}{3M_q}$, and if*

     $$\gamma \leq \min\{\epsilon_p - 3M_p\delta_p, D(p, \Phi_p), \epsilon_q - 3M_q\delta_q, D(q, \Phi_q)\}$$

     and

     $$M = \max\{M(p, F_p \cup F_q), M(q, F_p \cup F_q)\},$$

     then every $a \in F_p \cap F_q$ satisfies $\|\Phi_p(a) - \Phi_q(a)\| < \frac{\gamma}{18M}$.
   - *(c) There is a trivial embedding $\Theta : C^*(F_p \cup F_q) \to \mathcal{Q}(H)$ such that $\pi \circ \Phi_p = \Theta_{\restriction C^*(F_p)}$ and $\pi \circ \Phi_q = \Theta_{\restriction C^*(F_q)}$.*

Then p and q are compatible.

Proof. Write h for h_p and R for R_q. Let Φ be a faithful, essential, unital representation that lifts Θ to $\mathcal{B}(H)$. Since Φ_p and $\Phi_{\restriction F_p}$ agree modulo the compacts, and Φ_q and $\Phi_{\restriction F_q}$ agree modulo the compacts, there exists (by condition 1 of Definition 1.3.1) $k \in C$ such that $k \gg h$, $k \gg R$, and in addition the following holds: For all $a \in F_p$ we have

$$\|(\Phi_p(a) - \Phi(a))(1 - k^-)\| < \frac{\gamma}{36M},$$

and for all $a \in F_q$ we have

$$\|(\Phi_q(a) - \Phi(a))(1 - k^-)\| < \frac{\gamma}{36M}.$$

We shall denote k^- by k_s. Arguing as in the first part of the proof of Proposition 1.3.4 we can find $h_s \gg k_s$ (i.e. $h_s^- \geq k_s$) in C and a unitary w such that

1. w is a compact perturbation of the identity,
2. $wk_s = k_s w = k_s$,

and by letting $\Phi'_p = (\operatorname{Ad} w) \circ \Phi_p$, $\Phi'_q = (\operatorname{Ad} w) \circ \Phi_q$ and $\Phi' = (\operatorname{Ad} w) \circ \Phi$, we also have that

3. $\|(\Phi'_p(a) - \Phi_p(a))k_s\| < \frac{\gamma}{36M}$ for all $a \in F_p$,
4. $\|(\Phi'_q(a) - \Phi_q(a))k_s\| < \frac{\gamma}{36M}$ for all $a \in F_q$,
5. $\|(\Phi'(a) - \Phi(a))k_s\| < \frac{\gamma}{36M}$ for all $a \in F_p \cup F_q$,
6. $\Phi'_p(a)k_s[H] \subseteq h_s^-[H]$ and $\Phi'_p(a)h_s^-[H] \subseteq h_s^+[H]$ for all $a \in F_p$,
7. $\Phi'_q(a)k_s[H] \subseteq h_s^-[H]$ and $\Phi'_q(a)h_s^-[H] \subseteq h_s^+[H]$ for all $a \in F_q$,
8. $\Phi'(a)k_s[H] \subseteq h_s^-[H]$ and $\Phi'(a)h_s^-[H] \subseteq h_s^+[H]$ for all $a \in F_p \cup F_q$.

Let $R_s \in \mathcal{P}$ be such that $R_s \geq R$ and for all $a \in F_p$ and all $b \in F_q$ we have

$$\|(1 - R_s)\Phi'_p(a)h_s^+\| < \frac{\gamma}{18M},$$
$$\|(1 - R_s)\Phi'_q(b)h_s^+\| < \frac{\gamma}{18M}.$$

Given $a \in F_p$, consider the operator

$$\varphi(a) = \psi_p(a) + (1 - h^-)\Phi'_p(a)(h_s^- - h^+) + (1 - h^-)R_s\Phi'_p(a)(h_s^+ - h_s^-)$$

and for $a \in F_q \setminus F_p$

$$\varphi(a) = \psi_q(a) + (1 - h^-)\Phi'_q(a)(h_s^- - h^+) + (1 - h^-)R_s\Phi'_q(a)(h_s^+ - h_s^-).$$

Define now the function $\psi_s : F_p \cup F_q \to D_{R_s,h_s}$ as an approximation of φ in the same way it was done in the proof of Proposition 1.3.4. Suitably adapting the arguments in such proof to the present situation it is possible to show that

$$s = (F_p \cup F_q, \gamma/6, h_s, R_s, \psi_s)$$

is an element of $\mathbb{E}_A$ with promise (k_s, Φ'). We follow the proof of Claim 1.3.4.1 in order to check that the quantity $\|(\psi_s(a) - \Phi'(a))(h_s^+ - k_s)\|$ is small enough for $a \in F_p \cup F_q$, using in addition that for all $a \in F_p$

$$\|(\Phi_p(a) - \Phi(a))(1 - k_s)\| < \frac{\gamma}{36M}$$

and that for all $a \in F_q$

$$\|(\Phi_q(a) - \Phi(a))(1 - k_s)\| < \frac{\gamma}{36M}.$$

This entails the same inequality between Φ'_p and Φ' (and between Φ'_q and Φ') since the unitary w fixes k_s. The proofs of $s < p$ and $s < q$ go along the lines of those in Claim 1.3.4.3, 1.3.4.4 and 1.3.4.5, keeping the following caveat in mind: It might happen, for instance, that p and q are such that $a \in F_p \cap F_q$ and $b, ab \in F_q \setminus F_p$. In this case $\Delta^{q,\cdot}_{a,b}(h_s^- - h_q^-)$ can be approximated (following the proof of Claim 1.3.4.5) as $(\Phi_q(ab) - \Phi_p(a)\Phi_q(b))(h_s^- - h_q^-)$. This is where the condition $\Phi_p(a) \approx_{\frac{\gamma}{18M}} \Phi_q(a)$, required in item 3b of the statement of the present lemma, plays a key role, showing that the latter term is close to zero. The same argument applies for the analogous situations where Φ_p and Φ_q appear in the same formulas for the addition and the adjoint operation. $\square$

Property K is a strengthening of the countable chain condition (see section §1.1.2).

Proposition 1.3.7. *The poset $\mathbb{E}_A$ has property K and hence satisfies the countable chain condition.*

Proof. Let $\{p_\alpha : \alpha < \aleph_1\}$ be a set of conditions[5] in $\mathbb{E}_A$ and for each $\alpha < \aleph_1$ fix a promise (k_α, Φ_α) for the condition p_α. By passing to an uncountable subset if necessary, we may

[5]We suppress the notation and denote F_{p_α} by F_α, ϵ_{p_α} by ϵ_α, etc.

assume $\epsilon_\alpha = \epsilon$, $h_\alpha = h$, $R_\alpha = R$, $k_\alpha = k$ for all $\alpha < \aleph_1$. An application of the Δ-System Lemma (Lemma 1.1.6) yields a finite set $Z \Subset A$ such that $F_\alpha \cap F_\beta = Z$ for all $\alpha, \beta < \aleph_1$. Since Z is finite and $D_{R,h}$ is countable, we can furthermore assume that for all $\alpha, \beta < \aleph_1$ if $a \in F_\alpha \cap F_\beta$ then $\psi_\alpha(a) = \psi_\beta(a)$. Consider

$$F = \bigcup_{\alpha < \aleph_1} F_\alpha.$$

By [28] there is a locally trivial embedding $\Theta : C^*(F) \to \mathcal{Q}(H)$. For each $\alpha < \aleph_1$ fix a lift $\Theta_\alpha : C^*(F_\alpha) \to \mathcal{B}(H)$ of $\Theta_{\restriction C^*(F_\alpha)}$. Corollary 1.1.3 applied to Φ_α and Θ_α provides a faithful, essential, unital $\Phi'_\alpha : C^*(F_\alpha) \to \mathcal{B}(H)$ such that

1. $\Phi'_\alpha(a) - \Theta_\alpha(a) \in \mathcal{K}(H)$ for all $a \in F_\alpha$, hence $\pi \circ \Phi'_\alpha = \Theta_{\restriction C^*(F_\alpha)}$,
2. $\Phi'_\alpha(a) h^+_\alpha = \Phi_\alpha(a) h^+_\alpha$ for all $a \in F_\alpha$.

This entails that the pair (k_α, Φ'_α) is still a promise for p_α. Hence, with no loss of generality, we can assume $\pi \circ \Phi_\alpha = \Theta_{\restriction C^*(F_\alpha)}$ for every $\alpha < \aleph_1$. This in particular implies that

$$\Phi_\alpha(a) \sim_{K(H)} \Phi_\beta(a), \text{ for all } a \in Z.$$

Fix an arbitrary $\gamma > 0$. We can assume that for all $\alpha, \beta \in \aleph_1$ and all $a \in F_\alpha \cap F_\beta$

$$\|\Phi_\alpha(a) - \Phi_\beta(a)\| < \gamma.$$

Indeed, start by fixing $\delta < \aleph_1$. Then for each $\alpha < \aleph_1$ there is $P_\alpha \in \mathcal{P}$ such that

$$\|(\Phi_\alpha - \Phi_\delta)_{\restriction Z}(1 - P_\alpha)\| < \gamma/5$$

and $R_\alpha \in \mathcal{P}$ such that

$$\|(1 - R_\alpha)\Phi_{\alpha \restriction Z} P_\alpha\| < \gamma/5.$$

We can assume $R_\alpha = R$ and $P_\alpha = P$ for all $\alpha < \aleph_1$ and since $R\mathcal{B}(H)P$ is finite-dimensional we can also require that

$$\|R(\Phi_\alpha - \Phi_\beta)_{\restriction Z} P\| < \gamma/5$$

for all $\alpha, \beta < \aleph_1$. Thus, for $a \in Z$, we have that:

$$\begin{aligned}\|\Phi_\alpha(a) - \Phi_\beta(a)\| \leq \|(\Phi_\alpha - \Phi_\beta)_{\restriction Z} P\| + \|(\Phi_\alpha - \Phi_\delta)_{\restriction Z}(1-P)\| \\ + \|(\Phi_\beta - \Phi_\delta)_{\restriction Z}(1-P)\| < \gamma.\end{aligned}$$

Since the choice of γ was arbitrary, Lemma 1.3.6 implies that we can pass to an uncountable subset in which any two conditions p_α and p_β are compatible. □

We quickly recall that Martin's Axiom, MA, asserts that for every ccc poset $\mathbb{P}$ and every family $\mathcal{D}$ of fewer than $2^{\aleph_0}$ dense open subsets there exists a filter in $\mathbb{P}$ intersecting all sets in $\mathcal{D}$.

Corollary 1.3.8. *Assume MA. Then every* C^**-algebra with density character strictly less than* $2^{\aleph_0}$ *embeds into the Calkin algebra.*

Proof. By Proposition 1.1.1 it suffices to prove the statement for unital and simple C^*-algebras. For any unital and simple C^*-algebra A, the collection $\mathcal{D}$ of open, dense subsets of $\mathbb{E}_A$ (as defined prior to Proposition 1.3.5) has cardinality equal to the density character of A. Since the poset $\mathbb{E}_A$ is ccc, this implies that if the density character of A is strictly less than $2^{\aleph_0}$, then Martin's Axiom ensures the existence of a $\mathcal{D}$-generic filter for $\mathbb{E}_A$ and the corollary follows by Proposition 1.3.5. □

1.4 Concluding remarks on Theorem 1.0.2

The Calkin algebra is a fascinating object and the previous result is the first step in what we believe is a very promising direction of its study. A further step would be to have a simpler forcing notion in place of $\mathbb{E}_A$ defined in the course of the proof of Theorem 1.0.2. This would allow for an analysis of the names for C^*-subalgebras of $\mathcal{Q}(H)$ and better control of the structure of $\mathcal{Q}(H)$ in the extension. In particular, it would be a step towards proving that a given C^*-algebra can be 'gently placed' into $\mathcal{Q}(H)$ (cf. [87, p. 17–18]). In this regard, we conjecture the following.

Conjecture 1.4.1. Let A be an abelian and nonseparable C^*-algebra. If the density character of A is greater than $2^{\aleph_0}$, then $\mathbb{E}_A$ forces that A does not embed into ℓ_∞/c_0.

We now propose related directions of study, taking inspiration from the commutative setting.

1.4.1 Complete embeddings

From the very beginnings of forcing, it has been known that a given partial ordering E can be embedded into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ by a ccc forcing. The simplest such forcing notion was denoted $\mathcal{H}_E$ and studied in [22] where it was proved that $\mathcal{H}_E$ embeds E into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ in a minimal way: if a cardinal $\kappa > 2^{\aleph_0}$ is such that E does not have a chain of order type κ or κ^*, then in the forcing extension $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ does not have chains of order type κ or κ^* (this is a consequence of [22, Theorem 9.1]).

Given a forcing notion $\mathbb{P}$, its subordering $\mathbb{P}_0$ is a *complete subordering* of $\mathbb{P}$ if for every generic filter $G \subseteq \mathbb{P}_0$ one can define a forcing notion $\mathbb{P}/G$ such that $\mathbb{P}$ is forcing equivalent to the two-step iteration $\mathbb{P}_0 * \mathbb{P}/G$ (for an intrinsic characterization of this relation see [50, Definition III.3.65]). A salient property of the forcing notion $\mathcal{H}_E$ is that the map $E \mapsto \mathcal{H}_E$ is a covariant functor from the category of partial orderings and order-isomorphic embeddings as maps into the category of forcing notions with complete embeddings as morphisms. This is a consequence of [22, Proposition 4.2], where the compatibility relation in $\mathcal{H}_E$ has been shown to be 'local' in the sense that the conditions p and q are compatible in $\mathcal{H}_{\operatorname{supp}(p)\cup\operatorname{supp}(q)}$ if and only if they are compatible in $\mathcal{H}_E$.

Analogous arguments show that the mapping $B \mapsto \mathbb{P}_B$ defined on Section 1.2.1 is a covariant functor from the category of Boolean algebras and injective homomorphisms into the category of ccc forcing notions with complete embeddings as morphisms. As a result, if D is a Boolean subalgebra of B and G is $\mathbb{P}_D$-generic, then forcing with the poset $\mathbb{P}_B$ is equivalent to first forcing with $\mathbb{P}_D$ and then with $\mathbb{P}_B/G$.

It is not difficult to prove that the association $A \mapsto \mathbb{QD}_A$ as in Proposition 1.2.6 does not have this property, as $\mathbb{QD}_{\mathbb{C}}$, naturally considered as a subordering of $\mathbb{QD}_{M_2(\mathbb{C})}$, is not a complete subordering. More generally, if m is a proper divisor of n then the poset $\mathbb{QD}_{M_m(\mathbb{C})}$ is not a complete subordering of $\mathbb{QD}_{M_n(\mathbb{C})}$. We do not know whether there is an alternative definition of a functor $A \mapsto \mathbb{QD}_A$ that satisfies the conclusion of Proposition 1.2.6. The latter remark also applies to the poset $\mathbb{E}_A$ given in Theorem 1.0.2.

1.4.2 $2^{\aleph_0}$-universality

One line of research building on Theorem 1.0.1 would be to understand which C*-algebras of density character $2^{\aleph_0}$ embed into the Calkin algebra. Before discussing this matter, we introduce a definition. Given a cardinal λ, a C*-algebra A is *(injectively) λ-universal* if it has density character λ and all C*-algebras of density character λ embed into A. By

[41, Theorem 2.3 and Remark 2.10], there is no κ-universal C*-algebra in any density character $\kappa < 2^{\aleph_0}$ The results in [28] entail that the $2^{\aleph_0}$-universality of the Calkin algebra is independent from ZFC. On the one hand CH implies that $\mathcal{Q}(H)$ is $2^{\aleph_0}$-universal. Conversely, the Proper Forcing Axiom implies that $\mathcal{Q}(H)$ is not $2^{\aleph_0}$-universal because some abelian C*-algebras of density $2^{\aleph_0}$ do not embed into it (see [75, Corollary 5.3.14 and Theorem 5.3.15]), and this is part of a larger family of results on the rigidity of nonseparable quotient structures (see [54], [76]). Can the Calkin algebra be $2^{\aleph_0}$-universal even when the Continuum Hypothesis fails? The analogous fact for $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and linear orders, namely that there is a model of ZFC where CH fails and all linear orders of size $2^{\aleph_0}$ embed into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, has been proved in [52] (see also [4] for the generalization to Boolean algebras). We do not know whether these techniques can be generalized to provide a model in which CH fails and the Calkin algebra is a $2^{\aleph_0}$-universal C*-algebra, but the fact that $\mathbb{E}_A$ has property K is a step towards such a model. A poset with property K is *productively ccc*, in the sense that its product with any ccc poset is still ccc. A salient feature of the forcing iterations used in both [52] and [4] is that they are not 'freezing' any gaps in $\mathbb{N}^{\mathbb{N}}/\operatorname{Fin}$ and $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$[6].

Lemma 1.4.2. *For any* C*-*algebra A, the poset $\mathbb{E}_A$ cannot freeze any gaps in* $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

Proof. Every gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ or $\mathbb{N}^{\mathbb{N}}/\operatorname{Fin}$ that can be split without collapsing $\aleph_1$ can be split by a ccc forcing. This is well-known result of Kunen ([49]) not so easy to find in the literature.[7] Therefore if a gap can be split by a ccc forcing $\mathbb{P}$, then a poset which freezes it destroys the ccc-ness of $\mathbb{P}$. But $\mathbb{E}_A$ has property K, and is therefore productively ccc. □

While the gap spectra of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and $\mathbb{N}^{\mathbb{N}}/\operatorname{Fin}$ are closely related, the gap spectrum of the poset of projections in the Calkin algebra is more complicated. The following proposition was proved, but not stated, in [88], and we include a proof for reader's convenience.

Theorem 1.4.3. *Martin's Axiom implies that the poset of projections in the Calkin algebra contains a $(2^{\aleph_0}, 2^{\aleph_0})$-gap which cannot be frozen.*

Proof. By [88, Theorem 4], there exists (in ZFC) a gap in this poset whose sides are analytic and σ-directed. This gap cannot be frozen, and Martin's Axiom is used only to 'linearize' it. By the discussion following [88, Corollary 2], each of the sides of this gap is Tukey equivalent to the ideal of Lebesgue measure zero sets ordered by the inclusion. Since the additivity

[6] A gap is *frozen* if it cannot be split in a further forcing extension without collapsing $\aleph_1$.

[7] See e.g., [72, Fact on p. 76]. It is not difficult to see that a 'Suslin gap' as in [72, Definition 9.4] can be split by a natural ccc forcing whose conditions are finite K_0-homogeneous sets.

of the Lebesgue measure can be increased by a ccc poset ([50, Lemma III.3.28]), Martin's Axiom implies that this gap contains an $(2^{\aleph_0}, 2^{\aleph_0})$-gap and that any further ccc forcing that increases the additivity of the Lebesgue measure will split the gap. $\square$

Chapter 2

Borel Complexity of Non-Self-Adjoint AF Operator Algebras

In the setting of Borel complexity theory, one studies the relative complexity of various classification problems with the aid of tools and methods from descriptive set theory. When the objects of a category that is to be classified consist of countable or separable structures, in most cases there is a natural standard Borel space that parametrizes (up to a given notion of isomorphism) the class of objects of the category. From this point of view, a classification problem concretely consists of a pair (X, E), where X is a standard Borel space that corresponds to parameters for objects to be classified and E is the (usually analytic) equivalence relation on X, given by the isomorphism relation among the objects that X parametrizes. The key notion of comparison for such classification problems is that of *Borel reducibility*, which is used to assign appropriate degrees of complexity. If (X, E) and (Y, F) are classification problems in the above sense, then a *Borel reduction* of E to F is a Borel function $f : X \to Y$ that satisfies

$$xEx' \iff f(x)Ff(x'),$$

for all $x, x' \in X$. In this case we say that the equivalence relation E is *Borel reducible* to F and we view E as being "less complicated" than F.

There are various standard degrees in the complexity hierarchy that serve as benchmarks for classification problems. Notably, a classification problem (X, E) is said to be *classifiable by countable structures* if there is a category of countable structures $\mathcal{C}$ such that E is Borel reducible to the isomorphism relation $\cong_{\mathcal{C}}$ within the class of objects of $\mathcal{C}$. The interplay between descriptive set theory and functional analysis has been fruitful over the past years, however classification problems in the context of operator theory tend not to be classifiable by countable structures. For instance, using Hjorth's theory of turbulence developed in [39], it was shown in that the natural isomorphism relations for von Neumann factors of any type ([65]) and for unital, simple, separable, nuclear C*-algebras ([33]) are not classifiable by countable structures (see also the discussion in [24, Section 3]). On the opposite side of the spectrum, Elliot's classical K-theoretic classification of separable AF C*-algebras ([20]), along with the Borel computability of K-theory ([32]) imply that the isomorphism relation for separable AF C*-algebras is classifiable by countable structures.

In connection to the classification of AF C*-algebras, the question of whether the class of non-self-adjoint AF operator algebras can be similarly classified by the ordered K_0 groups, or by any other class of countable structures, remained open since the early stages of the development of operator-algebraic classification. In this chapter, we answer this question in the negative by showing that the canonical isomorphism relations for separable, non-self-adjoint AF operator algebras are not classifiable by countable structures. More concretely, to each separable operator space we associate a separable, non-self-adjoint AF operator algebra in a functorial way and show that this class of AF algebras acts as an isomorphism invariant for the class of separable operator spaces. Interestingly, the family of AF algebras under study arises in a particularly simple form, as inductive limits of 2-dimensional triangular algebras. The aforementioned identification between operator spaces and AF algebras, along with a combination of results from [55], [34], [21] and [2] regarding the Borel complexity of separable operator spaces allows us to prove the following theorem, which is the main result of the chapter (see Theorem 2.3.7 at the end of the chapter).

Theorem 2.0.1. *The equivalence relations of isomorphism, isometry, complete isomoprhism and complete isometric isomorphism for separable, non-self-adjoint AF operator algebras are not classifiable by countable structures.*

2.1 Preliminary Notions

By H we will always denote the complex, separable, infinite dimensional Hilbert space $\ell_2(\mathbb{N})$ and by $\mathcal{B}(H)$ the C*-algebra of bounded, linear operators on H. An *operator algebra* will be a (not necessarily self-adjoint) norm closed subalgebra of $\mathcal{B}(H)$. An operator algebra A is called *AF* if it contains a sequence $(A_n)_n$ of finite-dimensional operator subalgebras whose union is dense in A. This is equivalent to stating that for every $\varepsilon > 0$ and for every finite subset F of A, there exists a finite-dimensional subalgebra $B \subseteq A$ such that for all $a \in F$ there is $b \in B$ with $||a - b|| < \varepsilon$. An operator $a \in A$ is called an *idempotent* if $a^2 = a$. If $X \subseteq \mathcal{B}(H)$, we denote by $\mathrm{alg}(X)$ the operator algebra generated by the set X.

If $n \in \mathbb{N}$, we naturally identify the space $\mathcal{M}_n(\mathcal{B}(H))$ of n by n matrices over $\mathcal{B}(H)$ with $\mathcal{B}(H^n)$. If T is an n by n matrix with entries in $\mathcal{B}(H)$, we will denote its (i,j)-th entry by $T_{i,j}$. For $1 \leq i, j \leq n$, we denote by $e_{i,j}$ the *matrix units* of $\mathcal{M}_n(\mathcal{B}(H))$, i.e. the n by n matrices whose (i,j)-th entry is equal to the identity operator id_H and whose remaining entries are zero. If $T = [T_{i,j}] \in \mathcal{M}_n(\mathcal{B}(H))$, then we have the following canonical estimates for the norm of T:

$$\max_{1 \leq i,j \leq n} \{||T_{i,j}||\} \leq ||T|| \leq \sum_{1 \leq i,j \leq n} ||T_{i,j}||.$$

An *operator system* is a norm closed, unital, linear and self-adjoint subspace of $\mathcal{B}(H)$, while an *operator space* is a norm closed, linear subspace of $\mathcal{B}(H)$. The key fact that differentiates an operator space E from a Banach space is that E is equipped with a sequence of norms $(||\cdot||_n)_n$ defined on $M_n(E)$ for all $n \in \mathbb{N}$, each inherited by the canonical inclusions of $\mathcal{M}_n(E)$ into $\mathcal{M}_n(\mathcal{B}(H))$. If E_1 and E_2 are operator spaces and $\varphi : E_1 \to E_2$ is a linear map, then φ is *unital* if both E_1 and E_2 are unital and φ maps the identity operator to itself. We will write φ_n for the associated map

$$\begin{aligned} \varphi_n : \mathcal{M}_n(E_1) &\to \mathcal{M}_n(E_2) \\ [T_{i,j}] &\mapsto [\varphi(T_{i,j})]. \end{aligned}$$

This map is called the (n-th) *amplification* of φ and may be thought as the map $\mathrm{id}_n \otimes \varphi$ via the identification of $\mathcal{M}_n(E_i)$ with $\mathcal{M}_n(\mathbb{C}) \otimes E_i$ ($i = 1, 2$). We say that the map φ is *completely bounded* if

$$||\varphi||_{cb} := \sup_{n \in \mathbb{N}} ||\varphi_n|| < \infty,$$

it is *completely contractive* if $||\varphi||_{cb} \leq 1$ and *completely isometric* if each of the maps φ_n is isometric ($n \in \mathbb{N}$). The operator spaces E_1 and E_2 are called *completely isomoprhic* if there exists a completely bounded bijection $\varphi : E_1 \to E_2$ such that its inverse φ^{-1} is also completely bounded. Moreover, if this map φ is a linear, surjective, complete isometry, then E_1 and E_2 are called *completely isometrically isomorphic.* In the case when both E_1 and E_2 are operator systems, the map $\varphi : E_1 \to E_2$ will be called *unital completely positive* (abbreviated as *u.c.p.*) if it is unital and the associated maps φ_n are positive for each $n \in \mathbb{N}$. If φ is a map between operator algebras A and B, the notions of complete isomorphism and complete isometric isomorphism are defined similarly by requiring that φ is in addition a bijective algebra homomorphism. We denote by $\mathbf{OSp_{cb}}$, $\mathbf{OSp_{cc}}$ and $\mathbf{OSp_{ci}}$ the categories of operator spaces with completely bounded, completely contractive and completely isometric maps respectively. Similarly, we will use $\mathbf{OA_{cb}}$, $\mathbf{OA_{cc}}$ and $\mathbf{OSp_{ci}}$ to denote the categories of (not necessarily self-adjoint) Banach subalgebras of $\mathcal{B}(H)$ with the analogous maps.

2.2 Standard Borel Parametrizations

In this section, we provide standard Borel parametrizations for the classes of separable operator algebras and separable operator spaces. Following [45] and[33], since the open ball of $\mathcal{B}(H)$ of radius n is compact and metrizable with respect to the weak operator topology for each $n \in \mathbb{N}$, the space $\mathcal{B}(H)$ becomes a standard Borel space when equipped with the Borel σ-algebra generated by the weakly open sets. We define

$$\Gamma(H) = \mathcal{B}(H)^{\mathbb{N}},$$

which is a standard Borel space with the product Borel structure. If $\gamma = (\gamma)_{n\in\mathbb{N}} \in \Gamma(H)$, then we let $\mathrm{alg}(\gamma)$ to be the (separable) operator algebra generated by the sequence of operators γ. It is clear that this parametrizes all separable operator algebras in $\mathcal{B}(H)$, since any such algebra must be countably generated. Now we let

$$\Gamma(H)_{AF} = \{\gamma \in \Gamma(H) : \mathrm{alg}(\gamma) \text{ is AF}\},$$

which is a Borel set that parametrizes all separable AF operator algebras, in view of the finitary characterization of the AF algebras. Note that if $\cong$ represents the equivalence relation of either isomorphism, isometry, complete isomorphism or complete isometric isomorphism for separable AF operator algebras, then one defines an equivalence relation $\cong^{AF}$ on $\Gamma(H)_{AF}$ as follows:

$$\gamma \cong^{AF} \gamma' \iff \mathrm{alg}(\gamma) \cong \mathrm{alg}(\gamma'),$$

and therefore the pair $\left(\Gamma(H)_{AF}, \cong^{AF}\right)$ is a standard Borel parametrization of the category of separable AF operator algebras.

For the class of separable operator spaces, as in [2, Section 2] let $\Gamma(H)$ be once again the standard Borel space of sequences in $\mathcal{B}(H)$. For each $\gamma = (\gamma_n)_{n\in\mathbb{N}} \in \Gamma(H)$, let E_γ be the operator subspace of $\mathcal{B}(H)$ that is generated by the sequence γ, i.e.

$$E_\gamma = \overline{\operatorname{span}}\{\gamma_n : n \in \mathbb{N}\}.$$

Clearly this parametrizes all separable operator spaces, since any such space must be countably generated. As before, if $\cong$ represents the relation of either isomorphism, isometry, complete isomorphism or complete isometric isomorphism for separable operator spaces, then one defines an equivalence relation $\cong^{OS}$ on $\Gamma(H)$ as follows:

$$\gamma \cong^{OS} \gamma' \iff E_\gamma \cong E_{\gamma'},$$

and this means that the pair $\left(\Gamma(H), \cong^{OS}\right)$ is a standard Borel parametrization of the category of separable operator spaces.

We only mention at this point that in [33] and [2] a number of equivalent standard Borel parametrizations of various classes of C*-algebras and operator spaces were developed, however for our purposes we will not need more than the parametrizations presented above.

2.3 The Classification of Non-Self-Adjoint AF Algebras

In this section we associate to each operator space an AF operator algebra and discuss the functorial properties related to this association. Let us fix a separable operator space $E \subseteq \mathcal{B}(H)$. We define

$$\mathcal{U}(E) = \left\{ \begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, T \in E \right\} \subseteq \mathcal{M}_2\left(\mathcal{B}(H)\right),$$

where we write λ in place $\lambda \cdot \mathrm{id}_H$ for any $\lambda \in \mathbb{C}$, and where we naturally identify the space of 2 by 2 matrices over $\mathcal{B}(H)$ with the space of bounded, linear operators on the direct sum $H \oplus H$. It is immediate that $\mathcal{U}(E)$ is a norm closed subset of $\mathcal{M}_2\left(\mathcal{B}(H)\right)$ that contains the unit and is closed under addition and multiplication (therefore it is a unital operator algebra), however it is not closed with respect to the adjoint operators, unless $E = \{0\}$. Note that the algebra $\mathcal{U}(E)$ contains E as an operator subspace, via the canonical completely isometric linear injection $j : E \to \mathcal{U}(E)$ given by

$$j(T) = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \qquad (T \in E).$$

If $(x_n)_n$ is a countable dense subset of E, then let E_n be the finite-dimensional subspace of E spanned by the set $\{x_k : k \leq n\}$. It then follows that E is equal to the norm closure of the countable union of its subspaces E_n $(n \in \mathbb{N})$ and this immediately implies that

$$\mathcal{U}(E) = \overline{\bigcup_{n \in \mathbb{N}} \mathcal{U}(E_n)}.$$

Since each $\mathcal{U}(E_n)$ is finite-dimensional, we see that $\mathcal{U}(E)$ is a unital, separable, non-self-adjoint AF operator algebra.

To the operator space E, we also associate the following operator system:

$$\mathcal{S}(E) = \left\{ \begin{bmatrix} \lambda & T \\ S^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, T, S \in E \right\} \subseteq \mathcal{M}_2\left(\mathcal{B}(H)\right).$$

It is clear that $\mathcal{S}(E)$ is norm closed, unital and self-adjoint. Observe that $\mathcal{S}(E)$ contains $\mathcal{U}(E)$ as a unital, closed operator subspace.

Suppose that E_1 and E_2 are operator spaces and let $\varphi : E_1 \to E_2$ be a linear map. We define $\mathcal{U}(\varphi) : \mathcal{U}(E_1) \to \mathcal{U}(E_2)$ by setting

$$\mathcal{U}(\varphi)\left(\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix}\right) = \begin{bmatrix} \lambda & \varphi(T) \\ 0 & \mu \end{bmatrix},$$

for each $\lambda, \mu \in \mathbb{C}$ and $T \in E$. The map $U(\varphi)$ is a unital algebra homomorphism and it is injective (resp. surjective) precisely when φ is injective (resp. surjective). Using the canonical matrix norm estimates, it is clear that if φ is either bounded or contractive, then so is $U(\varphi)$.

Proposition 2.3.1. *If the map $\varphi : E_1 \to E_2$ is either completely bounded, or completely contractive or completely isometric, then the same holds for $\mathcal{U}(\varphi)$.*

Proof. Let us first assume that φ is completely contractive. Consider the natural map

$\mathcal{S}(\varphi) : \mathcal{S}(E_1) \to \mathcal{S}(E_2)$ given by

$$\mathcal{S}(\varphi)\left(\begin{bmatrix} \lambda & T \\ S^* & \mu \end{bmatrix}\right) = \begin{bmatrix} \lambda & \varphi(T) \\ \varphi(S)^* & \mu \end{bmatrix},$$

which is clearly unital and linear. Viewing the map φ as taking values in $\mathcal{B}(H)$ and the map $\mathcal{S}(\varphi)$ as taking values in $\mathcal{M}_2(\mathcal{B}(H))$, by [63, Lemma 8.1] we see that $\mathcal{S}(\varphi)$ is u.c.p. and, by [63, Proposition 3.6], it is completely contractive. Therefore, its restriction to $\mathcal{U}(E_1)$, which is equal to $\mathcal{U}(\varphi)$, will also be completely contractive.

If φ is completely isometric, then by applying the previous result to the maps φ and $\varphi^{-1} : \varphi[E_1] \to E_1$, we see that both maps $\mathcal{U}(\varphi)$ and $\mathcal{U}(\varphi^{-1}) = \mathcal{U}(\varphi)^{-1}$ are completely contractive and hence $\mathcal{U}(\varphi)$ is completely isometric.

Lastly, let us assume that φ is completely bounded and let $z = ||\varphi||_{cb}$. Then, the map $\varphi' = \frac{1}{z}\varphi$ is completely contractive and thus $\mathcal{U}(\varphi')$ is also completely contractive. Note that

$$\mathcal{U}(\varphi)\left(\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix}\right) = \mathcal{U}(\varphi')\left(\begin{bmatrix} \lambda & z \cdot T \\ 0 & \mu \end{bmatrix}\right)$$

for all $\lambda, \mu \in \mathbb{C}, T \in E$. Also, observe that the function $\psi : \mathcal{U}(E) \to \mathcal{U}(E)$ given by

$$\psi\left(\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix}\right) = \begin{bmatrix} \lambda & z \cdot T \\ 0 & \mu \end{bmatrix}$$

is completely bounded, and therefore $\mathcal{U}(\varphi)$ must be completely bounded, as the composition of the completely bounded maps ψ and $\mathcal{U}(\varphi')$. $\square$

The following Corollary is an immediate consequence of Proposition 2.3.1.

Corollary 2.3.2. *The mapping $E \mapsto \mathcal{U}(E)$ defines a (covariant) functor between the categories:*

(i) $\mathbf{OSp_{cb}}$ *and* $\mathbf{OA_{cb}}$*,*

(ii) $\mathbf{OSp_{cc}}$ *and* $\mathbf{OA_{cc}}$*,*

(iii) $\mathbf{OSp_{ci}}$ *and* $\mathbf{OA_{ci}}$*.*

We will now proceed to discuss the properties of the mapping $E \mapsto \mathcal{U}(E)$ as an invariant and, to this end, we begin by first discussing the structure of the idempotent operators in the algebra $\mathcal{U}(E)$.

Remark 2.3.3. Note that if E is an operator space and $\lambda, \mu \in \mathbb{C}$ and $T \in E$, then the matrix

$$\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix} \in \mathcal{U}(E)$$

is an idempotent precisely when $\lambda^2 = \lambda$, $\mu^2 = \mu$ and $(\lambda + \mu - 1) \cdot T = 0$. This implies that the proper, non-trivial idempotents in $\mathcal{U}(E)$ are matrices of the form

$$\begin{bmatrix} 1 & T \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & S \\ 0 & 1 \end{bmatrix},$$

for $T, S \in E$. Observe that in the case when T and S are non-zero, then both the matrices above have operator norm strictly greater that 1. In particular, the only proper, non-trivial idempotents in $\mathcal{U}(E)$ of norm at most 1 are the diagonal matrix units $e_{1,1}$ and $e_{2,2}$.

Lemma 2.3.4. *Let E_1, E_2 be operator spaces and suppose that $F : \mathcal{U}(E_1) \to \mathcal{U}(E_2)$ is an isomorphism. Also, consider the canonical injections $j_1 : E_1 \to \mathcal{U}(E_1)$ and $j_2 : E_2 \to \mathcal{U}(E_2)$. Then, for each $\lambda, \mu \in \mathbb{C}$ and $T \in E_1$, the following are equivalent:*

(i) $\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix} \in j_1(E_1)$,

(ii) $F\left(\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix}\right) \in j_2(E_2)$.

Proof. We may suppose that $E_1, E_2 \neq \{0\}$. Note that the matrix unit $e_{1,1}$ in $\mathcal{U}(E_1)$ satisfies the relation

$$\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix} \cdot e_{1,1} = \lambda \cdot e_{1,1},$$

and this shows that

$$F(e_{1,1}) \neq \begin{bmatrix} 0 & S \\ 0 & 1 \end{bmatrix},$$

for all $S \in E_2$. Indeed, if otherwise, let $0 \neq S' \in E_2$ and, since F is surjective, let $\lambda, \mu \in \mathbb{C}$ and $T \in E_1$ be such that

$$F\left(\begin{bmatrix} \lambda & T \\ 0 & \mu \end{bmatrix}\right) = \begin{bmatrix} 0 & S' \\ 0 & 0 \end{bmatrix}.$$

Then, we obtain that

$$F\left(\begin{bmatrix}\lambda & T\\ 0 & \mu\end{bmatrix}\cdot e_{1,1}\right)=\begin{bmatrix}0 & S'\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & S\\ 0 & 1\end{bmatrix}\iff\begin{bmatrix}0 & \lambda\cdot S\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}0 & S'\\ 0 & 0\end{bmatrix},$$

which contradicts our assumption on S' being non-zero. Similarly, using the matrix unit relation, we find that $F(e_{1,1})\notin\{e_{2,2},\mathrm{id}_{\mathcal{M}_2(\mathcal{B}(H))}\}$. Since $F(e_{1,1})$ is an idempotent, by the discussion in Remark 2.3.3, there must exist an operator $S_0\in E_2$ such that

$$F(e_{1,1})=\begin{bmatrix}1 & S_0\\ 0 & 0\end{bmatrix}.$$

Now, if $T\in E_1$ and $\lambda,\mu\in\mathbb{C},S\in E_2$ are such that

$$F\left(\begin{bmatrix}0 & T\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}\lambda & S\\ 0 & \mu\end{bmatrix},$$

then $\lambda=\mu=0$. Indeed, note that

$$F\left(\begin{bmatrix}0 & T\\ 0 & 0\end{bmatrix}\cdot e_{1,1}\right)=\begin{bmatrix}\lambda & S\\ 0 & \mu\end{bmatrix}\cdot\begin{bmatrix}1 & S_0\\ 0 & 0\end{bmatrix}\iff 0_{\mathcal{M}_2(\mathcal{B}(H))}=\begin{bmatrix}\lambda & \lambda\cdot S_0\\ 0 & 0\end{bmatrix},$$

which shows that $\lambda=0$. Moreover, we see that

$$F\left(e_{1,1}\cdot\begin{bmatrix}0 & T\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}1 & S_0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & S\\ 0 & \mu\end{bmatrix}\iff\begin{bmatrix}0 & S\\ 0 & \mu\end{bmatrix}=\begin{bmatrix}0 & S+\mu\cdot S_0\\ 0 & 0\end{bmatrix},$$

from which it follows that $\mu=0$ and this proves the forward implication of the Lemma. By applying the same argument to the inverse map $F^{-1}:\mathcal{U}(E_2)\to\mathcal{U}(E_1)$, one shows that the converse must hold as well. □

Corollary 2.3.5. *If $F:\mathcal{U}(E_1)\to\mathcal{U}(E_2)$ is an isomorphism, then there exists a unique linear bijection $\tilde{F}:E_1\to E_2$ that makes the diagram*

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{F}"] \arrow[d, "j_1"'] & E_2 \arrow[d, "j_2"] \\ \mathcal{U}(E_1) \arrow[r, "F"] & \mathcal{U}(E_2) \end{tikzcd}$$

Figure 2.1: Commutative diagram 1

commute. If F is in addition assumed to be bounded or isometric, then the same holds for the map$\tilde{F}$.

Proof. The map $\tilde{F} : E_1 \to E_2$ given by

$$\tilde{F}(T) = F\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right)_{1,2}$$

is well-defined and linear. Lemma 2.3.4 implies that it is also bijective and that the aforementioned diagram will commute, while the uniqueness of $\tilde{F}$ is ensured by the same Lemma and the commutativity of the diagram. Finally, since for any $T \in E_1$ we have that

$$F\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix},$$

for some $S \in E_2$, the canonical matrix norm estimates yield that if F is either bounded or isometric, then the same must hold for $\tilde{F}$. $\square$

Combining our results so far, we obtain the following theorem.

Theorem 2.3.6. *The spaces$\mathcal{U}(E_1)$ and$\mathcal{U}(E_2)$ are (completely) isomorphic (resp. (completely) isometrically isomorphic) as operator algebras if and only if E_1 and E_2 are (completely) isomorphic (reps. (completely) isometrically isomorphic) as operator spaces.*

Proof. The converse implication of the theorem is given by Proposition 2.3.1. Suppose that $F : \mathcal{U}(E_1) \to \mathcal{U}(E_2)$ is a linear, bijective homomorphism and, by Corollary 2.3.5, let $\tilde{F} : E_1 \to E_2$ be the unique linear bijection that makes the diagram

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{F}"] \arrow[d, "j_1"'] & E_2 \arrow[d, "j_2"] \\ \mathcal{U}(E_1) \arrow[r, "F"] & \mathcal{U}(E_2) \end{tikzcd}$$

Figure 2.2: Commutative diagram 2

commute. This implies that for all $n \in \mathbb{N}$, the diagram

$$\begin{tikzcd} \mathcal{M}_n(E_1) \arrow[r, "\tilde{F}_n"] \arrow[d, "j_1^n"'] & \mathcal{M}_n(E_2) \arrow[d, "j_2^n"] \\ \mathcal{M}_n(\mathcal{U}(E_1)) \arrow[r, "F_n"] & \mathcal{M}_n(\mathcal{U}(E_2)) \end{tikzcd}$$

Figure 2.3: Commutative diagram 3

will also commute (where $\tilde{F}_n, F_n, j_1^n, j_2^n$ denote the n-th amplifications of the corresponding maps) and therefore if F is either completely bounded or completely isometric, then the same holds for $\tilde{F}$. If F is assumed to be a complete isomorphism, then since $\left(\widetilde{F_n}\right)^{-1} = \widetilde{F_n^{-1}}$ for each $n \in \mathbb{N}$, we have that $\left(\widetilde{F}\right)^{-1}$ is completely bounded and hence the map $\tilde{F}$ induces a complete isomoprhism between E_1 and E_2. $\square$

We now restate and prove main result of this chapter.

Theorem 2.3.7. *For separable, non-self-adjoint AF operator algebras, neither one of the following equivalence relations*

(i) isomorphism,

(ii) isometry,

(iii) complete isomorphism,

(iv) complete isometric isomorphism,

is classifiable by countable structures.

Proof. By [34, Theorem 5] and [55, Theorem 3.2] the relations of isomorphism and isometry for separable operator spaces are not classifiable by countable structures (note that all separable Banach spaces can be isometrically identified with separable operator subspaces of $\mathcal{B}(H)$). Moreover, by [2, Theorem 3.1] and [21, Theorem 1.1] the relations of complete isomorphism and complete isometric isomorphism for separable operator spaces are also not classifiable by countable structures. Therefore, in view of Theorem 2.3.6, in order to complete the proof it is enough to find a Borel reduction from the parametrization of the category of separable operator spaces to the parametrization of the category of separable non-self-adjoint AF operator algebras (each equipped with the appropriate equivalence relations). To this end, let $\gamma = (\gamma_n)_{n\in\mathbb{N}} \in \Gamma(H)$ and define the operator space E_γ to be equal to the norm closure of the linear span of γ. Note that the countable set of bounded, linear operators on $H \oplus H$ given by

$$X_\gamma = \left\{ \begin{bmatrix} q_1 & \gamma_n \\ 0 & q_2 \end{bmatrix} \in \mathcal{M}_2(\mathcal{B}(H)) : q_1, q_2 \in \mathbb{Q} + i\mathbb{Q}, n \in \mathbb{N} \right\}$$

satisfies that $\mathrm{alg}(X_\gamma) = \mathcal{U}(E_\gamma)$ and, therefore, $X_\gamma \in \Gamma(H \oplus H)_{AF}$. As a result, since

$$\gamma \cong^{OS} \gamma' \iff X_\gamma \cong^{AF} X_{\gamma'}$$

for all $\gamma, \gamma' \in \Gamma(H)$, the assignment

$$\gamma \mapsto X_\gamma$$

from $\Gamma(H)$ to $\Gamma(H \oplus H)_{AF}$ is the required Borel reduction. $\square$

Part II : Bi-Free Probability Theory

Chapter 3

Bi-R-Diagonal Pairs of Operators

In the theory of free probability, an R-diagonal operator is an element of a non-commutative $*$-probability space (A, φ) whose $*$-distribution coincides with the $*$-distribution of a product of the form $u \cdot p$, where the sets $\{u, u^*\}$ and $\{p, p^*\}$ are freely independent and u is a Haar unitary, i.e. u is a unitary and $\varphi(u^n) = 0$, for all $n \in \mathbb{Z} \setminus \{0\}$. It is due to this free factorization property that the class of R-diagonal operators constitutes a particularly well-behaved class of non-normal operators. From a combinatorial point of view, R-diagonal elements are characterized by having all of their free $*$-cumulants that are either of odd order, or have entries that are not alternating in $*$-terms and non-$*$-terms equal to zero. This combinatorial approach has proved to be extremely fruitful in the development of the theory of R-diagonal operators (see [61] for an exposition of the combinatorics of free probability).

In [60], R-diagonal operators were found to satisfy a "free absorption" property, namely that for any elements a, b in some non-commutative $*$-probability space such that a is R-diagonal and a is $*$-free from b, the element ab is also R-diagonal. In [37], Brown's spectral distribution measure was computed for R-diagonal operators in finite von Neumann algebras, while in [51], powers of R-diagonal operators were shown to be R-diagonal and their determining sequences were computed (see also [61, Theorem 15.22] for a proof making use of combinatorial arguments).

In [59], a number of equivalent characterizations of R-diagonality were formulated, including conditions on $*$-moments, free cumulants and the freeness of certain self-adjoint matrices from the scalar matrices, with amalgamation over the diagonal scalar matrices, while in [6] similar results were obtained on B-valued R-diagonal elements in the operator-valued setting. Distributions of R-diagonal operators have found applications in the non-microstate approach to free entropy, answering questions regarding the minimization of the free Fisher information in the tracial framework (see [58]).

Bi-free probability theory originated in [82] as an extension of the free setting and involves the simultaneous study of left and right actions of algebras on reduced free product spaces. The corresponding notion of bi-free independence found its combinatorial characterization in [11] (see also [10] for the development of the combinatorics of bi-free probability in the operator-valued setting). This chapter is devoted to the study of the analogue of R-diagonal operators in the bi-free setting, namely bi-R-diagonal pairs of operators and, to this end, the combinatorial approach originally proposed in [68, Section 4] shall be adopted, which makes use of the bi-free cumulant functions. For the study of products and powers of bi-R-diagonal pairs, similar arguments are used as to those corresponding to the results in the free case, but more care is required due to the dealing with the lattice of bi-non-crossing partitions and the χ-order. Since products of pairs of operators are considered pointwise (i.e. left operators are multiplied by left operators and right operators are multiplied by right operators), caution ought to be exercised when it comes to the order in which the multiplication takes place and, for the most general cases, it is necessary that the order of the multiplication of right operators is reversed (see Theorem 3.2.2). However, this is found not to play a role in the case when both pairs in question are bi-R-diagonal and $*$-bi-free (Proposition 3.2.4). These results imply that bi-R-diagonal pairs of operators satisfy a corresponding "bi-free absorption" property and indicate that such pairs of operators exist in abundance.

The absence of characterizations of bi-free phenomena with conditions on moments is an unfortunate theme in the theory of bi-free probability (see, however, [9] for an equivalent formulation of bi-free independence in terms of alternating moments). In particular, a characterization of the condition of bi-R-diagonality in terms of $*$-moments was unable to be obtained. In the setting of free probability, one of the most salient features of the $*$-distribution of an R-diagonal operator is that it remains invariant after the multiplication by a freely independent Haar unitary, a result obtained with the use of freeness in terms of its characterization via moments (see [59, Theorem 1.2] and [61, Theorem 15.10]). Bi-Haar unitary pairs of operators constitute the bi-free analogue of Haar unitaries and their joint $*$-distribution is modelled by the left and right regular representations of groups on Hilbert spaces. Theorem 3.3.4 is the generalization of the aforementioned fact to the bi-free setting and displays the invariance of the joint $*$-distribution of any bi-R-diagonal pair of operators under the multiplication of a $*$-bi-free bi-Haar unitary pair. The proof follows the combinatorial approach instead, using the bi-free cumulant functions and hence a new proof follows for the free case as well. In the spirit of [59, Theorem 1.2], [6, Theorem 3.1] and by combining results from [68], we obtain Theorem 3.3.6, displaying equivalent formulations of the condition of bi-R-diagonality.

3.1 Preliminary Results

In this section we will develop the common preliminaries, fix the appropriate notation and state a number of lemmas to be used later in this chapter.

Our main framework will be that of a *non-commutative $*$-probability space*, i.e. a pair (A, φ) where A is a complex, unital $*$-algebra and $\varphi : A \to \mathbb{C}$ is a state, which is a unital, linear map such that

$$\varphi(a^*a) \geq 0,$$

for all $a \in A$.

For any $S \subseteq A$, we will denote by $\mathrm{alg}(S)$ the subalgebra of A generated by the set S. If $a_1, \ldots, a_n$ are elements of (A, φ), then:

(a) their *joint distribution* is given by the linear functional

$$\mu : \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathbb{C}$$

defined as

$$\mu(P) = \varphi(P(a_1, \ldots, a_n)), \quad (P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle)$$

where $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ denotes the unital algebra of polynomials in n-non-commuting indeterminates $X_1, \ldots, X_n$,

(b) their *joint $*$-distribution* is given by the joint distribution of the family

$$\{a_1, \ldots, a_n, {a_1}^*, \ldots, {a_n}^*\},$$

(c) the family of their *joint $*$-moments* is given by the action of their joint distribution μ on the monomials in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ and is therefore determined by the collection

$$\{\varphi(c_1 \cdots c_k) : k \geq 1, c_i \in \{a_1, \ldots, a_n, {a_1}^*, \ldots, {a_n}^*\} \text{ for all } 1 \leq i \leq k\}.$$

It is clear that for $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, in order to verify equality of joint $*$-distributions of the families $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$, it suffices to prove that all of their joint $*$-moments coincide.

For $a_1, \ldots, a_n \in A$ and $\emptyset \neq V = \{j_1 < j_2 < \ldots < j_s\} \subseteq \{1, \ldots, n\}$, the restriction of the sequence $(a_1, \ldots, a_n)$ to the set V is given by

$$(a_1, \ldots, a_n)|_V = (a_{j_1}, a_{j_2}, \ldots, a_{j_s}).$$

In this case, we define

$$\varphi((a_1, \ldots, a_n)|_V) = \varphi(a_{j_1} \cdot a_{j_2} \cdots a_{j_s}).$$

Also, if π is a partition of the set $\{1, \ldots, n\}$, then we use the following notation:

$$\varphi_\pi(a_1, \ldots, a_n) = \prod_{V \in \pi} \varphi((a_1, \ldots, a_n)|_V).$$

3.1.1 The Lattice of Bi-Non-Crossing Partitions

Familiarity with the collection of non-crossing partitions $\mathrm{NC}(n)$, multiplicative functions on $\mathrm{NC}(n)$ and free cumulants is assumed (see [61] for an exposition of the combinatorics of free probability).

For $n \in \mathbb{N}$, we will be using maps $\chi \in \{l, r\}^n$ to distinguish between left and right operators in a sequence of n-operators. Any such map gives rise to a permutation s_χ on $\{1, \ldots, n\}$ as follows:

If $\chi^{-1}(\{l\}) = \{i_1 < \ldots < i_p\}$ and $\chi^{-1}(\{r\}) = \{j_1 < \ldots < j_{n-p}\}$, then define:

$$s_\chi(k) = \begin{cases} i_k, & \text{if } k \leq p \\ j_{n+1-k}, & \text{if } k > p \end{cases}$$

From a combinatorial standpoint, the only differences between free and bi-free probability arise from dealing with s_χ.

The permutation s_χ naturally induces a total order on $\{1, \ldots, n\}$ (which we will henceforth be referring to as the χ-*order*) as follows:

$$i \prec_\chi j \iff s_\chi^{-1}(i) < s_\chi^{-1}(j).$$

Instead of reading $\{1, \ldots, n\}$ in the traditional order, this corresponds to first reading the elements of $\{1, \ldots, n\}$ labelled "l" in increasing order, followed by reading the elements labelled "r" in decreasing order. Note that if V is any non-empty subset of $\{1, \ldots, n\}$, the map $\chi|_V$ naturally gives rise to a map $s_{\chi|_V}$, which should be thought of as a permutation on $\{1, \ldots, |V|\}$.

Before we discuss the lattice of bi-non-crossing partitions, we fix some notation regarding general partitions. For $n \in \mathbb{N}$, the collection of all partitions on $\{1, \ldots, n\}$ is denoted by $\mathcal{P}(n)$, while the collection of non-crossing partitions on $\{1, \ldots, n\}$ is denoted by $\mathrm{NC}(n)$. The elements of any $\pi \in \mathcal{P}(n)$ are called the *blocks* of π and for $1 \leq i, j \leq n$, we write $i \sim_\pi j$ to mean that i and j belong to the same block of π, whereas $i \nsim_\pi j$ indicates that i and j belong to different blocks of π. For $\pi, \sigma \in \mathcal{P}(n)$, we write $\pi \leq \sigma$ if every block of π is contained in a block of σ. This defines the partial order of refinement on $\mathcal{P}(n)$. The maximal element of $\mathcal{P}(n)$ with respect to this partial order is the partition consisting of one block (denoted by 1_n), while the minimal element is the partition consisting of n-blocks (denoted by 0_n). This partial order induces a lattice structure on $\mathcal{P}(n)$, hence for $\pi, \sigma \in \mathcal{P}(n)$, the *join* $\pi \vee \sigma$ (i.e. the minimum element of the non-empty set $\{\rho \in \mathcal{P}(n) : \rho \geq \pi, \sigma\}$) of π and σ is well defined.

Definition 3.1.1. Let $n \in \mathbb{N}$ and $\chi \in \{l, r\}^n$. A partition $\tau \in \mathcal{P}(n)$ is called *bi-non-crossing with respect to* χ if the partition $s_\chi^{-1} \cdot \tau$ (i.e. the partition obtained by applying the permutation s_χ^{-1} to each entry of every block of τ) is non-crossing. Equivalently, τ is bi-non-crossing with respect to χ if whenever V, W are blocks of τ and $v_1, v_2 \in V, w_1, w_2 \in W$ are such that

$$v_1 \prec_\chi w_1 \prec_\chi v_2 \prec_\chi w_2,$$

then we necessarily have that $V = W$. The collection of bi-non-crossing partitions with respect to χ is denoted by $\mathrm{BNC}(\chi)$. It is clear that

$$\mathrm{BNC}(\chi) = \{\tau \in \mathcal{P}(n) : s_\chi^{-1} \cdot \tau \in \mathrm{NC}(n)\} = \{s_\chi \cdot \pi : \pi \in \mathrm{NC}(n)\}.$$

We will be referring to a partition τ simply as bi-non-crossing whenever it is clear from the context which map χ is used. Note that in the special case when the map χ is constant, one ends up with the collection of all non-crossing partitions on $\{1, \ldots, n\}$.

