

Multivariable Functions Introduced

Sometimes function outputs depend on multiple function inputs

Examples:

$$f(x, y, z) = x^2 y + z \cos(y) \quad \text{evaluated at } (e, \pi, \ln(\sqrt{2})) \rightarrow f(e, \pi, \ln(\sqrt{2})) = (e)^2 (\pi) + (\ln(\sqrt{2})) \cos(\pi)$$

$$g(u, v) = e^{uv} \quad \text{evaluated at } u = \ln(3), v = -2 \rightarrow g(\ln(3), -2) = e^{(\ln(3))(-2)} = (e^{\ln(3)})^{-2} = 3^{-2} = \frac{1}{9}$$

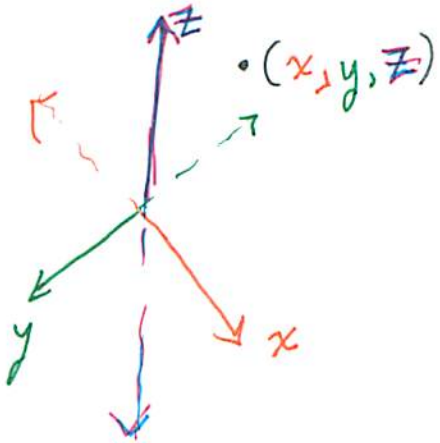
All input variables
listed here

Some Situations Where Arise

<u>Inputs</u>	<u>Output</u>
<u>Utility Function:</u> 1) $x_1, \dots, x_k$ represent quantities of k different goods	Level of "utility"
<u>Production Function:</u> 2) x = Labor, y = Capital	Level of "production"
<u>Altitude Function:</u> 3) x = Latitude, y = Longitude	Altitude
<u>Temperature:</u> 4) (x, y, z, t) (x, y, z) location in room t time	Temperature

Graphs of 2-Variable Functions

Graph in $\mathbb{R}^3$:

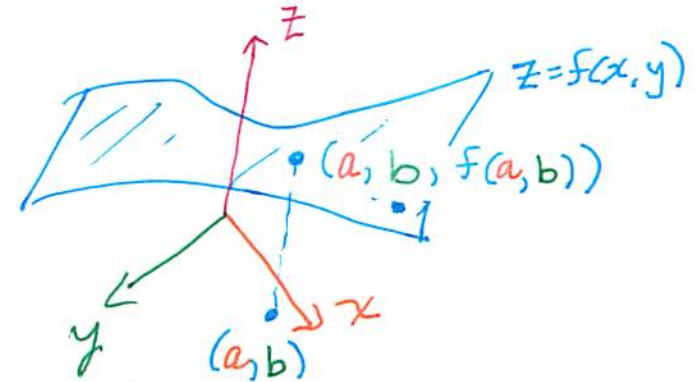


Points now have 3 coords

(x, y, z)

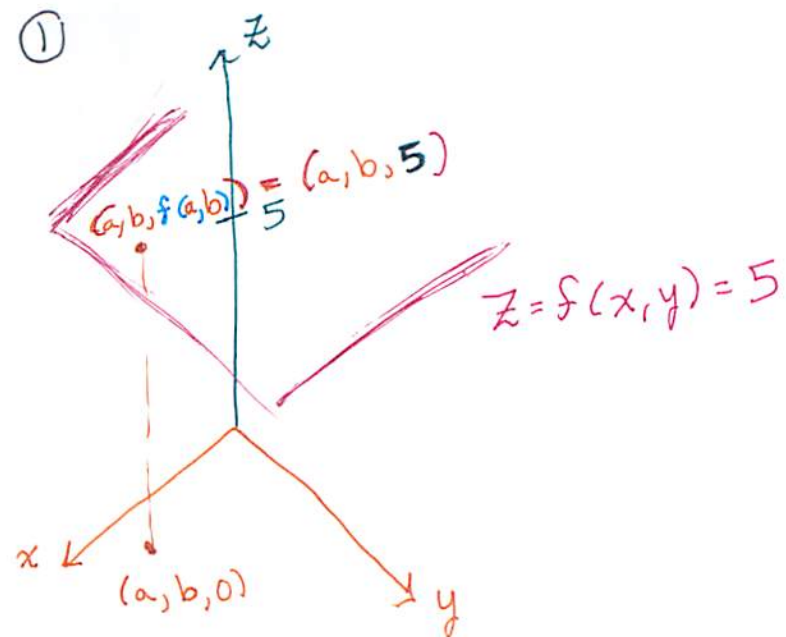
Says how far above xy -plane if positive,
how far below if negative

