

PfaffModule9L07

Mon, 2/21 2:37PM 16:15

SUMMARY KEYWORDS

limit, infinity, derivative, indeterminate form, top, quotient rule, bottom, function, rule, compute, separately, $2x$, chain rule, $3x$, mistake, equalities, tells, faster, divided, conditions

SPEAKERS

Catherine Pfaff

Welcome. In this lecture, we're going to introduce L'Hopital's Rule, which can be a great tool for comparing like the growth rate of two different functions. It can also be a great tool for computing limits of quotients. I always get a little nervous teaching it because it's, it's one of the things that students make the most mistakes in. And I'll tell you actually what they do, which is they apply the quotient rule, which you're not supposed to do. Okay, so watch carefully for where you're not going to apply the quotient rule, that you're going to do something different. Okay, so let's go ahead and get started with actually, there are some conditions, this is the other mistake students make, which is they forget that you have to have the conditions be satisfied in order for this to work. But there are these conditions. So let me get started with them. So if you have that the limit as X approaches some C of F of X divided by G of X .

So if you have that the limit oops, G of X , that's what I meant to write and I started thinking ahead to the next part, G of X . Okay, so if you have that this limit gives, this is where I meant right gives so gives an indeterminate form. And I'll tell you what that means. So an indeterminate form, i.e., so there are several different kinds, I'll give you an example of each one so that you kind of know what I'm talking about. So the first one is kind of is a zero over zero. Maybe I'll do this with the two colors just to kind of emphasize that you're taking the limit at the top, and you're taking the limit at the bottom. And let's look at kind of an example. One example of this is going to be if I was taking the limit as X goes to infinity, all of mine are going to be as X goes to infinity, because this is what we're really interested in anyway. So the limit as X goes to infinity of E to the negative πX . So I have E to the negative πX . And you'll see that's a very important kind of function that shows up a lot. Maybe not having the π there, but having some K there. This is totally legitimate. And then dividing by 1 over X , okay? So this is an example because if I took the limit of the top, right, I would get zero because this is like 1 over something getting really big. And then the same thing on the bottom, I would get zero. Okay? So this is one indeterminate form is where you have zero over zero when you take the limit of the top and bottom separately. The next one is, so we'd have plus minus infinity over, and then we have minus plus infinity. So for example, if I was going to take the limit as X goes to infinity of, and so here, I'm taking 1 minus X squared. So my, my top function, my numerator function is 1 minus X squared. My denominator function here is, I don't know, maybe E to the $5X$ I think I have there, E to the $5X$, okay? And when I'm taking, I'm just kind of closing these up, my infinities are getting a little sloppy, okay? So if I took the limit of the top, I would get negative infinity,

if I took the limit of the bottom, I would get infinity. And this is what I mean here is they're both infinite, positive or negative, but they're opposite. That's why I put plus on top here and minus on top, minus bottom plus on bottom. So it has to be kind of swapped like that. Because this one is equal to, and then I actually kind of wrote what it was here. So this was, right, this is equal to this one in particular is minus infinity over infinity. But I would have been, it's also in this kind of circumstance, if I count top infinity and minus infinity on the bottom, okay. And then the next one is the one where they're both infinite, but they match up inside, okay, so this is plus and minus infinity.

And then divided by, and then I have plus or minus infinity. So now the sign on the infinities match up. Okay? And here's the example would be if I took the limit as X goes to infinity of, and then I have E to the X over the natural log of X . Right? Because if you're looking at this, right, this is an infinity over infinity circumstance, just to be clear. So this is infinity over, oops, I'm going to keep with my, my scheme here. So this is infinity divided by infinity, infinity divided by infinity. Okay? And in fact, the top one, I didn't write it there, that's the start, but we can write it there now. Is that this is actually equal to, right, this is zero, because the limit at the top is zero. And then if I take the limit at the bottom I also got zero. So this is a zero over zero circumstance. Okay. But any of these are indeterminate forms. So if I'm in one of these forms, okay, then, so first you need to check you're actually in one of them. And then you know.

So here's the then, which is that the limit as X goes to C of, like we have there, if I wanted to take the limit of F of X divided by G of X , I could actually, and this is where be very careful to notice that I'm going to be taking the limits in the top and the bottom separately. You're going to think I'm seeing it so many times, then you're, you might end up making a mistake with it nonetheless, which is why I say it so many times that you're a little worried you're going to go crazy that I'm taking the derivative separately, that I'm taking the derivative of the top, and I'm taking the derivative of the bottom. Okay? But this actually works that this limit is going to be the same as this one. Maybe I'll put a like nice box around this, this is really the outcome of L'Hopital's Rule, but you have to start with an indeterminate form. And then you actually have that it might be, you don't really know what to do with this limit, but then you take the derivative of the top and the bottom and all of a sudden it's clear. And we'll go through an example in a moment. But before I go through the example, I want to kind of put this into words. I know that sometimes when you have a whole bunch of like symbols and like fancy math language, in a statement, it gets a little bit confusing as to what one's actually talking about. So sometimes I like to kind of actually say it in words. So i.e., if the limit gives an indeterminate form, so gives an indeterminate form, like one of those forms from up above, so I'm talking about this kind of situation here. Then we can compute the limit by, then we can compute the limit by.