Example 3.1.2. If $\chi \in \{l, r\}^6$ is such that $\chi^{-1}(\{l\}) = \{1, 2, 3, 6\}$ and $\chi^{-1}(\{r\}) = \{4, 5\}$, then $(s_\chi(1), \ldots, s_\chi(6)) = (1, 2, 3, 6, 5, 4)$ and the partition given by

$$\tau = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$$

is bi-non-crossing with respect to χ, even though $\tau \notin \mathrm{NC}(6)$. This may also be seen via the following diagrams:

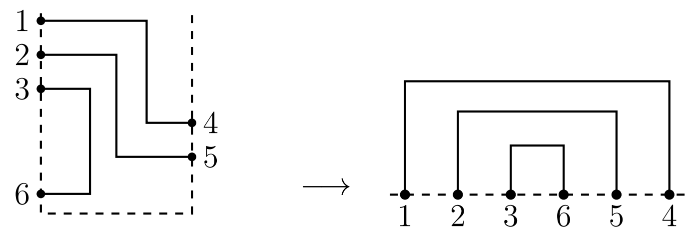

Figure 3.1: Relation between non-crossing and bi-non-crossing diagrams

The set of bi-non-crossing partitions with respect to a map $\chi \in \{l, r\}^n$ inherits a lattice structure from $\mathcal{P}(n)$ via the partial order of refinement (although the join operation in $\mathrm{BNC}(\chi)$ need not coincide with the restriction of the join operation in $\mathcal{P}(n)$). The minimal and maximal elements of $\mathrm{BNC}(\chi)$ will be denoted by 0_χ and 1_χ respectively (with $0_\chi = s_\chi(0_n) = 0_n$ and $1_\chi = s_\chi(1_n) = 1_n$). For $\emptyset \neq V \subseteq \{1, \ldots, n\}$, we denote by $\min_< V$ and $\min_{\prec_\chi} V$ the minimum element of V with respect to the natural order and the χ-order of $\{1, \ldots, n\}$ respectively. Similar notation will be used for such maximum elements.

Definition 3.1.3. The *bi-non-crossing Möbius function* is the map

$$\mu_{\mathrm{BNC}} : \bigcup_{n \in \mathbb{N}} \bigcup_{\chi \in \{l,r\}^n} \mathrm{BNC}(\chi) \times \mathrm{BNC}(\chi) \to \mathbb{C}$$

defined recursively by

$$\sum_{\substack{\rho \in \mathrm{BNC}(\chi) \\ \tau \leq \rho \leq \lambda}} \mu_{\mathrm{BNC}}(\tau, \rho) = \sum_{\substack{\rho \in \mathrm{BNC}(\chi) \\ \tau \leq \rho \leq \lambda}} \mu_{\mathrm{BNC}}(\rho, \lambda) = \begin{cases} 1, & \text{if } \tau = \lambda \\ 0, & \text{if } \tau < \lambda \end{cases}$$

whenever $\tau \leq \lambda$, while taking the zero value otherwise.

The connection between the bi-non-crossing Möbius function and the Möbius function on the lattice of non-crossing partitions μ_{NC} is given by the formula

$$\mu_{\mathrm{BNC}}(\tau, \lambda) = \mu_{\mathrm{NC}}(s_\chi^{-1} \cdot \tau, s_\chi^{-1} \cdot \lambda)$$

for all $\tau \leq \lambda \in \mathrm{BNC}(\chi)$ and hence μ_{BNC} inherits many of the multiplicative properties of μ_{NC} (see [11, Section 3]).

The *Catalan numbers* $\{C_n\}_{n\in\mathbb{N}}$ form a sequence of positive integers frequently used in the field of combinatorics; it is well known that the n-th Catalan number C_n equals the number of non-crossing partitions on a set of n-elements and, as a result, also equals the number of bi-non-crossing partitions with respect to any map $\chi \in \{l, r\}^n$ (see [61, Proposition 9.4]). This sequence will come up when we make reference to the joint $*$-distribution of bi-Haar unitary pairs of operators (Corollary 3.1.20). We state the following lemma tying the values of the bi-non-crossing Möbius function with the Catalan numbers.

Lemma 3.1.4. *Let $n \in \mathbb{N}$ and $\chi \in \{l, r\}^n$. Then, for all $\tau \in \mathrm{BNC}(\chi)$ we have that*

$$\mu_{\mathrm{BNC}}(0_\chi, \tau) = \prod_{V\in\tau} (-1)^{|V|-1} \cdot C_{|V|-1}.$$

In particular,

$$\mu_{\mathrm{BNC}}(0_\chi, 1_\chi) = (-1)^{n-1} \cdot C_{n-1},$$

where C_n denotes the n-th Catalan number.

Due to the connection between μ_{BNC} and μ_{NC}, the proof of the aforementioned lemma is based on facts regarding the behaviour of multiplicative functions on $\mathrm{NC}(n)$. More specifically, it relies on the canonical factorization of intervals in the lattice of non-crossing partitions and on the multiplicative properties of the Möbius function μ_{NC} (see [61, Theorem 9.29, Proposition 10.14 and 10.15]).

The *Kreweras complementation map* $K_{\mathrm{NC}} : \mathrm{NC}(n) \to \mathrm{NC}(n)$ defined in [47] is an important example of a lattice anti-isomorphism. For its descripition, we introduce new symbols $\overline{1}, \overline{2}, \dots, \overline{n}$ and consider them interlaced with $1, 2, \dots, n$ in the following manner:

$$1\,\overline{1}\,2\,\overline{2}\dots n\,\overline{n}.$$

For $\pi \in \mathrm{NC}(n)$, its Kreweras complement $K_{\mathrm{NC}}(\pi) \in \mathrm{NC}(\{\overline{1}, \overline{2}, \dots, \overline{n}\}) \cong \mathrm{NC}(n)$ is defined to be the largest non-crossing partition having the property

$$\pi \cup K_{\mathrm{NC}}(\pi) \in \mathrm{NC}(\{1, \overline{1}, 2, \overline{2} \dots n, \overline{n}\}).$$

The complementation map found its generalization for the lattice of bi-non-crossing partitions in [11, Section 5]. Specifically, for any $n \in \mathbb{N}, \chi \in \{l, r\}^n$ and $\tau \in \mathrm{BNC}(\chi)$, the Kreweras complement of τ in $\mathrm{BNC}(\chi)$, denoted by $K_{\mathrm{BNC}}(\tau)$, is defined as

$$K_{\mathrm{BNC}}(\tau) = s_\chi \cdot K_{\mathrm{NC}}(s_\chi^{-1} \cdot \tau),$$

i.e. is given by applying the permutation s_χ to the Kreweras complement of $s_\chi^{-1} \cdot \tau$ in $\mathrm{NC}(n)$. Note that in the special case when $\chi \in \{l, r\}^n$ gives the constant value "l", one obtains K_{NC}. In the following lemma, we list properties of K_{BNC} that we will be making use of.

Lemma 3.1.5. *Let $n \in \mathbb{N}$ and $\chi \in \{l, r\}^n$. Then:*

(a) $K_{\mathrm{BNC}} : \mathrm{BNC}(\chi) \to \mathrm{BNC}(\chi)$ is a bijection,

(b) $K_{\mathrm{BNC}}(0_\chi) = 1_\chi$ and $K_{\mathrm{BNC}}(1_\chi) = 0_\chi$,

(c) For all $\tau, \lambda \in \mathrm{BNC}(\chi)$ we have that

$$\tau \leq \lambda \iff K_{\mathrm{BNC}}(\lambda) \leq K_{\mathrm{BNC}}(\tau) \iff K_{\mathrm{BNC}}^{-1}(\lambda) \leq K_{\mathrm{BNC}}^{-1}(\tau).$$

All of these properties are easily verified by the definition of K_{BNC} and by the corresponding properties which hold for K_{NC}.

We shall now state a combinatorial lemma, which may be of independent interest and involves the following cancellation property for the lattice of bi-non-crossing partitions. A special case of this lemma will play a key role in the proof of Lemma 3.3.3.

Lemma 3.1.6. *Let $n \in \mathbb{N}, \chi \in \{l, r\}^n$ and consider a family $\{d_\tau\}_{\tau \in \mathrm{BNC}(\chi)}$ of indeterminates indexed by the bi-non-crossing partitions $\mathrm{BNC}(\chi)$. Then, the following holds:*

$$\sum_{\tau \in \mathrm{BNC}(\chi)} \Big(\mu_{\mathrm{BNC}}(0_\chi, \tau) \cdot \sum_{\substack{\lambda \in BNC(\chi) \\ \lambda \leq K_{\mathrm{BNC}}(\tau)}} d_\lambda \Big) = d_{1_\chi}$$

Proof. Re-arragning the left hand-side of the above expression yields:

$$\sum_{\tau \in \mathrm{BNC}(n)} \Big(\mu_{\mathrm{BNC}}(0_\chi, \tau) \cdot \sum_{\substack{\lambda \in \mathrm{BNC}(\chi) \\ \lambda \leq K_{\mathrm{BNC}}(\tau)}} d_\lambda \Big) = \sum_{\lambda \in \mathrm{BNC}(\chi)} \Big(d_\lambda \cdot \sum_{\substack{\tau \in \mathrm{BNC}(\chi) \\ \lambda \leq K_{\mathrm{BNC}}(\tau)}} \mu_{\mathrm{BNC}}(0_\chi, \tau) \Big)$$

With this remark in hand, it is immediate that to prove the conclusion of the lemma, it suffices to show that for all $\lambda \in \mathrm{BNC}(\chi)$, we have that

$$\sum_{\substack{\tau \in \mathrm{BNC}(\chi) \\ \lambda \leq K_{\mathrm{BNC}}(\tau)}} \mu_{\mathrm{BNC}}(0_\chi, \tau) = \begin{cases} 1, & \text{if } \lambda = 1_\chi \\ 0, & \text{if } \lambda < 1_\chi \end{cases}$$

We simply state that this condition must also be necessary, because the indeterminates $\{d_\tau\}$ satisfy no relations. Fix $\lambda \in \mathrm{BNC}(\chi)$ and let $\lambda' \in \mathrm{BNC}(\chi)$ be such that $\lambda = K_{\mathrm{BNC}}(\lambda')$. Observe that since

$$\lambda \leq K_{\mathrm{BNC}}(\tau) \iff K_{\mathrm{BNC}}(\lambda') \leq K_{\mathrm{BNC}}(\tau) \iff \tau \leq \lambda',$$

we have that

$$\{\tau \in \mathrm{BNC}(\chi) : \lambda \leq K_{\mathrm{BNC}}(\tau)\} = \{\tau \in \mathrm{BNC}(\chi) : \tau \leq \lambda'\}.$$

Elementary properties of the Möbius function on the lattice of bi-non-crossing partitions imply that

$$\sum_{\substack{\tau \in \mathrm{BNC}(\chi) \\ \lambda \leq K_{\mathrm{BNC}}(\tau)}} \mu_{\mathrm{BNC}}(0_\chi, \tau) = \sum_{\substack{\tau \in \mathrm{BNC}(\chi) \\ 0_\chi \leq \tau \leq \lambda'}} \mu_{\mathrm{BNC}}(0_\chi, \tau) = \begin{cases} 1, & \text{if } 0_\chi = \lambda' \\ 0, & \text{if } 0_\chi < \lambda' \end{cases}$$

Then, an application of Lema 3.1.5 yields:

$$0_\chi = \lambda' \iff K_{\mathrm{BNC}}^{-1}(1_\chi) = K_{\mathrm{BNC}}^{-1}(\lambda) \iff \lambda = 1_\chi$$

and

$$0_\chi < \lambda' \iff K_{\mathrm{BNC}}^{-1}(1_\chi) < K_{\mathrm{BNC}}^{-1}(\lambda) \iff \lambda < 1_\chi.$$

This completes the proof. □

Of course, when the map $\chi \in \{l, r\}^n$ gives the constant value "l", one obtains the analogous result for the lattice of non-crossing partitions.

3.1.2 Bi-Free Independence and Bi-Free Cumulants

We begin by recalling the notion of bi-free independence for pairs of faces in some non-commutative $*$-probability space, originally developed in [82].

Definition 3.1.7. Let (A, φ) be a non-commutative $*$-probability space.

(i) A *pair of faces* in (A, φ) consists of a pair (C, D) of unital subalgebras of A.

(ii) A family $\{(C_k, D_k)\}_{k \in K}$ of pairs of faces in (A, φ) is said to be *bi-freely independent* (or simply *bi-free*) if there exists a family of vector spaces with specified vector states $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, \xi_k)\}_{k \in K}$ and unital homomorphisms

$$l_k : C_k \to \mathcal{L}(\mathcal{X}_k) \text{ and } r_k : D_k \to \mathcal{L}(\mathcal{X}_k),$$

(where $\mathcal{L}(\mathcal{X}_k)$ denotes the space of all linear maps on $\mathcal{X}_k$) such that the joint distribution of the family $\{(C_k, D_k)\}_{k \in K}$ with respect to φ coincides with the joint distribution with respect to the vacuum state on the representation on $*_{k \in K}(\mathcal{X}_k, \mathring{\mathcal{X}}_k, \xi_k)$.

(iii) If S_k and V_k are subsets of A for all $k \in K$, then the family $\{(S_k, V_k)\}_{k \in K}$ will be said to be *bi-free* if the family of pairs of faces

$$\{(\mathrm{alg}(1_A \cup S_k), \mathrm{alg}(1_A \cup V_k))\}_{k \in K}$$

is bi-free.

(iv) If S_k and V_k are subsets of A for all $k \in K$, then the family $\{(S_k, V_k)\}_{k \in K}$ will be said to be $*$-*bi-free* if the family

$$\{(S_k \cup S_k^*, V_k \cup V_k^*)\}_{k \in K}$$

is bi-free.

The bi-free cumulant function is the main combinatorial tool used in bi-free probability theory and its definition is given below.

Definition 3.1.8. Let (A, φ) be a non-commutative $*$-probability space. The *bi-free cumulant function* is the map

$$\kappa : \bigcup_{n \in \mathbb{N}} \bigcup_{\chi \in \{l,r\}^n} \mathrm{BNC}(\chi) \times A^n \to \mathbb{C}$$

defined by

$$\kappa_{\chi,\tau}(a_1, \ldots, a_n) := \kappa(\tau, a_1, \ldots, a_n) = \sum_{\substack{\lambda \in BNC(\chi) \\ \lambda \leq \tau}} \varphi_\lambda(a_1, \ldots, a_n) \mu_{\mathrm{BNC}}(\lambda, \tau)$$

for each $n \in \mathbb{N}, \chi \in \{l, r\}^n, \tau \in \mathrm{BNC}(\chi)$ and $a_1, \ldots, a_n \in A$.

The previous formula is called the *moment-cumulant formula* and an application of Möbius inversion yields that we must also have that

$$\varphi(a_1 \cdots a_n) = \sum_{\tau \in \mathrm{BNC}(\chi)} \kappa_{\chi,\tau}(a_1, \ldots, a_n).$$

It is clear that for $n \in \mathbb{N}, \chi \in \{l, r\}^n$ and $\tau \in \mathrm{BNC}(\chi)$, the bi-free cumulant map

$$\kappa_{\chi,\tau} : A^n \to \mathbb{C}$$

is multilinear. In the special case when $\tau = 1_\chi$, we will denote $\kappa_{\chi,1_\chi}$ simply by κ_χ. Multiplicative properties of the bi-free cumulant function yield that

$$\kappa_{\chi,\tau}(a_1, \ldots, a_n) = \prod_{V \in \tau} \kappa_{\chi|_V}((a_1, \ldots, a_n)|_V),$$

for all $n \in \mathbb{N}, \chi \in \{l, r\}^n, \tau \in \mathrm{BNC}(\chi)$ and $a_1, \ldots, a_n \in A$. See [11] for proofs and discussions on all the aforementioned properties. Note that the result of reading the sequence $(a_1, \ldots, a_n)|_V$ with the indices in the induced $\chi|_V$-order coincides with first reading the sequence $(a_1, \ldots, a_n)$ with the indices in the χ-order and then restricting the resulting sequence to $s_\chi^{-1}(V)$. For a concrete example, let $a_1, a_2, a_3, a_4, a_5 \in A$, let $V = \{2, 3, 4\}$ and let $\chi \in \{l, r\}^5$ be such that $\chi^{-1}(\{l\}) = \{1, 4\}$. Then $(a_1, \ldots, a_5)|_V = (a_2, a_3, a_4)$ and the result of reading this sequence in the induced $\chi|_V$-order is (a_4, a_3, a_2) (since we are first listing the left entries in increasing order, followed by the right entries in decreasing order). Also, since $(a_{s_\chi(1)}, \ldots, a_{s_\chi(5)}) = (a_1, a_4, a_5, a_3, a_2)$ and $s_\chi^{-1}(V) = \{2, 4, 5\}$ (which corresponds to the fact that we will only be keeping the second, fourth and fifth terms of the aforementioned induced sequence), the coincidence of the two sequences follows.

For $a_1, \ldots, a_n \in A$, we will often make reference to the *bi-free cumulants of the tuple* $(a_1, \ldots, a_n)$, by which simply signify the collection

$$\{\kappa_\chi(c_1, \ldots, c_k) : k \in \mathbb{N}, \chi \in \{l, r\}^k, c_1, \ldots, c_k \in \{a_1, \ldots, a_n\}\}$$

of all bi-free cumulants with entries in the tuple $(a_1, \ldots, a_n)$. We will also make reference to the *bi-free $*$-cumulants of the tuple* $(a_1, \ldots, a_n)$ when talking about the bi-free cumulants of the tuple $(a_1, a_1^*, \ldots, a_n, a_n^*)$.

Observe that the moment-cumulant formula implies that for elements $X, Y, Z, W \in A$ the joint $*$-distribution of the pair (X, Y) coincides with the joint $*$-distribution of (Z, W) if and only if all bi-free $*$-cumulants of the pair (X, Y) coincide with the corresponding bi-free $*$-cumulants of the pair (Z, W).

The following theorem displays the equivalent combinatorial characterization of bi-free independence.

Theorem 3.1.9 ([11], Theorem 4.3.1)**.** *Let (A, φ) be a non-commutative $*$-probability space and let $\{(C_k, D_k)\}_{k\in K}$ be family of pairs of faces in A. The following are equivalent:*

(i) the family $\{(C_k, D_k)\}_{k\in K}$ is bi-free,

(ii) for all $n \in \mathbb{N}, \chi \in \{l, r\}^n, a_1, \ldots, a_n \in A$ and non-constant map $\epsilon : \{1, \ldots, n\} \to K$ such that

$$a_i \in \begin{cases} C_{\epsilon(i)}, & \text{if } \chi(i) = l \\ D_{\epsilon(i)}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

we have that

$$\kappa_\chi(a_1, \ldots, a_n) = 0.$$

Given that the majority of the results of this chapter involve computations of bi-free cumulants having products of operators as entries, we will recall from [10, Section 9] the necessary combinatorial notions.

Definition 3.1.10. Let $m, n \in N$, $\chi \in \{l, r\}^m$ and fix natural numbers

$$k(0) = 0 < k(1) < \ldots < k(m) = n.$$

We define $\widehat{\chi} \in \{l, r\}^n$ via

$$\widehat{\chi}(q) = \chi(p_q),$$

where p_q is the unique element of $\{1, \ldots, m\}$ such that $k(p_q - 1) < q \leq k(p_q)$. With this notation, we define an embedding

$$\begin{aligned} \mathrm{BNC}(\chi) &\to \mathrm{BNC}(\widehat{\chi}) \\ \pi &\mapsto \hat{\pi} \end{aligned}$$

that is obtained by replacing $p \in \{1, \ldots, m\}$ by the block $\{k(p-1)+1, \ldots, k(p)\}$. Note that $\widehat{\chi}$ is constant on each block $\{k(p-1)+1, \ldots, k(p)\}$ and its value is equal to $\chi(p)$.

It is easy to see that the mapping $\pi \mapsto \hat{\pi}$ is an injective order embedding, $1_{\widehat{\chi}} = \widehat{1_\chi}$ and that $\widehat{0_\chi}$ is the partition with blocks

$$\{\{k(p-1)+1, \ldots, k(p)\} : 1 \leq p \leq m\}.$$

Moreover, the image of $\mathrm{BNC}(\chi)$ under this map is

$$\widehat{\mathrm{BNC}(\chi)} = \left[\widehat{0_\chi}, \widehat{1_\chi}\right] = \left[\widehat{0_\chi}, 1_{\widehat{\chi}}\right] \subseteq \mathrm{BNC}(\widehat{\chi}),$$

and, since this map preserves the order, we obtain that $\mu_{\mathrm{BNC}}(\sigma, \pi) = \mu_{\mathrm{BNC}}(\hat{\sigma}, \hat{\pi})$.

The main idea behind introducing the notions in Definition 3.1.10 becomes clear when considering the following theorem, which we will be using frequently throughout this chapter.

Theorem 3.1.11 (Scalar case of [10], Theorem 9.1.5)**.** *Let (A, φ) be a non-commutative $*$-probability space, $m < n \in \mathbb{N}, \chi \in \{l, r\}^m$ and integers*

$$k(0) = 0 < k(1) < \ldots < k(m) = n.$$

Also, let $a_1, \ldots, a_n \in A$. Then with $\widehat{\chi} \in \{l, r\}^n$ as in Definition 3.1.10 we have that

$$\kappa_\chi(a_1 \cdots a_{k(1)}, a_{k(1)+1} \cdots a_{k(2)}, \ldots, a_{k(m-1)+1} \cdots a_{k(m)}) = \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \kappa_{\widehat{\chi}, \tau}(a_1, \ldots, a_n).$$

Note that in the case when there exists $t \in \mathbb{N}$ such that $k(i) = k(i-1) + t$ for all $i = 1, \ldots, m$, then $\widehat{0_\chi} = s_{\widehat{\chi}}(\widehat{0_\chi})$. We find it convenient to state and prove the following proposition, concerning bi-non-crossing partitions whose blocks have to connect consecutive indices in the χ-order. In sections 3.2 and 3.3, when discussing the behaviour of products of pairs of operators, the forward direction of this proposition will be used frequently in combination with Theorem 3.1.11.

Proposition 3.1.12. *Let $n \in \mathbb{N}$ and $\chi \in \{l, r\}^{2n}$ such that $\chi(2i-1) = \chi(2i)$ for every $i = 1, \ldots, n$. Also, let $\widehat{0_\chi} = \{\{2i-1, 2i\} : i = 1, \ldots, n\}$. Then, for a bi-non-crossing partition $\tau \in \mathrm{BNC}(\chi)$, the following are equivalent:*

(i) $\tau \vee \widehat{0_\chi} = 1_\chi$ *and every block of* τ *contains an even number of elements,*

(ii) $s_\chi(1) \sim_\tau s_\chi(2n)$ *and* $s_\chi(2i) \sim_\tau s_\chi(2i+1)$ *for every* $i = 1, \ldots, n-1$.

Proof. Since $\widehat{0_\chi} = s_\chi(\widehat{0_\chi})$, it is clear that clause (ii) above implies clause (i). Now, let $\tau \in \mathrm{BNC}(\chi)$ be such that $\tau \vee \widehat{0_\chi} = 1_\chi$ and every block of τ contains an even number of elements and let $V \in \tau$ such that $s_\chi(1) \in V$ (equivalently $1 = \min_< {s_\chi}^{-1}(V)$). Also, let $q \in \{1, \ldots, 2n\}$ such that $s_\chi(q) = \max_{\prec_\chi} V$(equivalently $q = \max_< {s_\chi}^{-1}(V)$). We claim that q must be an even number.

Indeed, by way of contradiction, suppose that $q = 2m-1$ for some $m \in \{2, \ldots, n\}$. We remark that V cannot be equal to $\{s_\chi(i) : 1 \leq i \leq 2m-1\}$ since V must contain an even number of elements. Notice that if $2 \leq p \leq 2m-2$ is such that $s_\chi(p) \notin V$ and $V' \in \tau$ is such that $s_\chi(p) \in V'$, then we necessarily must have that $V' \subseteq \{s_\chi(i) : 2 \leq i \leq 2m-2\}$; for if there exists $i \geq 2m$ with $s_\chi(i) \in V'$ then we obtain that

$$1 = \min_< {s_\chi}^{-1}(V)\ ,\ 2m-1 = \max_< {s_\chi}^{-1}(V)$$

and

$$p, i \in {s_\chi}^{-1}(V') \text{ with } 2 \leq p \leq 2m-2 \text{ and } 2m \leq i,$$

which contradicts the fact that ${s_\chi}^{-1} \cdot \tau \in \mathrm{NC}(2n)$. This shows that the set

$$\{s_\chi(i) : 1 \leq i \leq 2m-1\},$$

whose cardinality is obviously odd must be written as a union of blocks of τ, thus τ must contain at least one block with an odd number of elements, contradicting our initial assumption. Hence $q = 2m$ for some $m \in \{1, \ldots, n\}$. If $m < n$, then let

$$\widetilde{V} = \{s_\chi(i) : 1 \leq i \leq 2m\}$$

and define $\lambda = \{\widetilde{V}, \left(\widetilde{V}\right)^{\mathsf{c}}\}$. Since ${s_\chi}^{-1} \cdot \lambda = \{1, 2, \ldots, 2m\} \cup \{2m+1, \ldots, 2n\}$, we have that $\lambda \in \mathrm{BNC}(\chi), V \subseteq \widetilde{V}$ and that $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_\chi$, thus the condition $\tau \vee \widehat{0_\chi} = 1_\chi$ cannot be satisfied. Hence, we must have that $q = 2n$ and this implies that $s_\chi(1) \sim_\tau s_\chi(2n)$.

Now let $i \in \{1, \ldots, n-1\}$ and $V \in \tau$ such that $s_\chi(2i) \in V$. Assume that $s_\chi(2i+1) \notin V$ and we will distinguish between two possibilities:

First, let us suppose that $s_\chi(2i) = \max_{\prec_\chi} V$ and let $q \in \{1, \ldots, 2n\}$ be such that $s_\chi(q) = \min_{\prec_\chi}(V)$. Then, arguing as before, we deduce that we must have $q = 2p-1$

for some $1 \leq p \leq i$ (otherwise, if $q = 2p$ with $1 \leq p < i$, then the cardinality of the set

$$\{s_\chi(j) : 2p \leq j \leq 2i\}$$

is odd and thus τ contains at least one block with an odd number of elements which of course cannot happen). But then, by setting $\widetilde{V} = \{s_\chi(j) : 2p - 1 \leq j \leq 2i\}$ and $\lambda = \{\widetilde{V}, \big(\widetilde{V}\big)^{\mathsf{c}}\}$, since

$$s_\chi{}^{-1} \cdot \lambda = \{1, \ldots, 2(p-1)\} \cup \{2p-1, \ldots, 2i\} \cup \{2i+1, \ldots, 2n\},$$

we obtain that $\lambda \in \mathrm{BNC}(\chi), V \subseteq \widetilde{V}$ and $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_\chi$, a contradiction. This shows that it cannot be the case that $s_\chi(2i) = \max_{\prec_\chi} V$ and hence, there must exist $q \in \{1, \ldots, 2n\}$ such that $s_\chi(q) \in V$ and $s_\chi(2i) \prec_\chi s_\chi(q)$. Without loss of generality, we may assume that for every $v \in V \setminus \{s_\chi(2i), s_\chi(q)\}$, we either have that $v \prec_\chi s_\chi(2i)$ or $s_\chi(q) \prec_\chi v$ (i.e. we may assume that $s_\chi(q)$ is the χ-minimum element of V with this property). If $q = 2m$ for some $q \geq i+1$, then the set $\{s_\chi(j) : 2i + 1 \leq j \leq 2m - 1\}$ is non-empty and contains an odd number of elements. Thus, arguing as before, it must be written as a union of blocks of τ, which implies that at least one block of τ contains an odd number of elements, a contradiction.

If $q = 2m - 1$ for some $m > (i+1)$, then the set

$$\widetilde{V} = \{s_\chi(j) : 2i + 1 \leq j \leq 2(m-1)\}$$

is non-empty and contains an even number of elements. Let $\lambda = \widetilde{V} \cup \big(\widetilde{V}\big)^{\mathsf{c}}$. Then, since $V \subseteq \big(\widetilde{V}\big)^{\mathsf{c}}$, it follows that $\lambda \in \mathrm{BNC}(\chi)$ and $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_\chi$, a contradiction. This shows that we must have $s_\chi(2i) \sim_\tau s_\chi(2i+1)$ and this completes the proof. $\square$

3.1.3 Bi-R-Diagonal Pairs of Operators

In the context of free probability, R-diagonal operators are characterized by having all of their free $*$-cumulants that are either of odd order, or have entries that are not alternating in $*$-terms and non-$*$-terms equal to zero. With the aforementioned preliminaries on bi-free cumulants in hand and by adopting the combinatorial approach in the bi-free setting, we will now give the definition of bi-R-diagonal pairs of operators, which will be the central focus of this chapter. This definition was first proposed as the correct bi-free generalization of R-diagonal elements in [68, Section 4], but was only used to yield examples of R-cyclic pairs of matrices (see Proposition 3.1.23).

Definition 3.1.13. Let (A, φ) be a non-commutative $*$-probability space and $X, Y \in A$. We say that the pair (X, Y) is *bi-R-diagonal* if for every $n \in \mathbb{N}$, $\chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ such that

$$a_i \in \begin{cases} \{X, X^*\}, & \text{if } \chi(i) = l \\ \{Y, Y^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

we have that:

(i) $\kappa_\chi(a_1, \ldots, a_n) = 0$, if n is odd

(ii) $\kappa_\chi(a_1, \ldots, a_n) = 0$, if n is even and the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is *not* in one of the following forms:

  (a) $(Z, Z^*, \ldots, Z, Z^*)$, with $Z \in \{X, X^*, Y, Y^*\}$,

  (b) $(X, X^*, \ldots, X, X^*, Y, Y^*, \ldots, Y, Y^*)$,

  (c) $(X^*, X, \ldots, X^*, X, Y^*, Y, \ldots, Y^*, Y)$,

  (d) $(X, X^*, \ldots, X, X^*, X, Y^*, Y, \ldots, Y^*, Y, Y^*)$,

  (e) $(X^*, X, \ldots, X^*, X, X^*, Y, Y^*, \ldots, Y, Y^*, Y)$,

  i.e. whenever the sequence $(a_1, \ldots, a_n)$ is not alternating in $*$-terms and non $*$-terms when read with the indices in the χ-order, with any number of X-terms followed by any number of Y-terms.

It is clear from the definition that if the map χ is constant, then bi-free cumulants reduce to free cumulants and all free cumulants with entries in either $\{X, X^*\}$ (if the map χ yields the constant value "l") or $\{Y, Y^*\}$ (if the map χ yields the constant value "r") that are of odd order or are not alternating in $*$-terms and non-$*$-terms are equal to zero. In particular, if (X, Y) is a bi-R-diagonal pair, then both X and Y are R-diagonal operators. Also, it is immediate from the moment-cumulant formula that all joint $*$-moments of odd order of a bi-R-diagonal pair are equal to zero, i.e. if the pair (X, Y) is bi-R-diagonal, then for all $k \in \mathbb{N}$ and $a_1, \ldots, a_{2k+1} \in \{X, X^*, Y, Y^*\}$, it follows that

$$\varphi(a_1 \cdots a_{2k+1}) = 0.$$

Towards providing canonical examples of bi-R-diagonal pairs of operators, in analogy to the case of free probability and free Haar unitaries, we will define the notion of a bi-Haar unitary pair of operators and compute its bi-free $*$-cumulants. Bi-Haar unitary pairs will act as both the prototypical examples and building blocks of bi-R-diagonal pairs (see Theorem 3.3.4). First, we recall the definition of a free Haar unitary.

Definition 3.1.14. Let (B, ψ) be a non-commutative $*$-probability space. A unitary $v \in B$ is called a *Haar unitary* if for all $n \in \mathbb{Z}$ we have that:

$$\psi(v^n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The free $*$-cumulants of a Haar unitary are computed as follows:

Proposition 3.1.15 ([61], Proposition 15.1). *If $v \in (B, \psi)$ is a Haar unitary, then for every $n \in \mathbb{N}$, the non-vanishing free $*$-cumulants of v are given by:*

$$\kappa_{2n}(v, v^*, \ldots, v, v^*) = \kappa_{2n}(v^*, v, \ldots, v^*, v) = (-1)^{n-1} \cdot C_{n-1},$$

where C_n denotes the n-th Catalan number. All other free cumulants with entries in the set $\{v, v^*\}$ vanish.

The bi-free generalization of the notion of a Haar unitary was first proposed in [10, Definition 10.1.2] in the operator-valued setting.

Definition 3.1.16. Let (A, φ) be a non-commutative $*$-probability space and u_l, u_r be unitaries in A. The pair (u_l, u_r) is called a *bi-Haar unitary pair* if the following hold:

(i) the algebras $\mathrm{alg}(\{u_l, {u_l}^*\})$ and $\mathrm{alg}(\{u_r, {u_r}^*\})$ commute,

(ii) for all $n, m \in \mathbb{Z}$ we have that

$$\varphi(u_l^n \cdot u_r^m) = \begin{cases} 1, & \text{if } n + m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if the pair (u_l, u_r) is a bi-Haar unitary, then both u_l and u_r are free Haar unitaries.

Example 3.1.17. Let G be a group with identity e that contains an element of infinite order (i.e. there exists $g_0 \in G$ such that $g_0^n \neq e$ for all $n \in \mathbb{Z} \setminus \{0\}$). If $\lambda : G \to \mathcal{B}(\ell_2(G))$ and

$\rho : G \to \mathcal{B}(\ell_2(G))$ denote the left and right regular representations of G respectively, then it is straightforward to verify that the pair $(\lambda(g_0), \rho(g_0^{-1}))$ is a bi-Haar unitary pair, with respect to the vector state corresponding to the identity element of G. In particular, if u denotes the bilateral shift on $\ell_2(\mathbb{Z})$, then the pair (u, u) is a bi-Haar unitary. $\square$

The joint distributions of bi-Haar unitary pairs found applications in [10] where, in analogy to the setting of free probability, it was shown that conjugation by bi-Haar unitary pairs results in moving bi-free pairs of algebras into bi-free position, while maintaing their joint distributions. Concretely, in the scalar setting the authors obtained the following result:

Theorem 3.1.18 (Scalar case of [10], Theorem 10.1.3)**.** *Let (A, φ) be a non-commutative $*$-probability space and let (u_l, u_r) be a bi-Haar unitary pair in A. If (C, D) is a pair of subalgebras of A that is bi-free from the pair $(\mathrm{alg}(\{u_l, u_l^*\}), \mathrm{alg}(\{u_r, u_r^*\}))$, then the pairs of algebras $(u_l^* C u_l, u_r^* D u_r)$ and (C, D) are bi-freely independent and, moreover, the joint distibution of the pair $(u_l^* C u_l, u_r^* D u_r)$ coincides with the joint distribution of (C, D).*

The commutation assumption on the left and right operators of a bi-Haar unitary pair allows one to reduce the computation of its bi-free cumulants to computing free cumulants of a free Haar unitary. In particular, we have the following:

Proposition 3.1.19. *Let $(A, \varphi), (B, \psi)$ be non-commutative $*$-probability spaces and let (u_l, u_r) be a bi-Haar unitary pair in A and $v \in B$ a Haar unitary. For $n \in \mathbb{N}$ and $\chi \in \{l, r\}^n$, let $a_1, \ldots, a_n \in A$ such that for all $i = 1, \ldots, n$*

$$a_i \in \begin{cases} \{u_l, {u_l}^*\}, & \text{if } \chi(i) = l \\ \{u_r, {u_r}^*\}, & \text{if } \chi(i) = r \end{cases}$$

and define $b_1, \ldots, b_n \in B$ by

$$b_i = \begin{cases} v, & \text{if } a_{s_\chi(i)} \in \{u_l, u_r\} \\ v^*, & \text{if } a_{s_\chi(i)} \in \{{u_l}^*, {u_r}^*\} \end{cases}$$

for all $i = 1, \ldots, n$. Then, we have that

$$\kappa_\chi(a_1, \ldots, a_n) = \kappa_n(b_1, \ldots, b_n),$$

with the quantity on the left-hand side of the equation being a bi-free cumulant and the one on the right-hand side being a free cumulant.

Proof. For $n, m \in \mathbb{Z}$, the following relation between the joint $*$-moments of (u_l, u_r) and the $*$-moments of v is immediate by Definitions 3.1.14 and 3.1.16

$$\varphi({u_l}^n \cdot {u_r}^m) = \psi(v^{n+m}).$$

and, since the algebras $\mathrm{alg}(\{u_l, u_l^*\})$ and $\mathrm{alg}(\{u_r, u_r^*\})$ commute, every joint $*$-moment of the pair (u_l, u_r) factorizes in a moment that has a form similar to the left hand-side of the previous expression. The moment-cumulant formulas yield that

$$\kappa_\chi(a_1, \ldots, a_n) = \sum_{\tau \in \mathrm{BNC}(\chi)} \varphi_\tau(a_1, \ldots, a_n)\mu_{\mathrm{BNC}}(\tau, 1_\chi),$$

and

$$\kappa_n(b_1, \ldots, b_n) = \sum_{\pi \in \mathrm{NC}(n)} \psi_\pi(b_1, \ldots, b_n)\mu_{\mathrm{NC}}(\pi, 1_n).$$

The main observation needed to lead us to the conclusion of the proof is that for all $\tau \in \mathrm{BNC}(\chi)$ and for all $V \in \tau$, we have that

$$\varphi((a_1, \ldots, a_n)|_V) = \psi((b_1, \ldots, b_n)|_{s_\chi^{-1}(V)}).$$

Indeed, let $\tau \in \mathrm{BNC}(\chi)$ and $V \in \tau$. Define the sets

$$I_1 = \{i \in V : a_i = u_l\},\ I_2 = \{i \in V : a_i = u_l^*\},$$

and

$$I_3 = \{i \in V : a_i = u_r\},\ I_4 = \{i \in V : a_i = u_r^*\}.$$

Also, let $n_i \in \mathbb{N}$ to be equal to the cardinality of the set I_i, for all $i = 1, 2, 3, 4$. Then, by the definition of $b_1, \ldots, b_n$, we have that

$$\varphi((a_1, \ldots, a_n)|_V) = \varphi(u_l^{n_1-n_2}u_r^{n_3-n_4}) = \psi(v^{n_1+n_3-n_2-n_4}) = \psi((b_1, \ldots, b_n)|_{s_\chi^{-1}(V)}).$$

Hence, this implies that for any bi-non-crossing partition $\tau \in \mathrm{BNC}(\chi)$ we obtain

$$\varphi_\tau(a_1, \ldots, a_n)\mu_{\mathrm{BNC}}(\tau, 1_\chi) = \varphi_\tau(a_1, \ldots, a_n)\mu_{\mathrm{NC}}(s_\chi^{-1} \cdot \tau, 1_n) = \psi_{s_\chi^{-1} \cdot \tau}(b_1, \ldots, b_n)\mu_{\mathrm{NC}}(s_\chi^{-1} \cdot \tau, 1_n).$$

This completes the proof. $\square$

A combination of Propositions 3.1.15 and 3.1.19 gives a complete computation of the bi-free cumulants involving a bi-Haar unitary pair.

Corollary 3.1.20. *Let (A, φ) be a non-commutative $*$-probability space and (u_l, u_r) a bi-Haar unitary pair in A. Also, let $n \in \mathbb{N}, \chi \in \{l, r\}^{2n}$ and $a_1, \ldots, a_{2n} \in A$ such that*

(a) for all $i = 1, \ldots, 2n$ we have

$$a_i \in \begin{cases} \{u_l, {u_l}^*\}, & \text{if } \chi(i) = l \\ \{u_r, {u_r}^*\}, & \text{if } \chi(i) = r \end{cases}$$

(b) the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(2n)})$ is alternating in $*$-terms and non-$*$-terms.

Then,

$$\kappa_\chi(a_1, \ldots, a_{2n}) = (-1)^{n-1} \cdot C_{n-1},$$

where C_n denotes the n-th Catalan number. All other bi-free $*$-cumulants of the pair (u_l, u_r) vanish. In particular, the pair (u_l, u_r) is bi-R-diagonal.

3.1.4 Operator-Valued Bi-Free Independence and R-cyclic Pairs of Matrices

In the spirit of [10] and [68], we will present the basic definitions regarding operator-valued bi-free independence and a number of results concerning R-cyclic pairs of matrices. The results that are cited will be used in Section 3.3 to discuss an equivalent characterization of the condition of bi-R-diagonality, which will be formulated in terms of the bi-freeness of certain matrix pairs from scalar matrices with amalgamation over the diagonal scalar matrices (see Theorem 3.3.6).

Definition 3.1.21. Let B be a unital algebra.

(i) A *B-B-bimodule with specified B-vector state* is a triple $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ where $\mathcal{X}$ is a direct sum of B-B-bimodules

$$\mathcal{X} = B \oplus \mathring{\mathcal{X}}$$

and $p : \mathcal{X} \to B$ is the linear map given by

$$p(b \oplus \eta) = b,$$

for all $b \in B$ and $\eta \in \mathring{\mathcal{X}}$. On $\mathcal{L}(\mathcal{X})$, the space of linear maps on $\mathcal{X}$, we define the *expectation of* $\mathcal{L}(\mathcal{X})$ *onto* B which is the linear map given by

$$E_{\mathcal{L}(\mathcal{X})}(T) = p\left(T\left(1_B \oplus 0\right)\right),$$

for all $T \in \mathcal{L}(\mathcal{X})$.

(ii) A *B-B-non commutative probability space* is a triple (A, E_A, ε), where A is a unital algebra, $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ is a unital homomorphism such that both maps $\varepsilon|_{B\otimes 1_B}$ and $\varepsilon|_{1_B \otimes B^{\mathrm{op}}}$ are injective and $E_A : A \to B$ is a linear map such that

$$E_A(\varepsilon(b_1 \otimes b_2)Z) = b_1 E_A(Z) b_2$$

and

$$E_A(Z\varepsilon(b \otimes 1_B)) = E_A(Z\varepsilon(1_B \otimes b)),$$

for all $b, b_1, b_2 \in B$ and $Z \in A$. The unital subalgebras of A defined as

$$A_l = \{Z \in A : Z\varepsilon(1_B \otimes b) = \varepsilon(1_B \otimes b)Z \text{ for all } b \in B\}$$

and

$$A_r = \{Z \in A : Z\varepsilon(b \otimes 1_B) = \varepsilon(b \otimes 1_B)Z \text{ for all } b \in B\},$$

are called the *left* and *right algebras* of A respectively.

(iii) A *pair of B-faces* in a B-B-non commutative probability space (A, E_A, ε) consists of a pair (C, D) of unital subalgebras of A such that

$$\varepsilon(B \otimes 1_B) \subseteq C \subseteq A_l \text{ and } \varepsilon(1_B \otimes B^{\mathrm{op}}) \subseteq D \subseteq A_r.$$

(iv) A family $\{(C_k, D_k)\}_{k\in K}$ of pairs of B-faces in a B-B-non commutative probability space (A, E_A, ε) is said to be *bi-free with amalgamation over B* if there exist B-B-bimodules with specified B-vector states $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ and unital homomorphisms $l_k : C_k \to \mathcal{L}_l(\mathcal{X}_k)$ and $r_k : D_k \to \mathcal{L}_r(\mathcal{X}_k)$ such that the joint distribution of $\{(C_k, D_k)\}_{k\in K}$ with respect to E_A is equal to the joint distribution of the images of

$$\{((\lambda_k \circ l_k)(C_k), (\rho_k \circ r_k)(D_k))\}_{k\in K}$$

inside $\mathcal{L}(*_{k\in K}\mathcal{X}_k)$ with respect to $E_{\mathcal{L}(*_{k\in K}\mathcal{X}_k)}$, where λ_k and ρ_k denote the left and right regular representations onto $\mathcal{X}_k \subseteq *_{k\in K}\mathcal{X}_k$, respectively.

If $S_k \subseteq A_l$ and $V_k \subseteq A_r$ for all $k \in K$, we will say that the family $\{(S_k, V_k)\}_{k\in K}$ is bi-free with amalgamation over B if the family

$$\{(\mathrm{alg}(\varepsilon(B \otimes 1_B) \cup S_k), \mathrm{alg}(\varepsilon(1_B \otimes B^{\mathrm{op}}) \cup V_k))\}_{k\in K}$$

of pairs of B-faces is bi-free with amalgamation over B.

See [10, Section 3] for a discussion on why B-B-non-commutative probability spaces are the correct framework to formulate the notions of operator-valued bi-free probability. There, a combinatorial approach was adopted and the bi-multiplicative operator-valued bi-free cumulant maps were defined and used to characterize operator-valued bi-free independence.

Let (A, φ) be a non-commutative $*$-probability space and let $d \in \mathbb{N}$. In the algebra $\mathcal{M}_d(A)$ of all $d \times d$ matrices over A, consider the unital subalgebras $\mathcal{M}_d(\mathbb{C})$ and $\mathcal{D}_d$ consisting of all scalar matrices and all diagonal scalar matrices respectively and let $F_d : \mathcal{M}_d(\mathbb{C}) \to \mathcal{D}_d$ denote the conditional expectation onto the diagonal. We will recall from [68, Section 4] the process on how to turn $\mathcal{L}(\mathcal{M}_d(A))$, the space of all linear maps on $\mathcal{M}_d(A)$, into a $\mathcal{M}_d(\mathbb{C})$-$\mathcal{M}_d(\mathbb{C})$-non-commutative probability space. We will denote by $[a_{i,j}]$ a matrix whose $(i,j)^{\mathrm{th}}$ entry equals $a_{i,j}$.

Define the unital, linear map $\varphi_d : \mathcal{M}_d(A) \to \mathcal{M}_d(\mathbb{C})$ by

$$\varphi_d([T_{i,j}]) = [\varphi(T_{i,j})]$$

for all $[T_{i,j}] \in \mathcal{M}_d(A)$. Also, for $[a_{i,j}] \in \mathcal{M}_d(\mathbb{C})$, let

$$L_{[a_{i,j}]}([T_{i,j}]) = \left[\sum_{k=1}^{d} a_{i,k}T_{k,j}\right] \text{ and } R_{[a_{i,j}]}([T_{i,j}]) = \left[\sum_{k=1}^{d} a_{k,j}T_{i,k}\right],$$

for all $[T_{i,j}] \in \mathcal{M}_d(A)$. Then, if $\varepsilon : \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})^{\mathrm{op}} \to \mathcal{L}(\mathcal{M}_d(A))$ is defined as

$$\varepsilon([a_{i,j}] \otimes [a'_{i,j}]) = L_{[a_{i,j}]}R_{[a'_{i,j}]}, \qquad ([a_{i,j}], [a'_{i,j}] \in \mathcal{M}_d(\mathbb{C}))$$

and $E_d : \mathcal{L}(\mathcal{M}_d(A)) \to \mathcal{M}_d(\mathbb{C})$ is defined as

$$E(Z) = \varphi_d(Z(I_d)), \qquad (Z \in \mathcal{L}(\mathcal{M}_d(A)))$$

where I_d denotes the $d \times d$ identity matrix, we have that the triple $(\mathcal{L}(\mathcal{M}_d(A)), E_d, \varepsilon)$ is a $\mathcal{M}_d(\mathbb{C})$-$\mathcal{M}_d(\mathbb{C})$-non-commutative probability space. We will also need the unital homomorphisms $L : \mathcal{M}_d(A) \to \mathcal{L}(\mathcal{M}_d(A))_l$ and $R : \mathcal{M}_d(A^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{L}(\mathcal{M}_d(A))_r$ given by

$$L([Z_{i,j}])[T_{i,j}] = \left[\sum_{k=1}^{d} Z_{i,k}T_{k,j}\right] \text{ and } R([Z_{i,j}])[T_{i,j}] = \left[\sum_{k=1}^{d} Z_{k,j}T_{i,k}\right],$$

for all $[Z_{i,j}], [T_{i,j}] \in \mathcal{M}_d(A)$.

In the setting of free probability, there is a connection between R-diagonal operators and R-cyclic matrices (see [61, Example 20.5]). In the bi-free setting, R-cyclic pairs of matrices were first defined and studied in [68].

Definition 3.1.22. [68, Definition 4.4] Let (A, φ) be a non-commutative $*$-probability space, $d \in \mathbb{N}$, I, J be disjoint index sets and let $\{[Z_{k;i,j}]\}_{k\in I} \cup \{[Z_{k;i,j}]\}_{k\in J} \subseteq \mathcal{M}_d(A)$. The pair $(\{[Z_{k;i,j}]\}_{k\in I}, \{[Z_{k;i,j}]\}_{k\in J})$ is called *R-cyclic* if for all $n \in \mathbb{N}, \omega : \{1, \ldots, n\} \to I \sqcup J$ and $1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq d$, by defining $\chi \in \{l, r\}^n$ as

$$\chi(i) = \begin{cases} l, & \text{if } \omega(i) \in I \\ r, & \text{if } \omega(i) \in J \end{cases} \qquad (i = 1, \ldots, n)$$

we have that

$$\kappa_\chi(Z_{\omega(1);i_1,j_1}, Z_{\omega(2);i_2,j_2}, \ldots, Z_{\omega(n);i_n,j_n}) = 0$$

whenever at least one of the relations

$$j_{s_\chi(1)} = i_{s_\chi(2)}, j_{s_\chi(2)} = i_{s_\chi(3)}, \ldots, j_{s_\chi(n-1)} = i_{s_\chi(n)}, j_{s_\chi(n)} = i_{s_\chi(1)}$$

is *not* satisfied.

The following result was mentioned (but not proved) in [68, Section 4] and we include the proof for the convenience of the reader.

Proposition 3.1.23. *Let (A, φ) be a non-commutative $*$-probability space and let $X, Y \in A$. The following are equivalent:*

(i) the pair (X, Y) is bi-R-diagonal,

(ii) in $\mathcal{M}_2(A)$, the pair $([Z_{i,j}]_{1\le i,j\le 2}, [W_{i,j}]_{1\le i,j\le 2})$ defined as

$$[Z_{i,j}]_{1\le i,j\le 2} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \text{ and } [W_{i,j}]_{1\le i,j\le 2} = \begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix}$$

is R-cyclic.

Proof. The main observation that will make the equivalence of the proposition apparent is that the condition that at least one of the relations

$$j_{s_\chi(1)} = i_{s_\chi(2)}, j_{s_\chi(2)} = i_{s_\chi(3)}, \dots, j_{s_\chi(n-1)} = i_{s_\chi(n)}, j_{s_\chi(n)} = i_{s_\chi(1)}$$

is not satisfied is equivalent to the statement that the sequence

$$(a_{i_{s_\chi(1)}, j_{s_\chi(1)}}, \dots, a_{i_{s_\chi(n)}, j_{s_\chi(n)}})$$

is either not alternating in $*$-terms and non-$*$-terms, or is of odd length.

Indeed, first suppose that $j_{s_\chi(m)} \neq i_{s_\chi(m+1)}$ for some $m \in \{1, \dots, n-1\}$ and notice that this implies that we must have

$$i_{s_\chi(m)} = i_{s_\chi(m+1)} \text{ and } j_{s_\chi(m)} = j_{s_\chi(m+1)}.$$

But this is equivalent to stating that the elements $a_{i_{s_\chi(m)}, j_{s_\chi(m)}}$ and $a_{i_{s_\chi(m+1)}, j_{s_\chi(m+1)}}$ both correspond to either $*$-terms or non-$*$-terms and hence the sequence

$$(a_{i_{s_\chi(1)}, j_{s_\chi(1)}}, \dots, a_{i_{s_\chi(n)}, j_{s_\chi(n)}})$$

is not alternating in $*$-terms and non-$*$-terms.

Next, assume that $j_{s_\chi(n)} \neq i_{s_\chi(1)}$. As before, we must have that

$$i_{s_\chi(1)} = i_{s_\chi(n)} \text{ and } j_{s_\chi(1)} = j_{s_\chi(n)}.$$

This is equivalent to stating that the first and last terms of the sequence

$$(a_{i_{s_\chi(1)}, j_{s_\chi(1)}}, \dots, a_{i_{s_\chi(n)}, j_{s_\chi(n)}})$$

both correspond to either $*$-terms or non-$*$-terms, which means that this sequence either is not alternating in $*$-terms and non-$*$-terms, or is of odd length. $\square$

The main result we will need for Theorem 3.3.6 concerns the following equivalent characterization of R-cyclic pairs.

Theorem 3.1.24 ([68], Theorem 4.9)**.** *Let (A, φ) be a non-commutative $*$-probability space, $d \in \mathbb{N}$, I, J be disjoint index sets and let $\{[Z_{k;i,j}]\}_{k\in I} \cup \{[Z_{k;i,j}]\}_{k\in J} \subseteq \mathcal{M}_d(A)$. The following are equivalent:*

(i) the pair $\left(\{[Z_{k;i,j}]\}_{k\in I}, \{[Z_{k;i,j}]\}_{k\in J}\right)$ is R-cyclic,

(ii) the family $\left((\{L([Z_{k;i,j}])\}_{k\in I}, \{R([Z_{k;i,j}])\}_{k\in J}\right)$ is bi-free from $(L(\mathcal{M}_d(\mathbb{C})), R(\mathcal{M}_d(\mathbb{C})^{op}))$ with amalgamation over $\mathcal{D}_d$ with respect to $F_d \circ E_d$.

3.2 Operations Involving Bi-R-Diagonal Pairs

In this section, we will study the behaviour of bi-R-diagonal pairs of operators under the taking of sums, products and arbitrary powers, where, in most cases, a $*$-bi-free independence condition will be assumed. The proofs obtained will indicate that most of the results that hold for free R-diagonal elements (see [60] and [61, Lecture 15]) have corresponding generalizations in the bi-free setting. We begin with the following proposition regarding sums of $*$-bi-free bi-R-diagonal pairs.

Proposition 3.2.1. *Let (A, φ) be a non-commutative $*$-probability space and $X, Y, Z, W \in A$ such that:*

(a) the pairs (X, Y) and (Z, W) are both bi-R-diagonal,

(b) the pairs (X, Y) and (Z, W) are $*$-bi-free.

Then, the pair $(X + Z, Y + W)$ is also bi-R-diagonal.

Proof. Let $n \in \mathbb{N}, \chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ such that

$$a_i \in \begin{cases} \{X + Z, X^* + Z^*\}, & \text{if } \chi(i) = l \\ \{Y + W, Y^* + W^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

Define $b_1, \dots, b_n, c_1, \dots, c_n \in A$ by

$$b_i = \begin{cases} X, & \text{if } a_i = X + Z \\ X^*, & \text{if } a_i = X^* + Z^* \\ Y, & \text{if } a_i = Y + W \\ Y^*, & \text{if } a_i = Y^* + W^* \end{cases} \quad \text{and} \quad c_i = \begin{cases} Z, & \text{if } a_i = X + Z \\ Z^*, & \text{if } a_i = X^* + Z^* \\ W, & \text{if } a_i = Y + W \\ W^*, & \text{if } a_i = Y^* + W^* \end{cases}$$

for each $i = 1, \dots, n$. Then, the multi-linearity of the bi-free cumulants maps combined with the $*$-bi-free independence condition yield that

$$\kappa_\chi(a_1, \dots, a_n) = \kappa_\chi(b_1, \dots, b_n) + \kappa_\chi(c_1, \dots, c_n).$$

The conclusion of the proposition follows from the observation that the sequence

$$(a_{s_\chi(1)}, \dots, a_{s_\chi(n)})$$

is alternating in $*$-terms and non-$*$-terms if and only if both the sequences

$$(b_{s_\chi(1)}, \dots, b_{s_\chi(n)}) \text{ and } (c_{s_\chi(1)}, \dots, c_{s_\chi(n)})$$

are also alternating in $*$-terms and non-$*$-terms. $\square$

With the previous proof in mind, it is easy to see that if exactly one of the $*$-bi-free pairs (X, Y) and (Z, W) is bi-R-diagonal, then the pair $(X + Z, Y + W)$ cannot be bi-R-diagonal.