Graph of $f(x, y)$:



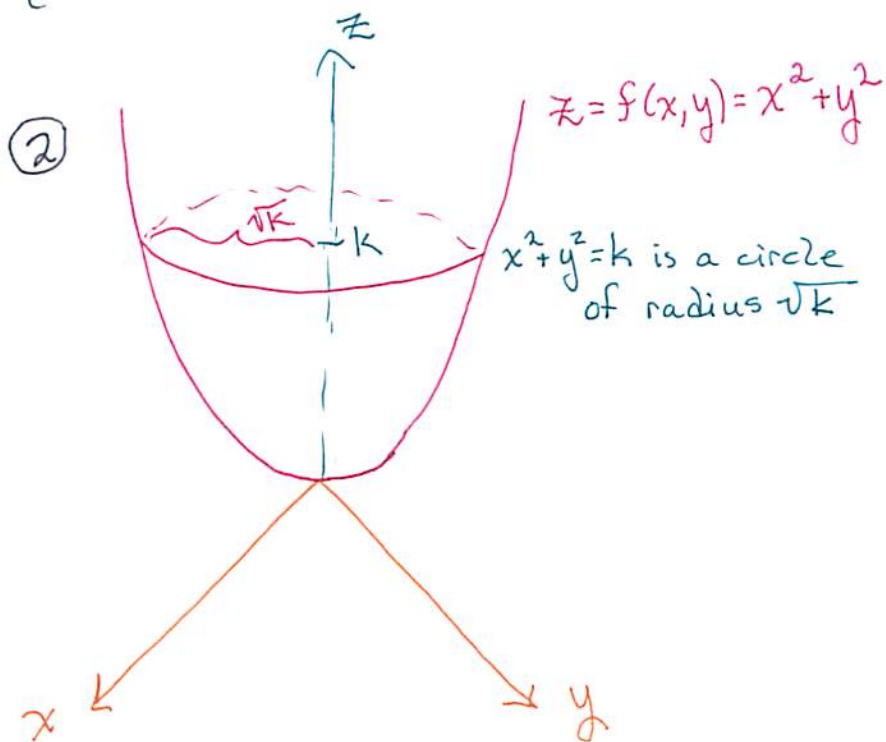
A Little Practice Graphing 2-Variable Functions

①



Graph of $f(x, y) = 5$

②

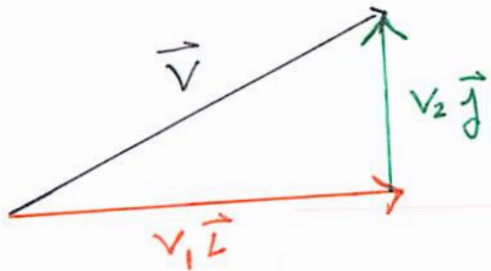
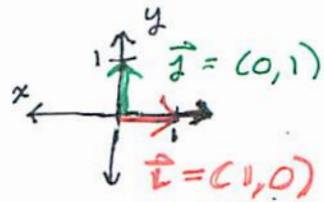


Graph of $f(x, y) = x^2 + y^2$

A Few Words on Vectors

In 2 dimensions:

$$\vec{V} = \underbrace{V_1}_{\text{distance in } x \text{ directions}} \vec{i} + \underbrace{V_2}_{\text{distance in } y \text{ direction}} \vec{j} \quad \text{where}$$



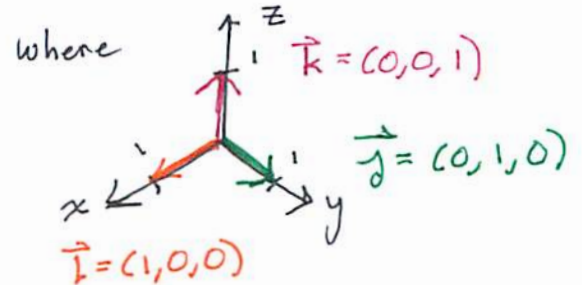
Other notation

$$\vec{V} = \langle V_1, V_2 \rangle$$

$$\text{Length: } \|\vec{V}\| = \sqrt{V_1^2 + V_2^2}$$

In 3 dimensions:

$$\vec{V} = V_1 \vec{i} + V_2 \vec{j} + V_3 \vec{k}$$



$$\text{Length} = \|\vec{V}\| = \sqrt{V_1^2 + V_2^2 + V_3^2}$$

Other Notation

$$\vec{V} = \langle V_1, V_2, V_3 \rangle$$

Underlying Ex: $f(x,y) = x^2y + x \cos(y)$ Partial Derivatives

Notation:

Partial derivative wrt x @ (x_0, y_0) :

$$f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$$

Computing:

Treat y as constant, differentiate wrt x
($\therefore$ plug in (x_0, y_0) if evaluated at (x_0, y_0))

$$\text{e.g. } \frac{\partial f}{\partial x} = 2x + \cos(y) \xrightarrow{\text{@}(1,2)} \frac{\partial f}{\partial x}(1,2) = 2(1) + \cos(2) = \boxed{4 + \cos(2)}$$

Partial derivative wrt y @ (x_0, y_0) :

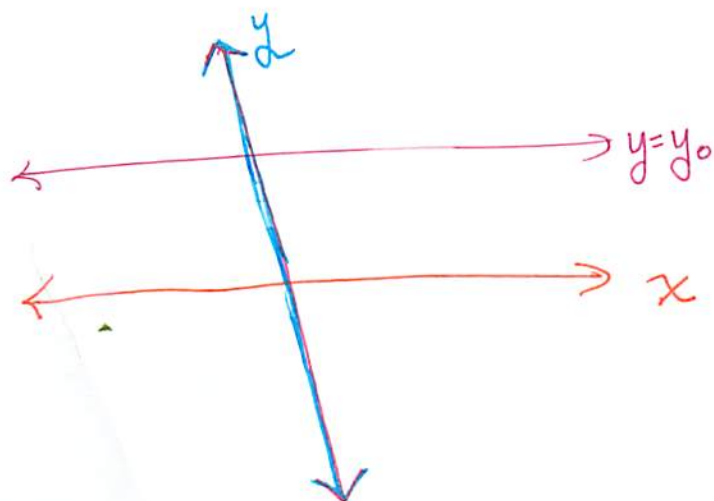
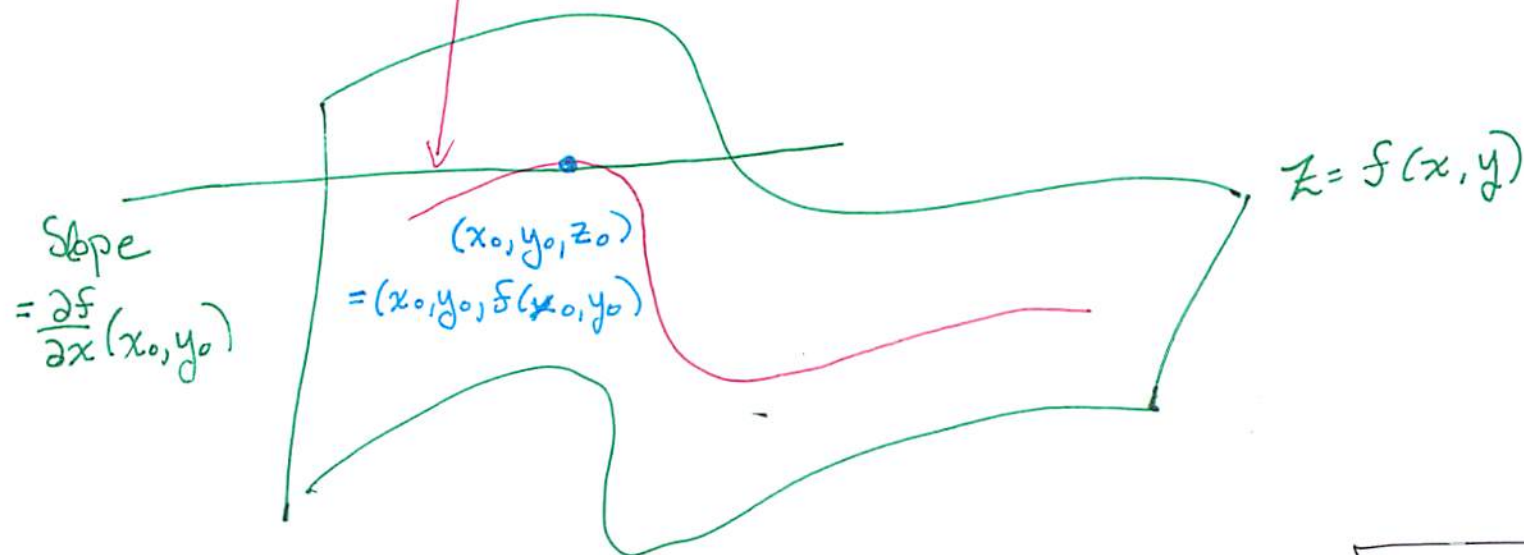
$$f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

