

**MULTIPLE RISK FACTOR DEPENDENCE STRUCTURES WITH
APPLICATIONS TO ACTUARIAL RISK MANAGEMENT**

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Abstract

Actuarial and financial risk management is one of the most important innovations of the 20th century, and modelling dependence risks is one of its central issues. Traditional insurance models build on the assumption of independence of risks. Criticized as one of the main causes of the recent financial crisis, this assumption has facilitated the quantification of risks for decades, but it has often lead to under-estimation of the risks and as a result under-pricing. Hence importantly, one of the prime pillars of the novel concept of Enterprise Risk Management is the requirement that insurance companies have a clear understanding of the various interconnections that exist within risk portfolios.

Modelling dependence is not an easy call. In fact, there is only one way to formulate independence, whereas the shapes of stochastic dependence are infinite. In this dissertation, we aim at developing interpretable practically and tractable technically probabilistic models of dependence that describe the adverse effects of multiple risk drivers on the risk portfolio of a generic insurer. To this end, we introduce a new class of Multiple Risk Factor (MRF) dependence structures. The MRF distributions are of importance to actuaries through their connections to the popular frailty models, as well as because of the capacity to describe dependent heavy-tailed risks. The new constructions are also linked to the factor models that lay in the very basis of the nowadays financial default measurement practice. Moreover,

we use doubly stochastic Poisson processes to explore the class of copula functions that underlie the MRF models. Then, motivated by the asymmetric nature of these copulas, we propose and study a new notion of the paths of maximal dependence, which is consequently employed to measure tail dependence in copulas.

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Chapter 1

Introduction

1.1 Practical background

No industry is immune to failures. The stability of the insurance industry, and hence of its major players, is a necessary factor for the macroeconomic growth of an entire country. Although the comparative frequency of insurer insolvencies has so far been relatively low, the US alone witnessed approximately 700 such failures between 1969 and 1998 ([Massey et al., 2003](#)). To assure solvency of insurance companies, sets of state-dependent laws, called insurance regulations, have been developed and constantly adjusted to the real world's changes. The recent financial crisis of 2008 left little doubt, that the existing regulations were rather inadequate and a significant change was inevitable.

As a response, the International Association of Insurance Supervisors (IAIS), of which Canada is a member, introduced the notions of Enterprise Risk Management (ERM) and Own Risk and Solvency Assessment (ORSA) as two key components of the regulatory reform. Loosely speaking, the ERM identifies, measures and monitors insurer risks while the ORSA manages them. The main differences between the new regulation and its predecessors

is perhaps the level and soundness of the quantitative methods involved as well as the extent of their integration in top-tier decision making. Hence the first pillar in the proposed new accords (e.g. Basel III and Solvency II) is devoted to the risk measurement and the calculation of capital requirements processes that are supposed to cushion unexpected losses. Details of the new regulatory accords are beyond the scope of this dissertation, and we only mention a few points that are of interest herein (e.g., [BIS, 2009](#); [Caouette et al., 2008](#); [Christiansen et al., 2012](#); [IAA, 2013](#)):

- financial institutions shall base capital needs on unexpected losses;
- financial institutions shall ensure that systemic risks and tail events are adequately captured in institutions' risk modelling and stress testing;
- financial institutions shall realize and understand the effects of correlation among the risk portfolios;
- new accords offer the option of using advanced approaches in that financial institutions can use their own internal models to quantify risks as long as the models are deemed satisfactory by the regulators.

According to the Canadian Office of the Superintendent of Financial Institutions (OSFI), the new regulation applies to all federally regulated life and property and casualty insurers in Canada, and it must be implemented starting from 2014 ([OSFI, 2014](#)).

1.2 Motivation

The new realm brings many open questions, and this thesis aims to investigate two issues imposed by the ERM and ORSA, that are extremes and dependence. Indeed, while the

assumption of light-tailed distributions and/or independence is a well-accepted allowance in traditional actuarial mathematics, in reality, it is rarely plausible and more of an exception. Are the two aforementioned notions at all important for an insurer? We give a brief answer to this question next.

1.2.1 Extremes

In 1998 freezing rains fell on eastern Ontario, south-western Quebec and lasted for six days. The event doubled the amount of precipitation in the area experienced in any prior ice storm, and resulted in a catastrophe that produced excess of 840,000 cases of insurance claims. This number is 20% more than that of the claims caused by the Hurricane Andrew - one of the largest natural disasters in the history of North America. After all, the catastrophe caused approximately 1.44 billion Canadian dollars insurance settlements which is the highest loss burden in the history of Canada ([Lecomte et al., 1998](#)).

In the context of insurance, a few heavy losses hitting a portfolio and then converting into claims usually represent the greatest part of the indemnities paid by insurance companies. The just-mentioned losses, also called ‘extremes’, are quantitatively modelled by the tails of the associated probability distributions.

Extremes are rather difficult to tackle analytically, and distributional models, which provide a good overall fit to actuarial data can perform rather poorly in fitting the tail. Relying on models with improper tails is of course daunting, e.g., insurance losses may occur with worse severity and periods of financial stress may appear with higher frequency than actuaries expect. This may further lead to poor reserving or even insolvency.

1.2.2 Dependence

Imagine a pool of 20-year term annually-paid joint life level annuities-due issued to married couples. The remaining lifetimes of the annuitants are per Complete Life Tables Canada 2000 – 2002, 2a and 2b, and the real interest rate is an average of .7%, as per World Bank's 2008 – 2012 Canadian data. Simple calculations show that under the standard assumption of independence of lifetimes (as opposed to a more realistic one of positive quadrant dependence (PQD)), the contracts are significantly under-priced. The spreads grow with age and are, for instance, 12% for a typical couple of a 65 old man and a woman aged 60. As the random payments due to the annuities in the pool, are themselves (inter)dependent, the impairment becomes rather daunting ([Denuit et al., 2005](#)).

In reality, structures similar to the aforementioned simple life annuity serve as the building block of many a.) insurance products such as registered retirement saving plans, registered retirement income funds and general pensions; and b.) various financial derivatives such as catastrophe bonds as well as basket credit default swap contracts. In fact, the share of such insurance products (premium-wise) exceeded a quarter of the entire Life and Health Insurance business in Canada in 2014 ([CLHIA, 2015](#)), while the derivatives above have been becoming the fastest growing sector in the financial market ([IOSCO, 2012](#)). In light of this, the notion of dependence is vital to the solvency of a stand-alone (life) insurer as well as to the stability of the entire financial sector and even to the overall health of the economy. Speaking shortly, neglecting dependencies within and among portfolios of risks can and often does lead to solvency issues and bankruptcy of institutions.

1.2.3 Thesis organization

In this dissertation, we aim at introducing practically interpretable and analytically tractable models of dependence, which are tolerant to extreme risks. The rest of this dissertation is organized as follows. In Chapter 2, a new multivariate distribution possessing arbitrarily parametrized univariate Pareto margins is introduced. Unlike the probability law of [Asimit et al. \(2010\)](#), the structure in this chapter is absolutely continuous with respect to the corresponding Lebesgue measure. The distribution is of importance to actuaries through its connections to the popular frailty models, CreditRisk⁺ as well as because of the capacity to describe dependent heavy-tailed risks. The genesis of the new distribution is linked to a number of existing probability models, and useful characteristic results are proved. Expressions for, e.g., the decumulative distribution function, (joint) moments and regressions are developed. The distributions of minima and maxima, as well as, some weighted risk measures are employed to exemplify possible applications of the distribution in insurance.

In Chapter 3, we introduce the main object of study in this dissertation. Namely, we introduce a new class of dependence structures, that we call the Multiple Risk Factor (MRF) dependence structures. The construction is connected to the one proposed in Chapter 2 and also to the factor models that lay in the very basis of the nowadays financial default measurement practice. On the one hand the new models extend the popular CreditRisk⁺ approach, and as such they formally describe default risk portfolios exposed to an arbitrary number of fatal risk factors with conditionally exponential hitting times that can be independent, positively dependent and fully comonotonic. There are numerous empirical evidences that CreditRisk⁺ tends to underestimate the real life default times, and we believe that our proposed approach can resolve this inconvenience. On the other hand, the MRF structures

can be viewed as a quite general family of multivariate probability distributions with Pareto of the 2nd kind univariate margins, and in this capacity, they can model risk portfolios of (insurance) losses with heavy-tailed and positively dependent risk components.

In Chapter 4, we use doubly stochastic Poisson processes to explore the class of copula functions that underlie the MRF dependence structures discussed in Chapter 3. Copulas have become an important element of best practice Enterprise Risk Management, displacing in many contexts other more conservative approaches to modelling stochastic dependence. A ‘good’ copula should conform to a problem at hand, be asymmetric and exhibit some extent of tail dependence. Herein, we introduce and study a new class of MRF copula functions, which we show are exactly such. Namely, we demonstrate that the MRF copulas arise from a number of default risk specifications with stochastic default barrier and are in general non-exchangeable in their domain of definition as well as in the tail behaviour. The MRF copulas cover the full range of non-negative dependence and are quite tractable analytically.

An important property that sets the MRF copulas apart from many other dependence structures employed in the modern theory and practice of default risk is the capability to allow for non-zero probabilities of simultaneous default. Speaking formally, this becomes feasible because the MRF copulas are not absolutely continuous with respect to the corresponding Lebesgue measure. Appealing to the language of credit risk, this is a direct outcome of the inclusion of the systemic risk factors having fully comonotonic hitting times. One way or the other, this special characteristic contributes to the effectiveness of the MRF copulas when modelling the clustering of real world defaults.

Motivated by the asymmetric dependencies studied in Chapters 2 to 4, in Chapter 5 we introduce a new notion of maximal dependence paths and then employ it to measure tail dependence in copulas. We demonstrate both analytically and numerically that the

existing methods for measuring tail dependence in copulas may sometimes underestimate the extent of extreme co-movements of dependent risks and, therefore, may not always comply with the new paradigm of prudent risk management. This phenomenon holds in the context of both symmetric and asymmetric copulas with and without singularities. As a remedy, we introduce a notion of *paths of maximal (tail) dependence* and utilize the notion to propose several new indices of tail dependence. The suggested new indices are conservative, conform with the basic concepts of modern quantitative risk management, and are capable of differentiating between distinct risky positions in situations when the existing indices fail to do so. Chapter 6 concludes the dissertation.

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Chapter 2

A form of multivariate Pareto distribution

The following paper introduces and studies a new class of absolutely continuous multivariate Pareto distributions. Applications in insurance are explored. This chapter has been submitted for publication with Edward Furman.

A FORM OF MULTIVARIATE PARETO DISTRIBUTION

Abstract. A new multivariate distribution possessing arbitrarily parametrized univariate Pareto margins is introduced. Unlike the probability law of Asimit et al. (2010) [Asimit, V., Furman, E. and Vernic, R. (2010). “On a multivariate Pareto distribution,” *Insurance: Mathematics and Economics* **46**(2), 308 – 316], the structure in this paper is absolutely continuous with respect to the corresponding Lebesgue measure. The distribution is of importance to actuaries through its connections to the popular frailty models, as well as because of the capacity to describe dependent heavy-tailed risks. The genesis of the new distribution is linked to a number of existing probability models, and useful characteristic results are proved. Expressions for, e.g., the decumulative distribution function, (joint) moments and regressions are developed. The distributions of minima and maxima, as well as, some weighted risk measures are employed to exemplify possible applications of the distribution in insurance.

Keywords and phrases: Multivariate Pareto distributions, characterizations, dependence, weighted Risk Measures, minima, maxima.

2.1 Introduction

We routinely fix the probability space $(\Omega, \Sigma, \mathbf{P})$ and define the random vector (r.v.) $\mathcal{X} \ni \mathbf{X} = (X_1, \dots, X_n)'$ as a map from (Ω, Σ) to the $n(\in \mathbf{N})$ -dimensional Borel space $(\mathbf{R}_+^n, \mathcal{B}(\mathbf{R}_+^n))$. The cumulative distribution function (c.d.f.) of $\mathbf{X}$ is in the sequel denoted by $F_{1,\dots,n}(x_1, \dots, x_n) = \mathbf{P}[X_1 \leq x_1, \dots, X_n \leq x_n]$, and the corresponding probability density function (p.d.f.) – by $f_{1,\dots,n}(x_1, \dots, x_n) = \partial/(\partial x_1 \cdots \partial x_n) F_{1,\dots,n}(x_1, \dots, x_n)$, where $(x_1, \dots, x_n)' \in \mathcal{R} \subseteq \mathbf{R}_{0,+}^n := [0, \infty)^n$. Finally, $F_i(x)$ and $f_i(x)$ denote, respectively, the marginal c.d.f. and p.d.f. of X_i , $i = 1, \dots, n$. Clearly r.v.'s à la $\mathbf{X} \in \mathcal{X}$ with mutually independent margins $X_i \sim F_i$ are characterized by the product c.d.f. $F_{1,\dots,n}^\Pi := \prod_{i=1}^n F_i$, while dependency of these margins readily yields an infinite Fréchet class of c.d.f.'s $\mathcal{F}_{1,\dots,n}(F_1, \dots, F_n)$. We further discuss a special subclass of $\mathcal{F}_{1,\dots,n}$ with univariate gamma margins. (For insurance applications of the aforementioned subclass, we refer to, e.g., [Vernic \(1997, 2000\)](#), [Boucher et al. \(2008\)](#), [Tsanakas \(2008\)](#), [Pfeifer and Nešlehová \(2004\)](#) and [Furman and Landsman \(2010\)](#); as well as to the references therein.) This then paves the way to introducing a new dependence structure with univariate Pareto-distributed margins in Section 2.2.

Let $\mathcal{Y} \ni \mathbf{Y} = (Y_1, \dots, Y_{n+1})'$ be an $(n+1)$ -variate r.v. with mutually independent univariate gamma-distributed margins. Namely, for $j = 1, \dots, n+1$, the p.d.f.'s of $Y_j \sim G_j(\gamma_j(\in \mathbf{R}_+), \alpha_j(\in \mathbf{R}_+))$ are given by

$$g_j(y; \gamma_j, \alpha_j) = e^{-\alpha_j y} \frac{y^{\gamma_j-1} \alpha_j^{\gamma_j}}{\Gamma(\gamma_j)}, \quad y \in \mathbf{R}_+ \quad (2.1.1)$$

with the Laplace transforms being well-defined on $x \in \mathbf{R}_{0,+}$ (the interval of interest herein) and given by

$$\hat{G}_j(x; \gamma_j, \alpha_j) = \int_0^\infty e^{-xy} g_j(y) dy = \left(1 + \frac{x}{\alpha_j}\right)^{-\gamma_j}. \quad (2.1.2)$$

Let $A \in \text{Mat}_{n \times (n+1)}(\mathbf{R}_0, +)$ denote a deterministic $n \times (n+1)$ matrix with non-negative entries.

Example 2.1.1 (Mathai and Moschopoulos, 1991; see also Cherian, 1941 and Krishnamoorthy and Parthasarathy, 1951). Let $Y_j \sim G_j(\gamma_j, \alpha_j)$, $j = 1, \dots, n+1$ be mutually independent gamma random variables with arbitrary parameters, and choose the matrix A such that, for $i = 1, \dots, n$ and $\sigma_i > 0$, it holds that $(A)_{i,n+1} = \alpha_{n+1}/\alpha_i$, $(A)_{i,i} \equiv 1$ and zero otherwise. Then $\mathbf{X} \sim G_{1,\dots,n}^A(\boldsymbol{\gamma}^*, \boldsymbol{\alpha})$, where $\boldsymbol{\gamma}^* = (\gamma_{n+1} + \gamma_1, \dots, \gamma_{n+1} + \gamma_n)'$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)'$ are two n -variate vectors of parameters.

Example 2.1.2 (Mathai and Moschopoulos, 1992; see also Furman, 2008). Let $Y_j \sim G_j(\gamma_j, \alpha_j)$, $j = 1, \dots, n$ be mutually independent gamma random variables with arbitrary parameters, and choose the matrix A such that, for $i = 1, \dots, n$, $j = 1, \dots, i$ and $\sigma_i > 0$, it holds that $(A)_{i,j} = \alpha_j/\sigma_i$ and zero otherwise. Then $\mathbf{X} \sim G_{1,\dots,n}^A(\boldsymbol{\gamma}^*, \boldsymbol{\sigma})$, where $\boldsymbol{\gamma}^* = (\gamma_1^*, \dots, \gamma_n^*)'$, $\gamma_i^* = \sum_{j=1}^i \gamma_j$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)'$ are two n -variate vectors of parameters.

In the present paper we employ the following modification of Examples 2.1.1 and 2.1.2 in order to introduce a simple yet general multivariate distribution with univariate margins distributed Pareto of the 2nd kind.

Example 2.1.3 (Furman, 2008). Let $Y_j \sim G_j(\gamma_j, \alpha_j)$, $j = 1, \dots, n+1$ be mutually independent gamma random variables with arbitrary parameters, and choose the matrix A such that, for $i = 1, \dots, n$, $j = 1, \dots, i$ and $\sigma_i > 0$, it holds that $(A)_{i,j} = \alpha_j/\sigma_i$, $(A)_{i,n+1} = \alpha_{n+1}/\sigma_i$ and zero otherwise. Then $\mathbf{X} \sim G_{1,\dots,n}^A(\boldsymbol{\gamma}^*, \boldsymbol{\sigma})$, where $\boldsymbol{\gamma}^* = (\gamma_1^*, \dots, \gamma_n^*)'$, $\gamma_i^* = \gamma_{n+1} + \sum_{j=1}^i \gamma_j$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)'$ are two n -variate vectors of parameters.

More generally, we arrive at the following definition.

Definition 2.1.1 (Furman, 2008; Furman and Landsman, 2010). For $i = 1, \dots, n$, let $t_i(\cdot)$, $u_i(\cdot) : \mathbf{R}_+^{n+1} \rightarrow \mathbf{R}_+$ be two deterministic positive functions and choose the entries of the matrix A such that $\mathbf{X} := A\mathbf{Y}$ has univariate gamma margins, then we say that $\mathbf{X}$ follows an n -variate gamma distribution with shape parameters $\gamma_i^* = t_i(\gamma_1, \dots, \gamma_{n+1})$ and rate parameters $\alpha_i^* = u_i(\alpha_1, \dots, \alpha_{n+1})$. Notationally, we write $\mathbf{X} \sim G_{1, \dots, n}^A(\boldsymbol{\gamma}^*, \boldsymbol{\alpha}^*)$, where $\boldsymbol{\gamma}^* = (\gamma_1^*, \dots, \gamma_n^*)' \in \mathbf{R}_+^n$ and $\boldsymbol{\alpha}^* = (\alpha_1^*, \dots, \alpha_n^*)' \in \mathbf{R}_+^n$ are vectors of parameters.

We note in passing that a real-valued r.v. is said to be distributed Pareto of the 2nd kind, notationally $X \sim Pa(II)(\mu, \sigma, \alpha)$, where $\mu \in \mathbf{R}$ is a location parameter, $\sigma \in \mathbf{R}_+$ is a scale parameter and $\alpha \in \mathbf{R}_+$ is a tail index, if its c.d.f. is given by

$$F_X(x; \mu, \sigma, \alpha) = 1 - \left(1 + \frac{x - \mu}{\sigma}\right)^{-\alpha}, \quad x \geq \mu \quad (2.1.3)$$

(see, e.g., Pareto, 1897; Arnold, 1983, 2015; Kotz et al., 2002). Similarly to Asimit et al. (2010), we often set $\mu = 0$, which conveniently makes the support of the distribution coincide with the non-negative half of the real line, i.e., $\text{supp}_F = \{x \in \mathbf{R} : f(x) \neq 0\} = \mathbf{R}_{0,+}$ and does not lead to any loss of generality. The resulting distribution (Lomax distribution), notationally $Pa(II)(\sigma, \alpha)$, enjoys a great variety of applications in all areas of applied mathematics in general and in actuarial science in particular, as it naturally arises in the extreme value theory as the limiting distribution of the excess-of-loss r.v. $X_d = X - d | X > d$ where $d(\in \mathbf{R}_+)$ denotes a threshold (see, e.g., Balkema and de Haan, 1974; Pickands III, 1975).

The rest of the paper is organized as follows: In Section 2.2 the multivariate Pareto of interest is defined and linked to a number of existing multivariate distributions, its probabilistic properties are derived and characterization results are proved. In Section 2.3 the new multivariate Pareto is reintroduced as a minima-based multiple risk factor model, and some applications to notions of actuarial interest are considered. In Section 2.4 the usefulness of

the model is elucidated with the help of a numerical example. Section 4.2 concludes the paper.

2.2 An absolutely continuous multivariate Pareto distribution

Let $\mathbf{Y} = (Y_1, \dots, Y_{n+1})'$ be a r.v. with mutually independent coordinates $Y_j \sim Ga(\gamma_j, 1)$, $\gamma_j \in \mathbf{R}_+$, and choose the matrix $A_c \in Mat_{n \times (n+1)}$ such that $(A_c)_{i,j} = c_{i,j}/\sigma_i$, where $c_{i,j} \in \{0, 1\}$ are deterministic constants, $\sigma_i \in \mathbf{R}_+$, $i = 1, \dots, n$ and $j = 1, \dots, n+1$. The following definition unifies Examples 2.1.1, 2.1.2 and 2.1.3 and serves as an auxiliary tool for constructing the multivariate Pareto distribution of interest.

Definition 2.2.1. *Let $\mathbf{X} = (X_1, \dots, X_n)' = A_c \mathbf{Y}$, then it follows an n -variate gamma distribution, notationally $\mathbf{X} \sim Ga_{1,\dots,n}(\boldsymbol{\gamma}_c^*, \boldsymbol{\sigma})$, where $\boldsymbol{\gamma}_c^* = (\gamma_{c,1}^*, \dots, \gamma_{c,n}^*)' \in \mathbf{R}_+^n$ with $\gamma_{c,i}^* = \sum_{j=1}^{n+1} c_{i,j} \gamma_j$, and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)' \in \mathbf{R}_+^n$ are two vectors of parameters.*

We note in passing that Definition 2.2.1 (auxiliary for the present paper) establishes an encompassing multivariate probability law with gamma-distributed univariate margins and an additive common-shock dependence structure (see, Gollier and Pratt, 1996; Tsanakas, 2008; Furman and Landsman, 2010 for applications of the additive common-shock structures in economics and actuarial science). More specifically, the following simple special cases of $Ga_{1,\dots,n}(\boldsymbol{\gamma}_c^*, \boldsymbol{\sigma})$ readily recover the models of Mathai and Moschopoulos (1991, 1992), as well as of Furman (2008):

- $c_{i,i} = c_{i,n+1} \equiv 1$ for $i = 1, \dots, n$ and zero otherwise - Example 1.1;
- $c_{i,j} \equiv 1$ for $1 \leq j \leq i \leq n$ and zero otherwise - Example 1.2;
- $c_{i,j} = c_{i,n+1} \equiv 1$ for $1 \leq j \leq i \leq n$ and zero otherwise - Example 1.3.

Some elementary but useful properties of $\mathbf{X} \sim G_{1,\dots,n}^A$ follow directly by definition or from the Laplace transform that is established next.

Theorem 2.2.1. *Let $\mathbf{X} \sim Ga_{1,\dots,n}(\gamma_c^*, \boldsymbol{\sigma})$ be the multivariate gamma r.v. as in Definition 2.2.1, then the corresponding Laplace transform is given by*

$$\hat{G}_{1,\dots,n}(\mathbf{t}) = \prod_{j=1}^{n+1} \left(1 + \sum_{i=1}^n \frac{c_{i,j}}{\sigma_i} t_i \right)^{-\gamma_j},$$

and it is well-defined on $\mathbf{t} = (t_1, \dots, t_n)' \in \mathbf{R}_{0,+}^n$.

Proof. By construction we readily have that

$$\hat{G}_{1,\dots,n}(\mathbf{t}) = \mathbf{E} \left[e^{-\mathbf{X}'\mathbf{t}} \right] = \mathbf{E} \left[\exp \left\{ - \sum_{i=1}^n \sum_{j=1}^{n+1} \frac{c_{i,j}}{\sigma_i} Y_j t_i \right\} \right] = \prod_{j=1}^{n+1} \hat{G}_j \left(\sum_{i=1}^n \frac{c_{i,j}}{\sigma_i} t_i \right),$$

which along with (2.1.2) completes the proof. $\square$

An immediate consequence of Theorem 2.2.1 is that, for $k, l = 1, \dots, n$,

- the distribution of $\mathbf{X} \sim Ga_{1,\dots,n}(\gamma_c^*, \boldsymbol{\sigma})$ is ‘marginally closed’, i.e., $X_k \sim Ga(\gamma_{c,k}^* (\in \mathbf{R}_+), \sigma_k (\in \mathbf{R}_+))$;
- the expectation of the k -th coordinate is $\mathbf{E}[X_k] = \gamma_{c,k}^* / \sigma_k$;
- the variance of the k -th coordinate is $\mathbf{Var}[X_k] = \gamma_{c,k}^* / \sigma_k^2$;
- for $k \neq l$, the covariance between the coordinates X_k and X_l is non-negative, and it is given by

$$\mathbf{Cov}[X_k, X_l] = \frac{\sum_{j=1}^{n+1} c_{k,j} c_{l,j} \gamma_j}{\sigma_k \sigma_l};$$

- for $k \neq l$, the Pearson’s linear correlation between the coordinates X_k and X_l is non-negative, and it is given by

$$\rho[X_k, X_l] = \frac{\sum_{j=1}^{n+1} c_{k,j} c_{l,j} \gamma_j}{\sqrt{\gamma_{c,k}^* \gamma_{c,l}^*}}.$$

We are now in a position to introduce the multivariate Pareto distribution of interest. To this end, we recall that for the decumulative distribution function (d.d.f.) of the univariate Lomax distribution, it trivially holds that

$$\overline{F}_X(x; \sigma, \alpha) = 1 - F_X(x; \sigma, \alpha) = \hat{G}(x; \alpha, \sigma), \text{ where } x \in \mathbf{R}_0, +$$

and $\sigma \in \mathbf{R}_+, \alpha \in \mathbf{R}_+$.

Definition 2.2.2. *We call the r.v. $\mathbf{X} = (X_1, \dots, X_n) \sim F_{1,\dots,n}$ having the d.d.f.*

$$\overline{F}_{1,\dots,n}(x_1, \dots, x_n) = \prod_{j=1}^{n+1} \left(1 + \sum_{i=1}^n \frac{c_{i,j}}{\sigma_i} x_i \right)^{-\gamma_j}, \text{ where } (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n \quad (2.2.1)$$

a multivariate Pareto of the 2nd kind. Notationally, we write $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$, where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)'$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)'$ are two deterministic vectors of positive parameters, and $\gamma_{n+1} \in \mathbf{R}_+$ and $c \in Mat_{n \times (n+1)}$ are scalar-valued and matrix-valued parameters, respectively.

Obviously, the form of d.d.f. (2.2.1) is invariant under linear transformations, and it therefore belongs to a (multivariate) location-scale family of distributions.

The probability law in Definition 2.2.2 is a generalization of the classic multivariate Pareto distribution of Arnold (1983, 2015) whose d.d.f. is obtained from (2.2.1) by setting $c_{i,j} = c_{\bullet,j}$, $i = 1, \dots, n$, $j = 1, \dots, n+1$ and zero otherwise. That being said, such well-known drawback of the classic multivariate Pareto distribution as 1.) common tail-index parameter to all univariate margins and 2.) incapability to model independence do not manifest in $Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$. Indeed, check Theorem 2.2.2 for the former drawback and set $c_{i,i} \equiv 1$, $i = 1, \dots, n$ and zero otherwise for the latter drawback. Furthermore, the form of multivariate Pareto presented in Definition 2.2.2 unifies both probability models studied recently in Chiragiev and Landsman (2009).

We further document several simple properties of the multivariate Pareto with d.d.f. (2.2.1). The proofs are straightforward and thus omitted.

Theorem 2.2.2. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as in Definition 2.2.2, then, for $i = 1, \dots, n$, the marginal d.d.f. of X_i is*

$$\bar{F}_i(x_i) = \left(1 + \frac{x_i}{\sigma_i}\right)^{-\gamma_{c,i}^*},$$

that is $X_i \sim Pa(II)(\sigma_i, \gamma_{c,i}^*)$, where $\gamma_{c,i}^* = \sum_{j=1}^{n+1} c_{i,j} \gamma_j$. Also, for $i = 1, \dots, n$ and setting $\gamma_{c,i}^* > 1$, we have that

$$\mathbf{E}[X_i] = \sigma_i / (\gamma_{c,i}^* - 1), \quad (2.2.2)$$

and furthermore setting $\gamma_{c,i}^* > 2$, we obtain that

$$\mathbf{Var}[X_i] = \sigma_i^2 \gamma_{c,i}^* / \left((\gamma_{c,i}^* - 1)^2 (\gamma_{c,i}^* - 2) \right). \quad (2.2.3)$$

Clearly, the substitution $c_{i,i} = c_{i,n+1} \equiv 1$, $i = 1, \dots, n$ and zero otherwise yields Theorem 1 of Chiragiev and Landsman (2009), whereas the substitution $c_{i,j} \equiv 1$ for $1 \leq j \leq i \leq n$ and zero otherwise results in their Theorem 5.