We now proceed to study various cases on products involving bi-R-diagonal pairs. The products of pairs will be considered pointwise, with the condition that the order of the right operators is reversed being necessary for the results concerning the more general cases (see Theorem 3.2.2 below and also Proposition 3.3.2). The proofs of these results will require more delicate arguments when compared to the cases of sums involving bi-R-diagonal pairs and, for this, the formula for bi-free cumulants with products of operators as arguments will play a key role. The next theorem states that the product of a bi-R-diagonal pair of operators by any $*$-bi-free pair is also bi-R-diagonal and exhibits the fact that bi-R-diagonal pairs exist in abundance.

Theorem 3.2.2. *Let (A, φ) be a non-commutative $*$-probability space and let $X, Y, Z, W \in A$ such that:*

(a) the pair (X, Y) is bi-R-diagonal,

(b) the pairs (X, Y) and (Z, W) are $*$-bi-free.

Then, the pair (XZ, WY) is also bi-R-diagonal.

Proof. Let $n \in \mathbb{N}$, $\chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ be such that

$$a_i \in \begin{cases} \{XZ, Z^*X^*\}, & \text{if } \chi(i) = l \\ \{WY, Y^*W^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

Define $\widehat{\chi} \in \{l, r\}^{2n}$ by $\widehat{\chi}(2i-1) = \widehat{\chi}(2i) = \chi(i)$ for each $i = 1, \ldots, n$ and $c_1, \ldots, c_{2n} \in A$ as follows:

$$c_{2i-1} = \begin{cases} X, & \text{if } a_i = XZ \\ Z^*, & \text{if } a_i = Z^*X^* \\ W, & \text{if } a_i = WY \\ Y^*, & \text{if } a_i = Y^*W^* \end{cases} \text{ and } c_{2i} = \begin{cases} Z, & \text{if } a_i = XZ \\ X^*, & \text{if } a_i = Z^*X^* \\ Y, & \text{if } a_i = WY \\ W^*, & \text{if } a_i = Y^*W^* \end{cases}$$

for each $i = 1, \ldots, n$. Then, an application of Theorem 3.1.11 yields:

$$\begin{aligned} \kappa_\chi(a_1, \ldots, a_n) &= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \kappa_{\widehat{\chi}, \tau}(c_1, \ldots, c_{2n}) && (1) \\ &= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \prod_{V \in \tau} \kappa_{\widehat{\chi}|_V}\big((c_1, \ldots, c_{2n})|_V\big) && (2) \end{aligned}$$

where $\widehat{0_\chi} = \{\{2i-1, 2i\} : i = 1, \ldots, n\} \in \mathrm{BNC}(\widehat{\chi})$.

To start, we make some remarks. First of all, if $\chi^{-1}(\{l\}) = \{i_1 < \ldots < i_p\}$ and $\chi^{-1}(\{r\}) = \{j_1 < \ldots < j_{n-p}\}$, the definition of $\widehat{\chi}$ implies that

$$\widehat{\chi}^{-1}(\{l\}) = \{2i_1 - 1 < 2i_1 < \ldots < 2i_p - 1 < 2i_p\}$$

and

$$\widehat{\chi}^{-1}(\{r\}) = \{2j_1 - 1 < 2j_1 < \ldots < 2j_{n-p} - 1 < 2j_{n-p}\}.$$

Thus, if $i \in \{1, \ldots, n\}$ with $a_{s_\chi(i)} = XZ$, then $c_{s_{\widehat{\chi}}(2i-1)} = X$ and $c_{s_{\widehat{\chi}}(2i)} = Z$ (a similar situation occurs when $a_{s_\chi(i)} = Z^*X^*$, since this corresponds to a left operator). Now if $a_{s_\chi(i)} = WY$, then $c_{s_{\widehat{\chi}}(2i-1)} = Y$ and $c_{s_{\widehat{\chi}}(2i)} = W$ (and a similar situation occurs when $a_{s_\chi(i)} = Y^*W^*$ since this corresponds to a right operator). Note that in the latter case, the right operators must appear reversed in the $\widehat{\chi}$-order.

Secondly, in order for a bi-non-crossing partition τ to contribute to the sum appearing in (2), we must have that for every $V \in \tau$, either $\{c_i : i \in V\} \subseteq \{X, X^*, Y, Y^*\}$ or $\{c_i : i \in V\} \subseteq \{Z, Z^*, W, W^*\}$; for if there exists $V \in \tau$ and $i \neq j \in V$ such that $c_i \in \{X, X^*, Y, Y^*\}$ and $c_j \in \{Z, Z^*, W, W^*\}$, then $\kappa_{\widehat{\chi}|_V}\big((c_1, \ldots, c_{2n})|_V\big) = 0$ due to the $*$-bi-free independence condition and thus $\kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{2n})$ vanishes. Note that this implies that if n is odd, then $\kappa_\chi(a_1 \ldots, a_n) = 0$, as then the cardinality of the set

$$\{j \in \{1, \ldots, 2n\} : c_j \in \{X, X^*, Y, Y^*\}\}$$

is odd and hence for any $\tau \in \mathrm{BNC}(\widehat{\chi})$ there exists $V \in \tau$ with odd cardinality that contains indices corresponding to elements in $\{X, X^*, Y, Y^*\}$. Since the pair (X, Y) is bi-R-diagonal, all bi-free cumulants of odd order with entries in $\{X, X^*, Y, Y^*\}$ vanish, thus $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V) = 0$.

We may now assume that n is even and that every block of a bi-non-crossing partition contains indices corresponding to elements either from $\{X, X^*, Y, Y^*\}$ or $\{Z, Z^*, W, W^*\}$. We must show that the cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes if the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is not alternating in $*$-terms and non-$*$-terms. When this occurs, by analysing individual cases, we will show that a given bi-non-crossing partition $\tau \in \mathrm{BNC}(\widehat{\chi})$ either yields zero contribution to the sum appearing in (1), or that the relation $\tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ cannot be satisfied.

Suppose the following situation occurs:

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, XZ, XZ, \ldots),$$

with $a_{s_\chi(m)} = a_{s_\chi(m+1)} = XZ$ for some $m \in \{1, \ldots, n-1\}$. This implies the following situation for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots, X, Z, X, Z, \ldots),$$

with $c_{s_{\widehat{\chi}}(2m-1)} = c_{s_{\widehat{\chi}}(2m+1)} = X$ and $c_{s_{\widehat{\chi}}(2m)} = c_{s_{\widehat{\chi}}(2m+2)} = Z$. For $\tau \in \mathrm{BNC}(\widehat{\chi})$, let $V \in \tau$ be such that $s_{\widehat{\chi}}(2m+1) \in V$. To start, consider the case when $s_{\widehat{\chi}}(2m+1) = \min_{\prec_{\widehat{\chi}}} V$ (equivalently, $2m+1 = \min_{<} s_{\widehat{\chi}}^{-1}(V)$). Let $q \in \{1, \ldots, 2n\}$ such that $s_{\widehat{\chi}}(q) = \max_{\prec_{\widehat{\chi}}} V$

(equivalently, $q = \max_{<} s_{\widehat{\chi}}^{-1}(V)$) and notice that we must have that $c_{s_{\widehat{\chi}}(q)} \in \{X^*, Y^*\}$. Indeed, if $c_{s_{\widehat{\chi}}(q)} \in \{X, Y\}$, then the sequence $(c_1, \ldots, c_{2n})|_V$ when read in the induced $\widehat{\chi}|_V$-order would have either one of the forms

$$(X, \ldots\ldots, X) \text{ or } (X, \ldots\ldots, Y)$$

and would thus not be alternating in $*$-terms and non-$*$-terms. Since the pair (X, Y) is bi-R-diagonal, this would imply that $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V) = 0$ and hence

$$\kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{2n}) = 0.$$

We assume that $c_{s_{\widehat{\chi}}(q)} = Y^*$ (with the case when $c_{s_{\widehat{\chi}}(q)} = X^*$ handled similarly). The following situation follows:

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots, X, Z, X, Z, \ldots, W^*, Y^*, \ldots)$$

and, as such, $q = 2p$ for some $p \in \{m+2, \ldots, n\}$. We will show that the relation $\tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ cannot be satisfied. Indeed, define $\widetilde{V} = \{s_{\widehat{\chi}}(i) : 2m+1 \leq i \leq 2p\}$ and let $\lambda = \{\widetilde{V}, \big(\widetilde{V}\big)^{\mathsf{c}}\}$. Since

$$s_{\widehat{\chi}}^{-1} \cdot \lambda = \{\{2m+1, \ldots, 2p\}, \{1, \ldots, 2m\} \cup \{2p+1, \ldots, 2n\}\} \in \mathrm{NC}(2n),$$

it follows that $\lambda \in \mathrm{BNC}(\widehat{\chi})$. It is easily seen that $\widehat{0_\chi} \leq \lambda$ and, moreover, $\tau \leq \lambda$ holds. To see this, first note that $V \subseteq \widetilde{V}$. For $V' \in \tau$ with $V \neq V'$, we must have that either $V' \subseteq \widetilde{V}$ or $V' \subseteq \big(\widetilde{V}\big)^{\mathsf{c}}$; for otherwise there would exist $i \neq j \in s_{\widehat{\chi}}^{-1}(V')$ such that

$$i \in \{1, \ldots, 2m\} \cup \{2p+1, \ldots, 2n\} \text{ and } j \in \{2m+2, \ldots, 2p-1\}.$$

But this cannot happen, since $\{2m+1, 2p\} \subseteq s_{\widehat{\chi}}^{-1}(V)$ and the partition $s_{\widehat{\chi}}^{-1} \cdot \tau$ is non-crossing. Hence, we have that $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_{\widehat{\chi}}$ and it follows that we cannot have that $s_{\widehat{\chi}}(2m+1) = \min_{\prec_{\widehat{\chi}}} V$.

So, suppose that there exists $q \in \{1, \ldots, 2n\}$ with $s_{\widehat{\chi}}(q) \in V$ and

$$s_{\widehat{\chi}}(q) \prec_{\widehat{\chi}} s_{\widehat{\chi}}(2m+1).$$

We may moreover assume that for all $v \in V \setminus \{s_{\widehat{\chi}}(2m+1), s_{\widehat{\chi}}(q)\}$, we either have that $v \prec_{\widehat{\chi}} s_{\widehat{\chi}}(q)$ or $s_{\widehat{\chi}}(2m+1) \prec_{\widehat{\chi}} v$ (i.e. that $s_{\widehat{\chi}}(q)$ is the $\widehat{\chi}$-maximum element of V with this property). Notice that it must necessarily be that $c_{s_{\widehat{\chi}}(q)} = X^*$. Indeed, if not, we would have that $c_{s_{\widehat{\chi}}(q)} = X$ and then the sequence $(c_1, \ldots, c_{2n})|_V$ when read in the induced $\widehat{\chi}|_V$-order would be of the form

$$(\ldots\ldots, X, X, \ldots\ldots),$$

with this implying that $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V) = 0$, since the pair (X, Y) is bi-R-diagonal.

Thus, $c_{s_{\widehat{\chi}}(q)} = X^*$ and this yields the following situation

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots, Z^*, X^*, \ldots, X, Z, X, Z, \ldots).$$

From this, one sees that $q = 2p$, for some $p \in \{1, \ldots, m-1\}$. We will show that once again the relation $\tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ cannot be satisfied. By defining $\widetilde{V} = \{s_{\widehat{\chi}}(i) : 2p+1 \leq i \leq 2m\}$ (and noting that this set is non-empty), let $\lambda = \{\widetilde{V}, (\widetilde{V})^{\mathsf{c}}\}$. Observe that $V \subseteq (\widetilde{V})^{\mathsf{c}}$ and, as before, it follows that $\lambda \in \mathrm{BNC}(\widehat{\chi})$ and $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_{\widehat{\chi}}$. Hence, when

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, XZ, XZ, \ldots),$$

we obtain that the bi-free cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes and the use of similar arguments shows that this is also the case when the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ has one of the following forms

$$(\ldots, XZ, WY, \ldots) \text{ or } (\ldots, WY, WY, \ldots).$$

Now, suppose that the following situation occurs:

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, Y^*W^*, Y^*W^*, \ldots),$$

with $a_{s_\chi(m)} = a_{s_\chi(m+1)} = Y^*W^*$ for some $m \in \{1, \ldots, n-1\}$. This implies the following situation for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots, W^*, Y^*, W^*, Y^*, \ldots),$$

with $c_{s_{\widehat{\chi}}(2m-1)} = c_{s_{\widehat{\chi}}(2m+1)} = W^*$ and $c_{s_{\widehat{\chi}}(2m)} = c_{s_{\widehat{\chi}}(2m+2)} = Y^*$. For $\tau \in \mathrm{BNC}(\widehat{\chi})$, let $V \in \tau$ be such that $s_{\widehat{\chi}}(2m) \in V$ and suppose that $s_{\widehat{\chi}}(2m) = \max_{\prec_{\widehat{\chi}}} V$ (equivalently, $2m = \max_{<} s_{\widehat{\chi}}^{-1}(V)$). Let $q \in \{1, \ldots, 2n\}$ such that $s_{\widehat{\chi}}(q) = \min_{\prec_{\widehat{\chi}}} V$ (equivalently, $q = \min_{<} s_{\widehat{\chi}}^{-1}(V)$) and notice that we must have that $c_{s_{\widehat{\chi}}(q)} \in \{X, Y\}$. Assume that $c_{s_{\widehat{\chi}}(q)} = Y$ (with the case when

$c_{s_{\widehat{\chi}}(q)} = X$ handled similarly). Hence, we have that

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots, Y, W, \ldots, W^*, Y^*, W^*, Y^*, \ldots)$$

and it follows that $q = 2p-1$ for some $p \in \{1, \ldots, m-1\}$. By defining

$$\widetilde{V} = \{s_{\widehat{\chi}}(i) : 2p-1 \leq i \leq 2m\}$$

and letting $\lambda = \{\widetilde{V}, \big(\widetilde{V}\big)^{\mathsf{c}}\}$, one sees that $V \subseteq \widetilde{V}$, $\lambda \in \mathrm{BNC}(\widehat{\chi})$ and $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_{\widehat{\chi}}$. Thus, the relation $\tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ cannot be satisfied.

This implies that there must exist $q \in \{1, \ldots, 2n\}$ with $s_{\widehat{\chi}}(q) \in V$ and

$$s_{\widehat{\chi}}(2m) \prec_{\widehat{\chi}} s_{\widehat{\chi}}(q)$$

and we may assume that $s_{\widehat{\chi}}(q)$ is the $\widehat{\chi}$-minimum element of V with this property. Notice that it must necessarily be that $c_{s_{\widehat{\chi}}(q)} = Y$ and this yields the following situation

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots, W^*, Y^*, W^*, Y^*, \ldots, Y, W, \ldots),$$

from which one sees that $q = 2p-1$, for some $p \in \{m+2, \ldots, n\}$. As before, by defining

$$\widetilde{V} = \{s_{\widehat{\chi}}(i) : 2m+1 \leq i \leq 2p-2\}$$

and letting $\lambda = \{\widetilde{V}, \big(\widetilde{V}\big)^{\mathsf{c}}\}$, one sees that $V \subseteq \big(\widetilde{V}\big)^{\mathsf{c}}$, $\lambda \in \mathrm{BNC}(\widehat{\chi})$ and $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_{\widehat{\chi}}$. Thus, the relation $\tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ cannot be satisfied. Hence, when

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, Y^*W^*, Y^*W^*, \ldots),$$

we obtain that the bi-free cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes and the use of similar arguments shows that this is also the case when the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ has one of the following forms

$$(\ldots, Z^*X^*, Z^*X^*, \ldots) \text{ or } (\ldots, Z^*X^*, Y^*W^*, \ldots).$$

This completes the proof. $\square$

The main technical difficulty that results in the length of the previous proof is that we cannot only deal with bi-non-crossing partitions whose blocks contain an even number of elements, thus Proposition 3.1.12 does not apply. This is because the pair (Z, W) need not be bi-R-diagonal and hence bi-free cumulants of odd order with entries in $\{Z, Z^*, W, W^*\}$ need not necessarily vanish.

We remark that for two $*$-bi-free pairs (X, Y) and (Z, W) with the first being bi-R-diagonal, it is not in general true that the pair (XZ, YW) will also be bi-R-diagonal, as the following example indicates. We will denote by "tr" the normalized trace on any matrix algebra.

Example 3.2.3. Let (A, φ) be a non-commutative $*$-probability space and (u_l, u_r) be a bi-Haar unitary pair in A. Also, consider the pair (Z, W) in $(\mathcal{M}_2(\mathbb{C}), \mathrm{tr})$ given as follows:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } W = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

In the free product space $(A * \mathcal{M}_2(\mathbb{C}), \varphi * \mathrm{tr})$ the pairs (u_l, u_r) and (Z, W) are $*$-bi-free, but for $\chi \in \{l, r\}^4$ with $\chi(1) = \chi(2) = l$ and $\chi(3) = \chi(4) = r$, the bi-free cumulant $\kappa_\chi(Z^* u_l{}^*, u_l Z, W^* u_r{}^*, u_r W)$ does not vanish, even though it is not alternating in $*$-terms and non-$*$-terms in the χ-order. Indeed, the moment-cumulant formula yields

$$\kappa_\chi(Z^* u_l{}^*, u_l Z, W^* u_r{}^*, u_r W) = \sum_{\tau \in \mathrm{BNC}(\chi)} (\varphi * \mathrm{tr})_\tau(Z^* u_l{}^*, u_l Z, W^* u_r{}^*, u_r W)\mu_{\mathrm{BNC}}(\tau, 1_\chi).$$

Using the characterization of free independence in terms of moments, it is seen that all operators that appear in the cumulant above are centred, i.e. the following holds

$$(\varphi * \mathrm{tr})(Z^* u_l^*) = (\varphi * \mathrm{tr})(W^* u_r^*) = (\varphi * \mathrm{tr})(u_l Z) = (\varphi * \mathrm{tr})(u_r W) = 0.$$

Hence, to find bi-non-crossing partitions that are to yield a non-zero contribution to the sum above, we may only consider partitions on $\{1, 2, 3, 4\}$ that are bi-non-crossing and all of whose blocks are not singletons. These are the following three bi-non-crossing partitions:

$$\tau_1 = \{\{1, 2\}, \{3, 4\}\}, \ \tau_2 = \{\{1, 2, 3, 4\}\} \text{ and } \tau_3 = \{\{1, 3\}, \{2, 4\}\}.$$

For τ_1, we have that

$$\begin{aligned}(\varphi * \mathrm{tr})_{\tau_1}(Z^* u_l^*, u_l Z, W^* u_r^*, u_r W) &= (\varphi * \mathrm{tr})(Z^* u_l^* u_l Z) \cdot (\varphi * \mathrm{tr})(W^* u_r^* u_r W) \\ &= \mathrm{tr}(Z^* Z) \cdot \mathrm{tr}(W^* W) = \frac{1}{4},\end{aligned}$$

while for τ_2 we obtain

$$\begin{aligned}(\varphi * \mathrm{tr})_{\tau_2}(Z^* u_l^*, u_l Z, W^* u_r^*, u_r W) &= (\varphi * \mathrm{tr})(Z^* u_l^* u_l Z W^* u_r^* u_r W) \\ &= \mathrm{tr}(Z^* Z W^* W) = \frac{1}{2}.\end{aligned}$$

For the case of τ_3, it follows that

$$(\varphi * \mathrm{tr})_{\tau_3}(Z^* u_l^*, u_l Z, W^* u_r^*, u_r W) = (\varphi * \mathrm{tr})(Z^* u_l^* W^* u_r^*) \cdot (\varphi * \mathrm{tr})(u_l Z u_r W),$$

and it is straightforward to show using the moment-cumulant formula that both terms appearing in the product above are equal to zero. Since

$$\mu_{\mathrm{BNC}}(\tau_1, 1_\chi) = \mu_{\mathrm{BNC}}(\tau_3, 1_\chi) = -1 \text{ and } \mu_{\mathrm{BNC}}(\tau_2, 1_\chi) = 1,$$

the bi-free cumulant is evaluated as follows

$$\kappa_\chi(Z^* {u_l}^*, u_l Z, W^* {u_r}^*, u_r W) = \frac{1}{2} - \frac{1}{4} - 0 = \frac{1}{4} \neq 0. \qquad \square$$

However, when the pairs (X, Y) and (Z, W) are *both* bi-R-diagonal and $*$-bi-free, then it is the case that the resulting pair (XZ, YW) is also bi-R-diagonal, as the following proposition shows.

Proposition 3.2.4. *Let (A, φ) be a non-commutative $*$-probability space and $X, Y, Z, W \in A$ such that:*

(a) the pairs (X, Y) and (Z, W) are both bi-R-diagonal,

(b) the pairs (X, Y) and (Z, W) are $*$-bi-free.

Then, the pair (XZ, YW) is also bi-R-diagonal.

Proof. Let $n \in \mathbb{N}$, $\chi \in \{l,r\}^n$ and $a_1, \ldots, a_n \in A$ such that

$$a_i \in \begin{cases} \{XZ, Z^*X^*\}, & \text{if } \chi(i) = l \\ \{YW, W^*Y^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

Define $\widehat{\chi} \in \{l,r\}^{2n}$ by $\widehat{\chi}(2i-1) = \widehat{\chi}(2i) = \chi(i)$ for each $i = 1, \ldots, n$ and $c_1, \ldots, c_{2n} \in A$ as follows:

$$c_{2i-1} = \begin{cases} X, & \text{if } a_i = XZ \\ Z^*, & \text{if } a_i = Z^*X^* \\ Y, & \text{if } a_i = YW \\ W^*, & \text{if } a_i = W^*Y^* \end{cases} \quad \text{and } c_{2i} = \begin{cases} Z, & \text{if } a_i = XZ \\ X^*, & \text{if } a_i = Z^*X^* \\ W, & \text{if } a_i = YW \\ Y^*, & \text{if } a_i = W^*Y^* \end{cases}$$

for each $i = 1, \ldots, n$. Then, an application of Theorem 3.1.11 yields:

$$\begin{aligned} \kappa_\chi(a_1, \ldots, a_n) &= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \kappa_{\widehat{\chi}}(c_1, \ldots, c_{2n}) & (1) \\ &= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \prod_{V \in \tau} \kappa_{\widehat{\chi}|_V}\big((c_1, \ldots, c_{2n})|_V\big) & (2) \end{aligned}$$

where $\widehat{0_\chi} = \{\{2i-1, 2i\} : i = 1, \ldots, n\} \in \mathrm{BNC}(\widehat{\chi})$. As in the proof of Theorem 3.2.2, we make the following remarks:

First of all, if $\chi^{-1}(\{l\}) = \{i_1 < \ldots < i_p\}$ and $\chi^{-1}(\{r\}) = \{j_1 < \ldots < j_{n-p}\}$, the definition of $\widehat{\chi}$ implies that $\widehat{\chi}^{-1}(\{l\}) = \{2i_1 - 1 < 2i_1 < \ldots < 2i_p - 1 < 2i_p\}$ and $\widehat{\chi}^{-1}(\{r\}) = \{2j_1 - 1 < 2j_1 < \ldots < 2j_{n-p} - 1 < 2j_{n-p}\}$. Thus, if $i \in \{1, \ldots, n\}$ is such that $a_{s_\chi(i)} = XZ$, then $c_{s_{\widehat{\chi}}(2i-1)} = X$ and $c_{s_{\widehat{\chi}}(2i)} = Z$ (a similar situation occurs when $a_{s_\chi(i)} = Z^*X^*$, since this corresponds to a left operator). Now if $a_{s_\chi(i)} = YW$, then $c_{s_{\widehat{\chi}}(2i-1)} = W$ and $c_{s_{\widehat{\chi}}(2i)} = Y$ (and a similar situation occurs when $a_{s_\chi(i)} = W^*Y^*$ since this corresponds to a right operator). Note that in the latter case, the right operators must appear reversed in the $\widehat{\chi}$-order.

Secondly, due to the $*$-bi-free independence condition, in order for a bi-non-crossing partition τ to contribute to the above sum, we must have that for every $V \in \tau$, either

$$\{c_i : i \in V\} \subseteq \{X, X^*, Y, Y^*\},$$

or

$$\{c_i : i \in V\} \subseteq \{Z, Z^*, W, W^*\}.$$

Observe that this implies that if n is odd, then $\kappa_\chi(a_1 \ldots, a_n) = 0$, as then the cardinality of the set

$$\{j \in \{1, \ldots, 2n\} : c_j \in \{X, X^*, Y, Y^*\}\}$$

is odd and hence for any $\tau \in \mathrm{BNC}(\widehat{\chi})$ there exists $V \in \tau$ with odd cardinality that contains indices corresponding to elements in $\{X, X^*, Y, Y^*\}$. Since the pair (X, Y) is bi-R-diagonal, all bi-free cumulants of odd order with entries in $\{X, X^*, Y, Y^*\}$ vanish, thus $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V) = 0$.

In addition, in order for a bi-non-crossing partition $\tau \in \mathrm{BNC}(\chi)$ to contribute to the sum appearing in (1), every block of τ must contain an even number of elements. Indeed, if $V \in \tau$ contains an odd number of elements, then we deduce that (additionally assuming that all indices in V correspond to elements either from $\{X, X^*, Y, Y^*\}$ or $\{Z, Z^*, W, W^*\}$) $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V)$ is a bi-free cumulant of odd order involving a bi-R-diagonal pair and thus vanishes.

Henceforth, when referring to a bi-non-crossing partition τ contributing to the sum appearing in (1), we will always assume that every block of τ contains indices all corresponding to elements either from $\{X, X^*, Y, Y^*\}$ or $\{Z, Z^*, W, W^*\}$ and, by Proposition 3.1.12, that $s_{\widehat{\chi}}(1) \sim_\tau s_{\widehat{\chi}}(2n)$ and $s_{\widehat{\chi}}(2i) \sim_\tau s_{\widehat{\chi}}(2i+1)$ for every $i = 1, \ldots, n-1$.

We will now show that if the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is not alternating in $*$-terms and non-$*$-terms, then the cumulant $\kappa_\chi(a_1, \ldots, a_n)$ must vanish. Suppose the following situation occurs:

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, YW, YW, \ldots),$$

with $a_{s_\chi(m)} = a_{s_\chi(m+1)} = YW$ for some $m \in \{1, \ldots, n-1\}$. This implies the following situation for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots, W, Y, W, Y, \ldots),$$

with $c_{s_{\widehat{\chi}}(2m-1)} = c_{s_{\widehat{\chi}}(2m+1)} = W$ and $c_{s_{\widehat{\chi}}(2m)} = c_{s_{\widehat{\chi}}(2m+2)} = Y$.

If τ is a bi-non-crossing partition contributing to the sum appearing in (1), then the block of τ containing $s_{\widehat{\chi}}(2m)$ must also contain $s_{\widehat{\chi}}(2m+1)$. But, since

$$c_{s_{\widehat{\chi}}(2m)} = Y \text{ and } c_{s_{\widehat{\chi}}(2m+1)} = W,$$

this is impossible, due to the $*$-bi-free independence condition. Hence, when

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, YW, YW, \ldots),$$

we obtain that the bi-free cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes and the use of similar arguments shows that this is also the case when the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ has either one of the following forms:

$$(\ldots, XZ, XZ, \ldots), (\ldots, Z^*X^*, Z^*X^*, \ldots) \text{ or } (\ldots, W^*Y^*, W^*Y^*, \ldots).$$

Next, suppose the following situation occurs:

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots\ldots, XZ, YW, \ldots\ldots),$$

with $a_{s_\chi(m)} = XZ$ and $a_{s_\chi(m+1)} = YW$ for some $m \in \{1, \ldots, n-1\}$. This implies the following situation for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)} \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots\ldots, X, Z, W, Y, \ldots\ldots),$$

with $c_{s_{\widehat{\chi}}(2m-1)} = X$, $c_{s_{\widehat{\chi}}(2m)} = Z$, $c_{s_{\widehat{\chi}}(2m+1)} = W$ and $c_{s_{\widehat{\chi}}(2m+2)} = Y$.

If τ is a bi-non-crossing partition contributing to the sum appearing in (1), then the block $V \in \tau$ containing $s_{\widehat{\chi}}(2m)$ must also contain $s_{\widehat{\chi}}(2m+1)$. As discussed in the beginning of the proof, in order for the cumulant $\kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{2n})$ not to vanish we must have that $V \subseteq \{j \in \{1, \ldots, 2n\} : s_j \in \{Z, Z^*, W, W^*\}\}$. But then the entries of the cumulant $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V)$ in the induced $\widehat{\chi}|_V$-order would be of the form:

$$(\ldots\ldots, Z, W, \ldots\ldots)$$

and this implies that the bi-free cumulant $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V)$ vanishes, as it is a cumulant involving the bi-R-diagonal pair (Z, W) that is not alternating in $*$-terms and non-$*$-terms in the induced $\widehat{\chi}|_V$-order. Since this is the case for every possible $\tau \in \mathrm{BNC}(\widehat{\chi})$, we deduce that

$\kappa_\chi(a_1, \ldots, a_n) = 0$. Hence, when

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, XZ, YW, \ldots),$$

we obtain that the bi-free cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes and the use of similar arguments shows that this is also the case when the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ has the form:

$$(\ldots, Z^*X^*, W^*Y^*, \ldots).$$

This completes the proof. $\square$

We now proceed to prove that the condition of bi-R-diagonality is preserved under the taking of arbitrary powers.

Theorem 3.2.5. *Let (A, φ) be a non-commutative $*$-probability space and let (X, Y) be a bi-R-diagonal pair in A. Then, for every $p \geq 1$ the pair (X^p, Y^p) is also bi-R-diagonal.*

Proof. Let $n \in \mathbb{N}, p \geq 1$, $\chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in \{X^p, (X^*)^p, Y^p, (Y^*)^p\}$ such that

$$a_i \in \begin{cases} \{X^p, (X^*)^p\}, & \text{if } \chi(i) = l \\ \{Y^p, (Y^*)^p\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

Define $\widehat{\chi} \in \{l, r\}^{np}$ and $c_1, \ldots, c_{np} \in A$ as follows:

$$\widehat{\chi}((i-1)p + j) = \chi(i)$$

and

$$c_{(i-1)p+j} = \begin{cases} X, & \text{if } a_i = X^p \\ X^*, & \text{if } a_i = (X^*)^p \\ Y, & \text{if } a_i = Y^p \\ Y^*, & \text{if } a_i = (Y^*)^p \end{cases}$$

for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p\}$. Then, an application of Theorem 3.1.11 yields:

$$\kappa_\chi(a_1, \ldots, a_n) = \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{np}) \tag{1}$$

$$= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \prod_{V \in \tau} \kappa_{\widehat{\chi}|_V}\left((c_1, \ldots, c_{np})|_V\right) \tag{2}$$

where $\widehat{0_\chi} = \{\{(i-1)p+1, \ldots, ip\} : i = 1, \ldots, n\} \in \mathrm{BNC}(\widehat{\chi})$.

To start, we remark that since the pair (X, Y) is bi-R-diagonal, in order for a bi-non-crossing partition $\tau \in \mathrm{BNC}(\widehat{\chi})$ to have non-zero contribution to the sum appearing in (1), every block of τ must contain indices corresponding to an equal number of $*$-terms and non-$*$-terms; for otherwise there would exist a block $V \in \tau$ with indices corresponding to an unequal number of $*$-terms and non-$*$-terms. This implies that the sequence $(c_1, \ldots, c_{np})|_V$ when read in the induced $\widehat{\chi}|_V$-order will not be alternating in $*$-terms and non-$*$-terms and hence $\kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{np}) = 0$.

We will first show that if the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is not alternating in $*$-terms and non-$*$-terms, then $\kappa_\chi(a_1, \ldots, a_n) = 0$. Suppose the following situation occurs:

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, X^p, Y^p, \ldots),$$

where $a_{s_\chi(m)} = X^p$ and $a_{s_\chi(m+1)} = Y^p$ for some $m \in \{1, \ldots, n\}$. This implies the following situation for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(np)}) = (\ldots, X, X \ldots, X, Y, Y, \ldots, Y, \ldots),$$

where $c_{s_{\widehat{\chi}}((m-1)p+k)} = X$ and $c_{s_{\widehat{\chi}}(mp+k)} = Y$, for all $k = 1, \ldots, p$. Let $\tau \in \mathrm{BNC}(\widehat{\chi})$ and $V \in \tau$ such that $s_{\widehat{\chi}}(mp+1) \in V$. Observe that $s_{\widehat{\chi}}(mp+k) \notin V$ for all $k = 2, \ldots, p$; for otherwise, the sequence $(c_1, \ldots, c_{np})|_V$ when read in the induced $\widehat{\chi}|_V$-order would be of the form

$$(\ldots\ldots, Y, Y, \ldots\ldots)$$

and this would imply that $\kappa_{\widehat{\chi}|_V}(c_1, \ldots, c_{np})|_V = 0$, since the pair (X, Y) is bi-R-diagonal.

Consider the case when $s_{\widehat{\chi}}(mp+1) = \min_{\prec_{\widehat{\chi}}} V$ (equivalently, $mp + 1 = \min_{<} s_{\widehat{\chi}}^{-1}(V)$) and let $q \in \{1, \ldots, np\}$ be such that $s_{\widehat{\chi}}(q) = \max_{\prec_{\widehat{\chi}}} V$ (equivalently, $q = \max_{<} s_{\widehat{\chi}}^{-1}(V)$). It is easy to see that $c_{s_{\widehat{\chi}}(q)} = Y^*$. We claim that we must necessarily have that $q = tp$, for some $t \in \{m+2, \ldots, n\}$.

To see this, suppose that $q = tp + k$ with $t \in \{m+2, \ldots, n-1\}$ and $k \in \{1, \ldots, p-1\}$. Define

$$A = \{s_{\widehat{\chi}}(i) : mp + 1 \leq i \leq tp + k\}$$

and notice that A has to be written as a union of blocks of τ, which means that if $V' \in \tau, V \neq V'$ is such that $V' \cap A \neq \emptyset$, then $V' \subseteq A$. Indeed, for such a block V', if there existed $i \neq j \in \{1, \ldots, np\}$ with $s_{\widehat{\chi}}(i) \in V' \cap A$ and $s_{\widehat{\chi}}(j) \in V' \setminus A$, then this would imply that

$$mp + 2 \leq i \leq tp + k - 1, \ \ j \in \{1, \ldots, mp+1\} \cup \{tp + k, \ldots, np\}$$

and

$$\{mp + 1, tp + k\} \subseteq s_{\widehat{\chi}}^{-1}(V),$$

which contradicts the fact that the partition $s_{\widehat{\chi}}^{-1} \cdot \tau$ is non-crossing. Thus, A has to be written as a union of blocks of τ and since A contains indices corresponding to an unequal number of $*$-terms and non-$*$-terms, there must exist a block $V' \in \tau$ with this same property. This yields that $\kappa_{\widehat{\chi}|_{V'}}(c_1, \ldots, c_{np})|_{V'} = 0$ and, as a result, the cumulant $\kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{np})$ vanishes.

We may now assume that $q = tp$, for some $t \in \{m+2, \ldots, n\}$. By defining

$$\widetilde{V} = \{s_{\widehat{\chi}}(i) : mp + 1 \leq i \leq tp\}$$

and letting $\lambda = \{\widetilde{V}, \big(\widetilde{V}\big)^{\mathsf{c}}\}$, one sees that $V \subseteq \widetilde{V}$, $\lambda \in \mathrm{BNC}(\widehat{\chi})$ and $\tau, \widehat{0_{\chi}} \leq \lambda \lneq 1_{\widehat{\chi}}$. Thus, the relation $\tau \vee \widehat{0_{\chi}} = 1_{\widehat{\chi}}$ cannot be satisfied. Hence, it cannot be the case that $s_{\widehat{\chi}}(mp+1) = \min_{\prec_{\widehat{\chi}}} V$.

So, suppose that there exists $q \in \{1, \ldots, np\}$ with $s_{\widehat{\chi}}(q) \in V$ and

$$s_{\widehat{\chi}}(q) \prec_{\widehat{\chi}} s_{\widehat{\chi}}(mp+1).$$

We may moreover assume that for all $v \in V \setminus \{s_{\widehat{\chi}}(mp+1), s_{\widehat{\chi}}(q)\}$, we either have that $v \prec_{\widehat{\chi}} s_{\widehat{\chi}}(q)$ or $s_{\widehat{\chi}}(2m+1) \prec_{\widehat{\chi}} v$ (i.e. that $s_{\widehat{\chi}}(q)$ is the $\widehat{\chi}$-maximum element of V with this property). Notice that it must necessarily be that $c_{s_{\widehat{\chi}}(q)} = X^*$ and, arguing as before, it must be the case that $q = tp$ for some $t \in \{1, \ldots, m-1\}$. Then, by defining

$$\widetilde{V} = \{s_{\widehat{\chi}}(i) : tp + 1 \leq i \leq mp\}$$

and letting $\lambda = \{\widetilde{V}, \big(\widetilde{V}\big)^{\mathsf{c}}\}$, one sees that $V \subseteq \big(\widetilde{V}\big)^{\mathsf{c}}$, $\lambda \in \mathrm{BNC}(\widehat{\chi})$ and $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_{\widehat{\chi}}$. Thus, the relation $\tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ once again cannot be satisfied.

This shows that when

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, X^p, Y^p, \ldots),$$

we obtain that the bi-free cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes and the use of similar arguments shows that this is also the case when the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ has one of the following forms:

(a) $(\ldots, X^p, X^p, \ldots)$,

(b) $(\ldots, Y^p, Y^p, \ldots)$,

(c) $(\ldots, (X^*)^p, (X^*)^p, \ldots)$,

(d) $(\ldots, (X^*)^p, (Y^*)^p, \ldots)$,

(e) $(\ldots, (Y^*)^p, (Y^*)^p, \ldots)$.

Hence, if the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is not alternating in $*$-terms and non-$*$-terms, we have that $\kappa_\chi(a_1, \ldots, a_n) = 0$. It remains to show that if the cumulant $\kappa_\chi(a_1, \ldots, a_n)$ is of odd order, then it must vanish.

Assume that n is an odd number. By the aforementioned considerations, we may assume that the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ does not contain consecutive elements that both correspond to either $*$-terms or non-$*$-terms. Suppose the following situation occurs:

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = ((X^*)^p, \ldots\ldots, (Y^*)^p),$$

where $a_{s_\chi(1)} = (X^*)^p$ and $a_{s_\chi(n)} = (Y^*)^p$. This implies the following situation for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(np)}) = (X^*, X^*, \ldots, X^*, \ldots\ldots, Y^*, Y^*, \ldots, Y^*),$$

where $c_{s_{\widehat{\chi}}(k)} = X^*$ and $c_{s_{\widehat{\chi}}((n-1)p+k)} = Y^*$, for all $k = 1, \ldots, p$. Let $\tau \in \mathrm{BNC}(\widehat{\chi})$ and $V \in \tau$ such that $s_{\widehat{\chi}}(1) \in V$. Also, let $q \in \{1, \ldots, np\}$ be such that $s_{\widehat{\chi}}(q) = \max_{\prec_{\widehat{\chi}}} V$. First of all, observe that it must be that $q \geq p+1$; for otherwise, since $c_{s_{\widehat{\chi}}(k)} = X^*$ for all $k = 1, \ldots, p$, the sequence $(c_1, \ldots, c_{np})|_V$ when read in the induced $\widehat{\chi}|_V$-order would be of the form

$$(X^*, \ldots\ldots, X^*),$$

and hence has either odd length or is not alternating in $*$-terms and non-$*$-terms. This implies that $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{np})|_V) = 0$, since the pair (X, Y) is bi-R-diagonal.

Secondly, note that it must necessarily be that $q = tp$ for some $t \in \{2, \ldots, n\}$. Indeed, if $q = tp + k$ for some $t \in \{2, \ldots, n-1\}$ and $k \in \{1, \ldots, p-1\}$, then the set

$$\{s_{\widehat{\chi}}(i) : 1 \leq i \leq tp + k\}$$

(which contains indices corresponding to an unequal number of $*$-terms and non-$*$-terms) must be written as a union of blocks of τ. Thus, there exists a block V' of τ containing indices that correspond to an unequal number of $*$-terms and non-$*$-terms and it follows that if that is the case, then $\kappa_{\widehat{\chi}|_{V'}}((c_1, \ldots, c_{np})|_{V'}) = 0$.

We will now show that $q = np$. If we assumed that $q = tp$, for some $t \in \{2, \ldots, n-1\}$, then by defining

$$\widetilde{V} = \{s_{\widehat{\chi}}(i) : 1 \leq i \leq tp\}$$

and letting $\lambda = \{\widetilde{V}, \left(\widetilde{V}\right)^{\mathsf{c}}\}$, one sees that $V \subseteq \widetilde{V}$, $\lambda \in \mathrm{BNC}(\widehat{\chi})$ and $\tau, \widehat{0_\chi} \leq \lambda \lneq 1_{\widehat{\chi}}$. Thus, the relation $\tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ cannot be satisfied.

This shows that when

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = ((X^*)^p, \ldots\ldots, (Y^*)^p),$$

we obtain that the bi-free cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes and the use of similar arguments shows that this is also the case when the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ has one of the following forms:

(a) $(X^p, \ldots\ldots, X^p)$,

(b) $((X^*)^p, \ldots\ldots, (X^*)^p)$,

(c) $((Y^*)^p, \ldots\ldots, (Y^*)^p)$,

(d) $(X^p, \ldots\ldots, Y^p)$,

(e) $(Y^p, \ldots\ldots, Y^p)$.

This completes the proof. $\square$

We now proceed to show that bi-R-diagonal pairs of operators yield examples of bi-free pairs that consist of self-adjoint operators.

Proposition 3.2.6. *Let (A, φ) be a non-commutative $*$-probability space and (X, Y) be a bi-R-diagonal pair in A. Then, the pairs*

$$(XX^*, Y^*Y) \text{ and } (X^*X, YY^*)$$

are bi-free.

Proof. Let $n \in \mathbb{N}, n \geq 2$, $\chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ such that

$$a_i \in \begin{cases} \{XX^*, X^*X\}, & \text{if } \chi(i) = l \\ \{Y^*Y, YY^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

Moreover, suppose that there exist $i \neq j \in \{1, \ldots, n\}$ such that $a_i \in \{XX^*, Y^*Y\}$ and $a_j \in \{X^*X, YY^*\}$. We will show that $\kappa_\chi(a_1, \ldots, a_n) = 0$, which will imply that the pairs (XX^*, Y^*Y) and (X^*X, YY^*) are indeed bi-free. Define $\widehat{\chi} \in \{l, r\}^{2n}$ by

$$\widehat{\chi}(2i - 1) = \widehat{\chi}(2i) = \chi(i),$$

for each $i = 1, \ldots, n$ and $c_1, \ldots, c_{2n} \in A$ as follows:

$$c_{2i-1} = \begin{cases} X, & \text{if } a_i = XX^* \\ Y^*, & \text{if } a_i = Y^*Y \\ X^*, & \text{if } a_i = X^*X \\ Y, & \text{if } a_i = YY^* \end{cases} \text{ and } c_{2i} = \begin{cases} X^*, & \text{if } a_i = XX^* \\ Y, & \text{if } a_i = Y^*Y \\ X, & \text{if } a_i = X^*X \\ Y^*, & \text{if } a_i = YY^* \end{cases}$$

for each $i = 1, \ldots, n$. Then, an application of Theorem 3.1.11 yields:

$$\kappa_\chi(a_1,\ldots,a_n) = \sum_{\substack{\tau\in\mathrm{BNC}(\widehat{\chi})\\ \tau\vee\widehat{0_\chi}=1_{\widehat{\chi}}}} \kappa_{\widehat{\chi},\tau}(c_1,\ldots,c_{2n}) \tag{1}$$

$$= \sum_{\substack{\tau\in\mathrm{BNC}(\widehat{\chi})\\ \tau\vee\widehat{0_\chi}=1_{\widehat{\chi}}}} \prod_{V\in\tau} \kappa_{\widehat{\chi}|_V}\big((c_1,\ldots,c_{2n})|_V\big) \tag{2}$$

where $\widehat{0_\chi} = \{\{2i-1,2i\} : i=1,\ldots,n\} \in \mathrm{BNC}(\widehat{\chi})$.

Since the pair (X,Y) is bi-R-diagonal, for a bi-non-crossing partition $\tau\in\mathrm{BNC}(\widehat{\chi})$ to have a non-zero contribution to the sum appearing in (1), every block of τ must contain an even number of elements, as every bi-free cumulant of odd order with entries in $\{X,X^*,Y,Y^*\}$ vanishes. Our initial assumptions imply that there exists $i\in\{1,\ldots,n\}$ such that either

$$a_{s_\chi(i)} \in \{XX^*, Y^*Y\} \text{ and } a_{s_\chi(i+1)} \in \{X^*X, YY^*\},$$

or

$$a_{s_\chi(i)} \in \{X^*X, YY^*\} \text{ and } a_{s_\chi(i+1)} \in \{XX^*, Y^*Y\}.$$

Assume that $a_{s_\chi(i)} = XX^*$ and $a_{s_\chi(i+1)} = YY^*$ (with the remaining cases handled similarly). Then, the following situation occurs for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)},\ldots,c_{s_{\widehat{\chi}}(2n)}) = (\ldots,X,X^*,Y^*,Y,\ldots),$$

where $c_{s_{\widehat{\chi}}(2i-1)} = X, c_{s_{\widehat{\chi}}(2i)} = X^*, c_{s_{\widehat{\chi}}(2i+1)} = Y^*$ and $c_{s_{\widehat{\chi}}(2i+2)} = Y$. Note that, due to the definition of the permutation $s_{\widehat{\chi}}$, the right operators must appear reversed in the $\widehat{\chi}$-order.

By Proposition 3.1.12, for $\tau\in\mathrm{BNC}(\widehat{\chi})$ such that $\tau\vee\widehat{0_\chi} = 1_{\widehat{\chi}}$ there exists $V\in\tau$ with $\{s_{\widehat{\chi}}(2i), s_{\widehat{\chi}}(2i+1)\}\subseteq V$. But then, the sequence $(c_1,\ldots,c_{2n})|_V$ when read in the induced $\widehat{\chi}|_V$-order would be of the form

$$(\ldots\ldots,X^*,Y^*,\ldots\ldots)$$

with this implying that $\kappa_{\widehat{\chi}|_V}((c_1,\ldots,c_{2n})|_V)=0$, since, as the pair (X,Y) is bi-R-diagonal, bi-free cumulants with entries in $\{X,X^*,Y,Y^*\}$ that are non-$*$-alternating in each of the corresponding χ-orders must vanish. Hence, $\kappa_{\widehat{\chi},\tau}(c_1,\ldots,c_{2n}) = 0$ and this finishes the proof. $\square$

We remark that if (X, Y) is a bi-R-diagonal pair in some non-commutative $*$-probability space, then it is not necessarily true that the pairs (XX^*, YY^*) and (X^*X, Y^*Y) are bi-free, as the following example indicates.

Example 3.2.7. Let (u_l, u_r) be a bi-Haar unitary pair in a non-commutative $*$-probability space (A, φ) and consider the pair (Z, W) in the space $(\mathcal{M}_2(\mathbb{C}), \mathrm{tr})$ defined as follows:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } W = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

In the free product space $(A * \mathcal{M}_2(\mathbb{C}), \varphi * \mathrm{tr})$, the pairs (u_l, u_r) and (Z, W) are $*$-bi-free and hence, by Theorem 3.2.2, the pair $(u_l Z, W u_r)$ is bi-R-diagonal. But, the pairs

$$(Z^* u_l^* u_l Z, u_r^* W^* W u_r) = (Z^* Z, u_r^* W^* W u_r),$$

and

$$(u_l Z Z^* u_l^*, W u_r u_r^* W^*) = (u_l Z Z^* u_l^*, W W^*)$$

are not bi-free, since the moment-cumulant formula yields

$$\kappa_\chi(Z^* Z, WW^*) = \mathrm{tr}(Z^* Z W W^*) - \mathrm{tr}(Z^* Z) \cdot \mathrm{tr}(WW^*) = -\frac{1}{4} \neq 0. \qquad \square$$

3.3 Joint $*$-Distributions of Bi-R-Diagonal Pairs

In this section, we will be concerned with proving that the joint $*$-distribution of a bi-R-diagonal pair of operators remains invariant under the multiplication with a $*$-bi-free bi-Haar unitary pair.

We begin by giving the definition of bi-even and $*$-bi-even pairs of operators, as well as display how this class of pairs of operators can yield examples of bi-R-diagonal pairs.

Definition 3.3.1. Let (A, φ) be a non-commutative $*$-probability space and $Z, W \in A$.

(i) The pair (Z, W) is called *bi-even* if for every $k \in \mathbb{N}$ and $a_1, \ldots, a_{2k+1} \in \{Z, W\}$ we have that

$$\varphi(a_1 \cdot \ldots \cdot a_{2k+1}) = 0,$$

that is, all of its joint moments of odd order vanish.

(ii) The pair (Z, W) is called $*$-*bi-even* if for every $k \in \mathbb{N}$ and $a_1, \ldots, a_{2k+1} \in \{Z, Z^*, W, W^*\}$ we have that

$$\varphi(a_1 \cdot \ldots \cdot a_{2k+1}) = 0,$$

that is, all of its joint $*$-moments of odd order vanish.

The moment-cumulant formula yields that the pair (X, Y) is $*$-bi-even if and only if all bi-free cumulants of odd order with entries in $\{X, X^*, Y, Y^*\}$ vanish. It clearly follows that every bi-R-diagonal pair is $*$-bi-even.

In the setting of free probability, it is observed that products of free, self-adjoint, even elements (i.e. self-adjoint elements of non-commutative $*$-probability spaces all whose moments of odd order vanish) result in R-diagonal elements ([61, Theorem 15.17]). Generalizing this to the bi-free setting, we will show that products of $*$-bi-even pairs (where the order of the right operators is reversed in the product) yield bi-R-diagonal pairs. For this, we have the following proposition, the proof of which will be similar to the proofs of Theorem 3.2.2 and Proposition 3.2.4.

Proposition 3.3.2. *Let (A, φ) be a non-commutative probability space and $X, Y, Z, W \in A$ such that:*

(a) the pairs (X, Y) and (Z, W) are both $*$-bi-even,

(b) the pairs (X, Y) and (Z, W) are $*$-bi-free.

Then, the pair (XZ, WY) is bi-R-diagonal.

Proof. Let $n \in \mathbb{N}$, $\chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ be such that

$$a_i \in \begin{cases} \{XZ, Z^*X^*\}, & \text{if } \chi(i) = l \\ \{WY, Y^*W^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

Define $\widehat{\chi} \in \{l, r\}^{2n}$ by $\widehat{\chi}(2i-1) = \widehat{\chi}(2i) = \chi(i)$ for each $i = 1, \ldots, n$ and $c_1, \ldots, c_{2n} \in A$ as follows:

$$c_{2i-1} = \begin{cases} X, & \text{if } a_i = XZ \\ Z^*, & \text{if } a_i = Z^*X^* \\ W, & \text{if } a_i = WY \\ Y^*, & \text{if } a_i = Y^*W^* \end{cases} \quad \text{and } c_{2i} = \begin{cases} Z, & \text{if } a_i = XZ \\ X^*, & \text{if } a_i = Z^*X^* \\ Y, & \text{if } a_i = WY \\ W^*, & \text{if } a_i = Y^*W^* \end{cases}$$

for each $i = 1, \ldots, n$. Then, an application of Theorem 3.1.11 yields:

$$\begin{aligned} \kappa_\chi(a_1, \ldots, a_n) &= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{2n}) && (1) \\ &= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \prod_{V \in \tau} \kappa_{\widehat{\chi}|_V}\big((c_1, \ldots, c_{2n})|_V\big) && (2) \end{aligned}$$

where $\widehat{0_\chi} = \{\{2i-1, 2i\} : i = 1, \ldots, n\} \in \mathrm{BNC}(\widehat{\chi})$. As in the proof of Theorem 3.2.2, we make the following remarks:

First of all, if for some $i \in \{1, \ldots, n\}$ we have that $a_{s_\chi(i)} = XZ$, then it follows that $c_{s_{\widehat{\chi}}(2i-1)} = X$ and $c_{s_{\widehat{\chi}}(2i)} = Z$ (with a similar situation occurring when $a_{s_\chi(i)} = Z^*X^*$, since this corresponds to a left operator). Now, if $a_{s_\chi(i)} = WY$, then $c_{s_{\widehat{\chi}}(2i-1)} = Y$ and $c_{s_{\widehat{\chi}}(2i)} = W$ (and a similar situation occurs when $a_{s_\chi(i)} = Y^*W^*$, since this corresponds to a right operator). Note that in the latter case, the right operators must appear reversed in the $\widehat{\chi}$-order.

Since the pairs (X, Y) and (Z, W) are $*$-bi-free, in order for a bi-non-crossing partition $\tau \in \mathrm{BNC}(\widehat{\chi})$ to contribute to the sum appearing in (1), we must have that for all $V \in \tau$, either

$$\{c_i : i \in V\} \subseteq \{X, X^*, Y, Y^*\},$$

or

$$\{c_i : i \in V\} \subseteq \{Z, Z^*, W, W^*\}.$$

Observe that this also implies that if n is odd, then $\kappa_\chi(a_1 \ldots, a_n) = 0$, as then the cardinality of the set

$$\{j \in \{1, \ldots, 2n\} : c_j \in \{X, X^*, Y, Y^*\}\}$$

is odd and hence for any $\tau \in \mathrm{BNC}(\widehat{\chi})$ there exists $V \in \tau$ with odd cardinality that contains indices corresponding to elements in $\{X, X^*, Y, Y^*\}$. Since the pair (X, Y) is $*$-bi-even, all bi-free cumulants of odd order with entries in $\{X, X^*, Y, Y^*\}$ vanish, thus $\kappa_{\widehat{\chi}|_V}\big((c_1, \ldots, c_{2n})|_V\big) = 0$.

In addition, in order for τ to contribute to the above sum, every block of τ must contain an even number of elements. Indeed, if $V \in \tau$ contains an odd number of elements, then we deduce that (additionally assuming that all indices in V correspond to elements either from $\{X, X^*, Y, Y^*\}$ or $\{Z, Z^*, W, W^*\}$) $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V)$ is a bi-free cumulant of odd order involving a $*$-bi-even pair and thus vanishes.

Henceforth, when referring to a bi-non-crossing partition τ contributing to the sum appearing in (1), we will assume that every block of τ contains indices all corresponding to elements either from $\{X, X^*, Y, Y^*\}$ or $\{Z, Z^*, W, W^*\}$ and, by Proposition 3.1.12, that $s_{\widehat{\chi}}(1) \sim_\tau s_{\widehat{\chi}}(2n)$ and $s_{\widehat{\chi}}(2i) \sim_\tau s_{\widehat{\chi}}(2i+1)$ for every $i = 1, \ldots, n-1$.

