“Retirement spending problem under a Habit Formation model ”

Snezhana Kirusheva

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Abstract

In the thesis we consider the problem of optimizing lifetime consumption under a habit formation model. Our work differs from previous results, because of incorporating mortality and pension income, using a fixed rather than a variable asset allocation, and adopting habit into the utility multiplicatively rather than additively, Lifetime utility of consumption makes the problem time inhomogeneous, because of the effect of ageing. Considering habit formation means increasing the dimension of the stochastic control problem, because one must track smoothed-consumption using an additional variable, habit $\bar{c}$. Including exogenous pension income π means that we cannot rely on a kind of scaling transformation to reduce the dimension of the problem as in earlier work, therefore we solve it numerically, using a finite difference scheme and then using a static programming approach. We also explore how consumption changes over time based on habit if the retiree follows the optimal strategy in the first part and a greedy strategy in the second part of the thesis. Also we explore how the optimal consumption and asset allocation change when pension varies. Finally, we answer the question of whether it is reasonable to annuitize wealth at the time of retirement or not by varying parameters, such as asset allocation θ and the smoothing factor η.

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List of Abbreviations	
AA	Asset allocation
BC	Boundary Conditions
BM	Brownian Motion
CRRA	Constant Relative Risk Aversion
EPP	Equity Premium Puzzle
EWA	Exponentially Weigthed Average
EWAC	Exponentially Weigthed Average of consumption
FDS	Finite Difference Scheme
FTH	Finite time horizon
ITH	Infinite time horizon
FTHM	Finite time horizon with mortality
HFM	Habit Formation Model
HJB	Hamilton-Jacobi-Bellman
OA	Order of Accuracy
OP	Optimization Problem
PDE	Partial Differential Equation
e.g.	exempli gratia (Latin), for example (English)
i.e.	id est (Latin), that is (English)

Introduction

Overview

Nowadays, we observe a growing interest in investment plans that give a potential client the confidence of a stable income over the course of retirement. The main goal of any such plan is to find the strategy that minimizes the risk of ruin and, at the same time, maximizes the level of consumption. Our current research deals with a retirement spending problem (RSP) under dynamics that include the individual's living standard. In other words, we take into consideration how much a retiree usually spends, i.e. we solve the problem under a habit formation model (HFM). This postulate makes the model much more difficult to solve.

In this thesis, our goal is to explore how the presence of exogenous income in a model that includes the client's habit will affect the optimal consumption and compare these results with different models, such as HFM without pension for two cases, with and without asset allocation. We also answer the question of how much an agent can spend during retirement, based on his initial wealth w and initial living standard $\bar{c}$. In addition, for an optimization problem (OP) with exogeneous income we can not reduce dimension by scaling, so solving the PDE numerically is slow, and our goal is to explore alternatives. At the same time, for the zero exogeneous income case, we have a PDE solution where asset allocation is the control variable and we want to see if there are assumptions under which we can get the similar results for alternative methods. Our choice is the martingale method. Based on our assumptions, we would rather call this approach a greedy heuristic than an optimal one. Finally, we explore how the retirement spending strategy changes as retirement income changes.

There is one more interesting option for a retiree that we also discuss in this thesis. The individual can convert some or all of his initial wealth into annuities. We briefly discuss this possibility, and if it is reasonble to do so at age of 65 depending on the client's habit. We assume that once he converts his wealth into annuities he can't reverse the transaction. We obtain numerical results for different parameters, such as habit $\bar{c}$, asset allocation θ, and smoothing factor η.

Literature review

Many articles have been written on this topic that consider various scenarios. Lately more researchers are paying more attention to the HFM when they deal with financial questions. There are several articles and books that solve similar problems, somehow related to the habit formation model, for example [Bodie, 2004], [Carrol, 2000], [Chetty, 2016], [Naryshkin & Davison, 2008], [Polkovnichenko, 2007], [Pliska, 2001], [Bodie etc., 2004], [Veron etc., 2017] and [Naryshkin, 2009]. All of them use different approaches and techniques. In this review we would like to mention papers that used two main approaches for solving optimization problem, the value function approach and martingale approach. First, we should mention one of the earliest papers [Constantinides, 1992]. In that article the author tried to solve the equity premium puzzle (EPP) which was first formalized in a study by Rajnish Mehra and Edward C. Prescott [Mehra & Prescott, 1985]. This is about an anomaly when historical real returns of stocks were higher than govenment bonds. Under the assumption of rational expectations this problem was resolved. One of the issues with that formulation is that the consumption should be always greater or equal to the exponentially weighted average (EWA) of consumption which is hard to implement in real situations. As a consequence, there are modifications of this work which are discussed, for instance in the book [Rogers, 2013] which introduces a novel form for the HFM utility. Another attempt to resolve the EPP was made by the authors [Xinfeng etc., 2013]. They considered optimal portfolio and consumption selection problems with habit formation in a jump diffusion incomplete market in continuous-time. One more pioneering work [Pollack, 1970] describes a model of consumer behaviour based on a specific class of utility functions, the so-called "modified Bergson family". There are many variations where also was implemented HFM for the cases with stochastic wages or incomplete financial markets, e.g. [Bodie etc., 2004] authors examine consumption and investment decisions in a life-cycle model with habit formation , stochastic wages and labor supply flexibility. As an example with incomplete markets we can refer on [Yu, 2015], [Yu, 2017], [Muraviev, 2011] or [Naryshkin & Davison, 2008]. In the book [Naryshkin, 2009] the author provides detailed discussion of HFM with transaction costs. In the paper [Yu, 2015] the author studies the continuous time utility maximization problem on consumption with addictive habit hormation in incomplete semimartingale markets.

Another example of using HFM was covered in the article [Polkovnichenko, 2007] which explores the implications of additive and endogenous habit formation preferences in the context of a life-cycle model of consumption and portfolio choice for an investor who has stochastic uninsurable labor income. In order to get a solution he derives analytically constraints for habit and wealth and explains the relationship between the worst possible path of future labor income and the habit strength parameter. He concludes that even a small possibility of a very low income implies more conservative portfolios and higher savings rates. The main implications of the model are robust to income smoothing through borrowing or flexible labor supply. In addition, we would like to mention one more paper [Detemple & Zapatero, 1992] where the authors proved existence of optimal consumption-portfolio policies for utility functions for which the marginal cost of consumption (MCC) interacted with the habit formation process and satisfied a recursive integral equation with a forward functional Lipschitz integrand and for utilities for which the MCC is independent of the standard of living and satisfied a recursive integral equation with locally Lipschitz integrand.

The martingale approach remains one of the most complicated approaches for solving optimization probelms. Combining this method with the habit formation model results in even more complicated problem. In order to understand better what the martingale approach is, we would refer on papers where authors solve various portfolio optimization problems, among them are [Egglezos & Karatzas, 2009], [Haugh etc., 2006], [Karatzas etc., 1986], [Karatzas etc., 1987], [Karatzas etc., 1991], [Detemple & Karatzas, 2003], [Munk, 2008] or [Yu, 2015]. One of the pioneer's work belongs to [Detemple & Zapatero, 1992]. In that paper the authors proved the existence of optimal consumption-portfolio policies for specific utility functions involving a general dependence on past consumption. Also they extended existing results to habit formation models with stochastic coefficients. In another paper [Liu etc., 2021] the authors solve portfolio management problem for an individual with a non-exponential discount function and habit formation in finite time. The authors considered case where investor receives a deterministic income, invests in risky assets, buys insurance and consumes continuously. They obtained analytical solution for two different strategies. For instance, in the paper [Karatzas etc., 1991] authors solve a problem of maximizing the expected utility from terminal wealth in an incomplete market containing bond and a finite number of stocks. There are some more papers, for example, [Liu etc., 2021] where the authors analytically solve the utility maximization problem for a consumption set with multiple habit formation of interaction where consumption is composed of habitual and nonhabitual components and habitual consumption represents the effect of past consumption. They further assume that the individual seeks to maximize his/her expected utility from nonhabitual consumption.

There are a lot of financial strategies which can help to plan how to spend money under different preferences but one of the most important targets is to arrange consumption during retirement. There are many works devoted to retirement spending plans, such as [Bodie, 2004], [Habib etc., 2017], [Huang etc., 2017], [Milevsky & Huang, 2011] or [Jeon & Park, 2020]. For example, in the paper [Bodie, 2004] the authors discuss consumption and investment decisions in a life-cycle model under a habit formation model incorporating stochastic wages and labor supply flexibility. One of the results shown was that utilities that exhibit habit formation and consumption-leisure complimentarities induce an impact of past wages on the consumption of retirees. Hence the authors showed that it is important to take into consideration habit and consumption-leisure complimentarities when formulating life-cycle investment plans. In the next article [Habib etc., 2017] the authors consider a model based on results from the article [Milevsky & Huang, 2011] where a similar problem was solved under assumption of deterministic investment returns. In [Habib etc., 2017] the authors accept stochastic returns and then compare optimal spending rates with the analytic approach from the article [Milevsky & Huang, 2011]. When a potential client starts to think about a retirement spending plan there is one more question that arises, namely under which conditions he can consider investment into annuities for part or all of his wealth. To be precise, when we say "annuities" we mean life annuities, insurance products that pay out a periodic amount for as long as the annuitant is alive, in exchange for a premium (see [Brown, 2001]). This question has been widely discussed in the literature, for example [Milevsky, 2020], [Milevsky & Huang, 2019], [Reichling & Smetters, 2015] or [Blake etc., 2001].

In the recent article [Habib etc., 2017] RSP was solved for fixed risky asset allocation $\theta = \text{const}$. Here we solve a similar problem following HFM, using the novel utility of [Rogers, 2013] [Kirusheva etc., under review].

In one of the most recent papers [Angoshtari etc., under review] the authors solve an infinite-horizon optimal consumption problem for an individual who forms a consumption habit based on an exponentially-weighted average of her past rate of consumption. The novelty of they approach is in introducing habit formation through a constraint, rather than through the objective function. In another paper that is also under review [Herdegen etc., under review] authors consider the infinite-horizon Merton investment-consumption problem in a constant-parameter Black–Scholes–Merton market for an agent with constant relative risk aversion. Along with some proofs they described the dual approach to the Merton problem.

In the end, we would like to mention some papers and books that provide the theoretical background necessary to solve an optimization problem like the one described in this thesis, and which has been widely used by the author, for example, articles [Strikwerda, 2004] where the author discussed the finite difference schemes, or [Mirica & Mirica, 2005] where the authors discuss different formulations of verification theorems, and books [Durret, 2013], [Øksendal, 2003], [Milevsky, 2006] or [Rogers, 2013].

Agenda

The thesis is organized as follows. In Chapter 1 we solve the reetirement spending OP using the value function approach which implementation requires to use a finite difference scheme. First, we explain what the habit formation model is and formulate our problem for two different cases, without (see Section 1.1.2) and with (see Section 1.1.3) pension. As with most of these models, this one doesn't have an analytical solution and has to be approximated numerically. Many algorithms have been developed through the years. Every algorithm has its own advantages and disadvantages, which differ in accuracy and efficiency. In Chapter 1 we chose a finite difference scheme for its simplicity and accuracy. The detailed description of the approximation scheme, some error analysis 1.2.3 as well as some theoretical background are provided in Section (see 1.2). In Section 1.3 we discuss the numerical results obtained for different cases, e.g. how the smoothing factor η affects the numerical solution and provide a comparison between two different cases, without pension (Section 1.3.1) and when the client has constant pension income (Section 1.3.2). Also in Section 1.3.3, we describe some numerical results for different sets of asset allocations θ and volatility σ for fixed smoothing factor $\eta = 1.0$, as the most interesting case for solving problem under HFM. Next Section 1.4 is devoted to discussing how the client can spend money during his retirement based on a given initial amount of wealth w and a certain habit $\bar{c}$. In the last Section 1.5 of the first chapter we analyze the possibility of annuitizing wealth, entire or partially, at the age of 65.

In Chapter2 we described a completely different approach for solving our problem. In the beginning Section 2.1, we provided some overview about martingale approach and some basic stochastic calculus that was used widely in this chapter. Then in Section 2.2 we discussed the classic Merton problem in order to use it as a "toy model". Finally, in Section 2.3 we solved our problem using greedy policy algorithm. Since the formulation of our problem in Chapter 2 was slightly different we provided some comparison between two approaches, PDE and martingale approaches in the paragraph 2.3.5. Finally, as in the previous chapter, by fixing the initial wealth and habit we provided some analyzis of how wealth and consumption change over time 2.3.6.

In the last chapter we made some concluding remarks and gave some insights about possible future reserach directions.

Chapter 1

Solving retirement spending problem under a habit formaton model using the value function approach

1.1 Model formulation

1.1.1 Theoretical Background

When we think about the model we should think about a client, more precisely about a retiree, who has a certain amount of wealth w and who wants to know what to do next with his endowment. In order to decide how much he can spend we should understand how wealth is changing over the time counting all possible income and expenses, as in the following:

$$\begin{aligned} \mathrm{d}w_t &= [\theta(\mu - r) + r]w_t \mathrm{d}t + \theta\sigma w_t \mathrm{d}W_t + \pi \mathrm{d}t - c_t \mathrm{d}t \\ \mathrm{d}\bar{c}_t &= \eta(c_t - \bar{c}_t)\mathrm{d}t. \end{aligned} \tag{1.1}$$

We can provide the following explanation of the equations (1.1). There is a part of wealth w which grows at the riskless rate r, there is a stochastic component represented by the parameter θ, which is the fraction of wealth invested into risky assets (in our case, we take it as a fixed parameter θ), drift μ, volatility σ and W_t a Brownian Motion (BM). Also assume that there is an exogenous fixed income, pension π. We solve our problem using a habit formation model. This means that the agent's utility depends on the consumption rate c_t and on an EWA $\bar{c}_t$ of consumption rates over previous time periods. We will consider the finite-horizon problem therefore the client's objective function is taken to be

$$\sup_{c_s} E\left[\int_t^T e^{-\rho s} {}_s p_x u\left(\frac{c_s}{\bar{c}_s}\right) \mathrm{d}s | w_t = w, \bar{c}_t = \bar{c}\right] \tag{1.2}$$

where ρ is the personal time preference or subjective discount rate, ${}_s p_x$ is the probability of survival from the retirement age x to $x + s$. We set up the probability of survival based on the Gompertz Law of Mortality ([Milevsky, 2006]), i.e.

$${}_s p_x = e^{-\int_0^s \lambda_{x+q} \mathrm{d}q}. \tag{1.3}$$

Here λ is the biological hazard rate $\lambda_{x+q} = \frac{1}{b} e^{(x+q-m)/b}$ where m is the modal value of life (see p47, [Milevsky, 2006]), b is the dispersion coefficient of the future lifetime random variable, u is the CRRA utility function

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1 \tag{1.4}$$

where $\gamma > 0$ is the risk aversion parameter. Note that the formulation of utility in (1.2) is due to [Rogers, 2013]. It differs from much of the HFM literature where authors mostly use an additive utility finction whereas in our case we consider a multiplicative form. This changes the nature of the solutions. One major difference is that the consumption c_t may now fall below the EWA of past consumption $\bar{c}_t$.

In order to solve this problem, we will use the value function approach that implies we should derive an Hamilton-Jacobi-Bellman (HJB) equation for our model. Let us consider two different cases. The problem that we are going to solve first, reflects an agent's expectations who does not have any pension income which means that his wealth's growth results partly from investing in a bank account growing at the risk free rate and partly from investing another portion of the wealth into risky assets. The second problem involves the presence of pension income. In addition, the asset allocation will be fixed.

But first, we need to understand if there exist a function which satisfies our constraints. In other words, there are some questions which should be answered before solving the problem:

1. Prove that a smooth solution of an HJB is a either supermartingale or martingale.

2. Prove that supermartingale or martingale solution solves the OP.

Let us start from proving first statement. Suppose we have a smooth function $\tilde{V}$ that solves the HJB equation. Define a stochastic process Y_t by

$$Y_t \equiv \tilde{V}(t,w_t,\bar{c}_t) + \int_0^t e^{-\rho t'}\,{}_{t'}p_x u\left(\frac{c_{t'}}{\bar{c}_{t'}}\right) \mathrm{d}t'. \tag{1.5}$$

The goal is to prove that the stochastic process Y_t is a supermartingale for any admissible choice of consumption c_t and martingale for the optimal value of consumption c^*. There are at least two approaches how this statement can be proved, for example, we can prove it by taking expectation $E[Y_t|\mathcal{F}_s]$ and then using Fatou's lemma [Durret, 2013]. Here, we will provide another approach. Take the differential of this expression (1.5) and compute in the drift and volatility terms. Then, plug the expression for wealth and consumption rate dynamics (1.1). After performing some calculations we get the following

$$\begin{aligned}\mathrm{d}Y_t = \Big\{&\tilde{V}_t + \tilde{V}_w(\theta(\mu - r) + r)w_t + \tilde{V}_w(\pi - c_t) + \tilde{V}_{\bar{c}}\eta(c_t - \bar{c}_t) + \\ &\tfrac{1}{2}\tilde{V}_{ww}\theta^2\sigma^2 w^2 + e^{-\rho t}\,{}_t p_x u\left(\frac{c_t}{\bar{c}_t}\right)\Big\}\mathrm{d}t + \tilde{V}_w\theta\sigma w_t \mathrm{d}W_t.\end{aligned} \tag{1.6}$$

Then integrate (1.6) over the interval $[0,t]$ assuming that function that represents drift term can be written as $f(t,w_t,\bar{c}_t)$ and volatility term as $g(t,w_t,\bar{c}_t)$.

$$\int_0^t \mathrm{d}Y_{t'} = \int_0^t f(t',w_{t'},\bar{c}_{t'})\mathrm{d}t' + \int_0^t g(t',w_{t'},\bar{c}_{t'})\mathrm{d}W_{t'} \tag{1.7}$$

Using the additivity property and taking expectation $E[\dots|\mathcal{F}_s] \quad \forall s \leq t$ after that, we obtain the following expression

$$E[Y_t|\mathcal{F}_s] = E[Y_s|\mathcal{F}_s] + E\Big[\int\limits_s^t f(t',w_{t'},\bar{c}_{t'})\mathrm{d}t'|\mathcal{F}_s\Big] + E\Big[\int\limits_s^t g(t',w_{t'},\bar{c}_{t'})\mathrm{d}W_{t'}|\mathcal{F}_s\Big]. \tag{1.8}$$

Now, recall that the drift term $f(t,w,\bar{c})$ is zero in the case of a martingale or negative if we have a supermartingale, then if we assume that integrand $g(t,w,\bar{c})$ is square integrable then

$$E\Big[\int\limits_s^t g(t',w_{t'},\bar{c}_{t'})\mathrm{d}W_{t'}|\mathcal{F}_s\Big] = 0. \tag{1.9}$$

Then $E[Y_t|\mathcal{F}_s] \leq Y_s$. The statement is proved. Now, let us reformulate the third statement.

Theorem 1.1.1. *(verification theorem): Suppose* $\exists\tilde{V} : [t,T] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *which is* $C^{1,2,1}$. *The objective has the form (1.2). Suppose that* $\forall c_t \in \mathscr{A}(w)$ *where* $\mathscr{A}(w) \equiv \{c : c \text{ is admissible with regards to } w\}$ *the stochastic process*

$$Y_t \equiv \tilde{V}(t,w_t,\bar{c}_t) + \int\limits_0^t e^{-\rho s}{}_s p_x u\left(\frac{c_s}{\bar{c}_s}\right)\mathrm{d}s \quad \textit{is a supermartingale} \tag{1.10}$$

and $\tilde{V}(T,w,\bar{c}) = 0$, *also* $\exists c^* \in \mathscr{A}(w)$ *such that the process* Y_t *is a martingale. Then* c^* *is optimal, and solution of the problem is*

$$\tilde{V}(t,w,\bar{c}) = \sup_{c_s} E\left[\int_t^T e^{-\rho s}{}_s p_x u\left(\frac{c_s}{\bar{c}_s}\right)\mathrm{d}s|w_t = w,\bar{c}_t = \bar{c}\right]. \tag{1.11}$$

The consumption $(c_t)_{t>0}$ *is admissible for wealth* w*, if* c_t *is adapted and if the wealth process* w_t *remains positive at all time.*

Proof. The supermartingale property states that $E[X_t|\mathcal{F}_s] \leq X_s$, $0 \leq s \leq t$, $E[X_s] < \infty$ (see, for example [Durret, 2013]). The goal is to prove that the following inequality holds

$$E[Y_T|\mathcal{F}_t] \leq Y_t. \tag{1.12}$$

By plugging values "T" and "t" consequently into equation (1.10) and performing simple calculations we can get

$$\tilde{V}(t,w,\bar{c}) \geq E\Big[\int_t^T e^{-\rho t'}\,{}_{t'}p_x u\left(\frac{c_{t'}}{\bar{c}_{t'}}\right) \mathrm{d}t'|\mathcal{F}_t\Big]. \tag{1.13}$$

The equality comes from the definition of optimal value. The theorem is proved. □

1.1.2 Model without pension.

In this paragraph we consider wealth and habit dynamics (1.1) without pension and with asset allocation as a parameter $\theta = const$ with the client's objective function described as follows (see Section 1.1.1). Since we use the value function approach, we need to derive an HJB equation using a verification theorem (1.1.1) but, first, let us change variables. This will allow us to reduce the dimension of our problem by analogy with one introduced in the book [Rogers, 2013]

$$
\begin{aligned}
x_t \equiv \frac{w_t}{\bar{c}_t}, \quad q_t = \frac{c_t}{\bar{c}_t}, \\
V(t,w,\bar{c}) = V\left(t,\frac{w}{\bar{c}},1\right) \equiv \nu\left(t,\frac{w}{\bar{c}}\right) = \nu(t,x)
\end{aligned}
\tag{1.14}
$$

where $V(t,w,\bar{c})$ is the unknown value function defined by equation (1.2). Then the dynamics of the new scaled wealth x will be the following

$$
\mathrm{d}x_t = \mathrm{d}\left(\frac{w_t}{\bar{c}_t}\right) = rx_t\mathrm{d}t + \theta((\mu-r)x_t\mathrm{d}t + \sigma x_t\mathrm{d}W_t) - (\eta x_t + 1)q_t\mathrm{d}t + \eta x_t\mathrm{d}t. \tag{1.15}
$$

New objective function can be expressed by equation

$$
\nu(t,x) = \sup_{q_s} E\left[\int_t^T e^{-\rho s}\, {}_sp_\xi u(q_s)ds | x_t = x\right]. \tag{1.16}
$$

If formulas (1.14) and (1.15) will be applied to the differential of the stochastic process (1.5) we can get the following HJB equation

$$
\sup_q \left[\nu_t - (\rho+\lambda)\nu + \nu_x\{(\theta(\mu - r) + r + \eta)x - (\eta x + 1)q\} + \frac{1}{2}\nu_{xx}\theta^2\sigma^2x^2 + u(q)\right] = 0.
$$

Next, let us find the optimal value of consumption q

$$
\begin{aligned}
-(\eta x + 1)\nu_x + (q^*)^{-\gamma} = 0 \\
q^* = [(\eta x + 1)\nu_x]^{-\frac{1}{\gamma}}.
\end{aligned}
\tag{1.17}
$$

The final HJB equation using the new variables

$$
\nu_t - (\rho + \lambda)\nu + \alpha\nu_x + \beta\nu_{xx} + u(q^*) = 0. \tag{1.18}
$$

and the coefficients $\alpha = \alpha(x)$ and $\beta = \beta(x)$ in the formula have the following form

$\alpha = (\theta(\mu - r) + r + \eta)x - (\eta x + 1)q^*, \quad \beta = \frac{1}{2}\theta^2\sigma^2 x^2.$

Boundary conditions for this problem are taken to be following: At the boundary $x = 0$ assume that optimal consumption is $q_t^* = 0$ since there is not any other income. Then at the other boundary where $x = x_{\max}$ assume that the value function gradually approaches zero which implies that the value function derivative is zero, i.e. $\nu_x = 0$.

1.1.3 Model with pension.

In this paragraph we expand on the previous problem and now, assume that the client not only invests part of his wealth into a bank account and makes profits from stocks, but also has pension income.

Now, suppose that the wealth dynamics and objective function can be described by equations (1.1)- (1.2). The goal is to maximize the value function by controlling the consumption c_t such that the wealth w_t remains non-negative with fixed asset allocation. In previous Section 1.1.1 we formulated and proved a verification theorem for the following HJB equation

$$\sup_{c_s}\Big[\tilde{V}_t + \tilde{V}_w(\theta(\mu - r) + r)w_t + \tilde{V}_w(\pi - c_t) + \tilde{V}_{\bar{c}}\eta(c_t - \bar{c}_t) + \frac{1}{2}\tilde{V}_{ww}\theta^2\sigma^2 w_t^2 + e^{-\rho t}{}_t p_x u\left(\frac{c_t}{\bar{c}_t}\right)\Big] = 0. \tag{1.19}$$

where the consumption rate c_t is the only our control variable. Now, we introduce new notation for our convenience

$$\tilde{V}(t,w_t,\bar{c}_t) = e^{-\rho t}\,{}_tp_x V(t,w_t,\bar{c}_t). \tag{1.20}$$

By performing standard calculations we can obtain the HJB equation

$$\begin{aligned} V_t - (\rho + \lambda_{t+x})V + V_w((\theta(\mu - r) + r)w + \pi - c^*) + \\ V_{\bar{c}}\eta(c^* - \bar{c}) + \frac{1}{2}V_{ww}\theta^2\sigma^2 w^2 + u\left(\frac{c^*}{\bar{c}}\right) = 0. \end{aligned} \tag{1.21}$$

where the optimal consumption has the form

$$c^* = \bar{c}^{\frac{\gamma-1}{\gamma}}(V_w - V_{\bar{c}}\eta)^{-\frac{1}{\gamma}}. \tag{1.22}$$

Since we deal with a nonlinear second order PDE (1.21) we should set up boundary conditions. At the terminal time T the integral in the formula that represents the value function $V(t,w,\bar{c})$ is zero. At the boundary with zero wealth, $w = 0$, we impose the constraint $c^* < \pi$. Then we will get a simple first-order PDE

$$V_t - (\rho + \lambda_{t+x})V + V_w(\pi - c^*) + V_{\bar{c}}\eta(c^* - \bar{c}) + u\left(\frac{c^*}{\bar{c}}\right) = 0 \tag{1.23}$$

If we assume that the value function is asymptotically proportional to wealth

$$V(t,w,\bar{c}) \sim f(t,\bar{c})w^{1-\gamma} \quad \text{then} \quad V_w \sim f(t,\bar{c})w^{-\gamma} \tag{1.24}$$

where $f(t,\bar{c})$ is some arbitrary function of time and the EWA of consumption. When the wealth is big enough, i.e. $w \to \infty$ the derivative of the value function goes to zero asymptotically

$$V_w = 0. \tag{1.25}$$

This means that changes in wealth w are not that important and do not affect the utility function as much as in the case when $w \to 0$.

The detailed discussion of the approximation scheme and discretization error is provided in the next two paragraphs (see 1.2 and 1.2.3).

1.2 Numerical Scheme

1.2.1 Finite Difference Scheme for the scaled problem

In this paragraph we explain what kind of numerical method we use and derive corresponding formulas. In order to solve PDE we will choose the implicit upwind method [Strikwerda, 2004]. The idea of this method is to use a forward difference for the time derivative and forward or backward diffference for other variables depending on the sign of the coefficient in front of every first order derivative. For this scheme we will implement a generalized upwind method where we add and subtruct the absolute value of the coefficient that changes sign. Let us set up the grid as follows $\nu(t,x) = \nu(t_n,x_j)$ and indices change $j = 1 \ldots M$, $n = 1 \ldots N$. We introduce a new time variable as follow: $t_n = T - n\Delta t$ where T is the terminal time. The approximation scheme for the equation will be the following

$$\frac{\nu_j^{n+1}-\nu_j^n}{\Delta t}+(\rho+\lambda_{t_{n+1}+age})\nu_j^{n+1}+\frac{\alpha_j^n+|\alpha_j^n|}{2}\frac{\nu_j^{n+1}-\nu_{j-1}^{n+1}}{\Delta x}+$$
$$\frac{\alpha_j^n-|\alpha_j^n|}{2}\frac{\nu_{j+1}^{n+1}-\nu_j^{n+1}}{\Delta x}+\beta_j^n\frac{\nu_{j+1}^{n+1}-2\nu_j^{n+1}+\nu_{j-1}^{n+1}}{\Delta x^2}-u\left(q_j^{*n}\right)=0 \quad (1.26)$$

where

$$\alpha_j^n=-\{(\theta_j^{*n}(\mu-r)+r+\eta)x_j-(1+\eta x_j)q_j^{*n}\}$$
$$\beta_j^n=-\frac{1}{2}\left(\theta_j^{*n}\sigma x_j\right)^2$$

where the optimal consumption q_j^{*n} and optimal asset allocation θ_j^{*n} can be computed as follows

$$q_j^{*n}=\left[(\eta x_j+1)\frac{\nu_j^n-\nu_{j-1}^n}{\Delta x}\right]^{-\frac{1}{\gamma}},\quad \theta_j^{*n}=-\frac{\kappa}{\sigma}\frac{\frac{\nu_j^n-\nu_{j-1}^n}{\Delta x}}{\frac{\nu_{j+1}^n-2\nu_j^n+\nu_{j-1}^n}{\Delta x^2}x},\quad \kappa=\frac{\mu-r}{\sigma}. \quad (1.27)$$

The rest of the parameters are constants and we will take the following values for them $\rho=0.02$, $\eta=[10^{-2}\ 10^{-1}\ 1]$, $\sigma=0.16$, $\mu=0.08$, $r=0.02$ and $\gamma=3$.

- Boundary condition at $x=x_1=0$.

  We can simplify equation (1.26) by plugging $x_1=0$, so $\alpha_j^n=q_1^{*n}=0$ and $\beta_1^n=0$ hence the equation will take the form

$$\frac{\nu_1^{n+1}-\nu_1^n}{\Delta t}+(\rho+\lambda_{t_{n+1}+age})\nu_1^{n+1}-u\left(q_1^{*n}\right)=0 \quad (1.28)$$

- Boundary condition at $x=x_{\max}=x_M$.

$$\nu_M^{n+1} = 0.$$

1.2.2 Finite Difference Scheme for the original problem

In this part we explain what kind of numerical method we use and derive the corresponding formulas. In order to solve PDEs we will choose the implicit upwind method. The idea of this method is to use a forward difference for the time derivative and forward or backward diffference for other variables depending on the sign of the coefficient in front of every first order derivative. For this scheme we will implement a generalized upwind method where we add and subtruct the absolute value of the coefficient that changes sign. Let us set up the grid as follows $V(t,w,\bar{c}) = V(t_n,w_j,\bar{c}_k)$ and indices change $j = 1 \dots M$, $n = 1 \dots N$ and $k = 1 \dots K$. We introduce a new time variable as follow: $t_n = T - n\Delta t$ where T is the terminal time.

The approximation scheme for the equation (1.21) will be the following

$$\begin{aligned}
&\frac{V_{j,k}^{n+1} - V_{j,k}^{n}}{\Delta t} + (\rho + \lambda_{t_{n+1}+x})V_{j,k}^{n+1} + \\
&\frac{\alpha_{j,k}^{n} + |\alpha_{j,k}^{n}|}{2}\frac{V_{j,k}^{n+1} - V_{j-1,k}^{n+1}}{\Delta w} + \frac{\alpha_{j,k}^{n} - |\alpha_{j,k}^{n}|}{2}\frac{V_{j+1,k}^{n+1} - V_{j,k}^{n+1}}{\Delta w} + \\
&\frac{\beta_{j,k}^{n} + |\beta_{j,k}^{n}|}{2}\frac{V_{j,k}^{n} - V_{j,k-1}^{n}}{\Delta \bar{c}} + \frac{\beta_{j,k}^{n} - |\beta_{j,k}^{n}|}{2}\frac{V_{j,k+1}^{n} - V_{j,k}^{n}}{\Delta \bar{c}} - \\
&\frac{1}{2}\theta^2\sigma^2 w_j^2 \frac{V_{j+1,k}^{n+1} - 2V_{j,k}^{n+1} + V_{j-1,k}^{n+1}}{\Delta w^2} - u\left(\frac{\zeta_{j,k}^{n}}{\bar{c}_k}\right) = 0
\end{aligned} \tag{1.29}$$

where

$$\alpha_{j,k}^n = -\{(\theta(\mu - r) + r)w_j + \pi - \zeta_{j,k}^n\}$$
$$\beta_{j,k}^n = -\eta(c^* - \bar{c}_k) = -\eta(\zeta_{j,k}^n - \bar{c}_k)$$

$$\zeta_{j,k}^n = c^* = \bar{c}_k^{\frac{\gamma-1}{\gamma}} \left(\frac{V_{j,k}^n - V_{j-1,k}^n}{\Delta w} - \eta \frac{V_{j,k+1}^n - V_{j,k}^n}{\Delta \bar{c}} \right)^{-\frac{1}{\gamma}}. \tag{1.30}$$

The rest of the parameters are constants and we will take the following values for them $\pi = 1$, $\rho = 0.02$, $\eta = [10^{-2}\ 10^{-1}\ 10^{0}]$, $\theta = [0.2\ 0.6\ 0.9]$, $\sigma = [0.16\ 0.50\ 0.75]$, $\mu = 0.08$, $r = 0.02$ and $\gamma = 3$. For every case study and test we specify which values we used.

- Boundary condition at $w = w_1 = 0$.

  The equation (1.23) can be approximated as follows

$$\frac{V_{1,k}^{n+1} - V_{1,k}^n}{\Delta t} + (\rho + \lambda_{t_{n+1}+x})V_{1,k}^{n+1} + \alpha_{1,k}^n \frac{V_{2,k}^{n+1} - V_{1,k}^{n+1}}{\Delta w} +$$
$$\frac{\beta_{1,k}^n + |\beta_{1,k}^n|}{2} \frac{V_{1,k}^n - V_{1,k-1}^n}{\Delta \bar{c}} + \frac{\beta_{1,k}^n - |\beta_{1,k}^n|}{2} \frac{V_{1,k+1}^n - V_{1,k}^n}{\Delta \bar{c}} - u\left(\frac{c^*}{\bar{c}_k}\right) = 0$$

  where $\alpha_{j,k}^n = -\pi + \zeta_{j,k}^n$ and $\beta_{1,k}^n = -\eta(\zeta_{1,k}^n - \bar{c}_k)$, $\zeta_{1,k}^n = \min(\pi, \bar{c}_k)$.

- Boundary condition at $w = w_{\max} = w_M$.

$$\frac{V_{M,k}^{n+1} - V_{M-1,k}^{n+1}}{\Delta w} = 0.$$

1.2.3 Error analysis

In this paragraph we are going to provide some intuition about the numerical error, the so-called discretization error. Since, in order to solve our optimization problem, we use the approximation scheme which is described in detail in Chapter 1.2, it is reasonable to check how big the errrors are and how changes in parameters can affect on them. So, we calculate an L_2- norm (see Tables 1.1 and 1.2). In our case, we have three variables and, as a consequence, the disretization error can be estimated by the following inequality:

$$e(\Delta t,\Delta w, \Delta\bar{c}) \leq c_1(\Delta t)^{p_1} + c_2(\Delta w)^{p_2} + c_3(\Delta\bar{c})^{p_3} \tag{1.31}$$

where Δt, Δw, $\Delta\bar{c}$ are step sizes over variables time t, wealth w and habit $\bar{c}$ respectively, c_1, c_2 and c_3 are finite constants. The time step can be calculated as follows

$$\Delta t = \frac{b-a}{N-1} \tag{1.32}$$

where time belongs to the interval $t \in [a \dots b]$ and N is the number of nodes. The other steps Δw (number of nodes M) and $\Delta\bar{c}$ (number of nodes K) can be computed analogously. Parameters p_1, p_2 and p_3 represent order of accuracy (OA). In the general case OA quantifies the rate of convergence of a numerical approximation of a differential equation to the exact solution. In our case we will compare two numerical solutions, with single and doubled step size. It can be said that numerical scheme is accurate of order (p_1,p_2,p_3), which means that a scheme is accurate of order p_1 in time, order p_2 in wealth and order p_3 in habit (see [Strikwerda, 2004]). Let us estimate the error ratio (ER), that is simply ratio of two consequtive norms, for our scheme. First, in the formula (1.32) we omit 1 in denominator because for N big enough it's not important. Then we choose parameters $p_1 = p_2 = p_3 = p = 1$ since we use corresponding approximations for a numerical scheme. Another simplification that we can accept is the following, we omit one term from the inequality (1.31) if we choose a fine enough grid, for example over the time t, namely

$$\Delta t \to 0 \quad \text{then} \quad c_1 \Delta t \ll c_2 \Delta w + c_3 \Delta \bar{c} \tag{1.33}$$

Then, if we assume the step, for example over the habit, is propotional to the wealth step $\Delta \bar{c} = h\Delta w$ where h is a constant then, by perfoming simple calculations we can derive the ER of our scheme. Create system of 2 equations and divide one by another, this operation will allow to get rid of the constant which will be the combination of another constants, namely $c_2' + hc_3'$ and the following equation for ER can be obtained:

$$\frac{e(M)}{e(2M)} = \left(\frac{\Delta w_K}{\Delta w_{2K}}\right)^p = \left(\frac{2M}{M}\right)^p = \left(\frac{4M}{2M}\right)^p = \ldots = 2^p. \tag{1.34}$$

Since we do not have the exact solution for this problem we calculated the L_2 matrix norm increasing the number of nodes in a particular direction. The error ratio (ER) can be computed based on the following formulas:

$$\begin{gathered} L_2(n) = norm(c^*(t_0{,}2^{5+n}{,}k) - c^*(t_0{,}2^{6+n}{,}k)), \quad n = 1{,}2{,}3{,}4 \\ \text{ER} = \frac{L_2(n)}{L_2(n+1)} \end{gathered} \tag{1.35}$$

where t_0 is a fixed time moment and k is the number of nodes over EWA of consumption. In other words we fix number of nodes over one variable, for example $\bar{c}$, and increase the number of nodes over another one. In our case it is wealth w, so fix the number of nodes over the wealth for which we got the best results and change the number of nodes over the habit $\bar{c}$. Results

Table 1.1: L_2 norm.

Error analysis for fixed # of nodes over $\bar{c}$.						
# of nodes	$\eta = 0.01$		$\eta = 0.1$		$\eta = 1.0$	
	L_2	ER	L_2	ER	L_2	ER
64/128	1.5	-	3.04	-	$1.3 \cdot 10^{1}$	-
128/256	$8.1 \cdot 10^{-1}$	1.84	1.63	1.86	7.6	1.76
256 /512	$4.3 \cdot 10^{-1}$	1.89	$8.58 \cdot 10^{-1}$	1.91	4.2	1.82
512/1024	$2.2 \cdot 10^{-1}$	1.93	$4.43 \cdot 10^{-1}$	1.94	2.2	1.86

are provided for three parameters $\eta = 10^{-2}$, 10^{-1} and 10^{0}.

1. Results for the first case are summarized in the Table (1.1)). The number of nodes over the habit $\bar{c}$ was chosen to be $K = 80$. Every table shows results for all three values of the smoothing factor η, L_2 represents values of L_2-norm, ER is the error ratio which was computed using formula (1.35). Numbers, for example 256/512, represent the number of nodes over the wealth w for norm calculation. As follows from the explanation $2^p = 2$ and since OA for our scheme equals 1 then we should expect that the ER approaches 2. As we can see from the tables values ER for L_2 norm vary from 1.76 for $\eta = 1$ (see Table (1.1), column 7) up to 1.94 for $\eta = 10^{-1}$ (see Table (1.1), column 5). As we can see from the tables, the results show stability and the value of ER approaches 2.0. In order to achieve better results we need to do the same procedure in the other direction, over the habit $\bar{c}$ keeping the fine grid over time.