The following lemma plays an important auxiliary role when deriving the expression for the covariance in the context of the multivariate Pareto distribution of interest, and it also establishes a useful characteristic relation; in the sequel ‘ $\stackrel{d}{=}$ ’ denotes equality in distribution. As a by-product, the lemma establishes a multiplicative common-shock representation of the multivariate probabilistic structure of main interest herein (see, Franke et al., 2006; Meyers, 2007 for applications of the multiplicative common-shock structures in economics and actuarial science).

Lemma 2.2.1. *Let $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)'$ be a r.v. with independent univariate marginal distributions $\Lambda_i \sim \text{Exp}(1)$, $i = 1, \dots, n$, and denote by $\boldsymbol{\Xi} = (\Xi_1, \dots, \Xi_n)' \sim Ga_{1,\dots,n}(\boldsymbol{\gamma}_c^*, \boldsymbol{\sigma})$ the*

n -variate gamma distribution introduced in Definition 2.2.1; here $\boldsymbol{\gamma}_c^* = (\gamma_{c,1}^*, \dots, \gamma_{c,n}^*)' \in \mathbf{R}_+^n$ with $\gamma_{c,i}^* = \sum_{j=1}^{n+1} c_{i,j} \gamma_j$, and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)' \in \mathbf{R}_+^n$ are vectors of parameters. Let $\boldsymbol{\Lambda}$ and $\boldsymbol{\Xi}$ be independent r.v.'s, then $\mathbf{X} = (X_1, \dots, X_n)' \sim F_{1,\dots,n}$ has d.d.f. (2.2.1) if and only if $X_i \stackrel{d}{=} \Lambda_i / \Xi_i$, $i = 1, \dots, n$.

Proof. The ‘if’ part is immediate from the following obvious relations

$$\begin{aligned} \overline{F}_{\mathbf{X}}(x_1, \dots, x_n) &= \mathbf{P}[\Lambda_1 > \Xi_1 x_1, \dots, \Lambda_n > \Xi_n x_n] \\ &= \int_{\mathbf{R}_+^n} \mathbf{P}[\Lambda_1 > \xi_1 x_1, \dots, \Lambda_n > \xi_n x_n] dF_{\boldsymbol{\Xi}}(\xi_1, \dots, \xi_n) \\ &= \int_{\mathbf{R}_+^n} \exp \left\{ - \sum_{i=1}^n \xi_i x_i \right\} dF_{\boldsymbol{\Xi}}(\xi_1, \dots, \xi_n) \end{aligned} \quad (2.2.4)$$

and by Theorem 2.2.1. The ‘only if’ part follows because (2.2.4) is the n -variate Laplace transform of $Ga_{1,\dots,n}(\boldsymbol{\gamma}_c^*, \boldsymbol{\sigma})$, and it is thus unique. This completes the proof. $\square$

It is worthwhile to make an additional observation before stating the next result of this section. Namely, we first note in passing that for $1 \leq k \neq l \leq n$, a r.v. $(\Xi_k, \Xi_l)'$ distributed the bivariate gamma as in Definition 2.2.1 and an $(n+1)$ -variate r.v. $\mathbf{Y} = (Y_1, \dots, Y_{n+1})'$ having mutually independent coordinates $Y_j \sim Ga(\gamma_j, 1)$, $\gamma_j \in \mathbf{R}_+$, the following stochastic representation holds

$$\sigma_k \Xi_k \stackrel{d}{=} Y_{c,(k,l)} + Y_{c,k} \text{ and } \sigma_l \Xi_l \stackrel{d}{=} Y_{c,(k,l)} + Y_{c,l}, \quad (2.2.5)$$

where $Y_{c,(k,l)} = \sum_{j=1}^{n+1} c_{k,j} c_{l,j} Y_j$, $Y_{c,k} = \sum_{j=1}^{n+1} c_{k,j} (1 - c_{l,j}) Y_j$ and $Y_{c,l} = \sum_{j=1}^{n+1} c_{l,j} (1 - c_{k,j}) Y_j$ are mutually independent gamma r.v.'s with the shape parameters $\gamma_{c,(k,l)} = \sum_{j=1}^{n+1} c_{k,j} c_{l,j} \gamma_j$, $\gamma_{c,k} = \sum_{j=1}^{n+1} c_{k,j} (1 - c_{l,j}) \gamma_j$ and $\gamma_{c,l} = \sum_{j=1}^{n+1} c_{l,j} (1 - c_{k,j}) \gamma_j$, respectively.

We next show that the covariance of two r.v.'s following the multivariate Pareto of interest in this paper can be formulated using the hypergeometric function ${}_3F_2$ that is defined for

$|z| < 1$ as the infinite power series

$${}_3F_2(a, b, c; d, e; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{z^k}{k!}, \quad (2.2.6)$$

where $(\cdot)_k$ denotes the Pochhammer symbol. For the values $|z| \geq 1$ the hypergeometric function is defined using analytic continuation (see, [Gradshteyn and Ryzhik, 2014](#)).

Theorem 2.2.3. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as in Definition 2.2.2, then, for $1 \leq k \neq l \leq n$, the covariance of X_k and X_l is, if finite, given by*

$$\text{Cov}[X_k, X_l] = \sigma_k \sigma_l \frac{1}{(\gamma_{c,k}^* - 1)(\gamma_{c,l}^* - 1)} \left({}_3F_2(\gamma_{c,(k,l)}, 1, 1; \gamma_{c,k}^*, \gamma_{c,l}^*; 1) - 1 \right).$$

Proof. We start by employing Lemma 2.2.1 and observation (2.2.5), then do change of variables and evoke Expressions (3.197(5)) and (7.512(5)) in [Gradshteyn and Ryzhik \(2014\)](#), that is

$$\begin{aligned} (\sigma_k \sigma_l)^{-1} \mathbf{E}[X_k X_l] &= (\sigma_k \sigma_l)^{-1} \mathbf{E} \left[\frac{1}{\Xi_k} \cdot \frac{1}{\Xi_l} \right] = \int_{\mathbf{R}_+^3} \frac{1}{(y_3 + y_1)(y_3 + y_2)} dG_{1,\dots,3}^{\Pi}(\mathbf{y}; \boldsymbol{\gamma}, \mathbf{1}) \\ &= \int_0^{\infty} \int_0^{\infty} (1+v)^{-\gamma_{c,l}} (1+u)^{-\gamma_{c,k}} (1+u+v)^{-\gamma_{c,(k,l)}} du dv \\ &= \int_0^{\infty} (1+v)^{-\gamma_{c,l}^*} \left(\int_0^{\infty} (1+u)^{-\gamma_{c,k}} \left(1 + \frac{u}{1+v} \right)^{-\gamma_{c,(k,l)}} du \right) dv \\ &= \int_0^{\infty} (1+v)^{-(\gamma_{c,l}^*)} \frac{1}{\gamma_{c,k}^* - 1} {}_2F_1 \left(\gamma_{c,(k,l)}, 1; \gamma_{c,k}^*; \frac{v}{1+v} \right) dv \\ &= \int_0^1 (1-u)^{(\gamma_{c,l}^*-1)-1} \frac{1}{\gamma_{c,k}^* - 1} {}_2F_1(\gamma_{c,(k,l)}, 1; \gamma_{c,k}^*; u) du \\ &= \frac{1}{(\gamma_{c,k}^* - 1)(\gamma_{c,l}^* - 1)} {}_3F_2(\gamma_{c,(k,l)}, 1, 1; \gamma_{c,k}^*, \gamma_{c,l}^*; 1), \end{aligned}$$

where the hypergeometric function converges since $\gamma_{c,k}^* + \gamma_{c,l}^* \geq \gamma_{c,(k,l)} + 2$ ([Olver et al., 2010](#), see, e.g.). This completes the proof. $\square$

An immediate consequence of Theorem 2.2.3 is that the maximal attainable correlation

in the context of the multivariate Pareto distribution introduced in the present paper is not one.

Corollary 2.2.1. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ while assume that both $\gamma_{c,k}^*$ and $\gamma_{c,l}^*$ are fixed and exceed two for $1 \leq k \neq l \leq n$. Then, for the Pearson's correlation, it holds that $\text{Corr}[X_k, X_l] \in [0, 1/2)$.*

Proof. The lower bound follows setting $\gamma_{c,(k,l)} \equiv 0$ and both of $\gamma_{c,l}^*$ and $\gamma_{c,k}^*$ to exceed two, because this corresponds to the independent case. To establish the upper bound, let $\gamma_{c,k} \rightarrow 0$ and $\gamma_{c,l} \rightarrow 0$ and assume that $\gamma_{c,(k,l)}$ exceeds two, then the related hyper-geometric function becomes well-posed and using the Dixon's theorem (see, [Dixon, 1902](#)), we have that

$$\lim_{\gamma_{c,k}, \gamma_{c,l} \rightarrow 0} {}_3F_2(\gamma_{c,(k,l)}, 1, 1; \gamma_{c,k}^*, \gamma_{c,l}^*; 1) = {}_3F_2(\gamma_{c,(k,l)}, 1, 1; \gamma_{c,(k,l)}, \gamma_{c,(k,l)}; 1) = \frac{\gamma_{c,(k,l)} - 1}{\gamma_{c,(k,l)} - 2},$$

and the covariance of interest reduces to

$$\text{Cov}[X_k, X_l] \rightarrow \frac{\sigma_l \sigma_k}{(\gamma_{c,(k,l)} - 1)^2 (\gamma_{c,(k,l)} - 2)}, \text{ where } 1 \leq k \neq l \leq n.$$

This, along with Theorem [2.2.3](#), completes the proof. $\square$

Another consequence of Theorem [2.2.3](#) pertains to two special cases of the multivariate Pareto introduced in this paper, and it is formulated as the following Corollary.

Corollary 2.2.2. *Let $\mathbf{X}_1 = (X_{1,1}, \dots, X_{1,n})' \sim Pa_{1,\dots,n}^{(I)}$ and $\mathbf{X}_2 = (X_{2,1}, \dots, X_{2,n})' \sim Pa_{1,\dots,n}^{(II)}$ be respectively the multivariate flexible Pareto of type I and II of [Chiragiev and Landsman \(2009\)](#). Then the corresponding covariances are readily obtained, for $1 \leq k \neq l \leq n$, as*

$$\begin{aligned} & \text{Cov}[X_{1,k}, X_{1,l}] \\ &= \frac{\sigma_k \sigma_l}{(\gamma_k + \gamma_{n+1} - 1)(\gamma_l + \gamma_{n+1} - 1)} ({}_3F_2(\gamma_{n+1}, 1, 1; \gamma_k + \gamma_{n+1}, \gamma_l + \gamma_{n+1}; 1) - 1), \end{aligned} \tag{2.2.7}$$

for $\gamma_k + \gamma_{n+1} > 2$ and $\gamma_l + \gamma_{n+1} > 2$, and as

$$\mathbf{Cov}[X_{2,k}, X_{2,l}] = \frac{\sigma_k \sigma_l}{(\gamma_k^* - 1)(\gamma_l^* - 1)(\gamma_l^* - 2)}, \quad (2.2.8)$$

for $\gamma_k^* = \sum_{j=1}^k \gamma_j > 2$ and $\gamma_l^* = \sum_{j=1}^l \gamma_j > 2$.

Proof. To obtain (2.2.7), we set $c_{i,i} = c_{i,n+1} \equiv 1$ and zero otherwise. This implies $\gamma_{c,k}^* = \gamma_k + \gamma_{n+1}$, $\gamma_{c,l}^* = \gamma_l + \gamma_{n+1}$ and $\gamma_{c,(k,l)} = \gamma_{n+1}$ (see, Example 2.1.1). Then the result directly follows from Theorem 2.2.3. To establish (2.2.8), let $c_{i,j} \equiv 1$ for $1 \leq j \leq i \leq n$ and zero otherwise. Then $\gamma_{c,(k,l)} = \gamma_{c,k}^* = \gamma_k^*$, $\gamma_{c,l}^* = \gamma_l^*$ (see, Example 2.1.2), and the desired statement follows because

$$\begin{aligned} \mathbf{Cov}[X_{2,k}, X_{2,l}] &= \frac{\sigma_k \sigma_l}{(\gamma_k^* - 1)(\gamma_l^* - 1)} ({}_3F_2(\gamma_{c,(k,l)}, 1, 1; \gamma_{c,(k,l)}, \gamma_l^*; 1) - 1) \\ &= \frac{\sigma_k \sigma_l}{(\gamma_k^* - 1)(\gamma_l^* - 1)} ({}_2F_1(1, 1; \gamma_l^*; 1) - 1) \\ &= \frac{\sigma_k \sigma_l}{(\gamma_k^* - 1)(\gamma_l^* - 1)(\gamma_l^* - 2)}, \end{aligned}$$

where the latter equality holds for $\gamma_k^* > 2$. This completes the proof of the Corollary. $\square$

We note in passing that expression (2.2.8) confirms the one derived in Chiragiev and Landsman (2009), while formula (2.2.7) complements the discussion therein. Also, the covariance of two Pareto r.v.'s coming from the Arnold's multivariate Pareto distribution (see, Arnold, 1983, 2015) is readily obtained from both (2.2.7) and (2.2.8). More specifically, we set $\gamma_k = \gamma_l \equiv 0$ for all $0 \leq k \neq l \leq n$ and verify that (2.2.7) reduces to

$$\frac{\sigma_k \sigma_l}{(\gamma_{n+1} - 1)(\gamma_{n+1} - 1)} ({}_2F_1(1, 1; \gamma_{n+1}; 1) - 1) = \frac{\sigma_k \sigma_l}{(\gamma_{n+1} - 1)^2(\gamma_{n+1} - 2)},$$

for $\gamma_{n+1} > 2$. The verification is straightforward in the case of (2.2.8).

Generalized hypergeometric function (2.2.6) plays an important role when deriving the centered regression function $r(y) = \mathbf{E}[X - \mathbf{E}[X] | Y = y]$ where $y \in \mathcal{R} \subseteq \mathbf{R}$ is in the range of

the r.v. Y (see, [Furman and Zitikis, 2008b, 2010](#), for applications of the function in insurance and finance).

Theorem 2.2.4. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as in Definition 2.2.2, the centred regression function of X_k on X_l , $1 \leq k \neq l \leq n$, is given, for $\gamma_{c,k}^* > 1$, by*

$$r_k(x_l) = m(x_l) \sum_{i=1}^2 a_i {}_2F_1 \left(\gamma_{c,k}, 1; \gamma_{c,k}^* + 2 - i; -\frac{x_l}{\sigma_l} \right) - \sigma_k / (\gamma_{c,k}^* - 1), \quad (2.2.9)$$

where

$$a_1 = \frac{\gamma_{c,(k,l)}^2}{\gamma_{c,k}^* \gamma_{c,l}^*}, \quad a_2 = \frac{\gamma_{c,l} \gamma_{c,(k,l)}}{\gamma_{c,l}^* (\gamma_{c,k}^* - 1)} \quad \text{and} \quad m(x_l) = \frac{\sigma_k}{\gamma_{c,(k,l)}} \left(1 + \frac{x_l}{\sigma_l} \right),$$

for $x_l \in \mathbf{R}_{0,+}$. The centered regression function is monotonically-increasing and concave.

Proof. Note that for the pair $(X_k, X_l)'$, $1 \leq k \neq l \leq n$, stochastic representation (2.2.5) is of utmost generality. Hence the proof is completed invoking Theorem 3 of [Chiragiev and Landsman \(2009\)](#) as well as Theorem 2.2.2. $\square$

The centered regression function of the new multivariate Pareto distribution is not linear, while it is well-known that the classic multivariate Pareto has linear regression (see, [Arnold, 1983, 2015](#)). Theorem 2.2.4 easily confirms the latter fact by setting $\gamma_{c,(k,l)} = \gamma_{c,k}^* = \gamma_{c,l}^* \equiv \gamma_{n+1}$ and $\gamma_{c,l} = \gamma_{c,k} \equiv 0$ in (2.2.9), which then reduces to the following linear form

$$r_k(x_l; \gamma_{c,(k,l)} = \gamma_{c,k}^* = \gamma_{c,l}^* \equiv \gamma_{n+1}, \gamma_{c,l} = \gamma_{c,k} \equiv 0) = \frac{\sigma_k}{\sigma_l} \left(x_l - \frac{\sigma_l}{\gamma_{n+1}(\gamma_{n+1} - 1)} \right),$$

for $\gamma_{n+1} > 1$ and $1 \leq k \neq l \leq n$.

We conclude this section with yet another characterization of the multivariate Pareto distribution of interest and its two implications. Let $X_- = \wedge_{i=1}^n X_i \sim F_-$ and $X_+ = \vee_{i=1}^n X_i \sim F_+$ denote, respectively the minima and the maxima r.v.'s, and let $X_i \sim F_i$, $i = 1, \dots, n$ be univariate coordinates of the multivariate Pareto r.v. of interest in this paper.

Theorem 2.2.5. *Let $\mathbf{X} = (X_1, \dots, X_n)'$ be distributed $Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as per Definition 2.2.2, then X_- admits the mixture representation as $X|\Lambda = \lambda \sim \text{Exp}(\lambda)$ and $\Lambda \stackrel{d}{=} Z_1 + \dots + Z_{n+1}$, where Z_j , $j = 1, \dots, n+1$ are univariate mutually independent gamma r.v.'s..*

Proof. For the proof, we readily have that

$$\begin{aligned} \bar{F}_-(x) &= \bar{F}_{1,\dots,n}(x, \dots, x) = \prod_{j=1}^{n+1} \left(1 + \sum_{i=1}^n \frac{c_{i,j}}{\sigma_i} x \right)^{-\gamma_j} \\ &= \int_0^\infty e^{-\lambda x} dF_{Z_1+\dots+Z_{n+1}}(\lambda), \text{ where } x \in \mathbf{R}_{0,+}, \end{aligned}$$

which establishes the mixture representation. $\square$

An important corollary of Theorem 2.2.5 is a random parameter representation (see, e.g., Feller, 1966) of the d.d.f.'s of X_- and X_+ . The following lemma is crucial in this context.

Lemma 2.2.2 (Moschopoulos, 1985; Furman and Landsman, 2005). *For $i = 1, \dots, n$, let $Z_i \sim Ga(\gamma_i(\in \mathbf{R}_+), \alpha_i(\in \mathbf{R}_+))$, denote independent gamma r.v.'s, and let $Z = Z_1 + \dots + Z_n$. Then the distribution of the convolution is gamma with a random shape parameter. More specifically, $Z \sim Ga(\gamma^* + K, \alpha_+)$, where $\gamma^* = \gamma_1 + \dots + \gamma_n$, $\alpha_+ = \vee_{i=1}^n \alpha_i$ and K is an integer-valued non-negative r.v. with the probability mass function (p.m.f.) given by*

$$p_k = \mathbf{P}[K = k] = c_+ \delta_k, \quad k = 0, 1, \dots, \quad (2.2.10)$$

where

$$c_+ = \prod_{i=1}^n \left(\frac{\alpha_i}{\alpha_+} \right)^{\gamma_i}, \quad \delta_0 = 1$$

and

$$\delta_k = k^{-1} \sum_{l=1}^k \sum_{i=1}^n \gamma_i \left(1 - \frac{\alpha_i}{\alpha_+} \right)^l \delta_{k-l} \text{ for } k > 0.$$

Theorem 2.2.6. Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as in Definition 2.2.2, then $X_- \sim Pa(II)(\alpha_+(\boldsymbol{\sigma}), \gamma^* + K)$, where $\alpha_+(\boldsymbol{\sigma}) = \vee_{j=1}^{n+1} \left(\sum_{i=1}^n \frac{c_{i,j}}{\sigma_i} \right)^{-1}$, K is an integer-valued r.v. with p.m.f. (4.4.3) and $\gamma^* = \gamma_1 + \dots + \gamma_{n+1} > 1$.

Proof. Employing Theorem 2.2.5 with $Z_j \sim Ga\left(\gamma_j, \left(\sum_{i=1}^n \frac{c_{i,j}}{\sigma_j}\right)^{-1}\right)$, $j = 1, \dots, n+1$, Lemma 4.4.1, changing the order of summation and integration and using equation (2.1.2), we have that

$$\bar{F}_-(x) = \int_0^\infty \sum_{k=0}^\infty e^{-\lambda x} p_k \frac{e^{-\lambda \sigma_+} \lambda^{\gamma^*+k-1} \alpha_+(\boldsymbol{\sigma})^{\gamma^*+k}}{\Gamma(\gamma^*+k)} d\lambda = \sum_{k=0}^\infty \left(1 + \frac{x}{\alpha_+(\boldsymbol{\sigma})}\right)^{-(\gamma^*+k)} p_k.$$

This completes the proof. $\square$

The next assertion is similar to Theorem 2 of Vernic (2011).

Corollary 2.2.3. Assume that $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as in Definition 2.2.2, then the d.d.f. of the maxima r.v. is given by

$$\bar{F}_+(x) = \sum_{S \subseteq \{1,\dots,n\}} (-1)^{|S|-1} \bar{F}_{S-}(x), \quad x \in \mathbf{R}_{0,+}, \quad (2.2.11)$$

where $X_{S-} = \bigwedge_{s \in S \subseteq \{1,\dots,n\}} X_s$ and $X_{S-} \sim F_{S-}$. Hence $\bar{F}_+$ is a linear combination of the d.d.f.'s of univariate Pareto r.v.'s of the 2nd kind with random shape parameters.

2.3 Applications to insurance

In the jargon of insurance mathematics, the multivariate Pareto distribution introduced in the previous section is a modification of the classic frailty model (see, e.g., Bowers et al., 1997). More specifically, let $\mathbf{X} = (X_1, \dots, X_n)'$ denote a risk portfolio (r.p.) with X_i , $i = 1, \dots, n$ representing its risk components (r.c.'s). Furthermore, let the r.c.'s be exposed to sets $\mathcal{R}_i = \{r \in \mathbf{N} : r \leq (n+1)\}$ of risk factors (r.f.'s), $i = 1, \dots, n$. Last but not

least, let the r.v. $\mathbf{Y} = (Y_1, \dots, Y_{n+1})'$ denote the randomness of actuarial interest associated with the corresponding r.f.'s. The following theorem specializes the set-up and establishes the minima-based multiple risk factor interpretation of the multivariate Pareto proposed in the present paper. We note in passing that $*$ stands for the mixture operator, i.e., given $X_\lambda \sim C(\cdot; \lambda)$ and $\Lambda \sim H$, it holds that $X_\lambda * \Lambda \stackrel{d}{=} X_\Lambda$.

Theorem 2.3.1. *Let $\mathbf{W}_i = (W_{i,1}, \dots, W_{i,n+1})'$ be r.v.'s with independent exponential margins, $W_{i,j} \sim \text{Exp}(\lambda_{i,j}(\in \mathbf{R}_+))$, $i = 1, \dots, n$, $j = 1, \dots, n+1$, and let A_c be a deterministic matrix of zero-one coefficients. Also, let $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{n+1})'$ be a r.v. having independent gamma-distributed margins with arbitrary shape $\gamma_j(\in \mathbf{R}_+)$ and scale parameters equal to 1 for $j = 1, \dots, n+1$. Set $X_i = \sigma_i \bigwedge_{j=1, c_{i,j} \neq 0}^{n+1} (W_{i,j} * \Lambda_j)$, $\sigma_i \in \mathbf{R}_+$, $i = 1, \dots, n$, then $\mathbf{X} = (X_1, \dots, X_n)' \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$.*

Proof. Note that

$$\bigwedge_{j=1}^{n+1} (X_{\lambda_j} * \Lambda_j) \stackrel{d}{=} (\bigwedge_{j=1}^{n+1} X_{\lambda_j}) * \mathbf{\Lambda},$$

and hence

$$\overline{F}_{\mathbf{X}}(x_1, \dots, x_n) = \int_{\mathbf{R}_+^{n+1}} \prod_{i=1}^n \exp \left\{ - \sum_{j=1}^{n+1} \frac{c_{i,j} x_i \lambda_j}{\sigma_i} \right\} dF_{\mathbf{\Lambda}}(\lambda_1, \dots, \lambda_{n+1}),$$

which, along with (2.1.2), completes the proof. $\square$

In view of Theorem 2.3.1, the new multivariate Pareto distribution meaningfully fills the gap between the multivariate probability distributions with independent and Arnold-dependent (see, Arnold, 1983, 2015) Pareto margins.

2.3.1 Actuarial risk measurement

Regulatory accords around the world require that insurance companies carry out a careful assessment of their risks. From now on, the r.v. $X : \Omega \rightarrow \mathbf{R}_+$ is interpreted as an insurance risk, and $\mathcal{X}$ denotes the collection of such risks.

Definition 2.3.1. *A risk measure is a functional map $H : \mathcal{X} \rightarrow [0, \infty]$.*

The literature on risk measures is vast and growing quickly. The following two indices are arguably the most popular amongst practitioners.

Definition 2.3.2. *Let $X \in \mathcal{X}$ and fix $q \in [0, 1)$, then the Value-at-Risk (VaR) and the Conditional Tail Expectation (CTE) risk measures are respectively given by*

$$VaR_q[X] = \inf\{x \in \mathbf{R} : \mathbf{P}[X \leq x] \geq q\} \quad (2.3.1)$$

and

$$CTE_q[X] = \mathbf{E}[X | X > VaR_q[X]]. \quad (2.3.2)$$

We note in passing that both VaR and CTE are distorted as well as weighted risk measures (see, respectively, [Wang, 1996](#); and [Furman and Zitikis, 2008a](#)).

Definition 2.3.3. *[Furman and Zitikis \(2008a\)](#), see also [Choo and de Jong \(2009, 2010\)](#).*

Let $w : \mathbf{R} \rightarrow \mathbf{R}_+$ be a non-decreasing Borel (weight) function, such that $0 < \mathbf{E}[w(X)] < \infty$, then the class of weighted risk measures is defined as

$$\pi_w[X] = \frac{\mathbf{E}[Xw(X)]}{\mathbf{E}[w(X)]} \text{ for } X \in \mathcal{X}. \quad (2.3.3)$$

Let $v_1, w_1 : \mathbf{R}^n \rightarrow \mathbf{R}_+$ be two legitimate weight functions such that all expectations in (2.3.4) are finite, and consider a generalized variant of (2.3.3)

$$\pi_{v_1, w_1}[\Lambda] = \frac{\mathbf{E}[v_1(\Lambda)w_1(\Lambda)]}{\mathbf{E}[w_1(\Lambda)]} \text{ for } \Lambda : \Omega \rightarrow \mathcal{R} \subseteq \mathbf{R}^n. \quad (2.3.4)$$

Theorem 2.3.2. For C and $H_{1,\dots,n}$ denoting univariate and n -variate distributions, respectively, let $X|\mathbf{\Lambda} = \boldsymbol{\lambda} \sim C(\cdot; \boldsymbol{\lambda}(\in \mathcal{R} \subseteq \mathbf{R}^n))$ and assume that $\mathbf{\Lambda} \sim H_{1,\dots,n}$. The functional $\pi_w[X]$ admits representation (2.3.4).

Proof. Notice that

$$\pi_w[X] = \frac{\mathbf{E}[\mathbf{E}[w(X)|\mathbf{\Lambda}]\pi_w[X|\mathbf{\Lambda}]]}{\mathbf{E}[\mathbf{E}[w(X)|\mathbf{\Lambda}]]} \text{ for } X \in \mathcal{X},$$

and set $w_1(\boldsymbol{\lambda}) = \mathbf{E}[w(X)|\boldsymbol{\lambda}]$ and $v_1(\boldsymbol{\lambda}) = \pi_w[X|\boldsymbol{\lambda}]$. This concludes the proof. $\square$

Corollary 2.3.1. Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as in Definition 2.2.2, then, for $i = 1, \dots, n$ and $q \in [0, 1)$, we have that

$$VaR_q[X_i] = \sigma_i \left((1 - q)^{-1/\gamma_{c,i}^*} - 1 \right).$$

Proof. Use Theorem 2.3.2 setting the weight function equal to the Dirac delta function, or alternatively evoke Theorem 2.2.2. This concludes the proof. $\square$

Corollary 2.3.2. For C and $H_{1,\dots,n}$ denoting univariate and n -variate distributions, respectively, let $X|\mathbf{\Lambda} = \boldsymbol{\lambda} \sim C(\cdot; \boldsymbol{\lambda}(\in \mathcal{R} \subseteq \mathbf{R}^n))$ and assume that $\mathbf{\Lambda} \sim H_{1,\dots,n}$. Then the CTE risk measure of $X \sim F$ is, if exists and for $q \in [0, 1)$, given by

$$CTE_q[X] = \frac{\mathbf{E}[\overline{C}(VaR_q[X]; \mathbf{\Lambda})CTE_{q^*}[X|\mathbf{\Lambda}]]}{\overline{F}(VaR_q[X])} \text{ for } X \in \mathcal{X},$$

where $q^* = C(VaR_q[X]; \boldsymbol{\lambda})$.