We will now show that if the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is not alternating in $*$-terms and non-$*$-terms, then the cumulant $\kappa_\chi(a_1, \ldots, a_n)$ must vanish. Suppose the following situation occurs:

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots\ldots, Z^*X^*, Y^*W^*, \ldots\ldots),$$

with $a_{s_\chi(m)} = Z^*X^*$ and $a_{s_\chi(m+1)} = Y^*W^*$ for some $m \in \{1, \ldots, n-1\}$. This implies the following situation for the $\widehat{\chi}$-order:

$$(c_{s_{\widehat{\chi}}(1)}, \ldots, c_{s_{\widehat{\chi}}(2n)}) = (\ldots\ldots, Z^*, X^*, W^*, Y^*, \ldots\ldots),$$

with

$$c_{s_{\widehat{\chi}}(2m-1)} = Z^*, c_{s_{\widehat{\chi}}(2m)} = X^*, c_{s_{\widehat{\chi}}(2m+1)} = W^* \text{ and } c_{s_{\widehat{\chi}}(2m+2)} = Y^*.$$

Now, if τ is a bi-non-crossing partition contributing to the sum appearing in (1), then the block of τ containing $s_{\widehat{\chi}}(2m)$ must also contain $s_{\widehat{\chi}}(2m+1)$. But, since

$$c_{s_{\widehat{\chi}}(2m)} = X^* \text{ and } c_{s_{\widehat{\chi}}(2m+1)} = W^*,$$

this is impossible, due to the $*$-bi-free independence condition. Hence, when

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)}) = (\ldots, Z^*X^*, Y^*W^*, \ldots),$$

we obtain that the bi-free cumulant $\kappa_\chi(a_1, \ldots, a_n)$ vanishes and the use of similar arguments shows that this is also the case when the sequence $(a_{s_\chi(1)}, \dots, a_{s_\chi(n)})$ has either one of the following forms:

(a) $(\dots, XZ, XZ, \dots)$,

(b) $(\dots, WY, WY, \dots)$,

(c) $(\dots, Z^*X^*, Z^*X^*, \dots)$,

(d) $(\dots, XZ, WY, \dots)$,

(e) $(\dots, Y^*W^*, Y^*W^*, \dots)$.

This completes the proof. $\square$

For a non-commutative $*$-probability space (A, φ) and $X_1, X_2, Y_1, Y_2 \in A$, consider the pair (Z, W) in the tensor product space $(\mathcal{M}_2(A), \varphi \otimes \mathrm{tr})$ defined by

$$Z = \begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix} \text{ and } W = \begin{bmatrix} 0 & Y_1 \\ Y_2 & 0 \end{bmatrix}.$$

Since any product with entries in $\{Z, Z^*, W, W^*\}$ containing an odd number of elements results in a matrix with zeroes across the diagonal, it follows that (Z, W) is a $*$-bi-even pair. Such pair is not necessarily bi-R-diagonal, since for instance

$$\kappa_\chi(Z, Z) = (\varphi \otimes \mathrm{tr})(Z \cdot Z) = \frac{1}{2}(\varphi(X_1X_2) + \varphi(X_2X_1))$$

which need not be equal to zero. However, the previous proposition implies that the product of two such pairs that are $*$-bi-free will always be bi-R-diagonal. Actually, matrix pairs arising in this manner can be used to characterize the condition of bi-R-diagonality (see Theorem 3.3.6).

We proceed with a lemma that contains the central combinatorial argument required for proving one of the main results of this section (see Theorem 3.3.4). At a key point, it makes use of the cancellation property observed in Lemma 3.1.6.

Lemma 3.3.3. *Let (A, φ) be a non-commutative $*$-probability space and $u_l, u_r, Z, W \in A$ such that:*

(a) the pair (u_l, u_r) is a bi-Haar unitary,

(b) the pair (Z,W) is $*$-bi-even,

(c) the pairs (u_l,u_r) and (Z,W) are $*$-bi-free.

Let $m\in\mathbb{N}$, $\chi\in\{l,r\}^{2m}$ and $a_1,\ldots,a_{2m}\in A$ with

$$a_i\in\begin{cases}\{u_lZ, Z^*{u_l}^*\}, & \text{if } \chi(i)=l\\[2ex] \{Wu_r, {u_r}^*W^*\}, & \text{if } \chi(i)=r\end{cases}\qquad (i=1,\ldots,2m)$$

such that the sequence $(a_{s_\chi(1)},\ldots,a_{s_\chi(2m)})$ is alternating in $*$-terms and non-$*$-terms. Define $b_1,\ldots,b_{2m}\in A$ as follows:

$$b_i=\begin{cases}Z, & \text{if } a_i=u_lZ\\ Z^*, & \text{if } a_i=Z^*{u_l}^*\\ W, & \text{if } a_i=Wu_r\\ W^*, & \text{if } a_i={u_r}^*W^*\end{cases}\qquad (i=1,\ldots,2m)$$

Then, we have that:

$$\kappa_\chi(a_1,\ldots,a_{2m})=\kappa_\chi(b_1,\ldots,b_{2m}).$$

Proof. Let $m\in\mathbb{N},\chi\in\{l,r\}^{2m}$ and $a_1,\ldots,a_{2m},b_1,\ldots,b_{2m}$ be given as in the statement of the lemma. Define $\widehat{\chi}\in\{l,r\}^{4m}$ by $\widehat{\chi}(2i-1)=\widehat{\chi}(2i)=\chi(i)$ for each $i=1,\ldots,2m$ and $c_1,\ldots,c_{4m}\in A$ as follows:

$$c_{2i-1}=\begin{cases}u_l, & \text{if } a_i=u_lZ\\ Z^*, & \text{if } a_i=Z^*{u_l}^*\\ W, & \text{if } a_i=Wu_r\\ {u_r}^*, & \text{if } a_i={u_r}^*W^*\end{cases}\quad\text{and } c_{2i}=\begin{cases}Z, & \text{if } a_i=u_lZ\\ {u_l}^*, & \text{if } a_i=Z^*{u_l}^*\\ u_r, & \text{if } a_i=Wu_r\\ W^*, & \text{if } a_i={u_r}^*W^*\end{cases}$$

for each $i=1,\ldots,2m$. Then, an application of Theorem 3.1.11 yields:

$$\kappa_\chi(a_1, \ldots, a_{2m}) = \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \kappa_{\widehat{\chi},\tau}(c_1, \ldots, c_{4m}) \tag{1}$$

$$= \sum_{\substack{\tau \in \mathrm{BNC}(\widehat{\chi}) \\ \tau \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \prod_{V \in \tau} \kappa_{\widehat{\chi}|_V}\big((c_1, \ldots, c_{4m})|_V\big) \tag{2}$$

where $\widehat{0_\chi} = \{\{2i-1, 2i\} : i = 1, \ldots, 2m\} \in \mathrm{BNC}(\widehat{\chi})$. As in the proof of Proposition 3.2.4, we make the following observations:

First of all, if for some $i \in \{1, \ldots, 2m\}$ we have that $a_{s_\chi(i)} = u_l Z$, then it follows that $c_{s_{\widehat{\chi}}(2i-1)} = u_l$ and $c_{s_{\widehat{\chi}}(2i)} = Z$ (with a similar situation occurring when $a_{s_\chi(i)} = Z^* X^*$, since this corresponds to a left operator). Now, if $a_{s_\chi(i)} = W u_r$, then $c_{s_{\widehat{\chi}}(2i-1)} = u_r$ and $c_{s_{\widehat{\chi}}(2i)} = W$ (and a similar situation occurs when $a_{s_\chi(i)} = Y^* W^*$, since this corresponds to a right operator). Note that in the latter case, the right operators must appear reversed in the $\widehat{\chi}$-order. This implies that since the sequence

$$(a_{s_\chi(1)}, \ldots, a_{s_\chi(2m)})$$

was assumed to be alternating in $*$-terms and non-$*$-terms, then both the sequences

$$\big(c_{s_{\widehat{\chi}}(1)}, c_{s_{\widehat{\chi}}(4)}, c_{s_{\widehat{\chi}}(5)}, \ldots, c_{s_{\widehat{\chi}}(4m-4)}, c_{s_{\widehat{\chi}}(4m-3)}, c_{s_{\widehat{\chi}}(4m)}\big)$$

and

$$\big(c_{s_{\widehat{\chi}}(2)}, c_{s_{\widehat{\chi}}(3)}, c_{s_{\widehat{\chi}}(6)}, c_{s_{\widehat{\chi}}(7)}, \ldots, c_{s_{\widehat{\chi}}(4m-2)}, c_{s_{\widehat{\chi}}(4m-1)}\big)$$

are also alternating in $*$-terms and non-$*$-terms (observe that for any $i \in \{1, \ldots, 2m\}$, the element $a_{s_\chi(i)}$ corresponds to a $*$-term if and only if both the elements $c_{s_{\widehat{\chi}}(2i-1)}$ and $c_{s_{\widehat{\chi}}(2i)}$ correspond to $*$-terms).

Since the pairs (u_l, u_r) and (Z, W) are $*$-bi-free, in order for a bi-non-crossing partition $\tau \in \mathrm{BNC}(\widehat{\chi})$ to contribute to the sum appearing in (1), we must have that for all $V \in \tau$, either

$$\{c_i : i \in V\} \subseteq \{u_l, {u_l}^*, u_r, {u_r}^*\},$$

or

$$\{c_i : i \in V\} \subseteq \{Z, Z^*, W, W^*\}.$$

In addition, in order for τ to contribute to the above sum, every block of τ must contain an even number of elements. Indeed, if $V \in \tau$ contains an odd number of elements, then we deduce that (additionally assuming that all indices in V correspond to elements either from $\{u_l, {u_l}^*, u_r, {u_r}^*\}$ or $\{Z, Z^*, W, W^*\}$) $\kappa_{\widehat{\chi}|_V}((c_1, \ldots, c_{2n})|_V)$ is a bi-free cumulant of odd order involving a $*$-bi-even pair and thus vanishes.

Henceforth, when referring to a bi-non-crossing partition τ contributing to the sum appearing in (1), we will assume that τ satisfies the following requirements:

(A) every block of τ contains indices all corresponding to elements either from $\{u_l, u_l^*, u_r, u_r^*\}$ or $\{Z, Z^*, W, W^*\}$,

(B) $s_{\widehat{\chi}}(1) \sim_\tau s_{\widehat{\chi}}(4m)$ and $s_{\widehat{\chi}}(2i) \sim_\tau s_{\widehat{\chi}}(2i+1)$ for every $i = 1, \ldots, 2m-1$ (this follows from an application of Proposition 3.1.12).

Define the sets

$$E_1 = \{s_{\widehat{\chi}}(1), s_{\widehat{\chi}}(4m)\} \text{ and } E_{i+1} = \{s_{\widehat{\chi}}(2i), s_{\widehat{\chi}}(2i+1)\}, \text{ for all } i = 1, \ldots, 2m-1.$$

We introduce new symbols $\overline{1}, \overline{2}, \ldots, \overline{m}$ and let

$$F_i = E_{2i-1} \text{ and } G_{\overline{i}} = E_{2i}, \text{ for all } i = 1, \ldots, m.$$

The notation $G_{\overline{i}}$ may seem unnatural, but it is being adopted for clarity for when we make use of Kreweras complementation map later in the proof. We claim that it must be the case that either

$$\{j \in \{1, \ldots, 4m\} : c_j \in \{u_l, {u_l}^*, u_r, {u_r}^*\}\} = \bigcup_{i=1}^{m} F_i,$$

or

$$\{j \in \{1, \ldots, 4m\} : c_j \in \{u_l, {u_l}^*, u_r, {u_r}^*\}\} = \bigcup_{i=1}^{m} G_{\overline{i}}.$$

Indeed, begin by assuming that $a_{s_\chi(1)} = u_l Z$. Since the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is alternating in $*$-terms and non-$*$-terms, we must have that $a_{s_\chi(2)} \in \{Z^* u_l^*, u_r^* W^*\}$. If $a_{s_\chi(2)} = u_l Z$, then for the $\widehat{\chi}$-order it is implied that

$$c_{s_{\widehat{\chi}}(1)} = u_l,\ c_{s_{\widehat{\chi}}(2)} = Z,\ c_{s_{\widehat{\chi}}(3)} = Z^*,\ c_{s_{\widehat{\chi}}(4)} = u_l^*,$$

while if $a_{s_\chi(2)} = u_r^* W^*$, then for the $\widehat{\chi}$-order it is implied that

$$c_{s_{\widehat{\chi}}(1)} = u_l,\ c_{s_{\widehat{\chi}}(2)} = Z,\ c_{s_{\widehat{\chi}}(3)} = W^*,\ c_{s_{\widehat{\chi}}(4)} = u_r^*,$$

hence in both cases we see that $\{c_{s_{\widehat{\chi}}(2)}, c_{s_{\widehat{\chi}}(3)}\} \subseteq \{Z, Z^*, W, W^*\}$. A straightforward induction argument then shows that for all $i = 1, \ldots, m$ and $j \in G_{\bar{i}}$, one must have that $c_j \in \{Z, Z^*, W, W^*\}$. Of course, this also implies that the union of $\{F_i : i = 1, \ldots, m\}$ must be equal to the set of all indices that correspond to elements in $\{u_l, u_l^*, u_r, u_r^*\}$. It clearly follows that similar arguments yield an analogous outcome in the case when $a_{s_\chi(1)} \in \{Z^* u_l^*, W u_r, u_r^* W^*\}$. Hence, we may assume that

$$\{j \in \{1, \ldots, 4m\} : c_j \in \{u_l, {u_l}^*, u_r, {u_r}^*\}\} = \bigcup_{i=1}^{m} F_i,$$

with the remaining case handled similarly. From this, it follows that

$$\{j \in \{1, \ldots, 4m\} : c_j \in \{Z, Z^*, W, W^*\}\} = \bigcup_{i=1}^{m} G_{\bar{i}}.$$

This assumption, along with requirement (A) above imply that for every $V \in \tau$, we have that

$$\text{either } V \subseteq \bigcup_{i=1}^{m} F_i, \text{ or } V \subseteq \bigcup_{i=1}^{m} G_{\bar{i}}.$$

Due to requirement (B) above and the definitions of the sets F_i and $G_{\bar{i}}$, it is easy to see that for any block $V \in \tau$ and any $i \in \{1, \ldots, m\}$, we have that

$$V \cap F_i \neq \emptyset \iff F_i \subseteq V \text{ and } V \cap G_{\bar{i}} \neq \emptyset \iff G_{\bar{i}} \subseteq V.$$

For all $V \in \tau$ with $V \subseteq \cup_{i=1}^{m} F_i$, define

$$I_V = \{i \in \{1, \ldots, m\} : V \cap F_i \neq \emptyset\}$$

and let

$$\pi_\tau = \left\{ I_V : V \in \tau, V \subseteq \bigcup_{i=1}^{m} F_i \right\}.$$

It is easy to see that $\pi_\tau \in \mathcal{P}(m)$ and we claim that $\pi_\tau \in \mathrm{NC}(m)$. Indeed, if not, there exist blocks $V \neq V' \in \tau$ with $V, V' \subseteq \cup_{i=1}^m F_i$, and integers $i_1, i_2, j_1, j_2 \in \{1, \ldots, m\}$ such that

$$i_1, i_2 \in I_V, \ \ j_i, j_2 \in I_{V'} \text{ and } i_1 < j_1 < i_2 < j_2.$$

Since $i_1, i_2 \in I_V$, it follows that $F_{i_1}, F_{i_2} \subseteq V$ and similarly $F_{j_1}, F_{j_2} \subseteq V'$. Initially, assume that $i_1 = 1$. By the definition of the sets $\{F_i : i = 1, \ldots, m\}$, it is implied that

$$\{s_{\widehat{\chi}}(1), s_{\widehat{\chi}}(4i_2 - 3)\} \subseteq V \text{ and } \{s_{\widehat{\chi}}(4j_i - 3), s_{\widehat{\chi}}(4j_2 - 3)\} \subseteq V',$$

or, equivalently,

$$\{1, 4i_2 - 3\} \subseteq s_{\widehat{\chi}}^{-1}(V) \text{ and } \{4j_1 - 3, 4j_2 - 3\} \subseteq s_{\widehat{\chi}}^{-1}(V').$$

But, since $1 = i_1 < j_1 < i_2 < j_2$, it follows that

$$1 < 4j_1 - 3 < 4i_2 - 3 < 4j_2 - 3,$$

which contradicts the fact that $s_{\widehat{\chi}}^{-1} \cdot \tau \in \mathrm{NC}(4m)$. Now, if we consider the case when $i_1 \geq 2$, then similarly we obtain

$$\{4i_1 - 3, 4i_2 - 3\} \subseteq s_{\widehat{\chi}}^{-1}(V) \text{ and } \{4j_1 - 3, 4j_2 - 3\} \subseteq s_{\widehat{\chi}}^{-1}(V'),$$

with the relations $i_1 < j_1 < i_2 < j_2$ implying that

$$4i_1 - 3 < 4j_1 - 3 < 4i_2 - 3 < 4j_2 - 3,$$

which once again contradicts the fact that $s_{\widehat{\chi}}^{-1} \cdot \tau \in \mathrm{NC}(4m)$.

Hence, we must have that $\pi_\tau \in \mathrm{NC}(m)$ and the use of similar arguments yields that if for all $V \in \tau$ with $V \subseteq \cup_{i=1}^m G_{\bar{i}}$ we define

$$J_V = \Big\{\bar{i} \in \{\bar{1}, \ldots, \overline{m}\} : V \cap G_{\bar{i}} \neq \emptyset\Big\},$$

then, by letting

$$\sigma_\tau = \Big\{J_V : V \in \tau, V \subseteq \bigcup_{i=1}^m G_{\bar{i}}\Big\},$$

it follows that $\sigma_\tau \in \mathrm{NC}(\{\bar{1}, \bar{2}, \ldots, \overline{m}\})$. We claim that we must necessarily have that $\pi_\tau \cup \sigma_\tau \in$

$\mathrm{NC}(\{(1,\overline{1},2,\overline{2},\ldots,m,\overline{m})\})$. Indeed, if not, there exist blocks $V \neq V' \in \tau$ such that

$$V \subseteq \bigcup_{i=1}^{m} F_i \text{ and } V' \subseteq \bigcup_{i=1}^{m} G_{\overline{i}}$$

and integers $i_1, i_2, j_1, j_2 \in \{1, \ldots, m\}$ with

$$i_1, i_2 \in I_V,\ \ \overline{j_1}, \overline{j_2} \in J_{V'} \text{ and } i_1 \leq j_1 < i_2 \leq j_2.$$

By the definitions of the sets I_V and $J_{V'}$, it follows that $F_{i_1}, F_{i_2} \subseteq V$ and $G_{\overline{j_1}}, G_{\overline{j_2}} \subseteq V'$. Consider the case when $i_1 \geq 2$ (with the case when $i_1 = 1$ treated analogously). This yields that

$$\{4i_1 - 3, 4i_2 - 3\} \subseteq s_{\widehat{\chi}}^{-1}(V) \text{ and } \{4j_1 - 1, 4j_2 - 1\} \subseteq s_{\widehat{\chi}}^{-1}(V').$$

But then, the relations $i_1 \leq j_1 < i_2 \leq j_2$ imply that

$$4i_1 - 3 < 4j_1 - 1 < 4i_2 - 3 < 4j_2 - 1,$$

which contradicts the fact that $s_{\widehat{\chi}}^{-1} \cdot \tau \in \mathrm{NC}(4m)$.

Hence, $\pi_\tau \cup \sigma_\tau \in \mathrm{NC}(\{1,\overline{1},2,\overline{2},\ldots,m,\overline{m}\})$ and by the definition of Kreweras complementation map and via the canonical identification

$$\mathrm{NC}(m) \cong \mathrm{NC}(\{\overline{1},\overline{2},\ldots,\overline{m}\}),$$

this is equivalent to $\sigma_\tau \leq K_{\mathrm{NC}}(\pi_\tau)$.

The previously described process implies that any $\tau \in \mathrm{BNC}(\widehat{\chi})$ that satisfies the requirements (A) and (B) uniquely determines two non-crossing partitions $\pi_\tau, \sigma_\tau \in \mathrm{NC}(m)$ such that $\sigma_\tau \leq K_{\mathrm{NC}}(\pi_\tau)$. Conversely, any two non-crossing partitions $\pi, \sigma \in \mathrm{NC}(m)$ with $\sigma \leq K_{\mathrm{NC}}(\pi)$ uniquely determine a bi-non-crossing partition $\tau_{(\pi,\sigma)} \in \mathrm{BNC}(\widehat{\chi})$ that satisfies the requirements (A) and (B) by defining

$$\tau_{(\pi,\sigma)} = \Big\{ \bigcup_{i \in V} F_i : V \in \pi \Big\} \bigcup \Big\{ \bigcup_{i \in V} G_{\overline{i}} : V \in \sigma \Big\}.$$

This yields a bijection between all bi-non-crossing partitions that satisfy the requirements (A) and (B) with the set of all bi-non-crossing partitions $\tau_{(\pi,\sigma)}$ obtained in the aforementioned manner. Thus, the sum appearing in (2) becomes:

$$
\begin{aligned}
\sum_{\substack{\tau\in \mathrm{BNC}(\widehat{\chi})\\ \tau\vee\widehat{0_\chi}=1_{\widehat{\chi}}}} \prod_{V\in\tau} \kappa_{\widehat{\chi}|_V}\big((c_1,\dots,c_{4m})|_V\big) &= \sum_{\substack{\tau_{(\pi,\sigma)}\in \mathrm{BNC}(\widehat{\chi})\\ \pi,\sigma\in \mathrm{NC}(m)\\ \sigma\le K_{\mathrm{NC}}(\pi)}} h_\pi\cdot d_\sigma \\
&= \sum_{\pi\in \mathrm{NC}(m)} h_\pi\cdot\left(\sum_{\substack{\sigma\in \mathrm{NC}(m)\\ \sigma\le K_{\mathrm{NC}}(\pi)}} d_\sigma\right) \quad (3)
\end{aligned}
$$

where we have used the notation

$$h_\pi = \prod_{V\in\pi} \kappa_{\widehat{\chi}|_{\bigcup_{i\in V} F_i}}\big((c_1,\dots,c_{4m})|_{\bigcup_{i\in V} F_i}\big)$$

and

$$d_\sigma = \prod_{V\in\sigma} \kappa_{\widehat{\chi}|_{\bigcup_{i\in V} G_{\bar{i}}}}\big((c_1,\dots,c_{4m})|_{\bigcup_{i\in V} G_{\bar{i}}}\big),$$

for any $\pi,\sigma\in \mathrm{NC}(m)$.

For a fixed $\pi\in \mathrm{NC}(m)$, we will compute the value of h_π. Since we assumed that

$$\{j\in\{1,\dots,4m\} : c_j\in\{u_l,u_l^*,u_r,u_r^*\}\} = \bigcup_{i=1}^m F_i,$$

and

$$\{j\in\{1,\dots,4m\} : c_j\in\{Z,Z^*,W,W^*\}\} = \bigcup_{i=1}^m G_{\bar{i}},$$

this implies that for all $V\in\pi$, the bi-free cumulant

$$\kappa_{\widehat{\chi}|_{\bigcup_{i\in V} F_i}}\big((c_1,\dots,c_{4m})|_{\bigcup_{i\in V} F_i}\big)$$

has entries in the set $\{u_l,u_l^*,u_r,u_r^*\}$ and the sequence

$$(c_1,\dots,c_{4m})|_{\bigcup_{i\in V} F_i}$$

is alternating in $*$-terms and non-$*$-terms when read in the induced $\widehat{\chi}|_{\bigcup_{i\in V} F_i}$-order. Moreover, notice that the cardinality of the union $\cup_{i\in V} F_i$ is equal to two times the cardinality of V.

Thus, by a combination of Corollary 3.1.20 and Lemma 3.1.4 we obtain

$$h_\pi = \prod_{V \in \pi} (-1)^{|V|-1} \cdot C_{|V|-1} = \mu_{\mathrm{NC}}(0_n, \pi).$$

Hence, equation (3) yields that

$$\kappa_\chi(a_1, \ldots, a_{2m}) = \sum_{\pi \in \mathrm{NC}(m)} \mu_{\mathrm{NC}}(0_n, \pi) \cdot \left(\sum_{\substack{\sigma \in \mathrm{NC}(m) \\ \sigma \leq K_{\mathrm{NC}}(\pi)}} d_\sigma \right),$$

with the right-hand side of the previous equation being equal to d_{1_m}, by Lemma 3.1.6. But then

$$\kappa_\chi(a_1, \ldots, a_{2m}) = d_{1_m} = \kappa_{\widehat{\chi}|_{\bigcup_{i=1}^m G_{\bar{i}}}}\big((c_1, \ldots, c_{4m})|_{\bigcup_{i=1}^m G_{\bar{i}}}\big) = \kappa_\chi(b_1, \ldots, b_{2m}),$$

where the elements $b_1, \ldots, b_{2m}$ are as in the statement of the lemma. This concludes the proof. $\square$

We are now in a position to state the following theorem (which is the generalization of [61, Theorem 15.10] to the bi-free setting), regarding the invariance of the joint $*$-distribution of a bi-R-diagonal pair under the multiplication by a $*$-bi-free bi-Haar unitary pair.

Theorem 3.3.4. *Let (A, φ) be a non-commutative $*$-probability space and $u_l, u_r, X, Y \in A$ such that:*

(a) the pair (u_l, u_r) is a bi-Haar unitary,

(b) the pairs (u_l, u_r) and (X, Y) are $*$-bi-free.

Then, the following are equivalent:

(i) the pair (X, Y) is bi-R-diagonal,

(ii) the joint $*$-distribution of the pair (X, Y) coincides with the joint $*$-distribution of $(u_l X, Y u_r)$.

Proof. By Theorem 3.2.2 the pair $(u_l X, Y u_r)$ is bi-R-diagonal and, since equality of joint $*$-distributions is equivalent to the equality of bi-free $*$-cumulants, it follows that the pair (X, Y) is also bi-R-diagonal. This yields the implication $(ii) \Rightarrow (i)$.

For the converse, we will show the equality of all $*$-bi-free cumulants involving the pairs (X, Y) and $(u_l X, Y u_r)$. Since (X, Y) is bi-R-diagonal, all bi-free cumulants with entries in $\{X, X^*, Y, Y^*\}$ that are either of odd order or that are not alternating in $*$-terms and non-$*$-terms in the χ-order must vanish. The same applies to the pair $(u_l X, Y u_r)$ since it is also bi-R-diagonal. Therefore, it is enough to show that for all even numbers $n \in \mathbb{N}, \chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ with

$$a_i \in \begin{cases} \{u_l X, X^* {u_l}^*\}, & \text{if } \chi(i) = l \\ \{Y u_r, {u_r}^* Y^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

such that the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is alternating in $*$-terms and non-$*$-terms, by setting

$$b_i = \begin{cases} X, & \text{if } a_i = u_l X \\ X^*, & \text{if } a_i = X^* {u_l}^* \\ Y, & \text{if } a_i = Y u_r \\ Y^*, & \text{if } a_i = {u_r}^* Y^* \end{cases} \qquad (i = 1, \ldots, n)$$

we have that

$$\kappa_\chi(a_1, \ldots, a_n) = \kappa_\chi(b_1, \ldots, b_n),$$

which is exactly what an application of Lemma 3.3.3 yields. □

We remark that the conclusion of the previous theorem no longer holds if the order of the multiplication of the right operators is not reversed, as the following example indicates.

Example 3.3.5. Let $(A, \varphi), (B, \psi)$ be two non-commutative $*$-probability spaces and let $u_l, u_r \in A, v_l, v_r \in B$ such that both pairs (u_l, u_r) and (v_l, v_r) are bi-Haar unitaries. In the free product space $(A * B, \varphi * \psi)$ these pairs are $*$-bi-free and clearly both bi-R-diagonal. But, the joint $*$-distribution of the pair (v_l, v_r) does not coincide with the joint $*$-distribution of $(u_l v_l, u_r v_r)$, since

$$\kappa_\chi(v_l, v_r^*) = \psi(v_l \cdot v_r^*) = 1,$$

while, by an application of Theorem 3.1.11, it is easily verified that

$$\kappa_\chi(u_l v_l, v_r^* u_r^*) = 0.$$

□

Gathering the results of this chapter, one can obtain a theorem similar to [59, Theorem 1.2] (and [6, Theorem 3.1] for the operator-valued setting).

Theorem 3.3.6. *Let (A, φ) be a non-commutative $*$-probability space and $X, Y \in A$. The following are equivalent:*

(i) the pair (X, Y) is bi-R-diagonal,

(ii) there exists an enlargement [8] $(\tilde{A}, \tilde{\varphi})$ of (A, φ) and $u_l, u_r \in \tilde{A}$ such that

(a) the pair (u_l, u_r) is a bi-Haar unitary,

(b) the pairs (u_l, u_r) and (X, Y) are $*$-bi-free,

(c) the joint $*$-distribution of the pair $(u_l X, Y u_r)$ coincides with the joint $*$-distribution of (X, Y),

(iii) for any enlargement $(\tilde{A}, \tilde{\varphi})$ of (A, φ) and any $u_l, u_r \in \tilde{A}$ such that

(d) the pair (u_l, u_r) is a bi-Haar unitary,

(e) the pairs (u_l, u_r) and (X, Y) are $*$-bi-free,

one has that the the joint $*$-distribution of the pair $(u_l X, Y u_r)$ coincides with the joint $*$-distribution of (X, Y),

(iv) consider the unital subalgebras $\mathcal{M}_2(\mathbb{C})$ and $\mathcal{D}_2$ of $\mathcal{M}_2(A)$ consisting of scalar matrices and diagonal scalar matrices respectively and let the maps

$$\varepsilon : \mathcal{D} \otimes \mathcal{D}^{op} \to \mathcal{L}(\mathcal{M}_2(A)),\ L, R : \mathcal{M}_2(A) \to \mathcal{L}(\mathcal{M}_2(A)),\ E_2 : \mathcal{L}(\mathcal{M}_2(A)) \to \mathcal{M}_2(\mathbb{C})$$

and

$$F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{D}$$

be as in section 3.1.4. Also, in $\mathcal{M}_2(A)$ consider the pair (Z, W) defined as

$$Z = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \text{ and } W = \begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix}.$$

Then, the pair $(L(Z), R(W))$ is bi-free from $(L(\mathcal{M}_2(\mathbb{C})), R(\mathcal{M}_2(\mathbb{C})^{op}))$ with amalgamation over $\mathcal{D}_2$ with respect to $F_2 \circ E_2$.

[8] An *enlargement* of a non-commutative $*$-probability space (A, φ) is a non-commutative $*$-probability space $(\tilde{A}, \tilde{\varphi})$ such that $A \subseteq \tilde{A}$ and $\tilde{\varphi}|_A = \varphi$.

Proof. The equivalence of (i) and (iii), as well as the implication $(ii) \Rightarrow (i)$ both follow from Theorem 3.3.4. Also, the equivalence of (i) and (iv) is a result of Proposition 3.1.23 and Theorem 3.1.24.

To see that (i) implies (ii), simply consider a non-commutative $*$-probability space (B, ψ) containing a bi-Haar unitary pair (u_l, u_r) and define $(\tilde{A}, \tilde{\varphi})$ to be the free product space $(A * B, \varphi * \psi)$. In $(\tilde{A}, \tilde{\varphi})$ the pairs (X, Y) and (u_l, u_r) are $*$-bi-free and thus, again by Theorem 3.3.4, the joint $*$-distribution of the pair $(u_l X, Y u_r)$ must coincide with the joint $*$-distribution of (X, Y). $\square$

Chapter 4

Analytical Operator-Valued Bi-Free Structures

Notions of free entropy were introduced in a series of papers by Voiculescu including [80, 81] that cemented the foundations of free probability and its applications to operator algebras. One the one hand, microstates free entropy measures the volumes of tuples of self-adjoint scalar matrices that approximate the distribution of tuples of self-adjoint operators in tracial von Neumann algebras, motivated by the connection between free probability and random matrix theory. The development of this notion led to several important results, giving answers to longstanding open problems regarding the structure of the free group factors, such as the absence of Cartan subalgebras ([79]) and the primeness of the free group factors ([35]).

On the other hand, the non-microstates approach to free entropy is motivated by the concept of Fisher information in classical probability and is based on conjugate variable systems defined with respect to non-commutative partial derivatives. The techniques developed throughout the advancement of this theory led to important applications in von Neumann algebras, as they were used to show that specific type II_1 factors do not have property Γ ([15]), as well as to show the absence of atoms and zero divisors from free product distributions (see [12] and [53]). These ideas were further extended to the operator-valued setting by Shlyakhtenko in [71] by modifying the conjugate variable formulae to involve a completely positive map on the algebra of amalgamation. One immediate application was [71, Proposition 7.14] that obtained a formula for the Jones index of a subfactor. Furthermore, free entropy with respect to a completely positive map was essential to the work in [58] which demonstrated that minimal values for the free Fisher information and maximal values for the non-microstate free entropy existed and were obtained at R-diagonal elements.

The notions of microstate and non-microstate entropy were recently generalized to the setting of bi-free probability (see [13, 14]). In chapters 4, 5 and 6 we will aim to extend the notion of non-microstate bi-free entropy to incorporate the existence of a completely positive map and examine applications of said theory. In particular, the main applications are Theorems 6.1.6 and 6.2.3 which examine the minimal value of the bi-free Fisher information and maximal value of the non-microstates bi-free entropy for collections of pairs of operators with similarities in their distributions and show that, modulo distributional conditions, these minimal and maximal values are attained at bi-R-diagonal pairs of operators. Presently, we will only be concerned with developing the technology that is required throughout the rest of this manuscript.

4.1 B-B-Non-Commutative Probability Spaces and Bi-Freeness

In this section we will recall the basic structures of bi-freeness with amalgamation. Even though these structures were briefly mentioned in chapter 3, they were defined under a purely algebraic lens (as in [10]), making reference only to complex, unital algebras. However, in this chapter we will aim to extend our core structures from an analytic perspective. This is necessary as in general expectations in operator-valued bi-free probability need not be positive and thus to perform analytical computations additional structures are required. We will see that our analytical structures will be modeled based on the left and right actions of a II_1 factor on its L_2-space (see Example 4.2.6). By adding a tracial state on the algebra of amalgamation that satisfies certain compatibility conditions, the appropriate L_2-spaces can be constructed and used to study operator-valued bi-free probability. Moreover, these will be used in order to extend the central combinatorial tools of operator-valued bi-free probability, namely the operator-valued bi-free moment and cumulant functions.

In order for us to build towards this goal and to keep this chapter as autonomous as possible for the convenience of the reader, we state a slightly altered definition of the core operator-valued bi-free structures so as to involve unital $*$-algebras in our considerations and give relevant examples.

Definition 4.1.1. Let B be a unital $*$-algebra. A *B-B-non-commutative probability space* consists of a triple (A, E, ε) where A is a unital $*$-algebra, $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ is a unital $*$-homomorphism such that the restrictions $\varepsilon|_{B\otimes 1_B}$ and $\varepsilon|_{1_B\otimes B^{\mathrm{op}}}$ are both injective, and

$E : A \to B$ is a unital linear map that such that

$$E(\varepsilon(b_1 \otimes b_2)a) = b_1 E(a) b_2 \qquad \text{and} \qquad E(a\varepsilon(b \otimes 1_B)) = E(a\varepsilon(1_B \otimes b)),$$

for all $b, b_1, b_2 \in B$ and $a \in A$. In addition, consider the unital $*$-subalgebras A_ℓ and A_r of A given by

$$A_\ell = \{a \in A \mid a\varepsilon(1_B \otimes b) = \varepsilon(1_B \otimes b)a \text{ for all } b \in B\}$$

and

$$A_r = \{a \in A \mid a\varepsilon(b \otimes 1_B) = \varepsilon(b \otimes 1_B)a \text{ for all } b \in B\}.$$

We call A_ℓ and A_r the *left and right algebras of* A respectively.

Note one can always assume that a B-B-non-commutative probability space is generated as a $*$-algebra by A_ℓ and A_r.

Example 4.1.2. Let $\mathcal{A}$ and B be unital $*$-algebras and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a unital, linear map. If $A = \mathcal{A} \otimes B \otimes B^{\mathrm{op}}$, if $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ is defined by $\varepsilon(b_1 \otimes b_2) = 1_{\mathcal{A}} \otimes b_1 \otimes b_2$ for all $b_1, b_2 \in B$, and $E : A \to B$ is defined by

$$E(a \otimes b_1 \otimes b_2) = \varphi(a) b_1 b_2$$

for all $a \in A$ and $b_1, b_2 \in B$, then (A, E, ε) is a B-B-non-commutative probability space. Indeed, clearly ε is a unital injective $*$-homomorphism. Furthermore, note for all $Z \in \mathcal{A}$ and $b, b_1, b_2, b_3, b_4 \in B$ that

$$E((1_{\mathcal{A}} \otimes b_1 \otimes b_2)(Z \otimes b_3 \otimes b_4)) = \varphi(Z) b_1 b_3 b_4 b_2 = b_1 E(Z \otimes b_3 \otimes b_4) b_2$$

and

$$E((Z \otimes b_1 \otimes b_2)(1_{\mathcal{A}} \otimes b \otimes 1_B)) = \varphi(Z) b_1 b b_2 = E((Z \otimes b_1 \otimes b_2)(1_{\mathcal{A}} \otimes 1_B \otimes b)).$$

Hence E satisfies the required properties.

For future use, notice that

$$\mathcal{A} \otimes B \otimes 1_B \subseteq \mathcal{A}_\ell \qquad \text{and} \qquad \mathcal{A} \otimes 1_B \otimes B^{\mathrm{op}} \subseteq \mathcal{A}_r.$$

Moreover, in the case $B = \mathbb{C}$, (A, E, ε) efficiently reduces down to $(\mathcal{A}, \varphi)$; the usual notion of a non-commutative $*$-probability space.

Example 4.1.3. Let $\mathfrak{M}$ be a finite von Neumann algebra with a tracial state $\tau : \mathfrak{M} \to \mathbb{C}$ and let $L_2(\mathfrak{M}, \tau)$ be the GNS Hilbert space generated by $(\mathfrak{M}, \tau)$. For $T \in \mathfrak{M}$, let L_T denote the left action of T on $L_2(\mathfrak{M}, \tau)$, and let R_T denote the right action of T on $L_2(\mathfrak{M}, \tau)$. Furthermore, let A be the algebra generated by $\{L_T, R_T \mid T \in \mathfrak{M}\}$.

Let B be a unital von Neumann subalgebra of $\mathfrak{M}$ and let $E_B : \mathfrak{M} \to B$ be the conditional expectation of $\mathfrak{M}$ onto B. Recall that if $P : L_2(\mathfrak{M}, \tau) \to L_2(B, \tau)$ is the orthogonal projection of $L_2(\mathfrak{M}, \tau)$ onto $L_2(B, \tau)$, then $E_B(Z) = P(Z1_{\mathfrak{M}})$ for all $Z \in \mathfrak{M}$.

Define $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ by $\varepsilon(b_1 \otimes b_2) = L_{b_1} R_{b_2}$ and define $E : A \to B$ by

$$E(Z) = P(Z1_{\mathfrak{M}})$$

for all $Z \in A$. Elementary von Neumann algebra theory implies that the range of E is indeed contained in B. To see that (A, E, ε) is a B-B-non-commutative probability space, first note that ε is clearly a unital $*$-homomorphism that is injective when restricted to $B \otimes 1_B$ and when restricted to $1_B \otimes B^{\mathrm{op}}$. Moreover, note for all $Z \in A$ and $b, b_1, b_2 \in B$ that

$$E(L_{b_1} R_{b_2} Z) = P(b_1(Z1_{\mathfrak{M}})b_2) = b_1 P(Z1_{\mathfrak{M}}) b_2 = b_1 E(Z) b_2$$

and

$$E(TL_b) = P(TL_b 1_{\mathfrak{M}}) = P(Tb) = P(TR_b 1_{\mathfrak{M}}) = E(TR_b).$$

Hence E satisfies the required properties.

The map $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ encodes the left and right elements of B in A. For notational purposes, for each $b \in B$ we will denote $\varepsilon(b \otimes 1_B)$ and $\varepsilon(1_B \otimes b)$ by L_b and R_b respectively and we denote

$$B_\ell = \varepsilon(B \otimes 1_B) = \{L_b \mid b \in B\} \quad \text{and} \quad B_r = \varepsilon(1_B \otimes B^{\mathrm{op}}) = \{R_b \mid b \in B\}.$$

To examine bi-free independence with amalgamation over B, it is necessary that left operators are contained in A_ℓ (i.e. commute with the right copy of B) and right operators are contained in A_r (i.e. commute with the left copy of B).

Definition 4.1.4 ([10])**.** Let (A, E, ε) be a B-B-non-commutative probability space.

(i) A *pair of B-algebras* is a pair (C, D) consisting of unital subalgebras of A such that

$$B_\ell \subseteq C \subseteq A_\ell \quad \text{and} \quad B_r \subseteq D \subseteq A_r.$$

(ii) A family $\{(C_k, D_k)\}_{k\in K}$ of pairs of B-algebras in A is called *bi-free with amalgamation over B* if there exist B-B-bimodules with specified B-vector states $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ and unital homomorphisms

$$l_k : C_k \to \mathcal{L}_\ell(\mathcal{X}_k) \quad \text{and} \quad r_k : D_k \to \mathcal{L}_r(\mathcal{X}_k),$$

such that the joint distribution of the family $\{(C_k, D_k)\}_{k\in K}$ with respect to E coincides with the joint distribution of the images

$$\{((\lambda_k \circ l_k)(C_k), (\rho_k \circ r_k)(D_k))\}_{k\in K}$$

in the space $\mathcal{L}(*_{k\in K}\mathcal{X}_k)$, with respect to $E_{\mathcal{L}(*_{k\in K}\mathcal{X}_k)}$, where $*_{k\in K}\mathcal{X}_k$ is the reduced free product of $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ with amalgamation over B.

Remark 4.1.5. Let $\mathcal{A}$ and B be unital $*$-algebras and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a unital linear map. Let (A, E, ε) be as in Example 4.1.2. By [13, 67], if $\{(C_k, D_k)\}$ are $*$-subalgebras of $\mathcal{A}$ that are bi-free with respect to φ, then $\{(C_k \otimes B \otimes 1_B, D_k \otimes 1_B \otimes B^{\mathrm{op}})\}_{k\in K}$ are bi-free with amalgamation over B with respect to E. Thus Example 4.1.2 is the correct notion of "inflating $(\mathcal{A}, \varphi)$ by B" in the bi-free setting.

Example 4.1.6. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be finite von Neumann algebras with a common von Neumann subalgebra B and tracial states τ_1 and τ_2 respectively such that $\tau_1|_B = \tau_2|_B$. Let $\mathfrak{M} = \mathfrak{M}_1 *_B \mathfrak{M}_2$ be the reduced free product von Neumann algebra with amalgamation over B, let $E_B : \mathfrak{M} \to B$ be the conditional expectation of $\mathfrak{M}$ onto B, and let $\tau = \tau_1 * \tau_2 = \tau_1|_B \circ E_B$ be the tracial state on $\mathfrak{M}$. If E and ε are as in Example 4.1.3 for $(\mathfrak{M}, \tau)$, then

$$\{(\{L_X \mid X \in \mathfrak{M}_1\}, \{R_Y \mid Y \in \mathfrak{M}_1\})\} \qquad \text{and} \qquad \{(\{L_X \mid X \in \mathfrak{M}_2\}, \{R_Y \mid Y \in \mathfrak{M}_2\})\}$$

are bi-free with amalgamation over B.

In order to study bi-free independence with amalgamation, the operator-valued bi-free moment and cumulant functions are key. These functions have specific properties that are described via the following concept.

Definition 4.1.7 ([10, Definition 4.2.1])**.** Let (A, E, ε) be a B-B-non-commutative probability space. A map

$$\Phi : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times A_{\chi(2)} \times \ldots \times A_{\chi(n)} \to B$$

is called *bi-multiplicative* if it is $\mathbb{C}$-linear in each of the $A_{\chi(k)}$ entries and for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $b \in B$, and $Z_k \in A_{\chi(k)}$ the following four conditions hold:

(i) Let

$$q = \max_{\leq}\{k \in \{1, \ldots, n\} \mid \chi(k) \neq \chi(n)\}.$$

If $\chi(n) = \ell$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n L_b) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q R_b, Z_{q+1}, \ldots, Z_n) & \text{if } q \neq -\infty, \\ \Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n)b & \text{if } q = -\infty. \end{cases}$$

If $\chi(n) = r$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n R_b) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q L_b, Z_{q+1}, \ldots, Z_n) & \text{if } q \neq -\infty, \\ b\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n) & \text{if } q = -\infty. \end{cases}$$

(ii) Let $p \in \{1, \ldots, n\}$ and let

$$q = \max_{\leq}\{k \in \{1, \ldots, n\} \mid \chi(k) = \chi(p), k < p\}.$$

If $\chi(p) = \ell$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, \ldots, Z_n) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q L_b, \ldots, Z_n) & \text{if } q \neq -\infty, \\ b\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n) & \text{if } q = -\infty. \end{cases}$$

If $\chi(p) = r$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{p-1}, R_b Z_p, \ldots, Z_n) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q R_b, \ldots, Z_n) & \text{if } q \neq -\infty, \\ \Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n)b & \text{if } q = -\infty. \end{cases}$$

(iii) Suppose $V_1, \ldots, V_m$ are unions of blocks of π that partition $\{1, \ldots, n\}$ with each being a χ-interval (i.e. an interval in the χ-ordering) and the sets $V_1, \ldots, V_m$ are ordered by $\preceq_\chi$ (i.e. $(\min_{\preceq_\chi} V_k) \prec_\chi (\min_{\preceq_\chi} V_{k+1})$ for all k). Then

$$\Phi_\pi(Z_1, \ldots, Z_n) = \Phi_{\pi|_{V_1}}((Z_1, \ldots, Z_n)|_{V_1}) \cdots \Phi_{\pi|_{V_m}}((Z_1, \ldots, Z_n)|_{V_m}).$$

(iv) Suppose that V and W are unions of blocks of π that partition $\{1, \ldots, n\}$, V is a χ-interval, and $s_\chi(1), s_\chi(n) \in W$. Let

$$p = \max_{\preceq_\chi} \left\{ k \in W \,\middle|\, k \prec_\chi \min_{\preceq_\chi} V \right\} \qquad \text{and} \qquad q = \min_{\preceq_\chi} \left\{ k \in W \,\middle|\, \max_{\preceq_\chi} V \prec_\chi k \right\}.$$

Then, we have that

$$\begin{aligned}\Phi_\pi(Z_1, \ldots, Z_n) &= \begin{cases} \Phi_{\pi|_W}\left((Z_1, \ldots, Z_{p-1}, Z_p L_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)}, \ldots, Z_n)|_W\right) & \text{if } \chi(p) = \ell, \\ \Phi_{\pi|_W}\left((Z_1, \ldots, Z_{p-1}, R_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)} Z_p, \ldots, Z_n)|_W\right) & \text{if } \chi(p) = r, \end{cases} \\ &= \begin{cases} \Phi_{\pi|_W}\left((Z_1, \ldots, Z_{q-1}, L_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)} Z_q, \ldots, Z_n)|_W\right) & \text{if } \chi(q) = \ell, \\ \Phi_{\pi|_W}\left((Z_1, \ldots, Z_{q-1}, Z_q R_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)}, \ldots, Z_n)|_W\right) & \text{if } \chi(q) = r. \end{cases}\end{aligned}$$

See [10, Section 4] for a discussion on the previous definition. Note that bi-multiplicative functions naturally expend the properties of the moment function in operator-valued free probability. Given a B-B-non-commutative probability space (A, E, ε), the moment and cumulant functions are well-defined bi-multiplicative functions.

Definition 4.1.8. Let (A, E, ε) be a B-B-non-commutative probability space.

(i) The *operator-valued bi-free moment function*

$$E : \bigcup_{n \in \mathbb{N}} \bigcup_{\chi \in \{\ell, r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \ldots \times A_{\chi(n)} \to B$$

is the bi-multiplicative function (see [10, Theorem 5.1.4]) that satisfies

$$E_{1_\chi}(Z_1, Z_2, \ldots, Z_n) = E(Z_1 Z_2 \cdots Z_n),$$

for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, and $Z_k \in A_{\chi(k)}$.

(ii) The *operator-valued bi-free cumulant function*

$$\kappa^B : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \ldots \times A_{\chi(n)} \to B$$

is the bi-multiplicative function (see [10, Corollary 6.2.2]) defined by

$$\kappa^B_\pi(Z_1,\ldots,Z_n) = \sum_{\substack{\sigma\in\mathrm{BNC}(\chi)\\ \sigma\leq\pi}} E_\sigma(Z_1,\ldots,Z_n)\mu_{\mathrm{BNC}}(\sigma,\pi),$$

for each $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^n$, $\pi\in\mathrm{BNC}(\chi)$, and $Z_k\in A_{\chi(k)}$. In the special case when $\pi = 1_\chi$, the map $\kappa^B_{1_\chi}$ is simply denoted by κ^B_χ. An instance of Möbius inversion yields that the equality

$$E_\sigma(Z_1,\ldots,Z_n) = \sum_{\substack{\pi\in\mathrm{BNC}(\chi)\\ \pi\leq\sigma}} \kappa^B_\pi(Z_1,\ldots,Z_n)$$

holds for all $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^n$, $\sigma\in\mathrm{BNC}(\chi)$, and $Z_k\in A_{\chi(k)}$.

The condition of bi-freeness with amalgamation over B for a family of pairs of B-faces is equivalent to the vanishing of their mixed operator-valued bi-free cumulants, as the following result indicates.

Theorem 4.1.9 ([10, Theorem 8.1.1])**.** *Let (A,E,ε) be a B-B-non-commutative probability space and let $\{(C_k,D_k)\}_{k\in K}$ be a family of pairs of B-algebras in A. The following are equivalent:*

(i) the family $\{(C_k,D_k)\}_{k\in K}$ is bi-free with amalgamation over B,

(ii) for all $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^n$, $Z_1,\ldots,Z_n\in A$, and non-constant maps $\gamma:\{1,\ldots,n\}\to K$ such that

$$Z_k \in \begin{cases} C_{\gamma(k)} & \text{if } \chi(k)=\ell \\ D_{\gamma(k)} & \text{if } \chi(k)=r \end{cases}$$

we have that

$$\kappa^B_\chi(Z_1,\ldots,Z_n) = 0.$$

4.2 Analytical B-B-Non-Commutative Probability Spaces

In order perform the more analytical computations necessary in this chapter, we will now discuss the additional conditions we will need to impose on the operator-valued bi-free structures. These structures are analogous to those observed in Example 4.1.3 and will be seen to be the correct enhancement of a B-B-non-commutative probability space to perform functional analysis.

Let (A, τ) be a non-commutative $*$-probability space. If $N_\tau = \{a \in A : \tau(a^*a) = 0\}$, then $L_2(A, \tau)$ will denote the Hilbert space completion of the quotient space A/N_τ with respect to the inner product induced by τ given by

$$\langle a_1 + N_\tau, a_2 + N_\tau \rangle = \tau(a_2^* a_1),$$

for all $a_1, a_2 \in A$ and $\| \cdot \|_\tau$ will denote the Hilbert space norm on $L_2(A, \tau)$.

Definition 4.2.1. Given a unital $*$-algebra B, an *analytical B-B-non-commutative probability space* consists of a tuple $(A, E, \varepsilon, \tau)$ such that

(i) (A, E, ε) is a B-B-non-commutative probability space,

(ii) $\tau : A \to \mathbb{C}$ is a state that is compatible with E; that is,

$$\tau(a) = \tau\left(L_{E(a)}\right) = \tau\left(R_{E(a)}\right)$$

for all $a \in A$,

(iii) the canonical state $\tau_B : B \to \mathbb{C}$ defined by $\tau_B(b) = \tau(L_b)$ for all $b \in B$ is tracial,

(iv) left multiplication of A on A/N_τ are bounded linear operators and thus extend to bounded linear operators on $L_2(A, \tau)$, and

(v) E is completely positive when restricted to A_ℓ and when restricted to A_r.

Remark 4.2.2. Given an analytical B-B-non-commutative probability space $(A, E, \varepsilon, \tau)$, note the following.

(i) The that fact that τ_B is a state immediately follows from the fact that τ is a state and ε is a $*$-homomorphism. Specifically, for positivity, notice for all $b \in B$ that

$$\tau_B(b^*b) = \tau(L_{b^*b}) = \tau((L_b)^* L_b) \geq 0.$$

(ii) Note for all $b \in B$ that

$$\|b + N_{\tau_B}\|_{\tau_B}^2 = \tau_B(b^*b) = \tau(L_{b^*b}) = \|L_b + N_\tau\|_\tau^2.$$

Hence the map from B/N_{τ_B} to $L_2(A, \tau)$ defined by

$$b + N_{\tau_B} \mapsto L_b + N_\tau$$

for all $b \in B$ is a well-defined, linear isometry. Therefore, a standard density argument yields that

$$L_2(B, \tau_B) \cong \overline{\{L_b + N_\tau \mid b \in B\}}^{\|\cdot\|_\tau} \subseteq L_2(A, \tau).$$

Henceforth, we shall only be making reference to the space $L_2(B, \tau_B)$ via this identification.

(iii) The state τ naturally extends to a linear functional on $L_2(A, \tau)$ by defining

$$\tau(\xi) = \langle \xi, 1_A + N_\tau \rangle_{L_2(A,\tau)}$$

for all $\xi \in L_2(A, \tau)$. Similarly, the scalar $\tau_B(\zeta) = \tau(\zeta)$ is well-defined for any $\zeta \in L_2(B, \tau_B)$.

(iv) As left multiplication by A on A/N_τ is bounded, we immediately extend the left multiplication map to obtain a unital $*$-homomorphism from A into $\mathcal{B}(L_2(A))$. Thus $a\xi$ is a well-defined element of $L_2(A, \tau)$ for all $a \in A$ and $\xi \in L_2(A, \tau)$.

(v) The requirement of the left multiplication inducing bounded operators is immediate in the case when A is a C*-algebra, however it also holds in more general situations. For instance, when A is a unital $*$-algebra generated its partial isometries, the left multiplication map is automatically bounded (see [61, Exercise 7.22]).

(vi) Since the state τ_B is assumed to be tracial, right multiplication of B on B/N_{τ_B} is also bounded. Thus, for any $b_1, b_2 \in B$ and $\zeta \in L_2(B, \tau_B)$, we have that $b_1\zeta b_2$ is a well-defined element of $L_2(B, \tau_B)$ and, in $L_2(A, \tau)$, $L_{b_1}R_{b_2}\zeta = b_1\zeta b_2$. Furthermore, note that left and right multiplication of B on $L_2(B, \tau_B)$ are commuting $*$-homomorphisms.

(vii) For all $a \in A$ and $b \in B$, we automatically have $\tau(aL_b) = \tau(aR_b)$, as τ is compatible with E. Indeed

$$\tau(aL_b) = \tau(L_{E(aL_b)}) = \tau(L_{E(aR_b)}) = \tau(aR_b),$$

as desired. Hence $L_b + N_\tau = R_b + N_\tau$ for all $b \in B$.

In some cases, property (v) of Definition 4.2.1 is redundant.