2. So, we will fix the number of nodes over the wealth for which results were the best, i.e. $M = 1024$ and increase the number of nodes in another direction. Results are summarized in Table 1.2 for three pa-

Table 1.2: L_2 norm.

Error analysis for fixed # of nodes over wealth w.						
# of nodes	$\eta = 0.01$		$\eta = 0.1$		$\eta = 1.0$	
	L_2	ER	L_2	ER	L_2	ER
64/128	1.58	-	5.02	-	-	-
128/256	$8.4 \cdot 10^{-1}$	1.88	2.56	1.96	4.29	-
256 /512	$4.4 \cdot 10^{-1}$	1.92	1.29	1.98	2.21	1.95

rameters $\eta = 10^{-2}$, 10^{-1} and 10^{0}. As in the previous case, we calculate an L_2-norm (see Table 1.2). As we can see from the table, the value of ER for this case is very similar to the previous one for parameter $\eta = 0.01$ despite the fact that in this case we chose a different time grid, more sparse then for other η values, namely for $\eta = 0.01$ the number of time nodes was approximately $1 \cdot 10^3$, for other η values $\sim 4 \cdot 10^4$. As the number of nodes goes up the time that is needed to do computations increases dramatically therefore it becomes more time consuming to perform calculations for very fine grids. For the last case when $\eta = 1.0$ some parameters were changed because of stability issues. So, the number of time nodes were increased $8 \cdot 10^4$, the number of wealth nodes was decreased to $M = 256$ but the final value of the error is good enough.

Overall, based on the ER results for all three values of the smoothing factor η we can see that the error gradually approaches to 2^1 as expected and we can conclude that our numerical scheme is stable.

1.3 Numerical Implementation

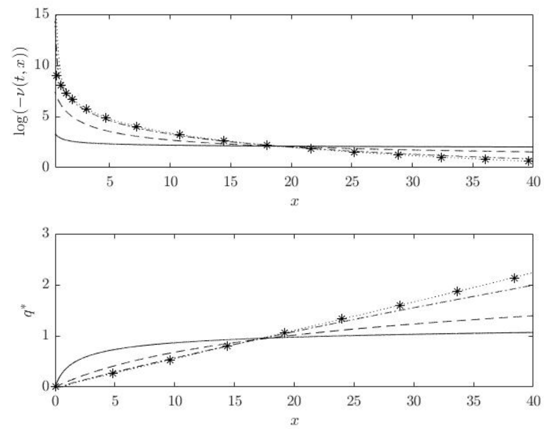
1.3.1 HFM w\o pension.

We start our discussion by presenting results where the retiree does not have any pension income. Since the problem does not have an analytical solution we have to solve it numerically. For that we choose an implicit upwind method described in detail in Section 1.2. Below we provide some numerical results for two cases. In the first picture (see Figure (1.1a)) we see the numerical results for fixed asset allocation $\theta = 0.6$, whereas in the second picture (see Figure 1.1b) we solved the optimization problem (OP) where asset allocation is a control variable θ_t (this is close to what is done in [Rogers, 2013]). In order to solve this problem we also set some parameters, namely: risk-free rate $r = 0.02$, drift $\mu = 0.08$, risk aversion parameter $\gamma = 3$ and volatility $\sigma = 0.16$. Figure (1.1) shows the solution with these parameters, at age 65.

In Figure (1.1a) the upper picture shows the relationship between the logarithm of the negative value function ($\log(-\nu(t,x))$) and the logarithm of the ratio of wealth to habit ($\log(w/\bar{c})$) for different values of the smoothing factor $\eta = 0, 0.01, 0.1, 1.0$. The solid line represents $\eta = 1.0$, the dashed line - $\eta = 0.1$ and the dash-dot line represents $\eta = 0.01$. The last case corresponds to the Merton problem $\eta = 0$ (dotted line). The star line represents another numerical solution for the later case with the assumption that value function has the form $\nu(t,x) = h(t)u(x)$ where the function $h(t)$ satisfies the following

ODE

$$h' + (-\rho + (1-\gamma)(r + \theta(\mu - r) - \frac{\gamma}{2}\theta^2\sigma^2))h + \gamma h^{\frac{\gamma-1}{\gamma}} = 0. \tag{1.36}$$



(a) $\theta = 0.6$ is fixed.

(b) θ is a control variable.

Figure 1.1: Solution for OP w\o pension for asset allocation θ in 1.1a as a fixed parameter $\theta = 0.6$ and as a control variable in 1.1b where the solid line represents $\eta = 1.0$, the dashed line - $\eta = 0.1$, the dash-dot line - $\eta = 0.01$ and the dotted line - $\eta = 0$ and $*-$ line represents an alternative numerical solution for the last case at the present time.

Then the solution will have the following form and also can be solved numerically

$$f_1 = -\rho + (1-\gamma)\left(\theta(\mu - r) + r - \frac{\gamma}{2}(\theta\sigma)^2\right), \quad f_2 = e^{\frac{x-m+T}{b}},$$

$$h(t) = \left(e^{\frac{1}{\gamma}\left(f_1(T-t) - f_2 + e^{(x-m+t)/b}\right)}\left(1 + \int_t^T e^{-\frac{1}{\gamma}\left(f_1(T-t') - f_2 + e^{(x-m+t')/b}\right)} \mathrm{d}t'\right)\right)^{\gamma}. \tag{1.37}$$

If we look at the upper pictures (see Figure 1.1a or 1.1b) they show that in the beginning the value function decreases dramatically and then gradually slows down and settles into asymptotic behaviour. This reflects the intuition about what the value function is. When wealth is small then its changes have a great impact on the value function. As soon as changes in wealth become significantly smaller than an absolute value of the wealth the value function slows down. The bottom picture in Figure (1.1a) represents the relationship between the optimal ratios of consumption-habit ($c/\bar{c}$) and wealth-habit ($w/\bar{c}$) for the same four values of the parameter η. As we can see in the case with $\eta = 1.0$ (solid line), the possibility for consumption c to exceed habit exists only in case of very big wealth, for other values the rule is following: the less responsive the consumption is to changes in habit, the higher the optimal consumption will be.

Figure 1.1b represents two graphs where θ_t is a control variable. All calculations can be done analogously with the only difference being that now, we need find an optimal value for asset allocation as well. It can be computed as follows

$$\theta^* = -\frac{\kappa}{\sigma}\frac{\nu'}{\nu'' x}, \qquad \kappa = \frac{\mu - r}{\sigma}. \tag{1.38}$$

We can also see the analogous behaviour for both functions, the value function (upper picture) and the optimal consumption (lower picture). Results also were obtained for the same values of the smoothing factor as before. Alternative solution for the case $\eta = 0$ also can be computed by analogy with previous case. Similar results for the smoothing parameter $\eta = 1.0$ were obtained in the book [Rogers, 2013].

1.3.2 HFM with pension.

Value function Here we provide some numerical results for the different parameters obtained by using MATLAB software. Parameters of the model are the following: $\gamma = 3$, $\sigma = 0.16$, $\theta = 0.6$, $\mu = 0.08$, $\rho = r = 0.02$.

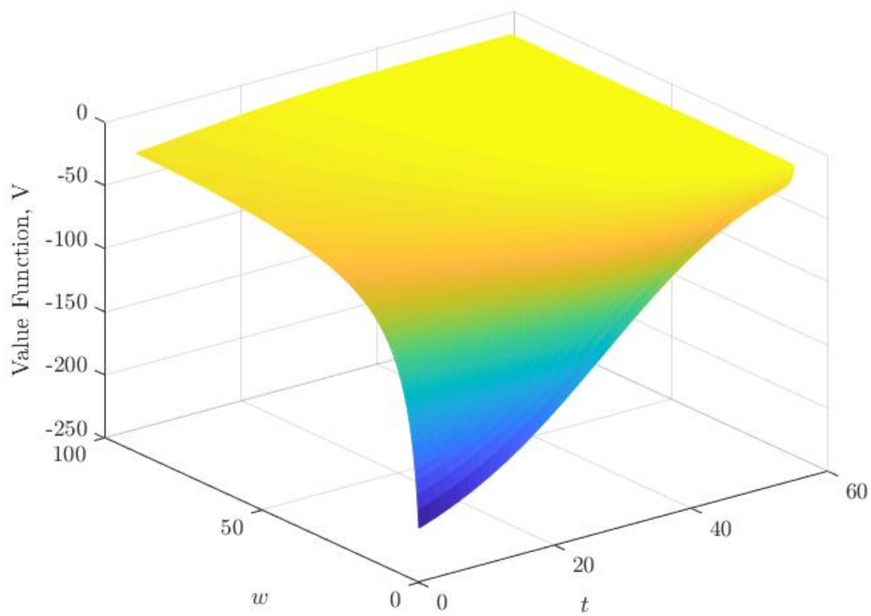


Figure 1.2: Value function V for parameter $\eta = 10^{-1}$, EWA of consumption $\bar{c} \approx 9$.

Let us choose pension $\pi = 1$ and remaining variables adopt proportionately relative to the selected pension. In the Figures (1.2- 1.3) is represented the value function $V(t,w,\bar{c})$ for two different axes, first one (1.2) shows the value function $V(t,w,\bar{c})$ vs. wealth w and time t with fixed value of habit $\bar{c} \approx 9$. The second one (1.3) shows the value function vs. wealth and EWA of consumption at the present time t. Because the value function shows how important changes in wealth w are, it was was expected to have this function concave in second argument, wealth w, and this is exactly what we can see from the picture (1.2). The intuition is the same as in the previous case, client who experience, for example \$100 constant change in wealth with \$100 as an initial wealth as well, as his wealth grows, will realize that when certain level achieved, for example \$10,000, the changes in wealth are not that important as it was in the begining therefore the value function slows down as wealth grows.

For the next picture (1.3) we can provide the following interpretation. In case if client has habit to consume a lot ($\bar{c}$) but does not have enough wealth w then the value function behaviour also should show very fast growth, in other words if client does not have money but usually spend a lot he should change his habit and it should be decreased.

Optimal consumption The next two pictures represent relationship between optimal consumption c^*, wealth w and habit $\bar{c}$. In the Figure (1.4a) We can see the optimal consumption c^* vs. wealth w for the different EWA of consumption $\bar{c}$ values ($\bar{c}$ values increase in ascend direction) whereas on the Figure (1.4b) shown relation between c^* and $\bar{c}$ for the different wealth w levels (w values increase in ascend direction). As we can see from the graph (1.4a), consumption shows steady growth, a little faster for small values of wealth then, for large wealth, we can see almost linear curve.

At the same time, for the small values of habit $\bar{c}$, optimal consumption remains low regardless on wealth w. It happens because if agent does not have habit to spend a lot it is not important how big his wealth is. For example, living on pension income, if somebody is satisfied with this level of wellbeing he may not use the possibility to spend more even despite that fact that he can afford more. On the second picture (1.4b) we can see almost linear increase of optimal consumption for the small values of EWA of consumption $\bar{c}$ and it significantly slows down for the large values of consumption's habit $\bar{c}$. In other words, it means that does not matter what your habit $\bar{c}$ is if your wealth w is small enough your optimal consumption will be also low. It explains why curve (Figure (1.4b)) shows very slow growth when habit $\bar{c}$ increases. This numerical solution was obtained for the similar intervals for both, the wealth w and EWA of consumption $\bar{c}$. For visual representation the value of smoothing parameter was chosen $\eta = 0.1$.

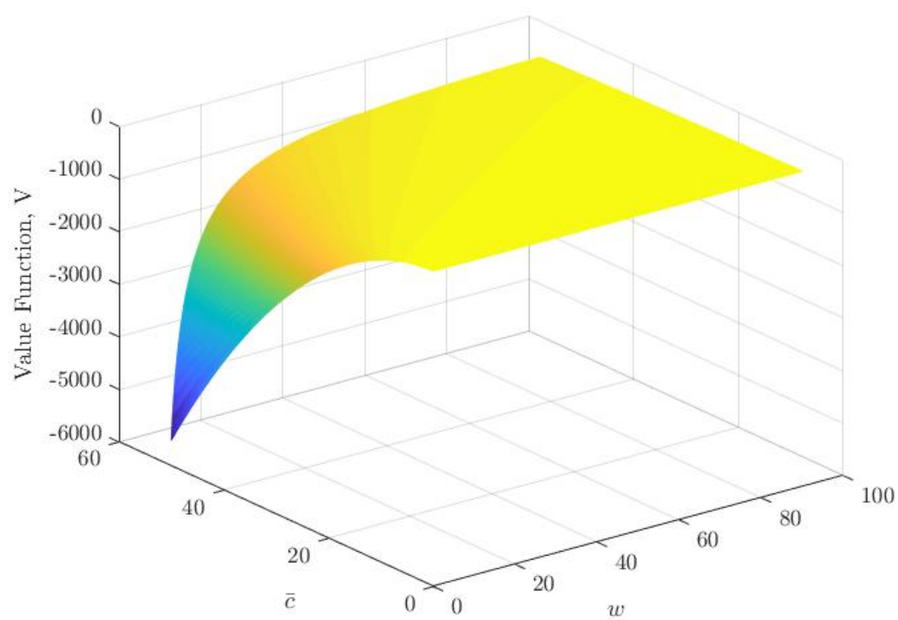
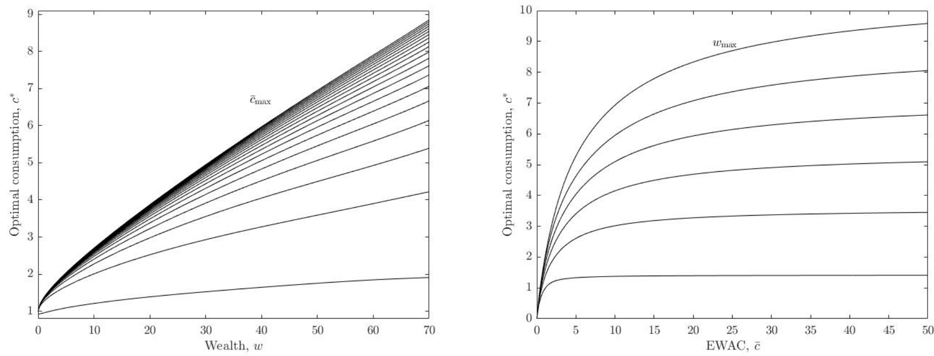


Figure 1.3: Value function V for parameter $\eta = 10^{-1}$ at the present time $t = 0$.



(a) c^* vs. w, habit interval is $\bar{c} \in [0 \ldots \bar{c}_{\max}]$.

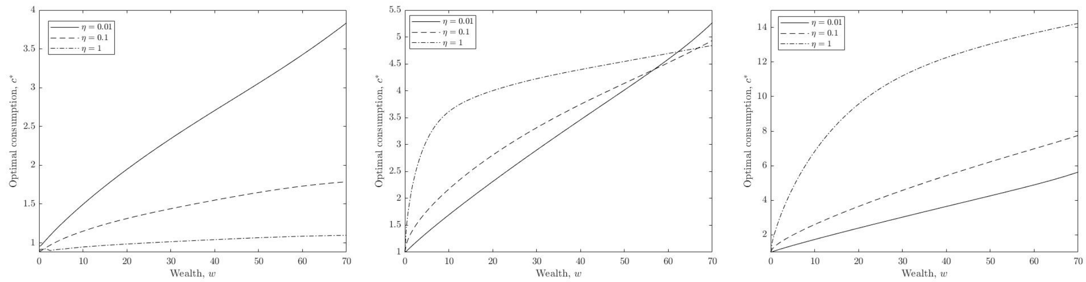
(b) c^* vs. $\bar{c}$, the wealth interval $w \in [0 \ldots w_{\max}]$.

Figure 1.4: There are two sets of graphs: optimal consumption c^* vs. wealth w (1.4a) and wealth w vs. habit $\bar{c}$ (1.4b). Smoothing factor is $\eta = 10^{-1}$.

In this part we are going to discuss numerical results obtained by solving PDE (1.21). The parameter set is taken to be the following. Based on papers where authors discuss what is the reasonable value of the risk aversion parameter (e.g. see [Blake etc., 2001]) $\gamma = 3$, risk-free rate $r = 0.02$, volatility $\sigma = 0.16$, and drift $\mu = 0.08$. Moreover, in this paragraph asset allocation is taken to be fixed, $\theta = 0.6$. Then choose pension $\pi = 1$. In the next few paragraphs we will discuss different variations of the solution of the existing problem. The goal is to see how optimal consumption changes for the different values of the smoothing parameter η.

Optimal consumption c^* vs. wealth w. First, we choose three different values of the smoothing factor η that characterizes how fast the habit changes, for example $\eta = 10^{-2}, 10^{-1}$ and 1.

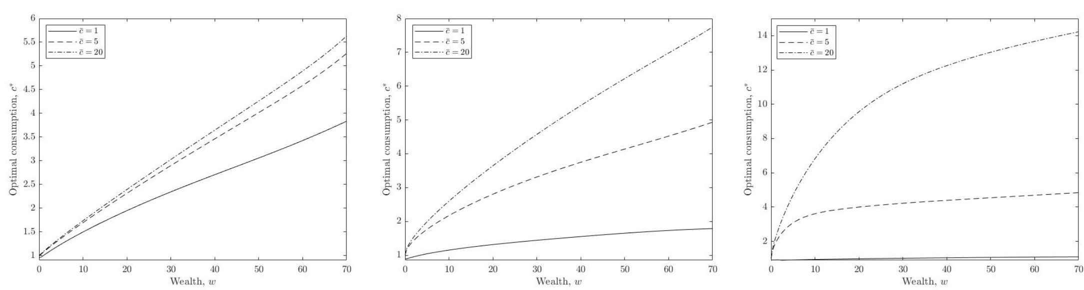
Figure 1.5 plots the relationship between optimal consumption c^* and wealth w for different values of habit $\bar{c}$, namely $\bar{c} = 1, 5, 20$.



(a) $\bar{c} \approx$ 1.0. (b) $\bar{c} \approx$ 5. (c) $\bar{c} \approx$ 20.

Figure 1.5: Optimal consumption c^* vs. wealth w for the fixed value of habit $\bar{c}$ at age 65.

When $\bar{c} = 1$, consumption stays modest with $\eta = 1$, and with $\eta = 0.1$ or 0.01, consumption is even smaller. With a higher habit ($\bar{c} = 5$), the $\eta = 1$ curve stays similar, but the $\eta = 0.1$ and $\eta = 0.01$ curves cross over, to display higher consumption. With the highest habit ($\bar{c} = 20$), the $\eta = 0.1$ and $\eta = 0.01$ consumption rises even faster with w.

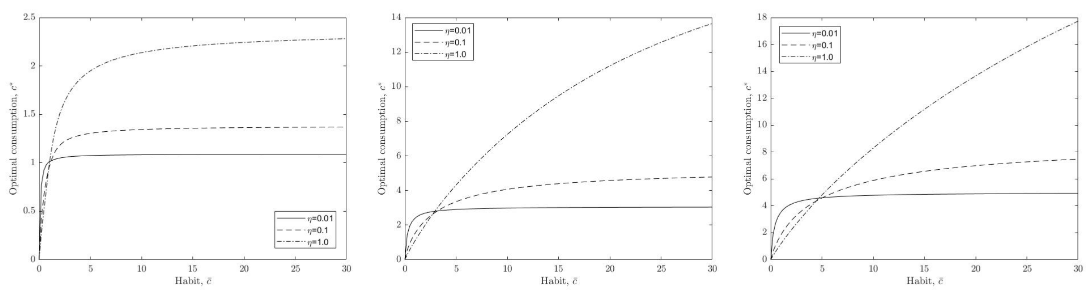


(a) $\eta = 10^{-2}$ (b) $\eta = 10^{-1}$ (c) $\eta = 1$

Figure 1.6: Optimal consumption c^* vs. wealth w for given $\bar{c}$. Every picture represents a different value of η.

Figure 1.6 shows how optimal consumption c^* changes with the wealth w for particular values of the parameter η. The variation in consumption with respect to $\bar{c}$ is the smallest when η is small, $\eta = 0.01$. Increasing η to 0.1 widens the spread, with lower consumption with $\bar{c} = 1$ and higher consumption with $\bar{c} = 20$. Increasing η to 1.0 widens the spread even further. Now, let us show our numerical results from different perspectives.

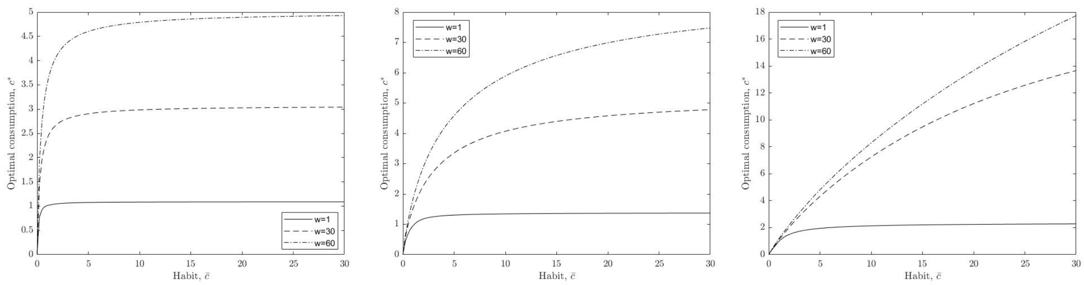
Optimal consumption c^* vs. habit $\bar{c}$. Figure 1.7 shows the relationship between the optimal consumption c^* and habit $\bar{c} \in [0 \ldots 30]$ for three different values of the smoothing factor $\eta = 0.01, 0.1, 1$. Every picture corresponds to a fixed value of wealth $w = 1, 30, 60$. For low wealth ($w = 1$), consumption levels off as $\bar{c}$ rises, although at different levels. For higher wealth ($w = 30$), consumption still levels off quickly when $\eta = 1$, but it keeps rising for longer when $\eta = 0.1$ or 0.01. With even higher wealth, this happens even more so.



(a) $w \approx 1$. (b) $w \approx 30$. (c) $w \approx 60$.

Figure 1.7: Optimal consumption c^* vs. habit $\bar{c}$ where every picture represents a fixed value of wealth.

The last set of pictures (see Figures (1.8a)-(1.8c) shows numerical results similar to the previous set. Each picture corresponds to a fixed value of the smoothing factor η. When $\eta = 0.01$, the three curves again level off at modest values of $\bar{c}$. For $\eta = 0.1$ the $w = 1$ curve levels off, while the others continue to rise for longer. And when $\eta = 1$, only the low wealth curve levels off, at least for moderate values of $\bar{c}$.

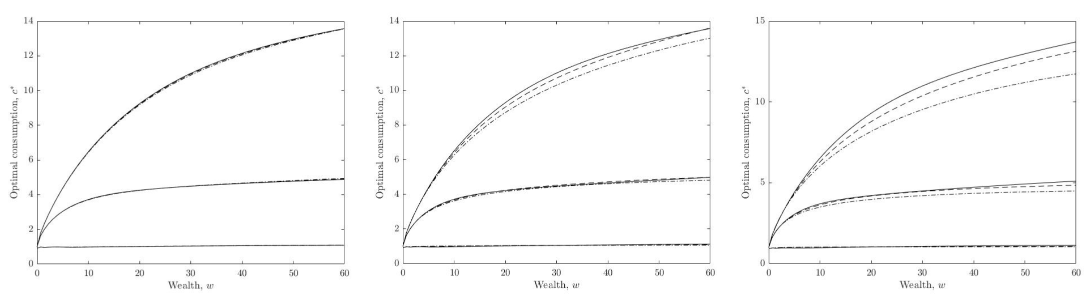


(a) $\eta = 10^{-2}$ (b) $\eta = 10^{-1}$ (c) $\eta = 1$

Figure 1.8: Optimal consumption c^* vs. habit $\bar{c}$ where every picture represents a fixed value of the smoothing factor η.

1.3.3 Numerical results for different asset allocation values and volatility.

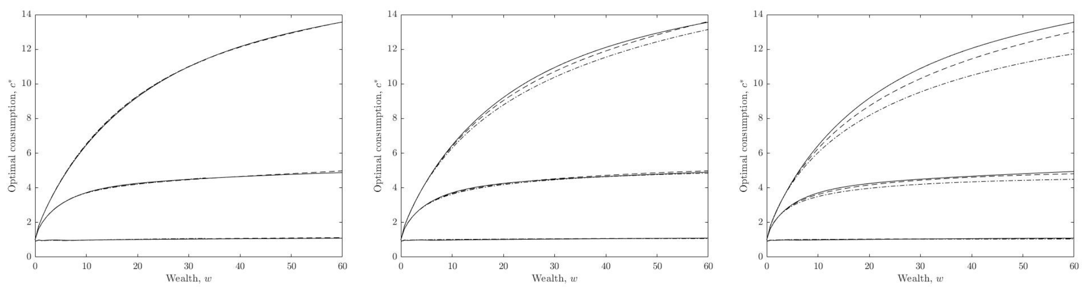
In this paragraph we explore the impact of varying asset allocation θ or volatility σ.



(a) $\theta = 0.2$ (b) $\theta = 0.6$ (c) $\theta = 0.9$

Figure 1.9: Numerical results for OP with pension, for $\eta = 1$, , $\bar{c} = 1$, 5 and 20 for fixed value of $\theta = 0.9$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line).

In Figure 1.9, each graph has a fixed value of θ and shows nine curves, corresponding to three choices of habit $\bar{c}$ and three choices of volatility σ. In Figure 1.10 the roles of θ and σ are reversed: each graph has a fixed σ, but three choices of $\bar{c}$ and three choices of θ. With low θ (Figure 1.9a), varying σ has little impact on consumption. But for higher θ (1.9b and 1.9c), the sensitivity to σ rises. Though it is still quite small in the case $\bar{c} = 1$.

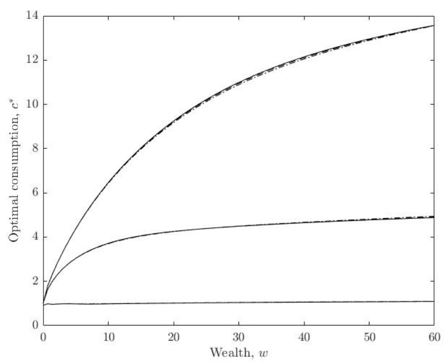


(a) $\sigma = 0.16$ (b) $\sigma = 0.5$ (c) $\sigma = 0.75$

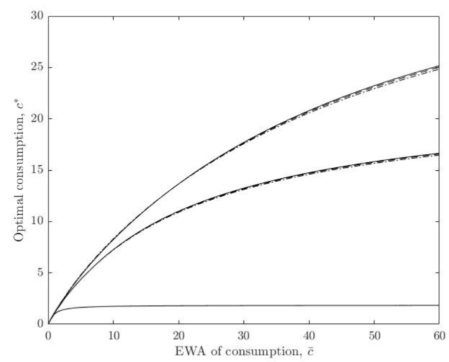
Figure 1.10: Numerical results for optimization problem with pension, for $\eta = 1$, $\bar{c} = 1,\ 5,\ 20$ and for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .

In the same way, with low σ (Figure 1.10a) there is little sensitivity to θ, but that sensitivity increases when σ is large Figures 1.10b and 1.10c .

For the next set of graphs we consider parameter $\eta = 1.0$. As we can see from the pictures (see Figures (1.11)-(1.16)), for all values of volatility σ and the asset allocation θ optimal consumption c^* shows stable growth, almost for all values of wealth w or habit $\bar{c}$. For some cases, as we discussed in previous paragraphs, for example for small values of wealth w, optimal consumption c^* is almost constant, regardless on habit $\bar{c}$ and opposite, it is almost constant for small values of habit $\bar{c}$ regardless on wealth w. At the same time, if we have small θ value, for example $\theta = 0.2$ (Figure (1.11)), we can see that there is no big difference between numerical solutions for different volatility σ. It happens because θ represents part of wealth invested into risky assets and if this part is small the volatility changes can not affect solution a lot.

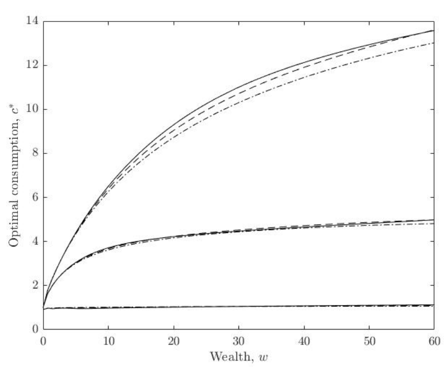


(a) $\bar{c} = 1$, 5 and 20.

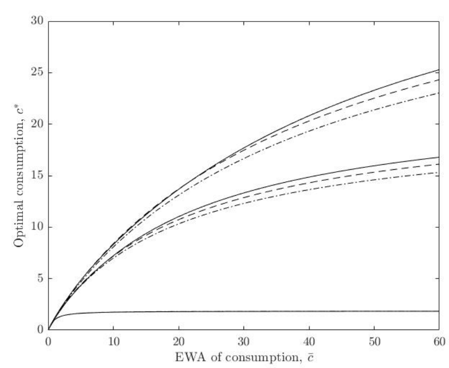


(b) $w = 1$, 30 and 60.

Figure 1.11: Numerical results for OP with pension, for $\eta = 1$, for fixed value of $\theta = 0.2$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line).

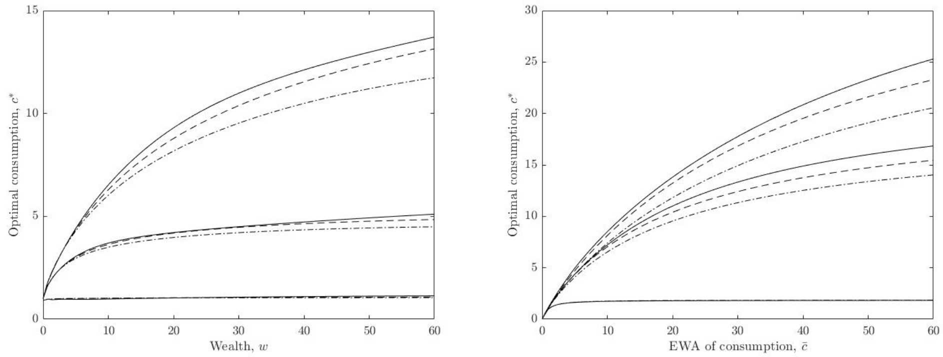


(a) $\bar{c} = 1$, 5 and 20.



(b) $w = 1$, 30 and 60.

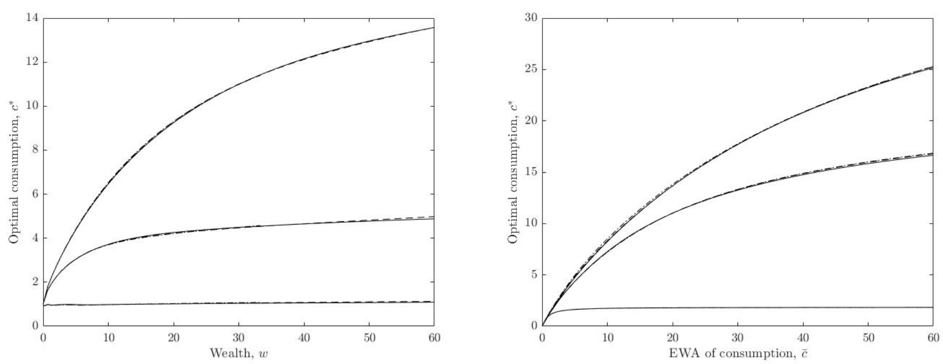
Figure 1.12: Numerical results for OP with pension, for $\eta = 1$, for fixed value of $\theta = 0.6$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line).



(a) $\bar{c} = 1, 5$ and 20. (b) $w = 1, 30$ and 60.

Figure 1.13: Numerical results for OP with pension, for $\eta = 1$, for fixed value of $\theta = 0.9$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line) .

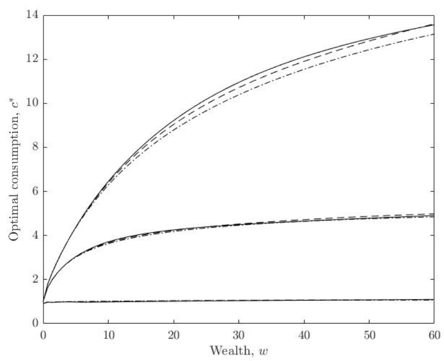
When part of wealth which is invested into risky assets increases, for example up to $\theta = 0.9$ (Figure (1.13)), we can see, higher volatility σ value is, smaller the optimal consumption c^* will be.



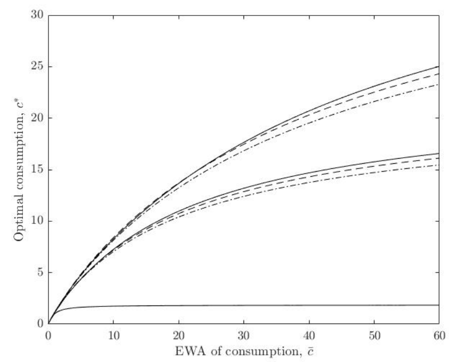
(a) $\bar{c} = 1, 5$ and 20. (b) $w = 1, 30$ and 60.

Figure 1.14: Numerical results for OP with pension, for $\eta = 1$, for fixed value of $\sigma = 0.16$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .

The next three graphs (see Figures (1.14)-(1.16)) show the behaviour of the optimal consumption c^* for different values of θ for fixed values of volatility parameter σ.

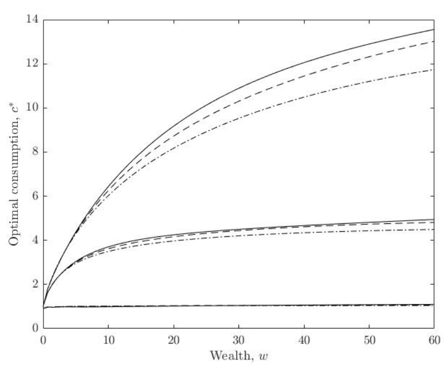


(a) $\bar{c} = 1$, 5 and 20.

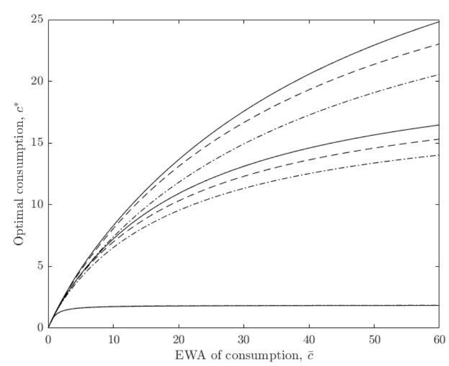


(b) $w = 1$, 30 and 60.

Figure 1.15: Numerical results for OP with pension, for $\eta = 1$, for fixed value of $\sigma = 0.50$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .



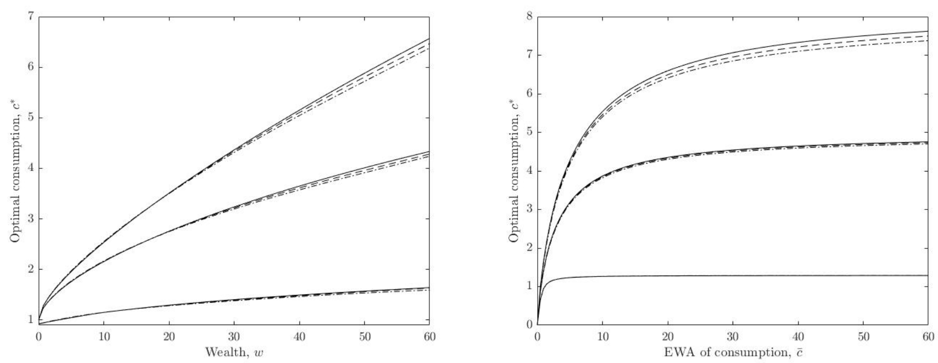
(a) $\bar{c} = 1$, 5 and 20.



(b) $w = 1$, 30 and 60.

Figure 1.16: Numerical results for OP with pension, for $\eta = 1$, for fixed value of $\sigma = 0.75$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .

Here we can see that higher σ values are, smaller optimal consumption c^* will be, e.g. for $\sigma = 0.16$ there is no big difference between numerical solutoins for all three parameters θ (Figure (1.14)) but for $\sigma = 0.75$ these changes can be seen clearly (Figure (1.16)). For better illustrarion, on every picture we chose three values of variables, for relationship $c^* - w$ values of habit are $\bar{c} = 1,\ 5,\ 20$ and for relationship $c^* - \bar{c}$, values of wealth $w = 1, 30, 60$.

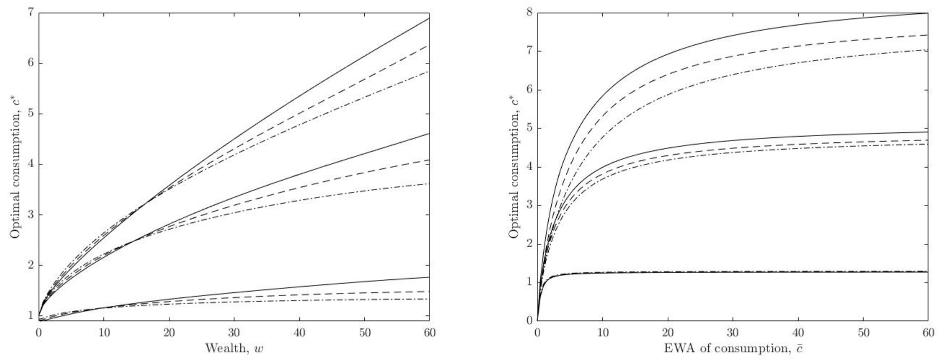


(a) $\bar{c} = 1, 5$ and 20.

(b) $w = 1, 30$ and 60.

Figure 1.17: Numerical results for OP with pension, for $\eta = 10^{-1}$, for fixed value of $\theta = 0.2$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line).

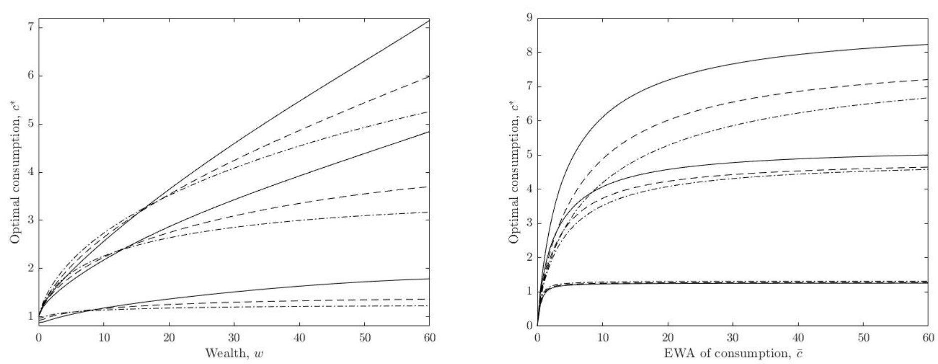
The same results we obtained for other parameter values, $\eta = 10^{-1}$ (see Figures (1.17) - (1.22)) and 10^{-2} (see Figures (1.23) - (1.28)). For example, numerical solution for parameter $\eta = 10^{-1}$ (Figure 1.18, 1.19 or 1.20), for every fixed value of the parameter θ, behaves similar to previous case with some minor differencies in amplitude. If we look at these results we can conclude that, as σ grows optimal consumption declines. In Figure (1.18), where the asset allocation parameter is $\theta = 0.6$, we can see that the greatest optimal consumption value client will have with greater habit $\bar{c} = 20$ and low volatility $\sigma = 0.16$.