So the first thing would be to compute the derivative of the top and the bottom separately not applying the quotient rule. So computing the derivative of the top and bottom separately. So the top and bottom, this is very melodramatic, so separately, not okay, no quotient rule. Okay? And then computing the limit. So then computing limit.

Okay, so let's go through an example here. Also kind of go like this so that it's clear what's on this side of the board and what's on this side of the board. Okay, so here's our example. So it'll be like our first example, which will be that I'm going to take the limit as X goes to infinity. There will be more

but the limit as X goes to infinity of, and I want to do E to the $3X$ over $2X$ plus 1. So E to the $3X$ and then on the bottom I have $2X$ plus 1. Right? And if I do these limits separately, so this, right, so this is going to end up equaling if I take the limit of the top I get infinity. If I take the limit on the bottom, I get infinity. Okay? So indeterminate forms, so keep going. So indeterminate form, which tells me I can keep going. So keep going with L'Hopital's Rule. Okay. So LH is a very common abbreviation of L'Hopital's Rule, just to make sure you're aware, that you know that's what I mean by that. Okay, so let's keep going. So what does L'Hopital's Rule tell me I can do? I check the conditions, I do indeed have this situation. I am in an indeterminate form. So I take this, and I've got to take the derivative of the top and the bottom separately. So I have the derivative with respect to X of the top, and the derivative with respect to X of the bottom. So on top, I'm going to get E to the $3X$. Right, the derivative of that, and on the bottom, I'm taking the derivative of $2X$ plus 1. And this is going to give me, right, I have to very carefully, so this is where we might make a mistake. Which is that we might forget to use the chain rule on the top. But we won't forget to use the chain rule on the top because we're careful calculus students. And so we're going to get, the other thing that I think is really important is that each time you write down that you're taking the limit, I know it's an extra second or so of work, and it's tiring on the hand, and so on. But it's a common mistake. If you don't actually write that, you start to make mistakes. It's very good to keep each thing exactly proper, so that you know what you're doing for sure. Okay? So I took the derivative of the top, I did not forget the chain rule. So that's a very happy moment. And the derivative of the bottom is, right, it just becomes very easy. It's just 2. Okay? And, right, the bottom is finite, the top goes to infinity. So this is definitely infinity.

Right? So this tells me that, going back to the start I look back at what I was trying to do. I make sure I did accomplish that, I did so I'm happy. So you see the limit as X goes to infinity of, and then I have of all the way back to the beginning, I was trying to compute the limit of E to the $3X$ over $2X$ plus 1. And through all these equalities, I got out infinity. So this has to equal infinity. Right? And what is that? That's going to tell me something about growth. So this also tells me something about growth, which is that because this is infinite, one of the things I know now is that and, so what else does this tell me? It tells me that E to the $3X$ grows faster? Right? It's going to grow faster, then, oops, that should be in red.

So it goes faster than if I had to, then $2X$ plus 1, which is on the bottom, right? Okay, so let's kind of recap what we went through with this example. In the first step, right, the first thing I had to do was to check I had an indeterminate form. How did I do that? I took the limit at the top, I got infinity, I took the limit of the bottom, I got infinity. So infinity over infinity and I check up here. Yes, that's an indeterminate form. Yes, I can apply L'Hopital's Rule. Next step, I took the derivative of the top, I took the derivative of the bottom, I did that separately. Okay, and I actually took those derivatives, I was very careful when I took this derivative. I remember to use the chain rule, and that's how I got a 3 in front. So I got $3E$ to the $3X$. The bottom, I took that derivative, I got 2. And so now this is something I can compute the limit of, right, because the bottom is finite, the top shooting off to infinity. So this is infinity. The bottom is finite positive. The top is shooting off to positive infinity, so I got infinity. So looking back at where I started through all my equalities, it's telling me that I was able to compute that this is actually equal to infinity. Now, then what does that tell me? If the limit when I take the ratio is infinity, that tells me that the top grows faster than the bottom. So this actually allowed me, I was able to use L'Hopital's Rule to, you know, and what am I really doing there? And this is kind of the key point in there, which is that what am I really doing is that, I know that the functions themselves, the limit doesn't, you know, capture which of them is kind of growing faster. But now I'm looking at their rates, like their derivatives, their rate of change, and I'm comparing the rate of change. And the

rate of change of this one really dominates the rate of change of this one. The rate of change of this one is a lot, is growing a lot faster than the rate of change of this one. So that's enough. And then you might actually, if that doesn't work, you can do L'Hopital's Rule again, and see that like the rate of change of the rate of change, and one of them is faster than the other. And that's really what L'Hopital allows us to do, is because if one of these rates of change is faster than the other, then definitely as you go towards infinity, the growth of that function is faster. Okay? So, I hope that made some sense, and I will see you in the next lecture.