

PfaffModule9L09

Mon, 2/21 3:00PM 22:33

SUMMARY KEYWORDS

equal, circumstance, equation, separable differential, constant, exponent, solve, side, integrate, kt, function, decay, radioactive decay, natural log, population, solutions, differential equation, respect, bit, sum

SPEAKERS

Catherine Pfaff

Welcome. In this lecture, we're going to talk about this situation where you have that your change of Y with respect to your input T is proportional to the value of Y at T . This is a very common circumstance, actually. And it's going to turn into an exponential growth and decay situation. So this is where we have, so the derivative of Y with respect to T , that function is the same function as K times Y . Okay? And then we're going to actually use so the question is, what are the solutions to this equation? And I'll kind of all explain to you how we actually find that. So what are the solutions to this equation that we had here where we had, so we have, right, we have dy/dt .

So we have dy/dt , oops, for some reason. So we have dy/dt is going to equal, and then we have KY . Okay? And so what that means is, you know, what is a function Y of T , so that we can take its derivative with respect to T , you get K times that function? In the way to figure this out is actually this is, this is called something called a separable differential equation. And I'll show you then how this means that we can actually get the solution. So let's go ahead and look at that. Okay, so a separable, this is a separable differential equation, which means that we can solve, and let's go ahead and do that. So we can solve. And this is how you solve with the separable differential equation. And so what we're going to want to do is, so we're going to start with, and I'm going to kind of get rid of the, I'm not going to get rid of actually the colors on that I like the colors there. So we want to have dy/dt .

So I have dy/dt equals KY . Now what we want to do with a separable differential equation, is I want to get everything with Y on it in it on one side and everything with T in it on the other side. Okay, so the goal now is we want to get so, so we want, right, we want everything. This is somehow not, I had less room over here than I realized. And so I kind of started that off in a way that wasn't going to work very well. So, so the first step here, so this is kind of like step one is when you want everything with Y in it on one side, on one side, and with T on the other side.

Okay, so that's the first thing you do if you had one of these. So we can go ahead and do that. So what do I have that has Y in it, I have dy and I have Y . So I can move over the Y , but then this is has T in it, so I move it to the other side. So what I end up getting is I get $1/Y$ over Y . So I have $1/Y$ over Y and

in it, so I move it to the other side. So what I end up getting is I get $1 \text{ over } Y$. So I have $1 \text{ over } Y$ and then DY , right? I just divided both sides by Y , and I'm going to multiply the other side by DT . I'm multiplying both sides by DT , so that ends up over here. So I just end up on the other side with this KDT .

Okay, so all I did was some multiplying and dividing to move everything that involved Y to the left and everything that involved T to the right. So now I have a new equation like this. And now the next thing that we do, if we're doing a separable differential equation, is that I want to integrate both sides. So let's go ahead and do that. So I'm going to take this, and I'm just going to take the integral of it on both sides. So I'm going to take the integral of $1 \text{ over } Y \text{ } DY$ on this side. And then I'm going to take the integral on this side also, but this one is going to be DT . So let's kind of put all this in. So I'm integrating on this side with respect to Y , right? And on this side, I'm integrating this constant with respect to T . Okay. So what did I do here? I just integrated both sides. So step two is I integrate both sides, but notice that they're being integrated with respect to different things. So integrate both sides.

Okay, now, I'm going to have, so let's go ahead and integrate these. So on the left hand side, so I get the natural log of Y plus $C1$ is going to equal, so this is the natural log of Y plus $C1$, is going to equal, and then I have that K times T plus $C2$. Right, this is just standard integration, right? This is a constant, so it picks up whatever I'm integrating with respect to in front of it, and then that's it. But now to actually solve for Y . So now what I want to do is I actually want to solve for Y . So this is what's happening next. So step three is solve for Y .

I keep on feeling like the board goes over further than it does. And this is my nemesis of the day. Okay, solve for Y . The board goes over further, the image does not where you can actually see it. Okay, so we have this. So how do we isolate Y here, is I need to take the exponential function of both sides, okay? So I'm going to get E to the natural log of Y plus $C1$. And then I'm going to get, I'm going to combine actually, before I do that, I'm going to combine these into. I did this in all in one step before, and now I'm looking at it and thinking that I want to combine. So I'm going to combine these two constants into one constant. Okay, so I'm going to take these, and now it's just going to be plus C . Okay? You can do that with constants. If I do constants, I get a constant, so that's okay. Because now, otherwise, it just gets a little bit too messy. It's still doable, it's just too messy. Or a little more messy than I'd like to deal with. Okay, so I'm going to take the exponential of the natural log, right, that's how these are inverse functions. So that's how we kind of, they undo each other in some sense. And then I take E to them. And then I have K times T plus C .

Right? But now, we know that, right, when we have a sum in the exponent, that's actually a product. Okay? And on this side, we know that these functions undo each other. So we just end up with, by the way, we know that in this circumstance, we're talking about positive objects, so this is actually just a positive Y . I'm just going to drop the absolute value. So this is Y is going to equal, and I'm going to kind of separate, so we have E to the KT right? This is, we have a sum, so I can write this as KT . Actually, let's do this here. This is going to equal, this is a sum here in the exponent, which means that it's a product. So we have E to the KT .

And we're going to multiply that by E to the C , right? This sum turns into a product of the exponents, right, but E to the C is again a constant, okay? So we get here that this is Y equals E to the KT times, right, this is a constant, which we're going to be able to solve for by setting, by setting T equal to zero, okay? But maybe actually also so this is just EC . And then I'm going to note here that this is a constant. Okay, so this is a constant that we can solve for. So this is a constant, can solve for by setting T equals zero, missing T equal to zero.