Lemma 4.2.3. *Let $(A, E, \varepsilon, \tau)$ satisfy assumptions (i), (ii), (iii), and (iv) of Definition 4.2.1. If B is a C*-algebra and τ_B is faithful, then property (v) of Definition 4.2.1 holds.*

Proof. To see that E is completely positive on A_ℓ, let $d \in \mathbb{N}$ and $A = [a_{i,j}] \in M_d(A_\ell)$. To verify that $E_d(A^*A) \geq 0$ in B, as B is a C*-algebra and τ_B is faithful, it suffices to show for all $h = (b_1, \ldots, b_d) \in B^d$ that

$$\langle E_d(A^*A)h, h\rangle_{L_2(B,\tau_B)^{\oplus d}} \geq 0.$$

Note that

$$\begin{aligned}
\langle E_d(A^*A)h, h\rangle_{L_2(B,\tau_B)^{\oplus d}} &= \sum_{i,j,k=1}^{d} \tau_B\left(b_i^* E(a_{k,i}^* a_{k,j}) b_j\right) \\
&= \sum_{i,j,k=1}^{d} \tau_B\left(E(R_{b_j} L_{b_i^*} a_{k,i}^* a_{k,j})\right) \\
&= \sum_{i,j,k=1}^{d} \tau_B\left(E(L_{b_i^*} a_{k,i}^* a_{k,j} R_{b_j})\right) \\
&= \sum_{i,j,k=1}^{d} \tau_B\left(E(L_{b_i}^* a_{k,i}^* a_{k,j} L_{b_j})\right) \\
&= \sum_{i,j,k=1}^{d} \tau\left(L_{b_i}^* a_{k,i}^* a_{k,j} L_{b_j}\right) \\
&= \sum_{k=1}^{d} \tau\left(c_k^* c_k\right)
\end{aligned}$$

where $c_k = \sum_{j=1}^{d} a_{k,j} L_{b_j}$. Hence, as τ is positive and the computation for A_r is similar, the result follows. □

At this point, let us revisit Examples 4.1.2 and 4.1.3 to provide the canonical examples of analytical B-B-non-commutative probability spaces.

Example 4.2.4. Let $\mathcal{A}$ and B be unital C*-algebras and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a state. Recall from Example 4.1.2 that (A, E, ε) is a B-B-non-commutative probability space where $A = \mathcal{A} \otimes B \otimes B^{\mathrm{op}}$, $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ is the natural embedding, and $E : A \to B$ is defined by

$$E(Z \otimes b_1 \otimes b_2) = \varphi(Z) b_1 b_2,$$

for all $Z \in \mathcal{A}$ and $b_1, b_2 \in B$.

Let $\tau_B : B \to \mathbb{C}$ be any tracial state. Extend τ_B to a linear map $\tau : A \to \mathbb{C}$ by defining

$$\tau(Z \otimes b_1 \otimes b_2) = \tau_B(E(Z \otimes b_1 \otimes b_2)) = \varphi(Z)\tau_B(b_1 b_2),$$

for all $Z \otimes b_1 \otimes b_2 \in A$. We claim that $(A, E, \varepsilon, \tau)$ is an analytical B-B-non-commutative probability space. To see this, it suffices to prove that τ is a state that is compatible with E, since $\mathcal{A}$ and B being unital C*-algebras automatically implies that left multiplication will be bounded on $L_2(A, \tau)$, and Lemma 4.2.3 implies that E is completely positive when restricted to A_ℓ or A_r (or one may simply use the fact that states are completely positive).

Clearly τ is a unital, linear map that is compatible with E. To see that τ is positive, let $(Z_i)_{i=1}^n \subseteq \mathcal{A}$, $(b_k)_{k=1}^n, (c_k)_{k=1}^n \subseteq B$, and

$$a = \sum_{k=1}^{n} Z_k \otimes b_k \otimes c_k \in A.$$

To see that $\tau(a^*a) \geq 0$, note that

$$\begin{aligned}
\tau(a^*a) &= \sum_{i,j=1}^{n} \tau(Z_i^* Z_j \otimes b_i^* b_j \otimes c_j c_i^*) \\
&= \sum_{i,j=1}^{n} \varphi(Z_i^* Z_j)\tau_B(b_i^* b_j c_j c_i^*) \\
&= \sum_{i,j=1}^{n} \varphi(Z_i^* Z_j)\tau_B(c_i^* b_i^* b_j c_j) \\
&= \sum_{i,j=1}^{n} \varphi(Z_i^* Z_j)\tau_B((b_i c_i)^*(b_j c_j)),
\end{aligned}$$

with the third equality being due to the fact that τ_B is tracial. Observe that the matrices

$$[Z_i^* Z_j] \qquad \text{and} \qquad [(b_i c_i)^*(b_j c_j)]$$

are positive in $M_n(\mathcal{A})$ and $M_n(B)$ respectively. Therefore, as states on C*-algebras are completely positive, this implies that the matrices

$$[\varphi(Z_i^* Z_j)] \qquad \text{and} \qquad [\tau_B((b_i c_i)^*(b_j c_j))]$$

are positive in $M_n(\mathbb{C})$. Consequently

$$[\varphi(Z_i^* Z_j)\tau_B((b_i c_i)^*(b_j c_j))]$$

is also positive being the Schur product of positive matrices (see, for instance, [61, Lemma 6.11]). Therefore, as the sum of all entries of a positive matrix equals a positive scalar, we obtain that $\tau(a^*a) \geq 0$. Hence $(A, E, \varepsilon, \tau)$ is an analytical B-B-non-commutative probability space.

Remark 4.2.5. Note Example 4.2.4 demonstrates E need not be a positive map on A since the product of two positive matrices need not be positive. Thus, even if $\tau_B : B \to \mathbb{C}$ is defined to be a state, $\tau_B \circ E$ may not be for an arbitrary A.

Example 4.2.6. For a finite von Neumann algebra $\mathfrak{M}$ with a unital von Neumann subalgebra B and tracial state τ, let (A, E, ε) be the B-B-non-commutative probability space as in Example 4.1.3. Note that τ extends to a unital linear map $\tau_A : A \to \mathbb{C}$ defined by

$$\tau_A(T) = \langle T1_{\mathfrak{M}}, 1_{\mathfrak{M}}\rangle_{L_2(\mathfrak{M},\tau)}$$

for all $T \in A$. Clearly τ_A is a state as $A \subseteq \mathcal{B}(L_2(\mathfrak{M}, \tau))$ and τ_A is a vector state. Furthermore, notice that

$$\tau_A(T) = \langle P(T1_{\mathfrak{M}}), 1_{\mathfrak{M}}\rangle_{L_2(\mathfrak{M},\tau)} = \langle L_{E(T)}1_{\mathfrak{M}}, 1_{\mathfrak{M}}\rangle_{L_2(\mathfrak{M},\tau)} = \tau_A(L_{E(T)})$$

for all $T \in A$ and $\tau_A(T) = \tau_A(R_{E(T)})$ by a similar computation. Finally as

$$\tau_A(L_b) = \langle b, 1_{\mathfrak{M}}\rangle_{L_2(\mathfrak{M},\tau)} = \tau(b)$$

for all $b \in B$, we see that $\tau_B = \tau_A \circ E$ is tracial on B as τ is tracial. Again, we automatically have that left multiplication will be bounded on $L_2(A, \tau)$ and that E is completely positive when restricted (as they are the conditional expectation of a copy of $\mathfrak{M}$ onto B). Hence $(A, E, \varepsilon, \tau_A)$ is an analytical B-B-non-commutative probability space.

As motivated by Example 4.2.6, it is natural in an analytical B-B-non-commutative probability space to extend the expectation $E : A \to B$ to a map from $L_2(A, \tau)$ to $L_2(B, \tau_B)$ via orthogonal projection. From this point onwards, for $a \in A$ we will often denote the coset $a + N_\tau$ simply by a and, for $b \in B$ we will often denote the coset $b + N_{\tau_b}$ by $\widehat{b}$. Note that if τ_B is faithful, then the map $b \mapsto \widehat{b}$ is a bijection.

Proposition 4.2.7. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space. If $\widetilde{E} : L_2(A, \tau) \to L_2(B, \tau_B)$ denotes the orthogonal projection, then*

$$\widetilde{E}(a) = \widehat{E(a)}$$

for all $a \in A$. In particular, when τ_B is faithful, $\widetilde{E}$ extends E.

Proof. Notice for all $a \in A$ and $b \in B$ that

$$\begin{aligned}
\left\langle a - \widehat{E(a)}, L_b \right\rangle_{L_2(B,\tau_B)} &= \langle L_{b^*}(a - L_{E(a)})1_A, 1_A \rangle_{L_2(A,\tau)} \\
&= \tau\left(L_{b^*}(a - L_{E(a)})\right) \\
&= \tau(L_{b^*}a) - \tau\left(L_{b*}L_{E(a)}\right) \\
&= \tau\left(L_{E(L_{b^*}a)}\right) - \tau\left(L_{b^*E(a)}\right) = 0.
\end{aligned}$$

Since b was arbitrary, the element $a - \widehat{E(a)}$ is orthogonal to $L_2(B, \tau_B)$ and hence $\widetilde{E}(a) = \widehat{E(a)}$. ☐

Remark 4.2.8. Notice in Proposition 4.2.7 that if B is finite-dimensional and the trace $\tau_B : B \to \mathbb{C}$ is faithful, then $L_2(B, \tau_B) \cong B$, so $E : A \to B$ extends to a map from $L_2(A, \tau)$ into B.

Of course $\widetilde{E}$ inherits many properties that E is required to have.

Proposition 4.2.9. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and let $\widetilde{E} : L_2(A, \tau) \to L_2(B, \tau_B)$ denote the orthogonal projection. For $a \in A$, $b, b_1, b_2 \in B$, $\xi, \xi_1, \xi_2 \in L_2(A, \tau)$, and $\zeta \in L_2(B, \tau_B)$, the following hold:*

(i) $\tau(\xi) = \tau_B\left(\widetilde{E}(\xi)\right)$,

(ii) $\widetilde{E}(aL_b) = \widetilde{E}(aR_b)$,

(iii) $\widetilde{E}(L_{b_1}R_{b_2}\xi) = b_1\widetilde{E}(\xi)b_2$,

(iv) if $a \in A_\ell$, *then* $\widetilde{E}(a\zeta) = E(a)\zeta$,

(v) if $a \in A_r$, *then* $\widetilde{E}(a\zeta) = \zeta E(a)$, *and*

(vi) if $\tau(L_b\xi_1) = \tau(L_b\xi_2)$ *for all* $b \in B$, *then* $\widetilde{E}(\xi_1) = \widetilde{E}(\xi_2)$.

(vii) if $\tau(R_b\xi_1) = \tau(R_b\xi_2)$ *for all* $b \in B$, *then* $\widetilde{E}(\xi_1) = \widetilde{E}(\xi_2)$.

Proof. For (i), since $L_{1_B} = 1_A$ as ε is unital, note that

$$\tau_B\left(\widetilde{E}(\xi)\right) = \left\langle \widetilde{E}(\xi), 1_B\right\rangle_{L_2(B,\tau_B)} = \left\langle \xi, \widetilde{E}(1_B)\right\rangle_{L_2(A,\tau)} = \langle \xi, 1_A\rangle_{L_2(A,\tau)} = \tau(\xi),$$

as desired.

For (ii), note for all $b_0 \in B$ that

$$\left\langle \widetilde{E}(aL_b), \widehat{b_0}\right\rangle_{L_2(B,\tau_B)} = \langle aL_b, L_{b_0}\rangle_{L_2(A,\tau)} = \tau\left(L_{b_0^*}aL_b\right) = \tau\left(L_{b_0^*}aR_b\right) = \left\langle \widetilde{E}(aR_b), \widehat{b_0}\right\rangle_{L_2(B,\tau_B)}.$$

Hence $\widetilde{E}(aL_b) = \widetilde{E}(aR_b)$.

For (iii), let $(a_n)_{n\geq 1}$ be a sequence of elements of A that converge to ξ in $L_2(A,\tau)$. Since left multiplication in $L_2(A,\tau)$ by elements of A are bounded and thus continuous, and since left and right multiplication in $L_2(B,\tau_B)$ by elements of B are bounded and thus continuous, we obtain that

$$\widetilde{E}(L_{b_1}R_{b_2}\xi) = \lim_{n\to\infty} E(L_{b_1}R_{b_2}a_n) + N_{\tau_B} = \lim_{n\to\infty} b_1E(a_n)b_2 + N_{\tau_B} = b_1\widetilde{E}(\xi)b_2$$

as desired.

For (iv) and (v), let $(c_n)_{n\geq 1}$ be a sequence of elements of B that converge to ζ in $L_2(B,\tau_B)$. Thus, by the inclusion of $L_2(B,\tau_B)$ into $L_2(A,\tau)$, we have that $(L_{c_n})_{n\geq 1}$ is a sequence of elements of $L_2(A, \tau)$ that converge to ζ in $L_2(A, \tau)$. Thus, if $a \in A_\ell$, then

$$\begin{aligned}\widetilde{E}(a\zeta) &= \lim_{n\to\infty} E(aL_{c_n}) + N_{\tau_B} \\ &= \lim_{n\to\infty} E(aR_{c_n}) + N_{\tau_B} \\ &= \lim_{n\to\infty} E(R_{c_n}a) + N_{\tau_B} \\ &= \lim_{n\to\infty} E(a)c_n + N_{\tau_B} = E(a)\zeta\end{aligned}$$

thereby proving (iv). Note (v) is similar using $(R_{c_n})_{n\geq 1}$ in place of $(L_{c_n})_{n\geq 1}$.

As (vi) and (vii) are similar, we prove (vii). Note by (iii) and the fact that τ_B is tracial that

$$\begin{aligned}\left\langle \widetilde{E}(\xi_1) - \widetilde{E}(\xi_2), \widehat{b}\right\rangle_{L_2(B,\tau_B)} &= \left\langle \widetilde{E}(\xi_1)b^*, \widetilde{E}(1_A)\right\rangle_{L_2(B,\tau_B)} - \left\langle \widetilde{E}(\xi_2)b^*, \widetilde{E}(1_A)\right\rangle_{L_2(B,\tau_B)} \\ &= \left\langle \widetilde{E}(R_{b^*}\xi_1), \widetilde{E}(1_A)\right\rangle_{L_2(B,\tau_B)} - \left\langle \widetilde{E}(R_{b^*}\xi_2), \widetilde{E}(1_A)\right\rangle_{L_2(B,\tau_B)} \\ &= \langle R_{b^*}\xi_1, 1_A\rangle_{L_2(A,\tau)} - \langle R_{b^*}\xi_2, 1_A\rangle_{L_2(A,\tau)} \\ &= \tau(R_{b^*}\xi_1) - \tau(R_{b^*}\xi_2) = 0\end{aligned}$$

As the above holds for all $b \in B$, (vii) follows. □

4.3 Analytical Bi-Multiplicative Functions

In this section, we extend the notion of bi-multiplicative functions on analytical B-B-non-commutative probability spaces in order to permit the last entry to be an element of $L_2(A, \tau)$. This is possible as the last entry can be treated as a left or right operator as [68] shows, or can be treated as a mixture of left and right operators as [14] shows. Extending the operator-valued bi-free cumulant function to permit the last entry to be an element of $L_2(A, \tau)$ is necessary in order to permit the simple development of conjugate variable systems in the next section.

We advise the reader that familiarity with specifics of bi-multiplicative functions, the construction of the operator-valued bi-free moment function, and the construction of the operator-valued bi-free cumulant function from [10] would be of great aid in comprehension of this section. As the proofs are nearly identical, to avoid clutter we will focus on that which is different and why the results of [10] extend.

Definition 4.3.1. Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and let Φ be a bi-multiplicative function on (A, E, ε). A function

$$\widetilde{\Phi} : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \ldots \times A_{\chi(n-1)} \times L_2(A,\tau) \to L_2(B, \tau_B)$$

is said to be *analytical extension of* Φ if $\widetilde{\Phi}_\pi$ is $\mathbb{C}$-multilinear function that does not change values if the last entry of χ is changed from an ℓ to an r and satisfies the following three properties: For all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A, \tau)$, $\zeta \in L_2(B, \tau_B)$, $b \in B$, and $Z_k \in A_{\chi(k)}$:

(i) If $\chi(k) = \ell$ for all $k \in \{1, 2, \ldots, n\}$ then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n\zeta) = \Phi_{1_\chi}(Z_1, \ldots, Z_n)\zeta,$$

and if $\chi(k) = r$ for all $k \in \{1, 2, \ldots, n\}$, then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n\zeta) = \zeta\Phi_{1_\chi}(Z_1, \ldots, Z_n).$$

In particular, by setting $\zeta = 1_B = 1_A$, we see $\widetilde{\Phi}$ does extend Φ.

(ii) Let $p \in \{1, \ldots, n\}$ and let

$$q = \max_{\leq} \{k \in \{1, \ldots, n\} \mid \chi(k) = \chi(p), k < p\}.$$

If $\chi(p) = \ell$, then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_bZ_p, \ldots, \xi) = \begin{cases} \widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_qL_b, \ldots, Z_{n-1}, \xi) & \text{if } q \neq -\infty, \\ b\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, \xi) & \text{if } q = -\infty, \end{cases}$$

and if $\chi(p) = r$, then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{p-1}, R_bZ_p, \ldots, \xi) = \begin{cases} \widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_qR_b, \ldots, Z_{n-1}, \xi) & \text{if } q \neq -\infty, \\ \widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, \xi)b & \text{if } q = -\infty. \end{cases}$$

(iii) Suppose $V_1, \ldots, V_m$ are unions of blocks of π that partition $\{1, \ldots, n\}$, with each being a χ-interval. Moreover, assume that the sets $V_1, \ldots, V_m$ are ordered by $\preceq_\chi$ (i.e.

$(\min_{\preceq_\chi} V_k) \prec_\chi (\min_{\preceq_\chi} V_{k+1}))$. Let $q \in \{1, \ldots, m\}$ be such that $n \in V_q$ and for each $k \neq q$ let

$$b_k = \Phi_{\pi|_{V_k}}\left((Z_1, \ldots, Z_{n-1}, \xi)|_{V_k} \right).$$

Then $b_k \in B$ for $k \neq q$ and

$$\widetilde{\Phi}_\pi(Z_1, \ldots, Z_{n-1}, \xi) = b_1 b_2 \cdots b_{q-1} \widetilde{\Phi}_{\pi|_{V_i}}((Z_1, \ldots, Z_{n-1}, \xi)|_{V_q}) b_{q+1} \cdots b_m.$$

(iv) Suppose that V and W are unions of blocks of π that partition $\{1, \ldots, n\}$ such that V is a χ-interval and $s_\chi(1), s_\chi(n) \in W$. Let

$$p = \max_{\preceq_\chi} \left\{ k \in W \,\middle|\, k \preceq_\chi \min_{\preceq_\chi} V \right\} \qquad \text{and} \qquad q = \min_{\preceq_\chi} \left\{ k \in W \,\middle|\, \max_{\preceq_\chi} V \preceq_\chi k \right\}.$$

Then one of the following cases holds:

a) If $n \in V$ and $k = \max_{\leq} W$, then

$$\widetilde{\Phi}_\pi(Z_1, \ldots, \xi) = \widetilde{\Phi}_{\pi|_W}\left((Z_1, \ldots, Z_{k-1}, Z_k \widetilde{\Phi}_{\pi|_V}((Z_1, \ldots, Z_{n-1}, \xi)|_V), \ldots, Z_{n-1}, \xi)|_W \right).$$

b) If $n \in W$ then

$$\begin{aligned}
\widetilde{\Phi}_\pi(Z_1, \ldots, Z_n) &= \begin{cases} \widetilde{\Phi}_{\pi|_W}\left((Z_1, \ldots, Z_{p-1}, Z_p L_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)}, \ldots, Z_n)|_W \right) & \text{if } \chi(p) = \ell, \\ \widetilde{\Phi}_{\pi|_W}\left((Z_1, \ldots, Z_{p-1}, R_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)} Z_p, \ldots, Z_n)|_W \right) & \text{if } \chi(p) = r, \end{cases} \\
&= \begin{cases} \widetilde{\Phi}_{\pi|_W}\left((Z_1, \ldots, Z_{q-1}, L_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)} Z_q, \ldots, Z_n)|_W \right) & \text{if } \chi(q) = \ell, \\ \widetilde{\Phi}_{\pi|_W}\left((Z_1, \ldots, Z_{q-1}, Z_q R_{\Phi_{\pi|_V}((Z_1,\ldots,Z_n)|_V)}, \ldots, Z_n)|_W \right) & \text{if } \chi(q) = r. \end{cases}
\end{aligned}$$

(Recall we can set $\chi(n) = \ell$ or $\chi(n) = r$.)

Remark 4.3.2. Note that the pair of a bi-multiplicative function and its extension are very reminiscent of the two expectation extensions of bi-multiplicative functions used for operator-valued conditional bi-free independence from [36]. The main difference is that the notion in [36] looks at interior versus exterior blocks of the partition whereas Definition 4.3.1 looks at the blocks containing the last entry. This is due to the fact that the $L_2(A, \tau)$ element is always the last entry and must be treated differently being a generalization of a mixture of left and right operators.

It is worth pointing out that treating the last entry as an element of $L_2(A, \tau)$ is no issue. In particular, the properties in Definition 4.3.1 are well-defined. Indeed, properties (i) and (ii) of Definition 4.3.1 are clearly well-defined and properties (iii) and (iv) in Definition 4.3.1 are well-defined as all terms where Φ is used over $\widetilde{\Phi}$ never involve an element of $L_2(A, \tau)$ and as elements from B have left and right actions on $L_2(B, \tau_B)$.

Remark 4.3.3. Note that property (i) of Definition 4.3.1 is clearly the correct generalization of property (i) from Definition 4.1.7, as an element of $L_2(B, \tau_B)$ is playing the role of L_b and R_b in this generalization and thus should be able to escape these expressions if only left operators or right operators are present. The absence of the full property (i) from Definition 4.1.7 causes no issues when attempting to reduce or rearrange the value of $\widetilde{\Phi}_\pi$ to an expression involving only $\widetilde{\Phi}_{1_\chi}$'s, as the last entry of any sequence input into $\widetilde{\Phi}$ is always in $L_2(A, \tau)$ which is then reduced to an element of $L_2(B, \tau)$ and an element of A_ℓ or A_r then acts on it via the left action of A on $L_2(A, \tau)$. Thus there is never any need to move the $L_2(A, \tau)$ entry to another position.

If property (i) is ever used, we note that if $\zeta \in L_2(B, \tau_B)$ is viewed as an element of $L_2(A, \tau)$, then $L_b\zeta$ is simply the element $b\zeta \in L_2(B, \tau_B)$ and $R_b\zeta$ is simply the element $\zeta b \in L_2(B, \tau_B)$. Thus, using (i) does not pose problems when trying to "move around L_b and R_b elements" in proofs when trying to show the equivalence of any reductions as the following example demonstrates.

Example 4.3.4. Let $\chi \in \{\ell, r\}^8$ be such that $\chi^{-1}(\{\ell\}) = \{5, 6\}$, let $\xi \in L_2(A, \tau)$, let $Z_k \in A_{\chi(k)}$, and let $\pi \in \mathrm{BNC}(\chi)$ be the partition

$$\pi = \{\{1, 2\}, \{3, 5\}, \{4, 7\}, \{6, 8\}\}.$$

Note the bi-non-crossing diagram of π can be represented as the following (with the convention now that the last entry is at the bottom instead of on its respective side):

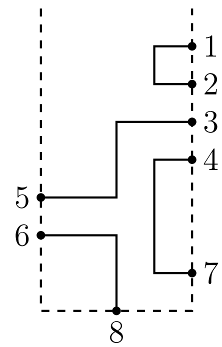

Figure 4.1: Bi-non-crossing diagram

When reducing $\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, \xi)$, we can clearly use property (iii) of Definition 4.3.1 first to obtain with $U = \{3, 4, \ldots, 8\}$ that

$$\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, \xi) = \widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi)\Phi_{1_{(r,r)}}(Z_1, Z_2).$$

To reduce the expression fully, we have to simply reduce $\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi)$ using property (iv) of Definition 4.3.1 of which there are three ways to do so.

The first way to reduce is to use $V = \{4, 6, 7, 8\}$ and $W = \{3, 5\}$. By applying property (iv) of Definition 4.3.1 we obtain that

$$\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) = \widetilde{\Phi}_{\pi|_W}\left(Z_3, Z_5\widetilde{\Phi}_{\pi|_V}(Z_4, Z_6, Z_7, \xi)\right).$$

Finally, by applying property (iii) of Definition 4.3.1 to $\widetilde{\Phi}_{\pi|_V}(Z_4, Z_6, Z_7, \xi)$ we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\Phi_{1_{(r,r)}}(Z_4, Z_7)\right)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right).\end{aligned}$$

The second way to reduce is to use $V = \{6, 8\}$ and $W = \{3, 4, 5, 7\}$. By applying property (iv) of Definition 4.3.1 we obtain that

$$\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) = \widetilde{\Phi}_{\pi|_W}\left(Z_3, Z_4, Z_5, Z_7\widetilde{\Phi}_{\pi|_V}(Z_6, \xi)\right).$$

By applying property (iv) of Definition 4.3.1 again as $\{4, 7\}$ is now a $\chi|_W$-interval), we obtain

that

$$\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) = \widetilde{\Phi}_{1_{r,\ell}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(r,r)}}(Z_4, Z_7\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right)\right).$$

However, as $Z_4, Z_7 \in A_r$, we obtain by property (i) that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\Phi_{1_{(r,r)}}(Z_4, Z_7)\right)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right),\end{aligned}$$

thereby agreeing with the above expression.

The third way to reduce is to use $V = \{4, 7\}$ and $W = \{3, 5, 6, 8\}$. By applying property (iv) of Definition 4.3.1 we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{\pi|_W}\left(Z_3, Z_5, Z_6, R_{\Phi_{\pi|_V}(Z_4,Z_7)}\xi\right) \\ &= \widetilde{\Phi}_{\pi|_W}\left(Z_3 R_{\Phi_{\pi|_V}(Z_4,Z_7)}, Z_5, Z_6, \xi\right),\end{aligned}$$

by using the two expressions in property (iv). Using either expression, we will now again property (iv) of Definition 4.3.1 as $\{6, 8\}$ is now $\chi|_W$-interval. For the first, we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\widetilde{\Phi}_{1_{(\ell,\ell)}}\left(Z_6, R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\xi\right)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\Phi_{1_{(r,r)}}(Z_4, Z_7)\right)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right)\end{aligned}$$

where the second equality follows from applying property (ii) of Definition 4.3.1, as $Z_6 \in A_\ell$. For the second expression, we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}, Z_5\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} Z_5\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right),\end{aligned}$$

where the second equality follows from applying property (ii) of Definition 4.3.1 as the last entry is now the $L_2(A, \tau)$ entry, and the third equality holds as $Z_5 \in A_\ell$ and thus commutes with R_b.

Hence Definition 4.3.1 is consistent in this example (and will be in all examples due to similar computations).

Using similar reductions for arbitrary expressions, one can prove the following.

Lemma 4.3.5. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let Φ be a bi-multiplicative function on (A, E, ε), and let $\widetilde{\Phi}$ be an analytic extension of Φ. Then, properties (i) and (ii) of Definition 4.3.1 hold when 1_χ is replaced with any $\pi \in \mathrm{BNC}(\chi)$.*

Proof. The proof is essentially the same as the proof that properties (i) and (ii) of Definition 4.1.7 hold for Φ when 1_χ is replaced with any $\pi \in \mathrm{BNC}(\chi)$ as in [10, Proposition 4.2.5]. To see that property (i) of Definition 4.3.1 extends, note when using (iii) and (iv) to reduce the expression for $\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_{n-1}, Z_n\zeta)$ that one is effectively using the bi-multiplicative properties of Φ and including ζ in the appropriate spot. To see that property (ii) of Definition 4.3.1 extends, indices that are always adjacent in the χ-ordering will remain in the correct ordering so that when L_b or R_b operators are considered, we can always move them outside the Φ- and $\widetilde{\Phi}$-expressions on the correct side to move them to the next operator (that is, things will always move around as they do in the free multiplicative functions from [71] after reordering by the χ-order). For example, in Example 4.3.4, we showed that

$$\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, \xi) = \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2).$$

If Z_3 were replaced with $R_b Z_3$, we would have

$$\begin{aligned}
\widetilde{\Phi}_\pi(Z_1, Z_2, R_b Z_3, Z_4 \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(R_b Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) b\Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2 R_b) \\
&= \widetilde{\Phi}_\pi(Z_1, Z_2 R_b, Z_3, Z_4 \ldots, Z_7, \xi).
\end{aligned}$$

If ξ were replaced with $L_b\xi$, then clearly the L_b can be moved to give $Z_6 L_b$ via (ii) with a 1_χ

as the expression $\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)$ is present. If ξ were replaced with $R_b\xi$, then

$$\begin{aligned}
\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, R_b\xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, R_b\xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi) b\right)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} R_b \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{b\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7 R_b)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_\pi(Z_1, \ldots, Z_6, Z_7 R_b, \xi),
\end{aligned}$$

as desired. Thus the result follows. $\square$

4.3.1 The Analytical Operator-Valued Bi-Moment Function

We will now construct the analytical extension of the operator-valued bi-moment function via recursion and the map $\widetilde{E} : L_2(A, \tau) \to L_2(B, \tau_B)$ from Section 4.2. Note that the recursive process in the following definition is different than that from [10, Definition 5.1] and [36, Definition 4.4], in order to facilitate the introduction of the $L_2(A, \tau)$ element. The same recursive process could have been used in [10, Definition 5.1] and [36, Definition 4.4], as these processes are equivalent in those settings. Note we use Ψ in the following to avoid confusion with $\widetilde{E}$ in Section 4.2, although Ψ is a multi-entry extension of $\widetilde{E}$.

Definition 4.3.6. Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space. The *analytical bi-moment function*

$$\Psi : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \ldots \times A_{\chi(n-1)} \times L_2(A, \tau) \to L_2(B, \tau_B)$$

is defined recursively as follows: Let $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A, \tau)$, and $Z_k \in A_{\chi(k)}$.

- If $\pi = 1_\chi$, then
$$\Psi_{1_\chi}(Z_1, Z_2, \ldots, Z_{n-1}, \xi) = \widetilde{E}(Z_1 Z_2 \cdots Z_{n-1}\xi).$$
- If $\pi \neq 1_\chi$, let V be the block in π such that $n \in V$. We divide discussion into two cases:

– Suppose that $\min_{\preceq_\chi} V = s_\chi(1)$ and $\max_{\preceq_\chi} V = s_\chi(n)$ and let

$$p = \min_{\preceq_\chi}\{i \in \{1, \ldots, n\} \mid i \notin V\}, \quad q = \min_{\preceq_\chi}\{j \in V \mid p \prec_\chi j\}$$

$$\text{and } m = \max_{\preceq_\chi}\{i \in V \mid i \prec_\chi p\}.$$

Set

$$W = \{i \in \{1, \ldots, n\} \mid p \preceq_\chi i \prec_\chi q\}.$$

Note by construction and the fact that $\pi \in \mathrm{BNC}(\chi)$ that W is equal to a union of blocks of π and $\chi(p) = \chi(j)$ for all $j \in W$. Thus, if $\chi(p) = \ell$ we define

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi) = \Psi_{\pi|_{W^c}}\left((Z_1, \ldots, Z_{p-1}, Z_m L_{E_{\pi|_W}((Z_1,\ldots,\xi)|_W)}, \ldots, Z_{n-1}, \xi)|_{W^c}\right)$$

and in the case when $\chi(p) = r$ we define

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi) = \Psi_{\pi|_{W^c}}\left((Z_1, \ldots, Z_{q-1}, Z_q R_{E_{\pi|_W}((Z_1,\ldots,\xi)|_W)}, \ldots, Z_{n-1}, \xi)|_{W^c}\right).$$

Since $n \notin W$, the quantity $E_{\pi|_W}((Z_1, \ldots, Z_{n-1}, \xi)|_W)$ is always a well-defined element of B in this case. Note that if $\chi(p) = \ell$, then $m \prec_\chi p \prec_\chi n$ and thus $Z_m \neq \xi$, so $\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi)$ is well-defined. Also, in the case when $\chi(p) = r$ observe that $n \prec_\chi q$ and thus $Z_q \neq \xi$, so $\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi)$ is well-defined.

– Otherwise, set

$$\widetilde{V} = \left\{i \in \{1, \ldots, n\} \;\middle|\; \min_{\preceq_\chi} V \preceq_\chi i \preceq_\chi \max_{\preceq_\chi} V\right\}.$$

Note $\widetilde{V}$ is a proper subset of $\{1, \ldots, n\}$ that is a union of blocks of π and is such that $n \in V \subseteq \widetilde{V}$. For $q = \max_{\leq} \widetilde{V}^c$ and define

$$\Psi_\pi(Z_1, \ldots, \xi) = \Psi_{\pi|_{\widetilde{V}^c}}\left((Z_1, \ldots, Z_{q-1}, Z_q \Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, \xi)|_{\widetilde{V}}), \ldots, Z_{n-1}, \xi)|_{\widetilde{V}^c}\right).$$

Note that the quantity $\Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, Z_{n-1}, \xi)|_{\widetilde{V}})$ is a well-defined element of $L_2(B, \tau_B)$ due to the recursive nature of our definition. Moreover,the last element of the sequence

$$(Z_1, \ldots, Z_{q-1}, Z_q \Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, Z_{n-1}, \xi)|_{\widetilde{V}}), \ldots, Z_{n-1}, \xi)|_{\widetilde{V}^c}$$

is equal to $Z_q\Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, Z_{n-1}, \xi)|_{\widetilde{V}})$, which is an element of $L_2(A, \tau)$ and therefore $\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi)$ is well-defined.

To aid in the comprehension of Definition 4.3.6, we provide an example using of bi-non-crossing diagrams to show the recursive construction. We note that ξ will always appear last in a sequence of operators and is an element of $L_2(A, \tau)$ and thus neither a left nor right operator. As such, we treat it as neither. This is reminiscent of [68, Lemma 2.17] where it was demonstrated that it does not matter whether we treat the last operator in a list as a left or as a right operator, and of [14, Lemma 2.29 and Proposition 2.30] where the last entry can be a mixture of left and right operators.

Example 4.3.7. Let $\chi \in \{\ell, r\}^{12}$ be such that $\chi^{-1}(\{\ell\}) = \{1, 5, 8, 9, 11, 12\}$, let $\xi \in L_2(A, \tau)$, let $Z_k \in A_{\chi(k)}$, and let $\pi \in \mathrm{BNC}(\chi)$ be the partition with blocks

$$V_1 = \{1, 3\}, \quad V_2 = \{2\}, \quad V_3 = \{4, 5, 11, 12\}, \quad V_4 = \{6, 10\}, \quad V_5 = \{7\}, \text{ and } V_6 = \{8, 9\}.$$

To compute $\Psi_\pi(Z_1, \ldots, Z_{11}, \xi)$, we note the second part of the second step of the recursive definition from Definition 4.3.6 applies first. In particular

$$\widetilde{V} = \bigcup_{k=3}^{6} V_k.$$

Thus, if

$$X = \Psi_{\pi|_{\widetilde{V}}}(Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, \xi),$$

then

$$\Psi_\pi(Z_1, \ldots, Z_{11}, \xi) = \Psi_{\pi|_{\widetilde{V}^c}}(Z_1, Z_2, Z_3 X).$$

Diagrammatically, this first reduction is seen as follows:

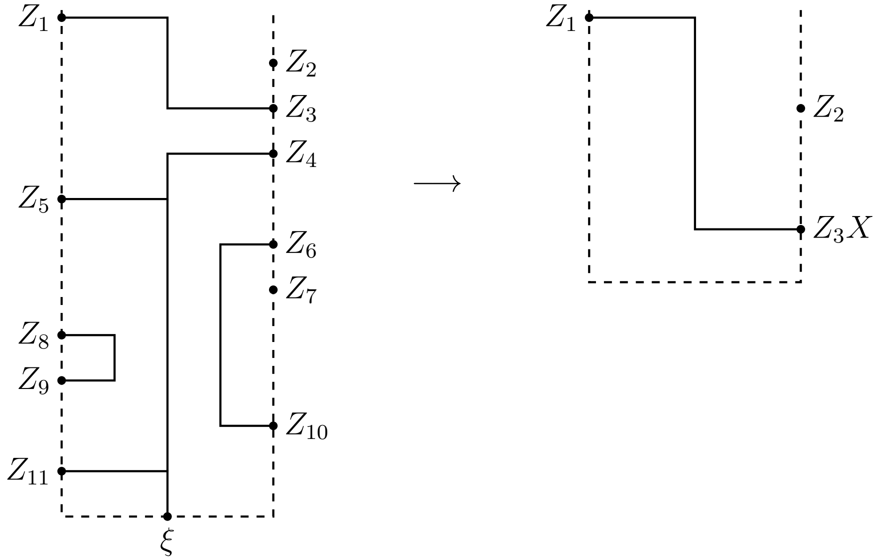

Figure 4.2: Bi-non-crossing reduction 1

Note

$$\Psi_{\pi|_{\widetilde{V}^c}}(Z_1, Z_2, Z_3X) = \Psi_{\pi|_{V_2}}(Z_2\Psi_{\pi|_{V_1}}(Z_1, Z_3X)) = \widetilde{E}\left(Z_2\widetilde{E}(Z_1Z_3X)\right),$$

where the first equality holds by the same recursive idea, whereas the second equality holds by the first step of Definition 4.3.6.

When computing the value of X, the minimal and maximal elements of $\{4, 5, \ldots, 11, 12\}$ in the $\chi|_{\widetilde{V}}$-order are 5 and 4 respectively and the block that contains the index corresponding to ξ contains both 5 and 4. Thus the first part of the second step of Definition 4.3.6 should be used. The algorithm in Definition 4.3.6 then calculates the value of X by “stripping out" the χ-intervals V_6 and $V_4 \cup V_5$ successively and this is seen via the following two diagrammatic reductions:

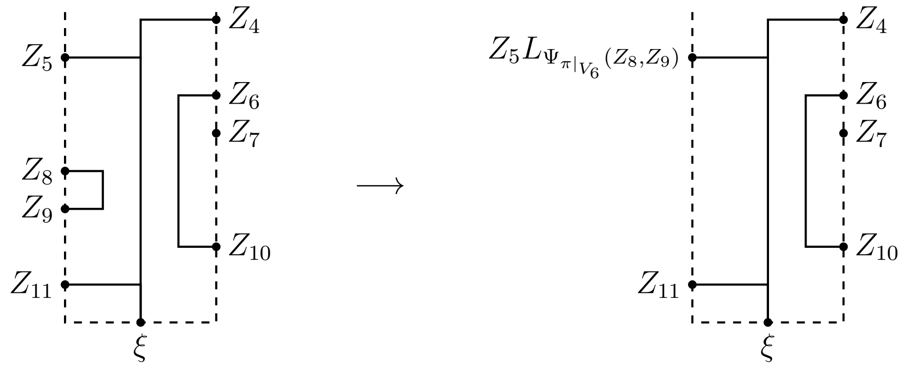

Figure 4.3: Bi-non-crossing reduction 2

and

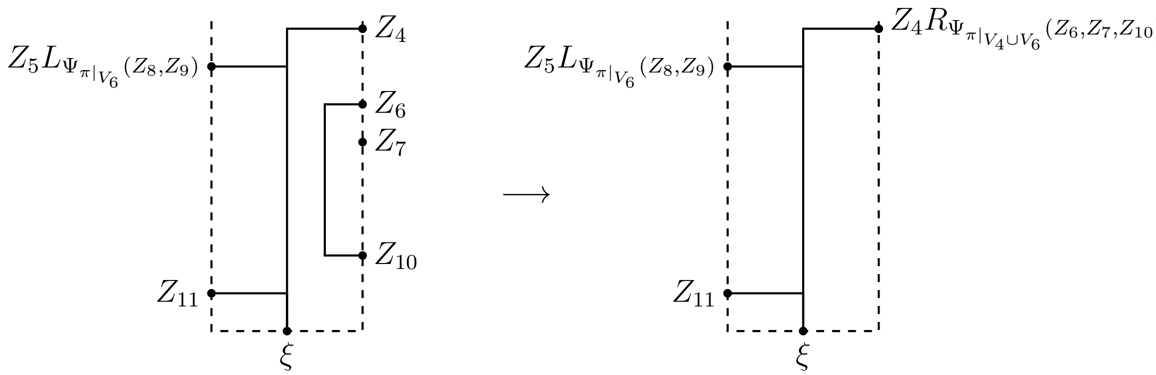

Figure 4.4: Bi-non-crossing reduction 3

It is readily verified using the fact that the operator-valued bi-free moment function is bi-multiplicative that $E_{\pi|_{V_6}}(Z_8, Z_9) = E(Z_8 Z_9)$ and $E_{\pi|_{V_4 \cup V_6}}(Z_6, Z_7, Z_{10}) = E(Z_6 R_{E(Z_7)} Z_{10})$. Thus, using the fact that $\pi|_{V_3} = 1_{\chi|_{V_3}}$, the first step in Definition 4.3.6 yields

$$X = \widetilde{E}\left(Z_4 R_{E(Z_6 R_{E(Z_7)} Z_{10})} Z_5 L_{E(Z_8 Z_9)} Z_{11} \xi\right).$$

Hence

$$\begin{aligned}\Psi_\pi(Z_1, \ldots, Z_{11}, \xi) &= \widetilde{E}\left(Z_2 \widetilde{E}\left(Z_1 Z_3 \widetilde{E}\left(Z_4 R_{E\left(Z_6 R_{E(Z_7)} Z_{10}\right)} Z_5 L_{E(Z_8 Z_9)} Z_{11} \xi\right)\right)\right) \\ &= \widetilde{E}\left(Z_1 Z_3 \widetilde{E}\left(Z_4 R_{E\left(Z_6 R_{E(Z_7)} Z_{10}\right)} Z_5 L_{E(Z_8 Z_9)} Z_{11} \xi\right)\right) E(Z_2),\end{aligned}$$

with the last equality following from Proposition 4.2.9.

Before investigating the bi-multiplicative properties inherited by the analytical bi-moment function, we note it is truly an extension of the operator-valued bi-moment function.

Theorem 4.3.8. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space. For any $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, and $Z_k \in A_{\chi(k)}$,*

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) = E_\pi(Z_1, \ldots, Z_{n-1}, Z_n) + N_{\tau_B}.$$

Proof. Note that each step in the recursive definition of Definition 4.3.6 is a step that can be performed to the operator-valued bi-free moment function as the operator-valued bi-free moment function is bi-multiplicative (see Definition 4.1.7). Therefore, as Proposition 4.2.7 implies that

$$\widetilde{E}(a) = E(a) + N_{\tau_B}$$

for all $a \in A$, by applying the same recusive properties to $E_\pi(Z_1, \ldots, Z_{n-1}, Z_n)$ as used to compute $\Psi_\pi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau)$, the result follows. □

Like with the construction of the operator-valued bi-free moment function in [10], although the construction of the analytical bi-moment function is done using specific rules from the operator-valued bi-free moment function in a specific order, we desire more flexibility in the reductions that can be done and the order they can be done in. In particular, we desire to show that the analytical bi-moment function is an analytic extension of the operator-valued bi-free moment function.

The main ideas used to prove this are similar to those utilized in the proof of [10, Theorem 5.1.4] and hence we shall be concerned with demonstrating that the inclusion of the $L_2(A, \tau)$ term and the slightly modified recursive definition are not issue and pose next to no changes. In particular, it may appear that Ψ behaves differently than the operator-valued bi-free moment function as entries in $L_2(A, \tau)$ can also act as mixtures of left and right operators, which was not dealt with in [10]. However, using the properties of $\widetilde{E}$ as developed in

Proposition 4.2.9, one familiar with [10] can easily see that the desired results will hold with simple adaptations. We note that similar adaptations were done in [36] without issue.

When examining the proof of the following, Example 4.3.7 serves as a good example to keep in mind, just as Example 4.3.4 aided in comprehending why analytic extensions of bi-multiplicative functions work.

Theorem 4.3.9. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and let*

$$\Psi : \bigcup_{n \in \mathbb{N}} \bigcup_{\chi \in \{\ell, r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \ldots \times A_{\chi(n-1)} \times L_2(A, \tau) \to L_2(B, \tau_B)$$

be the analytical bi-moment function. Then Ψ is an analytically extension of the operator-valued bi-free moment function.

Clearly the map Ψ_{1_χ} is $\mathbb{C}$-multilinear and it does not matter whether $\chi(n) = \ell$ or $\chi(n) = r$. A straightforward induction argument using the definition of Ψ shows that the map Ψ_π will be $\mathbb{C}$-multilinear. Thus we focus on the remaining four properties.

Proof of Theorem 4.3.9 property (i). This immediately follows from parts (ii) and (iii) of Proposition 4.2.9. □

Proof of Theorem 4.3.9 property (ii). To see (ii), fix $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A, \tau)$, $b \in B$, and $Z_k \in A_{\chi(k)}$, and let p and q be as in the statement of (ii). In the case that $\chi(p) = \ell$, note that

$$\Psi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) = \widetilde{E}(Z_1 \cdots Z_{p-1} L_b Z_p Z_{p+1} \cdots Z_{n-1} \xi).$$

If $q \neq \infty$, then $Z_{q+1}, \ldots, Z_{p-1} \in A_r$ and thus commute with L_b. Hence

$$\begin{aligned}\Psi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) &= \widetilde{E}(Z_1 \cdots Z_{q-1} Z_q L_b Z_{q+1} \cdots Z_{n-1} \xi) \\ &= \Psi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q L_b, Z_{q+1}, \ldots, Z_{n-1}, \xi)\end{aligned}$$

(and note $Z_q L_b \in A_\ell$).

If $q = \infty$, then $Z_1, \ldots, Z_{p-1} \in A_r$ and thus commute with L_b. Hence

$$\begin{aligned}\Psi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) &= \widetilde{E}(L_b Z_1 \cdots Z_{n-1}\xi) \\ &= b\widetilde{E}(Z_1 \cdots Z_{n-1}\xi) \\ &= b\Psi_{1_\chi}(Z_1, \ldots, Z_{n-1}, \xi)\end{aligned}$$

by Proposition 4.2.9. The case $\chi(p) = r$ is similar. □

Proof of Theorem 4.3.9 property (iii). To highlight how the proof works, we begin with the case that π consists of exactly three blocks that are χ-intervals and n is contained in the middle block under the χ-ordering. Suppose that $\pi = \{V_1, V_2, V_3\}$ and thus there exists $i \in \{2, \ldots, n-1\}$ and $j \in \{0, \ldots, n-2\}$ such that

$$\begin{aligned} V_1 &= \{s_\chi(1), s_\chi(2), \ldots, s_\chi(i-1)\}, \\ V_2 &= \{s_\chi(i), s_\chi(i+1), \ldots, s_\chi(i+j)\}, \text{ and} \\ V_3 &= \{s_\chi(i+j+1), s_\chi(i+j+2), \ldots, s_\chi(n)\}. \end{aligned}$$

Thus $n = s_\chi(k)$ for some $i \leq k \leq i+j$. This implies that $\chi(p) = \ell$ for all $p \in V_1$ and $\chi(p) = r$ for all $p \in V_3$. If $W = V_1 \cup V_3$, observe that the definition of the permutation s_χ yields that

$$q = \max_{\preceq} W = \max_{\preceq}\{s_\chi(i-1), s_\chi(i+j+1)\}.$$

Consider the case $q = s_\chi(i-1)$ so that $q \in V_1$, and let $m = s_\chi(i+j+1)$. Notice that if

$$X = \Psi_{\pi|_{V_2}}((Z_1, \ldots, Z_{n-1}, \xi)|_{V_2}),$$

then by Definition 4.3.6 we obtain that

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi) = \Psi_{\pi|_W}((Z_1, \ldots, Z_{q-1}, Z_q X, \ldots, Z_{n-1}, \xi)|_W).$$

Thus, if

$$Y = \Psi_{\pi|_{V_1}}((Z_1, \ldots, Z_{q-1}, Z_q X, \ldots, Z_{n-1}, \xi)|_{V_1}),$$

then again Definition 4.3.6 implies that

$$\begin{aligned}\Psi_\pi(Z_1,\dots,Z_{n-1},\xi) &= \Psi_{\pi|_{V_3}}((Z_1,\dots,Z_{m-1},Z_mY,\dots,Z_{n-1},\xi)|_{V_3}) \\ &= \widetilde{E}\left(Z_{s_\chi(n)}Z_{s_\chi(n-1)}\dots Z_{s_\chi(i+j+1)}Y\right).\end{aligned}$$

Therefore, since $Z_{s_\chi(n)}Z_{s_\chi(n-1)}\dots Z_{s_\chi(i+j+1)} \in A_r$, Proposition 4.2.9 implies that

$$\Psi_\pi(Z_1,\dots,Z_{n-1},\xi) = YE\left(Z_{s_\chi(n)}Z_{s_\chi(n-1)}\dots Z_{s_\chi(i+j+1)}\right) = YE_{\pi|_{V_3}}((Z_1,\dots,Z_{n-1},\xi)|_{V_3}).$$

Since $Z_{s_\chi(1)}, Z_{s_\chi(2)},\dots,Z_{s_\chi(i-1)} \in A_r$, Proposition 4.2.9 implies that

$$Y = E(Z_{s_\chi(1)}Z_{s_\chi(2)}\dots Z_{s_\chi(i-1)})X = E_{\pi|_{V_1}}((Z_1,\dots,Z_{n-1},\xi)|_{V_1})X,$$

so the result follows. Note the case $q = s_\chi(i+j+1)$ is handled similarly by interchanging the orders of the χ- intervals V_1 and V_3 . This argument can be extended via induction to any bi-non-crossing partition π all of whose blocks are χ-interval.

By the same argument as [10, Lemma 5.2.1], one need only consider the case in property (iii) that for each χ-interval, the χ-maximal and χ-minimal elements belong to the same block. When using the recursive procedure in Definition 4.3.6 to reduce Ψ_π, one of the χ-intervals (which will either be entirely on the left or entirely on the right) will have the $L_2(A,\tau)$ term added to the last entry as above. This $L_2(A,\tau)$ entry can be pulled out on the appropriate side leaving only the bi-moment function expression for the χ-interval, which can be undone as usual. By repetition, eventually all that remains is the expression for the χ-interval containing n as desired. □

Proof of Theorem 4.3.9 property (iv). The proof that property (iv) holds for the operator-valued bi-free moment function is one of the longest of [10] consisting of [10, Lemma 5.3.1], [10, Lemma 5.3.2], [10, Lemma 5.3.3], and [10, Lemma 5.3.4]. As such, we will only sketch the details here.

First one proceeds to show that properties (i) and (ii) of Definition 4.3.1 hold for Ψ when 1_χ is replaced with an arbitrary bi-non-crossing partition. This effectively makes use of the same arguments as in Lemma 4.3.5; that is, one uses the recursive algorithm to reduce down and then note the proofs of properties (i) and (ii) above still apply and lets one move elements around as needed. In particular, the same arguments used in [10, Lemma 5.3.2] and [10, Lemma 5.3.3] transfer with the use of Proposition 4.2.9.

Next, using property (iii), we need only prove property (iv) under the assumption that $s_\chi(1)$ and $s_\chi(n)$ are in the same block W_0 of W. One then follows many of the same ideas as [10, Lemma 5.3.1] and [10, Lemma 5.3.4] by applying the recursive definition from Definition 4.3.6, moving around the appropriate B-elements using the more general (i) and (ii), and combining the appropriate elements using (iii) as needed. $\square$

4.3.2 The Analytical Operator-Valued Bi-Free Cumulant Function

By convolving the analytical bi-moment function with the bi-non-crossing Möbius function, we obtain the following which is essential to our study of conjugate variables in the subsequent section.

Definition 4.3.10. Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and denote by Ψ the analytical bi-moment function. The *analytical bi-cumulant function*

$$\widetilde{\kappa} : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \ldots \times A_{\chi(n-1)} \times L_2(A,\tau) \to L_2(B,\tau_B)$$

is defined by

$$\widetilde{\kappa}_\pi(Z_1,\ldots,Z_{n-1},\xi) = \sum_{\substack{\sigma\in\mathrm{BNC}(\chi)\\ \sigma\leq\pi}} \Psi_\sigma(Z_1,\ldots,Z_{n-1},\xi)\mu_{\mathrm{BNC}}(\sigma,\pi),$$

for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A,\tau)$, and $Z_k \in A_{\chi(k)}$.

Remark 4.3.11. (i) In the case when $\pi = 1_\chi$, we will denote the map $\widetilde{\kappa}_{1_\chi}$ simply by $\widetilde{\kappa}_\chi$. By Möbius inversion, we obtain that

$$\Psi_\sigma(Z_1,\ldots,Z_n) = \sum_{\substack{\pi\in\mathrm{BNC}(\chi)\\ \pi\leq\sigma}} \widetilde{\kappa}_\pi(Z_1,\ldots,Z_{n-1},\xi)$$

for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\sigma \in \mathrm{BNC}(\chi)$, and $Z_k \in A_{\chi(k)}$.

(ii) In the case that $B = \mathbb{C}$, the analytical bi-cumulant function are precisely the $L_2(A,\tau)$-valued bi-free cumulants that were used in [14].

Unsurprisingly, the analytic extension of the operator-valued bi-free cumulant function lives upto its name.

Theorem 4.3.12. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space. Then the analytical bi-cumulant function is the analytic extension of the operator-valued bi-free cumulant function.*

Proof. Recall by [10, Theorem 6.2.1] that the convolution of a bi-multiplicative function with a scalar-valued multiplicative function on the lattice of non-crossing partitions (e.g. the bi-free Möbius function) produces a bi-multiplicative function. As the properties of an analytic extension of a bi-multiplicative function are analogous to those of a bi-multiplicative function, we obtain that the convolution of an analytic extension of a bi-multiplicative function with a scalar-valued multiplicative function on the lattice of non-crossing partitions (e.g. the bi-free Möbius function) produces the analytic extension of the corresponding bi-multiplicative function obtained via the same convolution. Hence the result follows. ☐

Theorem 4.3.13. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space. For all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, and $Z_k \in A_{\chi(k)}$, we have that*

$$\widetilde{\kappa}_\pi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) = \kappa^B_\pi(Z_1, \ldots, Z_{n-1}, Z_n) + N_{\tau_B}.$$

Proof. By Theorem 4.3.8, we know that

$$\widetilde{\Psi}_\pi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) = E_\pi(Z_1, \ldots, Z_{n-1}, Z_n) + N_{\tau_B}$$

for all $\pi \in \mathrm{BNC}(\chi)$. Therefore as $\widetilde{\kappa}$ and κ^B are the convolution of $\widetilde{\Psi}$ and E against the bi-free Möbius function respectively, the result follows. ☐

Of course, as [10, Theorem 8.1.1] demonstrated that bi-freeness with amalagmation over B is equivalent to the mixed operator-valued bi-free cumulants vanishing, Theorem 4.3.13 immediately implies the following.

Corollary 4.3.14. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space containing a family of pairs of B-algebras $\{(C_k, D_k)\}_{k\in K}$. Consider the following two conditions:*

1. *The family $\{(C_k, D_k)\}_{k\in K}$ is bi-free with amalgamation over B with respect to E.*
2. *For all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $Z_1, \ldots, Z_n \in A$, and non-constant maps $\gamma : \{1, \ldots, n\} \to K$ such that*

$$Z_k \in \begin{cases} C_{\gamma(k)} & \text{if } \chi(k) = \ell \\ D_{\gamma(k)} & \text{if } \chi(k) = r \end{cases},$$

it follows that

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) = 0.$$

Then (1) implies (2). In the case that $\tau_B : B \to \mathbb{C}$ is faithful, (2) implies (1).

Proof. Note (1) implies (2) follows from [10, Theorem 8.1.1] and Theorem 4.3.13. In the case that τ_B is faithful, (2) immediately implies $\kappa^B_\pi(Z_1, \ldots, Z_{n-1}, Z_n) = 0$ where $\{Z_k\}_{k=1}^n$ are as in (2) via Theorem 4.3.13. Hence [10, Theorem 8.1.1] completes the argument. □

4.3.3 Vanishing Analytical Cumulants

However, something stronger than Corollary 4.3.14 holds. Indeed, note that the analytic operator-valued bi-free cumulant function has the added benefit that the last entry can be an element of $L_2(A, \tau)$ and thus the L_2-image of a product of left and right operators. As such, it is possible to verify that additional analytic bi-cumulants vanish.