(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1$, 30 and 60.

Figure 1.18: Numerical results for OP with pension, for $\eta = 10^{-1}$, for fixed value of $\theta = 0.6$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line).

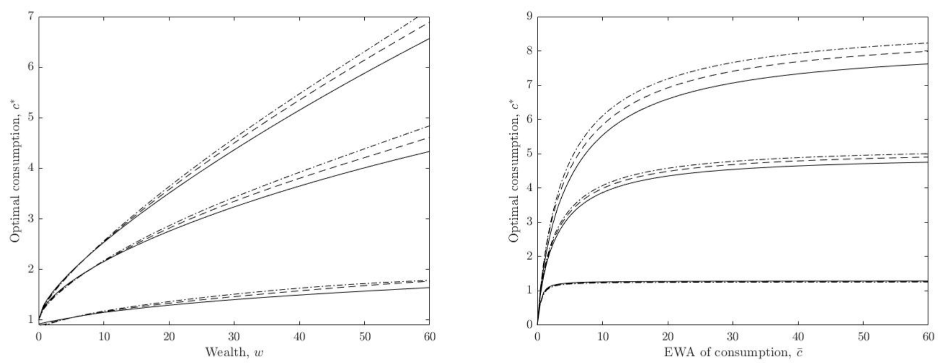
Moreover, for the small value of $\theta = 0.2$ Figure (1.17) the difference in optimal consumption for different habit $\bar{c}$ or wealth w values is not that big.



(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1$, 30 and 60.

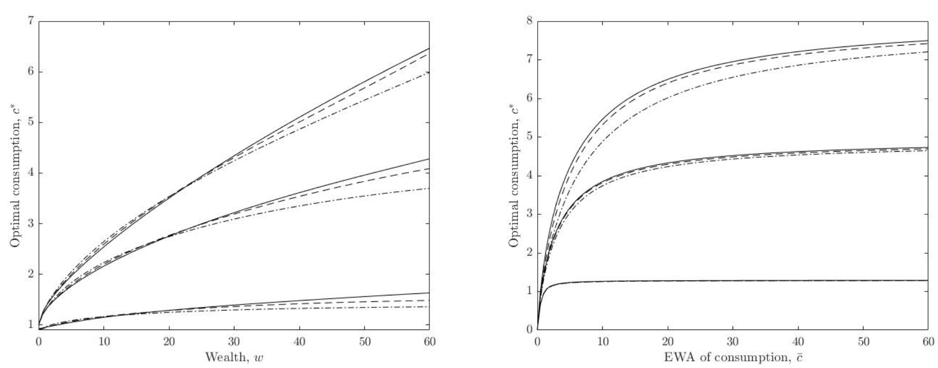
Figure 1.19: Numerical results for OP with pension, for $\eta = 10^{-1}$, for fixed value of $\theta = 0.9$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line) .

If we look at the next set where we fix σ and change θ (see Figures (1.20) - (1.22)), the picture is different from the previous cases since now the biggest values of optimal consumption c^* belong to solution with high asset allocation θ (see Figure (1.20)).



(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1$, 30 and 60.

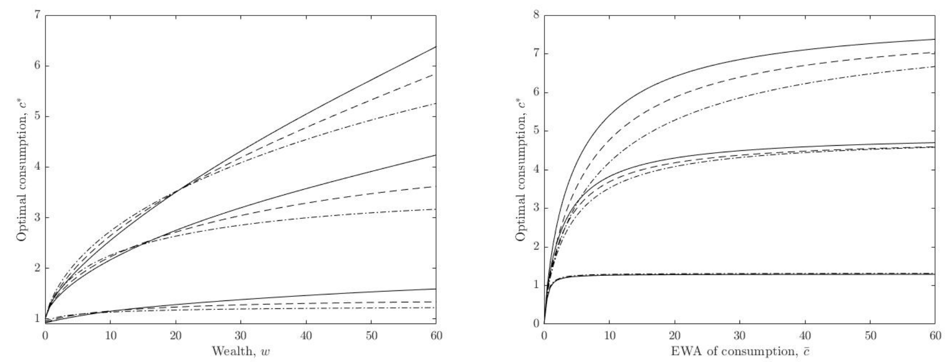
Figure 1.20: Numerical results for OP with pension, for $\eta = 10^{-1}$, for fixed value of $\sigma = 0.16$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .



(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1$, 30 and 60.

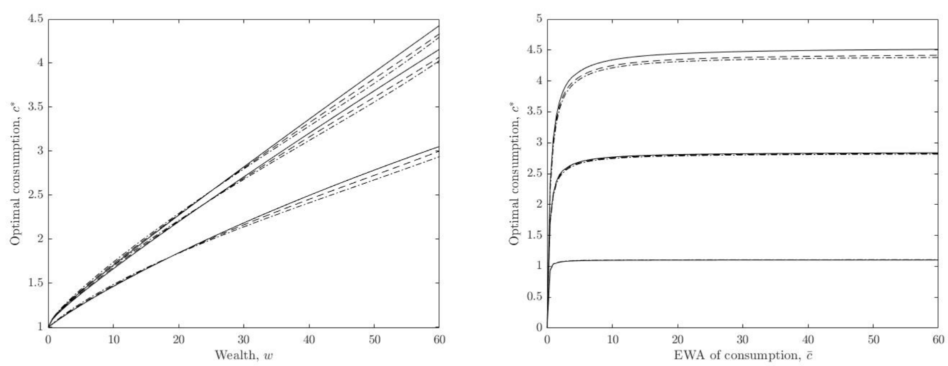
Figure 1.21: Numerical results for OP with pension, for $\eta = 10^{-1}$, for fixed value of $\sigma = 0.50$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .

As volatility goes up, $\sigma = 0.5$, 0.75 numerical solutions also become as in previous case.



(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1$, 30 and 60.

Figure 1.22: Numerical results for OP with pension, for $\eta = 10^{-1}$, for fixed value of $\sigma = 0.75$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .

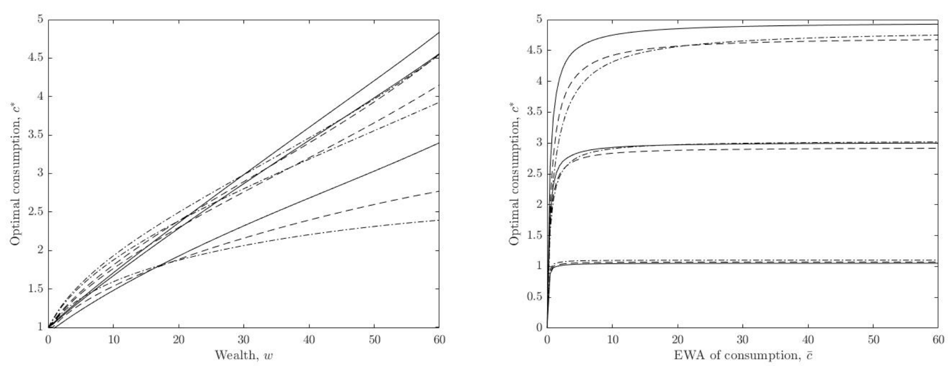


(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1$, 30 and 60.

Figure 1.23: Numerical results for OP with pension, for $\eta = 10^{-2}$, for fixed value of $\theta = 0.2$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line).

The explanation can be given in terms of PDE. If we recall that second derivative is responsible for the curvature then we can think about less impact or less contribution, of the second derivative in this particular case. This behaviour optimal consumption c^* shows on both graphs, $c^* - w$ (e.g. Figure (1.20a)-(1.22a)) and $c^* - \bar{c}$ (e.g. Figure (1.20b)-(1.22b)).

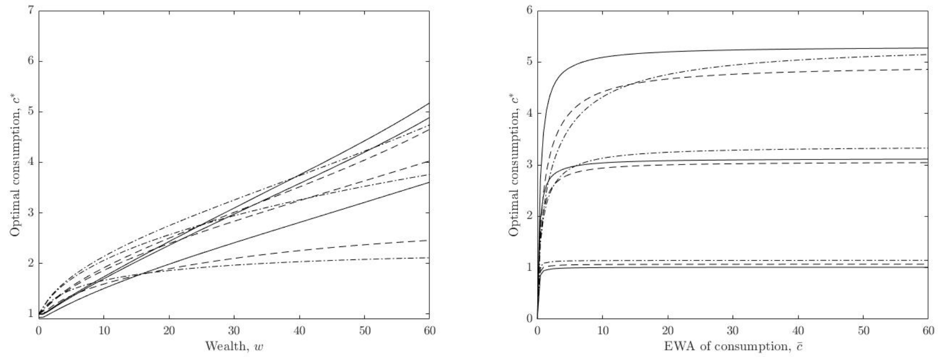
The last set of pictures represents numerical results for the parameter $\eta = 10^{-2}$ (Figures (1.23)-(1.28)). Similar to all other cases, we represent two types of graphs, relationship between optimal consumption and wealth $(c^* - w)$ and optimal consumption and habit $(c^* - \bar{c})$. As we can see from the first set where we as usual fix parameter θ and change σ (see Figures (1.23)-(1.25)) the solution behaves similar to previous case with $\eta = 10^{-1}$. For example, if we compare $\theta = 0.6$ (Figure 1.24) we can see that overall optimal consumption is smaller than for other η values (Figures 1.12 or 1.18).



(a) $\bar{c} = 1$, 5 and 20.

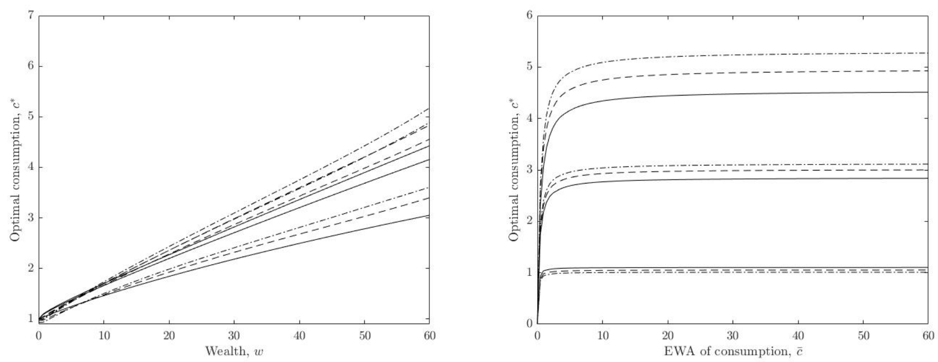
(b) $w = 1$, 30 and 60.

Figure 1.24: Numerical results for OP with pension, for $\eta = 10^{-2}$, for fixed value of $\theta = 0.6$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line).



(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1, 30$ and 60.

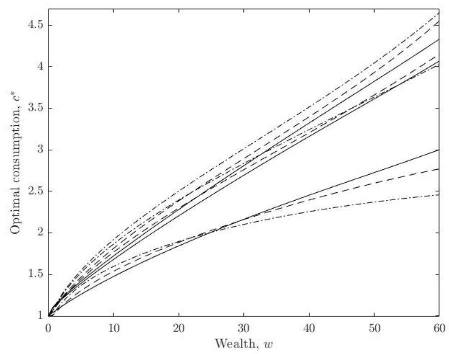
Figure 1.25: Numerical results for OP with pension, for $\eta = 10^{-2}$, for fixed value of $\theta = 0.9$ for three parameters $\sigma = 0.16$ (solid line), 0.50 (dashed line), 0.75 (dash-dot line) .



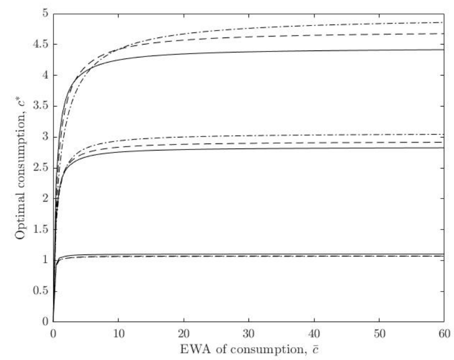
(a) $\bar{c} = 1$, 5 and 20. (b) $w = 1, 30$ and 60.

Figure 1.26: Numerical results for OP with pension, for $\eta = 10^{-2}$, for fixed value of $\sigma = 0.16$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .

Second set of graphs (see Figures (1.26)-(1.28) also shows values where the optimal consumption c^* for big values of $\theta = 0.6$, 0.9 has bigger values. The explanation is also related to the parameter $\eta = 10^{-2}$.

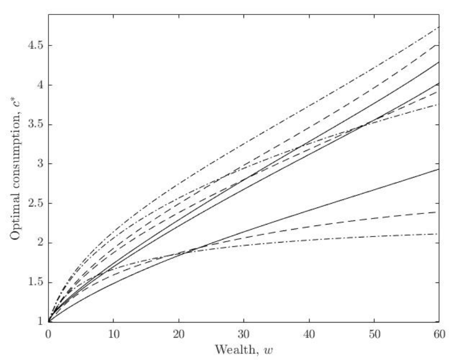


(a) $\bar{c} = 1$, 5 and 20.

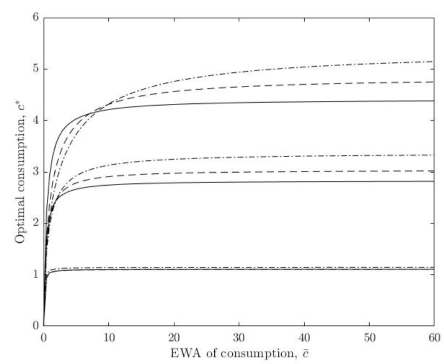


(b) $w = 1$, 30 and 60.

Figure 1.27: Numerical results for OP with pension, for $\eta = 10^{-2}$, for fixed value of $\sigma = 0.50$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .



(a) $\bar{c} = 1$, 5 and 20.



(b) $w = 1$, 30 and 60.

Figure 1.28: Numerical results for OP with pension, for $\eta = 10^{-2}$, for fixed value of $\sigma = 0.75$ for three parameters $\theta = 0.2$ (solid line), 0.6 (dashed line), 0.9 (dash-dot line) .

For this case averaging has great impact on our numerical solution since parameter is small, as a consequence, optimal consumption c^* has the least values, therefore among all three cases, $\eta = 1.0$, 10^{-1}, and 10^{-2} this one is the least interesting in terms of solving problem under HFM.

When giving advice to a client, we must take into account all parameters, such as habit, asset allocation and volatility, for example, regardless on his habit, optimal consumption is higher for greater value of θ and lower value of volatility σ or if σ is big then part of wealth inversted into risky assets should be smaller in case if we want to get higher consumption level. In terms of the smoothing parameter η, again, if you want to have higher level of consumption, it is preferable to pick higher value of η.

1.3.4 Comparison two models, with and w\o pension.

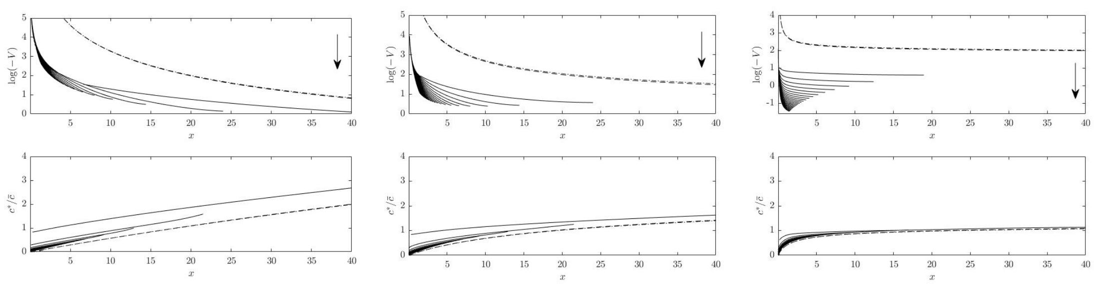
In this section we illustrate how changes in the smoothing factor η will affect our numerical solution. Furthermore, in order to better illustrate how the solutions differ, we compare results with and without pension (see 1.3.1).

Comparison based on the coordinate transformation $(t,w,\bar{c}) \mapsto (t,x)$.

In this case we scale our results with pension and compare them with those from the previous paragraph (Section 1.3.1). We provide results below (see Figure (1.29)) for our optimization problem where the solid lines represent how results with pension converge to those without one. Different solid lines represent particular habit values $\bar{c}$.

There are three sets of graphs for three values of the smoothing factor η. Every set consists of two pictures where the upper one represents the relationship between the logarithm of the scaled value function $\log(-V)$ and scaled wealth x. The arrow shows the direction of increasing habit and the second graph represents scaled optimal consumption $c^*/\bar{c}$ and scaled wealth

x. In the cases with pension, we show V and $c^*/\bar{c}$ in place of $\nu(t,x)$ and q^*, for various values of habit $\bar{c}$. The dashed line represents a problem where asset allocation is a control variable which proves empirically that an optimal strategy is better then any arbitrary strategy. In our case the dash-dot line represents a fixed parameter θ (see the bottom pictures on Figure (1.29)). These curves are very close, for reasons we will explain shortly. Parameters are as in Figure (1.1).



(a) $\eta = 0.01$ (b) $\eta = 0.1$ (c) $\eta = 1.0$

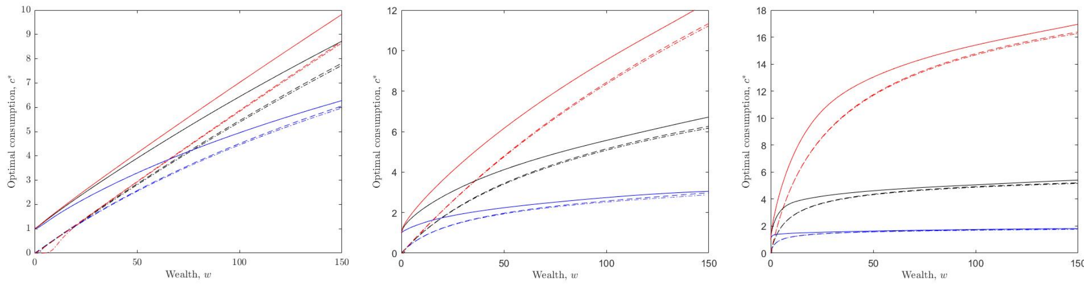
Figure 1.29: Scaled value function V (upper picture) and scaled optimal consumption $c^*/\bar{c}$ (bottom picture) vs. scaled weath x for two cases w\o pension, with (dashed line) and w\o (dash-dot line) asset allocation. In the latter, $\theta = 0.6$. The solid lines represent the value function V and scaled optimal consumption $c^*/\bar{c}$ for varioius values of habit $\bar{c}$.

Results were obtained for three values of the smoothing factor $\eta = 10^{-2}$ (see Figure (1.29a)), $\eta = 10^{-1}$ (see Figure (1.29b)) and 1.0 (see Figure (1.29c)). The first picture represents the case where $\eta = 0.01$ (Figure 1.29a). In this case we can see that curves that represent consumption approach to each other very slow because of great impact of averaging. For the next value of the smoothing factor, $\eta = 0.1$, consumption $c^*/\bar{c}$ adapts to the habit $\bar{c}$ faster (Figure 1.29b). Here we see a smaller gap between the solutions with (solid line) and without pension (dashed and dash-dot lines). This happens because the smoothing factor is significantly bigger which implies less impact from averaging and, as a consequence, faster response to changes in habit. In the last case when $\eta = 1.0$ (Figure 1.29c) we clearly see that changes happen very fast. For relatively big values of scaled wealth x solutions are very close. In order to see indistinguishable results, we would have to consider a bigger scale which is computationally expensive.

We can think about the case when our habit is the same order as pension, $\bar{c} \approx \pi$ or equivalently $x \approx w/\pi$. If this ratio is small we would expect to see that our numerical results differ from those that represent the case w\o pension. At the same time, as x increases, curves that represent the optimal consumption for all three cases approach to each other (see Figure 1.29c). If we consider an increasing habit $\bar{c}$ then the difference between our numerical results with and without pension will increase. On the graph habit increases in the downward direction. The next picture shows slightly different behaviour of our numerical solution (see Figure (1.29a)). Overall, we can say that consumption with pension income is definitely greater than without pension and it depends on habit and how fast the model reacts to changes, in other words how big the smoothing factor η is.

Comparison based on coordinate transformation $(t,x) \mapsto (t,w,\bar{c})$.

In the previous paragraph 1.3.4 we discussed results for scaled values of optimal consumption q^* and wealth x. Here we will consider the opposite transformation, $(t,x) \mapsto (t,w,\bar{c})$.



(a) $\eta = 0.01$ (b) $\eta = 0.1$ (c) $\eta = 1.0$

Figure 1.30: Optimal consumption $c^*(t,w,\bar{c})$ vs. weath w for multiple values of habit $\bar{c}$. Comparison between results with pension π for a fixed parameter θ (solid line), w\o pension and w\o asset allocation (dash-dot line) and w\o pension and with θ_t as a control variable (dashed line). The habit is fixed at $\bar{c} = 1$ (blue lines), 5 (black lines) and 20 (red lines).

In Figure 1.30 we provide results about optimal consumption c^* vs. wealth w for the following set of parameters: $\pi = 1$, $\gamma = 3$, $\mu = 0.08$, $r = 0.02$, $\rho = 0.02$, $\sigma = 0.16$ for the problem where the part of the wealth invested into risky assets is fixed, $\theta = 0.6$. As a result, we expect to see convergence results with pension π to results without pension for big enough wealth w. Also we see that the line which represents the solution with asset allocation (dashed line) is higher then line which characterizes the solution with fixed θ (dash-dot line). Though in fact these curves are so close it is hard to distinguish them. This confirms that the optimal strategy is better then any other strategy. At the same time, if we consider smaller values of the parameter η (1.30a or 1.30b), we should expect slower convergence of the results with pension to those without pension. This can be understood via PDE (1.21). When the parameter η is close to 1 the term that includes it dominates over the term with pension π. As a consequence, we see that results with pension converge to those without pension for big wealth and as the parameter η becomes smaller the term with pension π has more weight in the equation. In order to see convergence we would need to consider a significantly bigger scale. In terms of optimal consumption we can explain this the following way. When the parameter η is close to 1 (see Figure 1.30c), it means that the current value of consumption dominates. As the smoothing factor becomes smaller ($\eta << 1$) averaging has more weight and, as a result, the final answer will be smoother and will react less to current changes in values. Significantly slower convergence is anticipated for small value of the smoothing factor, for example $\eta = 0.01$ (see Figure 1.30a). This makes these results less interesting to explore.

Above, and in §1.3.4 we remarked on how close the curves were, corresponding to $\theta = 0.6$ and to variable θ. Figure 1.10a shows that for $\sigma = 0.16$ and $\eta = 1$, consumption is insensitive to asset allocation. For $\eta = 0.1$ or 0.01 it is more sensitive to θ, but only when θ varies a lot.

1.3.5 Numerical Tests

This section is devoted to testing the numerical method. Since we deal with free a boundary problem, which means that we do not know precise boundary conditions and have to choose them based on certain assumptions, then we need to create tests that will allow us to check how accurate results are, in other words we should answer the question how stable the numerical scheme is based on imposed boundary conditions.

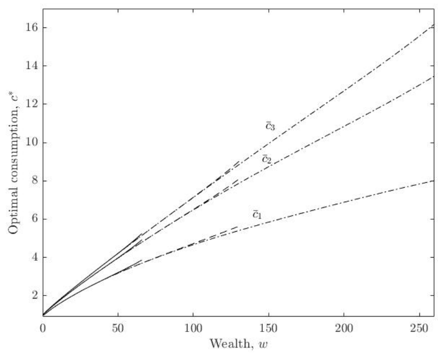


Figure 1.31: The relationship between the optimal consumption c^* and the wealth w. Comparison between different intervals w. Parameters of the model are the following: $\eta = 10^{-2}$, $\gamma = 3$. The values of EWA of consumption are $\bar{c}_1 = 1$, $\bar{c}_1 = 5$, $\bar{c}_1 = 20$.

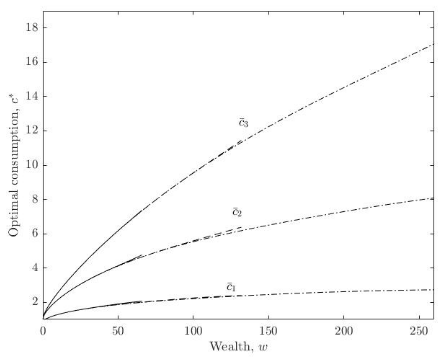


Figure 1.32: The relationship between the optimal consumption c^* and the wealth w. Comparison between different intervals w. Parameters of the model are the following: $\eta = 0.1$, $\gamma = 3$. The values of EWA of consumption are $\bar{c}_1 = 1$, $\bar{c}_1 = 5$, $\bar{c}_1 = 20$.

Changing intervals over wealth w. Let us change interval over the wealth w first, the interval over the habit $\bar{c} \in [0 \dots 30]$ will be fixed. We start from the initial interval $w \in [0 \dots 70]$ (solid line on the graph), then double it $w \in [0 \dots 140]$ (dashed line) and then, double it again $w \in [0 \dots 280]$ (dash-dot line). The results are shown on the Figure (1.31)-(1.33). We provide tests for all three values of parameter η and three fixed values of habit $\bar{c} = 1$, $\bar{c} = 5$ and $\bar{c} = 20$. It can be seen that for all values of η there is a good agreement between solutions.

Changing intervals over EWA of consumption $\bar{c}$. As a next step, we will change the EWA of consumption $\bar{c}$ range. The wealth $w \in [0 \dots 70]$ interval this time will be fixed. The initial interval $\bar{c} \in [0 \dots 30]$, then double

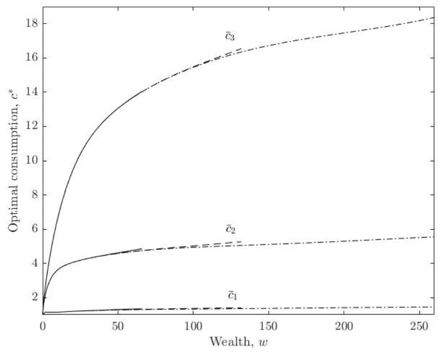


Figure 1.33: The relationship between the optimal consumption c^* and the wealth w. Parameters of the model are the following: $\eta = 1$, $\gamma = 3$. The values of EWA of consumption are $\bar{c}_1 = 1$, $\bar{c}_1 = 5$, $\bar{c}_1 = 20$.

it $\bar{c} \in [0 \ldots 60]$, and then double it again $\bar{c} \in [0 \ldots 120]$. The results are shown on the Figure ((1.34)-(1.36)) for three values of the parameter $\eta = 10^{-2}$, 10^{-1} and 1.0. Three different types of lines (solid, dashed and dash dot) represent different intervals between the EWA of consumption $\bar{c}$. On every picture there are three fixed wealth values, $w_1 = 1$, $w_1 = 10$ and $w_1 = 60$. As in the previous case, we can see that there is a good agreement between results.

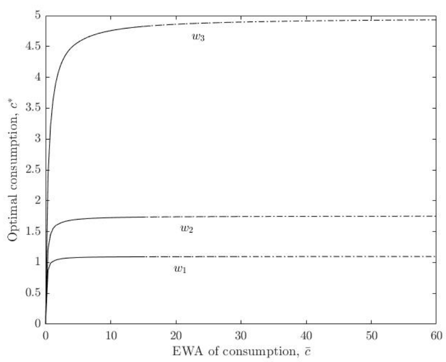


Figure 1.34: The relationship between the optimal consumption c^* and the wealth w. Comparison between different intervals $\bar{c}$ for the parameter $\eta = 10^{-2}$. The values of wealth are $w_1 = 1$, $w_2 = 10$, $w_3 = 60$.

Changing number of nodes over the wealth w. Next two paragraphs will be devoted to the tests where we will change number of nodes over the wealth Δw in the first case and over the EWA of consumption $\Delta\bar{c}$ in the second case. For the first experiment we will increase the number of nodes by power of two 2^6, 2^7, $2^8, 2^9, 2^{10}$. The rest parameters will be the same as in

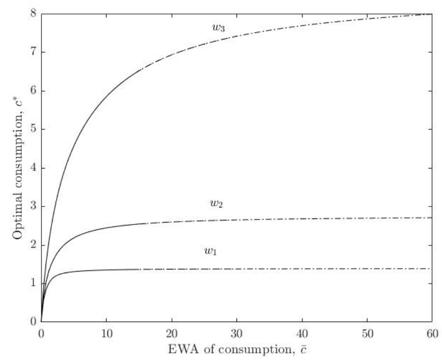


Figure 1.35: The relationship between the optimal consumption c^* and the wealth w. Comparison between different intervals $\bar{c}$ for the parameter $\eta = 10^{-1}$. The values of wealth are $w_1 = 1$, $w_2 = 10$, $w_3 = 60$.

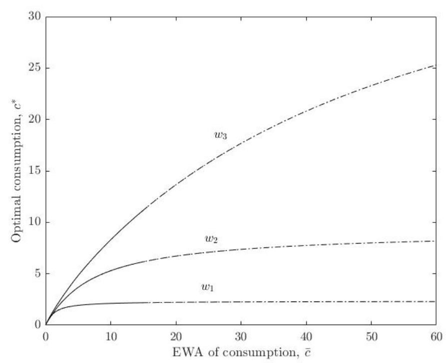


Figure 1.36: The relationship between the optimal consumption c^* and the wealth w. Comparison between different intervals $\bar{c}$ for the parameter $\eta = 1$. The values of wealth are $w_1 = 1$, $w_2 = 10$, $w_3 = 60$.

previous cases. The interval over the wealth $w \in [0 \dots 70]$, interval over the habit $\bar{c} \in [0 \dots 60]$ and they will be fixed. As we can see solutions converge as number on nodes over the wealth w increases. Results are shown on the Figures (1.37)-(1.39). Another step size $\Delta\bar{c}$ is fixed.

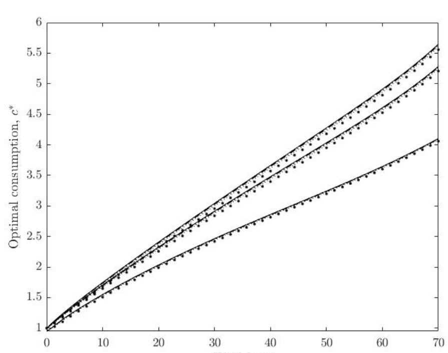


Figure 1.37: The relationship between the optimal consumption c^* and the wealth w. Comparison between different step sizes w for parameter $\eta = 10^{-2}$, number of nodes for wealth 2^6 (point line), 2^7 (dotted line), 2^8 (dash-dot line), 2^9 (dashed line), 2^{10} (solid line).

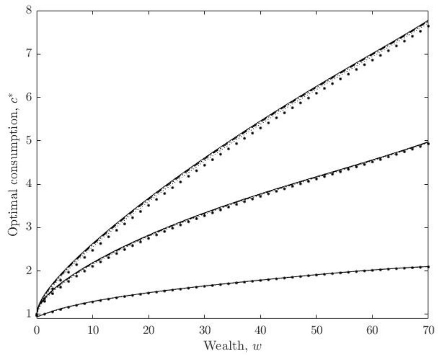


Figure 1.38: The relationship between the optimal consumption c^* and the wealth w. Comparison between different step sizes w for parameter $\eta = 10^{-1}$, number of nodes for wealth 2^6 (point line), 2^7 (dotted line), 2^8 (dash-dot line), 2^9 (dashed line), 2^{10} (solid line).

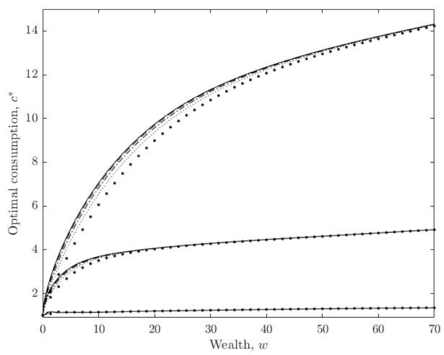


Figure 1.39: The relationship between the optimal consumption c^* and the wealth w. Comparison between different step sizes w for parameter $\eta = 1$, number of nodes for wealth 2^6 (point line), 2^7 (dotted line), 2^8 (dash-dot line), 2^9 (dashed line), 2^{10} (solid line).

Changing number of nodes over the EWA of consumption $\bar{c}$. In this test, we will change the step size $\Delta\bar{c}$ over the habit $\bar{c}$ interval, another one, Δw, will be fixed. Similar to previous case, we choose number of nodes over the habit by power two, namely $2^5, 2^6$, 2^7, $2^8, 2^9$. The rest parameters also will be the same as in previous case. The results are presented on the Figures (1.40)-(1.42). As we can see from the graph (1.40) as wealth goes to infinity ,$w \to \infty$, values are indistinguishable. The same behaviour shows the next graph (1.41) with respect to habit $\bar{c}$.

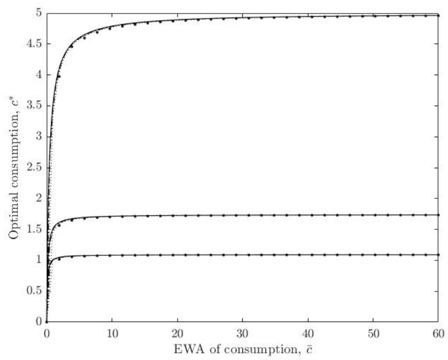


Figure 1.40: The relationship between the optimal consumption c^* and habit $\bar{c}$ for three fixed values of wealth $w = 1, 10, 60$ and parameter $\eta = 10^{-2}$. The number of nodes for habit $K = 2^5, 2^6, 2^7, 2^8, 2^9$.

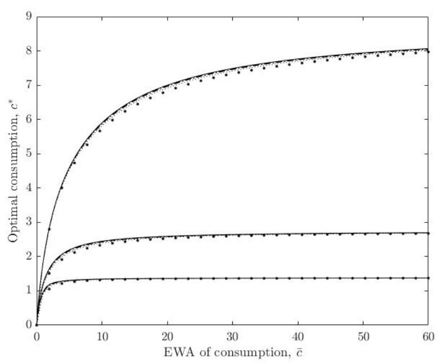


Figure 1.41: The relationship between the optimal consumption c^* and habit $\bar{c}$ for three fixed values of wealth $w = 1, 10, 60$ and parameter $\eta = 10^{-1}$. The number of nodes for habit $K = 2^5, 2^6, 2^7, 2^8, 2^9$.

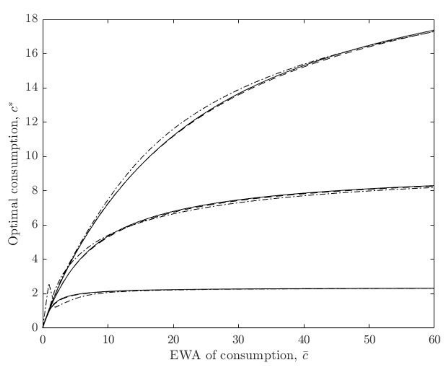


Figure 1.42: The relationship between the optimal consumption c^* and habit $\bar{c}$ for three fixed values of wealth $w = 1, 10, 30$ and parameter $\eta = 1$. The number of nodes for habit $K = 2^6, 2^7, 2^8$.

1.4 Retirement spending plan

1.4.1 Wealth depletion

In order to answer the question of how much a client can consume based on his habit during retirement, we need to simulate some representive scenarios. We assume that the agent follows the optimal strategy, and then we will use the results obtained in previous sections. Let us start by recalling the dynamics of wealth and habit

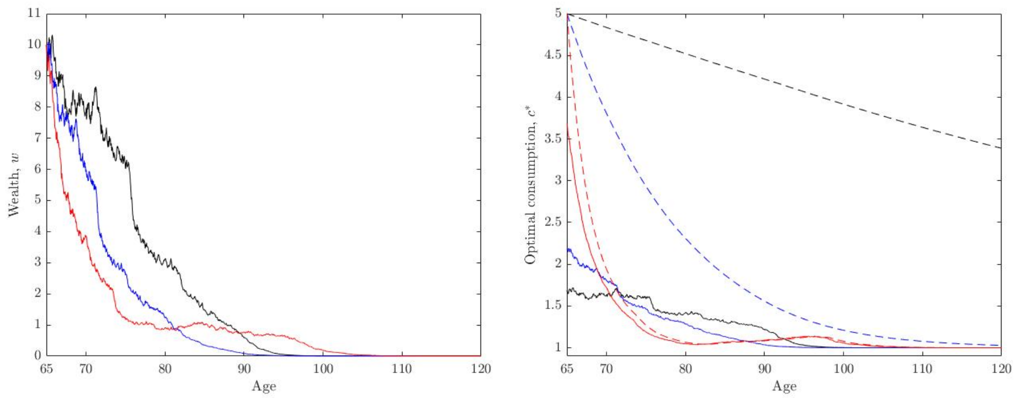
$$\begin{gathered} dw_t = [\theta(\mu - r) + r]w_t dt + \theta\sigma w_t dW_t + \pi dt - c_t dt \\ d\bar{c}_t = \eta(c_t - \bar{c}_t)dt. \end{gathered} \tag{1.39}$$

Then discretize them using Euler–Maruyama method for stochastic equation and simple Euler method for the second equation

$$\begin{gathered} w_{n+1} = w_n + [\theta(\mu - r) + r]w_n \Delta t + \theta\sigma w_n \Delta W_n + \pi\Delta t - c_n^* \Delta t \\ \bar{c}_{n+1} = \bar{c}_n + \eta(c_n^* - \bar{c}_n)\Delta t \end{gathered} \tag{1.40}$$

where $\theta, \mu, r, \sigma, \pi, \eta$ are constants and ΔW_n are the increments of the BM, $\Delta W_n = \sqrt{\Delta t} Z_n$ where $Z \sim N(0,1)$.

Below we provide some illustration of our numerical results for three different values of the smoothing factor $\eta = 10^{-2}, 10^{-1}$ and 1.0 (see Figures (1.43)-(1.17)). For all cases, numerical results were obtained for the following set of parameters, $\pi = 1$, $\gamma = 3$, $\sigma = 0.16$, $\theta = 0.6$, $\mu = 0.08$, $r = 0.02$. By changing initial wealth $w_0 = 10$, 30 and initial habit $\bar{c}_0 = 2$, 10, 20 we look at how the wealth and optimal consumption change over retirement.

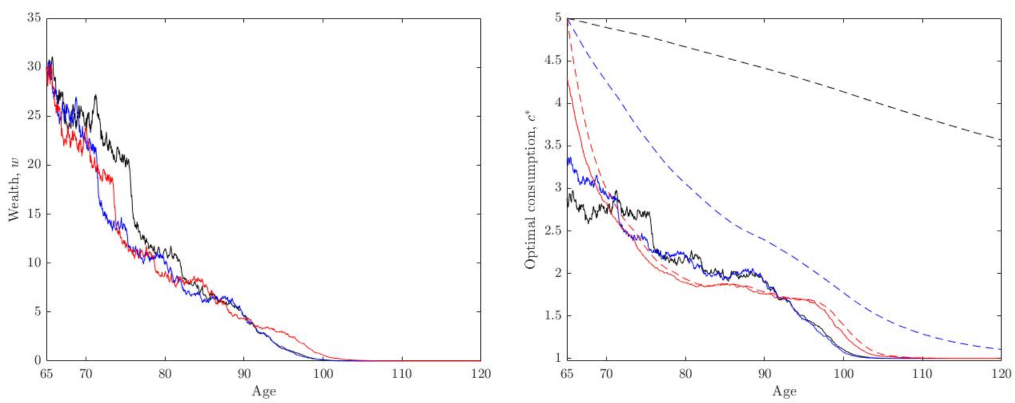


(a) (b)

Figure 1.43: Wealth and optimal consumption for initial value of wealth $w_0 = 10$ and initial EWA of consumption $\bar{c}_0 = 5$ for three parameters $\eta = 0.01$ (black line), $\eta = 0.1$ (blue line) $\eta = 1.0$ (red line).