Proof. Fix $w(x) = \mathbf{1}\{x > VaR_q[X]\}$ for $q \in [0, 1)$, the result follows from Theorem 2.3.2. $\square$

Corollary 2.3.3. Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \gamma_{n+1})$ as in Definition 2.2.2, we have that the CTE risk measure is, if exists and for $i = 1, \dots, n$ and $q \in [0, 1)$, given by

$$\begin{aligned} CTE_q[X_i] &= \mathbf{E}[X_i] \frac{\overline{F}_{X_i^*}(VaR_q[X_i])}{1 - q} + VaR_q[X_i] \\ &= \mathbf{E}[X_i] + VaR_q[X_i] \frac{\gamma_{c,i}^*}{\gamma_{c,i}^* - 1}, \end{aligned} \tag{2.3.5}$$

where $X_i^* \sim Pa(II)(\sigma_i, \gamma_{c,i}^* - 1)$.

Proof. As $X_i|\Lambda = \lambda \sim Exp(\lambda)$, $\lambda > 0$, $i = 1, \dots, n$, we readily have that

$$CTE_{q^*}[X_i | \Lambda = \lambda] = \frac{1}{\lambda} + VaR_q[X_i]$$

and

$$\overline{H}[VaR_q[X_i]; \lambda] = e^{-\lambda VaR_q[X_i]}$$

and the assertion holds by Theorem 2.3.2. This completes the proof. $\square$

The minima r.v. $X_- = \wedge_{i=1}^n X_i$ plays an important role in insurance mathematics (recall, e.g., the joint life policies in life insurance), as well as in general finance (think of, e.g., the first-to-default baskets).

Theorem 2.3.3. *In the context of the multivariate Pareto of interest, the CTE risk measure of the minima can be written, if finite, as*

$$CTE_q[X_-] = \mathbf{E}[X_-] \frac{\overline{F}_{X_-^*}(VaR_q[X_-])}{1 - q} + VaR_q[X_-],$$

where $X_-^* \sim Pa(\alpha_+(\boldsymbol{\sigma}), \gamma^* + Q - 1,)$ with $\alpha_+(\boldsymbol{\sigma}) = \vee_{j=1}^{n+1} \left(\sum_{i=1}^n \frac{c_{i,j}}{\sigma_i} \right)^{-1}$, $\gamma^* = \gamma_1 + \dots + \gamma_{n+1}$ and Q is an integer-valued r.v. with the p.m.f. obtained from the p.m.f. of K with the help of the following change of measure

$$q_k = \frac{1}{\mathbf{E}[X_-]} \frac{\alpha_+(\boldsymbol{\sigma})}{\gamma^* + k - 1} p_k, \quad k = 0, 1, \dots \quad (2.3.6)$$

Proof. We readily have that

$$\begin{aligned} CTE_q[X_-] &= VaR_q[X_-] + \frac{1}{1 - q} \int_{VaR_q[X_-]}^{\infty} \overline{F}_-(x) dx \\ &\stackrel{(1)}{=} VaR_q[X_-] + \frac{1}{1 - q} \int_{VaR_q[X_-]}^{\infty} \left(\sum_{k=0}^{\infty} \left(1 + \frac{x}{\alpha_+(\boldsymbol{\sigma})} \right)^{-(\gamma^* + k)} p_k \right) dx \\ &\stackrel{(2)}{=} VaR_q[X_-] + \frac{\mathbf{E}[X_-]}{1 - q} \int_{VaR_q[X_-]}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\gamma^* + k - 1}{\alpha_+(\boldsymbol{\sigma})} \left(1 + \frac{x}{\alpha_+(\boldsymbol{\sigma})} \right)^{-(\gamma^* + k - 1) - 1} q_k \right) dx, \end{aligned}$$

where $\stackrel{(1)}{=}$ follows because of Corollary 2.2.6 and $\stackrel{(2)}{=}$ holds since

$$\mathbf{E}[X_-] = \sum_{k=0}^{\infty} \frac{\alpha_+(\boldsymbol{\sigma})}{\gamma^* + k - 1} p_k.$$

This completes the proof. $\square$

Theorem 2.3.4. *In the context of the multivariate Pareto of interest, the CTE risk measure of the maxima can be written as the following linear combination*

$$CTE_q[X_+] = \frac{1}{1-q} \sum_{\mathcal{S} \subseteq \{1, \dots, n\}} (-1)^{|\mathcal{S}|-1} \mathbf{E}[X_{\mathcal{S}-} | X_{\mathcal{S}-} > VaR_q(X_+)] \bar{F}_{\mathcal{S}-}(VaR_q(X_+)),$$

where $X_{\mathcal{S}-} = \bigwedge_{s \in \mathcal{S} \subseteq \{1, \dots, n\}} X_s$ and $X_{\mathcal{S}-} \sim F_{\mathcal{S}-}$.

Proof. We observe that

$$\int_x^{\infty} \bar{F}(t) dt = \mathbf{E}[X - x | X > x] \bar{F}(x),$$

for all x in the range of the r.v. X . Then the assertion follows by Theorem 2.2.3 and changing the order of summation and integration in

$$CTE_q[X_+] = \frac{\int_{VaR_q[X_+]}^{\infty} \sum_{\mathcal{S} \subseteq \{1, \dots, n\}} (-1)^{|\mathcal{S}|-1} \bar{F}_{\mathcal{S}-}(x) dx}{1-q} + VaR_q[X_+]$$

This completes the proof. $\square$

Definition 2.3.4 (Furman and Zitikis, 2008b). *Let $w : \mathbf{R} \rightarrow \mathbf{R}_+$ be a non-decreasing Borel function, such that $0 < \mathbf{E}[w(Y)] < \infty$, then the functional $\Pi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ is referred to as the economic risk measure. Moreover, the special form of Π , given by*

$$\Pi_w[X, Y] = \frac{\mathbf{E}[Xw(Y)]}{\mathbf{E}[w(Y)]} \text{ for } X \in \mathcal{X} \text{ and } Y \in \mathcal{X}, \quad (2.3.7)$$

is called a weighted economic risk measure.

We further derive an expression for the economic CTE risk measure, which is a particular case of (2.3.7) with $w(y) = \mathbf{1}\{y > VaR_q[Y]\}$, $q \in [0, 1)$ and $y \in \mathbf{R}$.

Theorem 2.3.5. *In the context of the multivariate Pareto of interest, the economic CTE risk measure is given, for $\gamma_{c,k}^* > 1$, $q \in [0, 1)$ and $1 \leq k \neq l \leq n$, by*

$$\mathbf{E}[X_k | X_l > VaR_q[X_l]] = \frac{\sigma_k}{\gamma_{c,k}^* - 1} {}_2F_1 \left(\gamma_{c,(k,l)}, 1; \gamma_{c,k}^*; \frac{VaR_q[X_l]}{\sigma_l + VaR_q[X_l]} \right),$$

where the hypergeometric function converges absolutely.

Proof. Note that, for $1 \leq k \neq l \leq n$,

$$\begin{aligned} \mathbf{E}[X_k | X_l > VaR_q[X_l]] &= \int_0^\infty \mathbf{P}[X_k > x | X_l > VaR_q[X_l]] dx \\ &\stackrel{(1)}{=} \sigma_k \int_0^\infty (1+u)^{-\gamma_{c,k}} \left(1 + \frac{u}{1 + VaR_q[X_l]/\sigma_l} \right)^{-\gamma_{c,(k,l)}} du \end{aligned}$$

where ‘ $\stackrel{(1)}{=}$ ’ holds because of observation (2.2.5) and techniques similar to the ones used in Theorem 2.2.3. The assertion then follows by invoking Expression (3.197(5)) in Gradshteyn and Ryzhik (2014). Since $VaR_q[X_l]/(\sigma_l + VaR_q[X_l]) \in [0, 1)$, the convergence of the hypergeometric function is absolute. This completes the proof. $\square$

2.4 A numerical example

Consider two objects of actuarial interest (e.g., financial conglomerates in the context of default risk), each of which is exposed to two distinct categories of fatal risk factors, e.g., systemic (category A) and idiosyncratic (category B). Further assume that the risk factors from distinct risk categories are independent and that the hitting times (or occurrences) of the risk factors are exponentially-distributed with random gamma-distributed parameters. We are interested in modeling the joint default times of the aforementioned two objects of

interest. The set-up matches the framework outlined in the beginning of Section 3, as well as in Theorem 2.3.1, and the joint d.d.f. of the risk components of the portfolio of interest is thereby a special case of d.d.f. (2.2.1). We note in this respect that the idea of randomization can be traced back to Feller (1966), whereas times of default with the d.d.f.'s that are more heavy-tailed than the d.d.f. of the exponential r.v. can be motivated with the help of, e.g., Turcotte and Greene (1993), Malamud et al. (1998), Musson et al. (2002) and Benson et al. (2007).

We further explore four different exposures of the portfolio to the systemic and idiosyncratic risk factors. Varying exposures of the risk components to the distinct categories of risk factors are stipulated by appropriate choices of the c parameters gathered by matrices $A_c^{(k)}$, $k = 1, \dots, 4$ below. As hitherto, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)'$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)'$ are vectors of positive scale and shape parameters, respectively, and $\gamma_3 > 0$ is a scalar-valued parameter.

Case (1). Risk category A is empty, and risk category B contains two independent risk factors.

The risk components of the portfolio are thus uncorrelated, and the portfolio is represented schematically with the use of the following block matrix, with the vertical and horizontal blocks denoting the risk categories and risk components, respectively,

$$A_c^{(1)} = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right).$$

The joint d.d.f. is given by

$$\bar{F}^{(1)}(x_1, x_2) = \left(1 + \frac{x_1}{\sigma_1}\right)^{-\gamma_2} \left(1 + \frac{x_2}{\sigma_2}\right)^{-\gamma_3},$$

where $x_1 \geq 0$, $x_2 \geq 0$.

Case (2). Risk category A contains one risk factor, and risk category B is empty. The risk components of the portfolio are positively correlated with $\rho^{(2)} = 0.33$. Also, in this

case, we have that

$$A_c^{(2)} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array} \right),$$

and the joint d.d.f. of the risk components is given by

$$\overline{F}^{(2)}(x_1, x_2) = \left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2} \right)^{-\gamma_1},$$

where $x_1 \geq 0, x_2 \geq 0$.

Case (3). There are two uncorrelated idiosyncratic risk factors and one systemic risk factor.

The times of default due to the systemic risk factor are positively correlated with

$\rho^{(A)} = 0.33$, whereas the Pearson's correlation between portfolio's risk components is

$\rho^{(3)} = 0.07$. The matrix A_c for this case is

$$A_c^{(3)} = \left(\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array} \right),$$

and the joint d.d.f. of the risk components is given by

$$\begin{aligned} & \overline{F}^{(3)}(x_1, x_2) \\ &= \left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2} \right)^{-\gamma_1} \left(1 + \frac{x_1}{\sigma_1} \right)^{-\gamma_2} \left(1 + \frac{x_2}{\sigma_2} \right)^{-\gamma_3}, \end{aligned}$$

where $x_1 \geq 0, x_2 \geq 0$.

Case (4). Again there are two idiosyncratic risk factors and one systemic risk factor. The times

of default due to the systemic risk factor are positively correlated with $\rho^{(A)} = 0.33$,

whereas the times of default due to the idiosyncratic risk factors are positively corre-

lated with $\rho^{(B)} = 0.12$. The risk components of the portfolio are positively correlated

with $\rho^{(4)} = 0.103$. Then

$$A_c^{(4)} = \left(\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 1 & 1 \end{array} \right),$$

and the joint d.d.f. of the risk components becomes

$$\overline{F}^{(4)}(x_1, x_2) = \left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2}\right)^{-(\gamma_1+\gamma_2)} \left(1 + \frac{x_2}{\sigma_2}\right)^{-\gamma_3},$$

where $x_1 \geq 0, x_2 \geq 0$.

We note in passing that the expressions for the d.d.f.'s above readily follow from Theorem 2.3.1, whereas the values of Pearson's correlation between the risk components are obtained in non-trivial cases with the help of Theorem 2.2.3. It is also worthwhile to note that the marginal distributions of risk components generally vary for portfolios (1) – (4).

2.4.1 Expected times of the first default

Figure 2.1 compares the values of $CTE_q[X_-]$ for $q \in [0, 1)$, $X_- \in \mathcal{X}$ and portfolios (1) to (4). It is not difficult to see, e.g., from the corresponding d.d.f.'s, that

$$\overline{F}_-^{(2)} \geq_{st} \overline{F}_-^{(1)} \geq_{st} \overline{F}_-^{(3)} \geq_{st} \overline{F}_-^{(4)}, \quad (2.4.1)$$

where ' $\geq_{st}$ ' means 'dominates stochastically'. Hence

$$CTE_q^{(2)}[X_-] \geq CTE_q^{(1)}[X_-] \geq CTE_q^{(3)}[X_-] \geq CTE_q^{(4)}[X_-],$$

for all $q \in [0, 1)$ and $X_- \in \mathcal{X}$ (Wang, 1996). The ordering is confirmed in Figure 2.1. Two notes are worthwhile:

- when the univariate distributions of risk components are fixed, the assumption of independence of default times is conservative when compared to the assumption of Positive Quadrant Dependence (PQD) (compare cases (1) and (2));

- when the distributions of risk factors are fixed, then the conditional expected time of first default is in reciprocal relation to the number of risk factors (compare cases (3) and (4)).

We employ Theorem 2.3.3 to compute the curves in the left-hand panel of Figure 2.1.

2.4.2 Expected times of the last default

The right-hand panel of Figure 2.1 compares the values of $CTE_q[X_+]$ for $q \in [0, 1)$, $X_+ \in \mathcal{X}$ and portfolios (1) to (4). For an insight, recall that

$$\bar{F}_+(x) = \bar{F}_X(x) + \bar{F}_Y(x) - \bar{F}_-(x),$$

for $x \in \mathbf{R}_+$. This, along with (2.4.1), results in, e.g., $\bar{F}_+^{(1)} \geq_{st} \bar{F}_+^{(2)}$. Consequently, we have that

$$CTE_q^{(1)}[X_+] \geq CTE_q^{(2)}[X_+]$$

for all $q \in [0, 1)$ and $X_+ \in \mathcal{X}$. In a similar fashion, we observe that

$$\bar{F}_+^{(1)} \geq_{st} \bar{F}_+^{(2)} \geq_{st} \bar{F}_+^{(3)} \geq_{st} \bar{F}_+^{(4)},$$

and hence

$$CTE_q^{(1)}[X_+] \geq CTE_q^{(2)}[X_+] \geq CTE_q^{(3)}[X_+] \geq CTE_q^{(4)}[X_+],$$

for all $q \in [0, 1)$ and $X_+ \in \mathcal{X}$ (Wang, 1996). The ordering above agrees with the one in the right-hand panel of Figure 2.1. Again two notes are seemingly useful

- when the univariate distributions of risk components are fixed, the assumption of Positive Quadrant Dependence (PQD) of default times is conservative when compared to the assumption of independence (compare cases (1) and (2));

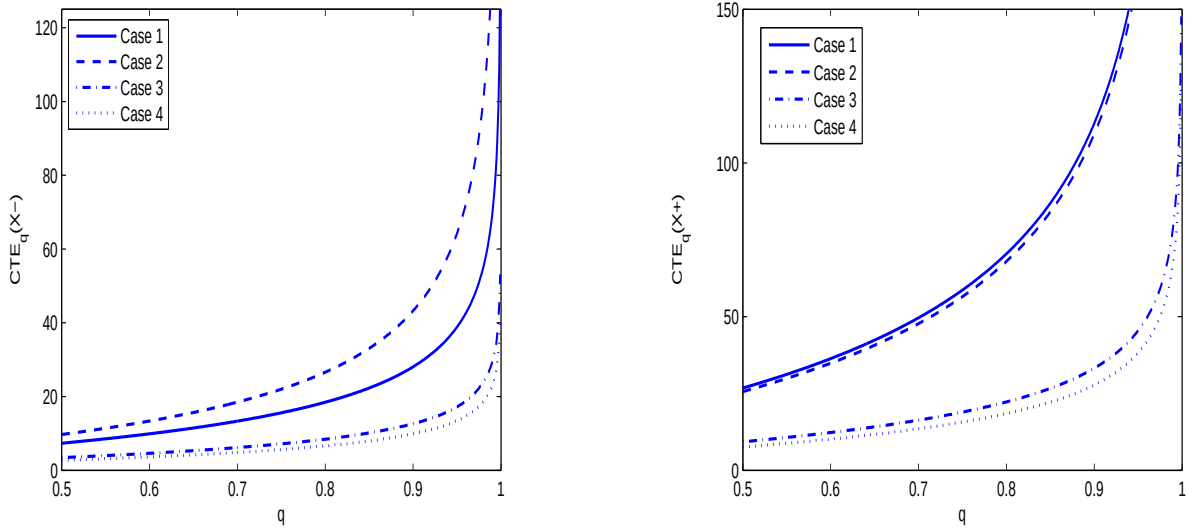


Figure 2.1: Conditional expected times of earliest default (left-hand panel) and latest default (right-hand panel) for portfolios (1) - (4).

- when the distributions of risk factors are fixed, then the conditional expected time of last default is in reciprocal relation to the number of risk factors (compare cases (3) and (4)).

We use Theorem 2.3.4 to compute the curves in the right-hand panel of Figure 2.1.

2.4.3 Solvency bonus function

Let $CTE_q[X_1, X_2] = \mathbf{E}[X_1 | X_2 > VaR_q[X_2]]$ for $q \in [0, 1)$ and $X_1, X_2 \in \mathcal{X}$, then the function $\beta(q; X_1, X_2) = CTE_q[X_1, X_2] - \mathbf{E}[X_1]$ denotes the increase in the default time of, say, business unit (X_1) given that business unit (X_2) has not defaulted and that X_1 and X_2 are PQD; clearly $\beta(q; X_1, X_2) \equiv 0$ if X_1 and X_2 are independent. Figure 2.2 depicts $\beta(q)$, $q \in [0, 1)$ for portfolios (1) to (4). As expected, stronger PQD dependent default times are associated with higher solvency bonuses. We use Theorem 2.3.5 to compute the

curves in Figure 2.2.

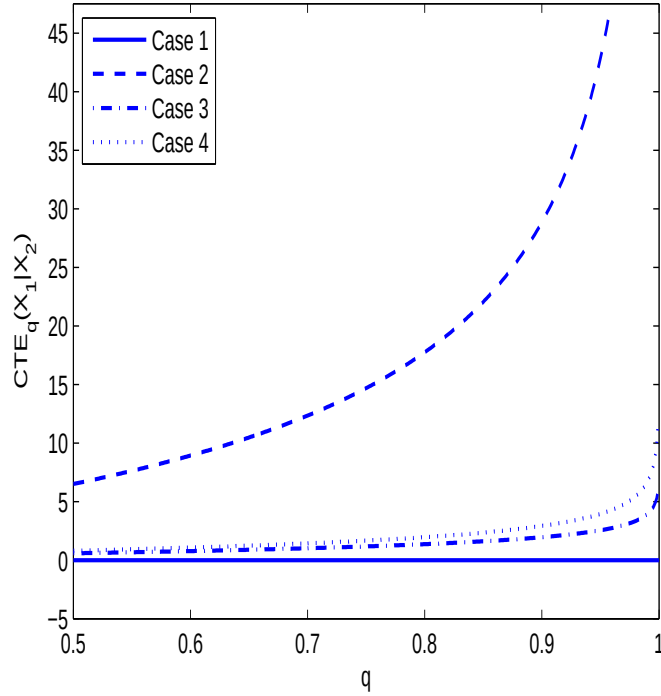


Figure 2.2: Solvency bonus function for portfolios (1) - (4).

2.5 Conclusions

We have introduced and studied a new form of an absolutely continuous with respect to the Lebesgue measure multivariate probability law with the univariate margins distributed Pareto of the 2nd kind. The genesis of our distribution is twofold: 1.) it originates as the Laplace transform of a multivariate gamma distribution with the dependence structure based on the additive form of the multivariate reduction method, and 2.) it admits the so-called multiple risk factor representation. We have meaningfully positioned our multivariate Pareto distribution in the general context of existing models. We have employed interesting char-

acteristic results to derive, e.g., the (conditional/product) moments of the new multivariate Pareto as well as the distributions of minima and maxima. Last but not least, we have developed expressions of some risk measure functionals of actuarial interest and elucidated our findings with the help of a numerical example.

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Chapter 3

Multiple risk factor dependence structures: Distributional properties

The main object of study in this dissertation is introduced in the following paper. The construction is connected to the one proposed in Chapter 2 and also to the factor models that lay in the very basis of the nowadays financial default measurement practice. This chapter has been submitted for publication with Edward Furman.

MULTIPLE RISK FACTOR DEPENDENCE STRUCTURES: DISTRIBUTIONAL PROPERTIES

Abstract.

The veins of excessive prudence has taken the practical quantitative risk management by storm, drawing the attention of the regulators specifically towards the rare but devastating financial risks. It is not a surprise therefore, that much of modern research efforts has been dedicated to exploring new multivariate probability distributions augmenting strongly positively correlated and heavy tailed univariate margins. The purpose of this paper is exactly that. Namely, we introduce a new class of dependence structures, that we call the Multiple Risk Factor (MRF) dependence structures. On the one hand the new models extend the popular CreditRisk⁺ approach, and as such they formally describe default risk portfolios exposed to an arbitrary number of fatal risk factors with conditionally exponential hitting times that can be independent, positively dependent and fully comonotonic. There are numerous empirical evidences that CreditRisk⁺ tends to underestimate the real life default times, and we believe that our proposed approach can resolve this inconvenience. On the other hand, the MRF structures can be viewed as a quite general family of multivariate probability distributions with Pareto of the 2nd kind univariate margins, and in this capacity, they can model risk portfolios of (insurance) losses with heavy-tailed and positively dependent risk components.

Keywords and phrases: Multivariate distributions, dependence, singularity,

Pareto distributions, credit risk, factor models, weighted risk functionals.

3.1 Introduction

Consider an $n(\in \mathbf{N})$ -variate random vector $\mathbf{X} := (X_1, \dots, X_n)'$ with coordinates $X_i := E_{\lambda_i} \wedge E_{\lambda_{n+1}}$, $i = 1, \dots, n$, where E_{λ_j} are exponentially distributed and stochastically independent random variables (r.v.'s) with parameters $\lambda_j \in \mathbf{R}_+$, $j = 1, \dots, n+1$. The decumulative distribution function (d.d.f.) of $\mathbf{X}$ along with its numerous properties was studied by Professors Albert Marshall and Ingram Olkin in a seminal paper ([Marshall and Olkin, 1967](#)). The d.d.f. was shown to be not absolutely continuous with respect to the Lebesgue measure, and its applications initially targeted reliability analysis and life testing.

Latter, actuaries found the multivariate distribution of Marshall and Olkin (MO) useful in the context of life insurance, where they interpreted the coordinates $X_1, \dots, X_n$ as dependent future life times of n insureds with E_{λ_i} , $i = 1, \dots, n$ and $E_{\lambda_{n+1}}$ denoting, respectively, the future life times due to the individual mortality and the ‘hitting’ time of a common shock risk factor (r.f.). The probability for the i -th insured to not survive one additional time unit was clearly $\mathbf{P}[X_i < 1] = 1 - \exp\{-(\lambda_i + \lambda_{n+1})\} > 1 - \exp\{-\lambda_i\}$, $i = 1, \dots, n$ where the bound held intuitively because of the inclusion of the additional (common shock) r.f. that affected the marginal mortality. The feasibility of having non-zero probabilities of simultaneous death were naturally welcomed in family insurance coverages as a result of common exposure to, e.g., aircraft and road accidents. We note in passing that the MO distribution has been consistently included in professional actuarial exams and taught to students for many years ([Bowers et al., 1997](#)), which is an indisputable evidence of its importance in insurance mathematics.

Recently, the MO distribution along with the underlying copula ([Nelsen, 2006](#)) were suggested as a reasonable model for describing dependent defaults ([Giesecke, 2003](#)). In this

respect, the r.v. X_i represented the default time of the i -th risk component (r.c.) $i = 1, \dots, n$ within the risk portfolio (r.p.) $\mathbf{X}$ exposed to n idiosyncratic and one systemic r.f.'s, having, correspondingly, exponentially distributed hitting times E_{λ_i} and $E_{\lambda_{n+1}}$. Non-zero probabilities of simultaneous default were motivated empirically by, e.g., the joint insolvency of 24 railway firms that defaulted on June 21, 1970 ([Azizpour and Giesecke, 2008](#)), whereas the exponentiality of the hitting times followed naturally if the occurrences of the risks were governed by $(n + 1)$ stochastically independent Poisson processes.

In fact, the aforementioned exponentiality assumption on E_{λ_j} , $j = 1, \dots, n + 1$ is rooted deeply in the modern practical default modelling. Speaking briefly, the CreditRisk⁺ approach ([Frey and Mcneil, 2003](#)) to modelling dependent defaults assumes that, for $i = 1, \dots, n$, the stochastically independent hitting times ${}_iE_{\lambda_j}$ are exponentially distributed conditionally on the r.v.'s $\Lambda_1, \dots, \Lambda_{n+1}$, which are mutually independent gamma distributed and explain the uncertainty associated with the corresponding r.f.'s $j \in \{1, \dots, n + 1\}$. In such a case, the probability that the i -th r.c. defaults before the end of the year, say, is given by $\mathbf{P}[X_i < 1 | \mathbf{\Lambda}] = 1 - \exp\{-\sum_{j=1}^{n+1} w_j \Lambda_j\}$, where $\mathbf{\Lambda} := (\Lambda_1, \dots, \Lambda_{n+1})'$ is a r.v. and $w_1, \dots, w_{n+1}$ are deterministic positive weights. The distributional properties of the CreditRisk⁺ approach were studied by us in [Furman and Su \(2015\)](#).

In the present paper we introduce and study what can be viewed as an extension of the CreditRisk⁺ approach. More generally, our set-up can be motivated by the ERM framework ([Sweeting, 2011](#)). In addition, our findings herein are seemingly of interest in distribution theory, as we end up with a new multivariate probability distribution having Pareto of the 2nd kind univariate margins, which unifies many existing stand alone results in, e.g., [Arnold \(1983, 2015\)](#), [Chiragiev and Landsman \(2009\)](#), [Asimit et al. \(2010\)](#). Distributions with Paretian tails have been applied in a multitude of areas. Herein we refer to: [Benson](#)

et al. (2007) for applications in catastrophic events; Koedijk et al. (1990), Longin (1996), Gabaix et al. (2003) for applications in general financial phenomena; Cebrián et al. (2003) for applications in insurance pricing; and Soprano et al. (2010), Chavez-Demoulin et al. (2015) for applications in risk management.

To prepare for the forthcoming discussion, we let $(X_1, \dots, X_n)'$ denote the default times of arbitrary financial units with labels in the set $\{1, \dots, n\}$, exposed to $d(\in \mathbf{N})$ exogenous r.f.'s having stochastically independent hitting times E_{λ_j} that are exponentially distributed conditionally on Λ_j , $j = 1, \dots, d$. We assume that the r.v.'s $\Lambda_1, \dots, \Lambda_d$ are gamma distributed and stochastically independent as in CreditRisk⁺. The exposure matrix $c \in \text{Mat}_{n \times d}(\mathbf{1})$ having entries $c_{i,j} \in \mathbf{1} := \{0, 1\}$, $i = 1, \dots, n$, $j = 1, \dots, d$ is deterministic and determined by the upper management. Then we say that the j -th r.f. hits the set of r.c.'s $\mathcal{RC}_j := \{i \in \{1, \dots, n\} : c_{i,j} = 1\}$ for any $j = 1, \dots, d$. In a similar fashion, we say that the i -th r.c. is hit by the set of r.f.'s $\mathcal{RF}_i := \{j \in \{1, \dots, d\} : c_{i,j} = 1\}$ for any $i = 1, \dots, n$. The rest of the paper is organized as follows: We revisit and generalize when necessary some existing multivariate dependence structures that are relevant to our present research in Section 3.2, and we introduce and study the main object of interest in the present paper - the Multiple Risk Factor (MRF) dependence models in Section 3.3. In Section 3.4 we specialize the discussion to the bivariate case, and in Section 3.5 we exemplify the usefulness of the MRF dependencies within the contexts of actuarial risk measurement and pricing. Section 3.6 concludes the paper.