The desired result is analogous to the scalar-valued result demonstrated in [14, Proposition 2.30] and proved in a similar manner. Thus we begin with a generalization of [10, Theorem 9.1.5] where we can expand out a cumulant involving products of operators. In [10, Theorem 9.1.5] only products of left and right operators were considered in the operator-valued setting whereas [14, Lemma 2.29] expanded out scalar-valued cumulants involving a product of left and right operator in the last entry.

To begin, fix $m < n \in \mathbb{N}$, $\chi \in \{\ell, r\}^m$, integers

$$k(0) = 0 < k(1) < \ldots < k(m) = n,$$

and any function $\widehat{\chi} \in \{\ell, r\}^n$ such that for all $q \in \{1, \ldots, n\}$ for which there exists a (necessarily unique) $p_q \in \{1, \ldots, m-1\}$ with $k(p_q - 1) < q \leq k(p_q)$, we have

$$\widehat{\chi}(q) = \chi(p_q).$$

Thus $\widehat{\chi}$ is constant from $k(p-1)+1$ to $k(p)$ whereas $\widehat{\chi}$ does not need to be constant from $k(m-1)+1$ to $k(m)$. This is also the main difference with the setting of Definition 3.1.10; since the last entry of the analytical bi-free cumulants will be an L_2-operator, we will need to make no assumptions on whether this last entry is treated as a left or right operator.

We may embed $\mathrm{BNC}(\chi)$ into $\mathrm{BNC}(\widehat{\chi})$ via $\pi \mapsto \widehat{\pi}$ where the blocks of $\widehat{\pi}$ are formed by taking each block V of π and forming a block

$$\widehat{V} = \bigcup_{p \in V} \{k(p-1)+1, \ldots, k(p)\}$$

of $\widehat{\pi}$. It is not difficult to see that $\widehat{\pi} \in \mathrm{BNC}(\widehat{\chi})$ since $\widehat{\chi}$ is constant on $\{k(p-1)+1, \ldots, k(p)\}$ for all $p \in V \setminus \{m\}$ and although the block containing $\{k(n-1)+1, \ldots, k(n)\}$ has both left and right entries, it occurs at the bottom of the bi-non-crossing diagram and thus poses no problem. Alternatively, this map can be viewed as an analogue of the map on non-crossing partitions from [61, Notation 11.9] after applying s_χ^{-1}.

It is easy to see that $\widehat{1_\chi} = 1_{\widehat{\chi}}$,

$$\widehat{0_\chi} = \bigcup_{p=1}^{m} \{k(p-1)+1, \ldots, k(p)\},$$

and that the map $\pi \mapsto \widehat{\pi}$ is injective and order-preserving. Furthermore, the image of $\mathrm{BNC}(\chi)$ under this map is

$$\widehat{\mathrm{BNC}}(\chi) = \left[\widehat{0_\chi}, \widehat{1_\chi}\right] = \left[\widehat{0_\chi}, 1_{\widehat{\chi}}\right] \subseteq \mathrm{BNC}(\widehat{\chi}).$$

Remark 4.3.15. Recall that since μ_{BNC} is the bi-non-crossing Möbius function, we have for each $\sigma, \pi \in \mathrm{BNC}(\chi)$ with $\sigma \leq \pi$ that

$$\sum_{\substack{\upsilon \in \mathrm{BNC}(\chi) \\ \sigma \leq \upsilon \leq \pi}} \mu_{\mathrm{BNC}}(\upsilon, \pi) = \begin{cases} 1 & \text{if } \sigma = \pi \\ 0 & \text{otherwise} \end{cases}.$$

Since the lattice structure is preserved under the map defined above, we see that $\mu_{\mathrm{BNC}}(\sigma, \pi) = \mu_{\mathrm{BNC}}(\widehat{\sigma}, \widehat{\pi})$.

It is also easy to see that the partial Möbius inversion from [61, Proposition 10.11] holds in the bi-free setting; that is, if $f, g : \mathrm{BNC}(\chi) \to \mathbb{C}$ are such that

$$f(\pi) = \sum_{\substack{\sigma \in \mathrm{BNC}(\chi) \\ \sigma \leq \pi}} g(\sigma)$$

for all $\pi \in \mathrm{BNC}(\chi)$, then for all $\pi, \sigma \in \mathrm{BNC}(\chi)$ with $\sigma \leq \pi$, we have the relation

$$\sum_{\substack{\upsilon \in \mathrm{BNC}(\chi) \\ \sigma \leq \upsilon \leq \pi}} f(\upsilon)\mu_{\mathrm{BNC}}(\upsilon, \pi) = \sum_{\substack{\omega \in \mathrm{BNC}(\chi) \\ \omega \vee \sigma = \pi}} g(\omega)$$

where $\pi \vee \sigma$ denotes the smallest element of $\mathrm{BNC}(\chi)$ greater than π and σ.

Thus, by following the proofs of either [61, Theorem 11.12], [10, Theorem 9.1.5], or [14, Lemma 2.29], we arrive at the following.

Proposition 4.3.16. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space. Under the above notation, if $\pi \in \mathrm{BNC}(\chi)$ and $Z_k \in A_{\widehat{\chi}(k)}$, then*

$$\widetilde{\kappa}_\pi(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) = \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \sigma \vee \widehat{0_\chi} = \widehat{\pi}}} \widetilde{\kappa}_\sigma(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau).$$

In particular, when $\sigma = 1_\chi$, we have

$$\widetilde{\kappa}_\chi(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) = \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \sigma \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \widetilde{\kappa}_\sigma(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau),$$

Proof. First, it is not difficult to verify using the recursive definition of the analytic operator-valued bi-moment function that

$$\Psi_\upsilon(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-2)+1} \cdots Z_{k(m-1)}, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) = \Psi_{\widehat{\upsilon}}(Z_1, \ldots, Z_n + N_\tau)$$

for all $\upsilon \in \mathrm{BNC}(\chi)$. Therefore, we have that

$$
\begin{aligned}
&\widetilde{\kappa}_\pi(Z_1\cdots Z_{k(1)},\ldots,Z_{k(m-2)+1}\cdots Z_{k(m-1)},Z_{k(m-1)+1}\cdots Z_{k(m)}+N_\tau)\\
&=\sum_{\substack{\upsilon\in \mathrm{BNC}(\chi)\\ \upsilon\leq\pi}}\Psi_\upsilon(Z_1\cdots Z_{k(1)},\ldots,Z_{k(m-2)+1}\cdots Z_{k(m-1)},Z_{k(m-1)+1}\cdots Z_{k(m)}+N_\tau)\mu_{\mathrm{BNC}}(\upsilon,\pi)\\
&=\sum_{\substack{\upsilon\in \mathrm{BNC}(\chi)\\ \upsilon\leq\pi}}\Psi_{\widehat{\upsilon}}(Z_1,\ldots,Z_{n-1},Z_n+N_\tau)\mu_{\mathrm{BNC}}(\widehat{\upsilon},\widehat{\pi})\\
&=\sum_{\substack{\sigma\in \mathrm{BNC}(\widehat{\chi})\\ \widehat{0_\chi}\leq\sigma\leq\widehat{\pi}}}\Psi_{\sigma}(Z_1,\ldots,Z_{n-1},Z_n+N_\tau)\mu_{\mathrm{BNC}}(\sigma,\widehat{\pi})\\
&=\sum_{\substack{\sigma\in \mathrm{BNC}(\widehat{\chi})\\ \sigma\vee\widehat{0_\chi}=\widehat{\pi}}}\widetilde{\kappa}_{\sigma}(Z_1,\ldots,Z_{n-1},Z_n+N_\tau)
\end{aligned}
$$

where the last line following from Remark 4.3.15. □

Theorem 4.3.17. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space containing a family of pairs of B-algebras $\{(C_k, D_k)\}_{k\in K}$ that are bi-free with amalgamation over B with respect to E. For each $k \in K$, let $L_2(A_k, \tau)$ be the closed subspace of $L_2(A, \tau)$ generated by*

$$\mathrm{alg}(C_k, D_k) + N_\tau.$$

Then for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, non-constant maps $\gamma : \{1, \ldots, n\} \to K$, $\xi \in L_2(A_{\gamma(n)}, \tau)$, and $Z_1, \ldots, Z_{n-1} \in A$ such that

$$Z_k \in \begin{cases} C_{\gamma(k)} & \text{if } \chi(k) = \ell \\ D_{\gamma(k)} & \text{if } \chi(k) = r \end{cases},$$

it follows that

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, \xi) = 0.$$

Proof. Fix an $m \in \mathbb{N}$, $\chi \in \{\ell, r\}^m$, non-constant map $\gamma : \{1, \ldots, m\} \to K$, and let $Z_1, \ldots, Z_{m-1} \in A$. For any $n \geq m$, if $Z_m, \ldots, Z_n \in C_{\gamma(m)} \cup D_{\gamma(m)}$ and $\widehat{\chi}$ is defined by

$$\widehat{\chi}(k) = \begin{cases} \chi(k) & \text{if } k \leq m \\ \ell & \text{if } k > m \text{ and } Z_k \in C_{\gamma(m)} \\ r & \text{if } k > m \text{ and } Z_k \in D_{\gamma(m)} \end{cases},$$

then by Proposition 4.3.16 and Theorem 4.3.8 implies that

$$\begin{aligned}\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{m-1}, Z_m \cdots Z_n + N_\tau) &= \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \sigma \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \widetilde{\kappa}_\sigma(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) \\ &= \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \sigma \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \kappa^B_\sigma(Z_1, \ldots, Z_{n-1}, Z_n) + N_{\tau_B}.\end{aligned}$$

As the conditions $\sigma \in \mathrm{BNC}(\widehat{\chi})$ and $\sigma \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ automatically imply that each block of σ containing one of $\{1, 2, \ldots, m-1\}$ must also contain an element of $\{m, \ldots, n\}$, the bi-multiplicative properties of the operator-valued bi-free cumulant function imply that each cumulant $\kappa^B_\sigma(Z_1, \ldots, Z_{n-1}, Z_n)$ appearing in the sum above can be reduced down to an expression involving a mixed κ^B term which must be 0 by [10, Theorem 8.1.1]. Hence

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{m-1}, Z_m \cdots Z_n + N_\tau) = 0.$$

Since $\widetilde{E}$ is a continuous function and left multiplication of A on $L_2(A, \tau)$ yields bounded operators, due to the recursive nature of Ψ we see that Ψ is continuous in the $L_2(A, \tau)$ entry. Therefore, by Möbius inversion, $\widetilde{\kappa}$ is continuous in the $L_2(A, \tau)$ entry. Hence the result follows. □

We end this chapter with the following Corollary, which demonstrates the vanishing of the analytical bi-free cumulants whenever at least one of the entries comes from B.

Corollary 4.3.18. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and let $n \geq 2$, $\chi \in \{\ell, r\}^n$, $\xi \in L_2(A, \tau)$, $b \in B$, and $Z_k \in A_{\chi(k)}$. Suppose that either there exists $p \in \{1, \ldots, n-1\}$ such that*

$$Z_p = \begin{cases} L_b & \text{if } \chi(p) = \ell \\ R_b & \text{if } \chi(p) = r \end{cases}$$

or that $\xi \in L_2(B, \tau_B)$. Then

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, \xi) = 0.$$

Proof. If $Z_p = L_b$ or $Z_p = R_b$ for some p, then we may proceed as in the proof of Theorem 4.3.17 by assuming that ξ is an element of A, expanding out the analytic operator-valued bi-free cumulant function with the aid of Proposition 4.3.16, and using the fact that non-singleton operator-valued bi-free cumulants involving L_b or R_b terms are zero by [10, Proposition 6.4.1] and then taking a limit at the end.

In the case where $\xi \in L_2(B, \tau_B)$, ξ is a limit of terms of the form $L_b + N_\tau$. As

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, L_b + N_\tau) = \kappa_\chi(Z_1, \ldots, Z_{n-1}, L_b) + N_{\tau_B} = 0 + N_{\tau_B},$$

by Theorem 4.3.13 and [10, Proposition 6.4.1] the result follows by taking a limit. $\square$

Chapter 5

Bi-Free Fisher Information and Entropy with Respect to Completely Positive Maps

With the technology of chapter 4 in hand, in this chapter we will develop the appropriate notions of bi-free conjugate variables, Fisher information and entropy with respect to completely positive maps. The conjugate variable systems we will present can be viewed as both an extension of the bi-free conjugate variables developed in [14] and of the free conjugate variables with respect to a completely positive map developed in [66]. Since our notion of bi-free entropy will be defined in terms of an integral of the bi-free Fisher information with respect to completely positive maps of perturbations of left and right operators by bi-freely independent operator-valued bi-semicircular pairs, in section 5.2 we will develop the theory of bi-semicircular pairs with completely positive covariance, which will be modeled by left and right creation and annihilation operators on full Fock spaces.

5.1 Bi-Free Conjugate Variables with respect to Completely Positive Maps

For the bi-free conjugate variables with respect to completely positive maps, we will focus on both their moment and cumulant characterizations, whereas [66] focused on the moment and derivation characterizations of free conjugate variables. Although in [14] the moment, cumulant, and bi-free difference quotient characterizations of bi-free conjugate variables were analyzed, we will not attempt to generalize the bi-free difference quotient characterization in this setting as it was the cumulant characterization that was found most effective and as the bi-module structures of [66] that were necessary for the derivation characterization using adjoints are less clear in this context. We refer the reader to [58, Definition 2.7] for an equivalent description to [66] of the free conjugate variables with respect to a completely positive map, which we follow closely below.

Definition 5.1.1. Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, let $Y \in A_r$, and let $\eta : B \to B$ be a completely positive map.

An element $\xi \in L_2(A, \tau)$ is said to satisfy the *left bi-free conjugate variable relations for X with respect to η and τ in the presence of* (C_ℓ, C_r) if for all $n \in \mathbb{N} \cup \{0\}$, $Z_1, \ldots, Z_n \in \{X\} \cup C_\ell \cup C_r$ we have

$$\tau(Z_1 \cdots Z_n \xi) = \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \tau\left(\left(\prod_{p \in V_k^c \setminus \{k, n+1\}} Z_p\right) L_{\eta\left(E\left(\prod_{p \in V_k} Z_p\right)\right)}\right),$$

where $V_k = \{k < m < n+1 \mid Z_m \in \{X\} \cup C_\ell\}$ and where all products are taken in numeric order (with the empty product being equal to 1). If, in addition,

$$\xi \in \overline{\mathrm{alg}(X, C_\ell, C_r)}^{\|\cdot\|_\tau},$$

we call ξ the *left bi-free conjugate variable for X with respect to η and τ in the presence of* (C_ℓ, C_r) and denote ξ by $J_\ell(X : (C_\ell, C_r), \eta)$.

Similarly, an element $\nu \in L_2(A, \tau)$ is said to satisfy the *right bi-free conjugate variable relations for Y with respect to η and τ in the presence of* (C_ℓ, C_r) if for all $n \in \mathbb{N} \cup \{0\}$, $Z_1, \ldots, Z_n \in \{Y\} \cup C_\ell \cup C_r$ we have

$$\tau(Z_1 \cdots Z_n \nu) = \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \tau\left(\left(\prod_{p \in V_k^c \setminus \{k, n+1\}} Z_p\right) R_{\eta\left(E\left(\prod_{p \in V_k} Z_p\right)\right)}\right),$$

where $V_k = \{k < m < n+1 \mid Z_m \in \{Y\} \cup C_r\}$. If, in addition,

$$\nu \in \overline{\mathrm{alg}(Y, C_\ell, C_r)}^{\|\cdot\|_\tau},$$

we call ν the *right bi-free conjugate variable for X with respect to η and τ in the presence of* (C_ℓ, C_r) and denote ν by $J_r(Y : (C_\ell, C_r), \eta)$.

Example 5.1.2. For an example of Definition 5.1.1, consider $X \in A_\ell$, $Y \in A_r$, $Z_2, Z_3 \in C_\ell$, and $Z_1, Z_4 \in C_r$. If $\xi = J_\ell(X : (C_\ell, \mathrm{alg}(C_r, Y)), \eta)$, then

$$\begin{aligned}\tau(XZ_1Z_2YXYZ_3XZ_4\xi) = \tau&\left(Z_1YYZ_4L_{\eta(E(Z_2XZ_3X))}\right)\\ &+\tau\left(XZ_1Z_2YYZ_4L_{\eta(E(Z_3X))}\right)\\ &+\tau\left(XZ_1Z_2YXYZ_3Z_4L_{\eta(E(1))}\right)\end{aligned}$$

This can be observed diagrammatically by drawing $X, Z_1, Z_2, Y, X, Y, Z_3, X, Z_4$ as one would in a bi-non-crossing diagram (i.e. drawing two vertical lines and placing the variables on these lines starting at the top and going down with left variables on the left line and right variables on the right line), drawing all pictures connecting the centre of the bottom of the diagram to any X, taking the product of the elements starting from the top and going down in each of the two isolated components of the diagram, taking the expectation of the bounded region and applying η to the result to obtain a $b \in B$, appending L_b to the end of the product of operators from the unbounded region, and applying τ to the result.

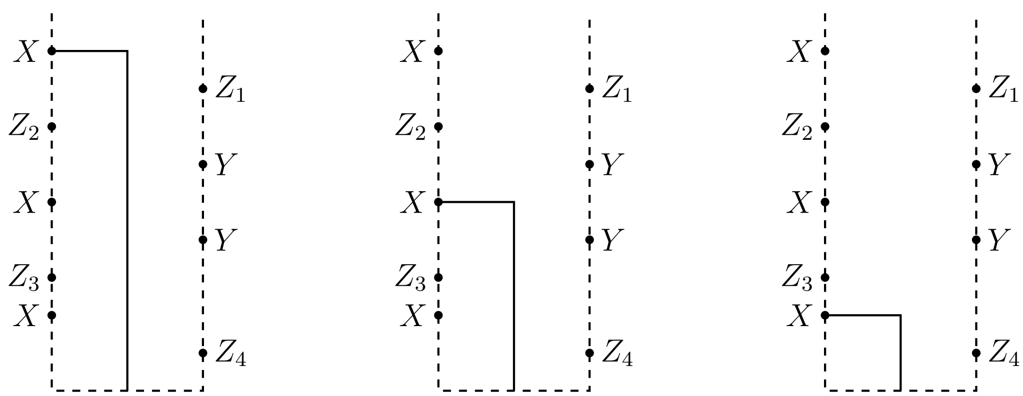

Figure 5.1: Left bi-free conjugate variable

This is analogous to applying the left bi-free difference quotient $\partial_{\ell,X}$ defined in [14] on a suitable algebraic free product to $XZ_1Z_2YXYZ_3XZ_4$ to obtain

$$Z_1YYZ_4 \otimes Z_2XZ_3Z + XZ_1Z_2YYZ_4 \otimes Z_3X + XZ_1Z_2YXYZ_3Z_4 \otimes 1,$$

applying $\mathrm{Id} \otimes (\eta \circ E)$, collapsing the tensor, and applying τ to the result.

Similarly, if $\nu = J_r(Y : (\mathrm{alg}(C_\ell, X), C_r), \eta)$, then

$$\tau(XZ_1Z_2YXYZ_3XZ_4\nu) = \tau\left(XZ_1Z_2XZ_3XR_{\eta(E(YZ_4))}\right) + \tau\left(XZ_1Z_2YXZ_3XR_{\eta(E(Z_4))}\right).$$

This can be observed diagrammatically in a similar fashion by drawing all pictures connecting the centre of the bottom of the diagram to any Y on the right.

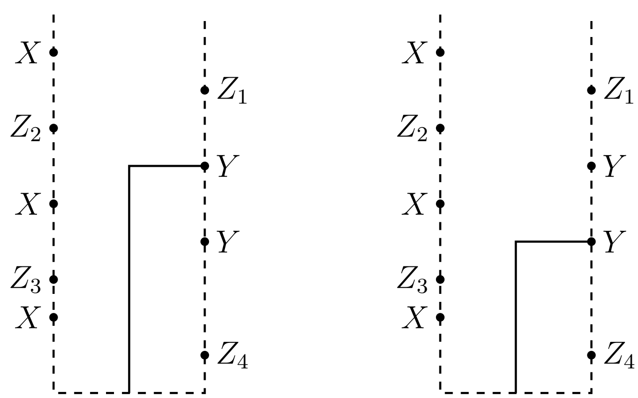

Figure 5.2: Right bi-free conjugate variable

This is analogous to applying the right bi-free difference quotient $\partial_{r,Y}$ defined in [14] on a suitable algebraic free product to $XZ_1Z_2YXYZ_3XZ_4$ to obtain

$$XZ_1Z_2XZ_3X \otimes YZ_4 + XZ_1Z_2YXZ_3X \otimes Z_4,$$

applying $\mathrm{Id} \otimes (\eta \circ E)$, collapsing the tensor, and applying τ to the result.

Remark 5.1.3. (i) As $\tau(aL_b) = \tau(aR_b)$ for all $a \in A$ and $b \in B$, one may use either $L_{\eta\circ E}$ or $R_{\eta\circ E}$ in either part of Definition 5.1.1. In fact, one may simply use $\eta \circ E$ if one views the resulting element of B as an element of $L_2(B, \tau_B) \subseteq L_2(A, \tau)$, since $\tau(aL_b) = \tau(a(b + N_\tau))$ for all $a \in A$ and $b \in B$ by construction.

(ii) The element $J_\ell(X : (C_\ell, C_r), \eta)$ is unique in the sense that if $\xi_0 \in \overline{\mathrm{alg}(X, C_\ell, C_r)}^{\|\cdot\|_\tau}$ satisfies the left bi-free conjugate variable relations for X with respect to (C_ℓ, C_r), then $\xi_0 = J_\ell(X : (C_\ell, C_r), \eta)$ as the left bi-free conjugate variable relations causes the inner products in $L_2(A, \tau)$ of both ξ_0 and $J_\ell(X : (C_\ell, C_r), \eta)$ against any element of $\mathrm{alg}(X, C_\ell, C_r)$ to be equal.

(iii) In the case where $B = \mathbb{C}$, E reduces down to a unital, linear map $\varphi : A \to \mathbb{C}$ and, as τ is compatible with E, one obtains that $\tau = \varphi$. As φ is linear, Definition 5.1.1 immediately reduces down to the left and right conjugate variables with respect to φ in the presence of (C_ℓ, C_r) as in [14], provided η is unital.

(iv) In the setting of Example 4.1.3, we note that $J_\ell(X : (B_\ell, B_r), \eta)$ exists if and only if the free conjugate variable of X with respect to (B, η) from [66] exists. This immediately follows as B_r commutes with X and B_ℓ and τ is tracial, so the expressions for either conjugate variable can be modified into the expressions of the other conjugate variable.

As with the bi-free conjugate variables in [14], any moment expression should be equivalent to certain cumulant expressions via Möbius inversion. Thus we obtain the following equivalent characterization of conjugate variables. Note in that which follows, it does not matter whether the last entry in the analytical operator-valued bi-free cumulant function is treated as a left or as a right operator by Definition 4.3.1.

Theorem 5.1.4. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, let $Y \in A_r$, and let $\eta : B \to B$ be a completely positive map. For $\xi \in L_2(A, \tau)$, the following are equivalent:*

(i) ξ satisfies the left bi-free conjugate variables relations for X (respectively ξ satisfies the right bi-free conjugate variables relations for Y) with respect to η and τ in the presence of (C_ℓ, C_r),

(ii) the following four cumulant conditions hold:

(a) $\widetilde{\kappa}_{1_{(\ell)}}(\xi) = 0 + N_{\tau_B}$,

(b) $\widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_b, \xi) = \eta(b) + N_{\tau_B}$ (respectively $\widetilde{\kappa}_{1_{(r,r)}}(YR_b, \xi) = \eta(b) + N_{\tau_B}$) for all $b \in B$,

(c) $\widetilde{\kappa}_{1_{(\ell,\ell)}}(c_1, \xi) = \widetilde{\kappa}_{1_{(r,\ell)}}(c_2, \xi) = 0 + N_{\tau_B}$ for all $c_1 \in C_\ell$ and $c_2 \in C_r$,

(d) for all $n \geq 3$, $\chi \in \{\ell, r\}^n$, and all $Z_1, Z_2, \ldots, Z_{n-1} \in A$ such that

$$Z_k \in \begin{cases} \{X\} \cup C_\ell & \text{if } \chi(k) = \ell \\ C_r & \text{if } \chi(k) = r \end{cases} \qquad \left(\text{respectively} \quad Z_k \in \begin{cases} C_\ell & \text{if } \chi(k) = \ell \\ \{Y\} \cup C_r & \text{if } \chi(k) = r \end{cases} \right)$$

we have that

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, \xi) = 0 + N_{\tau_B}.$$

Proof. We will prove the result for the left bi-free conjugate variable as the proof for the right bi-free conjugate variable is analogous.

Suppose that ξ satisfies (ii). To see that ξ satisfies the left bi-free conjugate variables relations, let $n \in \mathbb{N} \cup \{0\}$ and let $Z_1, \ldots, Z_n \in \{X\} \cup C_\ell \cup C_r$. Fix $\chi \in \{\ell, r\}^{n+1}$ such that

$$\chi(k) = \begin{cases} \ell & \text{if } Z_k \in \{X\} \cup C_\ell \\ r & \text{if } Z_k \in C_r \end{cases}$$

(note the value of $\chi(n+1)$ does not matter in that which follows). By the relation between the analytic extensions of the bi-moment and bi-cumulant functions, we obtain that

$$\widetilde{E}(Z_1 \cdots Z_n \xi) = \sum_{\pi \in \mathrm{BNC}(\chi)} \widetilde{\kappa}_\pi(Z_1, \ldots, Z_n, \xi).$$

Due to the cumulant conditions in (ii), the only way $\widetilde{\kappa}_\pi(Z_1, \ldots, Z_n, \xi)$ is non-zero is if the block of π containing $n+1$ contains a single other index k with $Z_k = X$. Moreover, there is a bijection between such partitions and partitions of the form

$$\pi = \{k, n+1\} \cup \pi_1 \cup \pi_2$$

where π_1 is a bi-non-crossing partition on $V_k = \{k < m < n+1 \mid Z_m \in \{X\} \cup C_\ell\}$ with respect to $\chi|_{V_k}$ and where π_2 is a bi-non-crossing partition on $W_k = V_k \setminus \{k, n+1\}$ with respect to $\chi|_{W_k}$. Using this decomposition, the properties of bi-analytic extensions of bi-multiplicative functions and the moment-cumulant formulas yield that

$$
\begin{aligned}
&\widetilde{E}(Z_1 \cdots Z_n \xi) \\
&= \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \sum_{\pi_2 \in \mathrm{BNC}(\chi|_{W_k})} \sum_{\pi_1 \in \mathrm{BNC}(\chi|_{V_k})} \widetilde{\kappa}_{\{k,n+1\} \cup \pi_1 \cup \pi_2}(Z_1, \ldots, Z_n, \xi) \\
&= \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \sum_{\pi_2 \in \mathrm{BNC}(\chi|_{W_k})} \widetilde{\kappa}_{\{k,n+1\} \cup \pi_1} \left(\left. \left(Z_1, \ldots, Z_{k-1}, Z_k L_{E\left(\prod_{p \in V_k} Z_p\right)} Z_{k+1}, \ldots, Z_n, \xi \right) \right|_{W_k \cup \{k,n+1\}} \right) \\
&= \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \widetilde{\kappa}_{\pi_1} \left(\left. \left(Z_1, \ldots, Z_{\max_{\preceq}(W_k)-1}, Z_{\max_{\preceq}(W_k)} \eta \left(E \left(\prod_{p \in V_k} Z_p \right) \right) \right) \right|_{W_k} \right) \\
&= \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \tilde{E} \left(\left(\prod_{p \in V_k^c \setminus \{k,n+1\}} Z_p \right) \eta \left(E \left(\prod_{p \in V_k} Z_p \right) \right) \right).
\end{aligned}
$$

Hence, by applying τ_B to both sides of this equation, the left bi-free conjugate variables relations from Definition 5.1.1 are obtained via part (i) of Proposition 4.2.9.

For the converse direction, suppose ξ satisfies the left bi-free conjugate variables relations for X. Thus for all $b \in B$, $\tau(L_b \xi) = 0$ by the conjugate variable relations. Hence $\widetilde{\kappa}_{1_{(\ell)}}(\xi) = \widetilde{E}(\xi) = 0$ by part (vi) of Proposition 4.2.9 and therefore (a) holds.

To see that (b) holds, note for all $b_0, b \in B$ that

$$
\begin{aligned}
b_0 \widetilde{\kappa}_{1_{(\ell,\ell)}}(X L_b, \xi) = \widetilde{\kappa}_{1_{(\ell,\ell)}}(L_{b_0} X L_b, \xi) &= \Psi_{1_{(\ell,\ell)}}(L_{b_0} X L_b, \xi) - \Psi_{0_{(\ell,\ell)}}(L_{b_0} X L_b, \xi) \\
&= \widetilde{E}(L_{b_0} X L_b \xi) - E(L_{b_0} X L_b) \widetilde{E}(\xi) = \widetilde{E}(L_{b_0} X L_b \xi).
\end{aligned}
$$

Therefore, by applying τ_B to both sides, we obtain that

$$
\tau_B \left(b_0 \widetilde{\kappa}_{1_{(\ell,\ell)}}(X L_b, \xi) \right) = \tau_B \left(\widetilde{E}(L_{b_0} X L_b \xi) \right) = \tau(L_{b_0} X L_b \xi).
$$

By the left bi-free conjugate variable relations we obtain that

$$
\tau_B \left(b_0 \widetilde{\kappa}_{1_{(\ell,\ell)}}(X L_b, \xi) \right) = \tau \left(L_{b_0} L_{\eta(E(L_b))} \right) = \tau(L_{b_0} L_{\eta(b)}) = \tau(L_{b_0 \eta(b)}) = \tau_B(b_0 \eta(b)).
$$

As this holds for all $b_0 \in B$, we obtain that $\widetilde{\kappa}_{1_{(\ell,\ell)}}(X L_b, \xi) = \eta(b) + N_{\tau_B}$ as desired.

To see that (c) holds, note for all $b \in B$ and $c_1 \in C_\ell$ that

$$\tau_B\left(b\widetilde{\kappa}_{1_{(\ell,\ell)}}(c_1, \xi)\right) = \tau(L_b c_1 \xi) = 0$$

by similar computations as above. Since this holds for all $b \in B$, we see that $\widetilde{\kappa}_{1_{(\ell,\ell)}}(c_1, \xi) = 0 + N_{\tau_B}$. Similarly, for all $c_2 \in C_r$ we see that

$$\begin{aligned}\widetilde{\kappa}_{1_{(r,\ell)}}(c_2, \xi)b = \widetilde{\kappa}_{1_{(r,\ell)}}(R_b c_2, \xi) &= \Psi_{1_{(r,\ell)}}(R_b c_2, \xi) - \Psi_{0_{(r,\ell)}}(R_b c_2, \xi) \\ &= \widetilde{E}(R_b c_2 \xi) - \widetilde{E}(\xi)E(R_b c_2) = \widetilde{E}(R_b c_2 \xi).\end{aligned}$$

Therefore, by applying τ_B to both sides, we obtain that

$$\tau_B\left(\widetilde{\kappa}_{1_{(r,\ell)}}(c_2, \xi)b\right) = \tau_B\left(\widetilde{E}(R_b c_2 \xi)\right) = \tau(R_b c_2 \xi).$$

By the left bi-free conjugate variable relations we obtain that

$$\tau_B\left(\widetilde{\kappa}_{1_{(r,\ell)}}(c_2, \xi)b\right) = 0.$$

Therefore, as τ_B is tracial and the above holds for all $b \in B$, we obtain that $\widetilde{\kappa}_{1_{(r,\ell)}}(c_2, \xi) = 0 + N_{\tau_B}$ as desired.

For (d), we proceed by induction on n. To do so, we will prove the base case $n = 3$ and the inductive step simultaneously. Fix $n \geq 3$ and suppose when $n > 3$ that (d) holds for all $m < n$. Let $\chi \in \{\ell, r\}^n$ and let $Z_1, Z_2, \ldots, Z_{n-1} \in A$ be as in the assumptions of (d). We will assume that $\chi(1) = r$ as the case $\chi(1) = \ell$ will be handled similarly. Thus for all $b \in B$ we know that

$$\begin{aligned}\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, \xi)b &= \widetilde{\kappa}_\chi(R_b Z_1, Z_2, \ldots, Z_{n-1}, \xi) \\ &= \widetilde{E}(R_b Z_1 Z_2 \cdots Z_{n-1}\xi) - \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \neq 1_\chi}} \widetilde{\kappa}_\pi(R_b Z_1, Z_2, \ldots, Z_{n-1}, \xi) \\ &= \widetilde{E}(R_b Z_1 Z_2 \cdots Z_{n-1}\xi) - \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \neq 1_\chi}} \widetilde{\kappa}_\pi(Z_1, Z_2, \ldots, Z_{n-1}, \xi)b.\end{aligned}$$

Using the fact that (a), (b), (c) hold and that (d) holds for all $m < n$, we obtain using the same arguments used in the other direction of the proof that

$$\sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \neq 1_\chi}} \widetilde{\kappa}_\pi(Z_1, \ldots, Z_{n-1}, \xi)b = \sum_{\substack{1 \leq k < n \\ Z_k = X}} \tilde{E}\left(R_b\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p\right)\eta\left(E\left(\prod_{p \in V_k} Z_p\right)\right)\right),$$

where $V_k = \{k < m < n \mid Z_m \in \{X\} \cup C_\ell\}$. Therefore, by applying τ_B to both sides of our initial equation, we obtain that

$$\begin{aligned}
&\tau_B\left(\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, \xi)b\right) \\
&= \tau_B\left(\widetilde{E}(R_b Z_1 Z_2 \cdots Z_{n-1}\xi) - \sum_{\substack{1 \leq k < n \\ Z_k = X}} \tilde{E}\left(R_b\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p\right)\eta\left(E\left(\prod_{p \in V_k} Z_p\right)\right)\right)\right) \\
&= \tau(R_b Z_1 Z_2 \cdots Z_{n-1}\xi) - \sum_{\substack{1 \leq k < n \\ Z_k = X}} \tau\left(R_b\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p\right)\eta\left(E\left(\prod_{p \in V_k} Z_p\right)\right)\right) = 0,
\end{aligned}$$

by the left bi-free conjugate variable relations. Therefore, as the above holds for all $b \in B$ and τ_B is tracial, the result follows. $\square$

The cumulant approach to conjugate variables has merits as it is very simple to check that most cumulants vanish and the values of others. For instance, an observant reader might have noticed that the operators X and Y in Definition 5.1.1 were not required to be self-adjoint. This is for later use and can be converted to studying self-adjoint operators as follows.

Lemma 5.1.5. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. The left bi-free conjugate variables*

$$J_\ell(X : (\mathrm{alg}(C_\ell, X^*), C_r), \eta) \qquad \textit{and} \qquad J_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta)$$

exist if and only if

$$J_\ell(\Re(X) : (\mathrm{alg}(C_\ell, \Im(X)), C_r), \eta) \qquad \textit{and} \qquad J_\ell(\Im(X) : (\mathrm{alg}(C_\ell, \Re(X)), C_r), \eta)$$

exist where $\Re(X) = \frac{1}{2}(X + X^*)$ *and* $\Im(X) = \frac{1}{2i}(X - X^*)$*. Furthermore,*

$$J_\ell(\Re(X) : (\mathrm{alg}(C_\ell, \Im(X)), C_r), \eta) = J_\ell(X{:}(\mathrm{alg}(C_\ell, X^*), C_r), \eta) + J_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta)$$
$$J_\ell(\Im(X) : (\mathrm{alg}(C_\ell, \Re(X)), C_r), \eta) = iJ_\ell(X{:}(\mathrm{alg}(C_\ell, X^*), C_r), \eta) - iJ_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta).$$

A similar result holds for right bi-free conjugate variables.

Proof. Suppose

$$\xi_1 = J_\ell(X : (\mathrm{alg}(C_\ell, X^*), C_r), \eta) \qquad \text{and} \qquad \xi_2 = J_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta)$$

exist. Hence ξ_1 and ξ_2 satisfy the appropriate analytic cumulant equations from Theorem 5.1.4. Let

$$h_1 = \xi_1 + \xi_2 \qquad \text{and} \qquad h_2 = i\xi_1 - i\xi_2.$$

As

$$\xi_1 \in \overline{\mathrm{alg}(X, \mathrm{alg}(C_\ell, X^*), C_r)}^{\|\cdot\|_\tau} \qquad \text{and} \qquad \xi_2 \in \overline{\mathrm{alg}(X^*, \mathrm{alg}(C_\ell, X), C_r)}^{\|\cdot\|_\tau},$$

we easily see that

$$h_1 \in \overline{\mathrm{alg}(\Re(X), \mathrm{alg}(C_\ell, \Im(X)), C_r)}^{\|\cdot\|_\tau} \qquad \text{and} \qquad \xi_2 \in \overline{\mathrm{alg}(\Im(X), \mathrm{alg}(C_\ell, \Re(X)), C_r)}^{\|\cdot\|_\tau}.$$

Thus, by Theorem 5.1.4, it suffices to show that h_1 and h_2 satisfy the appropriate conjugate variable formulae. Indeed, property (a) of Theorem 5.1.4 holds as

$$\widetilde{\kappa}_{1_{(\ell)}}(h_1) = \widetilde{\kappa}_{1_{(\ell)}}(h_2) = 0 + N_{\tau_B}.$$

Next, notice for all $b \in B$ that

$$\begin{aligned}
&\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Re(X)L_b, h_1) \\
&= \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, \xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, \xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, \xi_2\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, \xi_2\right) \\
&= \frac{1}{2}\eta(b) + 0 + 0 + \frac{1}{2}\eta(b) = \eta(b),
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Im(X)L_b, h_2)\\
&= \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2i}XL_b, i\xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(-\frac{1}{2i}X^*L_b, i\xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2i}XL_b, -i\xi_2\right)\\
&\quad + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(-\frac{1}{2i}X^*L_b, -i\xi_2\right)\\
&= \frac{1}{2i}i\eta(b) - 0 + 0 - \frac{1}{2i}(-i)\eta(b) = \eta(b).
\end{aligned}$$

Hence property (b) of Theorem 5.1.4 holds.

To see properties (c) and (d) of Theorem 5.1.4 hold, note for all $b \in B$ that

$$\begin{aligned}
&\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Re(X)L_b, h_2)\\
&= \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, i\xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, i\xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, -i\xi_2\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, -i\xi_2\right)\\
&= \frac{1}{2}i\eta(b) + 0 + 0 + \frac{1}{2}(-i)\eta(b) = 0,
\end{aligned}$$

and similarly $\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Im(X)L_b, h_1) = 0$. Therefore, Proposition 4.3.16 along with the linearity of the cumulants in each entry yield properties (c) and (d).

The converse direction is proved analogously. $\square$

Of course, many other results follow immediately from the cumulant definition of the conjugate variables.

Lemma 5.1.6. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. If*

$$\xi = J_\ell(X : (C_\ell, C_r), \eta)$$

exists, then for all $\lambda \in \mathbb{C} \setminus \{0\}$ the conjugate variable $J_\ell(\lambda X : (C_\ell, C_r), \eta)$ exists and is equal to $\frac{1}{\lambda}\xi$.

A similar result holds for right bi-free conjugate variables.

Proposition 5.1.7. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta_1, \eta_2 : B \to B$ be completely* *positive maps. If*

$$\xi_1 = J_\ell(X : (C_\ell, C_r), \eta_1) \qquad \text{and} \qquad \xi_2 = J_\ell(X : (C_\ell, C_r), \eta_2)$$

exist, then $\xi = J_\ell(X : (C_\ell, C_r), \eta_1 + \eta_2)$ exists and $\xi = \xi_1 + \xi_2$.

A similar result holds for right bi-free conjugate variables.

Proposition 5.1.8. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. For fixed $b_1, b_2 \in B$, define $\eta_{\ell,b_1,b_2} : B \to B$ by*

$$\eta_{\ell,b_1,b_2}(b) = \eta(bb_2)b_1$$

for all $b \in B$. If $\xi = J_\ell(X : (C_\ell, C_r), \eta)$ exists and η_{ℓ,b_1,b_2} is completely positive, then $J_\ell(X : (C_\ell, C_r), \eta_{\ell,b_1,b_2})$ exists and

$$J_\ell(X : (C_\ell, C_r), \eta_{\ell,b_1,b_2}) = R_{b_1} L_{b_2} \xi.$$

Similarly, if $Y \in A_r$ and $\eta_{r,b_1,b_2} : B \to B$ is defined by

$$\eta_{r,b_1,b_2} = b_2 \eta(b_1 b)$$

for all $b \in B$ is completely positive, and $J_r(Y : (C_\ell, C_r), \eta)$ exists, then $J_r(Y : (C_\ell, C_r), \eta_{r,b_1,b_2})$ exists and

$$J_r(Y : (C_\ell, C_r), \eta_{r,b_1,b_2}) = R_{b_1} L_{b_2} J_r(Y : (C_\ell, C_r), \eta).$$

Proof. By Theorem 5.1.4, it suffices to show that $R_{b_1} L_{b_2} \xi$ satisfies the appropriate analytical operator-valued bi-free cumulant formula. Indeed, clearly

$$\widetilde{\kappa}_{1_{(\ell)}}(R_{b_1} L_{b_2} \xi) = b_2 \widetilde{\kappa}_{1_{(\ell)}}(\xi) b_1 = 0$$

and for all $b \in B$

$$\widetilde{\kappa}_{1_{(\ell,\ell)}}(X L_b, R_{b_1} L_{b_2} \xi) = \widetilde{\kappa}_{1_{(\ell,\ell)}}(X L_b L_{b_2}, \xi) b_1 = \widetilde{\kappa}_{1_{(\ell,\ell)}}(X L_{bb_2}, \xi) b_1 = \eta(bb_2) b_1 = \eta_{b_1,b_2}(b).$$

To show that the other analytical operator-valued bi-free cumulants from Theorem 5.1.4 vanish, one simply needs to use the analytical extension properties of bi-multiplicative functions together with Proposition 4.3.16. The result for right bi-free conjugate variables is analogous. $\square$

Similarly, many results pertaining to conjugate variables from [14, 66, 80] immediately generalize to the conjugate variables in Definition 5.1.1. However, one result from [66] requires additional set-up. In the context of Example 4.1.3, one can always consider a further von Neumann subalgebra D of B and ask how the conjugate variables react. To analyze the comparable situation in our setting, we need the following example.

Example 5.1.9. Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and let D be a unital $*$-subalgebra of B (with $1_D = 1_B$). If $F : B \to D$ is a conditional expectation in the sense that $F(d) = d$ for all $d \in D$ and $F(d_1 b d_2) = d_1 F(b) d_2$ for all $d_1, d_2 \in D$ and $b \in B$, then $(A, F \circ E, \varepsilon|_{D \otimes D^{\mathrm{op}}})$ is a D-D-non-commutative probability space by [68, Section 3].

Note $\tau_D = \tau_B|_D : D \to \mathbb{C}$ is a tracial state being the restriction of a tracial state. Moreover, if τ_B is compatible with F in the sense that $\tau_B(F(b)) = \tau_B(b)$ for all $b \in B$, we easily see that τ is compatible with $F \circ E$, as for all $a \in A$ we have that

$$\tau(a) = \tau\left(L_{E(a)}\right) = \tau_B(E(a)) = \tau_B(F(E(a))) = \tau\left(L_{F(E(a))}\right)$$

and similarly $\tau(a) = \tau\left(R_{F(E(a))}\right)$. Hence $(A, F \circ E, \varepsilon|_{D \otimes D^{\mathrm{op}}}, \tau)$ is an analytical D-D-non-commutative probability space.

Proposition 5.1.10. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and $X \in A_\ell$. In addition, let D and F be as in Example 5.1.9 and let (C_ℓ, C_r) be a pair of B-algebras (and thus automatically a pair of D-algebras) and $\eta : D \to D$ be a completely positive map. Moreover, suppose that F is completely positive (and hence $\eta \circ F : B \to D \subseteq B$ is also completely positive). Then, the conjugate variable $J_\ell(X : (C_\ell, C_r), \eta \circ F)$ exists in the analytical B-B-non-commutative probability space $(A, E, \varepsilon, \tau)$ if and only if the conjugate variable $J_\ell(X : (C_\ell, C_r), \eta)$ exists in the analytical D-D-non-commutative probability space $(A, F \circ E, \varepsilon|_{D \otimes D^{op}}, \tau)$, in which case they are the same element of $L_2(A, \tau)$.*

A similar result holds for right bi-free conjugate variables.

Proof. As $(\eta \circ F) \circ E = \eta \circ (F \circ E)$, the bi-free conjugate variable relations from Definition 5.1.1 are precisely the same and thus there is nothing to prove. $\square$

With Proposition 5.1.10 out of the way, we turn our attention to proving that the expected generalizations of conjugate variable properties from [14, 66, 80] hold.

Lemma 5.1.11. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. Suppose further that $D_\ell \subseteq C_\ell$ and $D_r \subseteq C_r$ are such that (D_ℓ, D_r) is a pair of B-algebras in A. If*

$$\xi = J_\ell(X : (C_\ell, C_r), \eta)$$

exists then

$$\xi' = J_\ell(X : (D_\ell, D_r), \eta)$$

exists. In particular, if P is the orthogonal projection of $L_2(A, \tau)$ onto

$$\overline{\mathrm{alg}(X, D_\ell, D_r)}^{\|\cdot\|_\tau},$$

then $\xi' = P(\xi)$.

A similar result holds for right bi-free conjugate variables.

Proof. Notice if ξ satisfies the left bi-free conjugate variable relations for X with respect to E and η in the presence of (C_ℓ, C_r), ξ satisfies the left bi-free conjugate variable relations for X with respect to E and η in the presence of (D_ℓ, D_r). Therefore, since $\tau(ZP(\xi)) = \tau(Z\xi)$ for all $Z \in \mathrm{alg}(X, D_\ell, D_r)$, it follows that $P(\xi) = J_\ell(X : (D_\ell, D_r), \eta)$ as desired. □

The following generalizes [80, Proposition 3.6], [66, Proposition 3.8], and [14, Proposition 4.3].

Proposition 5.1.12. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. If (D_ℓ, D_r) is another pair of B-algebras such that*

$$(\mathrm{alg}(X, C_\ell), C_r) \qquad \textit{and} \qquad (D_\ell, D_r)$$

are bi-free with amalgamation over B with respect to E, then

$$\xi = J_\ell(X : (C_\ell, C_r), \eta)$$

exists if and only if

$$\xi' = J_\ell(X : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), \eta)$$

exists, in which case they are equal.

A similar result holds for right bi-free conjugate variables.

Proof. Note by Lemma 5.1.11 that if ξ' exists then ξ exists.

Conversely suppose that ξ exists. Hence ξ is an $\|\cdot\|_\tau$-limit of elements from $\mathrm{alg}(X, C_\ell, C_r)$. Since the analytical operator-valued bi-free cumulants are $\|\cdot\|_\tau$-continuous in the last entry, it follows that any analytical operator-valued bi-free cumulant involving ξ at the end and at least one element of D_ℓ or D_r must be zero by Theorem 4.3.17 as

$$(\mathrm{alg}(X, C_\ell), C_r) \qquad \text{and} \qquad (D_\ell, D_r)$$

are bi-free with amalgamation over B with respect to E. Therefore, as

$$\xi \in \overline{\mathrm{alg}(X, C_\ell, C_r)}^{\|\cdot\|_\tau} \subseteq \overline{\mathrm{alg}(X, \mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r))}^{\|\cdot\|_\tau},$$

it follows that ξ' exists and $\xi = \xi'$. □

The following generalizes [80, Proposition 3.7], [66, Proposition 3.11], and [14, Proposition 4.4]. In that which follows, we will use $\mathbf{Z}$ to denote a tuple of operators $(Z_1, \ldots, Z_k)$. Furthermore, given another tuple $\mathbf{Z}' = (Z'_1, \ldots, Z'_k)$, we will use $\mathbf{Z} + \mathbf{Z}'$ to denote the tuple $(Z_1 + Z'_1, \ldots, Z_k + Z'_k)$ and we will use $\widehat{\mathbf{Z}_p}$ to denote the tuple $(Z_1, \ldots, Z_{p-1}, Z_{p+1}, \ldots, Z_k)$ obtained by removing Z_p from the list.

Proposition 5.1.13. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space and let $\eta : B \to B$ be a completely positive map. Suppose $\mathbf{X}$ and $\mathbf{X}'$ are n-tuples of operators from A_ℓ, $\mathbf{Y}$ and $\mathbf{Y}'$ are m-tuples of operators from A_r, and (C_ℓ, C_r) and (D_ℓ, D_r) are pairs of B-algebras such that*

$$(\mathrm{alg}(\mathbf{X}, C_\ell), \mathrm{alg}(\mathbf{Y}, C_r)) \qquad \textit{and} \qquad (\mathrm{alg}(\mathbf{X}', D_\ell), \mathrm{alg}(\mathbf{Y}', D_r))$$

are bi-free with amalgamation over B with respect to E. If

$$\xi = J_\ell\left(X_1 : \left(\mathrm{alg}\left(\widehat{\mathbf{X}_1}, C_\ell\right), \mathrm{alg}(\mathbf{Y}, C_r)\right), \eta, \tau\right)$$

exists then

$$\xi' = J_\ell\left(X_1 + X'_1 : \left(\mathrm{alg}\left(\widehat{(\mathbf{X} + \mathbf{X}')_1}, C_\ell, D_\ell\right), \mathrm{alg}(\mathbf{Y} + \mathbf{Y}', C_r, D_r)\right)\right)$$

exists. Moreover, if P is the orthogonal projection of $L_2(\mathcal{A}, \varphi)$ onto

$$\overline{\mathrm{alg}(\mathbf{X}+\mathbf{X}', \mathbf{Y}+\mathbf{Y}', C_\ell, C_r, D_\ell, D_r)}^{\|\cdot\|_\tau},$$

then

$$\xi' = P(\xi).$$

A similar result holds for right bi-free conjugate variables.

Proof. Suppose ξ exists. For notation purposes, let $\mathcal{A} = \mathrm{alg}(\mathbf{X}+\mathbf{X}', \mathbf{Y}+\mathbf{Y}', C_\ell, C_r, D_\ell, D_r)$.

Since $\tau(L_b Z P(\xi)) = \tau(L_b Z\xi)$ for all $Z \in \mathcal{A}$ and $b \in B$ (as $B_\ell \subseteq C_\ell$), we obtain by Proposition 4.2.9 that $\widetilde{E}(ZP(\xi)) = \widetilde{E}(Z\xi)$ for all $Z \in \mathcal{A}$. Thus, as $B_\ell, B_r \subseteq \mathcal{A}$, we obtain for all $\chi \in \{\ell, r\}^p$ with $\chi(p) = \ell$, for all $\pi \in \mathrm{BNC}(\chi)$, and for all $Z_k \in \mathcal{A}$ with

$$Z_k \in \begin{cases} \mathrm{alg}(\mathbf{X}+\mathbf{X}', C_\ell, D_\ell) & \text{if } \chi(k) = \ell \\ \mathrm{alg}(\mathbf{Y}+\mathbf{Y}', C_r, D_r) & \text{if } \chi(k) = r \end{cases}$$

that $\Psi_\pi(Z_1, \ldots, Z_{p-1}P(\xi)) = \Psi_\pi(Z_1, \ldots, Z_{p-1}, \xi)$ and thus

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{p-1}, P(\xi)) = \widetilde{\kappa}_\chi(Z_1, \ldots, Z_{p-1}, \xi).$$

To show that $P(\xi)$ is the appropriate left bi-free conjugate variable, it suffices to consider expressions of the form $\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{p-1}, P(\xi))$ and show they obtain the correct values as dictated in Theorem 5.1.4. By the above, said cumulant is equal to an analytic operator-valued bi-free cumulant involving elements from C_ℓ, D_ℓ, C_r, D_r, $\mathbf{X}+\mathbf{X}'$, and $\mathbf{Y}+\mathbf{Y}'$ (in the appropriate positions) and a ξ at the end. By expanding using linearity, said cumulant can be modified to a sum of cumulants involving only elements from C_ℓ, D_ℓ, C_r, D_r, $\mathbf{X}$, $\mathbf{X}'$, $\mathbf{Y}$, and $\mathbf{Y}'$ with a ξ at the end. By a similar argument to that in Lemma 5.1.5, these cumulants then obtain the necessary values for $P(\xi)$ to be the appropriate left bi-free conjugate variable due to Theorem 5.1.4 applied to ξ and the fact that

$$(\mathrm{alg}(\mathbf{X}, C_\ell), \mathrm{alg}(\mathbf{Y}, C_r)) \qquad \text{and} \qquad (\mathrm{alg}(\mathbf{X}', D_\ell), \mathrm{alg}(\mathbf{Y}', D_r))$$

are bi-free with amalgamation over B with respect to E, so mixed cumulants vanish by Theorem 4.3.17. $\square$

5.2 Bi-Semicircular Operators with Completely Positive Covariance

One essential example of conjugate variables in [14, 71, 80] comes from central limit distributions. Thus this section is devoted to defining the operator-valued bi-semicircular operators with covariance coming from a completely positive map, showing that one may add in certain bi-semicircular operators into analytical B-B-non-commutative probability spaces, and showing the bi-free conjugate variables behave in the appropriate manner.

To begin, let B be a unital *-algebra and let K be a finite index set. For each $k \in K$ let Z_k be a symbol. Recall the *full Fock space* $\mathcal{F}(B, K)$ is the algebraic free product of B and $\{Z_k\}_{k \in K}$; that is

$$\mathcal{F}(B, K) = B \oplus H_1 \oplus H_2 \oplus \cdots$$

where

$$H_m = \{b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m \mid b_0, b_1, \ldots, b_m \in B, k_1, \ldots, k_m \in K\}.$$

Note $\mathcal{F}(B, K)$ is a B-B-bimodule with the obvious left and right actions of B on B and H_m. Moreover, as $\mathcal{F}(B, K)$ is a direct sum of B and another B-B-bimodule, $\mathcal{F}(B, K)$ is a B-B-bimodule with the specified vector state $p : \mathcal{F}(B, K) \to B$ (as in the sense of [10, Definition 3.1.1]) defined by taking the B-term in the above direct product. Therefore, the set A of linear maps on $\mathcal{F}(B, K)$ is a B-B-non-commutative probability space with respect to the expectation $E : A \to B$ defined by $E(T) = p(T1_B)$ (see [10, Remark 3.2.2]).

Let $\{\eta_{i,j}\}_{i,j \in K}$ be linear maps on B. For each $k \in K$, the *left creation and annihilation operators* l_k and l_k^* are the linear maps defined such that

$$\begin{aligned} l_k b &= 1_B Z_k b \\ l_k(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= 1_B Z_k b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m \\ l_k^* b &= 0 \\ l_k^*(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= \eta_{k,k_1}(b_0) b_1 \cdots Z_{k_m} b_m \end{aligned}$$

and the *right creation and annihilation operators* r_k and r_k^* are the linear maps defined such

that

$$
\begin{aligned}
r_k b &= b Z_k 1_B \\
r_k(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m Z_k 1_B \\
r_k^* b &= 0 \\
r_k^*(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= b_0 Z_{k_1} b_1 \cdots b_{m-1} \eta_{k_m,k}(b_m)
\end{aligned}
$$

It is elementary to see that $l_k, l_k^* \in A_\ell$ and $r_k, r_k^* \in A_r$. With these operators in hand, we make the following definition.

Definition 5.2.1. Using the above notation, write K as the disjoint union of two sets I and J. For each $i \in I$ and $j \in J$, let

$$S_i = l_i + l_i^* \qquad \text{and} \qquad D_j = r_j + r_j^*.$$

The pair $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ are called the *operator-valued bi-semicircular operators with covariance* $\{\eta_{i,j}\}_{i,j\in K}$.