In the picture on the left (see Figure (1.43a)) we can see how wealth changes over retirement between 65 and 120yrs whereas on the right we can see the optimal consumption c^* (solid lines) and habit $\bar{c}$ (dashed lines). For the small value of $\eta = 0.01$ habit changes very slowly. It explains why the two black lines in the picture (see Figure (1.43b)) are a significant distance from each other. This set of pictures (Figure (1.43)) shows the wealth w and optimal consumption c^* dynamics for initial wealth $w_0 = 10$ and initial habit $\bar{c} = 5$. As we can see the wealth's slope is relatively steep for all η values and, for example for $\eta = 0.01$ and 0.1, by the age 100 the wealth becomes zero. For $\eta << 1$ consumption remains low during the whole time period and asymptotically approaches to the level of the pension by age 100. In other words, it means that the agent who follows the optimal strategy in this particular case, i.e. has relatively low initial wealth and a low habit level, can not afford to spend more than twice the pension and has to remain approximately at this level during all the time period untill wealth ends around age 100. As the parameter η increases the wealth slope becomes steeper (Figure (1.43a)) because current values of consumption dominate over the averaging. This means the retiree consumes more early on. At the same time, the curve which represents optimal consumption (see Figure (1.43b)), is closer to the EWA of the consumption line. This means that agent's optimal consumption better reflects habit behaviour.

In addition, we can see that volatility is the highest when η is the smallest. It means that, for example the black lines which represent $\eta = 0.01$ have significantly bigger fluctuations then the red lines, which represent $\eta = 1.0$, on both pictures, Figure 1.43a and 1.43b.



(a)

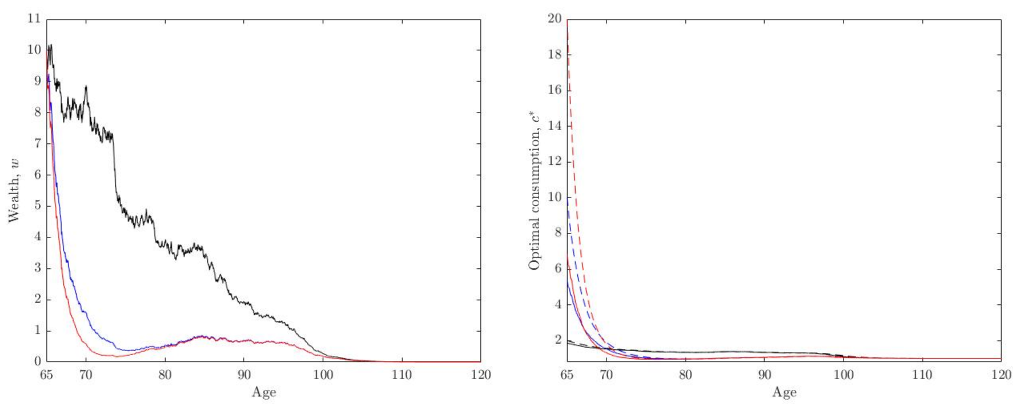
(b)

Figure 1.44: Wealth and optimal consumption for initial value of wealth $w_0 = 30$ and initial EWA of consumption $\bar{c}_0 = 5$ for three parameters $\eta = 0.01$ (black line), $\eta = 0.1$ (blue line) $\eta = 1.0$ (red line).

Now, if we increase initial wealth up to $w_0 = 30$ and leave habit at the same level we can see that the wealth slope shows a more moderate decline over time. In this case we see that the optimal consumption can fluctuate at the constant level approximately ~ 20yrs with the smoothing factor $\eta = 1.0$.

The explanation of this behaviour is simple. It happens because we accepted a low level of habit $\bar{c} = 5$ which means that the person did not consume a lot over previous time periods but has large enough wealth w, for his optimal consumption c^* to show positive dynamics over the next time period.

Based on results for different parameters η (Figure 1.43 or 1.44) we can see that changes in the agent's optimal consumption happen very fast for $\eta = 1.0$ which means that at every time the agent can consume approximately the same as his habit. Moreover, we see that the level of consumption in this case is also higher than for smaller η (solid red line). If we compare Figure 1.43a and 1.44a we conclude that the value of habit is very important. In cases where client does not consume a lot ($\bar{c}_0 = 5$) we see that higher initial wealth creates less difference between different models or parameters η.



(a) (b)

Figure 1.45: Wealth and optimal consumption for an initial value of wealth $w_0 = 10$, smoothing factor $\eta = 1.0$ and three values of initial habit $\bar{c}_0 = 2$ (black line), $\bar{c} = 10$ (blue line) $\bar{c} = 20$ (red line).

We can consider another case where we vary the habit level $\bar{c}_0$ for a smoothing factor $\eta = 1.0$ (see Figure 1.45). The overall observation is that with a higher initial habit $\bar{c}_0 = 20$, less time is needed for wealth to reach zero and for consumption go down to the pension level.

1.4.2 Wealth depletion time (WDT)

In the last section, we obtained simulation results using an Euler-Maruyama (EM) method. In this paragraph we will obtain WDT T_d using a PDE approach and will compare results with Euler's method. Following the same logic as in the article [Habib etc., 2017] let us define T_d as expected wealth depletion time $T^d(t,w,\bar{c}) = E[\tau | w_t = w, \bar{c}_t = \bar{c}, c_t^* = c^*]$ with an additional variable $\bar{c}$ that represents habit. Here τ is the first time that $w_t = 0$. We obtain a PDE

$$\frac{\partial T^d}{\partial t} + \frac{\partial T^d}{\partial w}((\theta(\mu - r) + r)w + \pi - c^*) + \frac{\partial T^d}{\partial \bar{c}}\eta(c^* - \bar{c}) + \frac{1}{2}\frac{\partial^2 T^d}{\partial w^2}\theta^2\sigma^2 w^2 + 1 = 0. \tag{1.41}$$

Boundary conditions for this equation are similar to the previous case except at the boundary with zero wealth, $w = 0$. WDT here is zero, $T^d(t,0,\bar{c}) = 0$. As in the previous case, in order to solve this equation, we use an implicit upwind method which is described in detail in Chapter 1.2.

As a next step, we run more simulations for the different sets of initial wealth and habit and compute their mean and variance in order to see how the time depletion changes if we fix the wealth value and take the average over all habit values. We summarize our results in the Table 1.3 where

Table 1.3: Comparison between Monte Carlo and PDE results for $\bar{c} = 10$.

w_0 values	EM			PDE		
	$\eta = 0.01$	$\eta = 0.1$	$\eta = 1.0$	$\eta = 0.01$	$\eta = 0.1$	$\eta = 1.0$
1	$81.8 \pm 2.1^*$	70.6 ± 4.6	68.3 ± 7.6	81.5	70.2	68.7
5	91.5 ± 2.6	78.8 ± 5.1	76.5 ± 13.1	91.0	80.8	84.9
10	95.5 ± 2.7	84.0 ± 5.5	86.1 ± 14.4	95.3	87.8	93.8
20	99.4 ± 2.6	90.3 ± 5.5	99.3 ± 4.0	99.7	95.5	100.0
35	102.6 ± 2.4	95.5 ± 5.3	100.9 ± 2.6	102.8	100.7	103.4
50	104.5 ± 2.3	99.0 ± 4.5	102.4 ± 2.7	104.6	103.4	104.9
75	105.8 ± 2.1	102.1 ± 3.9	103.7 ± 2.7	106.1	105.1	105.7

$*$ WDT range mean $\pm$std

every entry is the average age at which wealth depletes. By "age" we mean age = age of retirement + time. As before, Table 1.3 reflects an initial age of 65. As soon as wealth reaches level zero we record our depletion time. In other words, this means that the retiree has spent all his savings and does not have income except his pension.

To compare the EM and PDE methods, we choose three values of the smoothing factor $\eta = [0.01\ 0.1\ 1.0]$. This parameter reflects how fast habit reacts to changes in consumption. Based on the results from the Table 1.3 we can see that the wealth depletion time function shows nonlinear behaviour, which could be subject for the additional analysis in later research. When we set this parameter equal to 1 we see that averaging does not dominate. When this parameter decreases the averaging plays a greater role. In this paper we consider two more values of this parameter, namely $\eta = 0.1$ and 0.01. The first column in the table represents the initial wealth w_0 values. In our case we examined seven different values [1 5 10 20 35 50 75]. For instance, if we look at the third row and second column, the corresponding

Table 1.4: Numerical PDE results for $\eta = 0$.

w_0	1	5	10	20	35	50	75
PDE	85.7	93.0	96.8	100.6	103.4	105.1	106.4
WDT range mean $\pm$ std= 100.5 ± 9.6.							

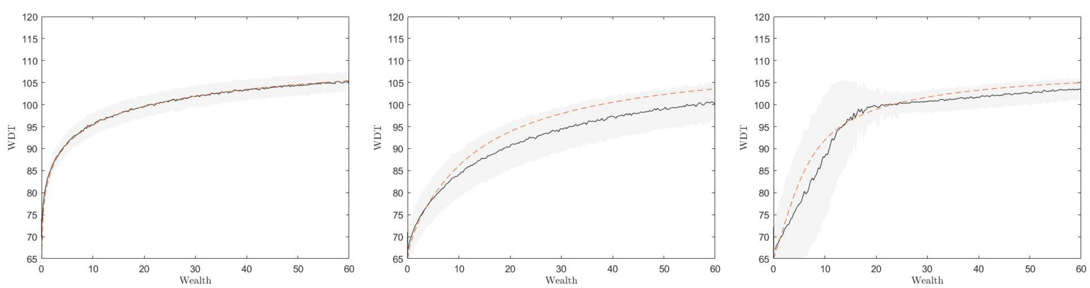
values (w_0, age) are (5, 91.5 ± 2.6) which means that if the initial wealth equals 5 then the average time when the wealth depletes is 91.49 yrs. with a standard deviation 2.6. The trend for every fixed smoothing factor η is, as initial wealth w_0 increases, depletion time also increases. Moreover, for small values of wealth for different values of the parameter η the difference in WDT is very big. For $w_0 = 1$ it is $\sim$ 67yrs for $\eta = 1.0$ and $\sim$ 82yrs for $\eta = 0.01$.

At the same time, as initial wealth grows, the difference in depletion time becomes less significant. For example, there is no significant difference between depletion age for big wealth 75 as η varies with average values varying within the range (102, 106) for EM and $(105, 106)$ for PDE. If we compare the EM and PDE results we see that PDE solution is included in our EM intervals.

There is one more case of smoothing parameter $\eta = 0$ which means that we do not take habit into consideration. Table 1.4 shows these results for the same set of initial wealth as in the previous case. The trend is the same as before, i.e. as soon as wealth increases WDT also increases. Comparing WDT for small values of wealth, with and without habit, it is greater in the second case because we do not count on habit and just follow the optimal strategy.

We can explain this as follows. Assume that we have two clients and one of them follows the optimal strategy ignoring habit. According to the table 1.4 if he has initial wealth 5 at the moment of his retirement then he will spend all his money by age 93. Assume that another client follows the optimal strategy under habit formation model. In this case if he has the same initial amount of wealth 5 he will spend all his money by age 91 if the parameter value is $\eta = 0.01$ or by age of 85 for $\eta = 1.0$. As we know the greater the value of the smoothing parameter we have, the faster consumption will react to habit changes which means that the person with a higher value of the parameter η can consume more. Therefore for big values of initial wealth the difference in WDT between η is not that big.

Now, we provide some graphs (see Figures 1.46) that illustrate how the results change for different smoothing factor values.



(a) $\eta = 0.01$ (b) for $\eta = 0.1$ (c) $\eta = 1.0$

Figure 1.46: WDT vs. initial wealth for fixed habit $\bar{c} = 10$.

For all three pictures, the solid line represents the EM solution, the dashed line the PDE solution and the shadowed area represents the standard deviation of the stochastic solution over all values of initial habit $\bar{c}_0$. For comparison we fixed habit at the level $\bar{c} = 10$. As we can see the case with $\eta = 0.01$ (Figure 1.46a) has the smallest shadow area and the two lines are almost indistinguishable which means both solutions, EM and PDE, are in the good agreement. As the smoothing factor increases to $\eta = 0.1$, the shadowed area becomes bigger as well (Figure 1.46b) and the lines are not as close as in the previous case. But still the PDE solution lies within the stochastic solution boundaries. The shadowed area reflects the stability of the particular solution, which is why for $\eta = 1.0$ (Figure 1.46c) we see an irregular shadow.

1.5 Liquid wealth vs. annuity

Suppose that there is a client who at the time of retirement, for example at age 65, has a certain amount of wealth w (non annuitized) and pension π, such as Social Security benefits. There are several papers where authors showed that under certain conditions it is better to convert their wealth into annuities (see for example [Reichling & Smetters, 2015]). In this paper we answer the question, should the endowment be annuitized at the moment of retirement or not, under HFM. First, let us define the annuity equivalent wealth (AEW). Based on the definition provided in the article [Milevsky & Huang, 2019], the AEW is the quantity $\hat{w}$ that satisfies the following equation:

$$V(0,\hat{w},\bar{c},\pi) = V(0,0,\bar{c},\pi + w/a_x) \tag{1.42}$$

where V is the maximized discounted lifetime utility function, w represents initial liquid wealth, a_x is the annuity factor and x is the client's current age. The idea is that a client with initial wealth w and a certain utility should retain the same utility level if he decides to annuitize the entire wealth. In this paper instead we explore whether it is reasonable to annuitize part of the retiree's wealth into an annuity under HFM. We compare two value functions, namely $V(0,w,\bar{c},\pi)$ and $V(0,w-\Delta w,\bar{c},\pi+\Delta w/a_x)$. Before we move forward let us say a couple words about the annuity factor a_x. We can define a_x as follows

$$a(r,x,m,b) = \frac{b\Gamma(-rb, exp(\frac{x-m}{b}))}{exp(r(m-x) - exp(\frac{x-m}{b}))} \tag{1.43}$$

where all values reflect the Gompertz formulation (see for example [Milevsky, 2006]) and all parameters are as described in §1.3.2. We can think about the following financial definition of the annuity factor a_x. In case the retiree converts some or all of his wealth into annuities, the annuity factor is their unit price a_x.

Now, let us introduce new notation. We define the difference ΔV on the graphs as follows

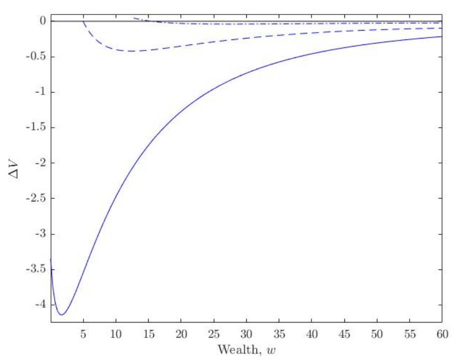
$$\Delta V = V(0,w,\bar{c},\pi) - V(0,w-\Delta w,\bar{c},\pi+\Delta w/a_x). \tag{1.44}$$

Below we provide some numerical results for our model where we take into consideration the client's habit $\bar{c}$. For the case with smoothing parameter $\eta = 0$, which means that we do not include habit, calculations were done in the paper [Milevsky & Huang, 2019] but for the case where the client does not invest anything into risky assets, i.e. $\theta = 0$. In this paper we will check if the inequality, that those authors derived in their article, holds for HFM

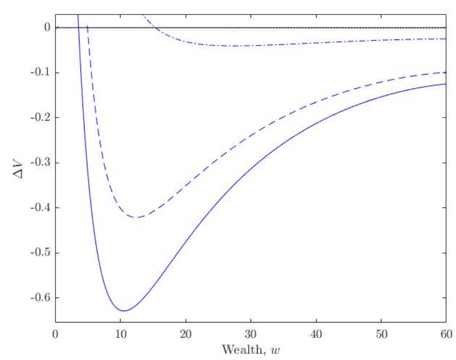
$$V(0,w,\bar{c},\pi) < V(0,w-\Delta w,\bar{c},\pi+\Delta w/a_x). \tag{1.45}$$

In other words, this inequality means that for the client at age 65 it is better to annuitize part of the wealth Δw when $\eta = 0$. At the same time, as the parameter η increases the picture changes. As soon as the retiree invests more into risky assets, for example 20% or 60% (see Figure 1.47) of his portfolio, the difference becomes positive for wealth below a certain level. On the graphs we see the relationship between the difference ΔV (1.44) and wealth w. Results were obtained for fixed asset allocation and habit value $\bar{c} \approx 10$ and for four values of the parameter $\eta = [0\ 0.01\ 0.1\ 1.0]$ (see Figure 1.47).

As we can see, for example from the Figure (1.47a) where only 20% of portfolio is invested into risky assets, and $\theta = 0.2$, when the effect of current consumption is strong (eg. $\eta \sim 1$) little benefit accrues from the converting wealth into annuities at a retirement age of 65yrs. For $\eta = 0.01$ the impact of annuitization is significant at large wealth.



(a) $\theta = 0.2$



(b) $\theta = 0.6$

Figure 1.47: Difference ΔV vs. wealth w for fixed habit $\bar{c} \approx 10$, smoothing parameter $\eta = 0$ (the blue solid line), 0.01 (the blue dashed line), 0.1 (the blue dash-dot line) and 1.0 (the blue dotted line). The black horizontal line represents zero level in ΔV.

Similar behaviour can be seen for the parameter value $\theta = 0.6$ (see Figure

Table 1.5: Wealth values that show when to annuitize.

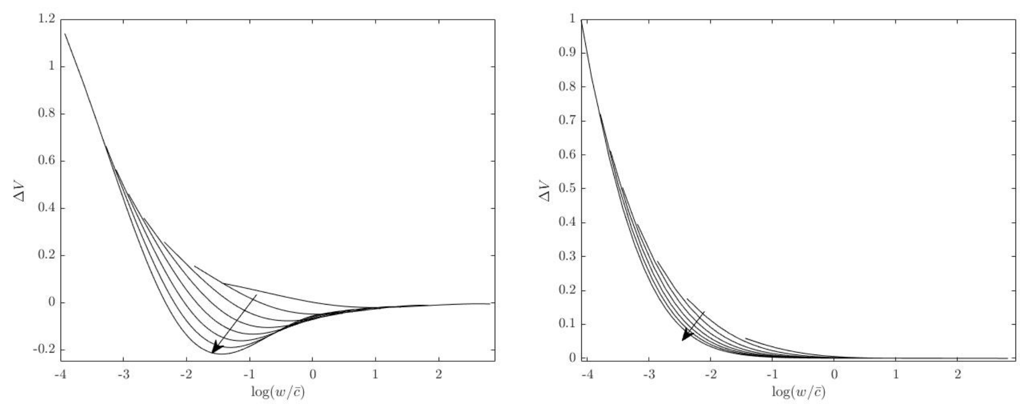
θ values	$\bar{c}=1$		$\bar{c}=5$		$\bar{c}=10$	
	$\eta=0.01$	$\eta=1.0$	$\eta=0.01$	$\eta=1.0$	$\eta=0.01$	$\eta=1.0$
0	$\star$	$\star$	$\star$	7.77	$\star$	14.47
0.2	$\star$	$\star$	$\star$	8.47	$\star$	15.53
0.6	3.18	2.82	4.94	12.35	5.29	21.17
$\star$ means that the difference ΔV is negative for all w						

1.47b). The observation that we can make after taking into consideration all numerical results is the following: as soon as the retiree decides to invest more into risky assets, for example $\theta = 0.6$, the behaviour of the function that represents the difference ΔV changes and becomes positive below a certain level, which means that client should not convert his endowment into annuities unless he has at least a minimum wealth level.

To summarize the results, Table 1.5 shows when it is better to annuitize wealth for two values of the smoothing factor $\eta = 0.01$ and 1.0, some values of asset allocation $\theta = 0$, 0.2, 0.6 and three values of habit $\bar{c} \approx 1$, 5, 10 (see Table 1.5). For instance, if we consider a small value of habit $\bar{c} = 1.0$ it is definitely better to annuitize wealth for almost all values of asset allocation whereas for $\bar{c} = 10$ this is not true.

For example, value 15 in the fifth row and fifth column means that for the smoothing factor $\eta = 1$, with habit $\bar{c} \approx 5$ and 60% of wealth invested into risky assets, the minimum value of the wealth when it is reasonable to annuitize at age 65 is approximately 15, which is significantly greater than pension income. The last column in the table shows that if the habit $\bar{c}$ is relatively high, in our case $\bar{c} = 10$, then it does not matter how much you invest into risky assets, the response is very fast, i.e. we have less impact from averaging ($\eta \approx 1$), and for all cases the value of wealth when it is reasonable to convert part of the wealth into annuities is greater than 15. In other words, the wealth should be 15 times greater than pension income, such as 17 for $\theta = 0.2$ or $w \approx 20$ for $\theta = 0.6$.

The last set of pictures (see Figure 1.48) represents the scaled version of our numerical results, where scaling was described in one of the previous sections. We use two values of the smoothing parameter $\eta = 0.01$ and 1.0. The y-axis represents the difference ΔV (1.44) and the x-axis is a logarithm of the ratio between wealth w and habit $\bar{c}$, $\log(w/\bar{c})$. We see two areas which show that the question about wealth annuitization does not have an unambiguous answer.



(a) $\eta = 0.01$ (b) $\eta = 1$.

Figure 1.48: Difference ΔV vs. scaled wealth w for fixed asset allocation $\theta = 0.6$. The arrow shows the direction of increasing habit $\bar{c}$.

Every line represents a different value of habit $\bar{c}$ that increases in the direction pointed to by the arrow. In the left Figure (1.48a) we see results for $\eta = 0.01$ which means that averaging dominates and habit adapts slowly.

Here there are regions where annuitization is favourable. Whereas in the right picture (1.48b) averaging has the least impact on our solution and, as a consequence, habit adapts rapidly. Here the answer is that for small wealth it is not optimal to annuitize part of the wealth.

Finally, we can say that all results show that the presence of habit makes annuitization less reasonable for small values of wealth w while it becomes a good choice for large wealth. At the same time, the more a retiree invests into risky assets the less favourable it will be to annuitize part of the wealth immediately at the age of 65.

Chapter 2

Solving retirement spending problem under a habit formation model using the static programming approach

2.1 Introduction

2.1.1 Overview

As in the previous chapter, we consider a retiree who wants to spend his wealth wisely by maximizing consumption. The goal of this chapter is to understand how to apply the martingale approach for solving a consumption optimization problem counting on retiree's living standard. This is a completely different method based on idea that instead on focusing on the solution at every time moment up to a terminal time T the method focuses on the solution at the terminal time only. Then by applying the Martingale representation theorem we can obtain information for the whole time interval. As a consequence, the formulation of the problem will differ and, in fact, we solve the retiree spending problem using a greedy algorithm which is feasible to compute but not necessarily optimal. The difference is that we find a locally optimal solution at every step. In some problems this approach can lead to the global maximum. Based on our formulation we claim that for the retiree spending problem we find only a locally optimal solution. As in the previous chapter we choose multiplicative CRRA utility function and consider constant pension income. The wealth X_t and habit $\bar{C}_t$ dynamics also will remain the same (1.1). For this chapter we change notation for some variables in order to emphasize the difference in our approach. In the previous chapter we solved a PDE and all variables were deterministic whereas in this chapter we deal with stochastic ones. At the same time, unlike the previous Chapter 1 we assume that the asset allocation θ_t will be a control variable except in some numerical tests whereas for the value function approach we assumed it to be constant all the time. In addition, we consider full consumption C_t^* as a sum of consumption over wealth C_t^w and pension $\pi = \text{const}$, i.e. $C_t^* = C_t^{w*} + \pi$. We optimize the part of consumption related to wealth and constrain the retiree to spend all pension at every time moment.

2.1.2 Stochastic Calculus

At the beginning we give an introduction to the general stochastic calculus, which has been used extensively in this chapter.

Brownian Motion

We start from the definition of the Brownian motion (or Wiener process).

Definition 2.1.1. *A Brownian motion or Wiener process is a stochastic process* B_t, or W_t $t \geq 0$, *which satisfies the following properties*

- *The process starts at the origin* $W_0 = 0$;
- W_t *has stationary independent increments;*
- *The process* W_t *is continous in* t;
- *The increments* $W_{t+s} - W_s$ *are normally distributed with mean zero and variance* t, *where* $0 \leq s < t$,

$$W_{t+s} - W_s \sim N(0,t). \tag{2.1}$$

	$\mathrm{d}t$	$\mathrm{d}W_t$
$\mathrm{d}t$	$(\mathrm{d}t)^2 = 0$	$\mathrm{d}t\mathrm{d}W_t = 0$
$\mathrm{d}W_t$	$\mathrm{d}W_t\mathrm{d}t = 0$	$(\mathrm{d}W_t)^2 = t$

Figure 2.1

Simulate the Wiener process as follows $W_{t+s} - W_s = \sqrt{t}\xi_t$, where $\xi_t \sim N(0,1)$. $\Delta W_{t_k} = \sqrt{\Delta t_k}\xi_k$.

"Box Algebra"

Throughout this chapter, we will use various rules of stochastic calculus, and one of the most useful is the so-called "Box Algebra" (see [Calin, 2022]).

This diagram 2.1 shows the basic relationships between time and Brownian motion increments. Let's explain all these relations.

- $(\mathrm{d}t)^2 = 0$

  In this case we deal with a regular Riemann integral with respect to $(\mathrm{d}t)^2$. Assume that we have equidistant time nodes

  $$t_k = t_0 + \frac{k}{n}(t - t_0) \tag{2.2}$$

  Accept $t_0 = 0$ then we can write the following chain of equalities

  $$\begin{aligned} \int_0^t (\mathrm{d}t)^2 &= \lim_{n\to\infty} \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(\frac{(k+1)t}{n} - \frac{kt}{n} \right)^2 \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{t^2}{n^2} = \lim_{n\to\infty} n\frac{t^2}{n^2} = \lim_{n\to\infty} \frac{t^2}{n} = 0 \end{aligned} \tag{2.3}$$

- $\mathrm{d}t\mathrm{d}W_t = \mathrm{d}W_t\mathrm{d}t = 0$

  $$\begin{aligned} \int_0^t \mathrm{d}t\mathrm{d}W_t &= \lim_{n\to\infty} \sum_{k=0}^{n-1} (t_{k+1} - t_k)(W_{t_{k+1}} - W_{t_k}) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} (t_{k+1} - t_k)\sqrt{t_{k+1} - t_k}\xi_k = \lim_{n\to\infty} \sum_{k=0}^{n-1} (t_{k+1} - t_k)^{3/2}\xi_k \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(\frac{(k+1)t}{n} - \frac{kt}{n} \right)^{3/2} \xi_k = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{t^{3/2}}{n^{3/2}} \xi_k \\
&= \frac{t^{3/2}}{n^{3/2}} \lim_{n\to\infty} \sum_{k=0}^{n-1} \xi_k = 0. \qquad (2.4)
\end{aligned}
$$

The mean and variance of independent random variables ξ_k are additive, so the sum of the ξ_k gives a Gaussian variable with mean 0 and variance n since ξ is a Gaussian (0,1).

- $(\mathrm{d}W_t)^2 = \mathrm{d}t$.

This is a one of the most important relations of Stochastic Calculus.

$$
\int_0^t \mathrm{d}W_s^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta W_{t_k}^2 = \lim_{n\to\infty} \sum_{k=1}^{n} (W_{t_k} - W_{t_{k-1}})^2 \qquad (2.5)
$$

Definition 2.1.2. *Let $\{X_n\}$ be a sequence of square integrable random variables defined on a sample space Ω. We say that $\{X_n\}$ is a mean-square convergent iff there exists a square integrable random variable X such that sequence $\{X_n\}$ converges to X, according to metric* $\mathrm{d}(X_n,X) = E[(X_n - X)^2]$ *that is*

$$
\lim_{n\to\infty} E[(X_n - X)^2] = 0. \qquad (2.6)
$$

In our case let us denote $X_n = \sum_{k=1}^{n} \Delta W_{t_k}$ and require $X = t$. Let's prove our statement under these assumptions.

$$\lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n}\Delta W_{t_k}^2 - t\right)^2\right] = \lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n}\Delta W_{t_k}^2\right)^2 - 2t\sum_{k=1}^{n}\Delta W_{t_k}^2 + t^2\right]$$
$$= \lim_{n\to\infty} E\left[\sum_{k=1}^{n}\Delta W_{t_k}^4 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1}\Delta W_{t_k}^2\Delta W_{t_j}^2 - 2t\sum_{k=1}^{n}\Delta W_{t_k}^2 + t^2\right] \quad (2.7)$$

We know that the following equalities hold

$$E[\Delta W_t^2] = \Delta t, \qquad E[\Delta W_t^4] = 3\Delta t^2. \quad (2.8)$$

Continuing with formula (2.7)

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}3\Delta t_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1}\Delta t_k\Delta t_j - 2t\sum_{k=1}^{n}\Delta t_k + t^2\right] \quad (2.9)$$

Now, we can recall that $\sum_{k=1}^{n}\Delta t_k = t$ and $\Delta t_k = \Delta t_j = \frac{t}{n}$. Recall the sum of an arithmetic progression (AP)

$$\sum_{k=1}^{n}\sum_{j=1}^{k-1}\left(\frac{t}{n}\right)^2 = \sum_{k=1}^{n}(k-1)\left(\frac{t}{n}\right)^2 = 0\cdot\left(\frac{t}{n}\right)^2 + \cdots + (n-1)\cdot\left(\frac{t}{n}\right)^2$$
$$\text{AP}: \ 0,1,2,\ldots,n-1 \Rightarrow \ S_n = \frac{a_1+a_n}{2}n = \frac{n(n-1)}{2}. \quad (2.10)$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}3\left(\frac{t}{n}\right)^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1}\left(\frac{t}{n}\right)^2 - t^2\right]$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left[3n\left(\frac{t}{n}\right)^2 + 2\frac{n(n-1)}{2}\left(\frac{t}{n}\right)^2 - t^2\right] \\
&= \lim_{n\to\infty}\left[3\overset{0}{\cancel{\left(\frac{t}{n}\right)^2}} + \Big(1 - \overset{0}{\cancel{\frac{1}{n}}}\Big)t^2 - t^2\right] = 0.
\end{aligned}
\tag{2.11}
$$

Hence, we can conclude that$(\mathrm{d}W_t)^2 = \mathrm{d}t$.

Finally, we proved all four relationships from the diagram 2.1 and in the next paragraphs we will use them in order to derive the solution for our optimization problem.

2.2 Consumption optimization under the Merton model using martingale approach.

2.2.1 Merton problem without mortality

In this paragraph we adopt a model which consists of risk-free investments (bonds) as well as risky assets and solve the maximiaztion problem for the well-known Merton problem (see [Karatzas etc., 1987]). We use it as "toy model" before we formulate the main problem under a habit formation model. The wealth process $\mathrm{d}X_t$ should satisfy the following stochastic differential equation

$$
\mathrm{d}X_t = \{\lambda_t X_t - C_t\}\mathrm{d}t + \sigma\theta_t X_t \mathrm{d}W_t, \quad \lambda_t = r + \theta_t(\mu - r) \tag{2.12}
$$

where the volatility σ, risk-free rate r and drift μ are all constants, W_t is a Brownian motion on a probablity space $(\Omega, \mathscr{F}, P)$. Here θ_t is the proportion of wealth invested into stocks. It and the consumption stream C_t are control variables. Define a consumption process $C = \{C_t, t \in [0,T]\}$ as a non-negative adapted stochastic process where $C_t : [0,T] \times \Omega \to \mathbb{R}_+ \cup \{0\}$ represents the rate of consumption at time t.

Definition 2.2.1. *A consumption and asset allocation process pair (θ, C) is said to be admissible for the initial wealth $v \geq 0$ if $X_0 = v$ and the wealth process remains nonnegative over entire time interval, i.e. $X_t \geq 0 \; \forall \; 0 \leq t \leq T$, i.e. $(\theta, C) \in \mathscr{A}(v)$.*

Define the objective function as follows

$$\sup_{(\theta,C)\in\mathscr{A}(v)} E\left[\int_0^T e^{-rt} u(C_t)\mathrm{d}t\right] \tag{2.13}$$

Our goal is to find the optimal admissible strategy for consumption C_t and asset allocation θ_t but in the beginning we will introduce the state-price density function based on Radon-Nykodim derivatives [Pliska, 2001]. We can write the following equality

$$E_Q\left[\int_0^T e^{-rt} C_t \mathrm{d}t\right] = E\left[\int_0^T \zeta_t C_t \mathrm{d}t\right] \tag{2.14}$$

where $\zeta_t : [0,T] \times \Omega \to \mathbb{R}$ represents the state-price density and can be defined as product of the exponential stochastic process $\xi_t : [0,T] \times \Omega \to \mathbb{R}$ and the discount process

$$\zeta_t = e^{-rt}\xi_t, \quad \xi_t = e^{-\kappa W_t - \frac{\kappa^2}{2}t}, \quad \mathrm{d}\xi_t = -\kappa\xi_t \mathrm{d}W_t, \quad \kappa = \frac{\mu - r}{\sigma}, \quad \xi_0 = 1. \tag{2.15}$$

Formula 2.14 represents budget constraint for our optimization problem. We can rewrite our OP formulation as follows

$$\sup_{\substack{C\geq 0\\ \text{adapted}}} E\left[\int_0^T e^{-rt}u(C_t)\mathrm{d}t\right] \quad \text{subject to} \quad E\left[\int_0^T e^{-rs}\xi_s C_s \mathrm{d}s\right] = v. \tag{2.16}$$

Discussion: Let $M_1(v)$ be the supremum in formula (2.13) and let $M_2(v)$ be the supremum in (2.16). Let W_t^Q be the Q-Brownian motion.

Suppose $(\theta, C) \in \mathscr{A}(v)$ for some θ. Then

$$e^{-rt}X_t + \int_0^t e^{-rs}C_s \mathrm{d}s = v + \int_0^t \sigma\theta_s X_s e^{-rs}\mathrm{d}W_s^Q \tag{2.17}$$

is a local martingale under risk-neutral probability Q. Since this process is nonnegative, it will be a supermartingale under probability measure Q as well, so

$$E_Q\left[\int_0^T e^{-rs}C_s\mathrm{d}s\right] \leq E_Q\left[e^{-rT}X_T + \int_0^T e^{-rs}C_s\mathrm{d}s\right] \leq X_0 = v. \tag{2.18}$$

Therefore, $E\left[\int_0^T e^{-rt}u(C_t)\mathrm{d}t\right] \leq M_2(v') \leq M_2(v)$, where

$$v' = E_Q\left[\int_0^T e^{-rs}C_s\mathrm{d}s\right]. \tag{2.19}$$

So, we have proved that $M_1(v) \leq M_2(v)$.

Conversely, let $C \geq 0$ be adapted and $E_Q[\int_0^T e^{-rs} C_s \mathrm{d}s] = v$. Define X_t by

$$e^{-rt} X_t = E_Q\left[\int_t^T e^{-rs} C_s \mathrm{d}s | \mathcal{F}_t\right] \geq 0. \tag{2.20}$$

Then exists a stochastic process

$$Y_t = e^{-rt} X_t + \int_0^t e^{-rs} C_s \mathrm{d}s \geq 0 \tag{2.21}$$

that is a $Q-$ martingale, since it equals to $E_Q\left[\int_0^T e^{-rs} C_s \mathrm{d}s | \mathcal{F}_t\right]$. Therefore, the stochastic process

$$Y_t = v + \int_0^t H_s \mathrm{d}W_s^Q \tag{2.22}$$

for some adapted H, and we may define θ_s by $H_s = \sigma \theta_s X_s e^{-rs}$ since $X_s \geq 0$.

It follows from Ito's formula that

$$\mathrm{d}X_t = rX_t \mathrm{d}t - C_t \mathrm{d}t + \sigma \theta_t X_t \mathrm{d}W_t^Q \tag{2.23}$$

and, therefore, that $(\theta, C) \in \mathscr{A}(v)$. Finally, $E\left[\int_0^T e^{-rt} u(C_t) \mathrm{d}t\right] \leq M_1(v)$ and, so, $M_2(v) \leq M_1(v)$.

To solve this problem,we can now proceed as follows

1. Optimal consumption C_t^*.

   By using Lagrange multiplier $\alpha > 0$ we can write the problem as an unconstrained problem, and we can immediately find the optimal consumption

$$\sup_{C\geq 0} E[\mathcal{L}(C)] = \sup_{C\geq 0} E\left[\int_0^T e^{-rt}u(C_t)\mathrm{d}t - \alpha\left[\int_0^T e^{-rt}\xi_t C_t \mathrm{d}t - v\right]\right], \tag{2.24}$$

Find the optimal value of consumption C_t^*.

$$e^{-rt}u'(C_t^*) = \alpha e^{-rt}\xi_t \;\Rightarrow\; (C_t^*)^{-\gamma} = \alpha\xi_t \;\Rightarrow\; C_t^* = (\alpha\xi_t)^{-\frac{1}{\gamma}}. \tag{2.25}$$

We claim that $\exists\ \alpha > 0$ such that

$$E\left[\int_0^T e^{-rt}\xi_t C_t^* \mathrm{d}t\right] = v. \tag{2.26}$$

In order to find the Lagrange multiplier α that gives the optimal consumption C_t^*, plug expression (2.25) into the equation for the constraint.

$$\begin{aligned} E\left[\int_0^T e^{-rt}\xi_t(\alpha\xi_t)^{-\frac{1}{\gamma}}\mathrm{d}t\right] &= v \\ \alpha &= \left(\frac{1}{v}E\left[\int_0^T e^{-rt}\xi_t^{\frac{\gamma-1}{\gamma}}\,\mathrm{d}t\right]\right)^{\gamma}. \end{aligned} \tag{2.27}$$

Plug expression 2.15 into equation (2.27)

$$\alpha = \left(\frac{1}{v}E\left[\int_0^T e^{-rt}(e^{-\kappa W_t - \frac{\kappa^2}{2}t})^{\frac{\gamma-1}{\gamma}}\,\mathrm{d}t\right]\right)^{\gamma}.$$

For this particular case, we can try to get the analytical solution. Let us define a stochastic process $Y_t = e^{-rt + (-\kappa W_t - \frac{\kappa^2}{2}t)\frac{\gamma-1}{\gamma}} = e^{-(r+\frac{\kappa^2}{2}\frac{\gamma-1}{\gamma})t - \kappa\frac{\gamma-1}{\gamma}W_t}$.