Treat x as constant, differentiate wrt y
($\therefore$ plug in (x_0, y_0) if evaluated at (x_0, y_0))

$$\text{e.g. } \frac{\partial f}{\partial y} = x^2 - x \sin(y) \xrightarrow{\text{@}(3,\pi)} \frac{\partial f}{\partial y}(3,\pi) = (3)^2 - (3) \sin(\pi) = \boxed{9}$$

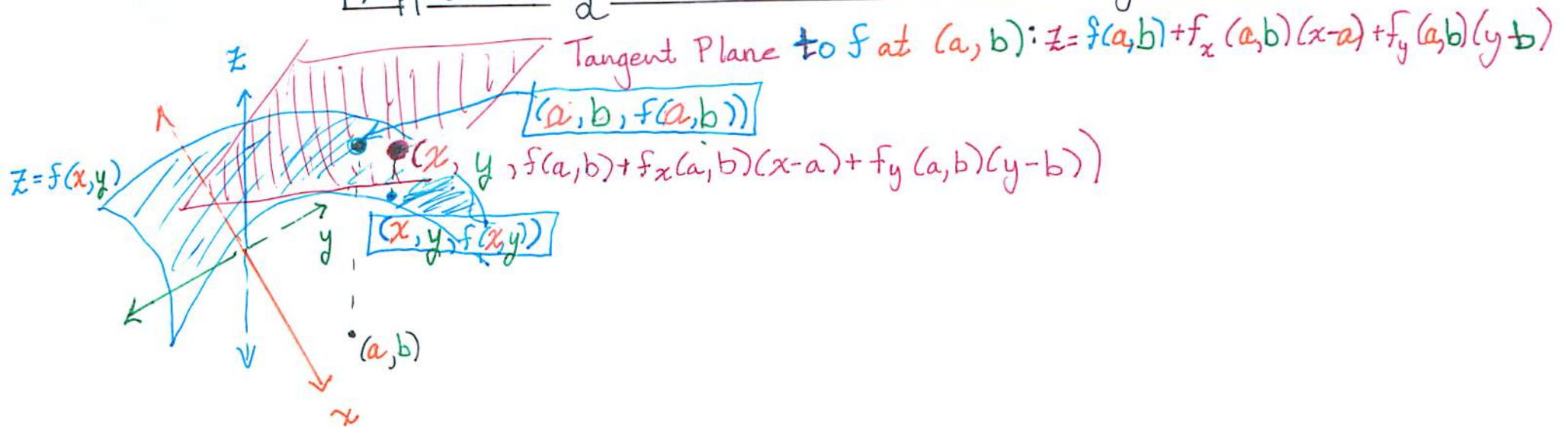
Partial Derivative $\frac{\partial f}{\partial x}$ Picture

Fixing $y=y_0$, i.e. slicing along $y=y_0$ plane gives this curve



$\frac{\partial f}{\partial x}(x_0, y_0)$ approximates $\frac{\Delta f}{\Delta x} @ (x_0, y_0)$