3.2 Relevant existing dependencies revisited

We start off with a number of simple but important notes. First, in the sequel we denote by $\Lambda \sim Ga(\alpha, \sigma)$ a r.v. that follows a gamma distribution and has the probability density function (p.d.f.)

$$f_\Lambda(x) = \frac{1}{\Gamma(\alpha)} e^{-\sigma x} x^{\alpha-1} \sigma^\alpha, \quad x \in \mathbf{R}_+,$$

where $\alpha \in \mathbf{R}_+$ and $\sigma \in \mathbf{R}_+$ are deterministic shape and rate parameters, respectively. Second, we denote by $E_\lambda \sim Exp(\lambda)$ an exponentially distributed r.v. with the p.d.f.

$$f_{E_\lambda}(x) = \lambda e^{-\lambda x}, \quad x \in \mathbf{R}_+,$$

where $\lambda \in \mathbf{R}_+$ is a deterministic parameter. Third, we denote by ‘ $*$ ’ the mixture operator such that given two appropriately jointly measurable r.v.’s $X_\beta \sim C(\beta)$ with $\beta \in \mathcal{B} \subseteq \mathbf{R}$ and $B \sim H$, it holds that $X_\beta * B$ has the same distribution as X_B , or notationally $X_\beta * B \stackrel{d}{=} X_B$, where we of course assume that the r.v. B has its range in $\mathcal{B}$. Then we readily have that

$$E_\Lambda \stackrel{d}{=} E_\lambda * \Lambda \sim Pa(II)(\sigma, \alpha),$$

that is E_Λ follows the Lomax distribution. We note in this respect that a Lomax distributed r.v. $X \sim Pa(II)(\sigma, \alpha)$ has the p.d.f. and the d.d.f., respectively,

$$f_X(x) = \frac{\alpha}{\sigma} \left(1 + \frac{x}{\sigma}\right)^{-\alpha-1} \quad \text{and} \quad \bar{F}_X(x) = \left(1 + \frac{x}{\sigma}\right)^{-\alpha}, \quad x \in \mathbf{R}_{0,+} := [0, \infty),$$

where $\sigma \in \mathbf{R}_+$ is a scale parameter and $\alpha \in \mathbf{R}_+$ is a power parameter. Last but not least, the classical extension of the univariate Lomax (and more generally Pareto of the 2nd kind) distributions to the multivariate context is arguably the one of [Arnold \(1983, 2015\)](#) with the d.d.f.

$$\bar{F}(x_1, \dots, x_n) = \left(1 + \sum_{i=1}^n \frac{x_i}{\sigma_i}\right)^{-\alpha}, \quad (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n := [0, \infty)^n \quad (3.2.1)$$

and the Pearson's correlation for, $1 \leq i \neq k \leq n$ and $\alpha > 2$,

$$\mathbf{Corr}[X_i, X_k] \equiv \frac{1}{\alpha} \in (0, 1/2). \quad (3.2.2)$$

Two recently introduced multivariate probability distributions are of central interest for the derivations herein. More specifically, these are the dependence structures with univariate Pareto margins introduced in [Asimit et al. \(2010\)](#) and in [Furman and Su \(2015\)](#). Further we discuss the just mentioned probability distributions and generalize them slightly where needed.

3.2.1 The model of [Asimit et al. \(2010\)](#)

Recall that E_{λ_j} , $j = 1, \dots, n+1$ denote $(n+1)$ stochastically independent exponentially distributed r.v.'s with parameters $\lambda_j \in \mathbf{R}_+$ and $n \in \mathbf{N}$. Further let $\Lambda_j \sim Ga(\alpha_j, 1)$ be gamma distributed r.v.'s all independent mutually and on E_{λ_j} , $j = 1, \dots, n+1$. Then the r.v. $\mathbf{X} = (X_1, \dots, X_n)'$ is said to follow the multivariate probability distribution of [Asimit et al. \(2010\)](#) if its coordinates admit the following stochastic representation

$$X_i \stackrel{d}{=} \sigma_i (E_{\Lambda_i} \wedge E_{\Lambda_{n+1}}), \quad (3.2.3)$$

with the corresponding joint d.d.f. given by

$$\bar{F}(x_1, \dots, x_n) = \left(1 + \bigvee_{i=1}^n \frac{x_i}{\sigma_i}\right)^{-\alpha_{n+1}} \prod_{i=1}^n \left(1 + \frac{x_i}{\sigma_i}\right)^{-\alpha_i}, \quad (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n. \quad (3.2.4)$$

Speaking practically, if E_{λ_j} , $j = 1, \dots, n+1$ are random and exponentially distributed (default) times, then (3.2.3) and (3.2.4) are akin to the famous common-shock set-up ([Bowers et al., 1997](#)) with $(n+1)$ exogenous risk factors, of which n are idiosyncratic and one is the common shock. By construction, the times of occurrence of the idiosyncratic risks are

independent, whereas the times of occurrence of the systemic risks are fully-commonotonic. The presence of the common-shock risk factor results in a singularity, that is $\mathbf{P}[X_1 = \dots = X_n] = \alpha_{n+1}/(\alpha_1 + \dots + \alpha_n + \alpha_{n+1})$. Thereby, d.d.f. (3.2.4) is not absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}_{0,+}^n$. The presence of singularity is in general desirable as it addresses the phenomenon of simultaneous death/default, but at the same time it reduces the tractability of (3.2.4) considerably (Asimit et al., 2015).

A possibility of having numerous systemic r.f.'s that may hit subsets of the coordinates of the r.p. $\mathbf{X}$ as well as the general lines of Sweeting (2011) motivate an extension of the set-up described above to allow for an arbitrary number, say $l(\in \mathbf{N})$ of r.f.'s. To this end, let $c^l \in \text{Mat}_{n \times l}(\mathbf{1})$ be an $n \times l$ matrix with entries in $\mathbf{1}$ and $n, l \in \mathbf{N}$. For the sake of the discussion in this section we assume that $n < l$. Also, let $\mathcal{RF}_i^l := \{j \in \{1, \dots, l\} : c_{i,j}^l = 1\}$ denote the set of all risk factors that 'attack' risk component $i \in \{1, \dots, n\}$.

Definition 3.2.1. Let $E_{\lambda_j}, j = 1, \dots, l$ be stochastically independent exponentially distributed r.v.'s with parameters $\lambda_j \in \mathbf{R}_+$. Also let $\Lambda_j \sim \text{Ga}(\alpha_j, 1)$ be gamma distributed r.v.'s all independent mutually and on $E_{\lambda_j}, j = 1, \dots, l$. Set, for $\sigma_i \in \mathbf{R}_+$,

$$X_i := \sigma_i \bigwedge_{j \in \mathcal{RF}_i^l} E_{\Lambda_j},$$

then the joint distribution of $X_1, \dots, X_n$ is a generalized variant of the one introduced in Asimit et al. (2010). Succinctly, we write $\mathbf{X} \sim \text{Pa}_{1, \dots, n}^{c^l}(II)(\boldsymbol{\sigma}, \boldsymbol{\alpha})$, where $\boldsymbol{\sigma} := (\sigma_1, \dots, \sigma_n)'$ is an n - dimensional vector of scale parameters and $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_l)'$ is an l - dimensional vector of power parameters, all the parameters are positive reals.

Clearly the distribution of the i -th coordinate is Lomax, i.e., we have that $X_i \sim \text{Pa}(II)(\sigma_i, \alpha_{c,i})$ where $\alpha_{c,i} := \sum_{j \in \mathcal{RF}_i^l} \alpha_j, i = 1, \dots, n$. The joint distribution is given in the following theorem. Therein we let $\alpha_{c,(i,k)} := \sum_{j \in \mathcal{RF}_{(i,k)}^l} \alpha_j$, where $\mathcal{RF}_{(i,k)}^l := \{j \in \{1, \dots, l\} : c_{k,j}^l c_{i,j}^l = 1\}$,

that is the set $\mathcal{RF}_{(i,k)}^l$ contains the risk factors that attack both the i -th and the k -th risk components, $1 \leq i \neq k \leq n$.

Theorem 3.2.1. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^{c_l}(II)(\boldsymbol{\sigma}, \boldsymbol{\alpha})$, then its d.d.f. is given by*

$$\overline{F}(x_1, \dots, x_n) = \prod_{j=1}^l \left(1 + \bigvee_{i \in \mathcal{RC}_j} \frac{x_i}{\sigma_i} \right)^{-\alpha_j}, \quad (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n. \quad (3.2.5)$$

Moreover, the index of Pearson's correlation is given by

$$\text{Corr}[X_i, X_k] = \frac{\alpha_{c,(i,k)}}{\alpha_{c,i} + \alpha_{c,k} - \alpha_{c,(i,k)} - 2} \sqrt{\frac{(\alpha_{c,i} - 2)(\alpha_{c,k} - 2)}{\alpha_{c,i}\alpha_{c,k}}}, \quad (3.2.6)$$

subject to $\alpha_{c,i} > 2$ and $\alpha_{c,k} > 2$.

The proof of the theorem is omitted, as it is a simple extension of Propositions 2.3 and 3.2 in [Asimit et al. \(2010\)](#).

A remark about the c^l -matrix is perhaps well-timed now. It is useful to realize that the importance of the matrix is in that it shapes the set of risk factors. More specifically, a risk factor $j \in \{1, \dots, l\}$ is idiosyncratic if and only if $\sum_{i=1}^n c_{i,j}^l = 1$, and it is the common-shock otherwise. Moreover, in the latter case the risk factor ($j \in \{1, \dots, l\}$ again) affects the subset of risk components $\mathcal{RC}_j = \{i \in \{1, \dots, n\} : c_{i,j}^l = 1\}$.

We conclude this subsection by noticing that the idea of randomization, and Definition 3.2.1 is nothing but that, has been well-studied and can be traced back to [Feller \(1966\)](#) - probability theory and [Bunke \(1969\)](#) - economics. Important recent references are [Geweke and Amisano \(2011\)](#) - economics, [McNeil et al. \(2005\)](#) and references therein - quantitative risk, generally and [Gordy \(2000\)](#) - credit risk, particularly. Also, as (3.2.6) can take on any value in $[0, 1]$, d.d.f. (3.2.5) covers the entire range of positive dependence when measured by the Pearson's index of correlation. We show in the next subsection that neither one of

the dependence structures investigated in [Arnold \(1983, 2015\)](#) and [Chiragiev and Landsman \(2009\)](#) enjoy the just mentioned important property. .

3.2.2 The model of [Furman and Su \(2015\)](#)

Let E_{λ_j} , $j = 1, \dots, n+1$ denote as hitherto exponentially distributed and stochastically independent r.v.'s with parameters $\lambda_j \in \mathbf{R}_+$ and $n \in \mathbf{N}$. Further let ${}_iE_{\lambda_j}$, $i = 1, \dots, n$ be independent copies of the r.v. E_{λ_j} , $j = 1, \dots, n+1$. In addition, let $\Lambda_j \sim Ga(\gamma_j, 1)$ be gamma distributed r.v.'s independent mutually and on ${}_iE_{\lambda_j}$, $i = 1, \dots, n$, $j = 1, \dots, n+1$. The r.v. $\mathbf{X} = (X_1, \dots, X_n)'$ follows the multivariate probability distribution of [Furman and Su \(2015\)](#) if its coordinates admit the following stochastic representation

$$X_i \stackrel{d}{=} \sigma_i ({}_iE_{\Lambda_i} \wedge {}_iE_{\Lambda_{n+1}}) \quad (3.2.7)$$

where $\sigma_i \in \mathbf{R}_+$ and $i = 1, \dots, n$. The joint d.d.f. is then given by

$$\bar{F}(x_1, \dots, x_n) = \prod_{i=1}^n \left(1 + \frac{x_i}{\sigma_i}\right)^{-\gamma_i} \left(1 + \sum_{i=1}^n \frac{x_i}{\sigma_i}\right)^{-\gamma_{n+1}}, \quad (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n. \quad (3.2.8)$$

Speaking plainly, (3.2.7) and (3.2.8) mimic the common-shock framework described in Subsection 3.2.1, but circumvent the rise of singularity by assuming that the times of occurrence of the common risk factor are not fully-comonotonic but rather conditionally independent. As in Subsection 3.2.1, we extend (3.2.7) and (3.2.8) to augment $m \in \mathbf{N}$ risk factors. To this end, let c^m be an $n \times m$ deterministic matrix with entries equal to either zero or one, and let $\mathcal{RF}_i^m := \{j \in \{1, \dots, m\} : c_{i,j}^m = 1\}$ for $i = 1, \dots, n$.

Definition 3.2.2. *Let ${}_iE_{\lambda_j}$, $i = 1, \dots, n$ be independent copies of the stochastically independent exponentially distributed r.v.'s E_{λ_j} , $\lambda_j \in \mathbf{R}_+$, $j = 1, \dots, m$. Also let $\Lambda_j \sim Ga(\gamma_j, 1)$ be*

gamma distributed r.v.'s all independent mutually and on ${}_iE_{\lambda_j}$, $i = 1, \dots, n$, $j = 1, \dots, m$.

Set, for $\sigma_i \in \mathbf{R}_+$,

$$X_i := \sigma_i \bigwedge_{j \in \mathcal{RF}_i^m} {}_iE_{\Lambda_j},$$

then the joint distribution of $X_1, \dots, X_n$ is a generalized variant of the one introduced in [Furman and Su \(2015\)](#). Succinctly, we write $\mathbf{X} \sim Pa_{1, \dots, n}^{c^m}(II)(\boldsymbol{\sigma}, \boldsymbol{\gamma})$, where $\boldsymbol{\sigma} := (\sigma_1, \dots, \sigma_n)'$ is an n - dimensional vector of scale parameters and $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_m)'$ is an m - dimensional vector of power parameters, all parameters are positive reals.

The distribution of X_i , $i = 1, \dots, n$ is clearly $Pa(II)(\sigma_i, \gamma_{c,i})$, where $\gamma_{c,i} = \sum_{j \in \mathcal{RF}_i^m} \gamma_j$, that is the coordinates of the random vectors in Definition 3.2.2 are Lomax distributed, similarly to the situation in Definition 3.2.1. The joint distribution of $X_1, \dots, X_n$ is formulated in the following theorem. We note in passing that by analogy with the discussion in the previous subsection, we let $\gamma_{c,(i,k)} := \sum_{j \in \mathcal{RF}_{(i,k)}^m} \gamma_j$ where $\mathcal{RF}_{(i,k)}^m := \{j \in \{1, \dots, m\} : c_{i,j}^m c_{k,j}^m = 1\}$, and that the $(q+1) \times q$ hypergeometric function is given by ([Gradshteyn and Ryzhik, 2014](#))

$${}_{q+1}F_q(a_1, \dots, a_{q+1}; b_1, \dots, b_q; z) := \sum_{k=0}^{\infty} \frac{(a_1)_k, \dots, (a_{q+1})_k}{(b_1)_k, \dots, (b_q)_k} \frac{z^k}{k!}, \quad (3.2.9)$$

where $(p)_n := p(p+1) \dots (p+n-1)$ for $n \in \mathbf{Z}_+$, $(p)_0 := 1$ and $q \in \mathbf{Z}_+$. For $a_1, \dots, a_{q+1}$ all positive, and these are the cases of interest in the present paper, the radius of convergence of the series is the open disk $|z| < 1$. On the boundary $|z| = 1$, the series converges absolutely if $d = b_1 + \dots + b_q - a_1 - \dots - a_{q+1} > 0$, and it converges except at $z = 1$ if $0 \geq d > -1$.

Theorem 3.2.2. *Let $\mathbf{X} \sim Pa_{1, \dots, n}^{c^m}(II)(\boldsymbol{\sigma}, \boldsymbol{\gamma})$, then its d.d.f. is given by*

$$\bar{F}(x_1, \dots, x_n) = \prod_{j=1}^m \left(1 + \sum_{i \in \mathcal{RC}_j} \frac{x_i}{\sigma_i} \right)^{-\gamma_j}, \quad (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n. \quad (3.2.10)$$

Furthermore, the index of Pearson's correlation is given, for $\gamma_{c,i} > 2$ and $\gamma_{c,k} > 2$, by

$$\mathbf{Corr}[X_i, X_k] = ({}_3F_2(\gamma_{c,(i,k)}, 1, 1; \gamma_{c,i}, \gamma_{c,k}; 1) - 1) \sqrt{\frac{(\gamma_{c,i} - 2)(\gamma_{c,k} - 2)}{\gamma_{c,i}\gamma_{c,k}}} \in [0, 1/2), \quad (3.2.11)$$

where $1 \leq i \neq k \leq n$.

We omit the proof of the theorem as it is very similar to the derivations in Theorem 2.1 in [Furman and Su \(2015\)](#). We only note in passing that the hypergeometric function in (3.2.11) converges absolutely as $\gamma_{c,i} \geq \gamma_{c,(i,k)}$ and also $\gamma_{c,k} \geq \gamma_{c,(i,k)}$, $1 \leq i \neq k \leq n$.

D.d.f. (3.2.10) is clearly absolutely continuous with respect to the corresponding Lebesgue measure, and it unifies the classic multivariate Pareto distribution of [Arnold \(1983, 2015\)](#) as well as the recently introduced ones of [Chiragiev and Landsman \(2009\)](#). Also, the fact that (3.2.11) is in $[0, 1/2)$ is rather unfortunate, as it makes d.d.f. (3.2.10) inappropriate for applications when the r.v.'s $X_1, \dots, X_n$ have strong positive correlations. As d.d.f. (3.2.10) serves as the basic building block of the popular in the modern practical default risk measurement CreditRisk⁺ approach, the aforementioned limited range of attainable Pearson's correlations conforms well to the empirical evidence showing that CreditRisk⁺ may underestimate defaults' clustering ([Das et al., 2007](#)).

To summarize the developments hitherto, we note that

- the formulations in terms of times of occurrence with random risk factors mimic the general set-up of default risk and as such are well suited for relevant practical applications and can be easily conveyed to upper management;
- simple stochastic representations in Definitions 3.2.1 and 3.2.2 are essential and allow for convenient simulation analysis and stresstesting;

- d.d.f.'s (3.2.5) and (3.2.10) are analytically tractable and quite general unifying a variety of existing multivariate structures.

The advantages described above lay the ground for gluing Definitions 3.2.1 and 3.2.2 into one general multivariate probability structure that would augment random times of occurrence that are a.) stochastically independent, b.) positively dependent, c.) fully-comonotonic as well as inherit the full range of attainable non-negative Pearson's correlations. This new general structure is referred to as the Multiple Risk Factor (MRF) dependence structure, and it is the main object of study in the present paper.

3.3 Multiple risk factors dependence structure

Naturally, the gap between the stochastically independent and fully-comonotonic times of occurrence in Definition 3.2.1 is somewhat too drastic. Next we fill the gap by unifying Definitions 3.2.1 and 3.2.2.

Let $\mathbf{Y} := (Y_1, \dots, Y_n)'$ and $\mathbf{Z} := (Z_1, \dots, Z_n)'$ be two r.v.'s of dimension $n(\in \mathbf{N})$. Next we formulate the MRF model of interest in this paper.

Definition 3.3.1. Assume that $\mathbf{Y} \sim Pa_{1,\dots,n}^{c_l}(\boldsymbol{\sigma}, \boldsymbol{\alpha})$ as in Definition 3.2.1 and $\mathbf{Z} \sim Pa_{1,\dots,n}^{c_m}(\boldsymbol{\sigma}, \boldsymbol{\gamma})$ as in Definition 3.2.2 be stochastically independent, then the r.v. $\mathbf{X} = (X_1, \dots, X_n)'$ with each coordinate $X_i := Y_i \wedge Z_i$, $i = 1, \dots, n$ is said to follow a Multiple Risk Factor dependence with Pareto of the 2nd kind univariate marginal distributions; notationally $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$. Here $c = (c^l, c^m)$ is an $n \times (l + m)$ -dimensional block matrix, $\boldsymbol{\sigma}$ is an n -dimensional vector of scale parameters and $\boldsymbol{\xi}$ is an $(l + m)$ - dimensional vector of power parameters, such that $\xi_j = \alpha_j$, $j = 1, \dots, l(\in \mathbf{N})$ and $\xi_j = \gamma_j$, $j = l + 1, \dots, l + m(\in \mathbf{N})$.

In what follows, we sometimes write $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$, if we wish to emphasize the presence of $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ vectors of parameters.

It is clear that Definition 3.3.1 unifies multiple risk factor dependence structures with and without singularities into one encompassing structure. One practical motivation for the MRF structures stems from Sweeting (2011) (also, Duffie and Singleton, 1999). More specifically, consider a portfolio of default times of n business units. Further assume that each component of the portfolio is exposed to a set of possibly $(l + m)$ fatal risk factors, of which some are idiosyncratic - $\{j \in \{1, \dots, l + m\} : \sum_{i=1}^n c_{i,j} = 1\}$, others are systemic - $\{j \in \{1, \dots, l\} : \sum_{i=1}^n c_{i,j} > 1\}$, and there exists yet another group of risk factors $\{j \in \{l + 1, \dots, l + m\} : \sum_{i=1}^n c_{i,j} > 1\}$ that attack the risk components at positively dependent but not perfectly dependent times. Then, assuming that the hitting times of the risk factors are stochastically independent group-wise and also that the risk factors are gamma distributed as in CreditRisk⁺, the MRF structure as in Definition 3.3.1 is obtained. Another motivation for the MRF dependencies stems from the recent trend of prudence that have taken the modern financial risk measurement by storm. Indeed, multivariate probability distributions having heavy tailed and positively dependent univariate margins are in a significant practical demand nowadays.

The joint d.d.f. of the MRF r.v.'s is given in the next theorem. Recall that $\mathcal{RC}_j$ has been defined as the set that contains all the risk components that are exposed to the j -th risk factor, $j \in \{1, \dots, l + m\}$.

Theorem 3.3.1. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$, then the joint d.d.f. of its coordinates is given*

by

$$\bar{F}(x_1, \dots, x_n) = \prod_{j=1}^l \left(1 + \bigvee_{i \in \mathcal{RC}_j} \frac{x_i}{\sigma_i} \right)^{-\xi_j} \prod_{j=l+1}^{l+m} \left(1 + \sum_{i \in \mathcal{RC}_j} \frac{x_i}{\sigma_i} \right)^{-\xi_j}, \quad (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n, \quad (3.3.1)$$

where $\boldsymbol{\sigma}$ is an n -dimensional vector of scale parameters, and $\boldsymbol{\xi}$ is an $(l+m)$ -dimensional vector of power parameters.

The proof of Theorem 3.3.1 is simply by construction of the MRF r.v.'s, and it is thus omitted.

On the one hand, d.d.f. (3.3.1) can be viewed as a new multivariate probability distribution with Lomax distributed univariate margins, which, for appropriately chosen c matrices, reduces to, e.g., the models in Arnold (1983, 2015), Chiragiev and Landsman (2009), Asimit et al. (2010) and Furman and Su (2015). As such it can be used to describe dependent (actuarial) risks with heavy tailed marginal behaviour. On the other hand, d.d.f. (3.3.1) enjoys the interpretation of a dependent default times model that encompasses an arbitrary number of exogenous risk factors having stochastically independent, positively orthant dependent (see, Denuit et al., 2005; Joe, 1997) or fully-comonotonic occurrence times. In either case, the following theorem is of basic importance. The proof is simple and thus omitted.

Theorem 3.3.2. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$, then, for $i = 1, \dots, n$, we have that each coordinate is Lomax distributed, i.e., $X_i \sim Pa(II)(\sigma_i, \xi_{c,i})$, where $\xi_{c,i} = \sum_{j \in \mathcal{RF}_i} \xi_j$ and therefore*

- the d.d.f. of X_i is

$$\bar{F}(x) = \left(1 + \frac{x}{\sigma_i} \right)^{-\xi_{c,i}}, \quad x \in \mathbf{R}_{0,+};$$

- the mathematical expectation of X_i is, for $\xi_{c,i} > 1$,

$$\mathbf{E}[X_i] = \frac{\sigma_i}{\xi_{c,i} - 1};$$

- the variance of X_i is, for $\xi_{c,i} > 2$,

$$\mathbf{Var}[X_i] = \frac{\sigma_i^2 \xi_{c,i}}{(\xi_{c,i} - 1)^2 (\xi_{c,i} - 2)}.$$

We note in passing that the expectation/variance of the r.v. X_i , $i = 1, \dots, n$ can be finite even if the expectations/variances of occurrence times of some risk factors $j \in \{1, \dots, l+m\}$ are infinite.

When the r.v.'s $X_1, \dots, X_n$ denote times of default of various financial units $i \in \{1, \dots, n\}$, the r.v.'s $X_- := \wedge_{i=1}^n X_i$ and $X_+ := \vee_{i=1}^n X_i$ denote, respectively, first and last default times. We say that X_- and X_+ have distributions F_- and F_+ , respectively, and they play a pivotal role in the general theory of credit risk (Fomby et al., 2008; Adalsteinsson, 2014) and in the mathematics of life contingencies (Bowers et al., 1997). We further investigate the d.d.f.'s of the two. In this respect, the following lemma is of central importance.

Lemma 3.3.1. *For $i = 1, \dots, n$, let $\Lambda_i \sim \text{Ga}(\gamma_i(\in \mathbf{R}_+), \sigma_i(\in \mathbf{R}_+))$, denote independent gamma distributed r.v.'s, and let $\Lambda^* := \Lambda_1 + \dots + \Lambda_n$ be their sum. Then $\Lambda^* \sim \text{Ga}(\gamma^* + K, \sigma_+)$, where $\gamma^* = \gamma_1 + \dots + \gamma_n$, $\sigma_+ = \vee_{i=1}^n \sigma_i$ and K is an integer-valued non-negative r.v. with the probability mass function given, for $k \in \mathbf{N}_0$, by*

$$p_k = \mathbf{P}[K = k] = c_+ \delta_k \tag{3.3.2}$$

where

$$c_+ = \prod_{i=1}^n \left(\frac{\sigma_i}{\sigma_+} \right)^{\gamma_i}$$

and

$$\delta_k = k^{-1} \sum_{l=1}^k \sum_{i=1}^n \gamma_i \left(1 - \frac{\sigma_i}{\sigma_+}\right)^l \delta_{k-l}, \text{ for } k > 0 \text{ and } \delta_0 = 1.$$

Moreover, the distribution of E_{Λ^*} is $Pa(II)(\sigma_+, \gamma^* + K)$.

Proof. The first part of the lemma, i.e., the distribution of Λ^* is proved in [Moschopoulos \(1985\)](#) (also, [Furman and Landsman, 2005](#)). To establish the second part, we note that, for $x \in \mathbf{R}_{0,+}$, the d.d.f. of E_{Λ^*} is given by

$$\bar{F}_{\Lambda^*}(x) = \int_{\mathbf{R}_+} e^{-xt} \sum_{k=0}^{\infty} \frac{p_k}{\Gamma(\gamma + k)} e^{-\alpha_+ t} t^{\gamma+k-1} \alpha_+^{\gamma+k} dt = \sum_{k=0}^{\infty} p_k \left(1 + \frac{x}{\alpha_+}\right)^{-(\gamma+k)},$$

where the interchange of the summation and the integration signs is justified because of the uniform convergence of the integrand. This completes the proof. $\square$

We are now ready to prove that in the context of the MRF dependencies, the distribution of the first default time is Lomax with a random power parameter.

Theorem 3.3.3. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$, then $X_- \sim Pa(II)(\sigma_+, \xi^* + K)$, where*

$$\sigma_+ = \left(\bigvee_{j=1}^l \bigwedge_{i \in \mathcal{RC}_j} \sigma_i \right) \vee \left(\bigvee_{j=l+1}^{l+m} \left(\sum_{i \in \mathcal{RC}_j} \frac{1}{\sigma_i} \right)^{-1} \right),$$

and $\xi^* + K = \xi_1 + \dots + \xi_{l+m} + K$ is a random power parameter with the integer valued r.v. K having probability mass function [\(4.4.3\)](#).

Proof. We observe that, for $x \in \mathbf{R}_{0,+}$, the d.d.f. of the first default r.v. is written as

$$\begin{aligned} \mathbf{P}[X_- > x] &= \mathbf{P}[\wedge_{i=1}^n X_i > x] \\ &= \prod_{j=1}^l \left(1 + x \bigvee_{i \in \mathcal{RC}_j} \frac{1}{\sigma_i}\right)^{-\xi_j} \prod_{j=l+1}^{l+m} \left(1 + x \sum_{i \in \mathcal{RC}_j} \frac{1}{\sigma_i}\right)^{-\xi_j} \\ &= \int_{\mathbf{R}_+^{l+m}} \exp\left\{-x \sum_{j=1}^{l+m} \lambda_j\right\} dG_{\mathbf{M}}(\lambda_1, \dots, \lambda_{l+m}) \end{aligned}$$

where $\mathbf{M} \sim G_{1,\dots,l+m}$ is an $(l+m)$ -dimensional r.v. with stochastically independent and gamma distributed coordinates. More specifically, we have that $M_j \sim Ga(\xi_j, \wedge_{i \in \mathcal{RC}_j} \sigma_i)$ for $j = 1, \dots, l$ and $M_j \sim Ga\left(\xi_j, \left(\sum_{i \in \mathcal{RC}_j} 1/\sigma_i\right)^{-1}\right)$ for $j = l+1, \dots, l+m$.