Use Ito's formula

$$\mathrm{d}Y_t = \left\{\left(-r - \frac{\kappa^2}{2}\frac{\gamma-1}{\gamma} + \frac{1}{2}\kappa^2\left(\frac{\gamma-1}{\gamma}\right)^2\right)\mathrm{d}t - \kappa\frac{\gamma-1}{\gamma}\mathrm{d}W_t\right\}Y_t,$$

$$-r - \frac{\kappa^2}{2}\frac{\gamma-1}{\gamma} + \frac{1}{2}\kappa^2\left(\frac{\gamma-1}{\gamma}\right)^2 = -r - \frac{\kappa^2}{2\gamma} + \frac{\kappa^2}{2\gamma^2}.$$

Integrating will give us the following expression

$$Y_t = Y_0 + \left(-r - \frac{\kappa^2}{2}\frac{\gamma-1}{\gamma} + \frac{1}{2}\kappa^2\left(\frac{\gamma-1}{\gamma}\right)^2\right)\int_0^t Y_{t'}\mathrm{d}t' - \kappa\frac{\gamma-1}{\gamma}\int_0^t Y_{t'}\mathrm{d}W_{t'}.$$

Then take expectation and differentiate $\frac{\mathrm{d}}{\mathrm{d}t}$ the final expression

$$E[Y_t] = E[Y_0] - \left(r + \frac{\kappa^2}{2\gamma} - \frac{\kappa^2}{2\gamma^2}\right)\int_0^t E\left[Y_t'\right]\mathrm{d}t', \quad E[Y_0] = 1,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}E[Y_t] = -\left(r + \frac{\kappa^2}{2\gamma} - \frac{\kappa^2}{2\gamma^2}\right)E\left[Y_t\right], \quad E[Y_t] = e^{-\left(r+\frac{\kappa^2}{2\gamma}-\frac{\kappa^2}{2\gamma^2}\right)t}.$$

Finally,

$$\alpha = \frac{1}{v^\gamma}\left(\int_0^T e^{-\left(r+\frac{\kappa^2}{2\gamma}-\frac{\kappa^2}{2\gamma^2}\right)t}\mathrm{d}t\right)^\gamma \tag{2.28}$$
$$= \frac{1}{v^\gamma}\frac{1}{\left(r+\frac{\kappa^2}{2\gamma}-\frac{\kappa^2}{2\gamma^2}\right)^\gamma}\left(1 - e^{-\left(r+\frac{\kappa^2}{2\gamma}-\frac{\kappa^2}{2\gamma^2}\right)T}\right)^\gamma. \tag{2.29}$$

As the terminal time goes to infinity, i.e. $T \to \infty$, the exponential term in the previous equation 2.29 will vanish

$$\alpha = \frac{1}{v^\gamma}\left(\int_0^T e^{-\left(r+\frac{\kappa^2}{2\gamma}-\frac{\kappa^2}{2\gamma^2}\right)t}\mathrm{d}t\right)^\gamma = \frac{1}{v^\gamma}\frac{1}{\left(r+\frac{\kappa^2}{2\gamma}-\frac{\kappa^2}{2\gamma^2}\right)^\gamma}. \tag{2.30}$$

The final expression for the Lagrange multiplier α can be obtained more easily by using the following Note:

Note 2.2.1.

$$X = e^{\mu+\sigma Z} \tag{2.31}$$

This r.v. has the log-normal distribution with parameters μ and σ. Moments of the log-normal distribution will be

$$E[X^n] = e^{n\mu+n^2\sigma^2/2}. \tag{2.32}$$

2. Portfolio process and asset allocation

   The next step is to maximize utility and find the optimal asset allocation. According to our formulation, the wealth process X corresponding to an admissible pair (θ, C) and the initial wealth v is a solution to the linear stochastic differential equation 2.12 under the initial condition $v > 0$.

   Assume $M = e^{-rT}\xi_T X_T + \int_0^T e^{-rs}\xi_s C_s \mathrm{d}s$ is a integrable random variable. Construct a stochastic process

$$M_t^P = E^P\left[M|\mathcal{F}_t\right]. \tag{2.33}$$

   Then for $s < t$ using the tower property, we will get

$$E[M_t|\mathcal{F}_s] = E[E[M|\mathcal{F}_t]|\mathcal{F}_s] = E[M|\mathcal{F}_s] = M_s. \tag{2.34}$$

   So, M_t is a martingale w.r.t. $\mathcal{F}_t$.

Now, find the differential of wealth process $\mathrm{d}X_t$ using the following formulas

$$\mathrm{d}\xi_t = -\kappa\xi_t \mathrm{d}W_t,\ \kappa = \frac{\mu - r}{\sigma},\ X_t = \frac{e^{rt}}{\xi_t}\left(M_t - \int_0^t e^{-rs}\xi_s C_s \mathrm{d}s\right) \quad (2.35)$$

$$M_t = v + \int_0^t \psi_s \mathrm{d}W_s, \qquad \mathrm{d}M_t = \psi_t \mathrm{d}W_t,\ \psi_t = e^{-rt}\xi_t X_t \phi_t. \quad (2.36)$$

Compute the portfolio process using two approaches.

(a) Using the differential of $e^{-rt}X_t\xi_t$.

$$\begin{aligned}
\mathrm{d}(e^{-rt}X_t\xi_t) &= -re^{-rt}X_t\xi_t\mathrm{d}t + e^{-rt}X_t\mathrm{d}\xi_t + e^{-rt}\xi_t\mathrm{d}X_t \\
+e^{-rt}\mathrm{d}X_t\mathrm{d}\xi_t &= -re^{-rt}X_t\xi_t\mathrm{d}t + e^{-rt}\xi_t((r + \theta_t(\mu - r))X_t\mathrm{d}t \\
&\quad -e^{-rt}\xi_t C_t\mathrm{d}t + \theta_t\sigma e^{-rt}X_t\xi_t\mathrm{d}W_t - \kappa e^{-rt}X_t\xi_t\mathrm{d}W_t \\
-\kappa\theta_t\sigma e^{-rt}X_t\xi_t\mathrm{d}t &= -e^{-rt}\xi_t C_t\mathrm{d}t + \left(\theta_t\sigma - \frac{\mu - r}{\sigma}\right) e^{-rt}\xi_t X_t\mathrm{d}W_t.
\end{aligned}$$

Using formula (2.35) we can compute the same differential as follows

$$e^{-rt}X_t\xi_t = M_t - \int_0^t e^{-rs}\xi_s C_s\mathrm{d}s$$
$$\mathrm{d}(e^{-rt}X_t\xi_t) = \mathrm{d}M_t - e^{-rt}\xi_t C_t\mathrm{d}t = e^{-rt}\xi_t X_t\phi_t\mathrm{d}W_t - e^{-rt}\xi_t C_t\mathrm{d}t$$

$$\begin{aligned}
\left(\theta_t\sigma - \frac{\mu - r}{\sigma}\right) e^{-rt}\xi_t X_t &= e^{-rt}\xi_t X_t\phi_t \\
\phi_t + \kappa &= \theta_t\sigma.
\end{aligned}$$

(b) Using $\mathrm{d}X_t$

Recall the original wealth process (2.12) and, now, compute it based on formula (2.35). Define a new stochastic process $H_t = \int_0^t e^{-rs}\xi_s C_s \mathrm{d}s$. Then let $f(t,M_t,\xi_t,H_t)$ be a differentiable function, then

$$
\begin{aligned}
&\mathrm{d}f(t,M_t,\xi_t,H_t) = \\
&\frac{\partial f_t}{\partial t}\mathrm{d}t+\frac{\partial f_t}{\partial \xi_t}\mathrm{d}\xi_t+\frac{\partial f_t}{\partial M_t}\mathrm{d}M_t+\frac{\partial f_t}{\partial H_t}\mathrm{d}H_t+\frac{1}{2}\frac{\partial^2 f_t}{\partial \xi_t^2}\mathrm{d}\xi_t^2+\frac{1}{2}\frac{\partial^2 f_t}{\partial M_t^2}\mathrm{d}M_t^2 \\
&+\frac{\partial^2 f_t}{\partial \xi_t \partial M_t}\mathrm{d}\xi_t\mathrm{d}M_t = \frac{re^{rt}}{\xi_t}\left(M_t - H_t\right)\mathrm{d}t - \frac{e^{rt}}{\xi_t^2}\left(M_t - H_t\right)\mathrm{d}\xi_t \\
&+\frac{e^{rt}}{\xi_t}\mathrm{d}M_t - \frac{e^{rt}}{\xi_t}\mathrm{d}H_t + \frac{e^{rt}}{\xi_t^3}(M_t - H_t)(\mathrm{d}\xi_t)^2 - \frac{e^{rt}}{\xi_t^2}\mathrm{d}M_t\mathrm{d}\xi_t \\
&=\frac{re^{rt}}{\xi_t}\left(M_t-\int_0^t e^{-rs}\xi_s C_s\mathrm{d}s\right)\mathrm{d}t+\frac{e^{rt}}{\xi_t^2}\left(M_t-\int_0^t e^{-rs}\xi_s C_s\mathrm{d}s\right)\kappa\xi_t\mathrm{d}W_t \\
&+\frac{e^{rt}}{\xi_t}e^{-rt}\xi_t X_t\phi_t\mathrm{d}W_t-\frac{e^{rt}}{\xi_t}e^{-rt}\xi_t C_t\mathrm{d}t+\frac{e^{rt}}{\xi_t^3}\left(M_t-\int_0^t e^{-rs}\xi_s C_s\mathrm{d}s\right)\kappa^2\xi_t^2\mathrm{d}t \\
&+\frac{e^{rt}}{\xi_t^2}\kappa\xi_t e^{-rt}\xi_t X_t\phi_t\mathrm{d}t=\{(r + \kappa^2+\kappa\phi_t)X_t-C_t\}\mathrm{d}t+(\kappa+\phi_t)X_t\mathrm{d}W_t.
\end{aligned}
\tag{2.37}
$$

Compare (2.12) and (2.37)

- Compare $\mathrm{d}t$ coefficients

$$
\begin{gathered}
(\cancel{r} + \kappa^2 + \kappa\phi_t)\cancel{X_t} - \cancel{C_t} = (\cancel{r} + \theta_t(\mu - r))\cancel{X_t} - \cancel{C_t} \\
\kappa^2 + \kappa\phi_t = \theta_t(\mu - r),\ \kappa = \frac{\mu - r}{\sigma} \quad \Rightarrow \quad \kappa + \phi_t = \theta_t\sigma.
\end{gathered}
\tag{2.38}
$$

- Compare $\mathrm{d}W_t$ terms

$$
\kappa + \phi_t = \theta_t\sigma. \tag{2.39}
$$

We can find explicit expression for process ϕ_t and portfolio process θ_t.

$$\begin{aligned}
M_t =& E_t\left[\int_0^T e^{-rs}\xi_s C_s^* \mathrm{d}s\right] \\
=& E_t\left[\int_0^t e^{-rs}\xi_s(\alpha\xi_s)^{-\frac{1}{\gamma}}\mathrm{d}s\right] + E_t\left[\int_t^T e^{-rs}\xi_s(\alpha\xi_s)^{-\frac{1}{\gamma}}\mathrm{d}s\right] \\
=& \int_0^t e^{-rs}(\alpha)^{-\frac{1}{\gamma}}\xi_s^{\frac{\gamma-1}{\gamma}}\mathrm{d}s + E_t\left[\int_t^T e^{-rs}\xi_s^{\frac{\gamma-1}{\gamma}}(\alpha)^{-\frac{1}{\gamma}}\mathrm{d}s\right].
\end{aligned} \tag{2.40}$$

Here and further in the chapter, E_t represents conditional expectation with respect to the probability measure P, given the σ-algebra $\mathcal{F}_t$, i.e. $E_t[\cdot] = E[\cdot|\mathcal{F}_t]$. Now, let us find the differential of the martingale $\mathrm{d}M_t$. In the beginning we will find the stochastic term then, in the second part we will show that drift terms will cancel out.

- First part or intuitive approach.

  Recall the formula for the wealth process

$$e^{-rt}X_t\xi_t = M_t - \int_0^t e^{-rs}\xi_s C_s \mathrm{d}s. \tag{2.41}$$

  We know that M_t is a martingale. As a consequence, the $\mathrm{d}t$ terms will cancel out. So, we need to figure out what the $\mathrm{d}W_t$ term looks like. Assume that $\xi_s = \xi_t \frac{\xi_s}{\xi_t} < \infty$.

$$\begin{aligned}
\mathrm{d}M_t &= \mathrm{d}\left[\int_0^t e^{-rs}(\alpha)^{-\frac{1}{\gamma}} \xi_s^{\frac{\gamma-1}{\gamma}} \mathrm{d}s\right] + \mathrm{d}\left[E_t\left[\int_t^T e^{-rs} \xi_s^{\frac{\gamma-1}{\gamma}} (\alpha)^{-\frac{1}{\gamma}} \mathrm{d}s\right]\right] \\
&= (\cdots)\mathrm{d}t + \mathrm{d}\underbrace{\left[e^{-rt}\xi_t^{\frac{\gamma-1}{\gamma}} E_t\left[\int_t^T e^{-rs}e^{rt}\left(\frac{\xi_s}{\xi_t}\right)^{\frac{\gamma-1}{\gamma}} (\alpha)^{-\frac{1}{\gamma}} \mathrm{d}s\right]\right]}_{\Rightarrow \text{ from 2.41 } e^{-rt}\xi_t X_t =}.
\end{aligned}$$

Since the Brownian motion has independent increments, we can rewrite the second term expression using new notation $g(t)$

$$g(t) = e^{rt} E\left[\int_t^T e^{-rs}\left(\frac{\xi_s}{\xi_t}\right)^{\frac{\gamma-1}{\gamma}} (\alpha)^{-\frac{1}{\gamma}} \mathrm{d}s\right] \tag{2.42}$$

then the expression for the differential becomes

$$\mathrm{d}M_t = \mathrm{d}\left[e^{-rt}\xi_t^{\frac{\gamma-1}{\gamma}} g(t)\right] = \frac{\gamma-1}{\gamma} e^{-rt} \xi_t^{-\frac{1}{\gamma}} g(t)\mathrm{d}\xi_t = -\kappa\frac{\gamma-1}{\gamma} e^{-rt}\xi_t^{\frac{\gamma-1}{\gamma}} g(t)\mathrm{d}W_t \tag{2.43}$$

and by definition (2.41) of the wealth process X_t we can define it as follows: $X_t = g(t)\xi_t^{-\frac{1}{\gamma}}$. Finally, the expression (2.43) will be

$$\mathrm{d}M_t = -\kappa\frac{\gamma-1}{\gamma} e^{-rt}\xi_t X_t \mathrm{d}W_t. \tag{2.44}$$

- Second part or drift terms $\mathrm{d}t$ calculations.

  Here we need to show how the $\mathrm{d}t$ terms cancel out. We will use the same representation $\xi_s = \xi_t \frac{\xi_s}{\xi_t} < \infty$.

$$M_t = \int_0^t e^{-rs}(\alpha)^{-\frac{1}{\gamma}} \xi_s^{\frac{\gamma-1}{\gamma}} \mathrm{d}s + E_t\left[\int_t^T e^{-rs}\xi_s(\alpha\xi_s)^{-\frac{1}{\gamma}} \mathrm{d}s\right]. \tag{2.45}$$

Using the intuitive approach we derived the $(\cdot)\mathrm{d}W_t$ term. Now let's show that $\mathrm{d}t$ terms cancel out.

$$\mathrm{d}M_t = e^{-rt}\xi_t C_t^*\mathrm{d}t + \mathrm{d}\left[e^{-rt}g(t)\xi_t^{\frac{\gamma-1}{\gamma}}\right] = e^{-rt}(\alpha)^{-\frac{1}{\gamma}}\xi_t^{\frac{\gamma-1}{\gamma}}\mathrm{d}t$$
$$-re^{-rt}g(t)\xi_t^{\frac{\gamma-1}{\gamma}}\mathrm{d}t + e^{-rt}g'(t)\xi_t^{\frac{\gamma-1}{\gamma}}\mathrm{d}t + \frac{1}{2}e^{-rt}g(t)\frac{\gamma-1}{\gamma}$$
$$\times\left(-\frac{1}{\gamma}\right)\xi_t^{-\frac{1}{\gamma}-1}(\mathrm{d}\xi_t)^2 + (\cdots)\mathrm{d}W_t. \quad (2.46)$$

First, compute explicitly $g(t)$ and it's derivative $g'(t)$.

$$g(t) = (\alpha)^{-\frac{1}{\gamma}}E\left[\int_t^T e^{-r(s-t)}\left(\frac{\xi_s}{\xi_t}\right)^{\frac{\gamma-1}{\gamma}}\mathrm{d}s\right]. \quad (2.47)$$

First, let us define a stochastic process

$$Y_{s-t} = e^{-r(s-t)+\left(-\kappa(W_s-W_t)-\frac{\kappa^2}{2}(s-t)\right)\frac{\gamma-1}{\gamma}}. \quad (2.48)$$

Find the differential of this process

$$\mathrm{d}Y_{s-t} = \left(-r - \frac{\kappa^2}{2}\frac{\gamma-1}{\gamma}\right)e^{(\cdot)}\mathrm{d}s + \frac{1}{2}\kappa^2\left(\frac{\gamma-1}{\gamma}\right)^2 e^{(\cdot)}\mathrm{d}s + (\cdots)\mathrm{d}W_s$$
$$\left(-r - \frac{\kappa^2}{2\gamma}\right)e^{(\cdot)}\mathrm{d}s + (\cdots)\mathrm{d}W_s.$$

Then we integrate, take expectations. As a consequence, the stochastic integral will disappear and we differentiate again. Finally, we will get

$$\frac{\mathrm{d}}{\mathrm{d}s}E[Y_{s-t}] = (-r - \frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma})E[Y_{s-t}] \quad (2.49)$$

and the solution will be the following

$$g(t) = (\alpha)^{-\frac{1}{\gamma}} \int_t^T e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(s-t)}\mathrm{d}s$$

$$= \frac{(\alpha)^{-\frac{1}{\gamma}}}{-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}} \left\{ e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)} - 1 \right\}. \tag{2.50}$$

The derivative will be the following $g'(t)$

$$g'(t) = -e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)}(\alpha)^{-\frac{1}{\gamma}}. \tag{2.51}$$

Continuing with 2.46 and replacing $g(t)$ and $g'(t)$ with (2.50) and (2.51)

$$
\begin{aligned}
&= e^{-rt}(\alpha)^{-\frac{1}{\gamma}}\xi_t^{\frac{\gamma-1}{\gamma}}\Bigg[1 - r\frac{1}{-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}}\left\{e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)} - 1\right\} \\
&-e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)} - \frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\frac{1}{-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}}\left\{e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)} - 1\right\}\Bigg]\mathrm{d}t \\
&+(\cdots)\mathrm{d}W_t = e^{-rt}(\alpha)^{-\frac{1}{\gamma}}\xi_t^{\frac{\gamma-1}{\gamma}}\left[1 - e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)} + \left(e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)} - 1\right)\right]\mathrm{d}t \\
&+(\cdots)\mathrm{d}W_t = e^{-rt}(\alpha)^{-\frac{1}{\gamma}}\xi_t^{\frac{\gamma-1}{\gamma}} \cdot 0 \cdot \mathrm{d}t + (\cdots)\mathrm{d}W_t = (\cdots)\mathrm{d}W_t.
\end{aligned}
$$

The rest of the calculations are the same as in the previous case. So, using formula (2.36) we can write

$$\mathrm{d}M_t = e^{-rt}\xi_t X_t \phi_t \mathrm{d}W_t, \tag{2.52}$$

$$\phi_t = -\kappa\frac{\gamma-1}{\gamma} \quad \Rightarrow \quad \theta_t = \frac{\kappa}{\sigma\gamma}. \tag{2.53}$$

3. Maximizing CRRA utility function

Now, compute the maximum of the utility function. We chose a CRRA utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$. In this case we can get an explicit formula for maximal utility

$$E\left[\int_0^T e^{-rt}u(C^*)\mathrm{d}t\right] = E\left[\int_0^T e^{-rt}u((\alpha\xi_t)^{-\frac{1}{\gamma}})\mathrm{d}t\right] \tag{2.54}$$

$$= \frac{1}{1-\gamma}E\left[\int_0^T e^{-rt}(\alpha\xi_t)^{\frac{\gamma-1}{\gamma}}\mathrm{d}t\right]. \tag{2.55}$$

Our constraint is the following

$$v = E\left[\int_0^T e^{-rt}\xi_t C^*\mathrm{d}t\right] = E\left[\int_0^T e^{-rt}\xi_t(\alpha\xi_t)^{-\frac{1}{\gamma}}\mathrm{d}t\right] \tag{2.56}$$

$$= E\left[\int_0^T e^{-rt}(\alpha)^{-1}(\alpha\xi_t)^{\frac{\gamma-1}{\gamma}}\mathrm{d}t\right] = \frac{1}{\alpha}E\left[\int_0^T e^{-rt}(\alpha\xi_t)^{\frac{\gamma-1}{\gamma}}\mathrm{d}t\right] \tag{2.57}$$

$$E\left[\int_0^T e^{-rt}(\alpha\xi_t)^{\frac{\gamma-1}{\gamma}}\mathrm{d}t\right] = \alpha v. \tag{2.58}$$

So, based on the last two equations (2.55) and (2.58) we can get the expression for the maximal utility

$$E\left[\int_0^T e^{-rt}u(C^*)\mathrm{d}t\right] = \frac{\alpha v}{1-\gamma}. \tag{2.59}$$

Special case $\mu = r$

In case the drift equals to risk-free rate, i.e. $\mu = r$, then the solution can be simplified significantly. The objective function will be

$$E\left[\int_0^T e^{-rt}u(C_t^*)\mathrm{d}t\right] \tag{2.60}$$

where $C_t^* = (\alpha\xi_t)^{-\frac{1}{\gamma}}$. Then plug expression for optimal consumption into (2.60)

$$E\left[\int_0^T e^{-rt}u((\alpha\xi_t)^{-\frac{1}{\gamma}})\mathrm{d}t\right] = \frac{1}{1-\gamma}E\left[\int_0^T e^{-rt}(\alpha\xi_t)^{\frac{\gamma-1}{\gamma}}\mathrm{d}t\right]$$

$$= \frac{(\alpha)^{\frac{\gamma-1}{\gamma}}}{1-\gamma}\int_0^T e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)t}\mathrm{d}t = \frac{(\alpha)^{\frac{\gamma-1}{\gamma}}}{1-\gamma}\frac{1}{r+\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}}\left(1-e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)T}\right). \quad (2.61)$$

As time $T\to\infty$ the exponential term goes to 0, i.e. $\lim\limits_{T\to\infty} e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)T} = 0$. Equation (2.61) can be simplified

$$E\left[\int_0^T e^{-rt}u(C_t^*)\mathrm{d}t\right] = \frac{(\alpha)^{\frac{\gamma-1}{\gamma}}}{1-\gamma}\frac{1}{r+\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}} = \frac{(\alpha)^{\frac{\gamma-1}{\gamma}}}{1-\gamma}\frac{2\gamma^2}{2r\gamma^2+\kappa^2(\gamma-1)}. \quad (2.62)$$

α, r, γ are constants. If $\mu\to r$ it means that $\kappa = \frac{\mu-r}{\sigma}$ decreases and therefore the ratio in equation (2.62) will increase. So, the value function will increase as μ approaches to risk-free rate. In this case, the portfolio process will be very simple $\mathrm{d}X_t = \{rX_t - C_t\}\mathrm{d}t + \sigma\theta_t X_t \mathrm{d}W_t$. The wealth process under risk-neutral probability and real world probability will be the same. In our case $\mu = r = const$, as a consequence, our objective function is

$$\sup_{C\geq 0}\int_0^T e^{-rt}u(C_t)\mathrm{d}t \quad \text{subject to} \quad \int_0^T e^{-rs}C_s\mathrm{d}s = v \quad (2.63)$$

where v is an initial wealth. We do not have expectations here because everything is deterministic. Construct a Lagrangian with multiplier $\alpha > 0$

$$\sup_{C\geq 0}\left\{\int_0^T e^{-rt}u(C_t)\mathrm{d}t - \alpha\left[\int_0^T e^{-rt}C_t\mathrm{d}t - v\right]\right\}. \quad (2.64)$$

Consumption can be computed using the general formula $C_t^* = (\alpha\xi_t)^{-\frac{1}{\gamma}}$. As $\mu \to r$ we can see that $\lim_{\mu\to r}(\alpha\xi_t)^{-\frac{1}{\gamma}} = (\alpha)^{-\frac{1}{\gamma}}$ and we can find the optimal value of consumption explicitly $c_t^* = (\alpha)^{-\frac{1}{\gamma}}$. Now, let's see what happens with the asset allocation θ_t when the drift μ approaches to risk-free rate r while holding σ constant. Using formula (2.53), we can conclude that, as soon as $\mu \to r$ asset allocation will go to zero, i.e. $\lim_{\mu\to r} \frac{\kappa}{\sigma\gamma} = 0$. Then we can find that $\theta^* = 0$ which coincides with other results [Rogers, 2013]. Also the relationship between optimal consumption and the portfolio process can be found using formulas for optimal consumption, (2) and (2.50). Let's rewrite them explicitly once again

$$g(t) = \frac{(\alpha)^{-\frac{1}{\gamma}}}{-r - \frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}} \cancelto{-1}{\left\{ e^{\left(-r-\frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}\right)(T-t)} - 1 \right\}}_{as\ T\to\infty}. \tag{2.65}$$

And as $\mu \to r$ the coefficient $\kappa \to 0$ and we can write the previous expression (2.65) as follows

$$\lim_{\mu\to r} g(t) = \lim_{\mu\to r} \frac{(\alpha)^{-\frac{1}{\gamma}}}{r + \frac{\kappa^2}{2\gamma}\frac{\gamma-1}{\gamma}} = \frac{(\alpha)^{-\frac{1}{\gamma}}}{r}. \tag{2.66}$$

Using the same chain of reasoning as in the previous paragraph $X_t = g(t)\xi_t^{-\frac{1}{\gamma}}$ and the optimal consumption will be $\lim_{T\to\infty} c^* = rX_t$. This coincides with the result in the book [Rogers, 2013]. In order to find the Lagrange multiplier α that corresponds to the optimal consumption c_t^*, plug the equation for optimal consumption into the equation for constraint (2.63).

$$\int_0^T e^{-rs}(\alpha)^{-\frac{1}{\gamma}} \mathrm{d}s = v \to \alpha = \left(\frac{1}{v}\int_0^T e^{-rt}\mathrm{d}t\right)^{\gamma} = \left(\frac{1}{rv}(1 - e^{-rT})\right)^{\gamma}. \tag{2.67}$$

As the terminal time goes to infinity, i.e. $T \to \infty$ the exponential term will vanish and we will get

$$\alpha = \left(\frac{1}{v}\int_0^T e^{-rt}\mathrm{d}t\right)^\gamma = \left(\frac{1}{rv}\right)^\gamma. \tag{2.68}$$

If we plug the formula for CRRA utility function into the objective function, we can get the maximal utility

$$\int_0^T e^{-rs}u(c^*)\mathrm{d}s = \int_0^T e^{-rs}u((\alpha)^{-\frac{1}{\gamma}})\mathrm{d}s = \frac{v^{1-\gamma}}{(1-\gamma)r^\gamma}(1-e^{-rT})^\gamma. \tag{2.69}$$

As $T \to \infty$ utility will be $\frac{v^{1-\gamma}}{(1-\gamma)r^\gamma}$ which coincides with results obtained by other researchers (see [Rogers, 2013]).

A different limit would be to hold K constant, so $\sigma \to 0$ when $\mu \to r$. In that case the optimal consumption C_t^* does not change, but $\theta_t^* \to \infty$.

2.2.2 Merton problem with mortality

We consider the same wealth dynamics as in the previous case (2.12) but in the objective function we incorporate a probability of survival based on Gompertz law of mortality.

$$\sup_{\theta_t, C_t} E\left[\int_0^T e^{-\rho s}\, {}_sp_x u(C_s)\mathrm{d}s | X_0 = v\right], \quad {}_sp_x = e^{-\int_0^s \lambda_{x+q}\mathrm{d}q}. \tag{2.70}$$

Here λ is the biological hazard rate $\lambda_{x+q} = \frac{1}{b}e^{(x+q-m)/b}$ where m is the modal value of life, since the parameter $\lambda_0 = 0$ (see p47, [Milevsky, 2006]), b is the dispersion coefficient of the future lifetime random variable. Since m, b and

x are constants we can compute the probability of survival explicitly, using

$$\int_0^s \lambda_{x+q}\mathrm{d}q = \int_0^s \frac{1}{b}e^{\frac{x+q-m}{b}}\mathrm{d}q = \frac{1}{b}e^{\frac{x-m}{b}}\int_0^s e^{\frac{q}{b}}\mathrm{d}q = e^{\frac{x-m}{b}}\left(e^{\frac{s}{b}}-1\right). \quad (2.71)$$

Then, the objective function becomes

$$\sup_{(\theta,C)\in\mathscr{A}(v)} E\left[\int_0^T e^{-\rho s}e^{e^{\frac{x-m}{b}}(1-e^{\frac{s}{b}})}u(C_s)\mathrm{d}s\right]. \quad (2.72)$$

Let us denote $f(s) = -\rho s + e^{\frac{x-m}{b}}(1-e^{\frac{s}{b}})$. Finally, the objective function will be rewritten as follows

$$\sup_{(\theta,C)\in\mathscr{A}(v)} E\left[\int_0^T e^{f(s)}u(C_s)\mathrm{d}s\right]. \quad (2.73)$$

The budget constraint remains the same as in the previous case, namely

$$E\left[\int_0^T \zeta_s C_s \mathrm{d}s\right] = v, \quad \zeta_t = e^{-rt}\xi_t, \quad \xi_t = e^{-\kappa W_t - \frac{\kappa^2}{2}t}, \quad \kappa = \frac{\mu - r}{\sigma}. \quad (2.74)$$

The unconstrained optimization problem will be

$$\sup_{\substack{C_t\geq 0\\ \text{adapted}}} E\left[\int_0^T e^{f(s)}u(C_s)\mathrm{d}s - \alpha\left[\int_0^T \zeta_s C_s\mathrm{d}s - v\right]\right]. \quad (2.75)$$

Optimal consumption for this problem will have the following form

$$C_t^* = (\alpha\zeta_t e^{-f(t)})^{-\frac{1}{\gamma}}. \quad (2.76)$$

We compute the Lagrange multilplier α using formulas (2.74) and (2.76)

$$E\left[\int_0^T \zeta_s(\alpha\zeta_s e^{-f(s)})^{-\frac{1}{\gamma}}\mathrm{d}s\right] = v. \tag{2.77}$$

Construct a stochastic process $Y_t = e^{\frac{f(t)}{\gamma}+\frac{\gamma-1}{\gamma}(-rt-\kappa W_t-\frac{\kappa^2}{2}t)}$. Then find the differential $\mathrm{d}Y_t$

$$\mathrm{d}Y_t = \left(\left\{\frac{1}{\gamma}\left(-\rho-\frac{1}{b}e^{\frac{x-m}{b}+\frac{t}{b}}\right)-r\frac{\gamma-1}{\gamma}-\frac{\kappa^2}{2}\frac{\gamma-1}{\gamma}\right\}\mathrm{d}t-\kappa\frac{\gamma-1}{\gamma}\mathrm{d}W_t\right)Y_t.$$

$$Y_t = Y_0 + \int_0^t \left\{\frac{1}{\gamma}\left(-\rho-\frac{1}{b}e^{\frac{x-m}{b}+\frac{s}{b}}\right)-r\frac{\gamma-1}{\gamma}-\frac{\kappa^2}{2}\frac{\gamma-1}{\gamma}\right\}Y_s\mathrm{d}s. \tag{2.78}$$

Take expectations and simplify the previous expression

$$E[Y_t] = E[Y_0] + \int_0^t \left\{\frac{1}{\gamma}\left(-\rho-\frac{1}{b}e^{\frac{x-m}{b}+\frac{t'}{b}}\right)-r\frac{\gamma-1}{\gamma}-\frac{\kappa^2}{2}\frac{\gamma-1}{\gamma}\right\}E\left[Y_{t'}\right]\mathrm{d}t'.$$

Then differentiate $\frac{\mathrm{d}}{\mathrm{d}t}$

$$\frac{\mathrm{d}}{\mathrm{d}t}E[Y_t] = \left\{\frac{1}{\gamma}\left(-\rho-\frac{1}{b}e^{\frac{x-m}{b}+\frac{t'}{b}}\right)-r\frac{\gamma-1}{\gamma}-\frac{\kappa^2}{2}\frac{\gamma-1}{\gamma}\right\}E\left[Y_{t'}\right]. \tag{2.79}$$

$$\alpha = \frac{1}{v^\gamma}\left(\int_0^T e^{-\frac{1}{\gamma}e^{\frac{x-m}{b}}(e^{\frac{s}{b}}-1)-(\frac{\rho}{\gamma}+(r+\frac{\kappa^2}{2})\frac{\gamma-1}{\gamma})s}\mathrm{d}s\right)^\gamma. \tag{2.80}$$

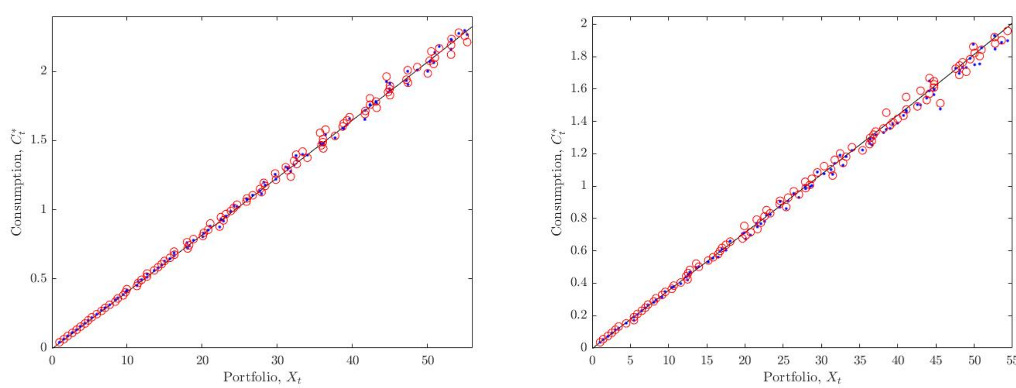
2.2.3 Numerical Results

Since this case is relatively simple and implies the existence of an analytical solution we can compare the analytical solution with the numerical results obtained using the PDE and martingale approaches of Section 2.2.1.Two sets of parameters were tested:1) risk-free rate $r = 0.02$, drift $\mu = 0.02$, subjective

Table 2.1: Numerical results for the first case study $\mu = 0.02$, $r = 0.02$.

# of time nodes	Mean Value	Theoretical value	Absolute Error
200	8.9	8.89699	0.00364
1000	8.89269	8.89197	0.00072
10000	8.89527	8.89520	0.00007

discount rate $\rho = 0.02$ and 2) $\mu = 0.08$ and $r = 0.02$. Volatility $\sigma = 0.16$ and risk aversion parameters $\gamma = 3$ were fixed. We also define the time interval as follows: $t_n = (n-1)\Delta t, t \in [0,T]$, the Brownian motion increments $\Delta W_t = Z\sqrt{\Delta t}, \quad Z \sim N(0,1)$. The numerical results are represented in the Table 2.1 for the first case and in the Table 2.2 for the second case.



(a) Finite-time horizon (b) Infinite-time horizon

Figure 2.2: Numerical solution of Merton problem. Comparison between the PDE solution (the black solid line), the probabilistic numerical solution (the blue dots) and the analytical solution (the red circles). Parameters for this model are $\eta = 0.0, \mu = 0.08, r = \rho = 0.02, \gamma = 3, \sigma = 0.16$.

Table 2.2: Numerical results for the second case study $\mu = 0.08$, $r = 0.02$.

# of MC samples	Portfolio Value	Theoretical value	Absolute Error
1000	9.9208	10.1127	0.1919
5000	9.7202	9.6457	0.07453
10000	10.1725	10.1936	0.0211

As we can see from the table, the numerical results have an excellent agreement with the analytical solution for both cases. In addition, in Figure 2.2 we represent the Merton problem solution for two cases, finite time horizon and infinite time horizon. The black solid line represents the PDE solution, the blue dots the Monte Carlo solution and the red circles represent the analytical solution. For both cases all three solutions show good agreement.

2.3 Consumption optimization problem under a habit formation model.

As we described at the beginning, we are going to solve an optimization problem under the habit formation model. The main idea of this approach is to take into account the client's living standard or habit $\bar{C}_t$. We consider two cases: the first one without pension income and the other one with the pension income π, which remains constant. We choose a CRRA multiplicative utility function u that depends on the habit $\bar{C}$ and our goal is to explore the martingale approach to a consumption strategy for retirement.

2.3.1 Optimal solution vs. greedy heuristic solution

As we will see in the next paragraph the approximation we obtain by using the martingale approach gives different results in comparison with the PDE solution from the previous chapter. In this section we will discuss what the difference between finding the optimal solution and our approach in this chapter will be. We will show that the numerical solution that we found is a greedy optimal. Let us start by formulating the global optimization problem.

- General formulation.

  First, we formulate our optimization problem for the case that implies a global optimal solution. Define the utility function $\mathcal{U}(C,H)$ as follows

$$\mathcal{U}(C,H) = E\left[\int_0^\infty e^{-\rho s}\,{}_sp_x u\left(\frac{C_s}{H_s}\right)\mathrm{d}s\right] \tag{2.81}$$

  where (θ,C) is a pair asset allocation - consumption stream from the admissible set $\mathscr{A}(v)$ (see Definition 2.2.1) and $H = \mathscr{H}(C,\bar{c})$ is the habit, $\mathscr{H}(C,\bar{c})$ associated to the consumption stream C and initial habit $\bar{c}$.

  As in Section 2.2.1 this is equivalent to maximizing $\mathcal{U}(C,H)$ over adapted C_t satisfying the budget constraint

$$E\left[\int_0^\infty \zeta_s C_s \mathrm{d}s\right] = v, \tag{2.82}$$

  together with $H = \mathscr{H}(C,\bar{c})$. Here $\zeta_t = e^{-rt}e^{-\kappa W_t - \frac{\kappa^2}{2}t}$, $\kappa = \frac{\mu - r}{\sigma}$ is the state price density, $C_t \geq 0$, $H_t \geq 0$ adapted consumption and habit streams respectively. In other words, we can write the global optimum problem for a given pair of initial values $(v,\bar{c})$ as follows

$$V = \sup\{\mathcal{U}(C,H) : E\left[\int_0^\infty \zeta_s C_s \mathrm{d}s\right] = v, H = \mathscr{H}(C,\bar{c})\}. \tag{2.83}$$

- Greedy formulation.

  As we mentioned before, in this chapter we apply a greedy heuristic algorithm which leads to a locally optimal solution. This is one of the simplest algorithms to implement. The main idea of the greedy algorithm is that it always chooses which element of a set seems to be the best at the moment. In general, a greedy algorithm may or may not lead to a globally optimal solution. There are cases where you can always get a globally optimal solution, for example, this is the case when we deal with so-called matroids. In our case, we were able to get locally optimal solution. So, we find an admissible consumption stream C_t^* that satisfies the budget constraint (2.82), such that

  $$\mathcal{U}(C^*,H) = \sup\{\mathcal{U}(C,H) : E\left[\int_0^\infty \zeta_s C_s \mathrm{d}s\right] = v, H = \mathscr{H}(C^*,\bar{c})\}. \tag{2.84}$$

  We say that such a consumption stream C_t is locally optimal, or a greedy optimum. In case the consumption C^* is locally optimal then the following inequality holds $\mathcal{U}(C^*,H) \leq V$. The following theorem will show why we are able to get only locally optimal solution for our problem.

Theorem 2.3.1. *Suppose there exists an adapted consumption stream $C_t^* \geq 0$ and Lagrange multiplier $\alpha > 0$ such that the following equality* *holds*

$$C_t^* = H_t^{\frac{\gamma-1}{\gamma}} \left(\alpha e^{-f(t)}\zeta_t\right)^{-\frac{1}{\gamma}} \text{ where } \quad f(t)=-\rho t+e^{\frac{x-m}{b}}\left(1-e^{\frac{t}{b}}\right), \quad (2.85)$$

$H = \mathscr{H}(C^*, \bar{c})$ habit associated with this consumption stream C^*, and, also C_t^* satisfies the budget constraint (2.82). Then this consumption stream C_t^* is a greedy optimum.