Okay? So we're going to get that, so now we're going to do this. So we're going to solve for this. And we'll solve for it right here. Okay, so this is solving for E to the C . This is what I'm going to do right here. Okay? Well, what do I have? I have, I'll erase this. So I have Y equals, and then I have E to the KT times C . Sorry, KT times E to the C . Right? And then that would tell me that Y of zero, right, is going to equal, well then I have T to the right, K times zero times E to the C , right? But this is the same thing as E to the zero which equals 1. Right? So this is telling me that Y of zero is going to equal E to the C . So when I plug in zero, I actually get E to the C . So I can take this. Right? I can take this here and plug it into here. Okay, and so my answer is actually going to be that, what is my function Y ? My function Y is going to be Y of zero times E to the KT .

I don't know what K is, but I know it's the K that I started out with. Okay? So this is if, so let's kind of go through what we've done so far. Which is that if we start out, like if I'm looking for the solutions to this, like functions Y of T , so that when I differentiate Y with respect to T , I get K times Y , those two functions are equal to each other. This is something called separable differential equation, which really exactly means that I can move all the Y stuff to one side and all the T stuff to the other side. And so that allows me to figure out what my function Y is like I did here. And the first step in solving a separable differential equation is I move all the Y stuff to one side, I move all the T stuff to the other side. So I did that. And then I integrated this side with respect to Y , this side with respect to T , and kind of combined my constants. At that point, I'm still trying to solve for Y . So I take the exponent, remember that exponent and natural logarithm are inverses. So I can, if I take E to both sides, then those will be equal. On this side, I can actually kind of write it as a product like that. Although actually, I did end up writing it in the next step anyway. So maybe this is a little extra for that to be there. So we can get rid of that just to unclutter a little bit. And right, because I have a sum in the exponent, which means I can break it apart as a product, right? So I get Y equals E to the K times E to the C . And we did leave it like that, but I'm actually able to solve for what this constant is by looking at what happens when Y equals zero. So, right because and the reason that kind of works out is because then I'm taking E to the zero, which is one. So everything drops out except for that constant. And so that's how I would have chosen to try to solve for Y of zero is because looking at this, it's like how do I make everything drop out except for the constant? Well I make the exponent becomes zero so that this part becomes 1. Okay? So that's how I actually chose to do that. That allowed me to solve for the constant, I could plug that back in. And now I have an equation that anything that satisfies this circumstance, which is actually a common circumstance, is going to look like this. Okay? So there are kind of two well known circumstances of when this happens. And these both have to do with what happens with this K , I didn't talk much about the K . Okay, so if K is greater than zero, okay, then this is actually so population growth is an example. So population growth is an example. Any kind of population, whether it's people, whether it's microbes, whether it's bunnies, whether it's actually, this is also a lot of how diseases spread, things like this, okay? So we have for.

Okay, and this would be for where we have that T equals time. So T equals time, and then P of T is going to equal the population. So this equals the population at my time T , so population at time T .

Okay, and then with this kind of, you know, population growth example, then my equation, so I ended up with the equation $\frac{dP}{dt} = K \times P$. And where we call, so where we have here, so K is going to be called the relative growth rate. So, this is called the relative growth rate.

So this is a new terminology or name in like a population growth circumstance, okay? And then my solution is, right, we have P of T is going to equal, I meant for that to be orange. So, P of T is going to equal here, so we do actually have this P naught, which is the same thing like having the Y as zero there. So, this is going to be my initial population, E to the KT . Okay? And as I said, this is my initial population. Or my population at time zero, so this is my initial population. And then that's going to be what's going to give me how my population is going to grow. The fact that there's an E to some positive thing times the time means this growth is going to happen fast. Okay? It's going to be something where at first it looks slow, you're like, oh I went from, you know, I started with 10, and then all of a sudden I had 100, and then I had 1000, and then I had 10,000. This is kind of how this works, it's like very very, it just can just explode. Okay, so then if K is less than zero okay, then the, here a good example of this. So radioactive decay.

What I'm going to do here is move this a little bit to the side, so that it stands out as to, this is the label, I always want. I always try to label I always tell students also to label your, label your boards or label what you're doing to help keep everybody organized. So I'm going to stick it out a little bit so you can tell I'm labelling a new section here. So this is radioactive decay is an example.

Okay? So what is this example here? So here we have for, and then we have so T , if T equals maybe the time in years. So T equals time in years. And then we have M naught equals the initial mass. Right? We care about radioactive decay. Well, first off because of unsafe circumstances but also this is used to determine like, so also if you wanted to determine how old something is, so this can show up in a number of fields. So we have M of T is going to equal the mass remaining after T years.

Okay? So then in this circumstance, our equation is going to look like. So well first off the differential equation is, so you have $\frac{dM}{dt}$, you do have like this decay constant, so this is going to equal, again you have like this proportionality constant, okay? So that's the differential equation and then again, so this becomes M of T is going to equal M naught E to the KT . Okay? Oops, I meant for that to be green. Okay. So that's our equation. So these are kind of one of them, right, kind of the highlight of this side over here is that if K is greater than zero, we're in a growth circumstance where things are getting larger, like populations or something like this. And then if K is less than zero, then you're actually decaying over time, as in, you're going to have less and less. Like maybe have a mass when you're talking about like, you know, the mass of the remaining sample. And this is important when we're in a lot of different circumstances, such as when you want to try to figure out how old something is, but also in terms of understanding like, you know, how much of a pollutant is going to be left or something like that. Okay? So, I hope that made some sense, and I will see you in the next lecture.