Approximating 2-Variable Function with Tangent Plane



Consequential Approximations

$$f(x, y) \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

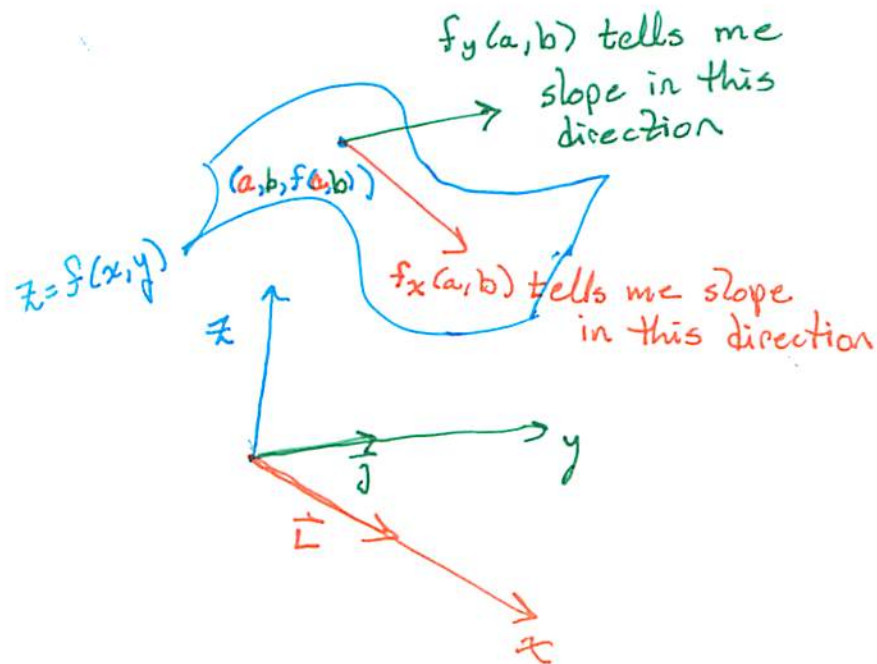
$$\underbrace{\Delta f}_{\text{Output Error}} \approx f_x(a, b) \underbrace{\Delta x}_{\text{x input error}} + f_y(a, b) \underbrace{\Delta y}_{\text{y input error}}$$

Gradient

Gradient of f at (a, b) :

$$\nabla f = \langle f_x(a, b), f_y(a, b) \rangle$$

- Gradient is a vector depending on point (a, b)
- ∇f tells me direction at (a, b) with largest slope (direction in which $f(x, y)$ increases fastest)
- $\|\nabla f(a, b)\|$ is rate $f(x, y)$ is increasing in the direction ∇f



Directional Derivative

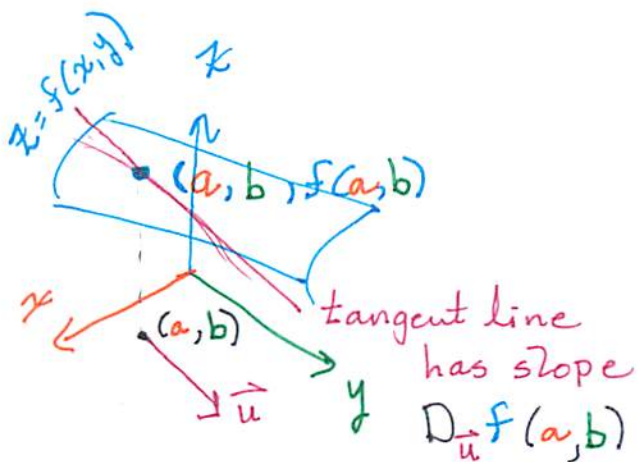
assuming $\|\vec{u}\|=1$

Q: How do we find the slope of the graph of f (rate of change of f) in some direction $\vec{u}$ from (a,b) ?
 $\langle u_1, u_2 \rangle$

Directional Derivative: $D_{\vec{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle$

i.e. $D_{\vec{u}} f(a,b) = (\nabla f(a,b)) \cdot \vec{u}$

Picture:



Inputs:

- ① Function f
- ② Point (a, b)
- ③ Length-1 vector $\vec{u}$ giving direction want slope in

Output: *

(giving the slope of the graph of $f(x, y)$ in that direction $\vec{u}$ starting at (a, b))