Clearly we also have that $\mathbf{P}[X_- > x] = \bar{F}_{E_{M^*}}(x)$, $x \in \mathbf{R}_{0,+}$, where M^* is an $(l+m)$ -fold convolution of the coordinates of $\mathbf{M} = (M_1, \dots, M_{l+m})'$, that is its distribution follows from the first part of Lemma 4.4.1. The statement of the theorem then follows by evoking the second part of Lemma 4.4.1. This completes the proof. $\square$

The next assertion is similar to Proposition 2 of Vernic (2011) as well as to Proposition 2.4 of Furman and Su (2015).

Corollary 3.3.1. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$, then the d.d.f. of the last default r.v. is a linear combination of the d.d.f.'s of univariate Pareto of the 2nd kind r.v.'s with random shape parameters, namely*

$$\bar{F}_+(x) = \sum_{\mathcal{S} \subseteq \{1,\dots,n\}} (-1)^{|\mathcal{S}|-1} \bar{F}_{\mathcal{S}-}(x), \quad x \in \mathbf{R}_{0,+}, \quad (3.3.3)$$

where $X_{\mathcal{S}-} = \wedge_{s \in \mathcal{S} \subseteq \{1,\dots,n\}} X_s$ and $X_{\mathcal{S}-} \sim F_{\mathcal{S}-}$.

Proof. Clearly $\bar{F}_+(x) = \mathbf{P}[\{\cup_{i=1}^n (X_i > x)\}]$, $x \in \mathbf{R}_{0,+}$, and then the statement of the Corollary follows using the inclusion-exclusion principle and employing Theorem 3.3.3. This completes the proof. $\square$

We conclude this section by stating yet another important property of the MRF dependence structures, which arguably distinguishes the construction from the majority of the multivariate probability distributions existing nowadays. In this respect, let $k \leq n$ and let $i_1, \dots, i_k \in \{1, \dots, n\}$ establish an index set, then the MRF dependence structures allow the probability $\mathbf{P}[X_{i_1} = \dots = X_{i_k}]$ to be non-zero. This phenomenon, which can be easily

motivated in practice by, e.g., the presence of a parent subsidiary or similar contractual relationships - in the context of default risk, and by occurrence of a catastrophe that affects a number of lives simultaneously - in the context of life contingencies, takes place because the MRF distributions may have singularities and thus are in general not absolutely continuous with respect to the Lebesgue measure. In the next theorem we study the probability mass that is assigned to the singularity.

Before stating and proving the next result, we extend some of the notations already used. First, the set of all risk factors that attack sub-portfolio $(X_{i_1}, \dots, X_{i_k})'$ is, for $k \geq 2$, in the sequel denoted by

$$\mathcal{RF}_{i_1, \dots, i_k} := \{j \in \{1, \dots, l + m\} : c_{i_h, j} = 1 \text{ for at least one } i_h \in \{i_1, \dots, i_k\}\}, \quad (3.3.4)$$

which is the union of two disjoint sets, that is of

$$\mathcal{RF}_{(i_1, \dots, i_k)} := \{j \in \{1, \dots, l + m\} : c_{i_h, j} = 1 \text{ for all } i_h \in \{i_1, \dots, i_k\}\} \quad (3.3.5)$$

and

$$\mathcal{RF}_{\overline{(i_1, \dots, i_k)}} := \mathcal{RF}_{i_1, \dots, i_k} \setminus \mathcal{RF}_{(i_1, \dots, i_k)}. \quad (3.3.6)$$

Also, let

$$\mathcal{RF}_{i_h, \overline{(i_1, \dots, i_k)}} := \mathcal{RF}_{i_h} \setminus \mathcal{RF}_{(i_1, \dots, i_k)}. \quad (3.3.7)$$

Based on the notation above, we can have, e.g., $\xi_{c, i_1, \dots, i_k} = \sum_{j \in \mathcal{RF}_{i_1, \dots, i_k}} \xi_j$ and in a similar fashion $\xi_{c, (i_1, \dots, i_k)} = \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}} \xi_j$, $\xi_{c, \overline{(i_1, \dots, i_k)}} = \sum_{j \in \mathcal{RF}_{\overline{(i_1, \dots, i_k)}}} \xi_j$ as well as $\xi_{c, i_h, \overline{(i_1, \dots, i_k)}} = \sum_{j \in \mathcal{RF}_{i_h, \overline{(i_1, \dots, i_k)}}} \xi_j$. Of course, by analogy, we may need to sum over the coordinates of γ or α , and in such cases we add superscripts 'l' and 'm', respectively, to sets (4.3.1) to (4.3.4).

Theorem 3.3.4. Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$, and let as before $i_1, \dots, i_k \in \{1, \dots, n\}$ for $2 \leq k \leq n$. Then we have that

$$\mathbf{P}[X_{i_1}/\sigma_{i_1} = \dots = X_{i_k}/\sigma_{i_k}] = \alpha_{c,(i_1,\dots,i_k)} \mathbf{E} \left[\frac{1}{\xi_{c,i_1,\dots,i_k} + K} \right], \quad (3.3.8)$$

where K is an integer-valued r.v. with the p.m.f. a lá (4.4.3).

Proof. First, since $\mathbf{P}[Y_{i_1} = \dots = Y_{i_k}] \equiv 0$, we arrive at

$$\begin{aligned} & \mathbf{P}[X_{i_1}/\sigma_{i_1} = \dots = X_{i_k}/\sigma_{i_k}] \\ &= \mathbf{P} \left[\left\{ \bigcap_{i_h=i_1}^{i_k} (Z_{i_h} \leq Y_{i_h}) \cap Z_{i_1}/\sigma_{i_1} = \dots = Z_{i_k}/\sigma_{i_k} \right\} \right] \\ &= \alpha_{c,(i_1,\dots,i_k)} \int_{\mathbf{R}_+} \prod_{j=l+1}^{l+m} \left(1 + z \sum_{i_h=1}^{i_k} c_{i_h,j} \right)^{-\gamma_j} (1+z)^{-\alpha_{c,i_1,\dots,i_k}-1} dz \end{aligned} \quad (3.3.9)$$

where we have the latter equality sign since, for $Z_{i_h, \overline{(i_1,\dots,i_k)}}^\bullet := \bigwedge_{j \in \mathcal{RF}^l_{i_h, \overline{(i_1,\dots,i_k)}}} E_{\Lambda_j}$, $i_h = i_1, \dots, i_k$ and $Z_{(i_1,\dots,i_k)}^\bullet := \bigwedge_{j \in \mathcal{RF}^l_{(i_1,\dots,i_k)}} E_{\Lambda_j}$, the following string of expressions follows

$$\begin{aligned} & \mathbf{P} \left[\frac{Z_{i_1}}{\sigma_{i_1}} = \dots = \frac{Z_{i_k}}{\sigma_{i_k}} > z \right] \\ &= \mathbf{P} \left[\left\{ Z_{i_h, \overline{(i_1,\dots,i_k)}}^\bullet > Z_{(i_1,\dots,i_k)}^\bullet \text{ for all } i_h \in \{i_1, \dots, i_k\} \right\} \cap \left\{ Z_{(i_1,\dots,i_k)}^\bullet > z \right\} \right] \\ &= \mathbf{P} \left[\left\{ Z_{i_h, \overline{(i_1,\dots,i_k)}}^\bullet > Z_{(i_1,\dots,i_k)}^\bullet > z \text{ for all } i_h \in \{i_1, \dots, i_k\} \right\} \right] \\ &= \mathbf{P} \left[\left\{ E_{\Lambda_j} > Z_{(i_1,\dots,i_k)}^\bullet > z \text{ for all } j \in \mathcal{RF}^l_{(i_1,\dots,i_k)} \right\} \right] \\ &= \alpha_{c,(i_1,\dots,i_k)} \int_z^\infty (1+x)^{-\alpha_{c,\overline{(i_1,\dots,i_k)}}} (1+x)^{-\alpha_{c,(i_1,\dots,i_k)}-1} dx \\ &= \alpha_{c,(i_1,\dots,i_k)} \int_z^\infty (1+x)^{-\alpha_{c,i_1,\dots,i_k}-1} dx \end{aligned}$$

and hence

$$-\frac{d}{dz} \mathbf{P}[Z_{i_1}/\sigma_{i_1} = \dots = Z_{i_k}/\sigma_{i_k} > z] = \alpha_{c,(i_1,\dots,i_k)} (1+z)^{-\alpha_{c,i_1,\dots,i_k}-1} dz.$$

Further, as the integrand in (3.3.9) is the d.d.f. of a first to default r.v. (Theorem 3.3.1), it is the d.d.f. of a Lomax r.v. with random shape parameter $\gamma_{c,i_1,\dots,i_k} + \alpha_{c,i_1,\dots,i_k} + K + 1$,

where K is an integer valued r.v. with p.m.f. á la (4.4.3) and scale parameter equal to one.

Hence by Lemma (4.4.1), we have that

$$\mathbf{P}[X_{i_1}/\sigma_{i_1} = \cdots = X_{i_k}/\sigma_{i_k}] = \alpha_{c,i_1,\dots,i_k} \int_{\mathbf{R}_+} \sum_{k=0}^{\infty} p_k (1+z)^{-(\gamma_{c,i_1,\dots,i_k} + \alpha_{c,i_1,\dots,i_k} + k) - 1} dz.$$

The proof is completed by changing the order of summation and integration signs. $\square$

Interestingly, if K is zero almost surely, then (3.3.8) reduces to

$$\mathbf{P}[X_{i_1}/\sigma_{i_1} = \cdots = X_{i_k}/\sigma_{i_k}] = \frac{\alpha_{c,(i_1,\dots,i_k)}}{\xi_{c,i_1,\dots,i_k}}.$$

This happens when there are no risk factors with positively dependent but not fully comonotonic default times present in the model. If in addition, $\mathcal{RC}_j$ is an empty set for all $j = l+1, \dots, l+m$, then (3.3.8) simplifies to

$$\mathbf{P}[Z_{i_1}/\sigma_{i_1} = \cdots = Z_{i_k}/\sigma_{i_k}] = \frac{\alpha_{c,(i_1,\dots,i_k)}}{\alpha_{c,i_1,\dots,i_k}},$$

which is a slightly more general expression than the one obtained in Asimit et al. (2010).

In summary, it is instructive to note that the probability mass that is assigned to the singular part of d.d.f. (3.3.1) is proportional to $\alpha_{c,(i_1,\dots,i_k)}$. In the language of default risk, this means that the stronger the contribution of the risk factors having fully-comonotonic hitting times is, the higher the probability of simultaneous default becomes.

3.4 The bivariate model of interest

In this section we study some properties of the bivariate variant of the MRF structure. More specifically, we derive joint (conditional) moments of $(X_i, X_k)' \sim Pa_{i,k}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$, where $\boldsymbol{\sigma} = (\sigma_i, \sigma_k)'$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{l+m})' = (\alpha_1, \dots, \alpha_l, \gamma_{l+1}, \dots, \gamma_{l+m})'$ are two deterministic vectors of scale and power parameters, respectively.

It is a simple matter to see that d.d.f. (3.3.1) reduces to

$$\begin{aligned} & \overline{F}(x, y) \\ &= \left(1 + \frac{x}{\sigma_i} \bigvee \frac{y}{\sigma_k}\right)^{-\alpha_{c,(i,k)}} \left(1 + \frac{x}{\sigma_i} + \frac{y}{\sigma_k}\right)^{-\gamma_{c,(i,k)}} \left(1 + \frac{x}{\sigma_i}\right)^{-\xi_{c,i,\overline{(i,k)}}} \left(1 + \frac{y}{\sigma_k}\right)^{-\xi_{c,k,\overline{(i,k)}}}, \end{aligned} \quad (3.4.1)$$

for $1 \leq i \neq k \leq n$ and $(x, y)' \in \mathbf{R}_{0,+}^2$. The next theorem states the Lebesgue decomposition of the d.d.f. above.

Theorem 3.4.1. *For $1 \leq i \neq k \leq n$, let $(X_i, X_k)' \sim Pa(II)_{i,k}^c(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$, then its d.d.f. can be decomposed as*

$$\overline{F}(x, y) = a\overline{F}_s(x, y) + (1 - a)\overline{F}_{ac}(x, y), \text{ where } a \in [0, 1] \text{ and } (x, y)' \in \mathbf{R}_{0,+}^2. \quad (3.4.2)$$

In (3.4.2), the singular component concentrates its mass on the line $\left\{(x, y)' \in \mathbf{R}_{0,+}^2 : \frac{x}{\sigma_i} = \frac{y}{\sigma_k}\right\}$ and is given by

$$\overline{F}_s(x, y) = \frac{\alpha_{c,(i,k)}}{a} \mathbf{E} \left[\frac{1}{\xi_{c,i,k} + K} \left(1 + \frac{x}{\sigma_i} \bigvee \frac{y}{\sigma_k}\right)^{-(\xi_{c,i,k} + K)} \right], \quad (3.4.3)$$

whereas the absolutely continuous component is given, for $(x, y)' \in \mathbf{R}_{0,+}^2$, by

$$\overline{F}_{ac}(x, y) = \frac{1}{1 - a} \left(\overline{F}(x, y) - a\overline{F}_s(x, y) \right), \quad (3.4.4)$$

where $a = \mathbf{P}[X_i/\sigma_i = X_k/\sigma_k]$ is given in Theorem 3.3.4, $\xi_{c,i,k} = \sum_{j \in \mathcal{R}_{\mathcal{F}_{i,k}}} \xi_j$ and K is an integer-valued non-negative r.v. with a p.m.f. à la (4.4.3).

Proof. Let $\mathcal{A} = \left\{(X_i, X_k)' \in \mathbf{R}_{0,+}^2 : \frac{X_i}{\sigma_i} = \frac{X_k}{\sigma_k}\right\}$ for $1 \leq i \neq k \leq n$ denote the set on which the singular component of the d.d.f. concentrates. Then, for $a = \mathbf{P}[\{\mathcal{A}\}]$ and $(x, y)' \in \mathbf{R}_{0,+}^2$,

$$\begin{aligned} \overline{F}(x, y) &= a\mathbf{P}[\{X_i > x \cap X_k > y \mid \mathcal{A}\}] + (1 - a)\mathbf{P}[\{X_i > x \cap X_k > y \mid \mathcal{A}^c\}] \\ &= \mathbf{P}[\{X_i > x \cap X_k > y \cap \mathcal{A}\}] + \mathbf{P}[\{X_i > x \cap X_k > y \cap \mathcal{A}^c\}], \end{aligned}$$

where

$$\begin{aligned}
\mathbf{P}[\{X_i > x \cap X_k > y \cap \mathcal{A}\}] &= \mathbf{P}\left[\left\{\frac{X_i}{\sigma_i} > \frac{x}{\sigma_i} \cap \frac{X_k}{\sigma_k} > \frac{y}{\sigma_k} \cap \mathcal{A}\right\}\right] \\
&= \mathbf{P}\left[\left\{\frac{X_i}{\sigma_i} = \frac{X_k}{\sigma_k} > \frac{x}{\sigma_i} \vee \frac{y}{\sigma_k}\right\}\right] \\
&= \alpha_{c,(i,k)} \int_{\frac{x}{\sigma_i} \vee \frac{y}{\sigma_k}}^{\infty} \sum_{h=0}^{\infty} (1+z)^{-\xi_{c,i,k}-h-1} p_h dz
\end{aligned}$$

with the latter expression following employing the techniques used in the proof of Theorem 3.3.4. It then follows that

$$\overline{F}_s(x, y) = \frac{\alpha_{c,(i,k)}}{a} \sum_{h=0}^{\infty} \frac{1}{\xi_{c,i,k} + h} \left(1 + \frac{x}{\sigma_i} \vee \frac{y}{\sigma_k}\right)^{-(\xi_{c,i,k} + h)} p_h,$$

which proves (3.4.3). The form of the absolutely continuous component then follows from the latter expression, (3.4.1) and evoking Theorem 3.3.4. This completes the proof. $\square$

If K is zero almost surely, then (3.4.3) simplifies to

$$\overline{F}_s(x, y) = \left(1 + \frac{x}{\sigma_i} \vee \frac{y}{\sigma_k}\right)^{-\xi_{c,i,k}},$$

but it is rather cumbersome otherwise. In fact, even the probability $\mathbf{P}[X_i/\sigma_i = X_k/\sigma_k]$ is not particularly easy to calculate. In this respect, the following alternative route is helpful.

Note that in the bivariate case, the integral in (3.3.9) becomes

$$\begin{aligned}
&\int_{\mathbf{R}_+} \prod_{j=l+1}^{l+m} (1+z(c_{i,j} + c_{k,j}))^{-\gamma_j} (1+z)^{-\alpha_{c,i,k}-1} dz \\
&= \int_{\mathbf{R}_+} (1+2z)^{-\gamma_{c,(i,k)}} (1+z)^{-(\gamma_{c,i,(\overline{i,k})} + \gamma_{c,k,(\overline{i,k})} + \alpha_{c,(i,k)} + 1)} dz \\
&= \frac{1}{\xi_{c,i,k}} {}_2F_1(\gamma_{c,(i,k)}, 1; \xi_{c,i,k} + 1; -1)
\end{aligned}$$

(check (3.197(5)) in Gradshteyn and Ryzhik, 2014, for the latter equality, in which the hypergeometric series converges absolutely for $\xi_{c,i,k} > \gamma_{c,(i,k)}$, $1 \leq i \neq k \leq n$). Hence we

have that

$$a = \mathbf{P} \left[\frac{X_i}{\sigma_i} = \frac{X_k}{\sigma_k} \right] = \frac{\alpha_{c,(i,k)}}{\xi_{c,i,k}} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,i,k} + 1; -1 \right),$$

which is easy to compute numerically.

In a similar fashion, we have the following string of expressions for the singular component of the d.d.f. of $(X_i, X_k)'$, for $v = \frac{x}{\sigma_i} \vee \frac{y}{\sigma_k}$ and $(x, y)' \in \mathbf{R}_{0,+}^2$,

$$\begin{aligned} \overline{F}_s(x, y) &= \frac{\alpha_{c,(i,k)}}{a} \int_v^\infty (1+2z)^{-\gamma_{c,(i,k)}} (1+z)^{-(\gamma_{c,i,\overline{(i,k)}} + \gamma_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)} + 1)} dz \\ &= 2^{-\gamma_{c,(i,k)}} \frac{\alpha_{c,(i,k)}}{a} \int_{\mathbf{R}_+} ((1+2v)/2 + z)^{-\gamma_{c,(i,k)}} (1+v+z)^{-(\gamma_{c,i,\overline{(i,k)}} + \gamma_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)} + 1)} dz \\ &= \frac{1}{a} \frac{\alpha_{c,(i,k)}}{\xi_{c,i,k}} (1+2v)^{-\gamma_{c,(i,k)}} (1+v)^{-(\gamma_{c,i,\overline{(i,k)}} + \gamma_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)})} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,i,k} + 1; -\frac{1}{1+2v} \right), \end{aligned}$$

where the latter expression is obtained using Equation (3.197(1)) in [Gradshteyn and Ryzhik \(2014\)](#).

We further derive the Pearson's index of correlation for a random pair with the MRF dependence. Speaking strictly, the Pearson's ρ has been criticized by many authors, yet it is still a ubiquitous measure of correlation when it comes to financial risk management and/or actuarial science. Let

$$h(x; a, b) := {}_3F_2(x-1, 1, a; x, b-1; -1). \quad (3.4.5)$$

where ${}_3F_2$ is a special form of the hypergeometric function in (4.4.7) and all of x, a, b are positive reals. Then we have the following result that formulates the joint product moment in the context of the MRF structures.

Theorem 3.4.2. *Let $(X_i, X_k)' \sim Pa(II)_{i,k}^c(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$, then the product moment is, if finite, formulated as follows*

$$\mathbf{E}[X_i X_k] = \sigma_1 \sigma_2 \frac{(\xi_{c,k} - 1)h(\xi_{c,i}; \gamma_{c,(i,k)}, \xi_{c,i,k}) + (\xi_{c,i} - 1)h(\xi_{c,k}; \gamma_{c,(i,k)}, \xi_{c,i,k})}{(\xi_{c,i,k} - 2)(\xi_{c,i} - 1)(\xi_{c,k} - 1)},$$

where $\xi_{c,i} = \alpha_{c,i} + \gamma_{c,i}$, $\xi_{c,k} = \alpha_{c,k} + \gamma_{c,k}$ and $\xi_{c,i,k} = \alpha_{c,i,k} + \gamma_{c,i,k}$. If in addition, we have that $\xi_{c,i} > 2$, $\xi_{c,k} > 2$, then the Pearson's correlation is

$$\begin{aligned} \text{Corr}[X_i, X_k] &= \sqrt{\frac{(\xi_{c,i} - 2)(\xi_{c,k} - 2)}{\xi_{c,i}\xi_{c,k}}} \\ &\times \frac{(\xi_{c,k} - 1)h(\xi_{c,i}; \gamma_{c,(i,k)}, \xi_{c,i,k}) + (\xi_{c,i} - 1)h(\xi_{c,k}; \gamma_{c,(i,k)}, \xi_{c,i,k}) - \xi_{c,i,k} + 2}{(\xi_{c,i,k} - 2)}, \end{aligned}$$

where $1 \leq i \neq k \leq n$.

Proof. Assume, without loss of generality, that $\sigma_i = \sigma_k = 1$ for $1 \leq i \neq k \leq n$. Then because of (3.4.1) the d.d.f. of $(X_i, X_k)'$ is given by

$$\begin{aligned} \overline{F}(x, y) &= (1 + x \vee y)^{-\alpha_{c,(i,k)}} (1 + x + y)^{-\gamma_{c,(i,k)}} (1 + x)^{-\xi_{c,i,\overline{(i,k)}}} (1 + y)^{-\xi_{c,k,\overline{(i,k)}}} \\ &= \begin{cases} (1 + x + y)^{-\gamma_{c,(i,k)}} (1 + x)^{-\xi_{c,i,\overline{(i,k)}}} (1 + y)^{-(\xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)})}, & x \leq y \\ (1 + x + y)^{-\gamma_{c,(i,k)}} (1 + x)^{-(\xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)})} (1 + y)^{-\xi_{c,k,\overline{(i,k)}}}, & x > y \end{cases}. \end{aligned}$$

Hence we readily have that

$$\begin{aligned} \mathbf{E}[X_i X_k] &= \int_{\mathbf{R}_+^2} \overline{F}(x, y) dx dy \\ &= \int_{\mathbf{R}_+} \int_x^\infty (1 + x + y)^{-\gamma_{c,(i,k)}} (1 + x)^{-\xi_{c,i,\overline{(i,k)}}} (1 + y)^{-(\xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)})} dy dx \\ &+ \int_{\mathbf{R}_+} \int_0^x (1 + x + y)^{-\gamma_{c,(i,k)}} (1 + x)^{-(\xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)})} (1 + y)^{-\xi_{c,k,\overline{(i,k)}}} dy dx \\ &= \int_{\mathbf{R}_+} \int_x^\infty (1 + x + y)^{-\gamma_{c,(i,k)}} (1 + x)^{-\xi_{c,i,\overline{(i,k)}}} (1 + y)^{-(\xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)})} dy dx \\ &+ \int_{\mathbf{R}_+} \int_y^\infty (1 + x + y)^{-\gamma_{c,(i,k)}} (1 + x)^{-(\xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)})} (1 + y)^{-\xi_{c,k,\overline{(i,k)}}} dx dy \\ &= I_1(\boldsymbol{\xi}) + I_2(\boldsymbol{\xi}). \end{aligned}$$

We further calculate $I_1(\boldsymbol{\xi})$, whereas the other integral can be tackled in a similar fashion.

More specifically, by change of variables and evoking Equation (3.197(1)) in [Gradshteyn and](#)

Ryzhik (2014), we have that

$$\begin{aligned} I_1(\boldsymbol{\xi}) &= \frac{1}{\xi_{c,k} - 1} \int_{\mathbf{R}_+} (1+2x)^{-\gamma_{c,(i,k)}} (1+x)^{-(\xi_{c,i,k} - \gamma_{c,(i,k)} - 1)} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,k}; \frac{x}{1+2x} \right) dx \\ &= \frac{1}{2(\xi_{c,k} - 1)} \int_0^1 (1-u/2)^{-(\xi_{c,i,k} - \gamma_{c,(i,k)} - 1)} (1-u)^{(\xi_{c,i,k} - 3)} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,k}; \frac{u}{2} \right) du. \end{aligned}$$

Furthermore, note that as the ${}_2F_1$ hypergeometric function has the following integral representation for all $\xi_{c,k} > 1$ and $u \in \mathbf{R}$,

$$\frac{1}{\xi_{c,k} - 1} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,k}; u/2 \right) = \int_0^1 (1-v)^{\xi_{c,k}-2} \left(1 - \frac{u}{2}v \right)^{-\gamma_{c,(i,k)}} dv$$

(Equation 9.111 in loc. cit.) we obtain the following string of integrals

$$\begin{aligned} I_1(\boldsymbol{\xi}) &= \frac{1}{2} \int_0^1 (1-v)^{\xi_{c,k}-2} \int_0^1 (1-u)^{\xi_{c,i,k}-3} (1-u/2)^{-(\xi_{c,i,k} - \gamma_{c,(i,k)} - 1)} (1-uv/2)^{-\gamma_{c,(i,k)}} dudv \\ &\stackrel{(1)}{=} \frac{1}{2(\xi_{c,i,k} - 2)} \int_0^1 (1-v)^{\xi_{c,k}-2} {}_2F_1 \left(1, \gamma_{c,(i,k)}, \xi_{c,i,k} - \gamma_{c,(i,k)} - 1, \xi_{c,i,k} - 1, \frac{v}{2}, \frac{1}{2} \right) dv \\ &\stackrel{(2)}{=} \frac{1}{(\xi_{c,i,k} - 2)} \int_0^1 (1-v)^{\xi_{c,k}-2} {}_2F_1 \left(1, \gamma_{c,(i,k)}; \xi_{c,i,k} - 1; v - 1 \right) dv \\ &= \frac{1}{(\xi_{c,i,k} - 2)} \int_0^1 v^{\xi_{c,k}-2} {}_2F_1 \left(1, \gamma_{c,(i,k)}; \xi_{c,i,k} - 1; -v \right) dv \\ &\stackrel{(3)}{=} \frac{1}{(\xi_{c,i,k} - 2)(\xi_{c,k} - 1)} {}_3F_2 \left(\xi_{c,k} - 1, 1, \gamma_{c,(i,k)}; \xi_{c,k}, \xi_{c,i,k} - 1; -1 \right), \end{aligned}$$

where F_1 is the bivariate hypergeometric function, and $\stackrel{(1)}{=}$, $\stackrel{(2)}{=}$ and $\stackrel{(3)}{=}$ hold by Equations (3.211), (9.182(1)) and (7.512(12)), respectively, in Gradshteyn and Ryzhik (2014). The expression for $I_2(\boldsymbol{\xi})$ is then by analogy

$$I_2(\boldsymbol{\xi}) = \frac{1}{(\xi_{c,i,k} - 2)(\xi_{c,i} - 1)} {}_3F_2 \left(\xi_{c,i} - 1, 1, \gamma_{c,(i,k)}; \xi_{c,i}, \xi_{c,i,k} - 1; -1 \right).$$

We note in passing that the hypergeometric functions in $I_1(\boldsymbol{\xi})$ and $I_2(\boldsymbol{\xi})$ converge for $\xi_{c,i,k} - \gamma_{c,(i,k)} > 0$ and converge absolutely for $\xi_{c,i,k} - \gamma_{c,(i,k)} > 1$, $1 \leq i \neq k \leq n$. This completes the proof. $\square$

Clearly, the Pearson's correlation coefficients for the special cases of the MRF structures having no risk factors with a.) fully comonotonic and b.) positively dependent but not fully comonotonic hitting times are recoverable from the general expression in Theorem 3.4.2.

Corollary 3.4.1. *Let $\alpha_{c,(i,k)} \equiv 0$ and assume that the rest of the conditions in Theorem 3.4.2 hold, then*

$$\mathbf{Corr}[X_i, X_k] = \sqrt{\frac{(\xi_{c,i} - 2)(\xi_{c,k} - 2)}{\xi_{c,i}\xi_{c,k}}} \left({}_3F_2\left(\gamma_{c,(i,k)}, 1, 1; \xi_{c,i}, \xi_{c,k}; 1\right) - 1 \right), \quad (3.4.6)$$

where $1 \leq i \neq k \leq n$.