Proof. Let C be any adapted consumption stream which satisfies the budget constraint (2.82) with $C_s \geq 0\ \forall s$. First, let's consider a simplified problem. Suppose that we want to maximize the following function

$$\int_0^\infty e^{-\rho s}\,{}_s p_x u\left(\frac{C_s}{H_s}\right) \mathrm{d}s \quad \text{s.t.} \quad \int_0^\infty \zeta_s C_s \mathrm{d}s = v. \quad (2.86)$$

We use the Lagrange multiplier method, the idea of which is that in order to find the maximum of our function, we construct an auxillary function, the Lagrangian, which transforms our constrained problem into an unconstrained one. Since we are fixing the habit H_t, our problem will be greedy rather than optimal. To be more precise, let us write the Lagrangian for this problem explicitly

$$\mathcal{L}(C,H) \equiv \int_0^\infty e^{-\rho s}\,{}_s p_x u\left(\frac{C_s}{H_s}\right) \mathrm{d}s - \alpha\left(\int_0^\infty \zeta_s C_s \mathrm{d}s - v\right). \quad (2.87)$$

Then, in order to find the global maximum we should find a stationary point for the Lagrange function, which means that all partial derivatives of Lagrangian are equal to zero. To solve this problem the following condition for gradient should be applied:

$$\nabla \mathcal{L}(C,H) = 0. \tag{2.88}$$

For the existence of a global maximum, this is a necessary but not sufficient condition. Since we maximize the utility function for fixed habit H we did not find conditions when the partial derivative over habit equlas to zero.

Simple calculus shows that

$$\begin{aligned}
&\int_0^\infty e^{-\rho s} {}_s p_x u\left(\frac{C_s}{H_s}\right) \mathrm{d}s - \alpha\left(\int_0^\infty \zeta_s C_s \mathrm{d}s - v\right) \\
\leq &\int_0^\infty e^{-\rho s} {}_s p_x u\left(\frac{C_s^*}{H_s}\right) \mathrm{d}s - \alpha\left(\int_0^\infty \zeta_s C_s^* \mathrm{d}s - v\right),
\end{aligned} \tag{2.89}$$

for C_s^* and habit H_s as in the statement of the Theorem 2.3.1. Taking expectations, we see that

$$\begin{aligned}
&\mathcal{U}(C,H) - \alpha\left(E\left[\int_0^\infty \zeta_s C_s \mathrm{d}s\right] - v\right) \\
\leq &\mathcal{U}(C^*,H) - \alpha\left(E\left[\int_0^\infty \zeta_s C_s^* \mathrm{d}s\right] - v\right) \\
= &\mathcal{U}(C^*,H)
\end{aligned} \tag{2.90}$$

because C^* satisfies formula 2.82. Therefore, if C_s also satisfies the budget constraint 2.82, we can conclude that $\mathcal{U}(C,H) \leq \mathcal{U}(C^*,H)$. $\square$

The greedy optimum therefore has the feature that we may compute it, using the formula of theorem 2.3.1 for any given value of the Lagrange multiplier

α, and then iterating the choice of α to satisfy the budget constraint. But there is no reason why the consumption C_s that maximizes the utility function $\mathcal{U}(C,H)$ for a fixed choice of habit H, should also maximize the utility $\mathcal{U}(C,\mathscr{H}(C,\bar{c}))$.

Therefore these methods are not expected to produce the globally optimal solution for consumption C_s. But we can expect them to be close to optimal, at least in some cases. Exploring when that is true is the purpose of this chapter.

2.3.2 Retirement spending problem without exogenous income

As in the Merton problem, we adopt a model that consists of risk-free investments (bonds) as well as risky assets, with the difference that now we have to take into account the living standard of a retiree. Suppose that the wealth process X_t, corresponding to the admissible pair (θ, C) and the initial wealth v, follows the dynamics

$$\mathrm{d}X_t = [\theta_t(\mu - r) + r]X_t\mathrm{d}t + \theta_t\sigma X_t\mathrm{d}W_t - C_t^*\mathrm{d}t \tag{2.91}$$

$$\mathrm{d}\bar{C}_t^* = \eta(C_t^* - \bar{C}_t^*)\mathrm{d}t, \quad \bar{C}_0^* = \bar{c} \tag{2.92}$$

and satisfies the constraint $X_0 = v > 0$. This is a linear stochastic differential equation (2.91), where μ is the drift, r is the risk-free rate, σ is the volatility. The second equation represents habit dynamics, where η is a parameter that represents how fast the client's living standard reacts to changes in consumption and $\bar{c}$ is the given initial value of habit. W_t is a geometric Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, where Ω is a sample space, $\mathcal{F}$ a $\sigma-$field of subsets of Ω and P defines the probability for each event. Next we define the client's objective function as follows

$$\sup_{(\theta,C)\in\mathscr{A}(v)} E\left[\int_0^T e^{-\rho s} u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s | \bar{C}_0 = \bar{c}\right] \tag{2.93}$$

where ρ is the subjective discount rate, ζ_t is the state-price density defined in the previous paragraph (2.15) and T is the terminal time. We also have two control variables: θ_t, the portion of wealth invested in risky assets, and C_t, consumption obtained by using the greedy policy algorithm. In the greedy formulation habit $\bar{C}_t$ is viewed as fixed while taking the above supremum, i.e. without imposing condition 2.92. Whereas in the global optimization problem we impose (2.92) This means that at every time moment a retiree can decide how much to invest in risky assets and how much to consume. As remarked ealier, we may requiring $C \geq 0$ to be adapted and to satisfy the budget constraint (2.82). In order to solve this optimization problem, we use a Lagrangian multiplier $\alpha > 0$ and, therefore, we can rewrite our greedy optimization problem as an unconstrained problem

$$\sup_{\substack{C_t \geq 0 \\ \text{adapted}}} E\left[\int_0^T e^{-\rho s} u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s - \alpha\left[\int_0^T \zeta_s C_s \mathrm{d}s - v\right]\right].$$

Consumption C_t^* can be found from formula 2.3.2

$$C_t^* = \left(\bar{C}_t^*\right)^{\frac{\gamma-1}{\gamma}} \left(\alpha\zeta_t e^{\rho t}\right)^{-\frac{1}{\gamma}}. \tag{2.94}$$

In this case C^* represents the consumption obtained by using the greedy policy and for a given α, we will see how to choose $\bar{C}_t$ to satisfy both conditions, 2.94 and 2.92. In order to compute the Lagrange multiplier α we then plug the expression for the optimal consumption (2.94) into constraint (2.93).

$$\alpha = \frac{1}{v^\gamma}\left(E\left[\int_0^T e^{-\frac{\rho s}{\gamma}}(\zeta_s \bar{C}_s^*)^{\frac{\gamma-1}{\gamma}} \mathrm{d}s\right]\right)^\gamma . \tag{2.95}$$

In this case there is no closed form solution for α and we will use Monte Carlo and iteration methods.

The next step is to get an explicit expression for the living standard $\bar{C}_t^*$ (2.92). In order to do this we will plug the expression for the consumption (2.94) into (2.92) and solve it. The differential equation we want to solve is the following

$$(\bar{C}_s^*)' + \eta \bar{C}_s^* - \eta f(s)(\bar{C}_s^*)^k = 0, \quad \text{where} \quad f(s) = (\alpha \zeta_s e^{\rho s})^{-\frac{1}{\gamma}}, \quad k = \frac{\gamma - 1}{\gamma}. \tag{2.96}$$

This is the Bernoulli's equation which can be solved with a standard substitution

$$\bar{C}_s^* = e^{-\eta s}\left(\frac{\eta}{\gamma}\int_t^s f(t')e^{\frac{\eta}{\gamma}t'}\mathrm{d}t' + (\bar{C}_t^*)^{\frac{1}{\gamma}} e^{\frac{\eta t}{\gamma}}\right)^\gamma \tag{2.97}$$

Let us rewrite this formula in a more convenient form

$$\zeta_t \bar{C}^*(s-t) = \left(\frac{\eta}{\gamma}\int_t^s \left(\alpha e^{\rho t'}\right)^{-\frac{1}{\gamma}} \left(\frac{\zeta_{t'}}{\zeta_t}\right)^{-\frac{1}{\gamma}} e^{\frac{\eta}{\gamma}(t'-s)}\mathrm{d}t' + \left(e^{-\eta(s-t)}\zeta_t \bar{C}_t^*\right)^{\frac{1}{\gamma}}\right)^\gamma \tag{2.98}$$

where $\bar{C}^*(s-t) = \bar{C}_s^* \ni s \geq t$. Once we have found the consumption C_t^*

and the living standard $\bar{C}_t^*$, we can find the last two parameters we need to see how greedy consumption and asset allocation change with wealth if the client follows the greedy strategy.

Theorem 2.3.2. *Consider the greedy optimization problem in the form (2.93). Then for the consumption process* $C_t^* = (\bar{C}_t^*)^{\frac{\gamma-1}{\gamma}} (\alpha\zeta_t e^{\rho t})^{-\frac{1}{\gamma}}$ *the continuous positive wealth process* $X_t : [0,T] \times \mathbb{R} \to \mathbb{R}$, $X_t \geq 0$ *has the form*

$$X_t = \zeta_t^{-1} F(t,Z_t) \quad \text{where } Z_t = \zeta_t \bar{C}_t^* \tag{2.99}$$

and

$$F(t,z) = \alpha^{-\frac{1}{\gamma}} E\left[\int_t^T e^{-\frac{\rho s}{\gamma}} \tilde{\zeta}_{s-t}^{\frac{\gamma-1}{\gamma}} \left(\frac{\eta}{\gamma}\int_t^s \left(\alpha\tilde{\zeta}_{t'-t} e^{\rho s} e^{\eta(s-t')}\right)^{-\frac{1}{\gamma}} \mathrm{d}t' + z^{\frac{1}{\gamma}} e^{-\frac{\eta(s-t)}{\gamma}}\right)^{\gamma-1} \mathrm{d}s\right] \tag{2.100}$$

satisfy the initial condition $X_0 = v$ *and* $\bar{C}_0^* = \bar{c}$. *Moreover, there exists a portfolio process* θ *such that* $(\theta, C^*) \in \mathscr{A}(v)$ *and the following equality holds*

$$\theta_t = \frac{\kappa}{\sigma}\left(1 - \frac{\bar{C}_t^* \zeta_t F_z(t,Z_t)}{F(t,Z_t)}\right). \tag{2.101}$$

Proof. By the martingale representation theorem we can write the following statement

$$M_t = \zeta_t X_t + \int_0^t \zeta_s C_s^* \mathrm{d}s \tag{2.102}$$

On the other hand, we can construct the following stochastic process, which is a local martingale

$$M_t = E_t\left[\int_0^T \zeta_s C_s^* \mathrm{d}s\right] = \int_0^t (\alpha e^{\rho s})^{-\frac{1}{\gamma}} \left(\bar{C}_s^* \zeta_s\right)^{\frac{\gamma-1}{\gamma}} \mathrm{d}s + E_t\left[\int_t^T \left(\bar{C}_s^* \zeta_s\right)^{\frac{\gamma-1}{\gamma}} (\alpha e^{\rho s})^{-\frac{1}{\gamma}} \mathrm{d}s\right]. \tag{2.103}$$

Comparing the equations (2.102) and (2.103) we can get the portfolio process X_t as follows

$$\zeta_t X_t = \alpha^{-\frac{1}{\gamma}} E_t \left[\int_t^T e^{-\frac{\rho s}{\gamma}} \tilde{\zeta}_{s-t}^{\frac{\gamma-1}{\gamma}} \left(\bar{C}_s^* \zeta_t \right)^{\frac{\gamma-1}{\gamma}} \mathrm{d}s \right] \tag{2.104}$$

where $\tilde{\zeta}_{s-t} = \frac{\zeta_s}{\zeta_t}$ and ζ_t is the state-price density at a fixed time t. Since $\tilde{\zeta}_{s-t}$ has indepedent increments of $\mathcal{F}_t$, we can write $E_t = E[\ldots|\mathcal{F}_t] = E[\ldots]$. Introduce a new stochastic variable $Z_t = \zeta_t \bar{C}_t^*$. The final expression for the portfolio process will be as follows

$$X_t = \zeta_t^{-1} F(t,Z_t) \tag{2.105}$$

The last step is to find the asset allocation θ_t. On the one hand the wealth process formula can be written as we assumed before (2.92).On the other hand, the formula that represents the wealth process is (2.105). Using Ito's lemma, we can write the wealth process as follows

$$\mathrm{d}X_t = (\cdots)\mathrm{d}t + \zeta_t^{-2} F(t,Z_t) \kappa \zeta_t \mathrm{d}W_t + \zeta_t^{-1} F_z(t,Z_t) \bar{C}_t^* (-\kappa) \zeta_t \mathrm{d}W_t \tag{2.106}$$

$$\mathrm{d}X_t = (\cdots)\mathrm{d}t + \kappa \left(\zeta_t^{-1} F(t,Z_t) - F_z(t,Z_t) \bar{C}_t^* \right) \mathrm{d}W_t \tag{2.107}$$

By equating stochastic terms $\mathrm{d}W_t$ we can get an expression for asset allocation

$$\kappa \left(\zeta_t^{-1} F(t,Z_t) - F_z(t,Z_t) \bar{C}_t^* \right) = \sigma \theta_t X_t \tag{2.108}$$

$$\theta_t = \frac{\kappa}{\sigma} \left(1 - \frac{\bar{C}_t^* \zeta_t F_z(t,Z_t)}{F(t,Z_t)} \right) \tag{2.109}$$

where $F_z(t,Z_t)$ is a derivative over variable Z_t that we can compute using standard calculus rules. $\square$

If we check the last formula for the case $\eta = 0$ we will get $\theta_t = \frac{\kappa}{\sigma\gamma}$, which coincides with the previous result.

A special case of small volatility σ.

To understand what happens in the case of zero volatility, we should recall what "volatility" means. When we say "volatility", we imply the amount of uncertainty or risk associated with changes in the value of risky assets. This immediately means that, for example $\sigma = 0$ implies a steady security value or that prices will be constant. As a consequence, the wealth dynamics (2.92) can be simplified as follows

$$\mathrm{d}X_t = (rX_t - C_t^*)\mathrm{d}t, \qquad \mathrm{d}\bar{C}_t^* = \eta(C_t^* - \bar{C}_t^*)\mathrm{d}t. \tag{2.110}$$

The client's objective function and the budget constraint also will be simplified

$$\sup_{\substack{C_t>0 \\ \text{adapted}}} \int_0^T e^{-\rho s} u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s \quad \text{s.t.} \quad \int_0^T e^{-rs} C_s \mathrm{d}s = v. \tag{2.111}$$

Finally, the unconstrained optimization problem becomes

$$\sup_{\substack{C_t>0 \\ \text{adapted}}} \int_0^T e^{-\rho s} u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s - \alpha \left[\int_0^T e^{-rs} C_s \mathrm{d}s - v\right] \tag{2.112}$$

Consumption C_t^* can be found easily from formula 2.3.2

$$C_t^* = (\bar{C}_t^*)^{\frac{\gamma-1}{\gamma}} \left(\alpha e^{(\rho-r)t}\right)^{-\frac{1}{\gamma}}. \tag{2.113}$$

In order to compute Lagrange multiplier α we plug the expression for the consumption (2.113) into constraint (2.111).

$$\alpha = \frac{1}{v^\gamma}\left(\int_0^T e^{-rs} e^{-\frac{(\rho-r)s}{\gamma}} (\bar{C}_s^*)^{\frac{\gamma-1}{\gamma}} \mathrm{d}s\right)^\gamma. \tag{2.114}$$

The portfolio process is calculated as follows

$$X_t = \int_t^T \alpha^{-\frac{1}{\gamma}} e^{-\frac{\rho s}{\gamma}} (\bar{C}_s^*)^{\frac{\gamma-1}{\gamma}} \mathrm{d}s. \tag{2.115}$$

There is another way to think about this case, if we assume that the volatility does not equal zero, but takes on a small value, i.e. $\sigma \to 0$. In this case we can consider different cases that depend on $\kappa = \frac{\mu-r}{\sigma}$. For example, if we fix $\mu > r$ and $t > \varepsilon$, then $\kappa \to \infty$ and

$$\lim_{\sigma\to 0} \zeta_t = \lim_{\sigma\to 0} e^{-rt-\kappa W_t - \frac{1}{2}\kappa^2 t} = \lim_{\sigma\to 0} e^{-rt-\frac{\mu-r}{\sigma}W_t - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t} = 0, \tag{2.116}$$

Limits for the portfolio process X_t and the asset allocation θ_t are not easy to find because of uncertainy that requires additional analysis based on specific conditions and we will omit this discussion here. One more case arises if we simply fix κ. As a consequence, if volatility $\sigma \to 0$ the expression for the state-price density won't contain σ and the limit will be different for every choice of κ. Moreover, X_t and C_t^* won't vary with σ either, but asset allocation will, $\theta_t \to \infty$. As can be seen from the Figure 2.3, there is no unique limit when σ goes to 0, e.g. when time is close to zero $t \to 0$ and volatility $\sigma \to 0$ the limit for ζ_t goes to 1. When time t is big enough but volatility $\sigma \to 0$ the limit goes to 0. There are other cases, e.g. the case when $\mu < r$ or $\mu \to r$ also give different answers for different time moments .

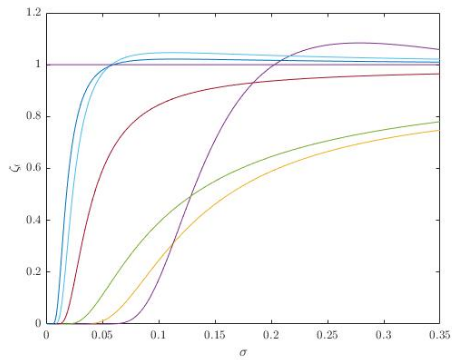


Figure 2.3: Test for $\sigma \to 0$ and different time moments $t = [0, 25]$.

2.3.3 Retirement spending problem with exogenous income

In this section, we generalize the previous case, namely, we assume the existence of a fixed income π. The wealth dynamics will be as follows

$$\mathrm{d}X_t = [\theta_t(\mu - r) + r]X_t\mathrm{d}t + \theta_t\sigma X_t\mathrm{d}W_t + \pi\mathrm{d}t - C_t^*\mathrm{d}t \tag{2.117}$$

$$\mathrm{d}\bar{C}_t^* = \eta(C_t^* - \bar{C}_t^*)\mathrm{d}t \tag{2.118}$$

Here $C_t^* = C_t^{w,*} + \pi$ where $C_s^{w,*}$ is a consumption that comes from wealth only. Also the following initial conditions hold $X_0 = v,\ \bar{C}_0^* = \bar{c}$. The use of martingale methods forces us to impose a constraint $C_t^w \geq 0$. As in the previous case, at every time moment we allow the retiree to decide what proportion of wealth θ_t to invest in risky assets. We want to solve a lifetime maximization problem with a CRRA utility function using a habit formation model

$$\sup_{(\theta, C^w)\in\mathscr{A}(v)} E\left[\int_0^T e^{-\rho s}\, {}_sp_x u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s | \bar{C}_0 = \bar{c}\right]. \tag{2.119}$$

where ρ is the subjective discount rate, ${}_sp_x$ is the probability of survival from the retirement age x to $x+s$. As before, for solving optimal problem using martingale methods we need to do additional analysis, therefore we adopt a greedy formulation in which the supremum is taken without imposing condition 2.118.

Remark. We set up the probability of survival based on the Gompertz Law of Mortality, i.e.

$${}_sp_x = e^{-\int_0^s \lambda_{x+q}\mathrm{d}q}. \tag{2.120}$$

Here λ is the biological hazard rate $\lambda_{x+q} = \frac{1}{b}e^{(x+q-m)/b}$ where m is the modal value of life since parameter $\lambda_0 = 0$ (see p47, [Milevsky, 2006]), b is the dispersion coefficient of the future lifetime random variable. Since m, b and x are constants we can compute the probability of survival explicitly

$$\int_0^s \lambda_{x+q}\mathrm{d}q = \int_0^s \frac{1}{b}e^{\frac{x+q-m}{b}}\mathrm{d}q = \frac{1}{b}e^{\frac{x-m}{b}} \int_0^s e^{\frac{q}{b}}\mathrm{d}q = e^{\frac{x-m}{b}}\left(e^{\frac{s}{b}} - 1\right). \tag{2.121}$$

Now, replace the probability of survival in formula (2.119) with the explicit expression

$$\sup_{(\theta,C^w)\in\mathscr{A}(v)} E\left[\int_0^T e^{-\rho s+e^{\frac{x-m}{b}}\left(1-e^{\frac{s}{b}}\right)} u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s|\bar{C}_0=\bar{c}\right]. \tag{2.122}$$

Let us denote the power in the exponent as $f(s) = -\rho s + e^{\frac{x-m}{b}}\left(1-e^{\frac{s}{b}}\right)$. Then the objective function will be

$$\sup_{(\theta,C^w)\in\mathscr{A}(v)} E\left[\int_0^T e^{f(s)} u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s|\bar{C}_0=\bar{c}\right]. \tag{2.123}$$

To solve our optimization problem, we need to set the so-called budget constraint.

Assumption 2.3.1. *The expected discounted consumption process* $C : [0,T]\times \Omega \to \mathbb{R}$ *over the entire time interval should not exceed the initial wealth* v*. In other words*

$$E\left[\int_0^T \zeta_s C_s^w \mathrm{d}s\right] = v. \tag{2.124}$$

Discussion: Recall the Radon-Nykodim derivative

$$\xi_t = \left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} \tag{2.125}$$

where $\xi_t : [0,T]\times\Omega \to \mathbb{R}$ is the stochastic process from the formula for the state-price density $\zeta_t = e^{-rt}\xi_t$ (2.15). We will write the differential of this exponential stochastic process as we are going to use it throughout this chapter

$$\mathrm{d}\zeta_t = (-r\mathrm{d}t - \kappa\mathrm{d}W_t)\xi_t e^{-rt} = (-r\mathrm{d}t - \kappa\mathrm{d}W_t)\zeta_t. \tag{2.126}$$

Then the following equalities hold

$$X_t \mathrm{d}Q = \xi_t X_t \mathrm{d}P \quad \text{and} \quad C_t \mathrm{d}Q = \xi_t C_t \mathrm{d}P. \tag{2.127}$$

In other words, in order to change probability measure from risk-neutral Q to real-world probability P we can simply multiply our stochastic process by density process ζ_t. Construct a stochastic process as follows

$$M_t = \zeta_t X_t + \int_0^t \zeta_s C_s^w \mathrm{d}s \tag{2.128}$$

where C_s^w is a consumption that comes from wealth only. Let's check if this process is a local $P-$martingale or not. In order to do that find the differential and check if the drift term $\mathrm{d}t$ equals to zero using formulas 2.125 and 2.118.

$$\begin{aligned}
\mathrm{d}M_t =& \zeta_t \mathrm{d}X_t + X_t \mathrm{d}\zeta_t + \mathrm{d}X_t \mathrm{d}\zeta_t + \zeta_t (C_t - \pi)\mathrm{d}t \\
=& \zeta_t([\theta_t(\mu - r) + r]X_t \mathrm{d}t + \theta_t \sigma X_t \mathrm{d}W_t + \pi \mathrm{d}t - C_t \mathrm{d}t) + X_t \zeta_t(-r\mathrm{d}t - \kappa \mathrm{d}W_t) \\
&+ (-r\mathrm{d}t - \kappa \mathrm{d}W_t)\zeta_t([\theta_t(\mu - r) + r]X_t \mathrm{d}t + \theta_t \sigma X_t \mathrm{d}W_t + \pi \mathrm{d}t - C_t \mathrm{d}t) + \zeta_t(C_t - \pi)\mathrm{d}t \\
=& -\cancel{r\zeta_t X_t \mathrm{d}t} + \cancel{r\zeta_t X_t \mathrm{d}t} + \cancel{\theta_t(\mu - r)\zeta_t X_t \mathrm{d}t} - \cancel{\zeta_t(C_t - \pi)\mathrm{d}t} \\
&+ \theta_t \sigma \zeta_t X_t \mathrm{d}W_t - \frac{\mu - r}{\sigma}\zeta_t X_t \mathrm{d}W_t - \cancel{\frac{\mu - r}{\sigma}\sigma\theta_t\zeta_t X_t \mathrm{d}t} + \cancel{\zeta_t(C_t - \pi)\mathrm{d}t} \\
=& \left(\theta_t \sigma \zeta_t X_t - \frac{\mu - r}{\sigma}\zeta_t X_t\right)\mathrm{d}W_t.
\end{aligned}$$

Hence, the sum of discounted current wealth and discounted consumption over the time period, is a local martingale.

The budget constraint under the real world probablity measure can be formulated as follows

$$E\left[\zeta_T X_T + \int_0^T \zeta_s C_s^w \mathrm{d}s\right] \leq v > 0. \tag{2.129}$$

If we have consumption we can claim that our terminal wealth should be $X_T = 0$ a.s. Finally, we can rewrite our budget constraint in the form (2.124).

Conversely, if $(\theta, C_t^w) \in \mathscr{A}(v)$ and we define the martingale

$$M_t = E\left[\int_0^T \zeta_s C_s^w \mathrm{d}s|\mathcal{F}_t\right], \tag{2.130}$$

we see that X_t can be defined as

$$\zeta_t X_t = E\left[\int_t^T \zeta_s C_s^w \mathrm{d}s|\mathcal{F}_t\right]. \tag{2.131}$$

Then the above steps can be reversed, using the martingale representation theorem, to obtain θ_t such that (2.117) holds. Provided we know that $X_t \geq 0\ \forall t$. For that reason we will require that consumption should remain nonnegative for any time moment, i..e $C_t^w \geq 0\ \forall t$. Therefore, formula (2.131) implies the desired inequality $X_t \geq 0$. In other words, we will require that consumption should greater than pension at any time moment, i.e. $C_t \geq \pi\ \forall t$ so that pension is always consumed. The optimization problem together with the budget constraint (Assumption 2.3.1) becomes

$$\sup_{\substack{C_t^w \geq 0 \\ \text{adapted}}} E\left[\int_0^T e^{f(s)} u\left(\frac{C_s}{\bar{C}_s^*}\right) \mathrm{d}s|\bar{C}_0 = \bar{c}\right] \quad \text{s.t. } E\left[\int_0^T \zeta_s C_s^w \mathrm{d}s\right] = v \tag{2.132}$$

where consumption $C_s^w \geq 0$ is a nonnegative stochastic process. By introducing a Lagrangian with $\alpha > 0$ we can rewrite our greedy optimization problem as an unconstrained problem

$$\mathcal{L}(C_t) = \int_0^T e^{f(s)} u\left(\frac{C_s^w + \pi}{\bar{C}_s^*}\right) \mathrm{d}s - \alpha\left[\int_0^T \zeta_s C_s^w \mathrm{d}s - v\right], \qquad (2.133)$$

$$\sup_{\substack{C_t^w \geq 0 \\ \text{adapted}}} E\left[\mathcal{L}(C_t) | \bar{C}_0 = \bar{c}\right]. \qquad (2.134)$$

A greedy optimum is now an adapted consumption stream C_t^* such that $C_t^{w*} = C_t^* - \pi$ is nonnegative, satisfies the budget constraint (2.124), and for habit $H = \mathscr{H}(C^*, \bar{c})$, C^* maximizes utility $\mathcal{U}(C^w + \pi, H)$ over all adapted stochastic processes $C_t^w \geq 0$ satisfying condition (2.124).

The same argument as in Theorem 2.3.1 now implies the following. Note that the Lagrange condition is only binding when the consumption stream is strictly positive $C_t^w > 0$.

Theorem 2.3.3. *Suppose there is an adapted consumption stream C_t^* and a Lagrange multiplier $\alpha > 0$ such that*

$$C_t^* = \pi + \max\left\{0, (\bar{C}_t^*)^{\frac{\gamma-1}{\gamma}} \left(\alpha e^{-f(t)} \zeta_t\right)^{-\frac{1}{\gamma}} - \pi\right\}, \qquad (2.135)$$

$H = \mathscr{H}(C^*, \bar{c})$ and also the consumption stream that comes from wealth only equals to the difference between full consumption and the fixed part (pension) π, i.e. $C_t^{w*} = C_t^* - \pi$ and satisfies the budget constraint (2.124). Then the consumption stream C_t^* is a greedy optimum.

Disscusion: In this method we optimize the consumption at every time moment. This is a so-called greedy policy rather than optimal one. Therefore, we expect our solution to converge to the local maximum. In order to compute consumption C_t^* we will use formula (2.134).

$$e^{f(t)}\frac{1}{\bar{C}_t^*}u'\left(\frac{C_t^{w*}+\pi}{\bar{C}_t^*}\right)=\alpha\zeta_t \;\Rightarrow\; \frac{1}{\bar{C}_t^*}\left(\frac{C_t^{w*}+\pi}{\bar{C}_t^*}\right)^{-\gamma}=\alpha e^{-f(t)}\zeta_t$$
$$\Rightarrow\; C_t^{w,*}=(\bar{C}_t^*)^{\frac{\gamma-1}{\gamma}}\left(\alpha e^{-f(t)}\zeta_t\right)^{-\frac{1}{\gamma}}-\pi \quad (2.136)$$

when the Lagrange condition is binding, i.e. $C_t^{w,*}>0$. In the case when optimal consumption reaches zero level, we assume that it remains nonnegative, which implies that we do spend all pension. Here $\bar{C}_t^*$ represents the standard of living that can be found from it's dynamics (the second equation in 2.118). As we have already mentioned C_t^* represents the total consumption. When pension equals zero, i.e. $\pi=0$, we solved our problem explicitly for habit $\bar{C}_t^*$. Now instead we will use the Euler-Maruyama method to do this numerically, and so compute the living standard and consumption at every time moment. Also in the formula (2.135) α represents the Lagrange multiplier, which can be found from the budget constraint (2.124) using a combination of bisection and Monte Carlo methods.

□

Theorem 2.3.4. *For the greedy optimal consumption processes C_t^* of Theorem 2.3.3, the continuous positive wealth process X_t has the form*

$$X_t=F(t,\zeta_t,\bar{C}_t^*) \quad (2.137)$$

where

$$F(t,\zeta,y) =E\left[\int_t^T \frac{\zeta_s}{\zeta_t} C^{w,*}_{s-t}\mathrm{d}s|\zeta_t = \zeta,\bar{C}^*_t = y\right]$$
$$=E\left[\int_t^T \tilde{\zeta}_{s-t} \max\left\{0, (\bar{C}^*_s)^{\frac{\gamma-1}{\gamma}} \left(\alpha e^{-f(s)}\tilde{\zeta}_{s-t}\right)^{-\frac{1}{\gamma}} \zeta^{-\frac{1}{\gamma}} - \pi\right\} \mathrm{d}s|\bar{C}^*_t = y\right]. \tag{2.138}$$

and satisfies the initial condition $X_0 = v$ *and* $\bar{C}^*_0 = \bar{c}$*. Moreover, the portfolio process* θ *has the form*

$$\theta_t = -\frac{\kappa\zeta_t}{\sigma X_t}\frac{\partial F}{\partial \zeta}. \tag{2.139}$$

Proof. In order to derive the portfolio process we should define a stochastic process as follows

$$M_t = \zeta_t X_t + \int_0^t \zeta_s C^{w,*}_s \mathrm{d}s \tag{2.140}$$

Then construct a stochastic process over time interval $[0,T]$.

$$M^P_t = E^P\left[M_t|\mathcal{F}_t\right]. \tag{2.141}$$

$$M_t = E_t\left[\int_0^T \zeta_s C^{w,*}_s \mathrm{d}s\right] = E_t\left[\int_0^t \zeta_s C^{w,*}_s \mathrm{d}s\right] + E_t\left[\int_t^T \zeta_s C^{w,*}_s \mathrm{d}s\right] \tag{2.142}$$
$$= \int_0^t \zeta_s C^{w,*}_s \mathrm{d}s + E_t\left[\int_t^T \zeta_s C^{w,*}_s \mathrm{d}s\right]. \tag{2.143}$$

Let $\zeta_s = \zeta_t\tilde{\zeta}_{s-t}$. By equating (2.140) and (2.143) we will get the following portfolio process formula

$$\zeta_t X_t = E_t\left[\int_t^T \zeta_s C^{w,*}_s \mathrm{d}s\right] = \zeta_t E_t\left[\int_t^T \tilde{\zeta}_{s-t} C^{w,*}_s \mathrm{d}s\right]. \tag{2.144}$$

$$X_t = E_t\left[\int_t^T \tilde{\zeta}_{s-t}\max\left\{0,(\bar{C}_s^*)^{\frac{\gamma-1}{\gamma}}\left(\alpha e^{-f(s)}\tilde{\zeta}_{s-t}\right)^{-\frac{1}{\gamma}}\zeta_t^{-\frac{1}{\gamma}}-\pi\right\}\mathrm{d}s\right]. \quad (2.145)$$

From the habit dynamics

$$\mathrm{d}\bar{C}_s = \eta(\pi + C_s^{w*} - \bar{C}_s^*)\mathrm{d}s$$
$$C_s^{w*} = \max\left\{0,(\bar{C}_s^*)^{\frac{\gamma-1}{\gamma}}\left(\alpha e^{-f(s)}\zeta_t\tilde{\zeta}_{s-t}\right)^{-\frac{1}{\gamma}}-\pi\right\} \quad (2.146)$$

for $s \geq t$, and independence of $\tilde{\zeta}_{s-t}$ from $\mathcal{F}_t$, we see that the conditional law of the process $(C_s^{w*},\bar{C}_s^*)_{s\geq t}$ given $\mathcal{F}_t$ depends only on ζ_t and $\bar{C}_t^*$.

For fixed values ζ and y, define

$$F(t,\zeta,y) = E\left[\int_t^T \tilde{\zeta}_{s-t}\max\{0,(\bar{C}_s^*)^{\frac{\gamma-1}{\gamma}}\left(\alpha e^{-f(s)}\tilde{\zeta}_{s-t}\right)^{-\frac{1}{\gamma}}\zeta^{-\frac{1}{\gamma}}-\pi\}\mathrm{d}s\right] \quad (2.147)$$

where $\bar{C}_s^*$ and C_s^{w*} evolve as above, from ζ and y. This can be evaluated by Monte Carlo simulation of $\tilde{\zeta}_{s-t}$. Then we have

$$X_t = F(t,\zeta_t,\bar{C}_t^*). \quad (2.148)$$

In order to compute habit $\bar{C}_s^*$ we will use the Euler-Maruyama method.

$$\bar{C}_{s_n}^* = \bar{C}_{s_{n-1}}^* + \eta\left(\pi + \max\left\{0,(\bar{C}_{n-1}^*)^{\frac{\gamma-1}{\gamma}}\left(\alpha e^{-f(s_{n-1})}\frac{\zeta_{s_{n-1}}}{\zeta_t}\right)^{-\frac{1}{\gamma}}\zeta_t^{-\frac{1}{\gamma}} - \pi\right\} - \bar{C}_{n-1}^*\right)\Delta s. \quad (2.149)$$

As the last step, we calculate the asset allocation θ. On the one hand, recall the formula for the wealth process (2.118). On the other hand, use

Ito's formula for $X_t = F(t,\zeta_t,\bar{C}_t^*)$.

$$\mathrm{d}X_t = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial \zeta}\mathrm{d}\zeta_t + \frac{\partial F}{\partial y}\mathrm{d}\bar{C}_t^* + \frac{1}{2}\frac{\partial^2 F}{\partial \zeta^2}(\mathrm{d}\zeta_t)^2.$$

Recall the expression $\mathrm{d}\zeta_t = \zeta_t(-r\mathrm{d}t - \kappa \mathrm{d}W_t)$.

$$\mathrm{d}X_t = (\cdots)\mathrm{d}t - \kappa\zeta_t\frac{\partial F}{\partial \zeta}\mathrm{d}\zeta_t\mathrm{d}W_t. \tag{2.150}$$

Finally, the asset allocation θ_t will be written as follows

$$\sigma\theta_t X_t = -\kappa\zeta_t\frac{\partial F}{\partial \zeta}, \tag{2.151}$$

$$\theta_t = -\frac{\kappa\zeta_t}{\sigma X_t}\frac{\partial F}{\partial \zeta}. \tag{2.152}$$

The last step is to compute the derivative $\frac{\partial F}{\partial \zeta}$. In order to do that we will use the following approximation

$$F(t,\zeta,y) = E\left[\int_t^T \tilde{\zeta}_{s-t}\left((\bar{C}_s^*)^{\frac{\gamma-1}{\gamma}}\left(\alpha e^{-f(s)}\tilde{\zeta}_{s-t}\right)^{-\frac{1}{\gamma}}\zeta^{-\frac{1}{\gamma}} - \pi\right)_+ \mathrm{d}s\right] \tag{2.153}$$

$$\frac{\partial F}{\partial \zeta} = \frac{F(t,\zeta,y) - F(t,\zeta - h,y)}{h}. \tag{2.154}$$

where we introduced new notation for state-price density $\zeta \in \mathbb{R}$ and habit $y \in \mathbb{R}$ at fixed time moment t. We will compute the derivative directly using Monte Carlo method applied to the formula (2.154). □

2.3.4 Numerical results

Numerical results for this problem will be presented in a three parts. First, we will discuss the results obtained for the problem under HFM without additional income 2.3.4, then we will present the solution for the problem under HFM with fixed pension income π. The problem formulation for the martingale approach differs from what we solved in the previous chapter in part because here we fix the habit at time moment t and solve optimization problem for this specific case whereas for PDE approach we solve the optimization problem for all values of habit and time simultaneously. Unless mention is made to the contrary we will use the following parameter values: **risk-free rate** $r = 0.02$, **volatility** $\sigma = 0.16$, **drift** $\mu = 0.08$, **subjective discount rate** $\rho = 0.02$ **and risk aversion parameter** $\gamma = 3$. We will vary the parameter that reflects how fast the habit formation model reacts to the client's consumption choices, i.e. the smoothing factor η, in order to see how the solution changes. In our calculations, we will take $\eta = 0.01$, 0.1 and 1. We define a time grid as follows $t_n = (n-1)\Delta t,\ t \in [0,T]$. Also we define the Brownian motion increments $\Delta W_t = Z\sqrt{\Delta t}, \quad Z \sim N(0,1)$. In order to find the Lagrange multiplier α using formula (2.114) we use a bisection method and the Euler-Maruyama formula to find the living standard $\bar{C}_t$ at every time step

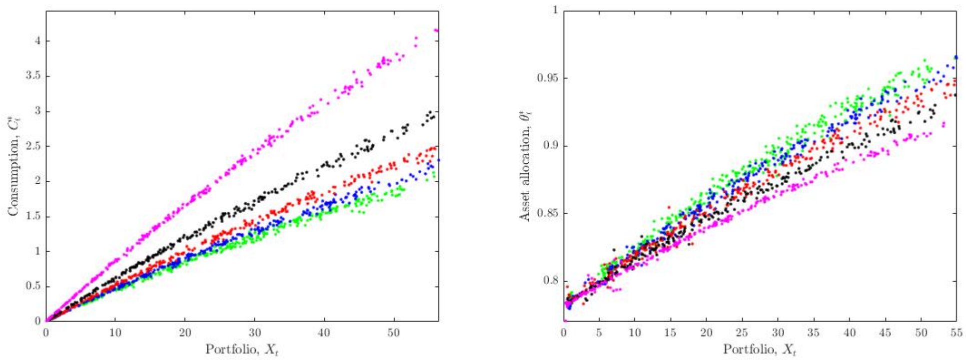
$$\bar{C}_n = \bar{C}_{n-1} + \eta \left(C^*_{n-1} - \bar{C}_{n-1} \right) \Delta t \tag{2.155}$$

where C^*_t is the consumption obtained by using the greedy policy (2.94), i.e. locally optimum.