In the case that $\eta_{k_1,k_2} = 0$ for all $k_1, k_2 \in K$ with $k_1 \neq k_2$, we say that $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ is a collection of $(\{\eta_{i,i}\}_{i\in I}, \{\eta_{j,j}\}_{j\in J})$ *bi-semicircular operators.*

Remark 5.2.2. It is natural to ask what are the necessary conditions for operator-valued bi-semicircular operators with covariance $\{\eta_{i,j}\}_{i,j\in K}$ to sit inside an analytical B-B-non-commutative probability space. One may hope that a condition similar to [71, Theorem 4.3.1] would work; that is, the answer is yes if τ_B is tracial and $\eta : M_{|K|}(B) \to M_{|K|}(B)$ defined by

$$\eta([b_{i,j}]_{i,j\in K}) = [\eta_{i,j}(b_{i,j})]_{i,j\in K}$$

is completely positive. However, if $i \in I$, $j \in J$, and $b_1, b_2 \in B$, it is not difficult to verify that

$$\tau_B\left(E((S_i D_j L_{b_1} R_{b_2})^*(S_i D_j L_{b_1} R_{b_2})\right) = \tau_B\left(b_2^* \eta_{i,i}(1_B) b_1 b_2 \eta_{j,j}(1_B) b_1^* + b_2^* \eta_{i,j}(\eta_{i,j}(b_1 b_2)) b_1^*\right)$$

and it is not clear if this is positive (even if the outer $\eta_{i,j}$ was a $\eta_{j,i}$).

Only certain operator-valued bi-semicircular operators are required in this manuscript. Indeed, we will need only the case where $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ are $(\{\eta_{i,i}\}_{i\in I}, \{\eta_{j,j}\}_{j\in J})$ bi-semicircular operators, as in this setting the pairs of algebras

$$\{(\mathrm{alg}(B_\ell, S_i), B_r)\}_{i\in I} \cup \{(B_\ell, \mathrm{alg}(B_r, D_j))\}_{j\in J}$$

are bi-free with amalgamation over B with respect to E, as the following result shows.

Theorem 5.2.3. *Using the above notation, if B is a $*$-algebra, τ_B is a tracial state on B, $\{\eta_i\}_{i\in I} \cup \{\eta_j\}_{j\in J}$ are completely positive maps from B to B, $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ are $(\{\eta_i\}_{i\in I}, \{\eta_j\}_{j\in J})$ bi-semicircular operators, $A_\ell = \mathrm{alg}(B_\ell, \{S_i\}_{i\in I})$, $A_r = \mathrm{alg}(B_r, \{D_j\}_{j\in J})$, A is generated as a $*$-algebra by A_ℓ and A_r and $\tau : A \to \mathbb{C}$ is defined by $\tau = \tau_B \circ E$, then $(A, E, \varepsilon, \tau)$ is an analytical B-B-non-commutative probability space. Moreover, all operator-valued bi-free cumulants involving $\{S_i\}_{i\in I}$ and $\{D_j\}_{j\in J}$ of order not two are zero and for all $i, i_1, i_2 \in I$ and $j, j_1, j_2 \in J$,*

$$\begin{aligned} &\kappa_{1_{(\ell,\ell)}}(S_{i_1}L_b, S_{i_2}) = \delta_{i_1,i_2}\eta_{i_1}(b) & \qquad &\kappa_{1_{(r,r)}}(D_{j_1}R_b, D_{j_2}) = \delta_{j_1,j_2}\eta_{j_1}(b) \\ &\kappa_{1_{(\ell,r)}}(S_iL_b, D_j) = 0 & &\kappa_{1_{(r,\ell)}}(D_jR_b, S_i) = 0. \end{aligned}$$

Proof. As shown above, (A, E, ε) is a B-B-non-commutative probability space. Next, note that E is completely positive when restricted A_ℓ and A_r by [71, Remark 4.3.2] as the expectations reduce to the free case. In fact, this same idea can be used to show that τ is positive. Indeed, first note that S_i and D_j commute. Thus every element of A can be written as sum of elements of the form

$$Z = L_{b_0}S_{i_1}L_{b_1}\cdots S_{i_n}L_{b_n}R_{b_{n+1}}D_{i_{n+1}}R_{b_{n+2}}\cdots D_{j_{n+m}}R_{b_{n+m}}$$

where $b_0, \ldots b_{n+m} \in B$. Moreover, we can write

$$\mathcal{F}(B, K) \cong \mathcal{F}(B, I) \otimes_B \mathcal{F}(B, J)$$

in such a way that Z acts via

$$L_{b_0}S_{i_1}L_{b_1}\cdots S_{i_n}L_{b_n} \otimes L_{b_{n+m}}S_{j_{n+m}}L_{b_{n+m-1}}\cdots S_{j_{n+1}}L_{b_{n+1}},$$

so that if $E_I : \mathcal{L}(\mathcal{F}(B, I)) \to B$ and $E_J : \mathcal{L}(\mathcal{F}(B, J)) \to B$ are the corresponding expectations, then

$$E(Z) = E_I(L_{b_0}S_{i_1}L_{b_1}\cdots S_{i_n}L_{b_n})E_J(L_{b_{n+m}}S_{j_{n+m}}L_{b_{n+m-1}}\cdots S_{j_{n+1}}L_{b_{n+1}}).$$

Therefore, if we have $Z = \sum_{k=1}^{d} X_k Y_k$ where X_k is a product of L_b's and S_i's and Y_k is a product of R_b's and D_j's, then

$$\tau(Z^*Z) = \sum_{k_1,k_2=1}^{d} \tau_B\left(E_I\left(X_{k_1}^* X_{k_2}\right) E_J\left(\widetilde{Y}_{k_2} \widetilde{Y}_{k_1}^*\right)\right),$$

where $\widetilde{Y}$ represents the monomial obtained by reversing the order and changing R's to L's and D's to S's. However, as E_I and E_J are completely positive on the algebras generated by L's and S's, we can find $b_{X,k_1,k_3}, b_{Y,k_1,k_4} \in B$ such that

$$E_I\left(X_{k_1}^* X_{k_2}\right) = \sum_{k_3=1}^{d_2} b_{X,k_1,k_3}^* b_{X,k_2,k_3} \quad \text{and} \quad E_J\left(\widetilde{Y}_{k_2} \widetilde{Y}_{k_1}^*\right) = \sum_{k_4=1}^{d_3} b_{Y,k_2,k_4} b_{Y,k_1,k_4}^*,$$

thus

$$\begin{aligned}
\tau(Z^*Z) &= \sum_{k_1,k_2=1}^{d} \sum_{k_3=1}^{d_2} \sum_{k_4=1}^{d_3} \tau_B(b_{X,k_1,k_3}^* b_{X,k_2,k_3} b_{Y,k_2,k_4} b_{Y,k_1,k_4}^*) \\
&= \sum_{k_3=1}^{d_2} \sum_{k_4=1}^{d_3} \tau_B\left(\left(\sum_{k_1=1}^{d} b_{X,k_1,k_3} b_{Y,k_1,k_4}\right)^* \left(\sum_{k_2=1}^{d} b_{X,k_2,k_3} b_{Y,k_2,k_4}\right)\right) \geq 0.
\end{aligned}$$

Hence τ is positive.

Next, one can verify in $L_2(A,\tau)$ that S_i and D_j are the sum of an isometry and its adjoint (see [71, Proposition 4.6.9]) and thus define bounded linear operators. Hence $(A, E, \varepsilon, \tau)$ is an analytical B-B-non-commutative probability space.

To see the cumulant condition, one can proceed in two ways. One can immediately realize that

$$(\mathrm{alg}(B_\ell, \{S_i\}_{i\in I}), B_r) \qquad \text{and} \qquad (B_\ell, \mathrm{alg}(B_r, \{D_j\}_{j\in J})$$

are bi-free over B due to the above tensor-product relation. This implies mixed cumulants are zero. The other cumulants then follow from the free case in [71]. Alternatively, one can analyze the actions of L_b, R_b, S_i, and D_j as one would on the operator-valued reduced free product space in an identical way to the LR-diagrams of [10, 11] to obtain a diagrammatic description of the elements of $\mathcal{F}(B, K)$ produced, note that the ones that contribute to a B-element are exactly the bi-non-crossing diagrams that correspond to pair bi-non-crossing partitions, and use induction to deduce the values of the operator-valued cumulants. $\square$

We immediately obtain the following using Theorem 5.2.3 and Theorem 5.1.4.

Lemma 5.2.4. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, and let $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ be $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators in A. Then*

$$J_\ell\left(S_1 : \left(\mathrm{alg}\left(B_\ell, \{S_i\}_{j=2}^n\right), \mathrm{alg}\left(B_r, \{D_j\}_{j=1}^m\right)\right), \eta_{\ell,1}\right) = S_1.$$

A similar result holds for the other left and the right conjugate variables.

In order to obtain more examples of bi-free conjugate variables, it would be typical to perturb by bi-semicircular operators and use Proposition 5.1.13. To do this we must have the collection of bi-semicircular operators in the same analytical B-B-non-commutative probability space. Thus, it is natural to ask whether given two analytical B-B-non-commutative probability spaces there is a bi-free product which causes the pairs of left and right algebras to be bi-freely independent over B and preserve the analytical properties.

Unfortunately we do not have an answer to this question. The proof of positivity in the operator-valued free case requires the characterization of the vanishing of alternating centred moments in [71, Proposition 3.3.3] to ensure positivity in the end. One may attempt to use the bi-free analogue of 'alternating centred moments vanish' from [9], however the bi-free formulae generalization of [71, Proposition 3.3.3] is far more complicated. In particular, the proof from [71] will not immediately generalize, as Example 4.2.4 shows E will not be positive and the traciality of τ_B will need to come into play.

Luckily, if we deal only with bi-semicircular operators, which is all that is required for our purposes, there is no issue. In fact, in the case one is working with von Neumann factors as in Example 4.1.3, the following is trivial as it we can add the corresponding collection of bi-semicircular operators using factors by [71].

Theorem 5.2.5. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space with A_ℓ and A_r generated by isometries, let $\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, and let $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ be $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators. Then, there exists an analytical B-B-non-commutative probability space $(A', E', \varepsilon', \tau')$ with $A \subseteq A'$, $E'|_A = E$, $\tau'|_A = \tau$, $A_\ell \subseteq A'_\ell$, $A_r \subseteq A'_r$, $\{S_i\}_{i=1}^n \subseteq A'_\ell$, $\{D_j\}_{j=1}^m \subseteq A'_r$ and such that the pairs of algebras*

$$(A_\ell, A_r) \qquad \textit{and} \qquad \left(\mathrm{alg}(B_\ell, \{S_i\}_{i=1}^n), \mathrm{alg}(B_r, \{D_j\}_{j=1}^m)\right)$$

are bi-free with amalgamation over B with respect to E'.

As for any $(\{\eta_{i,i}\}_{i\in I}, \{\eta_{j,j}\}_{j\in J})$ bi-semicircular operators $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ we know that

$$\{(\mathrm{alg}(B_\ell, S_i), B_r)\}_{i\in I} \cup \{(B_\ell, \mathrm{alg}(B_r, D_j))\}_{j\in J}$$

are bi-free over B, to prove Theorem 5.2.5 it suffices to use the following lemma and an analogous result on the right iteratively, or simply adapt the proof to multiple operators simultaneously.

Lemma 5.2.6. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space with A_ℓ and A_r generated by isometries, let $\eta : B \to B$ be a completely positive map and let S be an η-semicircular operator. Then, there exists an analytical B-B-non-commutative probability space $(A', E', \varepsilon', \tau')$ such that $A' = \mathrm{alg}(A_\ell, A_r, S)$, $E'|_A = E$, $\tau'|_A = \tau$, $A'_\ell = \mathrm{alg}(A_\ell, S)$, $A'_r = A_r$ and*

$$(A_\ell, A_r) \qquad \textit{and} \qquad (\mathrm{alg}(B_\ell, S), B_r)$$

are bi-free with amalgamation over B with respect to E'.

Proof. By taking the operator-valued bi-free product of B-B-non-commutative probability spaces, we obtain a B-B-non-commutative probability space (A', E', ε') such that $A' = \mathrm{alg}(A_\ell, A_r, S)$, $A'_\ell = \mathrm{alg}(A_\ell, S)$, $A'_r = A_r$, $E'|_A = E$ and

$$(A_\ell, A_r) \qquad \text{and} \qquad (\mathrm{alg}(B_\ell, S), B_r)$$

are bi-free with amalgamation over B with respect to E'. Note E' restricted to A_r is trivially completely positive and E' restricted to A'_ℓ is completely positive by the free result from [71]. Thus, to verify Definition 4.2.1 it suffices to verify that if $\tau' = \tau_B \circ E$, then τ' is positive and elements of A'_ℓ and A'_r define bounded operators on $L_2(A', \tau')$.

By analyzing the reduced free product construction, we can realize A_ℓ, A_r, and S as operators acting on

$$\mathcal{F} = B \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots,$$

where

$$\mathcal{H}_m = \{a_0 Z a_1 \cdots Z a_m \mid a_0, a_1, \ldots, a_{m-1} \in A_\ell, a_m \in A\}$$

and if $p : \mathcal{F} \to B$ is defined by taking the B-term in $\mathcal{F}$, then

$$E'(T) = p(T1_B).$$

Define a function $\langle\,\cdot,\cdot\,\rangle : \mathcal{F}\times\mathcal{F}\to A$ by setting B, $\mathcal{H}_1$, $\mathcal{H}_2$, ... to be pairwise orthogonal, $\langle b_1, b_2\rangle = L_{b_2^* b_1}$, and

$$\langle a_0' Z a_1' \cdots Z a_m', a_0 Z a_1 \cdots Z a_m\rangle = a_m^* L_\eta(a_{m-1}^* \cdots L_\eta(a_1^* L_\eta(a_0^* a_0') a_1') \cdots a_{m-1}') a_m',$$

where $L_\eta(T) = L_{\eta(E(T))}$. As η is completely positive and E is completely positive when restricted to A_ℓ, we obtain that $\langle\,\cdot,\cdot\,\rangle$ is an A-valued inner product by the same arguments as [71, Proposition 4.6.6]. To elaborate slightly, given $\sum_{k=1}^n a_{k,0} Z a_{k,1} \cdots Z a_{k,m}$, the matrix $[\eta(a_{i,0}^* a_{j,0})]$ is positive and thus can be written as $[\sum_{k=1}^n b_{k,i}^* b_{k,j}]$ for some $b_{i,j}\in B$. One then substitutes $L_\eta(a_{i,0}^* a_{j,0}) = \sum_{k=1}^n L_{b_{k,i}}^* L_{b_{k,j}}$ and continues until one ends with a sum of products of elements of A with their adjoints.

As $\tau : A \to \mathbb{C}$ is positive and as for all $T \in A'$,

$$\tau'(T^*T) = \tau(\langle T1_B, T1_B\rangle),$$

we obtain that τ' is positive as desired. To see that elements of A_ℓ' and A_r' define bounded operators on $L_2(A', \tau')$ note if $T \in A_r$, then, using the above description,

$$T(a_0 Z a_1 \cdots Z a_m) = a_0 Z a_1 \cdots Z T a_m.$$

As any of the terms

$$L_\eta(a_{m-1}^* \cdots L_\eta(a_1^* L_\eta(a_0^* a_0') a_1') \cdots a_{m-1}')$$

in the above A-valued inner product will be able to be written as sums involving terms of the form $L_{b_1}^* L_{b_2}$ and will then produce terms of the form

$$a_m^* T^* L_{b_1}^* L_{b_2} T a_m' = a_m^* L_{b_1}^* T^* T L_{b_2} a_m'$$

in the A-valued inner product when T acts on the left, the fact that A_r is generated by isometries yields that A_r acts as bounded operators on $L_2(A, \tau)$. The fact that A_ℓ is generated by isometries immediately yields that A_ℓ acts as bounded operators on $L_2(A, \tau)$, and it is not difficult to see that S acts as the sum of an isometry and its adjoint on $L_2(A, \tau)$ and thus is bounded. $\square$

With Theorem 5.2.5 establishing we can always assume our B-B-non-commutative probability spaces have $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators, we can proceed with the following.

Theorem 5.2.7. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^{n}$ and $\{\eta_{r,j}\}_{j=1}^{m}$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ be tuples of self-adjoint operators, let $(\{S_i\}_{i=1}^{n}, \{D_j\}_{j=1}^{m})$ be $(\{\eta_{\ell,i}\}_{i=1}^{n}, \{\eta_{r,j}\}_{j=1}^{m})$ bi-semicircular operators in A and let (C_ℓ, C_r) be pairs of B-algebras of A such that*

$$\{(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})))\} \cup \{(\mathrm{alg}(B_\ell, S_i), B_r)\}_{i=1}^{n} \cup \{(B_\ell, \mathrm{alg}(B_r, D_j))\}_{j=1}^{m}$$

are bi-free. If P is the orthogonal projection of $L_2(A, \varphi)$ onto

$$\overline{\mathrm{alg}\left(C_\ell, C_r, \mathbf{X} + \sqrt{\epsilon}\mathbf{S}, \mathbf{Y} + \sqrt{\epsilon}\mathbf{D}\right)}^{\|\cdot\|_\tau},$$

then

$$\xi = J_\ell\left(X_1 + \sqrt{\epsilon}S_1 : \left(\mathrm{alg}\left(C_\ell, (\widehat{\mathbf{X} + \sqrt{\epsilon}\mathbf{S}})_1\right), \mathrm{alg}\left(C_r, \mathbf{Y} + \sqrt{\epsilon}\mathbf{D}\right)\right), \eta_{\ell,1}\right) = \frac{1}{\sqrt{\epsilon}}P(S_1).$$

Thus

$$\|\xi\|_\tau \leq \frac{1}{\sqrt{\epsilon}}\sqrt{\tau_B(\eta(1_B))}.$$

A similar computation holds for the other entries of the tuples and the right conjugate variables.

Proof. By Lemma 5.2.4 and Lemma 5.1.6, we have that

$$J_\ell\left(\sqrt{\epsilon}S_1 : \left(\mathrm{alg}\left(B_\ell, \sqrt{\epsilon}\hat{\mathbf{S}}_1\right), \mathrm{alg}\left(B_r, \sqrt{\epsilon}\mathbf{D}\right)\right), \eta_{\ell,1}\right) = \frac{1}{\sqrt{\epsilon}}S_1.$$

The conjugate variable result then follows from Propositions 5.1.12 and 5.1.13, whereas the τ-norm computation is trivial. □

5.3 Bi-Free Fisher Information with Respect to a Completely Positive Map

With the above technology, the bi-free Fisher information with respect to completely positive maps can be constructed and has similar properties to the bi-free Fisher information from

[14] and the free Fisher information with respect to a completely positive map from [71]. We highlight the main results and properties in this section.

Definition 5.3.1. Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ and let (C_ℓ, C_r) be a pair of B-algebras of A. The *relative bi-free Fisher information of* $(\mathbf{X}, \mathbf{Y})$ *with respect to* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ *in the presence of* (C_ℓ, C_r) is

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \sum_{i=1}^n \|\xi_i\|_\tau^2 + \sum_{j=1}^m \|\nu_j\|_\tau^2$$

where for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$

$$\xi_i = J_\ell\left(X_i : \left(\mathrm{alg}\left(C_\ell, \widehat{\mathbf{X}}_i\right), \mathrm{alg}\left(C_r, \mathbf{Y}\right)\right), \eta_{\ell,i}\right)$$

and

$$\nu_j = J_r\left(Y_j : \left(\mathrm{alg}\left(C_\ell, \mathbf{X}\right), \mathrm{alg}\left(C_r, \widehat{\mathbf{Y}}_j\right)\right), \eta_{r,j}\right),$$

provided these variables exist, and otherwise is defined as ∞.

In the case that $\eta_{\ell,i} = \eta_{r,j} = \eta$ for all i and j, we use $\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), \eta)$ to denote the above bi-free Fisher information. In the case that $C_\ell = B_\ell$ and $C_r = B_r$, we use $\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))$. In the case both occur, we use $\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : \eta)$.

Note the bi-free Fisher with respect to completely positive maps exists in many settings due to Theorem 5.2.5 and Theorem 5.2.7. Furthermore, the properties of the bi-free Fisher with respect to completely positive maps are in analogy with those from [14, 71, 80] as the following shows.

Remark 5.3.2. (i) In the case that $B = \mathbb{C}$ and η is unital, Definition 5.3.1 immediately reduces down to the bi-free Fisher information in [14, Definition 5.1] by Remark 5.1.3.

(ii) In the case we are in the context of Example 4.1.3 with $m = 0$, $C_\ell = B_\ell$, and $C_r = B_r$, Definition 5.3.1 immediately reduces down to the free Fisher information with respect to a complete positive map from [71, Definition 4.1].

(iii) Note

$$\begin{aligned}
&\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\
&\quad = \sum_{i=1}^n \Phi^*\left(X_i \sqcup \emptyset : \left(\mathrm{alg}\left(C_\ell, \widehat{\mathbf{X}}_i\right), \mathrm{alg}\left(C_r, \mathbf{Y}\right)\right), \eta_{\ell,i}, \tau\right) \\
&\qquad + \sum_{j=1}^m \Phi^*\left(\emptyset \sqcup Y_j : \left(\mathrm{alg}\left(C_\ell, \mathbf{X}\right), \mathrm{alg}\left(C_r, \widehat{\mathbf{Y}}_j\right)\right), \eta_{r,j}, \tau\right).
\end{aligned}$$

(iv) If $\mathbf{X} = (X_1, X_1^*, \ldots, X_n, X_n^*)$ and $\mathbf{Y} = (Y_1, Y_1^*, \ldots, Y_m, Y_m^*)$, then Lemma 5.1.5 implies

$$\Phi^*(\mathbf{X}\sqcup\mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \frac{1}{2}\Phi^*(\mathbf{X}'\sqcup\mathbf{Y}' : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))$$

where

$$\mathbf{X}' = (\Re(X_1), \Im(X_1), \ldots, \Re(X_n), \Im(X_n)) \text{ and } \mathbf{Y}' = (\Re(Y_1), \Im(Y_1), \ldots, \Re(Y_m), \Im(Y_m)).$$

(v) In the context of Proposition 5.1.10 (i.e. reducing the B-B-non-commutative probability space to a D-D-non-commutative probability space),

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i} \circ F\}_{i=1}^n, \{\eta_{r,j} \circ F\}_{j=1}^m)).$$

(vi) By Lemma 5.1.6, for all $\lambda \in \mathbb{C} \setminus \{0\}$

$$\Phi^*(\lambda\mathbf{X} \sqcup \lambda\mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \tfrac{1}{|\lambda|^2}\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(vii) In the context of Lemma 5.1.11 (i.e. (D_ℓ, D_r) a smaller pair of B-algebras than (C_ℓ, C_r)),

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (D_\ell, D_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \leq \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(viii) In the context of Proposition 5.1.12 (i.e. adding in a bi-free pair of B-algebras),

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\mathrm{alg}(D_\ell, C_\ell), \mathrm{alg}(D_r, C_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(ix) If, in addition to the assumptions of Definition 5.3.1, we have $\mathbf{X}' \in A_\ell^{n'}$, $\mathbf{Y}' \in A_r^{m'}$, $(\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})$ is a collection of completely positive maps on B, and (D_ℓ, D_r) is a pair of B-algebras then, by (iii) and (vii),

$$\Phi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'}))$$

$$\geq \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \Phi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'}))$$

(x) In the context of (ix) with the additional assumption that

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})) \qquad \text{and} \qquad (\mathrm{alg}(D_\ell, \mathbf{X}'), \mathrm{alg}(D_r, \mathbf{Y}'))$$

are bi-free with amalgamation over B with respect to E, Proposition 5.1.12 implies that

$$\Phi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'}))$$

$$= \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \Phi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'}))$$

Unsurprisingly, more complicated properties of free Fisher information extend.

Proposition 5.3.3 (Bi-Free Stam Inequality). *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X}, \mathbf{X}' \in A_\ell^n$ and $\mathbf{Y}, \mathbf{Y}' \in A_r^m$, and let (C_ℓ, C_r) and (D_ℓ, D_r) be pairs of B-algebras of A such that*

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})) \qquad \textit{and} \qquad (\mathrm{alg}(D_\ell, \mathbf{X}'), \mathrm{alg}(D_r, \mathbf{Y}'))$$

are bi-free with amalgamation over B with respect to E. Then

$$\begin{aligned}
&\left(\Phi^*(\mathbf{X} + \mathbf{X}' \sqcup \mathbf{Y} + \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\right)^{-1} \\
&\geq \left(\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\right)^{-1} \\
&\quad + \left(\Phi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\right)^{-1}.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
P_0 &: L_2(\mathcal{A}, \varphi) \to L_2(B, \tau_B) \\
P_1 &: L_2(\mathcal{A}, \varphi) \to \overline{\mathrm{alg}(C_\ell, C_r, \mathbf{X}, \mathbf{Y})}^{\|\cdot\|_\tau} \\
P_2 &: L_2(\mathcal{A}, \varphi) \to \overline{\mathrm{alg}(D_\ell, D_r, \mathbf{X}', \mathbf{Y}')}^{\|\cdot\|_\tau}
\end{aligned}$$

be the orthogonal projections onto their co-domains. Note that if

$$Z \in \mathrm{alg}(C_\ell, C_r, \mathbf{X}, \mathbf{Y}) \quad \text{and} \quad \mathrm{alg}(D_\ell, D_r, \mathbf{X}', \mathbf{Y}'),$$

then bi-freeness implies

$$\widetilde{E}(ZZ') = \widetilde{E}\left(Z\widetilde{E}(Z')\right).$$

Indeed, this is easily seen as if Z and Z' are monomials, then any cumulant of the monomial ZZ' corresponding to a bi-non-crossing partition is non-zero if and only if it decomposes into a bi-non-crossing partition on Z union a bi-non-crossing partition on Z'. Thus $P_1P_2 = P_2P_1 = P_0$.

The remainder of the proof can then be read from [71, Proposition 4.5], [14, Proposition 5.8], or even [80, Proposition 6.5]. $\square$

Proposition 5.3.4 (Bi-Free Cramer-Rao Inequality)**.** *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ consist of self-adjoint operators and let (C_ℓ, C_r) be a pair of B-algebras of A. Then*

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\tau\left(\textstyle\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right) \geq \left(\textstyle\sum_{i=1}^n \tau_B(\eta_{\ell,i}(1)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1))\right)^2.$$

Moreover, equality holds if $(\mathbf{X}, \mathbf{Y})$ are $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$-bi-semicircular elements and

$$\{(C_\ell, C_r)\} \cup \{(\mathrm{alg}(B_\ell, X_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, Y_j))\}_{j=1}^m$$

are bi-free with amalgamation over B with respect to E. The converse holds when $C_\ell = B_\ell$ and $C_r = B_r$.

Proof. The result follow from the obvious modifications to [14, Proposition 5.10]. Also see [71, Proposition 4.6] and [80, Proposition 6.9]. $\square$

Similarly, limits behave as one expects based on [14, 71, 80].

Proposition 5.3.5. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ consist of self-adjoint operators and let (C_ℓ, C_r) be a pair of B-algebras of A. Suppose further for each $k \in \mathbb{N}$ that $\mathbf{X}^{(k)} \in A_\ell^n$ and $\mathbf{Y}^{(k)} \in A_r^m$ are tuples of self-adjoint elements in A such that*

$$\limsup_{k\to\infty} \left\| X_i^{(k)} \right\| < \infty,$$
$$\limsup_{k\to\infty} \left\| Y_j^{(k)} \right\| < \infty,$$
$$s\text{-}\lim_{k\to\infty} X_i^{(k)} = X_i, \textit{ and}$$
$$s\text{-}\lim_{k\to\infty} Y_j^{(k)} = Y_j$$

for all $1 \leq i \leq n$ *and* $1 \leq j \leq m$ *(where the strong limit is computed as bounded linear maps acting on* $L_2(A, \tau)$*). Then*

$$\liminf_{k\to\infty} \Phi^* \left(\mathbf{X}^{(k)} \sqcup \mathbf{Y}^{(k)} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right) \geq \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

The proof of Proposition 5.3.5 becomes identical to [14, Proposition 5.12] once the following lemma is established. Also see [71, Proposition 4.7] and [80, Proposition 6.10].

Lemma 5.3.6. *Under the assumptions of Proposition 5.3.5 along with the additional assumptions that*

$$\xi_k = J_\ell \left(X_1^{(k)} : \left(\operatorname{alg}\left(C_\ell, \widehat{\mathbf{X}}_1^{(k)} \right), \operatorname{alg}(C_r, \mathbf{Y}^{(k)}) \right), \eta_{\ell,1} \right)$$

exist and are bounded in L_2*-norm by some constant* $K > 0$*, it follows that*

$$\xi = J_\ell \left(X_1 : \left(\operatorname{alg}\left(C_\ell, \widehat{\mathbf{X}}_1 \right), \operatorname{alg}(C_r, \mathbf{Y}) \right), \eta_{\ell,1} \right)$$

exists and is equal to

$$w\text{-}\lim_{k\to\infty} P\left(\xi_k\right)$$

where P *is the orthogonal projection of* $L_2(A, \tau)$ *onto* $\overline{\operatorname{alg}(C_\ell, C_r, \mathbf{X}, \mathbf{Y})}^{\|\cdot\|_\tau}$.

If, in addition,

$$\limsup_{k\to\infty} \|\xi_k\|_2 \leq \|\xi\|_2$$

then

$$\lim_{k\to\infty} \|\xi_k - \xi\|_2 = 0.$$

The same holds with X_1 *replaced with* X_i*, and a similar result holds for the right.*

Proof. The proof of this result follows from the same sequence of steps as [14, Lemma 5.13] using the analytical operator-valued bi-free cumulants. ☐

Corollary 5.3.7. *Under the assumptions of Proposition 5.3.5, if in addition*

$$(B_\ell(\mathbf{X}), B_r(\mathbf{Y})) \qquad \text{and} \qquad \left(\operatorname{alg}\left(C_\ell, \mathbf{X}^{(k)}\right), \operatorname{alg}\left(C_r, \mathbf{Y}^{(k)}\right)\right)$$

are bi-free for all k and

$$\lim_{k\to\infty} \left\| X_i^{(k)} \right\| = \lim_{k\to\infty} \left\| Y_j^{(k)} \right\| = 0$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$, then

$\lim_{k\to\infty} \Phi^*\left(\mathbf{X} + \mathbf{X}^{(k)} \sqcup \mathbf{Y} + \mathbf{Y}^{(k)} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right) = \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$

Furthermore, if $C_\ell = B_\ell$, $C_r = B_r$, and

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) < \infty,$$

then

$$J_\ell\left(X_i^{(k)} : \left(B_\ell\left((\widehat{\mathbf{X} + \mathbf{X}^{(k)}})_i\right), B_r\left(\mathbf{Y} + \mathbf{Y}^{(k)}\right)\right), \eta_{\ell,i}\right)$$

tends to

$$J_\ell\left(X_i : \left(B_\ell\left(\hat{\mathbf{X}}_i\right), B_r\left(\mathbf{Y}\right)\right), \eta_{\ell,i}\right)$$

in the τ-norm. A similar result holds for right bi-free conjugate variables.

Proof. The proof is identical to [14, Corollary 5.14] and thus is omitted. □

Theorem 5.3.8. *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ be tuples of self-adjoint operators, let $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ be a collection of $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators in A and let (C_ℓ, C_r) be pairs of B-algebras of A such that*

$$\{(\operatorname{alg}(C_\ell, \mathbf{X}), \operatorname{alg}(C_r, \mathbf{Y})))\} \cup \{(\operatorname{alg}(B_\ell, S_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \operatorname{alg}(B_r, D_j))\}_{j=1}^m$$

are bi-free. Then, the map

$$h : [0, \infty) \ni t \mapsto \Phi^*\left(\mathbf{X} + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right)$$

is decreasing, right continuous, and

$$\frac{K_2^2}{K_1 + K_2 t} \leq h(t) \leq \frac{1}{t} K_3,$$

where

$$K_1 = \tau\left(\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2\right),$$
$$K_2 = \sum_{i=1}^{n} \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^{m} \tau_B(\eta_{r,j}(1_B)) \text{ and}$$
$$K_3 = \sum_{i=1}^{n} \tau_B(\eta_{\ell,i}(1_B))^2 + \sum_{j=1}^{m} \tau_B(\eta_{r,j}(1_B))^2.$$

Moreover if $(\mathbf{X}, \mathbf{Y})$ *is the* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$*-bi-semicircular distribution and*

$$\{(C_\ell, C_r)\} \cup \{(\mathrm{alg}(B_\ell, X_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, Y_j))\}_{j=1}^m$$

are bi-free with amalgamation over B *with respect to* E*, then* $h(t) = \frac{K_2^2}{K_1^2 + K_2 t}$ *for all* t*. Finally, if* $C_\ell = B_\ell$*,* $C_r = B_r$*, and* $h(t) = \frac{K_2^2}{K_1 + K_2 t}$ *for all* t*, then* $(\mathbf{X}, \mathbf{Y})$ *is the* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$*-bi-semicircular distribution.*

Proof. The proof becomes identical to [14, Theorem 5.15] using the above and the fact that

$$\tau\left(\left(X_i + \sqrt{t}S_i\right)^2\right) = \tau\left(X_i^2\right) + t\tau\left(S_i^2\right) = \tau\left(X_i^2\right) + t\tau_B\left(\eta_{\ell,i}(1_B)\right),$$

with an analogous computation on the right for use in the lower bound computation, in conjunction with the Bi-Free Cramer-Rao Inequality (Proposition 5.3.4). □

5.4 Bi-Free Entropy with Respect to Completely Positive Maps

With the construction and properties of the bi-free Fisher information with respect to completely positive maps complete, the construction and properties of bi-free entropy with respect to completely positive maps follows easily by extending results from [71] and [14] with similar proofs.

Definition 5.4.1. Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-non-commutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let (C_ℓ, C_r) be pairs of B-algebras of A, and let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ be tuples of self-adjoint operators. The *relative* *bi-free entropy of* $(\mathbf{X}, \mathbf{Y})$ *with respect to* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ *in the presence of* (B_ℓ, B_r) is defined to be

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))$$
$$= \tfrac{K}{2}\ln(2\pi e) + \tfrac{1}{2}\int_0^\infty \left(\tfrac{K}{1+t} - \Phi^*\left(\mathbf{X} + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right)\right) dt,$$

where

$$K = \sum_{i=1}^n \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1_B))$$

and $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ is a collection of $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators such that

$$\{(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})\rangle)\} \cup \{(B_\ell(S_i), B_r)\}_{i=1}^n \cup \{(B_\ell, B_r(D_j))\}_{j=1}^m$$

are bi-free (note such semicircular operators can be included in A by Theorem 5.2.5).

In the case that $C_\ell = B_\ell$ and $C_r = B_r$, we use $\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))$ to denote the bi-free entropy. If in addition $\eta_{\ell,i} = \eta_{r,j} = \eta$ for all i and j, we use $\chi^*(\mathbf{X} \sqcup \mathbf{Y} : \eta)$ to denote the bi-free entropy.

We note there is a slight change in the normalization used in Definition 5.4.1 over that used in [71, Definition 8.1]. Generally this makes no real difference other than making some of the bounds in this section nice, such as the following one.

Proposition 5.4.2. *In the context of Definition 5.4.1, if*

$$K_1 = \tau\left(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right),$$

then

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \leq \frac{K}{2}\ln\left(\frac{2\pi e}{K}K_1\right).$$

Moreover, equality holds when $(\mathbf{X}, \mathbf{Y})$ *are* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$*-bi-semicircular operators such that* $\{(C_\ell, C_r)\} \cup \{(\mathrm{alg}(B_\ell, X_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, Y_j))\}_{j=1}^m$ *are bi-free and if* $C_\ell = B_\ell$ *and* $C_r = B_r$*, this is the only setting where equality holds.*

Proof. The proof is identical to [14, Proposition 6.5] in conjunction with Theorem 5.3.8. □

Remark 5.4.3. (i) In the case that $B = \mathbb{C}$ and η is unital, Definition 5.4.1 produces the non-microstate bi-free entropy from [14, Definition 6.1].

(ii) In the setting of Example 4.1.3, when $C_r = B_r$ and $\eta_{\ell,i} = \eta_{r,j} = \eta$ for all i and j, Definition 5.4.1 produces the free entropy with respect to a completely positive map from [71, Definition 8.1] modulo an additive constant (which is 0 in the case η is unital).

Of course, due to the fact that the bi-free Fisher information from Section 5.3 behaves analogously to the Fisher informations considered in [14, 71], results for the behaviour of entropy automatically generalize.

Proposition 5.4.4. *Using Remark 5.3.2, the following hold:*

(v) In the context of Proposition 5.1.10 (i.e. reducing the B-B-non-commutative probability space to a D-D-non-commutative probability space),

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i} \circ F\}_{i=1}^n, \{\eta_{r,j} \circ F\}_{j=1}^m)).$$

(vi) For all $\lambda \in \mathbb{R} \setminus \{0\}$

$$\chi^*(\lambda\mathbf{X} \sqcup \lambda\mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))$$

$$= \left(\sum_{i=1}^n \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1_B))\right) \ln|\lambda| + \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(vii) In the context of Lemma 5.1.11 (i.e. (D_ℓ, D_r) a smaller pair of B-algebras than (C_ℓ, C_r)),

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (D_\ell, D_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \geq \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(viii) In the context of Proposition 5.1.12 (i.e. adding in a bi-free pair of B-algebras),

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (\mathrm{alg}(D_\ell, C_\ell), \mathrm{alg}(D_r, C_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(ix) If in addition to the assumptions of Definition 5.4.1 we have $\mathbf{X}' \in A_\ell^{n'}$, $\mathbf{Y}' \in A_r^{m'}$, $(\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})$ is a collection of completely positive maps on B, and (D_ℓ, D_r) is a pair of B-algebras, then

$$\chi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'}))$$

$$\leq \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \chi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})).$$

(x) In the context of (ix) with the additional assumption that

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})) \qquad \textit{and} \qquad (\mathrm{alg}(D_\ell, \mathbf{X}'), \mathrm{alg}(D_r, \mathbf{Y}'))$$

are bi-free with amalgamation over B with respect to E, Proposition 5.1.12 implies that

$$\begin{aligned}&\chi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'}))\\ &= \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \chi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})).\end{aligned}$$

Using Proposition 5.3.5, Theorem 5.3.8 and the same arguments as [14, Proposition 6.7], the following holds.

Proposition 5.4.5. *Under the assumptions of Definition 5.4.1, if for each $k \in \mathbb{N}$ there exists self-adjoint tuples $\mathbf{X}^{(k)} \in A_\ell^n$ and $\mathbf{Y}^{(k)} \in A_r^m$ such that*

$$\begin{aligned}&\limsup_{k\to\infty} \left\| X_i^{(k)} \right\| < \infty,\\ &\limsup_{k\to\infty} \left\| Y_j^{(k)} \right\| < \infty,\\ &s\text{-}\lim_{k\to\infty} X_i^{(k)} = X_i, \ \textit{and}\\ &s\text{-}\lim_{k\to\infty} Y_j^{(k)} = Y_j\end{aligned}$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$ (with the strong limit computed as bounded linear maps acting on $L_2(A, \tau)$), then

$$\limsup_{k\to\infty} \chi^*\left(\mathbf{X}^{(k)} \sqcup \mathbf{Y}^{(k)} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right) \leq \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

Using the previous proposition together with Theorem 5.3.8 and the same arguments as [14, Proposition 6.8], the following holds.

Proposition 5.4.6. *Under the assumptions of Definition 5.4.1, suppose $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ is a collection of $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators such that*

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y}))) \cup \{(\mathrm{alg}(B_\ell, S_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, D_j))\}_{j=1}^m$$

are bi-free. For $t \in [0, \infty)$, let

$$g(t) = \chi^*\left(\mathbf{X} + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right).$$

Then $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ is a concave, continuous, increasing function such that $g(t) \geq \frac{K}{2} \ln(2\pi e t)$ where

$$K = \sum_{i=1}^{n} \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^{m} \tau_B(\eta_{r,j}(1_B))$$

and, when $g(t) \neq -\infty$,

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon}(g(t+\epsilon) - g(t)) = \frac{1}{2}\Phi^*\left(\mathbf{X} + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^{n}, \{\eta_{r,j}\}_{j=1}^{m})\right).$$

Finally, using the Bi-Free Stam Inequality (Proposition 5.3.3) together with the same proof as [14, Proposition 6.11] yields the following.

Proposition 5.4.7. *Under the assumptions of Definition 5.4.1, if*

$$\Phi^*\left(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^{n}, \{\eta_{r,j}\}_{j=1}^{m})\right) < \infty,$$

then

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (B_\ell, B_r)) \geq \frac{K}{2} \ln\left(\frac{2\pi K e}{\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^{n}, \{\eta_{r,j}\}_{j=1}^{m}))}\right) > -\infty,$$

where

$$K = \sum_{i=1}^{n} \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^{m} \tau_B(\eta_{r,j}(1_B)).$$

Chapter 6

Minimization Problems for the Bi-Free Fisher Information

This chapter culminates the machinery developed in chapters 4 and 5 and presents applications of its theory. Specifically, we will focus on proving Theorem 6.1.6 which describes the minimal values of the bi-free Fisher information of non-self-adjoint pairs of operators under certain distributional conditions and asserts the these minimal values are achieved by bi-R-diagonal pairs of operators. Afterwards, a standard argument will show how minimization results about the bi-free Fisher information can be translated into corresponding maximization results concerning bi-free entropy. The general outline of the proof will follow that of [58, Theorem 1.1], but the necessary adaptions to the bi-free context require significant care.

6.1 Minimizing Bi-Free Fisher Information

Throughout the section, we will be working under the situation from Example 4.2.4 where $\mathcal{A}$ is a unital C*-algebra, $\varphi : \mathcal{A} \to \mathbb{C}$ is a state, $B = M_d(\mathbb{C})$ (the $d \times d$ matrices with complex entries) and $\tau_B = \mathrm{tr}_d$ (the normalized trace on $M_d(\mathbb{C})$). Thus $A_d = \mathcal{A} \otimes M_d(\mathbb{C}) \otimes M_d(\mathbb{C})^{\mathrm{op}}$, $E_d : A_d \to M_d(\mathbb{C})$ and $\tau_d : A_d \to \mathbb{C}$ are defined such that

$$E_d(Z \otimes b_1 \otimes b_2) = \varphi(Z) b_1 b_2 \qquad \text{and} \qquad \tau_d(Z \otimes b_1 \otimes b_2) = \varphi(Z)\, \mathrm{tr}_d(b_1 b_2),$$

for all $Z \in \mathcal{A}$ and $b_1, b_2 \in M_d(\mathbb{C})$. We recall the following result that aids in computing moments in (A_d, E_d, ε) where $\{E_{i,j}\}_{i,j=1}^n \subseteq M_d(\mathbb{C})$ are the canonical matrix units.

Lemma 6.1.1 ([67, Lemma 3.7])**.** *Let $(\mathcal{A}, \varphi)$ be a C^*-non-commutative probability space, let $\chi \in \{\ell, r\}^n$, and let*

$$Z_k = \begin{cases} \sum_{i,j=1}^d z_{k;i,j} \otimes E_{i,j} \otimes I_d & \text{if } \chi(k) = \ell \\ \sum_{i,j=1}^d z_{k;i,j} \otimes I_d \otimes E_{i,j} & \text{if } \chi(k) = r \end{cases}.$$

Then

$$E_d(Z_1 \cdots Z_n) = \sum_{\substack{i_1,\ldots,i_n=1 \\ j_1,\ldots,j_n=1}}^{d} \varphi(z_{1;i_1,j_1} \cdots z_{n;i_n,j_n}) E_\chi((i_1, \ldots, i_n), (j_1, \ldots, j_n))$$

where

$$E_\chi((i_1, \ldots, i_n), (j_1, \ldots, j_n)) := E_{i_{s_\chi(1)}, j_{s_\chi(1)}} \cdots E_{i_{s_\chi(n)}, j_{s_\chi(n)}} \in M_d(\mathbb{C}).$$

To discuss conjugate variables, we need to consider $L_2(A_d, \tau_d)$. It is not difficult to verify that $L_2(A_d, \tau_d)$ can be identified with the $d \times d$ matrices with entries in $L_2(\mathcal{A}, \varphi)$ where

$$(Z \otimes I_d \otimes I_d)[\xi_{i,j}] = [Z\xi_{i,j}],$$

$1_{\mathcal{A}} \otimes b \otimes I_d$ acts via left multiplication on $[\xi_{i,j}]$ and $1_{\mathcal{A}} \otimes I_d \otimes b$ acts via right multiplication on $[\xi_{i,j}]$ for all $[\xi_{i,j}] \in M_d(L_2(\mathcal{A}, \varphi))$ and $b \in M_d(\mathbb{C})$.

Next, we consider the subalgebra $D_d \subseteq M_d(\mathbb{C})$ of the diagonal matrices. Clearly if $F_d : M_d(\mathbb{C}) \to D_d$ is the canonical conditional expectation onto the diagonal, then $(A_d, F_d \circ E_d, \varepsilon|_{D_d \otimes D_d^{\mathrm{op}}}, \tau_d)$ is also an analytical D_d-D_d-non-commutative probability space (see Example 5.1.9).

To begin stating our main result, let $x, y \in \mathcal{A}$. For $d = 2$, let

$$X = x \otimes E_{1,2} \otimes I_2 + x^* \otimes E_{2,1} \otimes I_2 \qquad \text{and} \qquad Y = y \otimes I_2 \otimes E_{1,2} + y^* \otimes I_2 \otimes E_{2,1},$$

which are then self-adjoint elements of A_2. The pair (X, Y) is intimately related to whether or not (x, y) is bi-R-diagonal. Indeed, combining Proposition 3.1.23 and [68, Theorem 4.9], we obtain the following.

Proposition 6.1.2. *As described above, the following conditions are equivalent:*

(i) the pair (x, y) is bi-R-diagonal,

(ii) the pair (X, Y) is bi-R-cyclic,

(iii) the pair of algebras $(\mathrm{alg}(D_2, X), \mathrm{alg}(D_2, Y))$ *is bi-free from* $(M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)$ *with amalgamation over* D_2 *with respect to* $F_2 \circ E_2$.

In addition, the joint moments of (X, Y) with respect to τ_d are not too difficult to describe. Indeed, for any $n \in \mathbb{N}$, $Z_1, Z_2, \ldots, Z_n \in \{X, Y\}$, $\chi \in \{\ell, r\}^n$, and $z_1, \ldots, z_n \in \{x, y\}$ such that

$$\chi(k) = \begin{cases} \ell & \text{if } Z_k = X \\ r & \text{if } Z_k = Y \end{cases} \qquad \text{and} \qquad z_k = \begin{cases} x & \text{if } Z_k = X \\ y & \text{if } Z_k = Y \end{cases},$$

then

$$E_2(Z_1 \cdots Z_n) = \sum_{k=1}^{n} \sum_{p_k \in \{1,*\}} \varphi(z_1^{p_1} \cdots z_n^{p_n})(E_{1,2})^{p_{s_\chi(1)}}(E_{1,2})^{p_{s_\chi(2)}} \cdots (E_{1,2})^{p_{s_\chi(n)}}.$$

Thus, if n is odd, we see that $\tau_2(Z_1 \cdots Z_n) = 0$ and if n is even, we see that

$$\tau_2(Z_1 \cdots Z_n) = \frac{1}{2}\left(\varphi(z_1^{p_1} \cdots z_n^{p_n}) + \varphi(z_1^{q_1} \cdots z_n^{q_n})\right),$$

where

$$p_{s_\chi(k)} = \begin{cases} 1 & \text{if } k \text{ is odd} \\ * & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} * & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}$$

(that is, the 1's and $*$'s alternate in the χ-ordering). Hence, the joint moments of (X, Y) with respect to τ_d depend only on specific moments of (x, y). We let $\Delta_{X,Y}$ denote the set of all pairs (x_0, y_0) in a C*-non-commutative probability space $(\mathcal{A}_0, \varphi_0)$ such that if we apply the above procedure to (x_0, y_0) resulting in (X_0, Y_0), then (X_0, Y_0) has the same joint distribution as (X, Y) (so $\Delta_{X,Y} = \Delta_{X_0,Y_0}$).

One specific case worth mentioning is when x and y are both normal operators such that $[\mathrm{alg}(x, x^*), \mathrm{alg}(y, y^*)] = 0$, thus defining a probability measure μ on $\mathbb{C}^2$. Then, X and Y will be commuting self-adjoint operators and therefore their joint distribution gives rise to a compactly supported probability measure μ_0 on $\mathbb{R}^2$ with moments

$$\tau_2(X^n Y^m) = \begin{cases} 0 & \text{if } n + m \text{ is odd} \\ \varphi((x^*x)^i (y^*y)^j) & \text{if } n = 2i \text{ and } m = 2j \\ \frac{1}{2}\varphi((x^*x)^i (xy^* + x^*y)(y^*y)^j) & \text{if } n = 2i + 1 \text{ and } m = 2j + 1 \end{cases}.$$

For instance, when (x, y) is a bi-Haar unitary pair, the joint moments of the operators X

and Y are given by

$$\tau_2(X^nY^m) = \begin{cases} 0 & \text{if } n+m \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$$

and it follows in this case that $\mu_0 = \frac{1}{2}\left(\delta_{(1,1)} + \delta_{(-1,-1)}\right)$.

One additional property is required in this section. In particular, as we are attempting to generalize [58, Theorem 1.1] which makes heavy use of traciality, we need a condition that lets us bypass the issue that τ_2 is not tracial on A_2.

Definition 6.1.3. Let $(\mathcal{A}, \varphi)$ be a C*-non-commutative probability space and let $x, y \in \mathcal{A}$. We say that (x, y) is *alternating adjoint flipping with respect to* φ if for any $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^{2n}$, and $z_1, \ldots, z_{2n} \in \{x, y\}$ such that

$$z_k = \begin{cases} x & \text{if } \chi(k) = \ell \\ y & \text{if } \chi(k) = r \end{cases},$$

we have that

$$\varphi(z_1^{p_1} \cdots z_{2n}^{p_{2n}}) = \varphi(z_1^{q_1} \cdots z_{2n}^{q_{2n}})$$

where

$$p_{s_\chi(k)} = \begin{cases} 1 & \text{if } k \text{ is odd} \\ * & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} * & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

Remark 6.1.4. If (x, y) is alternating adjoint flipping, then the description of the joint moments of (X, Y) above reduce to a nicer expression. Furthermore we see that

$$\varphi((x^*x)^m) = \varphi((xx^*)^m) \qquad \text{and} \qquad \varphi((y^*y)^m) = \varphi((yy^*)^m)$$

for all $m \in \mathbb{N}$, so that x^*x and xx^* have the same distribution and y^*y and yy^* have the same distribution, which would be automatic if φ was tracial when restricted to $\mathrm{alg}(x, x^*)$ and when restricted to $\mathrm{alg}(y, y^*)$ (a common assumption in bi-free probability).

Of course, bi-Haar unitary pairs are trivially seen to be alternating adjoint flipping, since any joint moment with an equal number of adjoint and non-adjoint terms is 1 and any joint moment with a differing number of adjoint and non-adjoint terms is 0. Here is another example which is of use in this chapter.

Example 6.1.5. Let $\mathcal{H}$ be any Hilbert space of dimension at least 4, let $\mathcal{F}(\mathcal{H})$ denote the Fock space generated by $\mathcal{H}$, let φ_0 be the vacuum vector state on $\mathcal{B}(\mathcal{F}(\mathcal{H}))$ and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal set. For $i = 1, 2$ let $s_i = l(e_i) + l^*(e_i)$ (i.e. left creation plus annihilation by e_i) and for $j = 1, 2$ let $d_j = r(e_{j+2}) + r^*(e_{j+2})$ (i.e. right creation and annihilation by e_{j+2}). Thus $(\{s_1, s_2\}, \{d_1, d_2\})$ is a bi-free central limit distribution with variance 1 and covariance 0.

Let
$$c_\ell = \frac{1}{\sqrt{2}}(s_1 + is_2) \qquad \text{and} \qquad c_r = \frac{1}{\sqrt{2}}(d_1 + id_2).$$
We call the pair (c_ℓ, c_r) a *bi-free circular pair* (with mean 0, variance 1, and covariance 0).

We claim that (c_ℓ, c_r) is an example of a bi-R-diagonal pair that is alternating adjoint flipping with respect to φ_0. To see that (c_ℓ, c_r) is bi-R-diagonal, we note that any bi-free cumulant for $(\{s_1, s_2\}, \{d_1, d_2\})$ of order 1, of order greater than 3, or involving two different elements is 0. As
$$\kappa_{1_{(\ell,\ell)}}(c_\ell, c_\ell) = \frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_1, s_1) + (i)^2\frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_2, s_2) = 0,$$
$$\kappa_{1_{(\ell,\ell)}}(c_\ell^*, c_\ell^*) = \frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_1, s_1) + (-i)^2\frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_2, s_2) = 0,$$
and similar computations hold on the right, we have that (c_ℓ, c_r) is bi-R-diagonal.

To see that (c_ℓ, c_r) is alternating adjoint flipping with respect to φ_0, first note that φ_0 is tracial when restricted to $\mathrm{alg}(s_1, s_2)$ as s_1 and s_2 are freely independent with respect to φ_0. Hence for all $n \in \mathbb{N}$
$$\varphi_0((c_\ell^* c_\ell)^n) = \varphi_0((c_\ell c_\ell^*)^n).$$
Moreover, as any monomial of odd length involving freely independent semicircular variables is 0, we obtain that for all $n \in \mathbb{N}$ that
$$\varphi_0(c_\ell(c_\ell^* c_\ell)^n) = 0 = \varphi_0(c_\ell^*(c_\ell c_\ell^*)^n).$$
Similarly, for all $n \in \mathbb{N}$ we have that
$$\varphi_0((c_r^* c_r)^n) = \varphi_0((c_r c_r^*)^n) \qquad \text{and} \qquad \varphi_0(c_r(c_r^* c_r)^n) = 0 = \varphi_0(c_r^*(c_r c_r^*)^n).$$

To see the remaining moment conditions, first note that $\{c_\ell, c_\ell^*\}$ commutes with $\{c_r, c_r^*\}$. Thus, as the χ-ordering is not changed by commutation of left and right operators, it suffices to show that

$$\begin{aligned}
\varphi_0((c_\ell^* c_\ell)^n (c_r c_r^*)^m) &= \varphi_0((c_\ell c_\ell^*)^n (c_r^* c_r)^m),\\
\varphi_0(c_\ell (c_\ell^* c_\ell)^n (c_r c_r^*)^m) &= \varphi_0(c_\ell^* (c_\ell c_\ell^*)^n (c_r^* c_r)^m),\\
\varphi_0((c_\ell^* c_\ell)^n c_r^* (c_r c_r^*)^m) &= \varphi_0(c_\ell^* (c_\ell c_\ell^*)^n c_r (c_r^* c_r)^m) \text{ and}\\
\varphi_0(c_\ell (c_\ell^* c_\ell)^n c_r^* (c_r c_r^*)^m) &= \varphi_0(c_\ell^* (c_\ell c_\ell^*)^n c_r (c_r^* c_r)^m),
\end{aligned}$$

for all $n, m \in \mathbb{N} \cup \{0\}$. However, as $\{c_\ell, c_\ell^*\}$ is classically independent from $\{c_r, c_r^*\}$ since the joint bi-free cumulants vanish, each of the 8 above moment expressions simplifies to the φ_0-moment of the $\{c_\ell, c_\ell^*\}$ term times the φ_0-moment of the $\{c_r, c_r^*\}$. Thus, the desired moments are equal by the above knowledge of the φ_0-moments of the $\{c_\ell, c_\ell^*\}$ and the φ_0-moment of the $\{c_r, c_r^*\}$.