Proof. First notice that according to Theorem 3.4.2, we have that

$$\begin{aligned} & \int_0^\infty \int_u^\infty (1+u)^{-c}(1+v)^{-b}(1+u+v)^{-a} dv du \\ &= \frac{1}{(a+b+c-2)(a+c-1)} {}_3F_2(a+c-1, 1, c; a+c, a+b+c-1; -1), \end{aligned}$$

where a, b, c are all positive and such that $s_1 = a+b > 2$ and $s_2 = a+c > 2$. Then, for $s = a+b+c$,

$$\begin{aligned} & \int_0^\infty \int_0^\infty (1+u)^{-c}(1+v)^{-b}(1+u+v)^{-a} dv du \\ &= \int_0^\infty \int_v^\infty (1+u)^{-c}(1+v)^{-b}(1+u+v)^{-a} du dv + \int_0^\infty \int_u^\infty (1+u)^{-c}(1+v)^{-b}(1+u+v)^{-a} dv du \\ &= \frac{1}{s-2} \left(\frac{1}{s_1-1} {}_3F_2(s_1-1, 1, a; s_1, s-1; -1) + \frac{1}{s_2-1} {}_3F_2(s_2-1, 1, a; s_2, s-1; -1) \right) \\ &\stackrel{(1)}{=} \frac{1}{(s_1-1)(s_2-1)} {}_3F_2(a, 1, 1; s_1, s_2; 1), \end{aligned}$$

where the latter equality is by Theorem 2.1 in Furman and Su (2015), and the hypergeometric function converges absolutely since $\xi_{c,i} > 2$ and $\xi_{c,k} > 2$ by assumption. This, along with setting $\alpha_{c,(i,k)} \equiv 0$ completes the proof. $\square$

Formula (3.4.6) recovers the one derived in [Chiragiev and Landsman \(2009\)](#) as well as the recent one in [Furman and Su \(2015\)](#). In the simplest case, i.e., when $\alpha_{c,(i,k)}, \alpha_{c,i,\overline{(i,k)}}, \alpha_{c,k,\overline{(i,k)}}$ and also $\gamma_{c,i,\overline{(i,k)}}, \gamma_{c,k,\overline{(i,k)}}$ are all zero, we have that $\xi_{c,i} = \xi_{c,k} = \gamma_{c,(i,k)} > 2$ and hence

$${}_3F_2(\gamma_{c,(i,k)}, 1, 1; \xi_{c,i}, \xi_{c,k}; 1) = {}_2F_1(1, 1; \gamma_{c,(i,k)}; 1) = \frac{\gamma_{c,(i,k)} - 1}{\gamma_{c,(i,k)} - 2},$$

where $1 \leq i \neq k \leq n$. Consequently, we obtain that, for $1 \leq i \neq k \leq n$, correlation (3.4.6) reduces to

$$\mathbf{Corr}[X_i, X_k] = \frac{1}{\gamma_{c,(i,k)}},$$

which recovers the Pearson's correlation of the classical multivariate Pareto distribution of [Arnold \(1983, 2015\)](#).

Corollary 3.4.2. *Let $\gamma_{c,(i,k)} \equiv 0$ and assume that the rest of the conditions in Theorem 3.4.2 hold, then*

$$\mathbf{Corr}[X_i, X_k] = \sqrt{\frac{(\xi_{c,i} - 2)(\xi_{c,k} - 2)}{\xi_{c,i}\xi_{c,k}}} \frac{\alpha_{c,(i,k)}}{(\xi_{c,i,k} - 2)}, \quad (3.4.7)$$

where $1 \leq i \neq k \leq n$.

Proof. It is easy to check that, for $\gamma_{c,(i,k)} \equiv 0$, we have ${}_3F_2(a, b, 0; c, d; z) \equiv 1$, for any real a, b, c, d, z and also $\xi_{c,k} + \xi_{c,i} - \xi_{c,i,k} = \alpha_{c,(i,k)}$ for $1 \leq i \neq k \leq n$. This completes the proof. $\square$

Formula (3.4.7) conforms to the one derived in [Asimit et al. \(2010\)](#).

We conclude this section with a theorem that formulates the conditional decumulative distribution function of the MRF structures. To this end, we note that joint d.d.f. (3.4.1) can be written as

$$\overline{F}(x, y) = \begin{cases} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, \xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)}, \gamma_{c,(i,k)}), & x/\sigma_i < y/\sigma_k \\ \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, \xi_{c,k,\overline{(i,k)}}, \gamma_{c,(i,k)}), & x/\sigma_i \geq y/\sigma_k \end{cases}, \quad (3.4.8)$$

where, for $(x, y)' \in \mathbf{R}_{0,+}^2$ and b_1, b_2, b_3 all real,

$$\overline{G}(x, y; b_1, b_2, b_3) = (1+x)^{-b_1} (1+y)^{-b_2} (1+x+y)^{-b_3}. \quad (3.4.9)$$

Furthermore, the following holds

$$-\frac{\partial \overline{G}(x, y; b_1, b_2, b_3)}{\partial y} = b_2 \overline{G}(x, y; b_1, b_2 + 1, b_3) + b_3 \overline{G}(x, y; b_1, b_2, b_3 + 1).$$

Theorem 3.4.3. *Let $(X_i, X_k)' \sim Pa(II)_{i,k}^c(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$, then the d.d.f. of the conditional r.v. $X_i | X_k = y$ is given, for $y \in \mathbf{R}_{0,+}$, by*

$$\begin{aligned} & \overline{F}_{X_i | X_k = y}(x) \\ = & \begin{cases} \frac{\xi_{c,k,(i,k)} + \alpha_{c,(i,k)}}{\xi_{c,k}} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,(i,k)}, -\gamma_{c,(i,k)}, \gamma_{c,(i,k)}) \\ + \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,(i,k)}, -\gamma_{c,(i,k)} - 1, \gamma_{c,(i,k)} + 1), & x/\sigma_i < y/\sigma_k \\ \\ \frac{\xi_{c,k,(i,k)}}{\xi_{c,k}} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,(i,k)} + \alpha_{c,(i,k)}, -\xi_{c,(i,k)}, \gamma_{c,(i,k)}) \\ + \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,(i,k)} + \alpha_{c,(i,k)}, -\xi_{c,(i,k)} - 1, \gamma_{c,(i,k)} + 1), & x/\sigma_i \geq y/\sigma_k \end{cases}. \end{aligned}$$

Proof. To start off, note that the p.d.f. of the absolutely continuous component is formulated as

$$\begin{aligned} (1 - \beta) f_{ac}(x|y) &= (1 - \beta) \frac{\partial^2 \overline{F}(x, y)}{\partial x \partial y} \frac{1}{f_k(y)} \\ &= \begin{cases} (f_k(y) \sigma_i \sigma_k)^{-1} g \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,(i,k)}, \xi_{c,k,(i,k)} + \alpha_{c,(i,k)}, \gamma_{c,(i,k)} \right), & x/\sigma_i < y/\sigma_k \\ (f_k(y) \sigma_i \sigma_k)^{-1} g \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,(i,k)} + \alpha_{c,(i,k)}, \xi_{c,k,(i,k)}, \gamma_{c,(i,k)} \right), & x/\sigma_i > y/\sigma_k \end{cases}, \end{aligned}$$

where g denotes the mixed derivative of $\overline{G}$ and $\beta \in [0, 1]$. Therefore the probability associated

with the singular component is given by

$$\begin{aligned}
\beta &= \mathbf{P} \left[\frac{X_i}{\sigma_i} = \frac{X_k}{\sigma_k} \mid X_k = y \right] \\
&= \int_{\frac{\sigma_i y}{\sigma_k}}^{\infty} \frac{g \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, \xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)}, \gamma_{c,(i,k)} \right)}{\sigma_i \sigma_k f_k(y)} dx \\
&\quad - \int_{\frac{\sigma_i y}{\sigma_k}}^{\infty} \frac{g \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, \xi_{c,k,\overline{(i,k)}}, \gamma_{c,(i,k)} \right)}{\sigma_i \sigma_k f_k(y)} dx.
\end{aligned}$$

Let $x/\sigma_i \geq y/\sigma_k$ first, then, for $y \in \mathbf{R}_{0,+}$,

$$\begin{aligned}
&\mathbf{P}[X_i > x \mid X_k = y] \\
&= \int_x^{\infty} \frac{g \left(t/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, \xi_{c,k,\overline{(i,k)}}, \gamma_{c,(i,k)} \right)}{\sigma_i \sigma_k f_k(y)} dt \\
&= - \frac{\partial \overline{G} \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, \xi_{c,k,\overline{(i,k)}}, \gamma_{c,(i,k)} \right)}{f_k(y) \partial y} \\
&= \frac{\xi_{c,k,\overline{(i,k)}}}{\xi_{c,k}} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, -\xi_{c,(i,k)}, \gamma_{c,(i,k)}) \\
&\quad + \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, -\xi_{c,(i,k)} - 1, \gamma_{c,(i,k)} + 1),
\end{aligned}$$

where $1 \leq i \neq k \leq n$.

The case where $x/\sigma_i < y/\sigma_k$ is slightly more involved. More specifically, we have that

$$\begin{aligned}
& \overline{F}_{X_i|X_k=y}(x) \\
&= \mathbf{P} \left[\frac{X_i}{\sigma_i} = \frac{X_k}{\sigma_k} \mid X_k = y \right] + \int_x^{\frac{\sigma_i y}{\sigma_k}} \frac{g \left(t/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, \xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)}, \gamma_{c,(i,k)} \right)}{\sigma_i \sigma_k f_k(y)} dt \\
&+ \int_{\frac{\sigma_i y}{\sigma_k}}^\infty \frac{g \left(t/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, \xi_{c,k,\overline{(i,k)}}, \gamma_{c,(i,k)} \right)}{\sigma_i \sigma_k f_k(y)} dt \\
&= \int_x^\infty \frac{g \left(t/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, \xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)}, \gamma_{c,(i,k)} \right)}{\sigma_i \sigma_k f_k(y)} dt \\
&= - \frac{\partial \overline{G} \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, \xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)}, \gamma_{c,(i,k)} \right)}{f_k(y) \partial y} \\
&= \frac{\xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)}}{\xi_{c,k}} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, -\gamma_{c,(i,k)}, \gamma_{c,(i,k)}) \\
&+ \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \gamma_{c,(i,k)} \overline{G}(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, -\gamma_{c,(i,k)} - 1, \gamma_{c,(i,k)} + 1),
\end{aligned}$$

where $1 \leq i \neq k \leq n$. This completes the proof. $\square$

While the importance of the derivations in this section is clear from the distribution theory point of view, some connections to applications in (actuarial) risk management are perhaps desired. In summary:

- The Lebesgue decomposition derived in Theorem 3.4.1 becomes useful when developing formulas for, e.g., the joint moments of the MRF distributed random pairs. These moments are of central importance in (actuarial) risk management, as well as in actuarial and economic pricing, when the risk measure and pricing functionals belong to the classes of weighted (Furman and Zitikis, 2008a,b) or distorted (Wang, 1996) expectations.
- Theorem 3.4.2 provides an easy to compute expression for the Pearson's index of linear correlation for a random pair coming from the MRF structures. Noticeably, the

Pearson's correlation can attain any value in $[0, 1]$ and as such Theorem 3.4.2 emphasizes that the MRF models cover the entire range of positive dependence when it is measured by Pearson's ρ . It is well-known that defaults tend to cluster in reality, thus suggesting that they are positively dependent. Factor models à la the ones discussed in Subsection 3.2.2 have been employed extensively to describe the just-mentioned clustering in practice. The reason stems from a common belief that simultaneous defaults are rare and can be dismissed. The finding of Das et al. (2007) along with Theorem 3.4.2 suggest that the general MRF dependence structures may provide a better way to model real life default times.

- Last but not least, while the regression function of X_i on X_k is linear in the context of the classic model of Arnold (1983, 2015), Theorem 3.4.3 urges that the conditional expectation $\mathbf{E}[X_i | X_k = y]$ is not in general linear in $y (\in \mathbf{R}_{0,+})$, thereby the Pearson's index of correlation may not be appropriate in the context of the MRF dependencies.

3.5 Applications in actuarial mathematics

In the sequel we consider two possible venues to utilize the results derived hitherto. To this end, we note in passing that the functional $H : \mathcal{X} \rightarrow [0, \infty]$ is called a risk measure for any risk r.v. $X \in \mathcal{X}$. Moreover, the aforementioned functional is an actuarial premium calculation principle (p.c.p.) if the bound $H[X] \geq \mathbf{E}[X]$ holds for all $X \in \mathcal{X}$ with finite expectations.

3.5.1 Risk measurement

Regulatory accords around the world require that insurance companies carry out extensive and quantitatively sound assessments of their risks, and default risks are not an exception. In this subsection we report expressions for arguably the most popular risk measure functionals used nowadays in insurance industry when the risk portfolio is described by the MRF structures.

The literature on risk measures is vast and growing quickly. The following two indices have however earned an unprecedented amount of interest among both practitioners and theoreticians.

Definition 3.5.1. *Let $X \in \mathcal{X}$ and fix $q \in [0, 1)$, then the Value-at-Risk (VaR) and the Conditional Tail Expectation (CTE) risk measures are respectively given by*

$$VaR_q[X] := \inf\{x \in \mathbf{R} : \mathbf{P}[X \leq x] \geq q\} \quad (3.5.1)$$

and

$$CTE_q[X] := \mathbf{E}[X | X > VaR_q[X]]. \quad (3.5.2)$$

In the context of the MRF risk portfolios, the VaR and the CTE risk measures are readily obtained using Theorem 3.3.2. The proofs are omitted, as they are similar to the proofs of Corollary 3.1 and Corollary 3.3 in [Furman and Su \(2015\)](#).

Proposition 3.5.1. *Let $\mathbf{X} \sim Pa_{1,\dots,n}^c(II)(\boldsymbol{\sigma}, \boldsymbol{\xi})$ as in Definition 3.3.1, then, for $i = 1, \dots, n$, the Value-at-Risk risk measure is given by*

$$VaR_q[X_i] = \sigma_i \left((1 - q)^{-1/\xi_{c,i}} - 1 \right), \quad (3.5.3)$$

and the Conditional Tail Expectation risk measure is, for $\xi_{c,i} > 1$, given by

$$\begin{aligned} CTE_q[X_i] &= \mathbf{E}[X_i] \frac{\overline{F}_{X_i^*}(VaR_q[X_i])}{1-q} + VaR_q[X_i] \\ &= \mathbf{E}[X_i] + VaR_q[X_i] \frac{\xi_{c,i}}{\xi_{c,i} - 1}, \end{aligned} \quad (3.5.4)$$

where $X_i^* \sim Pa(II)(\sigma_i, \xi_{c,i} - 1)$.

If the r.v. $\mathbf{X} = (X_1, \dots, X_n)'$ represents a credit risk portfolio, then expressions (3.5.3) and (3.5.4) are interpreted as the q -th quantile time of default and the average excess-of-the q -th quantile time of default of the i -th risk component, $i = 1, \dots, n$.

As we have already mentioned, the minima and the maxima r.v.'s, X_- and X_+ , respectively, play an important role in the theory of credit risk and in insurance mathematics - think, e.g., of the first and last default times - for the former subject, and of the joint and last-survivor life statuses - for the latter subject. Next two propositions provide expressions for the average excess-of-the q -th quantile time of first and last default. The proofs are again similar to these in [Furman and Su \(2015\)](#) and are thus omitted.

Proposition 3.5.2. *In the context of the MRF risk portfolios, the CTE risk measure of the minima r.v. can be written, if finite and for $q \in [0, 1)$, as*

$$CTE_q[X_-] = \mathbf{E}[X_-] \frac{\overline{F}_{X_-^*}(VaR_q[X_-])}{1-q} + VaR_q[X_-],$$

where $X_-^* \sim Pa(\alpha_+(\boldsymbol{\sigma}), \xi_{c,1,\dots,n} + Q - 1)$, with

$$\alpha_+(\boldsymbol{\sigma}) = \left(\bigvee_{j=1}^l \bigwedge_{i \in \mathcal{RC}_j} \sigma_i \right) \vee \left(\bigvee_{j=l+1}^{l+m} \left(\sum_{i \in \mathcal{RC}_j} \frac{1}{\sigma_i} \right)^{-1} \right),$$

and Q is an integer-valued and non-negative r.v. with the p.m.f. obtained from the p.m.f. of K (Lemma 4.4.1) with the help of the following change of measure

$$q_k = \frac{1}{\mathbf{E}[X_-]} \frac{\sigma_+}{\xi_{c,1,\dots,n} + k - 1} p_k, \quad k = 0, 1, \dots \quad (3.5.5)$$

Proposition 3.5.3. *In the context of the MRF risk portfolios, the CTE risk measure of the maxima r.v. can be written, if finite and for $q \in [0, 1)$, as the following linear combination*

$$CTE_q[X_+] = \frac{1}{1-q} \sum_{\mathcal{S} \subseteq \{1, \dots, n\}} (-1)^{|\mathcal{S}|-1} \mathbf{E}[X_{\mathcal{S}-} | X_{\mathcal{S}-} > VaR_q(X_+)] \bar{F}_{\mathcal{S}-}(VaR_q(X_+)),$$

where $X_{\mathcal{S}-} = \bigwedge_{s \in \mathcal{S} \subseteq \{1, \dots, n\}} X_s$ and $X_{\mathcal{S}-} \sim F_{\mathcal{S}-}$.

We conclude this subsection by studying one more important index that aims at shedding light on the effect of the interdependence between two random default times. The index was referred to as the ‘solvency bonus’ in [Furman and Su \(2015\)](#).

Definition 3.5.2. *Let X_i and X_k denote two possibly correlated random default times of two business units i and k , where $1 \leq i \neq k \leq n$. Also, let $y \in \mathbf{R}_+$ be fixed, then*

$$\beta(y; X_i, X_k) := \mathbf{E}[X_i | X_k > y] - \mathbf{E}[X_i]$$

yields the change in the expected default time of i given that k has not defaulted and that i and k are parts of a risk portfolio rather than in the case when i is a stand alone business unit.

A simple observation about the solvency bonus index is perhaps appropriate. Namely, if the default times are independent (positively quadrant dependent), then $\beta(y; X_i, X_k) = 0 (\geq 0)$, respectively, for all $y \in \mathbf{R}_+$. The former part of the assertion is straightforward, whereas the latter part follows because

$$\mathbf{E}[X_i | X_k > y] = \frac{\mathbf{E}[X_i \mathbf{1}\{X_k > y\}]}{\mathbf{P}[X_k > y]} \geq \mathbf{E}[X_i],$$

for any $y \in \mathbf{R}_+$ ([Lehmann, 1966](#)).

Proposition 3.5.4. *Let i and k , $1 \leq i \neq k \leq n$ be two risk components with respective default times X_i and X_k . In the context of the MRF risk portfolios, the solvency bonus function for the risk component i given that the risk component k has not defaulted is, for $\xi_{c,i} > 1$,*

$$\begin{aligned}
& \beta(y; X_i, X_k) \\
&= \frac{\sigma_i}{\xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)} - 1} {}_2F_1 \left(\gamma_{c,(i,k)} \cdot 1; \xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)}, \frac{y/\sigma_k}{1 + y/\sigma_k} \right) \\
&- \frac{\sigma_i \left(1 + \frac{y}{\sigma_k}\right)^{-(\xi_{c,i,\overline{(i,k)}} - \gamma_{c,(i,k)}) + 1} \left(1 + 2\frac{y}{\sigma_k}\right)^{-\gamma_{c,(i,k)}}}{\xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)} - 1} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)}; \frac{y/\sigma_k}{1 + 2y/\sigma_k} \right) \\
&+ \frac{\sigma_i}{\xi_{c,i} - 1} \left(1 + \frac{y}{\sigma_k}\right)^{-(\xi_{c,i,\overline{(i,k)}} - \gamma_{c,(i,k)}) + 1} \left(1 + 2\frac{y}{\sigma_k}\right)^{-\gamma_{c,(i,k)}} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,i}; \frac{y/\sigma_k}{1 + 2y/\sigma_k} \right) \\
&- \frac{\sigma_i}{\xi_{c,i} - 1},
\end{aligned}$$

where $y \in \mathbf{R}_+$.

Proof. Employing (3.4.1), we obtain that, for $y \in \mathbf{R}_+$,

$$\begin{aligned}
\mathbf{E}[X_i | X_k > y] &= \int_{\mathbf{R}_+} \mathbf{P}[X_i > x | X_k > y] dx \\
&= \int_{\mathbf{R}_+} \overline{G} \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, -\gamma_{c,(i,k)}, \gamma_{c,(i,k)} \right) dx \tag{3.5.6}
\end{aligned}$$

$$- \int_{\frac{\sigma_i y}{\sigma_k}}^{\infty} \overline{G} \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}}, -\gamma_{c,(i,k)}, \gamma_{c,(i,k)} \right) dx \tag{3.5.7}$$

$$+ \int_{\frac{\sigma_i y}{\sigma_k}}^{\infty} \overline{G} \left(x/\sigma_i, y/\sigma_k; \xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}, -\xi_{c,(i,k)}, \gamma_{c,(i,k)} \right) dx. \tag{3.5.8}$$

In order to compute (3.5.6), we use Expression (3.197(5)) in Gradshteyn and Ryzhik (2014)

and obtain that

$$\begin{aligned}
\int_{\mathbf{R}_+} \overline{G}(x, y; b_1, -b_2, b_2) dx &= \int_{\mathbf{R}_+} (1+x)^{-b_1} \left(1 + \frac{x}{1+y}\right)^{-b_2} dx \\
&= \frac{1}{b_1 + b_2 - 1} {}_2F_1 \left(b_2, 1; b_1 + b_2; \frac{y}{1+y} \right),
\end{aligned}$$

whereas to compute the other two integrals, i.e., (3.5.7) and (3.5.8), we use Expression (3.197(1)) in [Gradshteyn and Ryzhik \(2014\)](#) and have the following string of expressions

$$\begin{aligned}
\int_z^\infty \overline{G}(x, y; b_1, -b_2, b_2) dx &= (1+y)^{b_2} \int_z^\infty (1+x)^{-b_1} (1+x+y)^{-b_2} dx \\
&= (1+y)^{b_2} \int_0^\infty (1+x+z)^{-b_1} (1+x+y+z)^{-b_2} dx \\
&= \frac{1}{b_1 + b_2 - 1} (1+y)^{b_2} (1+z)^{1-b_1} (1+y+z)^{-b_2} \\
&\times {}_2F_1\left(b_2, 1; b_1 + b_2; \frac{y}{1+y+z}\right),
\end{aligned}$$

where $z \in \mathbf{R}_+$ and $y \in \mathbf{R}_+$. This, along with the formula for $\mathbf{E}[X]$ that has been derived in Theorem 3.3.2, completes the proof. $\square$

3.5.2 Pricing

Actuarial p.c.p.'s traditionally hinge only on the c.d.f. of $X \in \mathcal{X}$. This property has been well-known in the literature as ‘objectiveness’ ([Denuit et al., 2005](#)) or ‘conditional state independence’ ([Bühlmann, 1980, 1984](#)). Such approach, which frequently allows for quantitative tractability, is somewhat unrealistic as it neglects the dependence of the premium on, e.g., insurer’s financial position and attitude, general conditions of the economy, other risks. As a result some authors [Bühlmann \(1980, 1984\)](#); [Deprez and Gerber \(1985\)](#); [Furman and Zitikis \(2009\)](#) have suggested to use the so-called economic p.c.p. functional $\Pi : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$, where the first coordinate $X \in \mathcal{X}$ in the functional $\Pi[X, Y]$ is the risk to be priced, and the second coordinate $Y \in \mathcal{X}$ is the leverage risk, an overall risk or, speaking generally, any random variable of interest to the insurer that affects the price of X .

Definition 3.5.3. *Let X_i, X_k be two risks in $\mathcal{X}$, then the Economic CTE p.c.p. is formulated*

as

$$CTE_q[X_i, X_k] := \mathbf{E}[X_i | X_k > VaR_q[X_k]]$$

where $q \in [0, 1)$ and X_k has finite expectation.

For the following proposition, the MRF r.v. $\mathbf{X} = (X_1, \dots, X_n)'$ is interpreted as a risk portfolio with heavy tailed risk components measured in money units. All the rest is essentially the same as in Proposition 3.5.4.

Proposition 3.5.5. *Let $(X_i, X_k)' \sim Pa_{1, \dots, n}^c(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$, then the economic CTE p.c.p. of X_i with the r.v. X_k serving as the leverage or background risk is, for $\xi_{c,i} > 1$ and $q \in [0, 1)$,*

$$\begin{aligned} & CTE_q[X_i, X_k] \\ &= \frac{\sigma_i}{\xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)} - 1} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)}, \frac{VaR_q[Y]/\sigma_k}{1 + VaR_q[Y]/\sigma_k} \right) \\ &- \frac{\sigma_i \left(1 + \frac{VaR_q[Y]}{\sigma_k} \right)^{-(\xi_{c,i,\overline{(i,k)}} - \gamma_{c,(i,k)}) + 1} \left(1 + 2 \frac{VaR_q[Y]}{\sigma_k} \right)^{-\gamma_{c,(i,k)}}}{\xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)} - 1} \\ &\times {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,i,\overline{(i,k)}} + \gamma_{c,(i,k)}; \frac{VaR_q[Y]/\sigma_k}{1 + 2VaR_q[Y]/\sigma_k} \right) \\ &+ \frac{\sigma_i}{\xi_{c,i} - 1} \left(1 + \frac{VaR_q[Y]}{\sigma_k} \right)^{-(\xi_{c,i,\overline{(i,k)}} - \gamma_{c,(i,k)}) + 1} \left(1 + 2 \frac{VaR_q[Y]}{\sigma_k} \right)^{-\gamma_{c,(i,k)}} \\ &\times {}_2F_1 \left(\gamma_{c,(i,k)}, 1; \xi_{c,i}; \frac{VaR_q[Y]/\sigma_k}{1 + 2VaR_q[Y]/\sigma_k} \right). \end{aligned}$$

3.6 Conclusions

The latest Solvency II directives require the insurers to recognize the interdependences within and among different liability classes. This new paradigm arguably brings to an end the indisputable role of the assumption of independence that has shaped both research and applications in actuarial science and general quantitative risk management in the 20th century. The

choice of an appropriate probability dependence model is however not an easy call. Indeed, while there exists only one way to describe stochastic independence, the forms of stochastic dependence are infinite.

In this paper we have introduced a new class of Multiple Risk Factor dependence structures. On the one hand, these structures emerge as an extension of the popular CreditRisk⁺ approach, and as such they formally describe default risk portfolios exposed to an arbitrary number of fatal risk factors with conditionally exponential hitting times that can be independent, positively dependent and fully comonotonic. On the other hand, the MRF structures can be viewed as a quite general family of multivariate probability distributions with Pareto of the 2nd kind univariate margins, and in this capacity, they can model risk portfolios of (insurance) losses with heavy-tailed and positively dependent risk components.

It often happens in mathematical sciences that generalizing an object highlights the essence of the matter and helps to understand it better. By generalizing the classical multivariate Pareto distribution of [Arnold \(1983, 2015\)](#) as well as the recently introduced ones of [Chiragiev and Landsman \(2009\)](#), in this paper we have extended the range of attainable Pearson’s correlations to the entire $[0, 1]$ interval and complemented some formal dependence analysis. As factor models have been known to produce under-correlated default times, the MRF structures introduced and studied herein may provide a possible route to ‘inject the required amounts of extra correlation’, and they may thereby become of interest to banks, credit unions and insurance companies. In addition, as the new realms of excessively prudent risk management make particular effort to model non-hedgeable and heavy tailed risks, the family of MRF multivariate distributions may be of interest well beyond the context of credit risk.

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Chapter 4

Multiple risk factor dependence structures: Related copulas and applications in de- fault risk

In this chapter, we use doubly stochastic Poisson processes to explore the class of copula functions that underlie the multiple risk factor dependence structures discussed in Chapter 3. This chapter has been submitted for publication with Edward Furman.

MULTIPLE RISK FACTOR DEPENDENCE STRUCTURES: RELATED COPULAS AND APPLICATIONS IN DEFAULT RISK

Abstract.

Copulas have become an important element of best practice Enterprise Risk Management, displacing in many contexts other more conservative approaches to modelling stochastic dependence. Choosing the ‘right’ copula is however not an easy task, and the temptation to make a decision in favour of a tractable rather than a meaningful copula is high. The use of the Gaussian copula to price the collateralized debt obligations is arguably the most illuminating example.

A ‘good’ copula should conform to a problem at hand, be asymmetric and exhibit some extent of tail dependence. In this paper we introduce and study a new class of Multiple Risk Factor (MRF) copula functions, which we show are exactly such. Namely, we demonstrate that the MRF copulas arise from a number of default risk specifications with stochastic default barrier and are in general non-exchangeable in their domain of definition as well as in the tail behaviour. The MRF copulas cover the full range of non-negative dependence and are quite tractable analytically.