2nd Order Partial Derivatives by Example

SUPPOSE $f(x,y) = xe^y + \frac{y}{x}$

$$\frac{\partial f}{\partial x} = \frac{d}{dx} [xe^y + \frac{y}{x}] = e^y - yx^{-2}$$

$$\begin{cases} \xrightarrow{f_{xx}} \frac{d}{dx} \left[\frac{\partial^2 f}{\partial x^2} \right] = \frac{d}{dx} \left[\frac{\partial f}{\partial x} \right] = \frac{d}{dx} [e^y - yx^{-2}] = 2yx^{-3} \\ \xrightarrow{f_{yx}} \frac{d}{dy} \left[\frac{\partial^2 f}{\partial y \partial x} \right] = \frac{d}{dy} \left[\frac{\partial f}{\partial x} \right] = \frac{d}{dy} [e^y - yx^{-2}] = e^y - x^{-2} \end{cases}$$

$$\frac{\partial f}{\partial y} = \frac{d}{dy} [xe^y + \frac{y}{x}] = xe^y + x^{-1}$$

$$\begin{cases} \xrightarrow{f_{xy}} \frac{d}{dx} \left[\frac{\partial^2 f}{\partial x \partial y} \right] = \frac{d}{dx} \left[\frac{\partial f}{\partial y} \right] = \frac{d}{dx} [xe^y + x^{-1}] = e^y - x^{-2} \\ \xrightarrow{f_{yy}} \frac{d}{dy} \left[\frac{\partial^2 f}{\partial y^2} \right] = \frac{d}{dy} \left[\frac{\partial f}{\partial y} \right] = \frac{d}{dy} [xe^y + x^{-1}] = xe^y \end{cases}$$

Not a coincidence
that the same!
Always have f_{xy} and f_{yx}

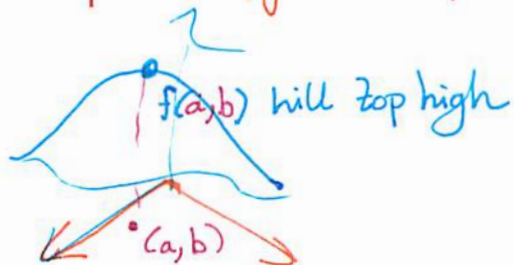
Local & Global Extrema

SUPPOSE (a,b) is not on the domain's boundary

Local max at (a,b)

IF $f(a,b) \geq f(x,y)$

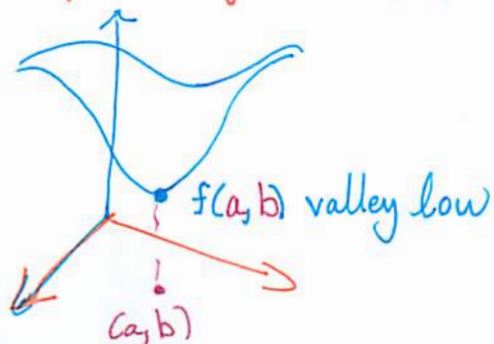
for all points (x,y) near (a,b)



Local min at (a,b)

IF $f(a,b) \leq f(x,y)$

for all points (x,y) near (a,b)



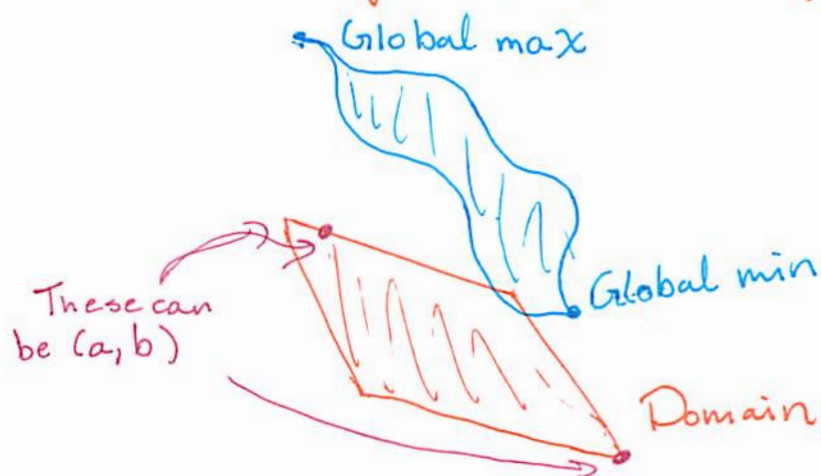
For global max/min allow to be on boundary

Global max at (a,b)

IF $f(a,b) \geq f(x,y)$ for all points (x,y)

Global min at (a,b)

IF $f(a,b) \leq f(x,y)$ for all points (x,y)