With the above definitions, notation and constructions out of the way, the main result is at hand.

Theorem 6.1.6. *Let $(\mathcal{A}, \varphi)$ be a C^*-non-commutative probability space and let $x, y \in \mathcal{A}$ be such that x^*x and xx^* have the same distribution with respect to φ and y^*y and yy^* have the same distribution with respect to φ. With X and Y as described above*

$$\min\left\{ \Phi^*(\{x_0, x_0^*\} \sqcup \{y_0, y_0^*\} : (\mathbb{C}, \mathbb{C}), \varphi) \,|\, (x_0, y_0) \in \Delta_{X,Y} \right\} \geq 2\Phi^*(X \sqcup Y)$$

and equality holds and is achieved for any pair (x_0, y_0) that is alternating adjoint flipping and bi-R-diagonal.

Remark 6.1.7. Note that Theorem 6.1.6 is a generalization of [58, Theorem 1.1] to the bi-free setting. Prior to the acknowledgements of [58] it is mentioned that the minimum in the free result can only be reached by an R-diagonal element via the result from [81] that $\Phi^*(x_1, \ldots, x_n : B) = \Phi^*(x_1, \ldots, x_n) < \infty$ implies $\{x_1, \ldots, x_n\}$ is free from B. As there is no such known analogous result in the bi-free case, we leave Theorem 6.1.6 as stated.

To begin the proof of Theorem 6.1.6, we note the following connecting the bi-free Fisher informations of $(\{x, x^*\}, \{y, y^*\})$ and (X, Y) (and thereby demonstrating the necessity of considering bi-free Fisher information with respect to completely positive maps in this construction). We note that the following is a generalization of [58, Proposition 3.6] with a similar but more complicated proof due to the χ-ordering and additional variables present.

Proposition 6.1.8. *Under the assumptions of Theorem 6.1.6, if $\eta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is defined by*

$$\eta\left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\right) = \begin{bmatrix} a_{2,2} & 0 \\ 0 & a_{1,1} \end{bmatrix},$$

then

$$\Phi^*(\{x, x^*\} \sqcup \{y, y^*\} : (\mathbb{C}, \mathbb{C}), \varphi) = 2\Phi^*(X \sqcup Y : (M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r), \eta).$$

Proof. First suppose the bi-free Fisher information $\Phi^*(\{x, x^*\} \sqcup \{y, y^*\})$ is finite. Thus there exist

$$\xi_1, \xi_2 \in \overline{\mathrm{alg}(x, x^*, y, y^*)}^{\|\cdot\|_\varphi}$$

such that ξ_1 is the left bi-free conjugate variable for x with respect to φ in the presence of $(x, \{y, y^*\})$ and ξ_2 is the left bi-free conjugate variable for x^* with respect to φ in the presence of $(x^*, \{y, y^*\})$. Let

$$\Xi = \begin{bmatrix} 0 & \xi_2 \\ \xi_1 & 0 \end{bmatrix} \in M_2(L_2(A_2, \tau_2)).$$

We claim that $\Xi = J_\ell\left(X : (M_2(\mathbb{C})_\ell, \mathrm{alg}\left(M_2(\mathbb{C})_r, Y\right)), \eta\right)$. Since a similar result holds on the right and since

$$\|\Xi\|_{\tau_2}^2 = \frac{1}{2}\left(\|\xi_1\|_\varphi^2 + \|\xi_2\|_\varphi^2\right),$$

the result will follow in this case.

First we claim that $\Xi \in \overline{\mathrm{alg}\left(X, Y, M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r\right)}^{\tau_2}$. Indeed, it is not difficult to verify that

$$z \otimes E_{i_1, j_1} \otimes E_{i_2, j_2} \in \mathrm{alg}\left(X, Y, M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r\right)$$

for all $z \in \{x, x^*, y, y^*\}$ and $i_1, i_2, j_1, j_2 \in \{1, 2\}$. Thus, since $\xi_1, \xi_2 \in \overline{\mathrm{alg}(x, x^*, y, y^*)}^{\|\cdot\|_\varphi}$, the claim follows.

To complete the claim that Ξ is the appropriate left bi-free conjugate variable, we must show Ξ satisfies the left bi-free conjugate variable relations; that is, for all $n \in \mathbb{N}$, $b_0, b_1, \ldots, b_n \in M_2(\mathbb{C})$, $\chi \in \{\ell, r\}^n$ with $\chi(n) = \ell$, and $Z_1, \ldots, Z_{n-1} \in A_2$ and $C_1, \ldots C_{n-1} \in 1_{\mathcal{A}} \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{\mathrm{op}}$ where

$$Z_k = \begin{cases} X & \text{if } \chi(k) = \ell \\ Y & \text{if } \chi(k) = r \end{cases} \qquad \text{and} \qquad C_k = \begin{cases} L_{b_k} & \text{if } \chi(k) = \ell \\ R_{b_k} & \text{if } \chi(k) = r \end{cases},$$

we have that

$$\tau_2\left(L_{b_0}R_{b_n}Z_1C_1\cdots Z_{n-1}C_{n-1}\Xi\right)=\sum_{\substack{1\leq k<n\\ \chi(k)=\ell}}\tau_2\left(L_{b_0}R_{b_n}\left(\prod_{p\in V_k^c\setminus\{k,n\}}Z_pC_p\right)L_{\eta\left(E_2\left(C_k\prod_{p\in V_k}Z_pC_p\right)\right)}\right),\tag{6.1}$$

where $V_k=\{k<m<n\mid\chi(m)=\ell\}$. By linearity, it suffices to consider $b_k=E_{i_k,j_k}$ for all k where $i_k,j_k\in\{1,2\}$. In that which follows, the proof is near identical to that of [58, Proposition 3.6] taking into account the χ-order. For notational purposes, for $k\in\{1,2\}$ let $\overline{k}=3-k$.

Let $q=s_\chi^{-1}(n)$ (i.e. Ξ appears q^{th} in the χ-ordering). We begin by computing the left-hand side of (6.1). Using Lemma 6.1.1 (and recalling $\tau_2=\mathrm{tr}_2\circ E_2$), proceeding via χ-order using commutation, we obtain that

- the only way the product produces a non-zero trace is if $i_0=j_n$,
- the term $L_{b_0}XL_{b_{s_\chi(1)}}$ can be made to appear in the product and is non-zero only if $j_0=\overline{i_{s_\chi(1)}}$,
- the term $L_{b_{s_\chi(k-1)}}XL_{b_{s_\chi(k)}}$ can be made to occur for all $2\leq k<q$ and is non-zero only if $j_{s_\chi(k-1)}=\overline{i_{s_\chi(k)}}$,
- the term $L_{b_{s_\chi(q-1)}}R_{b_{s_\chi(q+1)}}\Xi$ can be made to occur and is non-zero only if $j_{s_\chi(q-1)}=\overline{i_{s_\chi(q+1)}}$,
- the term $R_{b_{s_\chi(k+1)}}YR_{b_{s_\chi(k)}}$ can be made to occur for all $q<k<n$ and is non-zero only if $j_{s_\chi(k)}=\overline{i_{s_\chi(k+1)}}$ (recall the opposite multiplication), and
- the term $R_{b_n}YR_{b_{s_\chi(n)}}$ can be made to occur and is non-zero only if $j_{s_\chi(n)}=\overline{i_n}$.

Note the discrepancy in notation around the Ξ term due to the labelling of the left and right B-operators (i.e. $b_{s_\chi(q)}=b_n$ is in the wrong spot). Thus with

$$(X)_{1,2}=x,\quad (X)_{2,1}=x^*,\quad (Y)_{1,2}=y,\quad (Y)_{2,1}=y^*,\quad (\Xi)_{1,2}=\xi_2,\quad\text{and}\quad (\Xi)_{2,1}=\xi_1,$$

and

$$\overline{Z_k} = \begin{cases} (X)_{j_0,\overline{j_0}} & \text{if } s_\chi(k) = 1 \\ (X)_{j_{s_\chi(k-1)},\overline{j_{s_\chi(k-1)}}} & \text{if } 1 < s_\chi(k) < q \\ (\Xi)_{j_{s_\chi(q-1)},\overline{j_{s_\chi(q-1)}}} & \text{if } s_\chi(k) = q \\ (Y)_{j_{s_\chi(k)},\overline{j_{s_\chi(k)}}} & \text{if } q < s_\chi(k) < n \\ (Y)_{j_{s_\chi(n)},\overline{j_{s_\chi(n)}}} & \text{if } s_\chi(k) = n \end{cases}$$

we see that the left-hand side of (6.1) is

$$\frac{1}{2}\delta_{j_n,i_0}\delta_{j_0,\overline{i_{s_\chi(1)}}}\delta_{j_{s_\chi(1)},\overline{i_{s_\chi(2)}}}\cdots\delta_{j_{s_\chi(q-2)},\overline{i_{s_\chi(q-1)}}}\delta_{j_{s_\chi(1-2)},\overline{i_{s_\chi(q+1)}}}\delta_{j_{s_\chi(q+1)},\overline{i_{s_\chi(q+2)}}}\cdots\delta_{j_{s_\chi(n-1)},\overline{i_{s_\chi(n)}}}\delta_{j_{s_\chi(n)},\overline{i_n}} \\ \times\varphi\left(\overline{Z_1}\cdots\overline{Z_n}\right). \tag{6.2}$$

where $\delta_{j,i}$ is the Kronecker delta. Moreover, using the conjugate variable relations for ξ_1 and ξ_2, we see that

$$\varphi\left(\overline{Z_1}\cdots\overline{Z_n}\right) = \sum_{\substack{1\leq k<n \\ \chi(k)=\ell}} \delta_{j_{s_\chi(k-1)},\overline{j_{s_\chi(q-1)}}}\varphi\left(\prod_{p\in V_k^c\setminus\{k,n\}}\overline{Z_p}\right)\varphi\left(\prod_{p\in V_k}\overline{Z_p}\right) \tag{6.3}$$

where the $\delta_{j_{s_\chi(k-1)},\overline{j_{s_\chi(q-1)}}}$ should be $\delta_{j_{s_\chi(k-1)},\overline{j_{s_\chi(q-1)}}}$ when $k = s_\chi^{-1}(1)$.

To complete the proof that equation (6.1) holds, we compute the right-hand side of equation (6.1) and show the k^{th} term in the sum equals the k^{th} term obtain in equation (6.2) using equation (6.3). Indeed, for a fixed $1 \leq k < n$ for which $\chi(k) = \ell$, we can compute

$$M_k = E_2\left(C_k\prod_{p\in V_k} Z_pC_p\right),$$

in a similar fashion to the above. Thus, to obtain a non-zero value, the relations $j_{s_\chi(p-1)} = \overline{i_{s_\chi(p)}}$ for all $p \in V_k$ must hold. Moreover, one immediately obtains when $M_k \neq 0$ that

$$M_k = \varphi\left(\prod_{p\in V_k}\overline{Z_p}\right)T_k,$$

for some $T_k \in M_2(\mathbb{C})$.

Next, notice $\eta(M_k)$ is equivalent to multiplying M_k on the left by $U = E_{1,2} + E_{2,1}$ (for right conjugate variables, one would multiply on the right) and thus we consider UM_k in place of $\eta(M_k)$. At this point, notice by Lemma 6.1.1 that UT_k can be written as a product of b_p's with $b_{s_\chi(q-1)}$ being the right-most term. By commutation, $R_{b_{s_\chi(q+1)}}$ will act on the right of UT_k thereby multiplying by $b_{s_\chi(q+1)}$ on the right and forcing $j_{s_\chi(q-1)} = \overline{i_{s_\chi(q+1)}}$ for a non-zero value to be obtained. One then proceeds as above to show that a non-zero value is obtained only if the above relations are satisfied and that the term that is produced agrees with the k^{th} term of (6.3). Hence the proof is complete in the case that $\Phi^*(\{x, x^*\} \sqcup \{y, y^*\}) < \infty$.

To prove the result in the case that $\Phi^*(\{x, x^*\} \sqcup \{y, y^*\}) = \infty$, it suffices to show that if $\Phi^*(X \sqcup Y : \eta) < \infty$ then $\Phi^*(\{x, x^*\} \sqcup \{y, y^*\} : (\mathbb{C}, \mathbb{C}), \varphi) < \infty$. Thus, suppose that $\Phi^*(X \sqcup Y : \eta) < \infty$. Hence $\Xi = J_\ell(X : (M_2(\mathbb{C})_\ell, \operatorname{alg}(M_2(\mathbb{C})_r, Y)), \eta)$ exists and can be written as

$$\Xi = \begin{bmatrix} \xi_{1,1} & \xi_{1,2} \\ \xi_{2,1} & \xi_{2,2} \end{bmatrix}.$$

We claim that

$$\xi_{2,1} = J_\ell(x : (x^*, \{y, y^*\}), \varphi) \qquad \text{and} \qquad \xi_{1,2} = J_\ell(x^* : (x, \{y, y^*\}), \varphi).$$

As a similar result will hold on the right, we will obtain that $\Phi^*(\{x, x^*\} \sqcup \{y, y^*\} : (\mathbb{C}, \mathbb{C}), \varphi) < \infty$ as desired.

As $\Xi \in \overline{\operatorname{alg}(X, Y, M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)}^{\tau_2}$, it is not difficult to see that $\xi_{i,j} \in \overline{\operatorname{alg}(x, x^*, y, y^*)}^{\|\cdot\|_\varphi}$. To see that $\xi_{2,1}$ and $\xi_{1,2}$ satisfy the appropriate left bi-free conjugate variable relations, one need only use equation (6.1), choose $b_k = E_{i_k, j_k}$ satisfying the above required relations for a non-zero value and expand both sides of equation (6.1) in an identical way to that above. The resulting equations are exactly the left bi-free conjugate variable relations required. $\square$

Using the results from chapter 5, there is some immediate knowledge about the bi-free Fisher information with respect to η from Proposition 6.1.8. We note that the following is a generalization of [58, Proposition 3.7] with a similar but more complicated proof due to the χ-ordering and additional variables present.

Proposition 6.1.9. *Under the assumptions and notation of Proposition 6.1.8,*

$$\Phi^*(X \sqcup Y : \eta) \geq \Phi^*(X \sqcup Y : \eta|_{D_2})$$

and equality holds when $(\operatorname{alg}((D_2)_\ell, X), \operatorname{alg}((D_2)_r, Y))$ *is bi-free from* $(M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)$ *with* *amalgamation over D_2 with respect to F_2. Moreover*

$$\Phi^*(X \sqcup Y : \eta|_{D_2}) \geq \Phi^*(X \sqcup Y)$$

and equality holds if (x, y) is alternating adjoint flipping.

Proof. Since $\eta = \eta \circ F$, we have that

$$\Phi^*(X \sqcup Y : \eta) = \Phi^*(X \sqcup Y : ((M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r), \eta \circ F)$$

by Remark 5.3.2 part (v). Moreover

$$\Phi^*(X \sqcup Y : ((M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r), \eta \circ F) \geq \Phi^*(X \sqcup Y : \eta|_{D_2})$$

by Remark 5.3.2 part (vii). Furthermore, equality holds if $(\mathrm{alg}((D_2)_\ell, X), \mathrm{alg}((D_2)_r, Y))$ is bi-free from $(M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)$ over D_2 with respect to F_2 by Remark 5.3.2 part (viii).

To see that $\Phi^*(X \sqcup Y : \eta|_{D_2}) \geq \Phi^*(X \sqcup Y)$ we assume that

$$\Xi = J_\ell\left(X : ((D_2)_\ell, \mathrm{alg}((D_2)_r, Y))\,, \eta|_{D_2}\right) \in \overline{\mathrm{alg}(X, Y, (D_2)_\ell, (D_2)_r)}^{\|\cdot\|_{\tau_2}}$$

exists and show that Ξ satisfies the left bi-free conjugate variable relations for X in the presence of Y. Thus if P is the orthogonal projection of $L_2(A_2, \tau_2)$ onto $\overline{\mathrm{alg}(X, Y)}^{\|\cdot\|_{\tau_2}}$ then $P(\Xi)$ will also satisfy the left bi-free conjugate variable relations for X in the presence of Y. As a similar result will hold on the right, the inequality $\Phi^*(X \sqcup Y : \eta|_{D_2}) \geq \Phi^*(X \sqcup Y)$ will be demonstrated.

By the defining property of Ξ, we know for all $n \in \mathbb{N}$, $b_0, b_1, \ldots, b_n \in D_2$, $\chi \in \{\ell, r\}^n$ with $\chi(n) = \ell$, and $Z_1, \ldots, Z_{n-1} \in A_2$ and $C_1, \ldots C_{n-1} \in 1_{\mathcal{A}} \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{\mathrm{op}}$ where

$$Z_k = \begin{cases} X & \text{if } \chi(k) = \ell \\ Y & \text{if } \chi(k) = r \end{cases} \qquad \text{and} \qquad C_k = \begin{cases} L_{b_k} & \text{if } \chi(k) = \ell \\ R_{b_k} & \text{if } \chi(k) = r \end{cases},$$

that

$$\tau_2\left(L_{b_0}R_{b_n}Z_1C_1\cdots Z_{n-1}C_{n-1}\Xi\right) = \sum_{\substack{1\le k<n\\ \chi(k)=\ell}} \tau_2\left(L_{b_0}R_{b_n}\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_pC_p\right) L_{\eta\left((F\circ E_2)\left(C_k\prod_{p\in V_k} Z_pC_p\right)\right)}\right), \tag{6.4}$$

where $V_k = \{k < m < n \mid \chi(m) = \ell\}$. We will use equation (6.4) where $b_k = I_2$ for all k. To begin, notice that

$$\eta\left((F\circ E_2)\left(C_k\prod_{p\in V_k} Z_pC_p\right)\right) = \tau_2\left(X^{|V_k|}\right) I_2,$$

as odd moments of X are zero and as x^*x and xx^* have the same distribution with respect to φ. Therefore

$$\varphi((x^*x)^m) = \varphi((xx^*)^m) = \tau_2(X^{2m}),$$

for all $m \in \mathbb{N}$. Hence, equation (6.4) reduces to

$$\tau_2\left(Z_1\cdots Z_{n-1}\Xi\right) = \sum_{\substack{1\le k<n\\ \chi(k)=\ell}} \tau_2\left((Z_1,\ldots,Z_{n-1})|_{V_k^c\setminus\{k,n\}}\right)\tau_2\left(X^{|V_k|}\right),$$

which is exactly the desired formula.

To prove $\Phi^*(X\sqcup Y : \eta|_{D_2}) \le \Phi^*(X\sqcup Y)$ when (x, y) is alternating adjoint flipping thereby completing the proof, we proceed in a similar (but more complicated) fashion. Suppose

$$\Xi = J_\ell\left(X : (\mathbb{C}, \mathrm{alg}(Y))\right) \in \overline{\mathrm{alg}(X,Y)}^{\|\cdot\|_{\tau_2}} \subseteq \overline{\mathrm{alg}(X,Y,(D_2)_\ell,(D_2)_r)}^{\|\cdot\|_{\tau_2}}$$

exists. We will demonstrate that Ξ satisfies the left bi-free conjugate variable relations for X with respect to η in the presence of $((D_2)_\ell, \mathrm{alg}((D_2)_r, Y))$. As an analogous result will hold on the right, this will complete the proof.

Write

$$\Xi = \begin{bmatrix}\xi_{1,1} & \xi_{1,2}\\ \xi_{2,1} & \xi_{2,2}\end{bmatrix} \in M_2(L_2(\mathcal{A},\varphi)) = L_2(A_2,\tau_2).$$

First we will demonstrate that $\xi_{1,1} = \xi_{2,2} = 0$. To begin, let

$$\mathcal{H}_e = \overline{\mathrm{span}(Z_1 \cdots Z_{2n} \mid n \in \mathbb{N}, Z_k \in \{X, Y\})}^{\|\cdot\|_{\tau_2}} \text{ and}$$
$$\mathcal{H}_o = \overline{\mathrm{span}(Z_1 \cdots Z_{2n-1} \mid n \in \mathbb{N}, Z_k \in \{X, Y\})}^{\|\cdot\|_{\tau_2}}.$$

By the defining property of Ξ, we know that for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$ with $\chi(n) = \ell$, and $Z_1, \ldots, Z_{n-1} \in A_2$ where

$$Z_k = \begin{cases} X & \text{if } \chi(k) = \ell \\ Y & \text{if } \chi(k) = r \end{cases}$$

that

$$\tau_2 \left(Z_1 \cdots Z_{n-1}\Xi\right) = \sum_{\substack{1 \leq k < n \\ \chi(k) = \ell}} \tau_2 \left((Z_1, \ldots, Z_{n-1})|_{V_k^c \setminus \{k,n\}}\right) \tau_2 \left(X^{|V_k|}\right), \tag{6.5}$$

where $V_k = \{k < m < n \mid \chi(m) = \ell\}$. Note as τ_2 evaluates any odd product involving X and Y to 0 by Lemma 6.1.1, if $n - 1$ is even, then $\tau_2 \left(Z_1 \cdots Z_{n-1}\Xi\right) = 0$. Therefore, since $\Xi \in \mathcal{H}_e + \mathcal{H}_o$, we obtain that $\Xi \in \mathcal{H}_o$.

Note for $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^{2n-1}$, $Z_1, \ldots, Z_{2n-1} \in \{X, Y\}$, and $z_1, \ldots, z_{2n-1} \in \{x, y\}$ where

$$Z_k = \begin{cases} X & \text{if } \chi(k) = \ell \\ Y & \text{if } \chi(k) = r \end{cases} \qquad \text{and} \qquad z_k = \begin{cases} x & \text{if } \chi(k) = \ell \\ y & \text{if } \chi(k) = r \end{cases}$$

that in $M_2(L_2(\mathcal{A}, \varphi))$ we have

$$Z_1 \cdots Z_{2n-1} = \begin{bmatrix} 0 & z_1^{p_1} z_2^{p_2} \cdots z_{2n-1}^{p_{2n-1}} \\ z_1^{q_1} z_2^{q_2} \cdots z_{2n-1}^{q_{2n-1}} & 0 \end{bmatrix},$$

where

$$p_{s_\chi(k)} = \begin{cases} 1 & \text{if } k \text{ is odd} \\ * & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} * & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

Therefore, as $\mathcal{H}_o$ is the $\|\cdot\|_{\tau_2}$-limit of matrices of the above form and as $\Xi \in \mathcal{H}_o$, we obtain that $\xi_{1,1} = \xi_{2,2} = 0$ as desired.

Let

$$H_1 = \operatorname{span}\left(z_1 \cdots z_{2n-1} \,\middle|\, \begin{array}{c} n\in\mathbb{N}, z_k\in\{x,x^*,y,y^*\} \\ \text{the powers of the } z_k\text{'s alternate between 1 and * in the } \chi\text{-ordering} \\ \text{and the first and last element in the } \chi\text{-ordering have power 1} \end{array} \right) \quad \text{and}$$

$$H_* = \operatorname{span}\left(z_1 \cdots z_{2n-1} \,\middle|\, \begin{array}{c} n\in\mathbb{N}, z_k\in\{x,x^*,y,y^*\} \\ \text{the powers of the } z_k\text{'s alternate between 1 and * in the } \chi\text{-ordering} \\ \text{and the first and last element in the } \chi\text{-ordering have power *} \end{array} \right).$$

Using the above and the notation $\xi = \xi_{1,2}$ and $\xi^* = \xi_{2,1}$ (note we do not claim that there is an involution operation on $L_2(\mathcal{A}, \varphi)$ as we do not know φ is tracial), we see that $\xi \in \overline{H_1}^{\|\cdot\|_\varphi}$, $\xi^* \in \overline{H_*}^{\|\cdot\|_\varphi}$, and if we have a $\|\cdot\|_\varphi$-limiting sequence using $\{x, x^*, y, y^*\}$ producing ξ we can obtain an a $\|\cdot\|_\varphi$-limiting sequence using $\{x, x^*, y, y^*\}$ producing ξ^* by exchanging $x \leftrightarrow x^*$ and $y \leftrightarrow y^*$. This, in conjunction with the alternating adjoint flipping condition lets us show if $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^{2n}$, $Z_1, \ldots, Z_{2n-1} \in \{X, Y\}$ and $z_1, \ldots, z_{2n-1} \in \{x, y\}$ where

$$Z_k = \begin{cases} X & \text{if } \chi(k) = \ell \\ Y & \text{if } \chi(k) = r \end{cases} \qquad \text{and} \qquad z_k = \begin{cases} x & \text{if } \chi(k) = \ell \\ y & \text{if } \chi(k) = r \end{cases}$$

that

$$\varphi(z_1^{p_1} z_2^{p_2} \cdots z_{2n-1}^{p_{2n-1}} \xi^{p_{2n}}) = \varphi(z_1^{q_1} z_2^{q_2} \cdots z_{2n-1}^{q_{2n-1}} \xi^{q_{2n}}), \tag{6.6}$$

where

$$p_{s_\chi(k)} = \begin{cases} 1 & \text{if } k \text{ is odd} \\ * & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} * & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

Indeed, consider $z_1^{p_1} \cdots z_{2n-1}^{p_{2n-1}} \xi^{p_{2n}}$ with $p_{2n} = 1$ (the case $p_{2n} = *$ is analogous). As the terms preceding ξ in the χ-ordering both must have $*$'s on them and as ξ is a $\|\cdot\|_\varphi$-limit of elements of H_1, we see that $z_1^{p_1} \cdots z_{2n-1}^{p_{2n-1}} \xi^{p_{2n}}$ is a $\|\cdot\|_\varphi$-limit of a linear combination of monomials in $\{x, x^*, y, y^*\}$ that alternate between $*$ and non-$*$-terms in the χ-ordering. As $x \leftrightarrow x^*$ and $y \leftrightarrow y^*$ produces the same φ-moment by the alternating adjoint flipping condition (as $z_1^{p_1} \cdots z_{2n-1}^{p_{2n-1}}$ and every element of H_1 is of odd length) and produces a sequence that converges to $z_1^{q_1} z_2^{q_2} \cdots z_{2n-1}^{q_{2n-1}} \xi^{q_{2n}}$ with respect to $\|\cdot\|_\varphi$, the claim is complete.

Returning to showing Ξ satisfies the left bi-free conjugate variable relations for X with respect to η in the presence of $((D_2)_\ell, \operatorname{alg}((D_2)_r, Y))$, it suffices to demonstrate that equation (6.4) holds for this Ξ. Furthermore, it suffices to verify that equation (6.4) holds when $b_k = E_{i_k,i_k}$ for all k. By the same computations as done in the proof of Proposition 6.1.8 with

$j_k = i_k$ for all k, we see with $q = s_\chi^{-1}(n)$ that

$$\begin{aligned}
&\tau_2\left(L_{b_0}R_{b_n}Z_1C_1\cdots Z_{n-1}C_{n-1}\Xi\right)\\
&\quad= \begin{cases} \frac{1}{2}\varphi(z_1^{p_1}z_2^{p_2}\cdots z_{n-1}^{p_{n-1}}\xi^{p_n}) & \text{if } n \text{ is even, } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(n)}\right) = (1,2,\ldots,1,2) \\ \frac{1}{2}\varphi(z_1^{q_1}z_2^{q_2}\cdots z_{n-1}^{q_{n-1}}\xi^{q_n}) & \text{if } n \text{ is even, } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(n)}\right) = (2,1,\ldots,2,1) \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{1\leq k<n\\ \chi(k)=\ell}} \tau_2\left(L_{b_0}R_{b_n}\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_pC_p\right) L_{\eta\left((F\circ E_2)\left(C_k\prod_{p\in V_k} Z_pC_p\right)\right)}\right)\\
&= \begin{cases} \frac{1}{2}\sum_{\substack{1\leq k<n\\ \chi(k)=\ell\\ |V_k| \text{ even}}} \varphi\left((z_1^{p_1},\ldots,z_{n-1}^{p_{n-1}})|_{V_k^c\setminus\{k,n\}}\right)\varphi\left((z_1^{p_1},\ldots,z_{n-1}^{p_{n-1}})|_{V_k}\right) & \\ \qquad\text{if } n \text{ is even and } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(q-1)}, i_{s_\chi(q+1)}, \ldots, i_{s_\chi(n)}\right) = (1,2,\ldots,1,2) & \\ \frac{1}{2}\sum_{\substack{1\leq k<n\\ \chi(k)=\ell\\ |V_k| \text{ even}}} \varphi\left((z_1^{q_1},\ldots,z_{n-1}^{q_{n-1}})|_{V_k^c\setminus\{k,n\}}\right)\varphi\left((z_1^{q_1},\ldots,z_{n-1}^{p_{q_{n-1}}})|_{V_k}\right) & \\ \qquad\text{if } n \text{ is even and } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(q-1)}, i_{s_\chi(q+1)}, \ldots, i_{s_\chi(n)}\right) = (2,1,\ldots,2,1) & \\ 0, \qquad \text{otherwise} & \end{cases}
\end{aligned}$$

where $V_k = \{k < m < n \mid \chi(m) = \ell\}$ (note only the terms where $|V_k|$ is even survive from the $\eta \circ F \circ E_2$ expression due to the form of X) and z_k, p_k, and q_k are defined as usual in this proof. Hence, it suffices to show when n is even that

$$\varphi(z_1^{p_1}z_2^{p_2}\cdots z_{n-1}^{p_{n-1}}\xi^{p_n}) = \sum_{\substack{1\leq k<n\\ \chi(k)=\ell\\ |V_k| \text{ even}}} \varphi\left((z_1^{p_1},\ldots,z_{n-1}^{p_{n-1}})|_{V_k^c\setminus\{k,n\}}\right)\varphi\left((z_1^{p_1},\ldots,z_{n-1}^{p_{n-1}})|_{V_k}\right), \tag{6.7}$$

$$\varphi(z_1^{q_1}z_2^{q_2}\cdots z_{n-1}^{q_{n-1}}\xi^{q_n}) = \sum_{\substack{1\leq k<n\\ \chi(k)=\ell\\ |V_k| \text{ even}}} \varphi\left((z_1^{q_1},\ldots,z_{n-1}^{q_{n-1}})|_{V_k^c\setminus\{k,n\}}\right)\varphi\left((z_1^{q_1},\ldots,z_{n-1}^{p_{q_{n-1}}})|_{V_k}\right). \tag{6.8}$$

Note equations (6.7) and (6.8) are the same equation by the alternating adjoint flipping condition and equation (6.6). Moreover, due to the defining property of Ξ, we know with n

even that

$$\begin{aligned}\tau_2(Z_1\cdots Z_{n-1}\Xi) &= \sum_{\substack{1\leq k<n\\ \chi(k)=\ell}} \tau_2\left(\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_p\right) L_{\eta\left((F\circ E_2)\left(\prod_{p\in V_k} Z_p\right)\right)}\right)\\ &= \sum_{\substack{1\leq k<n\\ \chi(k)=\ell}} \tau_2\left(\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_p\right)\right) \tau_2\left(X^{|V_k|}\right).\end{aligned}$$

Due to the form of X and the alternating adjoint flipping condition, we immediately see that

$$\tau_2\left(X^{|V_k|}\right) = \begin{cases} 0 & \text{if } |V_k| \text{ is odd}\\ \varphi\left((z_1^{p_1},\ldots,z_{n-1}^{p_{n-1}})|_{V_k}\right) & \text{if } |V_k| \text{ is even}\end{cases}$$

and, for n even and k such that $|V_k|$ is even, we have

$$\begin{aligned}\tau_2\left(\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_p\right)\right) &= \frac{1}{2}\left(\varphi\left((z_1^{p_1},\ldots,z_{n-1}^{p_{n-1}})|_{V_k^c\setminus\{k,n\}}\right) + \varphi\left((z_1^{q_1},\ldots,z_{n-1}^{q_{n-1}})|_{V_k^c\setminus\{k,n\}}\right)\right)\\ &= \varphi\left((z_1^{p_1},\ldots,z_{n-1}^{p_{n-1}})|_{V_k^c\setminus\{k,n\}}\right),\end{aligned}$$

thereby completing the proof. □

Proof of Theorem 6.1.6. The proof follows immeditaley by combining Propositions 6.1.2, 6.1.8, and 6.1.9. □

6.2 Maximizing Bi-Free Entropy

In this section, we will prove Theorem 6.2.3 obtaining an upper bound for the bi-free entropy of a pair of operators and their adjoints based on the entropy of a pair of matrices and demonstrate when equality is obtained. In particular, this generalizes an essential result from [58, Section 5].

To begin, we must establish a formula for the bi-free entropy of non-self-adjoint operators.

Definition 6.2.1. Let $(\mathcal{A},\varphi)$ be a non-commutative C*-probability space and let $\{X_i,X_i^*\}_{i=1}^n\cup\{X_i'\}_{i=1}^{n'}\cup\{Y_j,Y_j^*\}_{j=1}^m\cup\{Y_j'\}_{j=1}^{m'}\subseteq\mathcal{A}$ where X_i' and Y_j' are self-adjoint for all i and j. The *bi-free entropy of* $(\{\mathbf{X}, \mathbf{X}^*, \mathbf{X}'\}, \{\mathbf{Y}, \mathbf{Y}^*, \mathbf{Y}'\})$ is defined to be

$$\begin{aligned}
&\chi^*(\mathbf{X}, \mathbf{X}^*, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}^*, \mathbf{Y}') \\
&\quad = \frac{2n + 2m + n' + m'}{2} \ln(2\pi e) + \frac{1}{2} \int_0^\infty \left(\frac{2n + 2m + n' + m'}{1 + t} - g(t) \right) dt,
\end{aligned}$$

where

$$g(t) = \Phi^* \left(\mathbf{X} + \sqrt{t}\mathbf{C}_\ell, \mathbf{X}^* + \sqrt{t}\mathbf{C}_\ell^*, \mathbf{X}' + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{C}_r, \mathbf{Y}^* + \sqrt{t}\mathbf{C}_r^*, \mathbf{Y}' + \sqrt{t}\mathbf{D} \right),$$

with $\mathbf{S}$ and $\mathbf{D}$ consisting of semicircular variables of mean 0, variance 1, covariance 0 and $\mathbf{C}_\ell$ and $\mathbf{C}_r$ consist of circular variables of mean 0, variance 1 and covariance 0 such that

$$(\{\mathbf{X}, \mathbf{X}^*, \mathbf{X}'\}, \{\mathbf{Y}, \mathbf{Y}^*, \mathbf{Y}'\}) \cup \{(S_i, 1)\}_{i=1}^{n'} \cup \{(1, D_j)\}_{j=1}^{m'} \cup \{(\{C_{\ell,i}, C_{\ell,i}^*\}, 1)\}_{i=1}^{n} \cup \{(1, \{C_{r,j}, C_{r,j}^*\})\}_{j=1}^{m}$$

are bi-free.

Remark 6.2.2. Given any C*-non-commutative probability space $(\mathcal{A}, \varphi)$, it is always possible to find a larger C*-non-commutative probability space that contains the necessary bi-free elements from Definition 6.2.1. Indeed, one need only consider the scalar reduced free product of the appropriate spaces and use Definition 4.1.4 to obtain bi-freeness. The fact that the state is positive follows as it will be a vector state.

In the simplest case, one may ask why we do not simply define

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) = \chi^* \left(\{\Re(x), \Im(x)\} \sqcup \{\Re(y), \Im(y)\} \right)$$

to trivially reduce to the self-adjoint case in a similar fashion to Remark 5.3.2 part (iv) and why the integrand in Definition 6.2.1 is well-defined. Both of these questions are answered via Remark 5.3.2 part (iv) as

$$\begin{aligned}
&\Phi^* \left(\left\{ x + \sqrt{t}c_\ell, x^* + \sqrt{t}c_\ell^* \right\} \sqcup \left\{ y + \sqrt{t}c_r, y^* + \sqrt{t}c_r^* \right\} \right) \\
&= \frac{1}{2}\Phi^* \left(\left\{ \Re(x) + \sqrt{t}\Re(c_\ell), \Im(x) + \sqrt{t}\Im(c_\ell) \right\} \sqcup \left\{ \Re(y) + \sqrt{t}\Re(c_r), \Im(y) + \sqrt{t}\Im(c_r) \right\} \right) \\
&= \frac{1}{2}\Phi^* \left(\left\{ \Re(x) + \frac{\sqrt{t}}{\sqrt{2}}s_1, \Im(x) + \frac{\sqrt{t}}{\sqrt{2}}s_2 \right\} \sqcup \left\{ \Re(y) + \frac{\sqrt{t}}{\sqrt{2}}d_1, \Im(y) + \frac{\sqrt{t}}{\sqrt{2}}d_2 \right\} \right) \\
&= \Phi^* \left(\left\{ \sqrt{2}\Re(x) + \sqrt{t}s_1, \sqrt{2}\Im(x) + \sqrt{t}s_2 \right\} \sqcup \left\{ \sqrt{2}\Re(y) + \sqrt{t}d_1, \sqrt{2}\Im(y) + \sqrt{t}d_2 \right\} \right)
\end{aligned}$$

where s_1, s_2, d_1, and d_2 are as in Example 6.1.5. Hence the integrand in Definition 6.2.1 is well-defined with

$$\begin{aligned}\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) &= \chi^*\left(\left\{\sqrt{2}\Re(x), \sqrt{2}\Im(x)\right\} \sqcup \left\{\sqrt{2}\Re(y), \sqrt{2}\Im(y)\right\}\right) \\ &= \chi^*\left(\{\Re(x), \Im(x)\} \sqcup \{\Re(y), \Im(y)\}\right) + 4\ln(\sqrt{2}).\end{aligned}$$

We normalize Definition 6.2.1 so that the following holds and generalizes [58, Theorem 1.4] in the case $d = 1$.

Theorem 6.2.3. *Let $(\mathcal{A}, \varphi)$ be a C*-non-commutative probability space and let $x, y \in \mathcal{A}$ be such that x^*x and xx^* have the same distribution with respect to φ and y^*y and yy^* have the same distribution with respect to φ. With X and Y as in Section 6.1,*

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) \leq 2\chi^*(X \sqcup Y)$$

and equality holds whenever the pair (x, y) is bi-R-diagonal and alternating adjoint flipping.

To prove Theorem 6.2.3, we need two technical lemmata. For the first, note the following does not immediately follow from Remark 4.1.5 as being bi-free over $M_2(\mathbb{C})$ with respect to E_2 does not imply being bi-free with respect to τ_2.

Lemma 6.2.4. *Let $(\mathcal{A}, \varphi)$ be a C*-non-commutative probability space, let $x, y \in \mathcal{A}$ be such that x^*x and xx^* have the same distribution with respect to φ and y^*y and yy^* have the same distribution with respect to φ, and let (c_ℓ, c_r) be a bi-free circular pair in $\mathcal{A}$ with mean 0, variance 1 and covariance 0 such that*

$$(\{x, x^*\}, \{y, y^*\}) \cup \{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$$

are bi-free with respect to φ. Using the notation of Section 6.2, if

$$S_\ell = c_\ell \otimes E_{1,2} \otimes I_2 + c_\ell^* \otimes E_{2,1} \otimes I_2 \in A_2 \quad \text{and} \quad S_r = c_r \otimes I_2 \otimes E_{1,2} + c_r^* \otimes I_2 \otimes E_{2,1} \in A_2,$$

then S_ℓ and S_r have semicircular distributions with respect to τ_2 of mean 0 and variance 1 and

$$\{(X, Y)\} \cup \{(S_\ell, 1_{A_2})\} \cup \{(1_{A_2}, S_r)\}$$

are bi-free with respect to τ_2.

Proof. As $\{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$ are bi-free with respect to φ by Example 6.1.5, Remark 4.1.5 implies that $(S_\ell, 1)$ and $(1, S_r)$ are bi-free with respect to E_2. Moreover, as c_ℓ and c_r commute, s_ℓ and s_r commute. Hence, by Example 6.1.5 and the alternating adjoint flipping condition we see for all $n, m \in \mathbb{N}$ that

$$\tau_2(s_\ell^n s_r^m) = \mathrm{tr}_2(E_2(s_\ell^n s_r^m)) = \mathrm{tr}_2(E_2(s_\ell^n)E_2(s_r^m)),$$

with

$$\tau_2(s_\ell^n s_r^m) = \begin{cases} 0 & \text{if } n \text{ or } m \text{ is odd} \\ \varphi\left((c_\ell^* c_\ell)^{\frac{n}{2}}\right)\varphi\left((c_r^* c_r)^{\frac{m}{2}}\right) & \text{if } n \text{ and } m \text{ are even.} \end{cases}$$

Therefore, as $c_\ell^* c_\ell$ and $c_r^* c_r$ are known to have the same distributions as the square of a semicircular element of mean 0 and variance 1 (see [83, Section 5.1]), we obtain that (s_ℓ, s_r) is the bi-free central limit distribution with mean 0, variance 1 and covariance 0 with respect to τ_2. Hence $\{(S_\ell, 1_{A_2})\} \cup \{(1_{A_2}, S_r)\}$ are bi-free with respect to τ_2.

To complete the proof, it suffices to show that $\{(X, Y)\} \cup \{(S_\ell, S_r)\}$ are bi-free with respect to τ_2. Therefore, by [11], it suffices to show for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, non-constant $\gamma \in \{1, 2\}^n$, and $Z_k \in \{X, Y, S_\ell, S_r\}$ where

$$Z_k = \begin{cases} X & \text{if } \chi(k) = \ell \text{ and } \gamma(k) = 1 \\ Y & \text{if } \chi(k) = r \text{ and } \gamma(k) = 1 \\ S_\ell & \text{if } \chi(k) = \ell \text{ and } \gamma(k) = 2 \\ S_r & \text{if } \chi(k) = r \text{ and } \gamma(k) = 2 \end{cases}$$

that

$$\tau_2(Z_1 \cdots Z_n) = \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^{\tau_2}(Z_1, \ldots, Z_n) \tag{6.9}$$

where γ is representing the partition $\{\{k \mid \gamma(k) = 1\}, \{k \mid \gamma(k) = 2\}\}$. Note if $z_1, \ldots, z_n \in$

$\{x, y, c_\ell, c_r\}$ are such that

$$z_k = \begin{cases} x & \text{if } Z_k = X \\ y & \text{if } Z_k = Y \\ c_\ell & \text{if } Z_k = S_\ell \\ c_r & \text{if } Z_k = S_r \end{cases}$$

then by Lemma 6.1.1 and the fact that $(\{x, x^*\}, \{y, y^*\}) \cup \{(c_\ell, 1)\} \cup \{(1, c_r)\}$ are bi-free with respect to φ, we have that

$$\begin{aligned} \tau_2(Z_1 \cdots Z_n) &= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{2}\left(\varphi\left(z_1^{p_1} \cdots z_n^{p_n}\right) + \varphi\left(z_1^{q_1} \cdots z_n^{q_n}\right)\right) & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{2} \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^\varphi\left(z_1^{p_1}, \ldots, z_n^{p_n}\right) + \kappa_\pi^\varphi\left(z_1^{q_1}, \ldots, z_n^{q_n}\right) & \text{if } n \text{ is even} \end{cases}, \end{aligned}$$

where

$$p_{s_\chi(k)} = \begin{cases} 1 & \text{if } k \text{ is odd} \\ * & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} * & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

To show this agrees with the right-hand side of equation (6.9), we divide the discussion into several cases. To this end, let

$$I_{X,Y} = \{k \mid \gamma(k) = 1\} \qquad \text{and} \qquad I_S = \{k \mid \gamma(k) = 2\}.$$

First suppose n is odd. If $|I_S|$ is odd, then the right-hand side of equation (6.9) is zero as there must be a cumulant involving an odd number of S_ℓ and S_r and $\{(S_\ell, S_r)\}$ is a bi-free central limit distribution with 0 mean. Otherwise, $|I_{X,Y}|$ is odd. In this case, we may rearrange the sum on the right-hand side of equation (6.9) to add over all $\pi \in \mathrm{BNC}(\chi)$ with $\pi \leq \gamma$ that form the same partition when restricted to I_S. Since summing over such partitions yields a product of moment terms in the X's and Y's where the sum of the lengths of the moments is $|I_{X,Y}|$ and since all odd moment terms involving only X's and Y's is zero by Lemma 6.1.1, this portion of the sum yields zero. Hence, equation (6.9) holds when n is odd.

In the case n is even, note if $|I_S|$ is odd, then the right-hand side of equation (6.9) is still zero. However,

$$\frac{1}{2} \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^\varphi \left(z_1^{p_1}, \ldots, z_n^{p_n}\right) + \kappa_\pi^\varphi \left(z_1^{q_1}, \ldots, z_n^{q_n}\right) = 0$$

as there must be a cumulant involving an odd number of $(\{c_\ell, c_\ell^*\}, \{c_r, c_r^*\})$ and (c_ℓ, c_r) is a bi-free circular pair. Thus, we may assume that n, $|I_S|$, and $|I_{X,Y}|$ are even.

Under these assumptions, we claim that

$$\frac{1}{2} \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^\varphi \left(z_1^{p_1}, \ldots, z_n^{p_n}\right) + \kappa_\pi^\varphi \left(z_1^{q_1}, \ldots, z_n^{q_n}\right) = \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^{\tau_2}(Z_1, \ldots, Z_n).$$

To see this, again we need only consider $\pi \in \mathrm{BNC}(\chi)$ that form pair partitions when restricted to I_S and no block of π contains both an element of $\{k \mid \chi(k) = \ell\}$ and of $\{k \mid \chi(k) = r\}$, since $\{(S_\ell, 1)\} \cup \{(1, S_r)\}$ are bi-free with respect to τ_2 and $\{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$ are bi-free with respect to φ. For such a partition π, if we let $\widehat{\pi}$ be the largest partition on $I_{X,Y}$ such that $\widehat{\pi} \cup \pi|_{I_S}$ is an element of $\mathrm{BNC}(\chi)$, then by adding over all $\sigma \in \mathrm{BNC}(\chi)$ with $\sigma \leq \gamma$ and $\sigma|_{I_S} = \pi|_{I_S}$, it suffices to show that

$$\begin{aligned}
&\tfrac{1}{2}\left(\varphi_{\widehat{\pi}}\left((z_1^{p_1}, \ldots, z_n^{p_n})|_{I_{X,Y}}\right) \kappa^\varphi_{\pi|_{I_S}}\left((z_1^{p_1}, \ldots, z_n^{p_n})\,|_{I_S}\right) + \varphi_{\widehat{\pi}}\left((z_1^{q_1}, \ldots, z_n^{q_n})|_{I_{X,Y}}\right) \kappa^\varphi_{\pi|_{I_S}}\left((z_1^{q_1}, \ldots, z_n^{q_n})\,|_{I_S}\right)\right) \\
&\quad = (\tau_2)_{\widehat{\pi}}\left((Z_1, \ldots, Z_n)\,|_{I_{X,Y}}\right).
\end{aligned}$$

Note that

$$\kappa^\varphi_{\pi|_{I_S}}\left((z_1^{p_1}, \ldots, z_n^{p_n})\,|_{I_S}\right) = 0 \quad \text{or} \quad \kappa^\varphi_{\pi|_{I_S}}\left((z_1^{q_1}, \ldots, z_n^{q_n})\,|_{I_S}\right) = 0$$

if and only if π has a block with two $*$-terms or two non-$*$-terms, as $\pi|_{I_S}$ is a pair partition and (c_ℓ, c_r) is a bi-circular pair. In this case we would have that $\widehat{\pi}$ has a block of odd length and thus the right-hand side of the equation above is also zero, as any odd τ_2-moment involving X and Y is zero. Otherwise, both φ-cumulants are 1 and this forces every block of $\widehat{\pi}$ to be of even length and alternate between 1 and $*$ in the χ-ordering. Since

$$\varphi((x^*x)^m) = \varphi((xx^*)^m) = \tau_2(X^{2m}) \quad \text{and} \quad \varphi((y^*y)^m) = \varphi((yy^*)^m) = \tau_2(Y^{2m})$$

and since (by the assumption that π does not contain a block containing elements of $\{k \mid \chi(k) = \ell\}$ and of $\{k \mid \chi(k) = r\}$) there is a single block of $\widehat{\pi}$ containing elements of $\{k \mid \chi(k) = \ell\}$ and $\{k \mid \chi(k) = r\}$, adding the two φ-terms together produces exactly the desired τ_2 term. $\square$

Lemma 6.2.5. *Let $(\mathcal{A}, \varphi)$ be a C^*-non-commutative probability space, let $x, y \in \mathcal{A}$, and let (c_ℓ, c_r) be a bi-free circular pair in $\mathcal{A}$ with mean 0, variance 1 and covariance 0 such that*

$$(\{x, x^*\}, \{y, y^*\}) \cup \{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$$

are bi-free with respect to φ. Then:

(i) If (x, y) is bi-R-diagonal, then $\left(x + \sqrt{t}c_\ell, y + \sqrt{t}c_r\right)$ is bi-R-diagonal for all $t \in (0, \infty)$.

(ii) If (x, y) is alternating adjoint flipping, then $\left(x + \sqrt{t}c_\ell, y + \sqrt{t}c_r\right)$ is alternating adjoint flipping for all $t \in (0, \infty)$.

(iii) If x^*x and xx^* (respectively y^*y and yy^*) have the same distribution with respect to φ, then $(x + \sqrt{t}c_\ell)^*(x + \sqrt{t}c_\ell)$ and $(x + \sqrt{t}c_\ell)(x + \sqrt{t}c_\ell)^*$ (respectively $(y + \sqrt{t}c_r)^*(y + \sqrt{t}c_r)$ and $(y + \sqrt{t}c_r)(y + \sqrt{t}c_r)^*$) have the same distribution with respect to φ.

Proof. As (c_ℓ, c_r) is bi-R-diagonal by Example 6.1.5 and as sums and scalar multiples of bi-R-diagonal pairs are bi-R-diagonal by [42, Proposition 3.1], (i) follows.

To see that (ii) holds, first we claim for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^{2n}$, and $z_1, \ldots, z_n \in \{x, y, c_\ell, c_r\}$ such that

$$z_k \in \begin{cases} \{x, c_\ell\} & \text{if } \chi(k) = \ell \\ \{y, c_r\} & \text{if } \chi(k) = r \end{cases},$$

we have that

$$\varphi(z_1^{p_1} \cdots z_{2n}^{p_{2n}}) = \varphi(z_1^{q_1} \cdots z_{2n}^{q_{2n}}),$$

where

$$p_{s_\chi(k)} = \begin{cases} 1 & \text{if } k \text{ is odd} \\ * & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} * & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

Recall that

$$\varphi(z_1^{p_1} \cdots z_{2n}^{p_{2n}}) = \sum_{\pi \in \mathrm{BNC}(\chi)} \kappa_\pi(z_1^{p_1}, \ldots, z_{2n}^{p_{2n}})$$

and the bi-free cumulant is zero if any block of π contains both an element of $\{x, x^*, y, y^*\}$ and an element of $\{c_\ell, c_\ell^*, c_r, c_r^*\}$ As the only cumulants involving $c_\ell, c_\ell^*, c_r, c_r^*$ with non-zero values are

$$\kappa_{1_{(\ell,\ell)}}(c_\ell, c_\ell^*) = 1 = \kappa_{1_{(\ell,\ell)}}(c_\ell^*, c_\ell) \qquad \text{and} \qquad \kappa_{1_{(r,r)}}(c_r, c_r^*) = 1 = \kappa_{1_{(r,r)}}(c_r^*, c_r),$$

for any fixed $\pi \in \mathrm{BNC}(\chi)$ for which the blocks containing $\{c_\ell, c_\ell^*, c_r, c_r^*\}$ do not cause the bi-free cumulant to be zero, we may add over all elements of $\mathrm{BNC}(\chi)$ with the same blocks as π for those indices corresponding to elements of $\{c_\ell, c_\ell^*, c_r, c_r^*\}$ to obtain a product of moments involving $\{x, x^*, y, y^*\}$, each of which is of even length and alternates between 1 and $*$ in the χ-ordering. We may then use the alternating adjoint flipping condition on (x, y) to exchange the powers and reverse this cumulant reduction process to obtain $\varphi(z_1^{q_1} \cdots z_{2n}^{q_{2n}})$, thereby completing the claim. Thus (ii) then follows by linearity .

To see that (iii) holds, we desire to show that

$$\varphi\left(\left((x+\sqrt{t}c_\ell)^*(x+\sqrt{t}c_\ell)\right)^n\right) = \varphi\left(\left((x+\sqrt{t}c_\ell)(x+\sqrt{t}c_\ell)^*\right)^n\right)$$

for all $n \in \mathbb{N}$. To see how the left-hand side can be changed into the right-hand side, arguments similar to the proof of Lemma 6.2.4 are used. First, we expand out the product and expand the moment using linearity. Then, for each moment term, we expand via the free cumulants and use the fact that mixed free cumulants vanish. Cumulants involving an odd number of c_ℓ and c_ℓ^* vanish and thus we can consider only pair partitions when restricted to entries involving c_ℓ and c_ℓ^*. Any cumulant involving just c_ℓ or just c_ℓ^* vanishes and can be ignored. By adding over all partitions with the same blocks on c_ℓ and c_ℓ^* that do not vanish yields a product of moment terms of the form $\varphi((x^*x)^m)$ and $\varphi((xx^*)^m)$. For any such terms, viewing the $(2n)^{\text{th}}$ term as the first term doesn't change the value, as the distributions of x^*x and xx^* are the same, thereby effectively moving the x or c_ℓ term at the end to the beginning. One then reverses the above process and obtains the right-hand side as desired. □

Proof of Theorem 6.2.3. As per Remark 6.2.2, we may assume without loss of generality that there exists a bi-free circular pair (c_ℓ, c_r) (with mean 0, variance 1 and covariance 0) in $\mathcal{A}$ such that

$$\{(\{x, x^*\}, \{y, y^*\})\} \cup \{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$$

are bi-free. Therefore, as $\{(X, Y)\} \cup \{(S_\ell, S_r)\}$ are bi-free with respect to τ_2 by Lemma 6.2.4, we obtain that

$$\chi^*(X \sqcup Y) = \ln(2\pi e) + \frac{1}{2}\int_0^\infty \left(\frac{2}{1+t} - \Phi^*\left(X + \sqrt{t}S_\ell \sqcup Y + \sqrt{t}S_r\right)\right) dt.$$

However, as

$$\begin{aligned}
X + \sqrt{t}S_\ell &= (x + \sqrt{t}c_\ell) \otimes E_{1,2} \otimes I_2 + (x + \sqrt{t}c_\ell)^* \otimes E_{2,1} \otimes I_2, \\
Y + \sqrt{t}S_r &= (y + \sqrt{t}c_r) \otimes I_2 \otimes E_{1,2} + (y + \sqrt{t}c_r)^* \otimes I_2 \otimes E_{2,1}
\end{aligned}$$

and as Lemma 6.2.5 part (iii) shows that the assumptions of Theorem 6.1.6 as satisfied, we obtain that

$$\Phi^*\left(\{x + \sqrt{t}c_\ell, (x + \sqrt{t}c_\ell)^*\} \sqcup \{y + \sqrt{t}c_r, (y + \sqrt{t}c_r)^*\}\right) \geq 2\Phi^*\left(X + \sqrt{t}S_\ell \sqcup Y + \sqrt{t}S_r\right) \tag{6.10}$$

for all $t \in (0, \infty)$. Hence the inequality

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) \leq 2\chi^*(X \sqcup Y)$$

follows by comparing the above bi-free entropy formula with that from Definition 6.2.1.

In the case that (x, y) is bi-R-diagonal and alternating adjoint flipping, Lemma 6.2.5 implies $(x + \sqrt{t}c_\ell, y + \sqrt{t}c_r)$ is bi-R-diagonal and alternating adjoint flipping for all $t \in (0, \infty)$, thus equality holds in equation (6.10) by Theorem 6.1.6. $\square$

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