An important property that sets the MRF copulas apart from many other dependence structures employed in the modern theory and practice of default risk is the capability to allow for non-zero probabilities of simultaneous default. Speaking formally, this becomes feasible because the MRF copulas are not absolutely continuous with respect to the corresponding Lebesgue measure. Appealing to

the language of credit risk, this is a direct outcome of the inclusion of systemic risk factors having fully comonotonic hitting times. One way or the other, this special characteristic contributes to the effectiveness of the MRF copulas when modelling the clustering of real world defaults. As the CreditRisk⁺ approach has been criticized for underestimating the just-mentioned clustering, we hope that the MRF copulas may serve as a reasonable substitution.

Keywords and phrases: Multivariate distributions, (tail) dependence, singularity, generalized Archimedean copula, Marshall-Olkin copula, credit risk, factor models.

4.1 Introduction

There is only one way to formulate stochastic independence, whereas the shapes of stochastic dependence are infinite. Nowadays, dependence is often modelled using copulas, which have become a well established mathematical tool in actuarial and financial research and practice (Frees et al., 1996; McNeil et al., 2005; Frees and Wang, 2006; Embrechts, 2009; Frees, 2009; Jaworski et al., 2013, 2010; and references therein), as well as in many other areas such as economics (Patton, 2012), medicine (Nikoloulopoulos and Karlis, 2008), reliability engineering and life sciences (Balakrishnan and Lai, 2009), and so on. Briefly, an $n(\in \mathbf{N})$ -variate function $C : [0, 1]^n \rightarrow [0, 1]$ is a copula if it is grounded, n -increasing, and have uniform margins (Nelsen, 2006).

Copulas are beautiful mathematical constructions, however with the tractability comes a price. While we must choose a copula depending on the problem at hand, this choice is rather vague. As a result, practitioners often choose copulas based on mathematical convenience, rather than meaningful connections to the phenomena they model. This is because these connections are often very difficult to find. Luckily there are exceptions.

Consider spouses that purchase a life insurance. The joint future lifetimes of the couple may be modelled by the bivariate random variable (r.v.) $(\tau_1, \tau_2)'$, such that $\tau_1 = {}_1E_{\lambda_1} \wedge E_{\lambda_0}$ and $\tau_2 = {}_2E_{\lambda_2} \wedge E_{\lambda_0}$, where ${}_1E_{\lambda_1}$, ${}_2E_{\lambda_2}$ and E_{λ_0} are all exponentially distributed and stochastically independent r.v.'s. The set-up is known as ‘common-shock’, and it is quite popular in life insurance mathematics. The formal structure is well connected to reality, as the spouses may die either at independent future times ${}_1E_{\lambda_1}$ and ${}_2E_{\lambda_2}$ - as a result of individual mortality, or simultaneously (fully comonotonically) - as a result of, e g., a natural disaster, car accident or another common shock risk factor. The joint survivorship

probability, that is the decumulative distribution function (d.d.f.) is then [Marshall and Olkin \(1967\)](#)

$$\mathbf{P}[\tau_1 > s, \tau_2 > t] = \exp\{-\lambda_1 s - \lambda_2 t - \lambda_0(s \vee t)\}, \text{ where } s, t \in \mathbf{R}_{0,+},$$

and, a routine application of Sklar's theorem ([Sklar, 1959](#)), shows that the corresponding bivariate copula function ([Cherubini et al., 2013](#)) is given by

$$C(u, v) = u^{1-\lambda_0/(\lambda_1+\lambda_0)} v \wedge v^{1-\lambda_0/(\lambda_2+\lambda_0)} u, \text{ where } u, v \in [0, 1].$$

In summary, the bivariate Marshall-Olkin copula above and more generally its multivariate counterpart, can be mapped to a meaningful stochastic representation, which, in turn, describes a real world phenomenon of interest to actuaries.

Alternatively, let $(\tau_1, \tau_2)'$ represent a risk portfolio of two conditionally independent and exponentially distributed default times. That is, assume that $\tau_1 = {}_1E_\lambda$ and $\tau_2 = {}_2E_\lambda$ conditionally on a r.v. Λ , which is independent on both exponentials and distributed gamma with shape parameter $\gamma \in \mathbf{R}_+$ and unit scale parameter. The unconditional times of occurrence, in the sequel - hitting times, are thus positively quadrant dependent (PQD) ([Lehmann, 1966](#)). The set-up thereby shapes to a simplification of the CreditRisk⁺ approach to modelling the risk of default ([Bielecki and Rutkowski, 2004](#)). The corresponding joint d.d.f. is then ([Arnold, 1983, 2015](#))

$$\mathbf{P}[\tau_1 > s, \tau > t] = (1 + s + t)^{-\gamma}, \text{ where } s, t \in \mathbf{R}_{0,+},$$

whereas a direct use of Sklar's theorem yields the following Clayton copula ([Joe, 1997](#))

$$C(u, v) = (u^{-\gamma} + v^{-\gamma} - 1)^{-1/\gamma}, \text{ where } u, v \in [0, 1].$$

The ideas can be easily extended to higher dimensions. Hence, similarly to the Marshall-Olkin copulas, the Clayton copulas admit a stochastic representation that is meaningful to

(credit) risk professionals.

The goal of this paper is to introduce and study yet another class of copula functions that arise from the realms of Enterprise Risk Management (ERM). More specifically, we generalize the somewhat ‘distributional’ ideas in [Furman and Su \(2015b\)](#) and translate them into the language of copulas. Speaking shortly, the resulting Multiple Risk Factor (MRF) dependencies emerge as an extension of the CreditRisk⁺ approach already mentioned above. As elucidated by their stochastic representation, the MRF dependencies describe default risk portfolios exposed to an arbitrary number of fatal risk factors having conditionally exponential hitting times that can be independent, fully comonotonic or positively orthant dependent (POD) ([Lehmann, 1966](#)). Besides the meaningful stochastic representations, the class of MRF copulas is quite encompassing, and it is able to unify the unrelated at the first glance classes of Marshall-Olkin and Archimedean copulas. Also, the MRF copulas are flexible in that they are in general non-exchangeable, cover the entire range of non-negative dependence and allow for a great variety of tail dependences. Importantly, in spite of the impressive generality, the MRF copulas are often sufficiently tractable analytically.

The rest of the paper is organized as follows. In [Section 4.2](#) we generalize the ideas in [Furman and Su \(2015b\)](#) and introduce the MRF copulas in their most general form along with the various links to default specifications having stochastic default barriers. One of the interesting facets of the MRF copulas, is the fact that they are not absolutely continuous, thus allowing for a non-zero probability of simultaneous default. Using the language of credit risk, this is achieved by the inclusion of the risk factors with fully comonotonic hitting times. We study the phenomenon of simultaneous default generally in [Section 4.3](#), and we specialize the discussion to the context of the Clayton subclass of the MRF copulas in [Section 4.4](#), where we also study the dependence properties of these copulas thoroughly. Last but

not least, we explore the extremal (tail) dependence behaviour of the Clayton MRF copulas in Section 4.5, where we make use of both the classic indices of tail dependence and the new notion of maximal tail dependence introduced recently in [Furman et al. \(2015b\)](#). Section 4.6 concludes the paper.

4.2 Construction of the multiple risk factor copula functions and some basic properties

Imagine a risk portfolio (r.p.) that consists of $n(\in \mathbf{N})$ risk components (r.c.'s) with the labels in the set $\{1, \dots, n\}$. Let $\mathbf{X} := (X_1, \dots, X_n)'$ denote a r.v. with the i -th coordinate interpreted as the default time of the i -th r.c. with $i = 1, \dots, n$, and assume that each r.c. is exposed to some of (or all of) $(l + m)$ fatal risk factors (r.f.'s) of which $l(\in \mathbf{N})$ r.f.'s have fully-comonotonic hitting times and $m(\in \mathbf{N})$ r.f.'s have POD hitting times.

In [Furman and Su \(2015b\)](#) we introduced the block matrix $c := (c^l, c^m) \in \text{Mat}_{n \times (l+m)}(\mathbf{1})$ with entries in $\mathbf{1} := \{0, 1\}$, which described the exposure of the r.p. $\{1, \dots, n\}$ to the distinct r.f.'s $\{1, \dots, l + m\}$ (c was deterministic and in practice set by the senior risk management). In addition, we introduced the sets $\mathcal{RF}_i^l := \{j \in \{1, \dots, l\} : c_{i,j}^l = 1\}$, $\mathcal{RF}_i^m := \{j \in \{l + 1, \dots, l + m\} : c_{i,j}^m = 1\}$ and $\mathcal{RF}_i := \mathcal{RF}_i^l \cup \mathcal{RF}_i^m$ that contained the r.f.'s that 'hit' the i -th r.c., $i = 1, \dots, n$. Similarly, we denoted by $\mathcal{RC}_j := \{i \in \{1, \dots, n\} : c_{i,j} = 1\}$ the set that contained all the r.c.'s that were hit by the j -th r.f., $j = 1, \dots, l + m$.

To make the distributional structure underlying the aforementioned set-up tractable, in [Furman and Su \(2015b\)](#) we assumed that the hitting times of the r.f.'s in $\{1, \dots, l\}$ and the hitting times of the r.f.'s in $\{l + 1, \dots, l + m\}$ were, respectively, conditionally fully-comonotonic and conditionally independent exponentially distributed r.v.'s, succinctly

$E_{\lambda_1}, \dots, E_{\lambda_l}$ and ${}_i E_{\lambda_{l+1}}, \dots, {}_i E_{\lambda_{l+m}}$, where $i = 1, \dots, n$. The desired dependent hitting times were then obtained with the help of the so called mixture operator ‘ $*$ ’, which was such that given two appropriately jointly measurable r.v.’s $X_\beta \sim C(\beta)$ with $\beta \in \mathcal{B} \subseteq \mathbf{R}$ and $B \sim H$, the r.v. $X_\beta * B$ had the same distribution as X_B , or notationally $X_\beta * B \stackrel{d}{=} X_B$, where we of course assumed that the r.v. B had its range in $\mathcal{B}$.

Keeping the set-up above in mind and driven by the general interest to explore encompassing yet tractable multivariate dependence structures, in [Furman and Su \(2015b\)](#) we introduced and studied the class of MRF models that admitted the following stochastic representation, for $Y_i := \bigwedge_{j \in \mathcal{RF}_i^m} {}_i E_{\Lambda_j}$ and $Z_i := \bigwedge_{j \in \mathcal{RF}_i^l} E_{\Lambda_j}$,

$$X_i \stackrel{d}{=} \sigma_i (Y_i \wedge Z_i), \quad (4.2.1)$$

where $\Lambda_j \sim Ga(\xi_j, 1)$ were stochastically independent and gamma distributed r.v.’s with shape parameters $\xi_j (\in \mathbf{R}_+)$ and unit scale parameters, $\sigma_i \in \mathbf{R}_+$ and $i = 1, \dots, n$, $j = 1, \dots, l + m$. Among other findings, we showed that the joint d.d.f. of $\mathbf{X} = (X_1, \dots, X_n)'$ was given by

$$\bar{F}(x_1, \dots, x_n) = \prod_{j=1}^l \left(1 + \bigvee_{i \in \mathcal{RC}_j} \frac{x_i}{\sigma_i} \right)^{-\xi_j} \prod_{j=l+1}^{l+m} \left(1 + \sum_{i \in \mathcal{RC}_j} \frac{x_i}{\sigma_i} \right)^{-\xi_j}, \quad (x_1, \dots, x_n)' \in \mathbf{R}_{0,+}^n, \quad (4.2.2)$$

which unified, e.g., the classical multivariate Pareto of [Arnold \(1983, 2015\)](#) as well as the structures with univariate Pareto margins introduced in [Chiragiev and Landsman \(2009\)](#) and [Asimit et al. \(2010\)](#). Also, as stochastic representation (4.2.1) revealed the common-shock facet of d.d.f. (4.2.2), we made links to notions of interest in actuarial risk management (ARM) and, more generally, ERM.

An alternative way to arrive at (4.2.2) is via doubly stochastic Poisson processes ([Kingman, 1964](#)), and this is exactly the way that we explore in the present work. More specif-

ically, for $i = 1, \dots, n$, let $N_{\lambda_j t}$, $j = 1, \dots, l$ and ${}_i N_{\lambda_j t}$, $j = l+1, \dots, l+m$ denote stochastically independent homogeneous Poisson processes with intensities $\lambda_j t$ such that $\mathbf{P}[N_{\lambda_j t} = 0] = \mathbf{P}[E_{\lambda_j} > t]$ and $\mathbf{P}[{}_i N_{\lambda_j t} = 0] = \mathbf{P}[{}_i E_{\lambda_j} > t]$ for any time $t \in \mathbf{R}_{0,+}$. Then set

$$\tau_i := \inf \left\{ t \in \mathbf{R}_{0,+} : \sum_{j \in \mathcal{RF}_i^l} {}_i N_{\lambda_j t} + \sum_{j \in \mathcal{RF}_i^m} N_{\lambda_j t} > 0 \right\} \quad (4.2.3)$$

to represent the default time of the i -th r.c., where Λ_j , $j \in \mathcal{RF}_i$ are positive r.v.'s and $i = 1, \dots, n$. Given that the coordinates of the vector $\mathbf{\Lambda} := (\Lambda_1, \dots, \Lambda_{l+m})'$ are stochastically independent, it is easy to show that, for $t \in \mathbf{R}_{0,+}$, the marginal survival probability of τ_i is

$$\begin{aligned} S_i(t) := \mathbf{P}[\tau_i > t] &= \mathbf{E} \left[\mathbf{P} \left[\sum_{j \in \mathcal{RF}_i^l} N_{\lambda_j t} + \sum_{j \in \mathcal{RF}_i^m} {}_i N_{\lambda_j t} = 0 \middle| \mathbf{\Lambda} \right] \right] \\ &= \prod_{j \in \mathcal{RF}_i^l} \mathbf{P}[E_{\lambda_j} > t] \prod_{j \in \mathcal{RF}_i^m} \mathbf{P}[{}_i E_{\lambda_j} > t] \\ &= \prod_{j \in \mathcal{RF}_i^l} \psi_{\Lambda_j}(t) \prod_{j \in \mathcal{RF}_i^m} \psi_{\Lambda_j}(t) = \psi_{\sum_{j \in \mathcal{RF}_i} \Lambda_j}(t), \end{aligned} \quad (4.2.4)$$

where $\psi_{\Lambda_j}(t) = \mathbf{E}[e^{-\Lambda_j t}]$ denotes the Laplace-Stieltjes transform of Λ_j , $j = 1, \dots, l+m$. In a similar fashion and with a bit of an effort, we show that, for $t_i \in \mathbf{R}_{0,+}$, $i = 1, \dots, n$ and $\mathbf{\Lambda}$ as before, the joint survival probability is

$$\begin{aligned} S(t_1, \dots, t_n) &:= \mathbf{P}[\tau_1 > t_1, \dots, \tau_n > t_n] \\ &= \mathbf{E} \left[\mathbf{P} \left[\sum_{i=1}^n \sum_{j \in \mathcal{RF}_i^l} N_{\lambda_j t_i} + \sum_{i=1}^n \sum_{j \in \mathcal{RF}_i^m} {}_i N_{\lambda_j t_i} = 0 \middle| \mathbf{\Lambda} \right] \right] \\ &= \mathbf{E} \left[\mathbf{P} \left[\sum_{j=1}^l \sum_{i \in \mathcal{RC}_j} N_{\lambda_j t_i} + \sum_{j=l+1}^{l+m} \sum_{i \in \mathcal{RC}_j} {}_i N_{\lambda_j t_i} = 0 \middle| \mathbf{\Lambda} \right] \right] \\ &= \prod_{j=1}^l \mathbf{P} \left[E_{\Lambda_j} > \bigvee_{i \in \mathcal{RC}_j} t_i \right] \prod_{j=l+1}^{l+m} \mathbf{P} \left[\bigcap_{i \in \mathcal{RC}_j} {}_i E_{\Lambda_j} > t_i \right] \\ &= \prod_{j=1}^l \psi_{\Lambda_j} \left(\bigvee_{i \in \mathcal{RC}_j} t_i \right) \prod_{j=l+1}^{l+m} \psi_{\Lambda_j} \left(\sum_{i \in \mathcal{RC}_j} t_i \right). \end{aligned} \quad (4.2.5)$$

As a result, since for $\Lambda_j \sim Ga(\xi_j, 1)$, the Laplace transform is $\psi_{\Lambda_j}(t) = (1+t)^{-\xi_j}$, $t \in \mathbf{R}_{0,+}$, we readily have that the r.v.'s $(X_1/\sigma_1, \dots, X_n/\sigma_n)'$ and $(\tau_1, \dots, \tau_n)'$ belong to the same multivariate family of distributions.

In practice, the mixed (doubly stochastic) Poisson processes that generate defaults must not be necessarily homogeneous. Namely, we may be interested in $\Lambda_j(t)$, $t \in \mathbf{R}_{0,+}$, $j = 1, \dots, l+m$ being right continuous and increasing stochastic processes such that $\Lambda_j(0) = 0$. In this section, we are interested in the following generalization of (4.2.3)

$$\tau_i := \inf \left\{ t \in \mathbf{R}_{0,+} : \sum_{j \in \mathcal{RF}_i^l} N_{\Lambda_j(t)} + \sum_{j \in \mathcal{RF}_i^m} N_{\Lambda_j(t)} > 0 \right\}, \quad (4.2.6)$$

where $i = 1, \dots, n$. The d.d.f. of τ_i as well as the joint d.d.f. of $(\tau_1, \dots, \tau_n)'$ are formulated in the next theorem. The proofs are omitted, as they mimic the derivations that led to (4.2.4) and (4.2.5). Here and throughout $\psi_X(x)$ denotes the Laplace-Stieltjes transform of the r.v. X at $x \in \mathcal{X} \subseteq \mathbf{R}$.

Theorem 4.2.1. *For default specification (4.2.6) and assuming that $\Lambda_j(t)$, $t \in \mathbf{R}_{0,+}$ are right continuous and increasing stochastic processes with $\Lambda_j(0) = 0$, $j = 1, \dots, l+m$, the marginal survival probability of the i -th r.c. is given by*

$$S_i(t) = \psi_{\sum_{j \in \mathcal{RF}_i} \Lambda_j(t)}(1), \quad (4.2.7)$$

for $i = 1, \dots, n$. Also, the joint survivorship probability of the portfolio $\{1, \dots, n\}$ is formulated as

$$S(t_1, \dots, t_n) = \prod_{j=1}^l \psi_{\Lambda_j(\vee_{i \in \mathcal{RC}_j} t_i)}(1) \prod_{j=l+1}^{l+m} \psi_{\sum_{i \in \mathcal{RC}_j} \Lambda_j(t_i)}(1), \quad (4.2.8)$$

where $t_i \in \mathbf{R}_{0,+}$, $i = 1, \dots, n$.

We next show that the general form of the MRF dependencies, and hence (4.2.6), admit the so called default specification with stochastic default barrier. To this end, let, for $i = 1, \dots, n$ and $t \in \mathbf{R}_{0,+}$,

$$\Theta_i(t) := \sum_{j \in \mathcal{RF}_i^l} \mathbf{I}_{\{N_{\Lambda_j(t)} > 0\}}^\infty + \sum_{j \in \mathcal{RF}_i^m} \Lambda_j(t), \quad (4.2.9)$$

where (Jacod and Shiryaev, 1987)

$$\mathbf{I}_{\{N_{\Lambda_j(t)} > 0\}}^\infty := \begin{cases} 0, & N_{\Lambda_j(t)} = 0 \\ \infty, & N_{\Lambda_j(t)} > 0 \end{cases}.$$

Theorem 4.2.2. *Let ${}_iE_1$ and U_i , $i = 1, \dots, n$ denote stochastically independent r.v.'s that are, respectively, exponentially distributed with unit scale parameters and uniformly distributed on $[0, 1]$. Then the following stochastic default specifications are equivalent mutually as well as to (4.2.6)*

D1.) exponential default barrier representation

$$\tau_i = \inf \{t \in \mathbf{R}_{0,+} : \Theta_i(t) \geq {}_iE_1\}; \quad (4.2.10)$$

D2.) uniform default barrier representation

$$\tau_i = \inf \{t \in \mathbf{R}_{0,+} : \exp\{-\Theta_i(t)\} \leq U_i\}. \quad (4.2.11)$$

Proof. As the equivalence of D2 and D1 is trivial, we only prove that D1 is equivalent to

(4.2.6). By conditioning, we have that, for $t_i \in \mathbf{R}_{0,+}$, $i = 1, \dots, n$,

$$\begin{aligned}
& S\left(t_1, \dots, t_n \middle| \Lambda_j(t_i), N_{\Lambda_j(t_i)}, i = 1, \dots, n, j = 1, \dots, l+m\right) \\
&= \mathbf{P}\left[{}_1E_1 > \Theta_1(t_1), \dots, {}_nE_1 > \Theta_n(t_n) \middle| \Lambda_j(t_i), N_{\Lambda_j(t_i)}, i = 1, \dots, n, j = 1, \dots, l+m\right] \\
&= \prod_{i=1}^n \exp\left\{-\sum_{j \in \mathcal{RF}_i^l} \mathbf{I}_{\{N_{\Lambda_j(t_i)} > 0\}}^\infty - \sum_{j \in \mathcal{RF}_i^m} \Lambda_j(t_i)\right\} \\
&= \prod_{j=1}^l \exp\left\{-\sum_{i \in \mathcal{RC}_j} \mathbf{I}_{\{N_{\Lambda_j(t_i)} > 0\}}^\infty\right\} \prod_{j=l+1}^{l+m} \exp\left\{-\sum_{i \in \mathcal{RC}_j} \Lambda_j(t_i)\right\}.
\end{aligned}$$

Consequently, the unconditional joint survival function is given, for $t_i \in \mathbf{R}_{0,+}$, $i = 1, \dots, n$,

by

$$\begin{aligned}
S(t_1, \dots, t_n) &= \mathbf{E}\left[S_{1,\dots,n}\left(t_1, \dots, t_n \middle| \Lambda_j(t_i), N_{\Lambda_j(t_i)}, i = 1, \dots, n, j = 1, \dots, l+m\right)\right] \\
&= \prod_{j=1}^l \mathbf{E}\left[\exp\left\{-\Lambda_j\left(\bigvee_{i \in \mathcal{RC}_j} t_i\right)\right\}\right] \prod_{j=l+1}^{l+m} \mathbf{E}\left[\exp\left\{-\sum_{i \in \mathcal{RC}_j} \Lambda_j(t_i)\right\}\right], \tag{4.2.12}
\end{aligned}$$

since

$$\begin{aligned}
& \mathbf{E}\left[\exp\left\{-\sum_{i \in \mathcal{RC}_j} \mathbf{I}_{\{N_{\Lambda_j(t_i)} > 0\}}^\infty\right\}\right] = \mathbf{P}\left[\bigcap_{i \in \mathcal{RC}_j} \{\mathbf{I}_{\{N_{\Lambda_j(t_i)} > 0\}}^\infty = 0\}\right] \\
&= \mathbf{P}\left[\left\{N_{\Lambda_j}\left(\bigvee_{i \in \mathcal{RC}_j} t_i\right) = 0\right\}\right] = \mathbf{E}\left[\exp\left\{-\Lambda_j\left(\bigvee_{i \in \mathcal{RC}_j} t_i\right)\right\}\right].
\end{aligned}$$

Finally, by rewriting (4.2.12) in terms of the Laplace-Stieltjes transforms of $\Lambda_j(t)$, we obtain joint survival function (4.2.8). This completes the proof. $\square$

We note in passing that default specifications with stochastic barriers à la D1 have been discussed in, e.g., Lando (2004), Escobar et al. (2012), Skoglund and Chen (2015) and references therein. Interestingly, D1 reduces to an extent the complexity involved in simulating the r.v.'s $\tau_1, \dots, \tau_n$. Indeed, note that according to (4.2.9) and concentrating on

the r.f.'s with POD hitting times, we have that in order to simulate τ_i , $i = 1, \dots, n$ it is only necessary to simulate at most m sample paths of the stochastic processes $\Lambda_j(t)$, $j \in \mathcal{RF}_{1, \dots, n}^m$ as well as one exponential r.v. with unit scale, whereas the stochastic representations in [Furman and Su \(2015b\)](#) required an $n \times m$ array of such exponentials.

On a different note, default specification D2 is of special interest in the present paper as the next theorem hints. Let $S_i^- : [0, 1] \rightarrow \overline{\mathbf{R}}_{0,+} := [0, \infty]$ be a function, such that

$$S_i^-(u) := \inf\{x \in \overline{\mathbf{R}}_{0,+} : S_i(x) \leq u\},$$

where $u \in [0, 1]$ and $\inf\{\emptyset\} = \infty$ by convention. The function S_i^- is called the generalized inverse of S_i , and as such it is equal to the usual inverse $S_i^{-1}(u)$ if the survival function is continuous ([Embrechts and Hofert, 2013](#)).

Theorem 4.2.3. *The copula function $C : [0, 1]^n \rightarrow [0, 1]$ that corresponds to the general MRF dependence structures is given, for $u_i \in [0, 1]$, $i = 1, \dots, n$, by*

$$C(u_1, \dots, u_n) = \prod_{j=1}^l \psi_{\Lambda_j(\bigvee_{i \in \mathcal{RC}_j} S_i^-(u))}(1) \prod_{j=l+1}^{l+m} \psi_{\sum_{i \in \mathcal{RC}_j} \Lambda_j(S_i^-(u))}(1), \quad (4.2.13)$$

where $S_i^-(u)$ is a (generalized) inverse of $S_i(t) = \psi_{\sum_{j \in \mathcal{RF}_i} \Lambda_j(t)}(1)$ and $t \in \mathbf{R}_{0,+}$.

The proof of the theorem is omitted, as it is a direct consequence of Sklar's theorem ([Sklar, 1959](#)) and Theorem [4.2.1](#).

General forms of MRF copulas can be used in practice to model dependent default times or, more generally, dependent risks in the context of the ERM. Indeed, copulas [\(4.2.13\)](#) admit default time specifications [\(4.2.6\)](#), [\(4.2.10\)](#) and [\(4.2.11\)](#) and cover the full range of non-negative dependence (Section [4.4](#)). More specifically, they reduce to the product copula and the Fréchet upper bound copula, if there are only idiosyncratic r.f.'s $j \in \{1, \dots, l+m : \sum_{i \in \mathcal{RC}_j} = 1\}$ and only systemic r.f.'s $j \in \{1, \dots, l : \sum_{i \in \mathcal{RC}_j^l} = n\}$, respectively, included.

In addition, copulas (4.2.13) are non-exchangeable unless the exposure matrix c is such that $c_{1,j} = \dots = c_{n,j}$ for all $j = 1, \dots, l + m$.

Notwithstanding, the MRF copulas in their most general form are somewhat too abstract to be tackled analytically. As it often happens, some simplifying assumptions are necessary. For instance, it is possible to consider a class of linear stochastic processes $\Lambda_j t$, $j = 1, \dots, l + m$, only. In such a case, (4.2.13) yields the following class of copula functions

$$C(u_1, \dots, u_n) = \prod_{j=1}^l \psi_{\Lambda_j} \left(\bigvee_{i \in \mathcal{RC}_j} \psi_{\sum_{j \in \mathcal{RF}_i} \Lambda_j}^{-1}(u_i) \right) \prod_{j=l+1}^{l+m} \psi_{\Lambda_j} \left(\sum_{i \in \mathcal{RC}_j} \psi_{\sum_{j \in \mathcal{RF}_i} \Lambda_j}^{-1}(u_i) \right), \quad (4.2.14)$$

where $u_i \in [0, 1]$ and $i = 1, \dots, n$. A simplification of (4.2.13), the subclass of MRF copulas in (4.2.14) is rich enough to unify the well-known Archimedean and Marshall-Olkin classes of copula functions. In fact, we have that (4.2.14) introduces a class of non-exchangeable Archimedean copulas and coincides with the class of the Marshall-Olkin copulas, if the sets $\mathcal{RC}_j$ contain at least two elements for some $j \in \{l + 1, \dots, l + m\}$ and $j \in \{1, \dots, l\}$, respectively. Moreover, we have that (4.2.14) simplifies to the product copula if the sets $\mathcal{RC}_j$ have at most one element for all $j \in \{1, \dots, l + m\}$, and it reaches the Fréchet upper bound copula if the cardinalities of the sets $\mathcal{RC}_j$, $j \in \{1, \dots, l\}$ coincide with the dimension of the copula whereas these sets are empty for all other risk factors.

The following theorem establishes a characteristic representation of MRF copulas (4.2.14) à la the popular common-shock (Asimit et al., 2010; Furman and Su, 2015a,b).