Finding Global Extrema

SUPPOSE $f(x, y)$ defined on closed, bounded domain D
including boundary inputs can only get so big

THEN the global extrema occur either

① On the boundary of D

OR

② At a point where $\nabla f = \langle 0, 0 \rangle$

Methodology for Finding Local Min/Max of $f(x,y)$

① Need a 2-variable function $f(x,y)$ to maximize/minimize
(May need to use more than 1 equation to reduce down to 2 variables)

② Critical points are (a,b) w/ $\nabla f(a,b) = \langle 0,0 \rangle$
 $= \langle f_x(a,b), f_y(a,b) \rangle$

So want (a,b) with

$$\left\{ \begin{array}{l} f_x(a,b) = 0 \\ \text{AND} \\ f_y(a,b) = 0 \end{array} \right\}$$

④ Check whether actually have local max/min:
 $D = f_{xx}(a,b) \cdot f_{yy}(a,b) - (f_{xy}(a,b))^2$

③ Reread question for whether want
• where max/min is AT, so (a,b) is answer
OR
• max/min VALUE, so $f(a,b)$ is answer

D	$f_{xx}(a,b)$	Local min/max
>0	>0	Local Min
>0	<0	Local Max
<0		Saddle Point
$=0$		Test Fails



Finding Critical Points for $f(x,y)$ Example

Task: Critical points for $f(x,y) = x^3 - 3xy^2$

Need $\nabla f(x,y) = 0$, i.e. $\begin{cases} f_x(x,y) = 0 \\ \text{AND} \\ f_y(x,y) = 0 \end{cases}$

$$f_x(x,y) = 0$$

$$\frac{d}{dx}[x^3 - 3xy^2] = 0$$

$$3x^2 - 3y^2 = 0$$

$$3x^2 = 3y^2$$

$$x^2 = y^2$$

$$x = \pm y$$

Need $(x,y) = (0,0)$

Critical Point = $(0,0)$

$$f_y(x,y) = 0$$

$$\frac{d}{dy}[x^3 - 3xy^2] = 0$$

$$-6xy = 0$$

Either $x=0$ or $y=0$

Classifying Local Minima, Local Maxima, Saddle Points Example

Suppose Know:

$$\left\{ \begin{array}{l} f_x(x,y) = 8y - (x+y)^3 \text{ AND} \\ f_y(x,y) = 8x - (x+y)^3 \text{ AND} \\ \text{Critical points are } (0,0), (-1,1), (1,1) \end{array} \right\}$$

Task: Classify the critical points

Step 3:

	D	f_{xx}
$(0,0)$	$-64 < 0$	$f_{xx}(0,0) = -3(0+0)^2 = 0$
$(-1,-1)$	$128 > 0$	$f_{xx}(-1,-1) = 3(-1+1)^2 = 12 > 0$
$(1,1)$	$128 > 0$	$f_{xx}(1,1) = 3(1+1)^2 = 12 > 0$

Step 1: For $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ need
 f_{xx}, f_{yy}, f_{xy}

$$\begin{aligned} \cdot f_{xx} &= \frac{d}{dx} [f_x] = \frac{d}{dx} [8y - (x+y)^3] = -3(x+y)^2 \frac{d}{dx} [x+y] = -3(x+y)^2 \\ \cdot f_{yy} &= \frac{d}{dy} [f_y] = \frac{d}{dy} [8x - (x+y)^3] = -3(x+y)^2 \frac{d}{dy} [x+y] = -3(x+y)^2 \\ \cdot f_{xy} &= \frac{d}{dx} [f_y] = \frac{d}{dx} [8x - (x+y)^3] = 8 - 3(x+y)^2 \end{aligned}$$

Step 4:

$f(0,0)$ is a saddle point
 $f(-1,-1)$ is a local maximum
 $f(1,1)$ is a local maximum

So $D(x,y) = (-3(x+y)^2)^2 - (8 - 3(x+y)^2)^2$

Step 2:
 $D(0,0) = (-3(0+0)^2)^2 - (8 - 3(0+0)^2)^2 = -64 < 0$

$D(-1,-1) = (-3(-1+1)^2)^2 - (8 - 3(-1+1)^2)^2 = 128 > 0$

$D(1,1) = (-3(1+1)^2)^2 - (8 - 3(1+1)^2)^2 = 128 > 0$