HFM without pension income

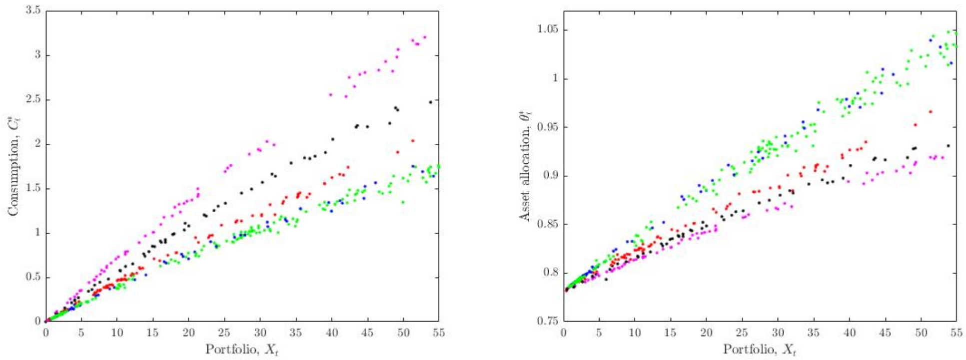
In this paragraph we will show numerical results for the greedy policy under HFM without pension income. In comparison with PDE solution, in this case we optimize our consumption at the fixed time moment and for fixed habit $\bar{C}_t$, so, as a result, our approach more likely should be called a "greedy policy algorithm" rather than the "optimal policy". The resulting solution for the greedy algorithm will not converge to the global maximum.

First case study with smoothing factor $\eta = 0.01$. We start from the smallest value of the smoothing factor $\eta = 0.01$. The client's habit does not adapt to consumption rapidly, therefore this numerical solution will be closer to the Merton problem (see Figure 2.2). The numerical results presented in Figure 2.4 show the relationship between the greedy policy C^* and portfolio X_t in the left pictures and asset allocation θ_t^* vs. portfolio X_t in the right pictures. The results were divided into three sets of images that show different cases, namely: Figure 2.4a represents the case with a finite time horizon and no mortality, Figure 2.4b shows the case for an infinite time horizon also without mortality, the last picture 2.4c represents the case with mortality and for a finite time horizon. We provide results for five time moments: $t = 0$(green), 10(blue), 20(red), 30(black) and 40(magenta) if we consider a finite time-horizon, and for an infinie time horizon the time moments are $t = 0$(green), 30(blue), 150(red), 170(black) and 180(magenta) and the initial living standard $\bar{c} = 1$.

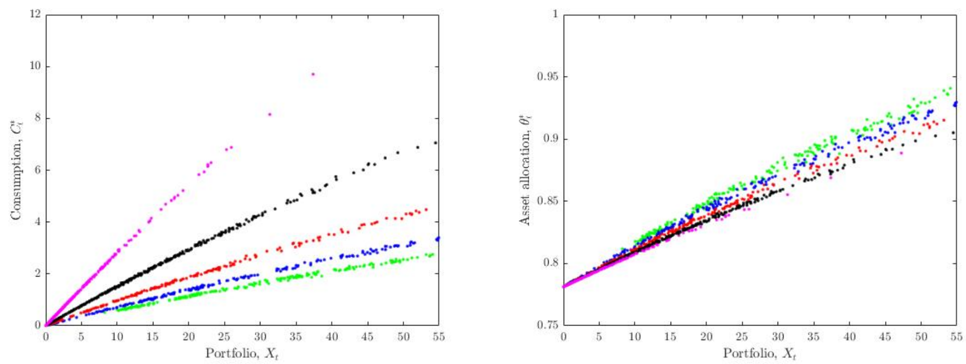


(a) Finite time horizon.

The results on both sides, left and right pictures, show almost linear growth. The highest level of consumption a retiree will have, comes if we solve the lifetime problem (i.e. finite time horizon). At the same time, assuming an infinite lifespan, the retiree will get the lowest consumption level (see Figure 2.4b). If we look at the asset allocation, we can say that the asset allocation is not that sensitive to the model and shows approximately the same behaviour for all cases. Now, let's look at the second case.



(b) Infinite time horizon.



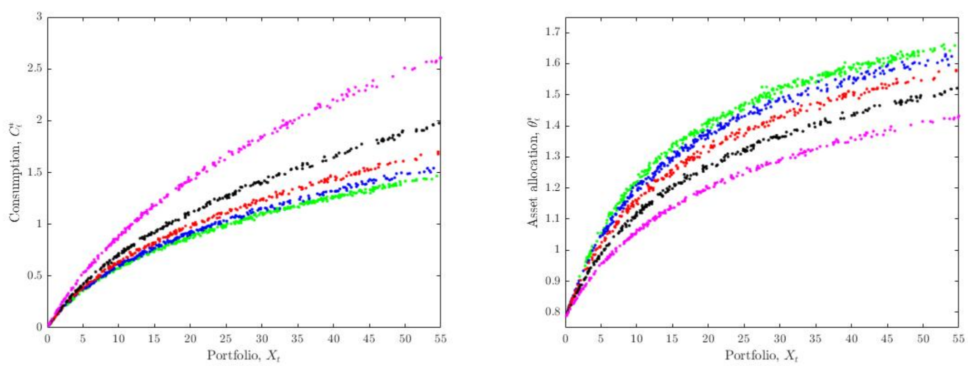
(c) Lifetime problem.

Figure 2.4: Numerical solution of the greedy policy under HFM for $\eta = 0.01$ for different time moments.

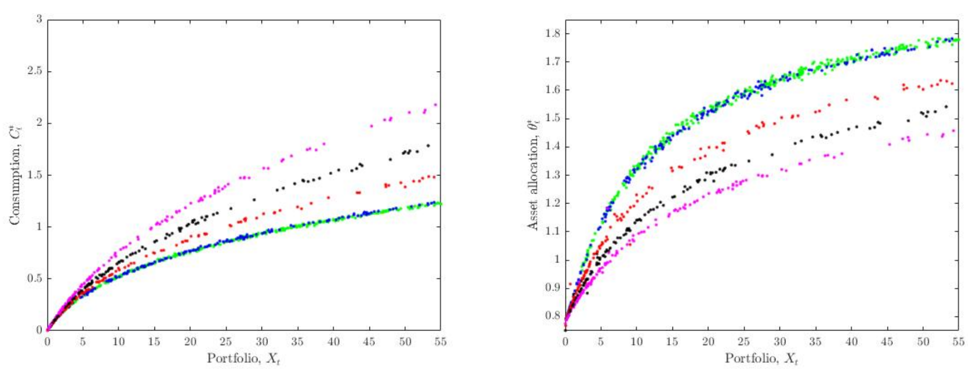
Second case study with smoothing factor $\eta = 0.1$. In this case we assume that the impact of habit should affect results more. We increase the smoothing factor to $\eta = 0.1$. In this case the client's habit will react quicker to changes in consumption. As a result, the numerical solution will depend on the client's living standard more and we will see a difference between this and the previous case where $\eta = 0.01$.

In Figure 2.5 we have the same set of pictures as in the previous case.

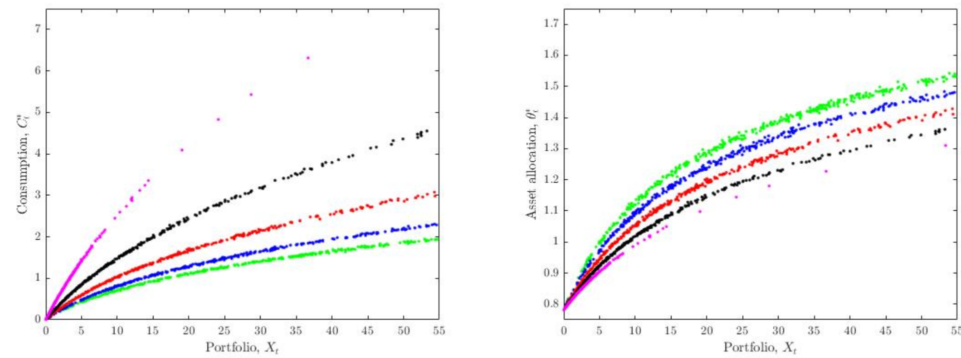
As we can see, the numerical solution depends more on the retiree's habit. Consumption and asset allocation grow faster when wealth is small and then gradually slow down as wealth goes to infinity. Similar to the previous case the highest level of consumption we can see is for the lifetime greedy policy (see Figure 2.5c). Also, as time grows (shown by different dots colours, where the magenta line represents the greatest time value $t = 40$), the asset allocation declines which means that the portion of stocks in the portfolio will decrease.



(a) Finite time horizon.



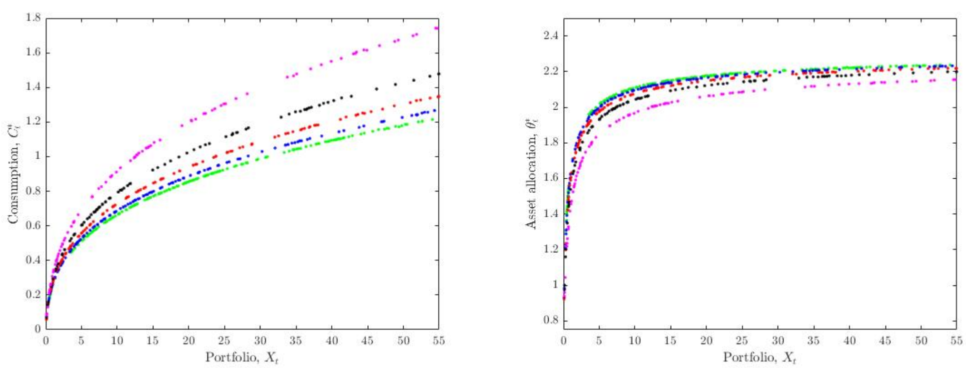
(b) Infinite time horizon.



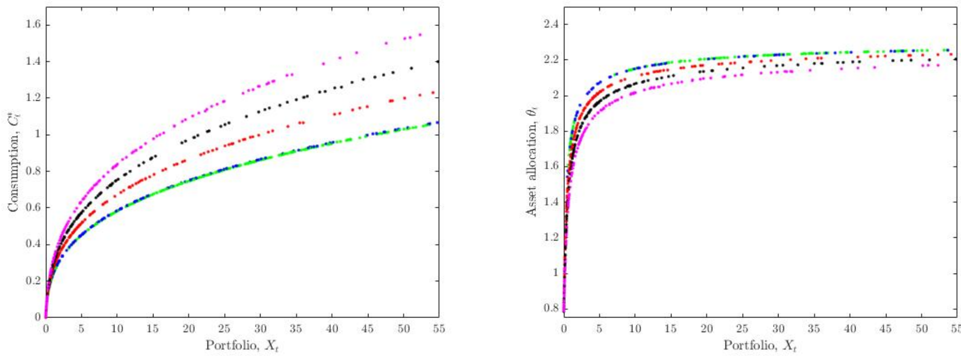
(c) Lifetime problem.

Figure 2.5: Numerical solution of the greedy policy under HFM for $\eta = 0.1$ for different time moments and initial habit $\bar{C} = 1$.

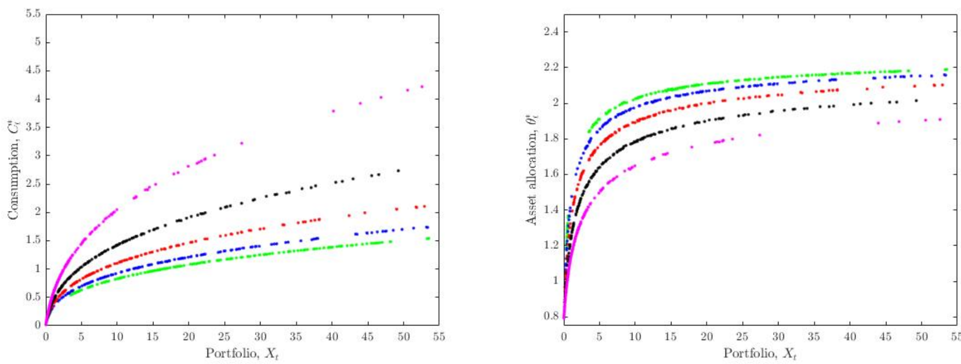
Third case study with smoothing factor $\eta = 1$. The last case that we are going to describe is $\eta = 1$. It means a model in which habit responds very quicly to changes in consumption. As a consequence, we will get consumption (left pictures) and asset allocation (right pictures) that grow very fast for small values of wealth and then slow down quickly.



(a) Finite time horizon.



(b) Infinite time horizon.



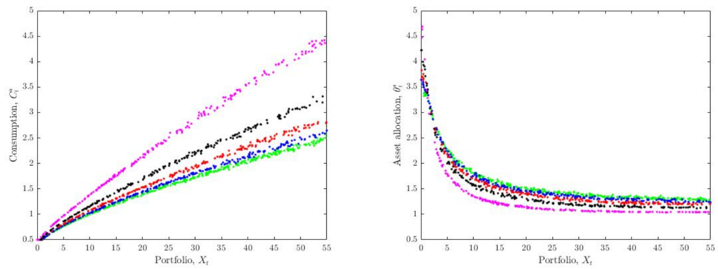
(c) Lifetime problem.

Figure 2.6: Numerical solution of the greedy policy under HFM for $\eta = 1$ for intial habit $\bar{C} = 1$ and for different time moments.

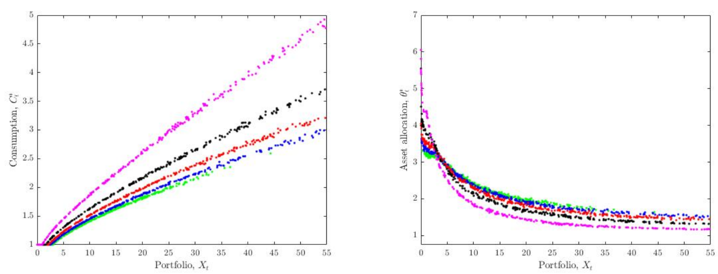
HFM with pension income π

In the previous paragraph we summarized numerical results for the greedy policy without pension. Here we will be discussing the greedy policy with fixed pension income. In order to understand how the solution behaves we represent results for three values of the smoothing factor $\eta = [0.01\ 0.1\ 1]$ and four values of pension $\pi = [0.5\ 1\ 1.5\ 2]$ for a finite time horizon, an infinite time horizon and a lifetime problem.

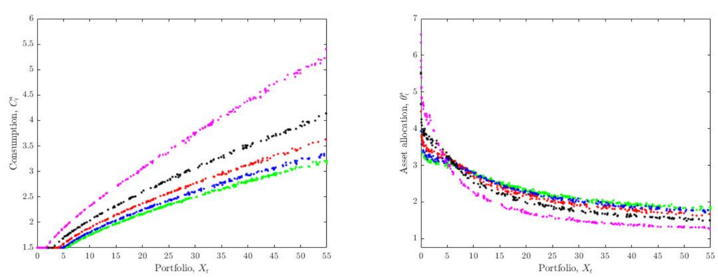
First case study with smoothing factor $\eta = 0.01$. Again, as in the previous paragraph, we begin our discussion by summarizing the numerical results for the smallest value of the smoothing factor $\eta = 0.01$. In Figure 2.7 we see four sets of pictures, where every set represents a specific value of pension π, namely $\pi = 0.5$ (see Figure 2.7a and 2.8a), $\pi = 1$ (see Figure 2.7b and 2.8b), $\pi = 1.5$ (see Figure 2.7c and 2.8c) and $\pi = 2$ (see Figure 2.7d and 2.8d). Also, each case contains two graphs: the relationship between the consumption C^* obtained by using the greedy policy and the portfolio X_t (left picture) and the asset allocation vs. portfolio (right picture) for five time moments $t = 0$(the green dots), $t = 10$(the blue dots), $t = 20$(the red dots), $t = 30$(the black dots) and $t = 40$(the magenta dots). We provide results for two cases: a solution for the problem without mortality with a finite time horizon (see Figure 2.7) and a lifetime problem (see Figure 2.8). From the graphs on the left side (consumption vs. portfolio) it can be seen that the consumption level grows as the level of pension rises. At the same time, if we look at the case without mortality (see Figure 2.7) we can see some "lag"' in consumption for small wealth values. The delay is greater for the larger pension values. This means that the optimal solution is to consume at the pension level, but the part of consumption related to wealth is close to zero. Recall that we constrained consumption from wealth to be nonnegative $X_t \geq 0$.



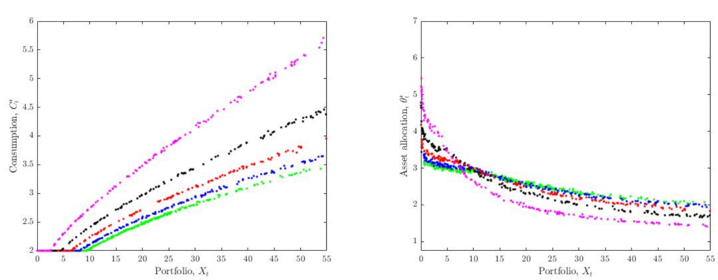
(a) $\pi = 0.5$



(b) $\pi = 1$



(c) $\pi = 1.5$

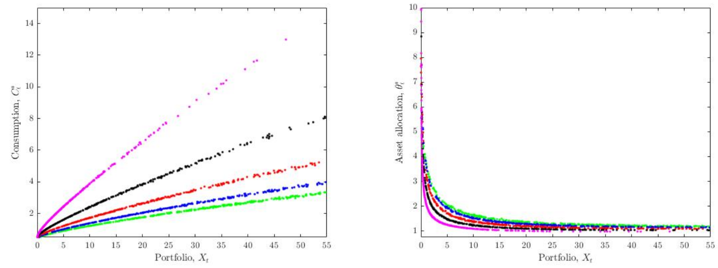


(d) $\pi = 2$

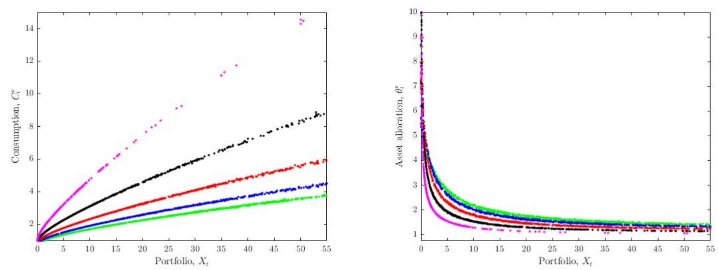
Figure 2.7: Numerical solution for an infinite time horizon using the greedy policy with $\eta = 0.01$ and intial habit $\bar{C} = 1$.

If we look at the asset allocation (Figure 2.7 right pictures), we see that the behaviour of asset allocation over time is more complicated. Recall that the asset allocation is the percentage of the portfolio invested into risky assets, and not the actual amount of stocks. There are five curves where each one represents the fixed time moment. The green line represents time $t = 0$ and the magenta curve represents the largest time moment $t = 40$. It can be seen that asset allocation decreases over time for small wealth. As wealth goes to infinity the behaviour changes and as the time is smaller, the asset allocation is larger. When wealth is small, the relatively large income from pension means a highly leveraged portfolio is sustainable. But for larger wealth, pension income is less significant, and a smaller asset allocation is called for. For the lifetime problem, the results are somewhat different (see Figure 2.8). If we look at the left pictures, which represent consumption, we can see that there is almost no delay in consumption. This happens because mortality gives heavier discounting than for the case without mortality where discounting will be at risk-free rate only. As a result, the overall level of consumption is higher.

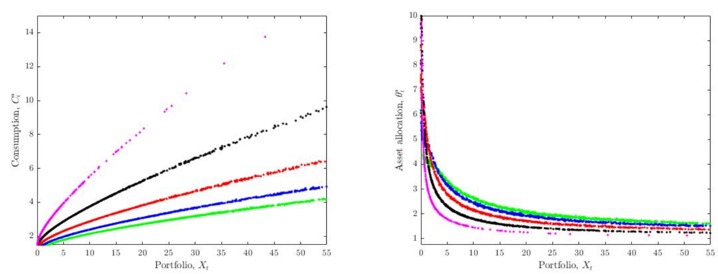
Second case study with smoothing factor $\eta = 0.1$. For the next set of results we increased the smoothing factor to $\eta = 0.1$. As we discussed before, this means the model will react faster to changes in consumption, so habit will track consumption closer. As in the previous case, in Figure 2.9 we see four sets of pictures, where every set represents a specific value of pension π, namely $\pi = 0.5$ (Figure 2.9a and 2.10a), $\pi = 1$ (Figure 2.9b and 2.10b), $\pi = 1.5$ (Figure 2.9c and 2.10c) and $\pi = 2$ (Figure 2.9d and 2.10d).



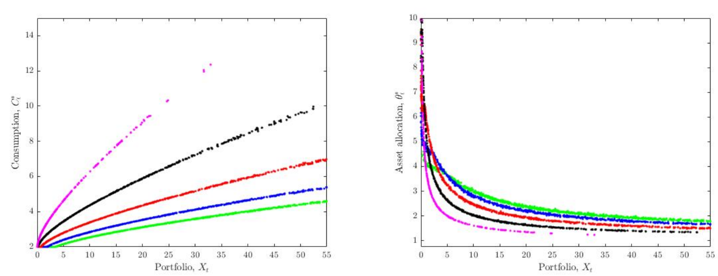
(a) $\pi = 0.5$



(b) $\pi = 1$



(c) $\pi = 1.5$

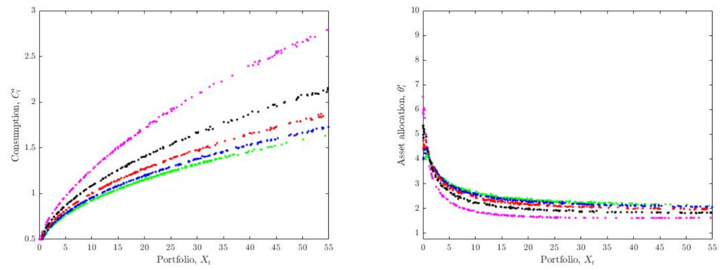


(d) $\pi = 2$

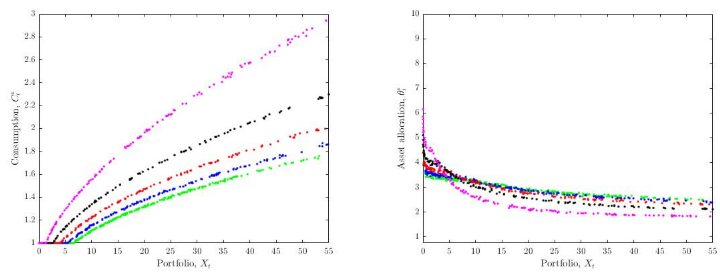
Figure 2.8: Numerical solution for a lifetime problem using the greedy policy under HFM with $\eta = 0.01$ and intial habit $\bar{C} = 1$.

Each case contains two graphs: the relationship between the consumption C^* and the portfolio X_t (left picture) and the asset allocation vs. portfolio (right picture) for five time moments $t = 0$(the green dots), $t = 10$(the blue dots), $t = 20$(the red dots), $t = 30$(the black dots) and $t = 40$(the magenta dots). We provide results for two cases: a solution for the problem without mortality and with a finite time horizon (see Figure 2.9), and a lifetime problem (see Figure 2.10). In general, the numerical results are similar to the previous case. For example, we also have some "lag"' in consumption for small wealth values (left pictures). Comparing the asset allocation at different time moments (right pictures), it is larger for small values of wealth since in this case the impact of pension on the portfolio is larger and there is a possibility for leverage. The main difference between these and the results in the previous paragraph is that the growth in consumption and asset allocation is faster for the small values of wealth and slows down as wealth goes to infinitity, while in the previous case the growth was almost linear.

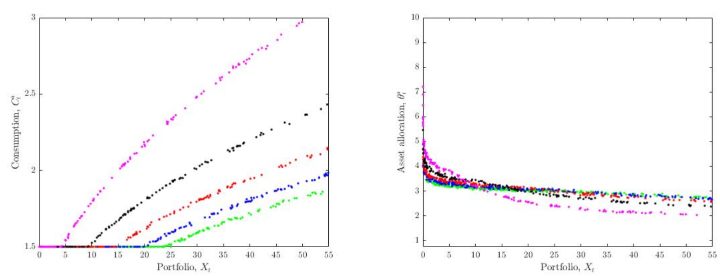
Third case study with smoothing factor $\eta = 1$. The last set of numerical results was obtained for the smoothing factor $\eta = 1$. As we already know, this means the client's living standard reacts immediately to changes in consumption, and, therefore consumption and habit will be close for any portfolio or time values. As in all previous cases, in Figures 2.11 and 2.12 we see four sets of pictures, where every set represents a specific value of pension π, namely $\pi = 0.5$ (Figure 2.11a and 2.12a), $\pi = 1$ (Figure 2.11b and 2.12b), $\pi = 1.5$ (Figure 2.11c and 2.12c) and $\pi = 2$ (Figure 2.11d and 2.12d).



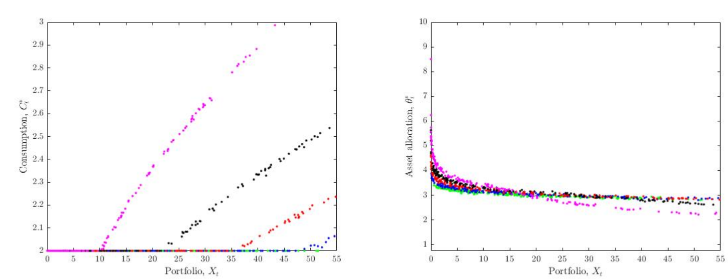
(a) $\pi = 0.5$



(b) $\pi = 1$

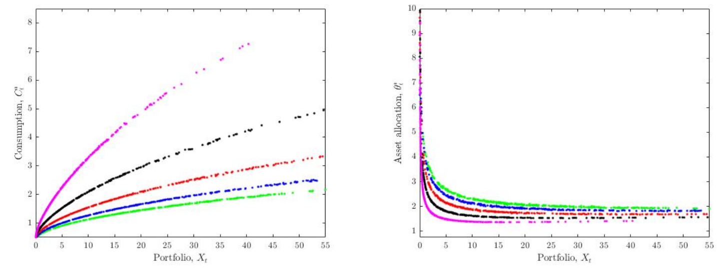


(c) $\pi = 1.5$

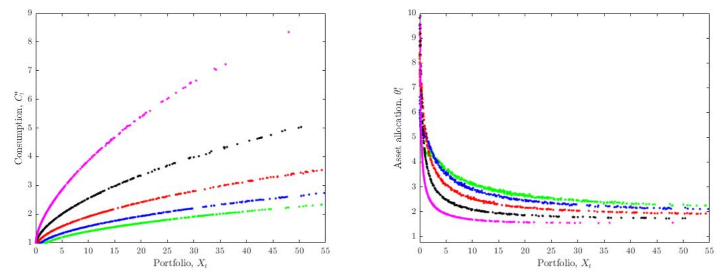


(d) $\pi = 2$

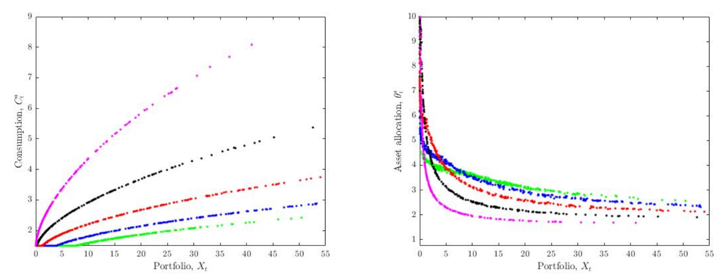
Figure 2.9: Numerical solution of the greedy policy for $\eta = 0.1$ and the intial habit $\bar{C} = 1$.



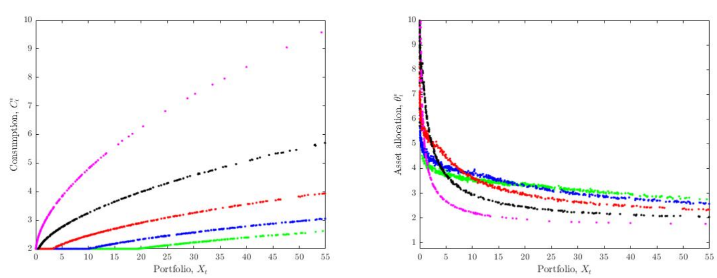
(a) $\pi = 0.5$



(b) $\pi = 1$



(c) $\pi = 1.5$



(d) $\pi = 2$

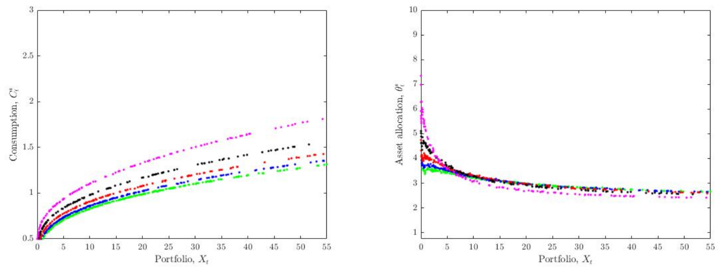
Figure 2.10: Numerical solution for the lifetime greedy policy for $\eta = 0.1$ and the intial habit $\bar{C} = 1$.

Each case contains two graphs: the relationship between the consumption C^* and the portfolio X_t (left picture) and the asset allocation vs. portfolio (right picture) for five time moments $t = 0$(the green dots), $t = 10$(the blue dots), $t = 20$(the red dots), $t = 30$(the black dots) and $t = 40$(the magenta dots). We provide results for two cases: a solution for a problem without mortality with a finite time horizon (see Figure 2.11), and a lifetime problem (see Figure 2.12). Qualitatively, the numerical results are similar to all previous cases, but since we have a greater smoothing factor value, the consumtion and the asset allocatioon will grow even faster for small wealth values and will slow down faster as wealth goes to infinity.

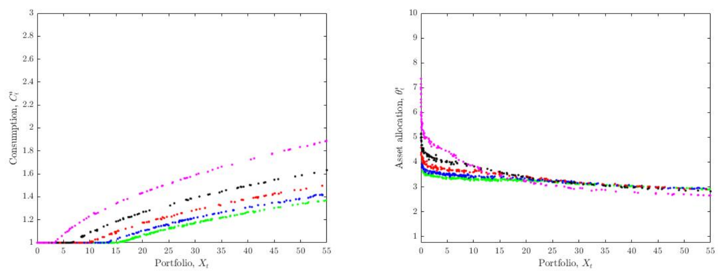
In the next two paragraphs, we are going to provide a comparison between the PDE (optimal) and probabilistic (greedy) solutions and analyze how wealth depletes over time for certain values of initial living standard $\bar{C}_{ini}$ and wealth v.

2.3.5 Comparison of PDE solution with probabilistic solution.

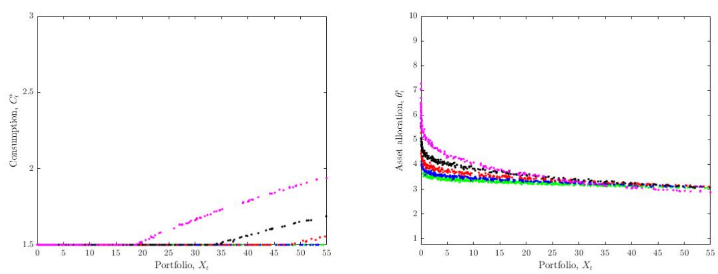
In this paragraph we compare the numerical results obtained from the previous Chapter 1 and the results of this chapter obtained using the martingale approach. Due to the difference in formulation for each case we do not expect our results to be exactly the same, but we will try to understand under what settings we can get close solutions. We will analyze results for three values of the smoothing factor $\eta = [0.01\ 0.1\ 1]$ and for fixed initial wealth v and living standard $\bar{C}_{ini}$.



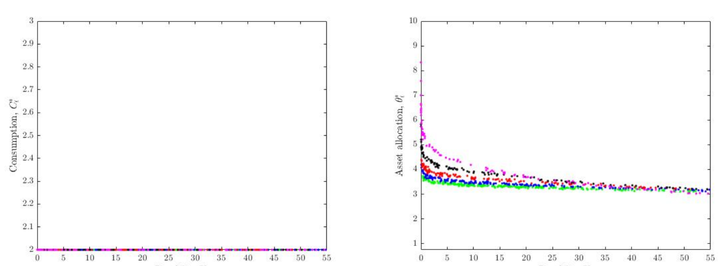
(a) $\pi = 0.5$



(b) $\pi = 1$

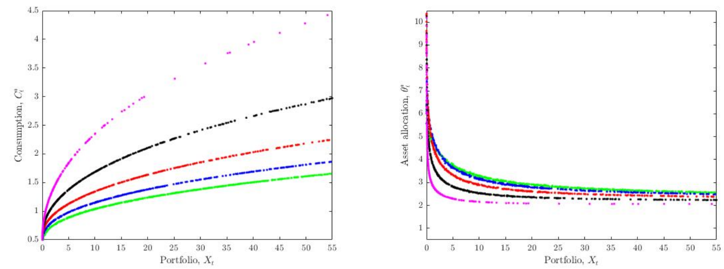


(c) $\pi = 1.5$

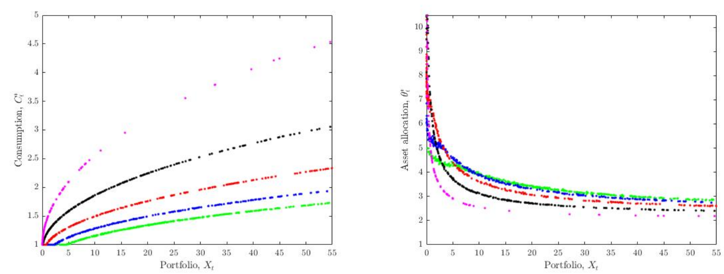


(d) $\pi = 2$

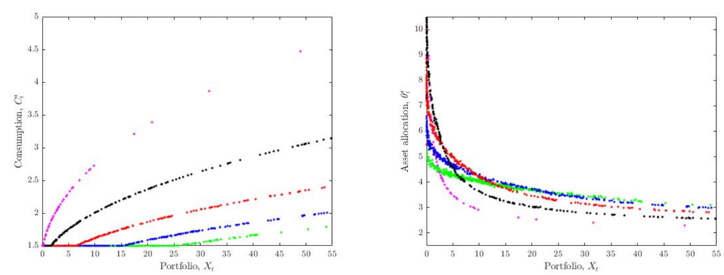
Figure 2.11: Numerical solution of the greedy policy for $\eta = 1$ and the intial habit $\bar{C} = 1$.



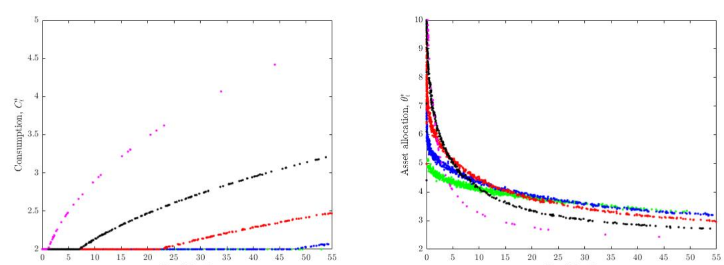
(a) $\pi = 0.5$



(b) $\pi = 1$



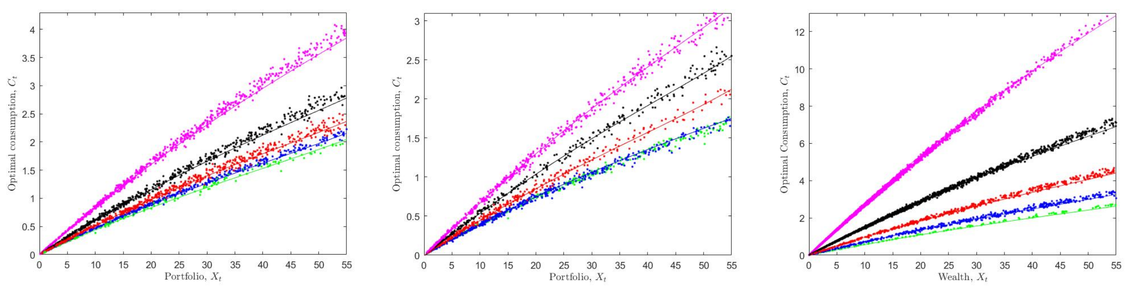
(c) $\pi = 1.5$



(d) $\pi = 2$

Figure 2.12: Numerical solution for a lifetime greedy policy for $\eta = 1$ and the intial habit $\bar{C} = 1$.

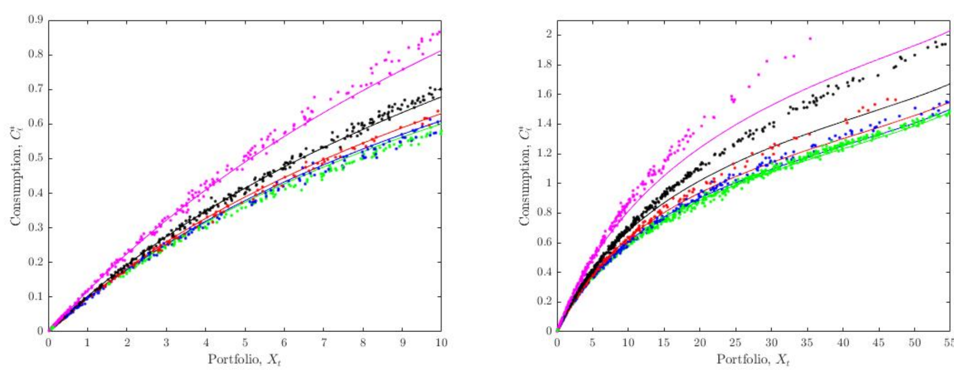
First case study with smoothing factor $\eta = 0.01$. We start with the model where habit does not significantly affect the results, which means that the smoothing factor is small, $\eta = 0.01$. We consider three cases: a finite time horizon, an infinite time horizon and a lifetime problem where the mortality rate follows a Gompertz law. In this case, we can see from Figure 2.13 that the numerical results obtained by the value function approach (the solid lines) (see [Kirusheva etc., under review]) and the probabilistic approach (the dots) are very close for all three cases. Every line, solid or dotted, represents a specific time moment, namely for a finite time horizon problem $t = 0$(green), 10(blue), 20(red), 30(black) and 40(magenta) and for an infinite time horizon problem $t = 0$(green), 30(blue), 150(red), 170(black) and 180(magenta). Let's increase the smoothing factor and see if the results are still close or not.



(a) FTH. (b) ITH. (c) FTHM.

Figure 2.13: Numerical solution of HFM for $\eta = 0.01$. Comparison between the PDE solution (the solid lines) and the probabilistic numerical solution (the dots).

Second case study with smoothing factor $\eta = 0.1$. In the value function approach we have optimized our solution over the entire grid, wealth w_t, habit $\bar{c}_t$ and time t, simultaneously. Using a probabilistic approach, we changed the problem. We compute the solution for fixed time moments t and fixed habit $\bar{C}_t$. Since our assumptions for this problem are different, as a result, the numerical solution obtained using the martingale approach is also different. Below we provide an analysis of how this solution differs from the PDE approach for the chosen smoothing factor. The first set of graphs (see Figure 2.14) represents a finite time horizon problem. The difference between the left and right pictures is only in scale. In Figure 2.14a we see that for modest values of wealth, the solution still gives results that are relatevely close to the PDE numerical solution. In orther words, the greedy solution is close to optimal. While in right Figure 2.14b we show the entire wealth interval that we mostly used in the previous chapter, and the results for large wealth values show differences. So the greedy policy gives us significantly higher consumption than the PDE solution, when wealth is large, except when time t is small (when the two solutions remain close).



(a) (b)

Figure 2.14: Numerical solution of HFM for $\eta = 0.1$. Comparison between the PDE solution (the solid line) and the probabilistic numerical solution (the dots) for finite time horizon without mortality.

We can make similar observation when we consider an infinite time horizon problem (see Figure 2.15). If wealth is relatively small the numerical results are close to the PDE solution. For larger wealth the agreement is not so close. Now at small times, greedy consumption lies below the optimallevel, and rises above it for large time moments.

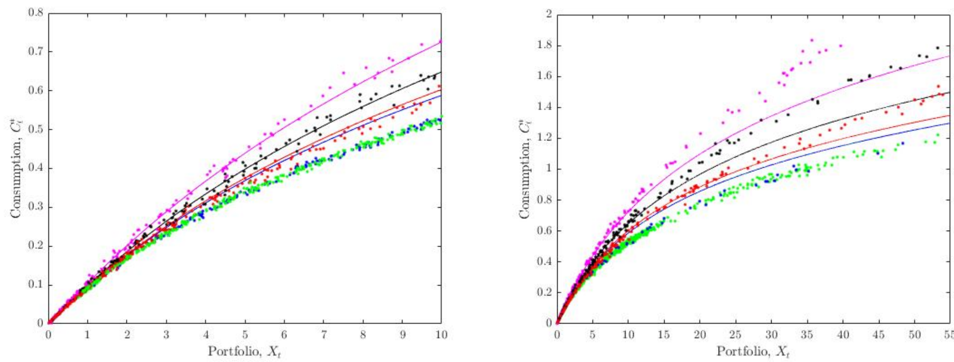


Figure 2.15: Numerical solution of HFM for $\eta = 0.1$. Comparison between the PDE solution and the probabilistic numerical solution (the dots) for infinite time horizon and for differne time moments.

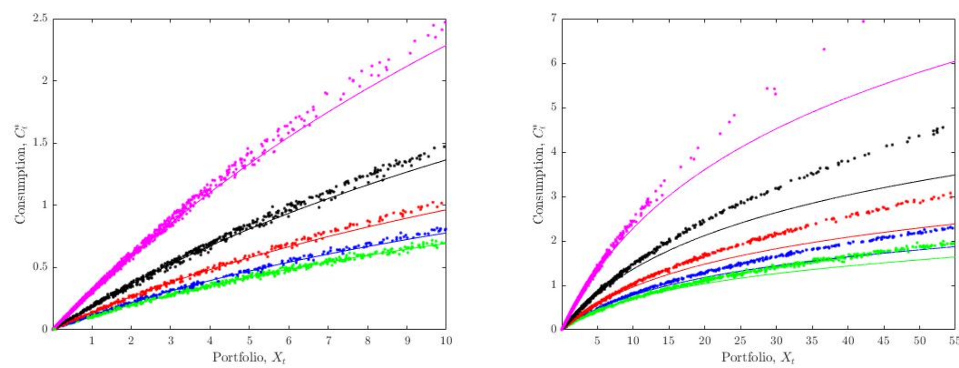
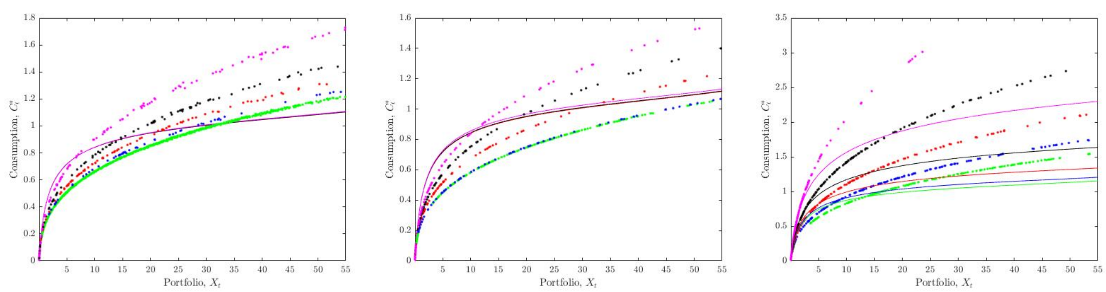


Figure 2.16: Numerical solution of HFM for $\eta = 0.1$. Comparison between the PDE solution (the solid line) and the probabilistic numerical solution (the dots) for finite time horizon with mortality for different time moments.

The final set of pictures represents the numerical results for the lifetime problem and we see that for modest values of wealth both solutions are in close agreement. But for large wealth (eg. 50 times habit) they differ. So, based on our numerical results, we can conclude that for the smoothing factor $\eta = 0.1$ and for modest values of wealth relative to habit, eg. $0 < w/\bar{c} < 10$, we have good agreement between the PDE and martingale approaches. But when $\frac{w}{\bar{c}}$ is more (which then raises habit), while the optimal solution raises consumption more conservatively.

Third case study with smoothing factor $\eta = 1$. The last case, when we increase the smoothing factor to 1, show that the numerical results obtained using the martingale approach are very different from the value function approach.



(a) FTH. (b) ITH. (c) FTHM.

Figure 2.17: Numerical solution of HFM for $\eta = 1$. Comparison between the PDE solution (the solid lines) and the probabilistic numerical solution (the dots).

We conclude that a greedy policy does not perform optimally when changes very rapidly with consumption (eg. within one year). But when habit adapts more gradually (over several years), the greedy policy can be a good match, provided the discrepancy between habit and wealth is not too large. So as increasingly realistic problems get considered, and PDE methods slow down due to the rise in complexity, martingale (Monte Carlo) methods should behave well under these circumstances. In Figure 2.17 there are three cases: a finite time horizon, an infinite time horizon and a lifetime problem for five time moments $t = 0$(green), 10(blue), 20(red), 30(black) and 40(magenta) for Figures 2.17a and 2.17c and $t = 0$(green), 30(blue), 150(red), 170(black) and 180(magenta) for Figure 2.17b.

2.3.6 Wealth depletion

In this paragraph we discuss how much a client can afford to spend during retirement based on our model.

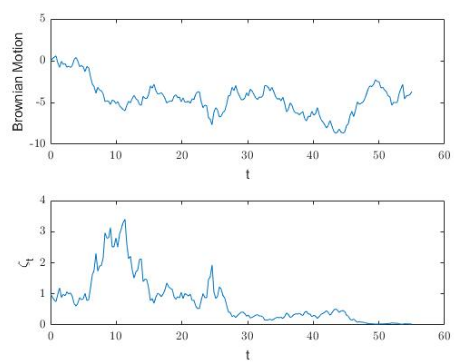
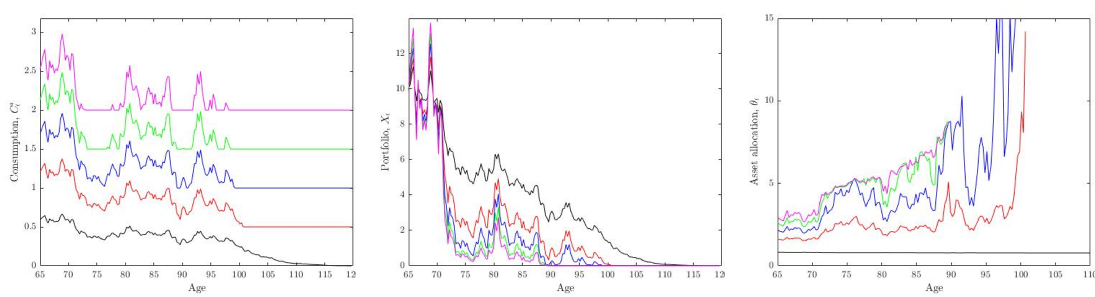


Figure 2.18: Brownian motion (upper picture) and state price density (lower picture).

We give results based on the greedy policy for two values of the living standard $\bar{C} = 1$ and 5, for three values of the smoothing factor $\eta = [0.01\ 0.1\ 1]$, and three values of the initial wealth $v = [10\ 30\ 60]$. We will show that the numerical results depend significantly on the value of the smoothing factor η. The higher the smoothing factor η is, the closer habit and consumption will be over time. But first, in Figure 2.18 we show what Brownian motion was used in order to compute state-price density for fixed time moment ζ_t only. This explains the shape of the curves that represent our numerical results.

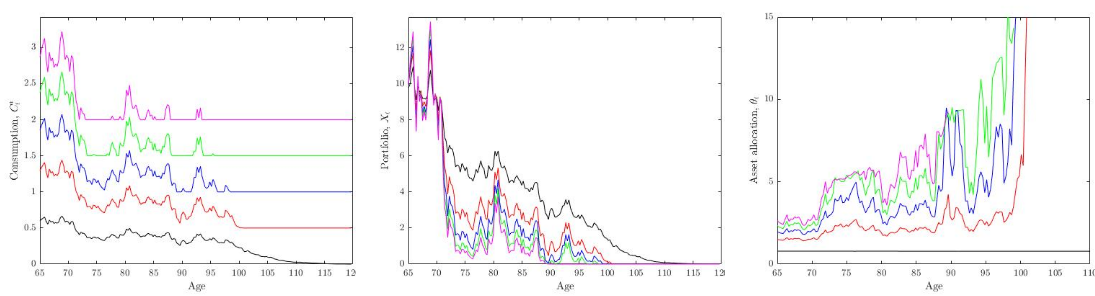
First case study with smoothing factor $\eta = 0.01$. We start our discussion from the smallest value of the smoothing factor $\eta = 0.01$. As we know, in this case the habit $\bar{C}_t$ has a small effect on the client's consumption strategy. Below there are some numerical results for the initial wealth $v = 10$ and the initial habit $\bar{C} = 1$ (see Figure 2.19) and 5 (see Figure 2.20). In Figure 2.19a we show the relationship between consumption and age starting from the retirement, 65yrs in a simulations study.



(a) (b) (c)

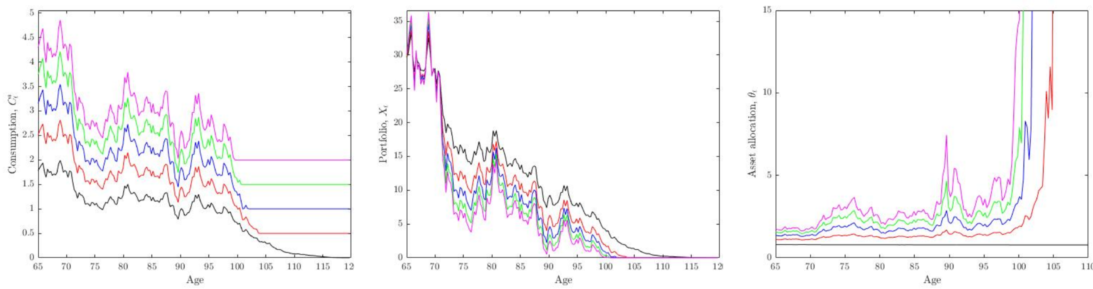
Figure 2.19: Numerical solution for the lifetime problem with smoothing factor $\eta = 0.01$ and the initial habit $\bar{C} = 1$ and initial wealth $v = 10$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

In Figure 2.19b we show the relationship between Portfolio and age and in Figure 2.19c we show asset allocation vs. age. Every curve represents specific pension value, namely $\pi = 0$ (the black line), $\pi = 0.5$ (the red line), 1 (the blue line), 1.5 (the green line) and 2 (the magenta line). Recall that we assume that the client spends all pension at every time moment. We use the same driving Brownian motion for each curve. As we can see the level of consumption grows with pension. The consumption "peak" is typically when age is lowest. When $\pi > 0$, wealth eventually gets very small, and consumption plateaus at the level of pension. Similar behaviour occurs for the higher living standard $\bar{C} = 5$. In Figure 2.20 for higher values of habit, consumption reaches the pension level faster. Asset allocation rises with pension, and rises as wealth declines. The reason behind that - as wealth declines the propotion of pension in portfolio grows and, since we have risk-free income, this allows us to leverage the portfolio.



(a) (b) (c)

Figure 2.20: Numerical solution for the lifetime problem with smoothing factor $\eta = 0.01$ and the initial habit $\bar{C} = 5$ and initial wealth $v = 10$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

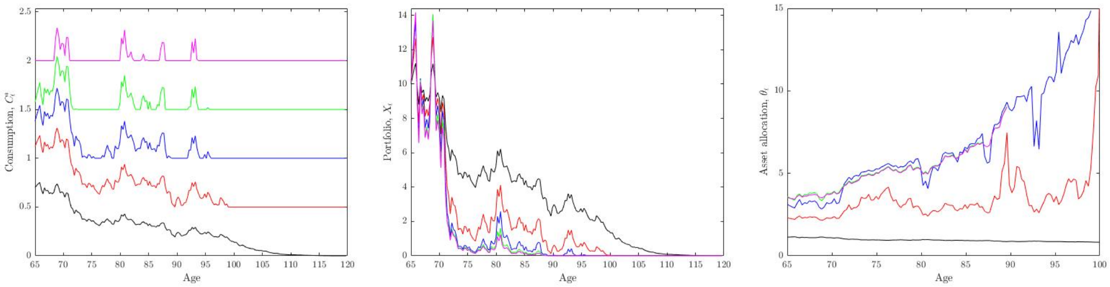


(a) (b) (c)

Figure 2.21: Numerical solution for the lifetime problem with smoothing factor $\eta = 0.01$ and the initial habit $\bar{C} = 5$ and initial wealth $v = 30$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

For the last set of numerical results we increased the initial wealth up to $v = 30$. In Figure 2.21a we can see that the optimal consumption level is higher then in the previous picture Figure 2.20a. Despite that, it takes longer to deplete wealth, particularly for lower values of the pension.

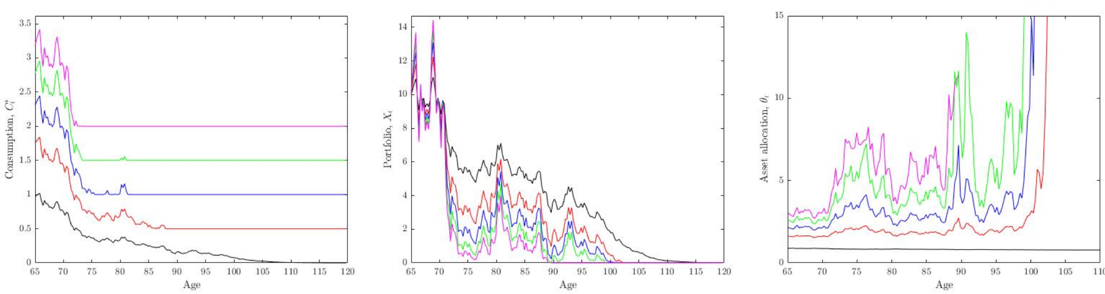
Second case study with smoothing factor $\eta = 0.1$. In the second case with the smoothing factor $\eta = 0.1$ the impact of habit formation is significantly greater. Figure 2.22 shows three plots, as before. The level of consumption is slightly higher than in the previous case as a reflection of the smoothing factor η. Since we are only optimizing portion of consumption related to wealth, client will consume at least at the pension level (see Figure 2.22a for habit $\bar{C} = 1$ or Figure 2.23a for initial habit $\bar{C} = 5$). As in previous case, the specific colour of lines corresponds to the different pension values $\pi = 0$ (the black line), $\pi = 0.5$ (the red line), 1 (the blue line), 1.5 (the green line) and 2 (the magenta line).



(a) (b) (c)

Figure 2.22: Numerical solution for lifetime problem under HFM with smoothing factor $\eta = 0.1$, the initial habit $\bar{C} = 1$ and initial wealth $v = 10$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

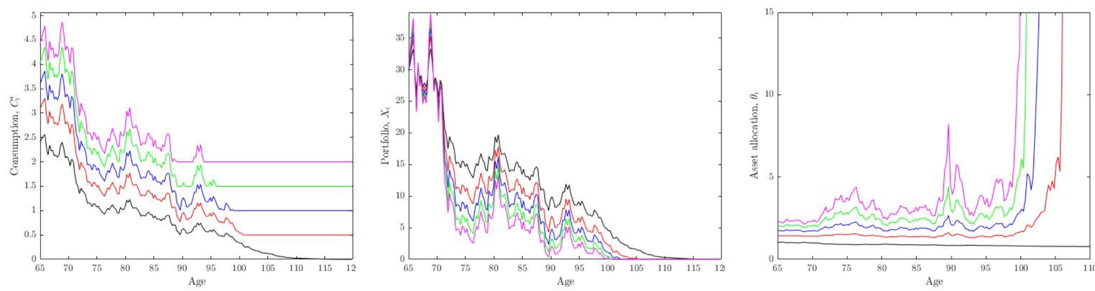
If we compare the consumption pictures for habit 1 and 5 there is a significant difference in results for pension $\pi = 2$.



(a) (b) (c)

Figure 2.23: Numerical solution for the lifetime problem under HFM with smoothing factor $\eta = 0.1$, the initial habit $\bar{C} = 5$ and initial wealth $v = 10$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

The reason is that the level of consumption is initially significantly lower in Figure 2.22a than in Figure 2.23a. In other words, if client has lower initial living standard his consumption at the beginning, under our assumptions, also will be lower. But that also means that consumption can be higher later. In Figure 2.22c we see market recoveries bumping consumption above the level of pension, even for $\pi = 2$, which we don't see in Figure 2.23c.



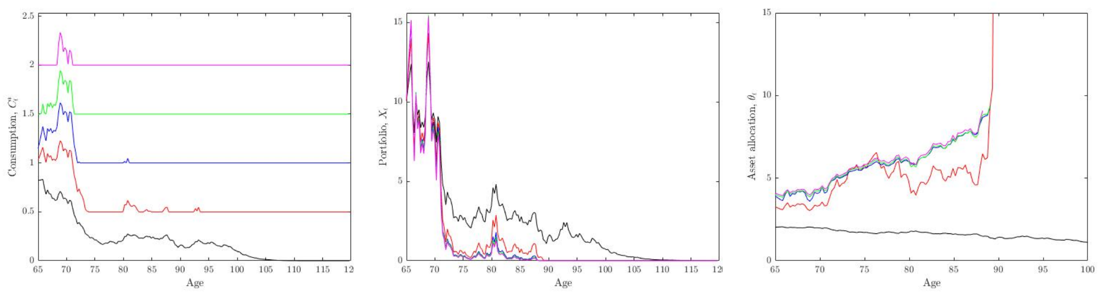
(a) (b) (c)

Figure 2.24: Numerical solution for the lifetime problem under HFM with smoothing factor $\eta = 0.1$, the initial habit $\bar{C} = 5$ and initial wealth $v = 30$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

As in the previous paragraph, the last set of numerical results represent the case with initial wealth value $v = 30$ and initial habit $\bar{C} = 5$ but for a larger smoothing factor. As we have increased the initial wealth, the consumption level also has increased significantly (Figure 2.24a). Second picture in all our sets in this paragraph shows the relationship between Portfolio and age.

We should remark about why wealth depletion has such behaviour. If we look at the Figures 2.22b, 2.23b or 2.24b, we can see that after consumption depletion, the wealth still fluctuate and it is not equal to zero. The reason is the stochasticity of our approach, in other words, portfolio from 2.145 can be not equal to zero with positive probability. Therefore, asset allocations are still meaningful at all times. Note also that while θ is quite smooth when $\pi = 0$, it is not actually constant. In general, wealth shows very fast depletion (Figure 2.24b) with some fluctuations in the end. Another pattern we can find by analyzing the relationship between asset allocation and age, Figures 2.22c, 2.23c or 2.24c. Asset allocation rises allowing high leverage in our portfolio (Figure 2.24c). According to the theory 2.152 if wealth goes to zero, we should expect asset allocation go to infinity.

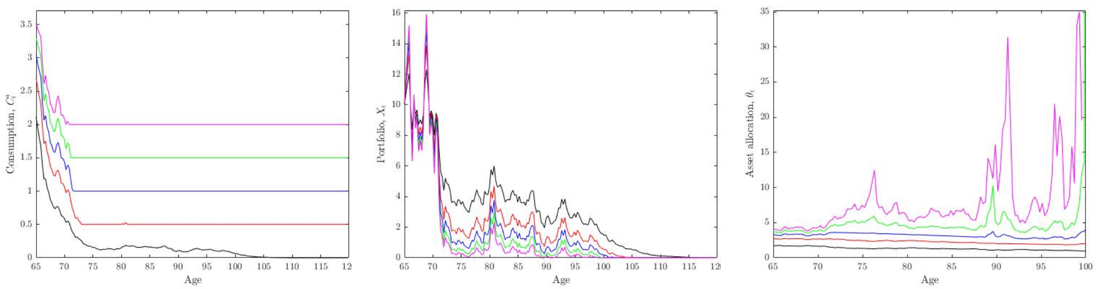
Third case study with smoothing factor $\eta = 1$. The last case shows numerical results for the model with the highest value of the smoothing factor $\eta = 1$. This means that our model responds to changes in the client's consumption very quickly, as a result, the graphs that represent habit and consumption should be close to each other and we will see this behaviour in the next paragraph. Here we provide the numerical results for two values of an initial habit $\bar{C}_t = 1$ (see Figure 2.25) and 5 (see Figure 2.26). The specific colour of lines represents the fixed pension value π, namely $\pi = 0$ (the black line), $\pi = 0.5$ (the red line), 1 (the blue line), 1.5 (the green line) and 2 (the magenta line). The first set of pictures in Figure 2.25 shows the case where the client has a very low living standard $\bar{C}_t$, and, as a result, we can see a very low consumption level in Figure 2.25a. As pension grows the consumption level goes down for small values of wealth which means that the greedy consumption lies below the pension level. At the same time, in the beginning, we assumed that a client will consume at least the pension income therefore all pictures eventually have a flat line. Figure 2.25a and Figure 2.26a show slightly different behaviours of consumption. The reason is that in this case we have higher living standard $\bar{C}_t = 5$ and, therefore, consumption will be higher. For both cases the asset allocation shows similar behaviour (see Figure 2.25c or 2.26c).



(a) (b) (c)

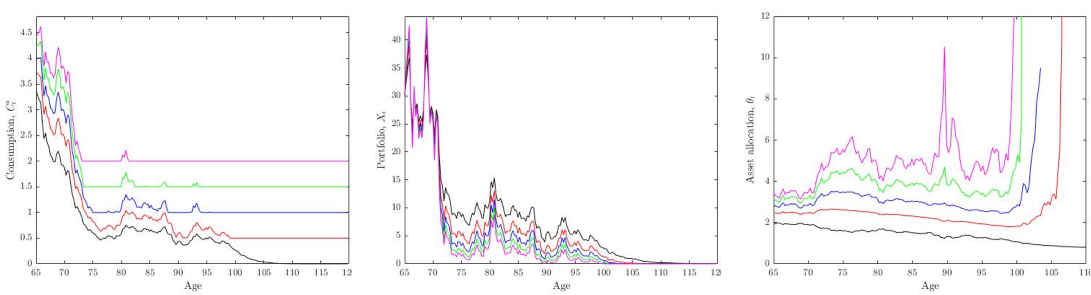
Figure 2.25: Numerical solution for the lifetime problem with $\eta = 1$, the initial habit $\bar{C} = 1$ and initial wealth $v = 10$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

Note in Figure 2.25c that consumption from wealth for high values of pension π is very small. We might have seen a rise in consumption once habit had moved up towards the pension level. But in this simulation, we can see market dropped significantly by age 75 which is why consumption did not recover. Because of pension income, the client can afford to use a more leveraged portfolio when wealth is small, which is why the asset allocation in both figures gets large over time. When pension is zero (black line) the asset allocation will decline over time. Consumption and asset allocation behaviour coincides with portfolio behaviour in Figure 2.26b. How the portfolio declines depends on the initial habit and the smoothing factor η. For example, if we look at the Figure 2.26c or 2.27c we can see that when pension is small, $\pi = 0.5$ or 1 the asset allocation over time declines and the reason is that the amount of pension is not big enough in order to allow high leverage.



(a) (b) (c)

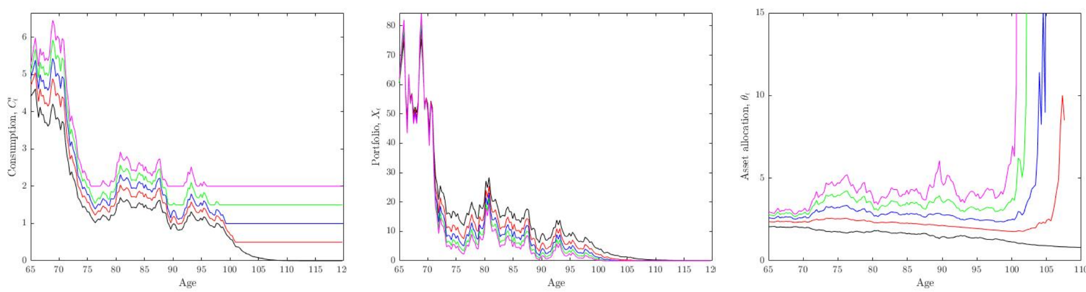
Figure 2.26: Numerical solution for the lifetime problem with $\eta = 1$ and the initial habit $\bar{C} = 5$ and initial wealth $v = 10$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).



(a) (b) (c)

Figure 2.27: Numerical solution for the lifetime problem under HFM with smoothing factor $\eta = 1$ and the initial habit $\bar{C} = 5$ and initial wealth $v = 30$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

At the same time, if we look at the black line in relationship asset allocation versus age (for example Figure 2.25c, 2.26c or 2.27c), which represents the case without a pension, we can see that asset allocation decreases over time. The reason is that without additional income, all consumption depends only on wealth and there is no possibility to apply high leverage.



(a) (b) (c)

Figure 2.28: Numerical solution for the lifetime problem under HFM with smoothing factor $\eta = 1$ and the initial habit $\bar{C} = 5$ and initial wealth $v = 60$. Relationship between consumption vs. age (a), wealth vs. age (b) and asset allocation vs. age (c).

Another comparison between Figure 2.26, Figure 2.27 and Figure 2.28 shows that increase in initial wealth from 10 to 60 does not change the numerical results qualitatively. The difference is in increasing consumption level only. The main drawback for this solution is that it is computationally expensive.

Comparison portfolio and wealth depletion calculations. In the last paragraph we will implement numerical tests that can give us understanding how accurate our results are. In order to implement this we recall the wealth dynamics

$$\mathrm{d}X_t = [\theta_t(\mu - r) + r]X_t\mathrm{d}t + \theta_t\sigma X_t\mathrm{d}W_t + \pi\mathrm{d}t - C_t\mathrm{d}t. \tag{2.156}$$

We will use the same approach as in Chapter 1. Unlike the previous chapter, here we will assume that the client follows the greedy strategy. Also we will use the numerical results obtained in the previous paragraphs. We discretize and plug in the values of asset allocation θ_t^* and consumption C_t^* obtained by using the greedy algorithm.

$$X_{n+1} = X_n + [\theta_n^*(\mu - r) + r]X_n\Delta t + \theta_n^*\sigma X_n Z\sqrt{\Delta t} + \pi\Delta t - C_n^*\Delta t \tag{2.157}$$

where $Z \sim N(0,1)$. For habit we did discretization in the previous paragraph (2.155). As we mentioned before, the main drawback of the probabilistic approach is that it is time consuming to compute portfolio values. As a result, in order to verify results, we will provide just a few numerical tests. For example, in Figure 2.29 we compare numerical results for a very small value of the smoothing factor $\eta = 0.01$, initial habit $\bar{C} = 5$ and the initial wealth $v = 10$. On the left we compare portfolio (the black solid line) computed based on the main formula (2.145) and time ("Age=time+65") and wealth ((the blue dashed line)) computed based on formula (2.157). Calculations were done with time nodes $N = 900$ and Monte Carlo samples $MC = 10000$. As we can see, both solutions are in good agreement.

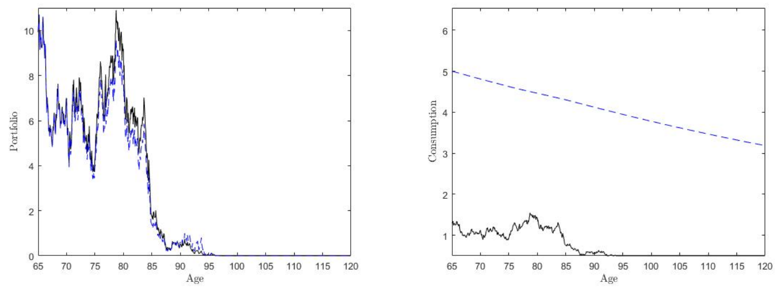


Figure 2.29: Comparison of the numerical solutions for fixed pension $\pi = 0.5$, $\eta = 0.01$, the initial habit $\bar{C} = 5$, the initial wealth $v = 10$.

On the right we show consumption (the black solid line) and age and the average living standard (the blue dashed line). Since the smoothing factor η is very small the model does not react fast to changes in the client's consumption and therefore we can see a gap between the consumption and habit lines. There is a pattern: the smaller the smoothing factor, the greater the gap between these lines. This behaviour is also in agreement with the optimal PDE solution. Similar behaviour is seen in the next Figure (2.30) that represents comparison for a slightly bigger smoothing factor $\eta = 0.1$.

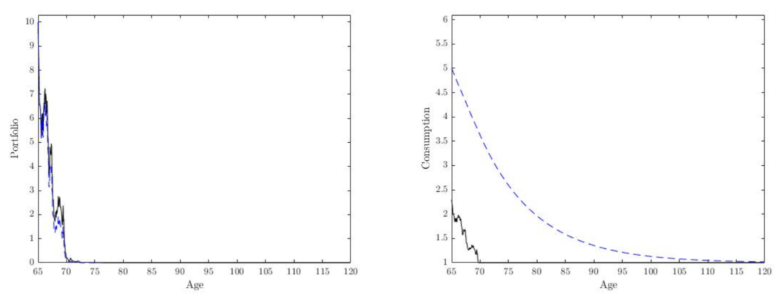
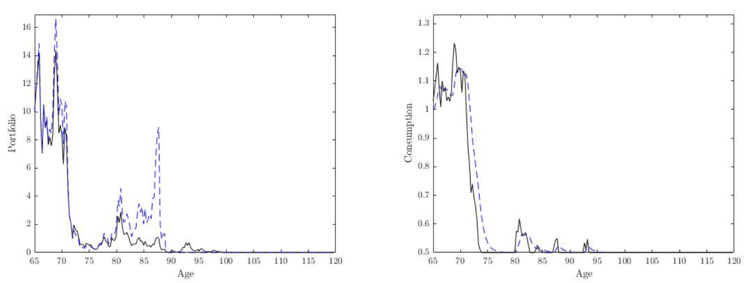
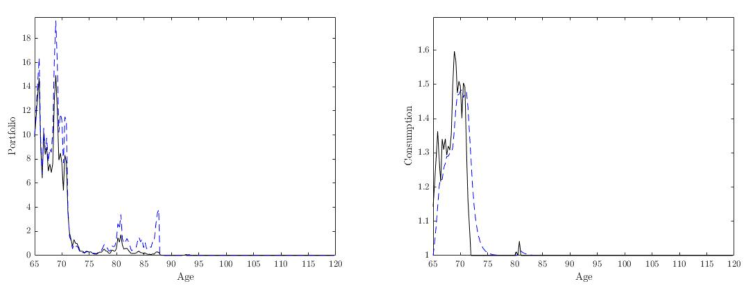


Figure 2.30: Comparison of the numerical solutions for fixed pension $\pi = 1$, $\eta = 0.1$, the initial habit $\bar{C} = 5$, the initial wealth $v = 10$.

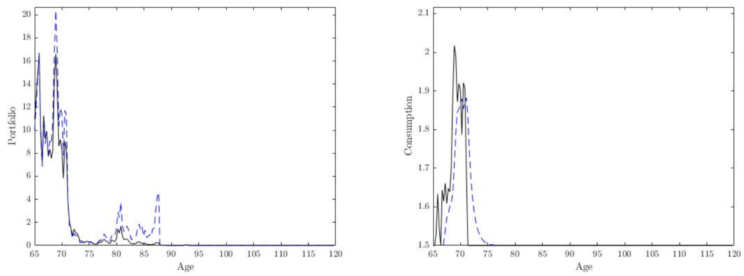
The next set of numerical tests (2.31) shows four pairs of graphs which correspond to the different values of pension, namely $\pi = 0.5$ (a), 1 (b), 1.5 (c) and 2 (d), the initial habit $\bar{C} = 1$ and the initial wealth $v = 10$. On the left we compare, as in the previous case, the relationship between portfolio (the black solid curve) and wealth (the blue dashed curve) vs. age and it clearly can be seen that these curves stay close to each other. Because the smoothing factor is big $\eta = 1$ we see that on the right side the curve that represents consumption (the black solid curve) and habit (the blue dashed curve) are very close. This means that the model responds quickly to changing consumption and also coincides with our expectations. For these calculations the number of time nodes were taken to be $N = 200$ and Monte Carlo samples $MC = 700$.



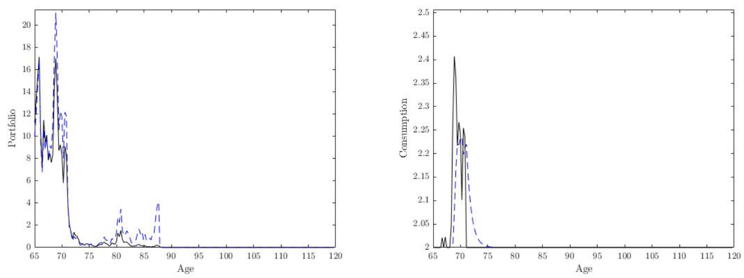
(a)



(b)



(c)



(d)

Figure 2.31: Comparison of the numerical solutions for four pension values: $\pi = 0.5$ (a), $\pi = 1$ (b), $\pi = 1.5$ (c) and $\pi = 2$ (d) with $\eta = 1$.

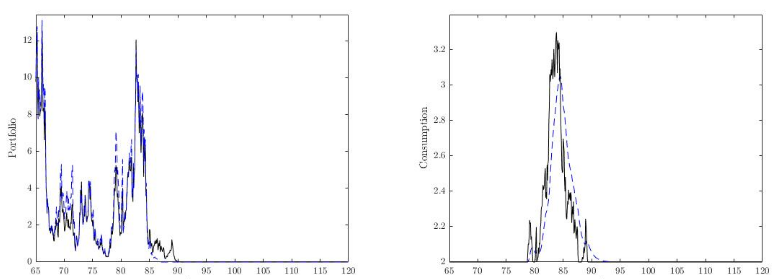


Figure 2.32: Comparison of the numerical solutions for fixed pension $\pi = 2$, $\eta = 1$, the initial habit $\bar{C} = 1$, the initial wealth $v = 10$.

If we analyze all aforementioned graphs, we can see certain errors. The accuracy will increase with the number of nodes in both, time nodes and

Monte Carlo samples. For example, if we consider the last graph with pension $\pi = 2$ and increase $N = 700$ and $MC = 1000$ we will immediately see an improvement in our solution (Figure 2.32).

The last set of pictures for comparison represents the numerical results when we increased the initial wealth $v = 60$ and habit $\bar{C} = 5$ and fixed pension $\pi = 1$. As we can see the results show good agreement, consumption with habit in the left picture Figure 2.33 and portfolio vs. wealth in the right one.

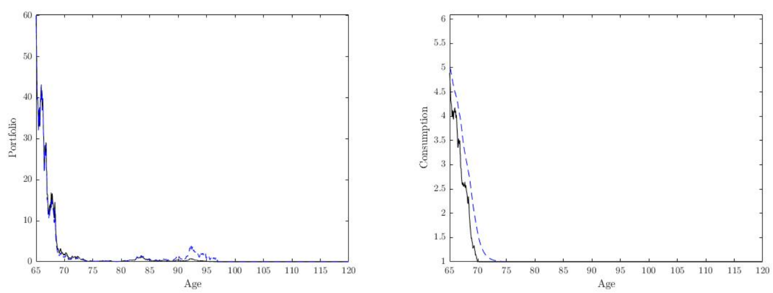


Figure 2.33: Comparison of the numerical solutions for fixed pension $\pi = 1$, $\eta = 1$, the initial habit $\bar{C} = 5$, the initial wealth $v = 60$.

Conclusion and future work

Conclusion

In this thesis, we considered two different approaches for solving a retirement spending optimization problem under the presence of a habit in a complete financial market. One of the main goals of our work was to find an optimal consumption strategy in order to show how a potential client can spend money during retirement, taking into account his living standard and explore different numerical techniques that would allow us to solve this problem numerically. In Chapter 1, we discussed the value function approach where we were able to find the optimal consumption strategy by solving a nonlinear Hamilton-Jacobi-Bellman partial differential equation. We also provided some analyses to validate the chosen numerical method. In addition, by fixing initial wealth v and living standard $\bar{c}$ we analyzed how both, wealth and consumption, change over time, under the assumption that the asset allocation θ is fixed. There is another possibility for the client to invest money, namely to buy so-called annuities. In Chapter 1, we also explored this possibility within our model and discussed whether it is reasonable to buy annuities or not.

The results of the thesis indicate that considering HFM as an underlying model can qualitatively change the optimal solution and provide a better consumption strategy for retirement than Merton's problem. Also, these results show that the presense of a pension in a long-term inverstment can significantly increase consumption in the case of a high living standard and suggests a modest level of consumption if we are dealing with a low habit.

In the second chapter, we presented a probabilistic approach for approximating our optimization problem. Here we have solved the retirement spending problem under slightly different assumptions. The client was able to change the asset allocation during retirement, but also received a pension that must be spent. More precisely, in Chapter 2 we obtained a greedy policy algorithm. This explaines why the numerical solutions obtained by both methods did not coincide for all habit values. As in the previous chapter, by fixing the initial wealth and habit, we found a strategy for spending money during retirement for the client, but unlike the previous results, we provided a solution for different pension levels and some fixed time moments. In addition, we were able to compare both numerical approaches and analyze when the greedy policy is close to optimal.

In the end, we can conclude, the results on how the asset allocation sensitive to changes in, for example, wealth suggest that the presence of pension income enables a potential client to invest in riskier assets and, as a consequence, may anticipate higher expected returns. Also, as an implicit result, it may be a trigger to think about employer-sponsored pension plans in order to increase reskless income during the retirement, as a result, expectations to increase consumption level in the future.

Future work

To better understand the implications of our results, future studies could include, for example, the analysis of different pension levels for optimal problem or introducing asset allocation as another control variable. There are many other variations of this problem that have been left for the future. We can choose a different utility function or different wealth dynamics. As an example, one of the possible modifications in the wealth dynamics can be the presence of jumps. Another variation can be considering an incomplete financial market, i.e. the presence of transaction costs.

Nowadays, there is a growing interest to solve optimization problems using reinforcement learning techniques. Reinforcement learning is learning what to do or how to map our settings to actions so as to get optimal solution. The learner is not told which actions to take, but instead must discover which actions yield the most reward by trying them [Sutton & Barto, 2020]. Under this approach we can try to solve the original optimization problem and see if we can get any advantages by using this technique.

Finally, both numerical approaches have proven to be very time consuming, so it would be interesting to think about improving the efficiency of these methods by reducing the computational time.

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