Theorem 4.2.4. *For $i = 1, \dots, n$, let V_j , $j = 1, \dots, l$ and ${}_i V_j$, $j = l + 1, \dots, l + m$ denote a sequence of independent Uniform $[0, 1]$ r.v.'s. Then the r.v. $\mathbf{U} := (U_1, \dots, U_n)'$ has*

cumulative distribution function (c.d.f.) (4.2.14) if and only if

$$U_i \stackrel{d}{=} \left(\bigvee_{j \in \mathcal{RF}_i^l} \psi_i \left(-\frac{\ln(V_j)}{\Lambda_j} \right) \right) \bigvee \left(\bigvee_{j \in \mathcal{RF}_i^m} \psi_i \left(-\frac{\ln(V_{i,j})}{\Lambda_j} \right) \right). \quad (4.2.15)$$

Proof. To prove the ‘if’ part, note that, for $u_i \in [0, 1]$, $i = 1, \dots, n$, the c.d.f. of $\mathbf{U}$ is

$$\begin{aligned} & \mathbf{P}[U_1 \leq u_1, \dots, U_n \leq u_n] \\ &= \mathbf{P} \left[\left\{ \bigcap_{j \in \mathcal{RF}_i^l} \frac{\ln(V_j)}{\Lambda_j} \leq -\psi_i^{-1}(u_i) \right\} \text{ and } \left\{ \bigcap_{j \in \mathcal{RF}_i^m} \frac{\ln({}_i V_j)}{\Lambda_j} \leq -\psi_i^{-1}(u_i) \right\} \text{ for all } i = 1, \dots, n \right] \\ &= \mathbf{P} \left[\left\{ \bigcap_{j=1}^l \frac{\ln(V_j)}{\Lambda_j} \leq -\bigvee_{i \in \mathcal{RC}_j} \psi_i^{-1}(u_i) \right\} \right] \mathbf{P} \left[\left\{ \bigcap_{j=l+1}^{l+m} \bigcap_{i \in \mathcal{RC}_j} \frac{\ln({}_i V_j)}{\Lambda_j} \leq -\psi_i^{-1}(u_i) \right\} \right], \end{aligned}$$

where

$$\begin{aligned} & \mathbf{P} \left[\left\{ \bigcap_{j=1}^l \frac{\ln(V_j)}{\Lambda_j} \leq -\bigvee_{i \in \mathcal{RC}_j} \psi_i^{-1}(u_i) \right\} \right] = \mathbf{E} \left[\mathbf{P} \left[\left\{ \bigcap_{j=1}^l \frac{\ln(V_j)}{\Lambda_j} \leq -\bigvee_{i \in \mathcal{RC}_j} \psi_i^{-1}(u_i) \right\} \middle| \mathbf{\Lambda} \right] \right] \\ &= \mathbf{E} \left[\prod_{j=1}^l \exp \left\{ -\bigvee_{i \in \mathcal{RC}_j} \Lambda_j \psi_i^{-1}(u_i) \right\} \right] = \prod_{j=1}^l \psi_{\Lambda_j} \left(\bigvee_{i \in \mathcal{RC}_j} \psi_i^{-1}(u_i) \right) \end{aligned}$$

and similarly

$$\begin{aligned} & \mathbf{P} \left[\left\{ \bigcap_{j=l+1}^{l+m} \bigcap_{i \in \mathcal{RC}_j} \frac{\ln({}_i V_j)}{\Lambda_j} \leq -\psi_i^{-1}(u_i) \right\} \right] = \mathbf{E} \left[\mathbf{P} \left[\left\{ \bigcap_{j=l+1}^{l+m} \bigcap_{i \in \mathcal{RC}_j} \frac{\ln({}_i V_j)}{\Lambda_j} \leq -\psi_i^{-1}(u_i) \right\} \middle| \mathbf{\Lambda} \right] \right] \\ &= \mathbf{E} \left[\prod_{j=l+1}^{l+m} \exp \left\{ -\Lambda_j \sum_{i \in \mathcal{RC}_j} \psi_i^{-1}(u_i) \right\} \right] = \prod_{j=l+1}^{l+m} \psi_{\Lambda_j} \left(\sum_{i \in \mathcal{RC}_j} \psi_i^{-1}(u_i) \right). \end{aligned}$$

Hence the joint c.d.f. of $\mathbf{U}$ coincides with (4.2.14). The ‘only if’ part follows by the uniqueness of the Laplace-Stieltjes transform. This completes the proof. $\square$

We conclude this section with some references. Namely, we note that the class of Archimedean copulas has been extensively used in the context of credit risk in, e.g., [Hull and White \(2006\)](#), [Choudhry \(2010\)](#), [Puzanova \(2011\)](#), [Rösch and Scheule \(2013\)](#) and references

therein, as well as in general ERM in, e.g., [Kole et al. \(2007\)](#), [Staudt \(2010\)](#), [Sandström \(2010\)](#). Also, the class of Marshall-Olkin copulas has been recently suggested for applications in credit risk in [Cherubini et al. \(2013\)](#), and its applications to insurance mathematics were presented in [Bowers et al. \(1997\)](#).

4.3 Probability of simultaneous default

D.d.f. (4.2.2) is not absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}_{0,+}^n$, and as a result, default specifications (4.2.3), (4.2.6), (4.2.10) and (4.2.11) as well as copula functions (4.2.13) and (4.2.14) allow for non-zero probabilities of simultaneous default. One empirical justification for accommodating this phenomenon is the famous example of 24 railway firms defaulting on June 21, 1970 ([Azizpour and Giesecke, 2008](#)), another justification, that is somewhat more theoretical, is the conclusion of [Das et al. \(2007\)](#) that the mixed Poisson processes approach tends to underestimate the clustering of real world defaults.

We next formulate the probability of simultaneous default for the general MRF dependencies discussed in the previous section. To this end, we need to introduce some more notations ([Furman and Su, 2015b](#)). First, we find useful to break the vector parameter $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{l+m})'$ as following $\boldsymbol{\xi} := (\boldsymbol{\alpha}', \boldsymbol{\gamma}')'$ where $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_l)'$ and $\boldsymbol{\gamma} := (\gamma_{l+1}, \dots, \gamma_{l+m})'$. Then, for $2 \leq k \leq n$, we denote the set of all risk factors that ‘attack’ the sub-portfolio $\{i_1, \dots, i_k\}$ by

$$\mathcal{RF}_{i_1, \dots, i_k} := \{j \in \{1, \dots, l+m\} : c_{i_h, j} = 1 \text{ for at least one } i_h \in \{i_1, \dots, i_k\}\}, \quad (4.3.1)$$

and we note that it is a union of two disjoint sets, that is of

$$\mathcal{RF}_{(i_1, \dots, i_k)} := \{j \in \{1, \dots, l+m\} : c_{i_h, j} = 1 \text{ for all } i_h \in \{i_1, \dots, i_k\}\} \quad (4.3.2)$$

and

$$\mathcal{RF}_{(\overline{i_1, \dots, i_k})} := \mathcal{RF}_{i_1, \dots, i_k} \setminus \mathcal{RF}_{(i_1, \dots, i_k)}. \quad (4.3.3)$$

Also, let, for $1 \leq h \leq k$ and $2 \leq k \leq n$,

$$\mathcal{RF}_{i_h, \overline{(i_1, \dots, i_k)}} := \mathcal{RF}_{i_h} \setminus \mathcal{RF}_{(i_1, \dots, i_k)}. \quad (4.3.4)$$

Based on the notations above, we can have general sums of the form $\square_{\bullet} = \sum_{j \in \mathcal{RF}_{\bullet}} \square_j$, where ‘ $\square$ ’ can be a parameter, e.g., α , γ , ξ or a r.v., e.g., Λ , and ‘ $\bullet$ ’ is any one of $i_1, \dots, i_k$, $(i_1, \dots, i_k)$, $\overline{(i_1, \dots, i_k)}$ and $i_h, \overline{(i_1, \dots, i_k)}$. We note in passing that if the sum of interest is over the coordinates of $\alpha = (\alpha_1, \dots, \alpha_l)'$ or $\gamma = (\gamma_{l+1}, \dots, \gamma_{l+m})'$, we add superscripts ‘ l ’ or ‘ m ’, respectively to sets (4.3.1) - (4.3.4).

Let, for $t \in \mathbf{R}_{0,+}$.

$$\mathcal{A}(t) := \left\{ \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} N_{\Lambda_j(t)} > 0 \text{ and } \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} N_{\Lambda_j(t-)} = 0 \right\}. \quad (4.3.5)$$

Theorem 4.3.1. *Consider default specification (4.2.6) with $\Lambda_j(t)$ such that, for all $j = 1, \dots, l+m$, it holds that $\mathbf{P}[\Lambda_j(t) < \infty, t \in \mathbf{R}_{0,+}] \equiv 1$, and let $\{i_1, \dots, i_k\}$ establish an index set with $2 \leq k \leq n$, then the probability of simultaneous default is given by*

$$\mathbf{P}[\tau_{i_1} = \dots = \tau_{i_k}] = \int_{\mathbf{R}_{0,+}} \mathbf{P}[\mathcal{A}(t)] \prod_{j \in \mathcal{RF}_{(\overline{i_1, \dots, i_k})}^l} \psi_{\Lambda_j(t)}(1) \prod_{j \in \mathcal{RF}_{i_1, \dots, i_k}^m} \psi_{\Lambda_j(t)}(|\mathcal{RC}_j|) dt, \quad (4.3.6)$$

where $|\cdot|$ denotes set’s cardinality.

Proof. Since the non-zero probability of simultaneous default can only come from the risk factors in $\mathcal{RF}_i^l$, $i = 1, \dots, n$, we obtain, for any $t \in \mathbf{R}_{0,+}$ and by conditioning on $\Lambda(t) :=$

$(\Lambda_1(t), \dots, \Lambda_{l+m}(t))'$, that

$$\begin{aligned}
& \mathbf{P}[\tau_{i_1} = \dots = \tau_{i_k} = t \mid \Lambda(t)] \\
&= \mathbf{P} \left[\left\{ \bigcap_{i \in \{i_1, \dots, i_k\}} \left(\left(\sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} N_{\Lambda_j(t)} + \sum_{j \in \mathcal{RF}_i^m} {}_i N_{\Lambda_j(t)} = 0 \right) \cap \mathcal{A}(t) \right) \right\} \mid \Lambda(t) \right] \\
&= \prod_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \mathbf{P} \left[N_{\Lambda_j(t)} = 0 \mid \Lambda(t) \right] \prod_{j \in \mathcal{RF}_{i_1, \dots, i_k}^m} \mathbf{P} \left[\sum_{i \in \mathcal{RC}_j^m} {}_i N_{\Lambda_j(t)} = 0 \mid \Lambda(t) \right] \mathbf{P}[\mathcal{A}(t) \mid \Lambda(t)].
\end{aligned}$$

Consequently, for any $t \in \mathbf{R}_{0,+}$,

$$\mathbf{P}[\tau_{i_1} = \dots = \tau_{i_k} = t] = \mathbf{P}[\mathcal{A}(t)] \prod_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \psi_{\Lambda_j(t)}(1) \prod_{j \in \mathcal{RF}_{i_1, \dots, i_k}^m} \psi_{\Lambda_j(t)}(|\mathcal{RC}_j|).$$

The proof is then completed by integration over all $t \in \mathbf{R}_{0,+}$. $\square$

Under an additional assumption of linearity of the stochastic intensity processes, the probability of simultaneous default can be simplified as follows.

Corollary 4.3.1. *Let $\Lambda_j(t) = \Lambda_j t$ for all $j = 1, \dots, l+m$ and $t \in \mathbf{R}_{0,+}$ and leave the rest of the set-up of Theorem 4.3.1 unchanged, then the probability of simultaneous default of the sub-portfolio $\{i_1, \dots, i_k\}$ is given by*

$$\mathbf{P}[\tau_{i_1} = \dots = \tau_{i_k}] = \mathbf{E} \left[\frac{\Lambda_{(i_1, \dots, i_k)}^l}{\Lambda_{i_1, \dots, i_k}^l + \tilde{\Lambda}_{i_1, \dots, i_k}^m} \right], \quad (4.3.7)$$

where $\Lambda_{(i_1, \dots, i_k)}^l = \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \Lambda_j$, $\Lambda_{i_1, \dots, i_k}^l = \sum_{j \in \mathcal{RF}_{i_1, \dots, i_k}^l} \Lambda_j$, $\tilde{\Lambda}_{i_1, \dots, i_k}^m = \sum_{j \in \mathcal{RF}_{i_1, \dots, i_k}^m} \Lambda_j |\mathcal{RC}_j|$ and $|\cdot|$ stands for set's cardinality.

Proof. Under the assumption of linearity, we obviously have that, for $j = 1, \dots, l+m$ and $t \in \mathbf{R}_{0,+}$,

$$\psi_{\Lambda_j(t)}(1) = \psi_{\Lambda_j}(t) = \mathbf{P}[E_{\Lambda_j} > t],$$

as well as that

$$\mathbf{P}[\{\mathcal{A}(t)\}] = \mathbf{E} \left[-\frac{d}{dt} \exp \left\{ - \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \Lambda_j t \right\} \right] = \mathbf{E} \left[\exp \left\{ - \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \Lambda_j t \right\} \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \Lambda_j \right].$$

Consequently, denoting by L_j the c.d.f. of the r.v. Λ_j , the integrand in (4.3.6) reduces to

$$\begin{aligned} & \int_{\mathbf{R}_{0,+}} \int_{\mathbf{R}_{0,+}^{|\mathcal{RF}_{i_1, \dots, i_k}|}} \sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \lambda_j \exp \left\{ - \left(\sum_{j \in \mathcal{RF}_{i_1, \dots, i_k}^l} \lambda_j + \sum_{j \in \mathcal{RF}_{i_1, \dots, i_k}^m} \lambda_j |\mathcal{RC}_j| \right) t \right\} d \prod_{j \in \mathcal{RF}_{i_1, \dots, i_k}} L_j(\lambda_j) dt \\ &= \int_{\mathbf{R}_{0,+}^{|\mathcal{RF}_{i_1, \dots, i_k}|}} \frac{\sum_{j \in \mathcal{RF}_{(i_1, \dots, i_k)}^l} \lambda_j}{\sum_{j \in \mathcal{RF}_{i_1, \dots, i_k}^l} \lambda_j + \sum_{j \in \mathcal{RF}_{i_1, \dots, i_k}^m} \lambda_j |\mathcal{RC}_j|} d \prod_{j \in \mathcal{RF}_{i_1, \dots, i_k}} L_j(\lambda_j). \end{aligned}$$

This completes the proof. $\square$

Unless very special cases are of interest, e.g., when the set of the r.f.'s having fully-comonotonic hitting times is empty ($\mathcal{RF}_{(i_1, \dots, i_k)}^l = \emptyset$) or when the set of the r.f.'s having POD hitting times is empty ($\mathcal{RF}_{(i_1, \dots, i_k)}^m = \emptyset$) and in addition the r.v.'s Λ_j , $j \in \mathcal{RF}_{i_1, \dots, i_k}$ follow favourable probability distributions (Section 4.4), even expression (4.3.7) is somewhat involved to be handled analytically. It is however possible to use Corollary 4.3.1 to compute the probability of simultaneous default employing Monte-Carlo simulations.

We have hitherto derived a number of important results in the context of the general form of the MRF dependencies as well as in the special case when the stochastic intensity processes are linear. However, in order to obtain insights into such higher level characteristics of the new dependence structures, as, e.g., measures of rank correlation and indices of tail dependence, further simplifications are required. In the sequel, we assume that the r.v.'s Λ_j are stochastically independent and distributed gamma with shape parameters ξ_j and unit scales, $j = 1, \dots, l + m$. The above choice of the distribution of Λ_j may seem ad hoc at the first glance, but it is well motivated by the CreditRisk⁺ approach, which serves as one

of the most popular ways to model default risk in the modern practice of credit risk. This assumption gives birth to the Clayton subclass of the MRF dependencies.

4.4 Clayton Multiple Risk Factor copula functions

Let $\Lambda_j \sim Ga(\xi_j, 1)$, $j = 1, \dots, l + m$ denote $(l + m)$ stochastically independent r.v.'s distributed gamma. Then the probability density function of Λ_j is

$$f_{\Lambda_j}(\lambda; \xi_j, 1) = e^{-\lambda} \frac{\lambda^{\xi_j-1}}{\Gamma(\xi_j)}, \quad \lambda \in \mathbf{R}_+, \quad (4.4.1)$$

and the corresponding Laplace transform is

$$\psi_{\Lambda_j}(x) = (1 + x)^{-\xi_j}, \quad x \in \mathbf{R}_{0,+}. \quad (4.4.2)$$

This immediately establishes that gamma distributions are infinitely divisible and so closed under convolutions, i.e., in our case, we have that $\Lambda := \Lambda_1 + \dots + \Lambda_n$ is distributed $Ga(\xi, 1)$, where $\xi = \xi_1 + \dots + \xi_n$. Importantly, even if the scale parameters are not equal, the convolution will still be a gamma but with a random shape parameter. This is formulated in the following lemma.

Let K be an integer valued non-negative r.v. with the probability mass function (p.m.f.) $p_k := \mathbf{P}[K = k]$, which is given by

$$p_k = c_+ \delta_k, \quad k = 0, 1, \dots, \quad (4.4.3)$$

where

$$c_+ = \prod_{i=1}^n \left(\frac{\sigma_i}{\sigma_+} \right)^{\xi_i}$$

and

$$\delta_k = \begin{cases} 1, & k = 0 \\ k^{-1} \sum_{l=1}^k \sum_{i=1}^n \xi_i \left(1 - \frac{\sigma_i}{\sigma_+} \right)^l \delta_{k-l}, & k = 1, 2, \dots \end{cases}.$$

Lemma 4.4.1 (Moschopoulos, 1985, also, Furman and Landsman, 2005). *Let $\Lambda_i \sim Ga(\xi_i(\in \mathbf{R}_+), \sigma_i(\in \mathbf{R}_+))$, $i = 1, \dots, n$ denote gamma distributed and independent stochastically r.v.'s with arbitrary shape and scale parameters, and let $\Lambda = \Lambda_1 + \dots + \Lambda_n$ be their convolution. Then $\Lambda \sim Ga(\xi + K, \sigma_+)$, where $\xi = \xi_1 + \dots + \xi_n$, $\sigma_+ = \vee_{i=1}^n \sigma_i$ and K is an integer valued non-negative r.v. with p.m.f. (4.4.3).*

We note in passing that if $\sigma_1 = \dots = \sigma_n$, then $K = 0$ almost surely, and the findings of the lemma reduce to the simple convolution of gamma distributed r.v.'s with a common scale parameter.

We further introduce the Clayton subclass of MRF dependencies. The definition below follows from (4.2.14) and (4.4.2). We remind that the doubly stochastic Poisson approach with gamma distributed intensity processes has been adapted in CreditRisk⁺, and, as such, it is arguably one of the most popular ways to model dependent defaults in nowadays credit risk practice. The method has been often criticized for underestimating the clustering of defaults' occurrences (Das et al., 2007). We note that the Clayton MRF dependencies augment the POD hitting times of the r.f.'s in CreditRisk⁺ with the fully comonotonic hitting times of the so called systemic r.f.'s. This allows for a mechanism to model the clustering of defaults more accurately and may thus resolve to an extent the aforementioned drawback.

Definition 4.4.1. *Copula functions $C_{\boldsymbol{\xi}} : [0, 1]^n \rightarrow [0, 1]$, parametrized by the deterministic vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{l+m})'$ with $\xi_j \in \mathbf{R}_+$, $j = 1, \dots, l+m$, are called the Clayton MRF copulas if*

$$C_{\boldsymbol{\xi}}(u_1, \dots, u_n) = \prod_{j=1}^l \bigwedge_{i \in \mathcal{RC}_j} u_i^{\frac{\xi_j}{\xi_{c,i}}} \prod_{j=l+1}^{l+m} \left[1 + \sum_{i \in \mathcal{RC}_j} \left(u_i^{-\frac{1}{\xi_{c,i}}} - 1 \right) \right]^{-\xi_j}, \quad (4.4.4)$$

where $u_i \in [0, 1]$ and $\xi_{c,i} = \sum_{j \in \mathcal{RF}_i} \xi_j$ for $i = 1, \dots, n$.

It is easy to see that, for a fixed dimension and $(u_1, \dots, u_n)' \in [0, 1]^n$,

- if the sets $\mathcal{RC}_j$ contain at most one element for all $j \in \{1, \dots, l + m\}$, then

$$C_{\xi}(u_1, \dots, u_n) = \prod_{i=1}^n u_i := \Pi(u_1, \dots, u_n) - \text{the product copula;}$$

- if $|\mathcal{RC}_j| = n$ for some $j \in \{1, \dots, l\}$ and are zero otherwise, then

$$C_{\xi}(u_1, \dots, u_n) = \bigwedge_{i=1}^n u_i := M(u_1, \dots, u_n) - \text{the Fréchet upper bound copula;}$$

- if $|\mathcal{RC}_j| = n$ for some $j \in \{l + 1, \dots, m + l\}$ and are zero otherwise, then

$$C_{\xi}(u_1, \dots, u_n) = \left[1 + \sum_{i=1}^n \left(u_i^{-\frac{1}{\xi_{c,(1,\dots,n)}}} - 1 \right) \right]^{-\xi_{c,(1,\dots,n)}} := C_{\gamma}(u_1, \dots, u_n)$$

- the Clayton Archimedean copula;

- if the sets $\mathcal{RC}_j$ have at least one element for $j \in \{1, \dots, l\}$ and are empty sets otherwise, then

$$C_{\xi}(u_1, \dots, u_n) = \prod_{j=1}^l \bigwedge_{i \in \mathcal{RC}_j} u_i^{\frac{\alpha_j}{\alpha_{c,i}}} := C_{\alpha}(u_1, \dots, u_n) - \text{the Marshall-Olkin copula.}$$

The closure under convolutions property of the r.v.'s distributed gamma, and more generally Lemma 4.4.1, facilitate yet additional simplification of (4.3.6). Recall that we set $\xi_j = \alpha_j$ and $\xi_j = \gamma_j$ for $j = 1, \dots, l$ and $j = l + 1, \dots, l + m$, respectively. A very special case of Theorem 4.3.1 and Corollary 4.3.1, the following proposition establishes at a stroke two stand alone results obtained independently in Marshall and Olkin (1967) and Asimit et al. (2010).

Proposition 4.4.1. *Within the Clayton subclass of the MRF copulas and with $\{i_1, \dots, i_k\}$, $k \leq n$ establishing an index set, the probability of simultaneous default is given by*

$$\mathbf{P} \left[U_{i_1}^{1/\xi_{c,i_1}} = \dots = U_{i_k}^{1/\xi_{c,i_k}} \right] = \alpha_{(i_1, \dots, i_k)} \mathbf{E} \left[\frac{1}{\xi_{c,i_1, \dots, i_k} + K} \right], \quad (4.4.5)$$

where $\alpha_{c,(i_1,\dots,i_k)} = \sum_{j \in \mathcal{RF}_{(i_1,\dots,i_k)}^l} \xi_j$, $\xi_{c,i_1,\dots,i_k} = \sum_{j \in \mathcal{RF}_{i_1,\dots,i_k}} \xi_j$, and K is an integer-valued r.v. having p.m.f. à la (4.4.3).

Proof. First note that $\Lambda_{c,i_1,\dots,i_k}^l \sim Ga(\alpha_{c,i_1,\dots,i_k}, 1)$ and $\tilde{\Lambda}_{i_1,\dots,i_k}^m \sim Ga(\gamma_{c,i_1,\dots,i_k} + K, 1)$ appealing to Lemma 4.4.1. Then, conditionally on $K = h(\in \mathbf{Z}_0)$, the distribution of $\Lambda_{c,i_1,\dots,i_k}^l + \tilde{\Lambda}_{i_1,\dots,i_k}^m$ is $Ga(\xi_{c,i_1,\dots,i_k} + h, 1)$, and the conditional probability of simultaneous default is equal to the expectation of a beta distributed r.v. with parameters $\alpha_{c,i_1,\dots,i_k}$ and $\gamma_{c,i_1,\dots,i_k} + h$. The assertion of the proposition follows evoking the law of iterated expectation. $\square$

In the rest of this section we derive the Spearman's rho measure of rank correlation in the context of the Clayton subclass of MRF dependencies. It is well-known that the Pearson's measure of correlation can produce somewhat counter-intuitive results when the dependence is not linear, i.e., beyond the class of multivariate elliptical distributions (Fang et al., 1990; Branco and Dey, 2001; Genton, 2004). The Spearman's rho, succinctly ρ_S , provides a natural extension for arbitrary dependencies.

Definition 4.4.2 (Nelsen, 2006). *Let the r.v.'s U and V have a copula C . Then the Spearman's rho measure of rank correlation is given by*

$$\rho_S(C) = 12 \int \int_{[0,1]^2} uv dC(u,v) - 3.$$

A number of notes are instrumental before formulating the expression for ρ_S for the Clayton MRF dependencies. First, we are interested in the bivariate copula functions only and thus (4.4.4) reduces to

$$C_{\xi}(u_i, u_k) = u_i^{\frac{\xi_{c,i,(i,k)}}{\xi_{c,i}}} u_k^{\frac{\xi_{c,k,(i,k)}}{\xi_{c,k}}} \left(u_i^{\frac{\alpha_{c,(i,k)}}{\xi_{c,i}}} \wedge u_k^{\frac{\alpha_{c,(i,k)}}{\xi_{c,k}}} \right) \left(u_i^{-\frac{1}{\xi_{c,i}}} + u_k^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}} \quad (4.4.6)$$

where u_i and u_k are in $[0, 1]$ for $1 \leq i \neq k \leq n$.

Second, we recall that the $(q+1) \times q$ hypergeometric function ([Gradshteyn and Ryzhik, 2014](#)) is formulated as

$${}_{q+1}F_q(a_1, \dots, a_{q+1}; b_1, \dots, b_q; z) := \sum_{k=0}^{\infty} \frac{(a_1)_k, \dots, (a_{q+1})_k}{(b_1)_k, \dots, (b_q)_k} \frac{z^k}{k!}, \quad (4.4.7)$$

where $(p)_n := p(p+1) \cdots (p+n-1)$ for $n \in \mathbf{Z}_+$, $(p)_0 := 1$ and $q \in \mathbf{Z}_+$. For $a_1, \dots, a_{q+1}$ all positive, and these are the cases of interest in the present paper, the radius of convergence of the series is the open disk $|z| < 1$. On the boundary $|z| = 1$, the series converges absolutely if $d = b_1 + \cdots + b_q - a_1 - \cdots - a_{q+1} > 0$, and it converges except at $z = 1$ if $0 \geq d > -1$. The hypergeometric function above is essential when computing the Spearman's rank correlation for the Clayton subclass of MRF copulas.

Let

$$h(x) := {}_3F_2(2x, 1, \gamma_{c,(i,k)}; 2x+1, 2\xi_{c,i} + 2\xi_{c,k} - \xi_{c,(i,k)} + 1; -1),$$

where $x \in \mathbf{R}_+$ and such that $h(x)$ is well-defined.

Theorem 4.4.1. *Consider the Clayton subclass of the MRF copulas, then the Spearman's measure of rank correlation is given, for $1 \leq i \neq k \leq n$,*

$$\rho_S(C_{\boldsymbol{\xi}}) = \frac{6}{2\xi_{c,i} + 2\xi_{c,k} - \xi_{c,(i,k)}} (\xi_{c,k} h(\xi_{c,i}) + \xi_{c,i} h(\xi_{c,k})) - 3. \quad (4.4.8)$$

Proof. By definition, we have that, for $1 \leq i \neq k \leq n$,

$$\begin{aligned}
\rho_S(C) &= \int \int_{[0,1]^2} u_i u_k dC_{\boldsymbol{\xi}}(u_i, u_k) = \int \int_{[0,1]^2} C_{\boldsymbol{\xi}}(u_i, u_k) du_i du_k \\
&= \int_0^1 \int_0^{u_k^{\xi_{c,i}/\xi_{c,k}}} u_i^{\frac{\xi_{c,i,(\overline{i,k})} + \alpha_{c,(i,k)}}{\xi_{c,i}}} u_k^{\frac{\xi_{c,k,(\overline{i,k})}}{\xi_{c,k}}} \left(u_i^{-\frac{1}{\xi_{c,i}}} + u_k^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}} du_i du_k \\
&+ \int_0^1 \int_{u_k^{\xi_{c,i}/\xi_{c,k}}}^1 u_i^{\frac{\xi_{c,i,(\overline{i,k})}}{\xi_{c,i}}} u_k^{\frac{\xi_{c,k,(\overline{i,k})} + \alpha_{c,(i,k)}}{\xi_{c,k}}} \left(u_i^{-\frac{1}{\xi_{c,i}}} + u_k^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}} du_i du_k \\
&= \int_0^1 \int_0^{u_k^{\xi_{c,i}/\xi_{c,k}}} u_i^{\frac{\xi_{c,i,(\overline{i,k})} + \alpha_{c,(i,k)}}{\xi_{c,i}}} u_k^{\frac{\xi_{c,k,(\overline{i,k})}}{\xi_{c,k}}} \left(u_i^{-\frac{1}{\xi_{c,i}}} + u_k^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}} du_i du_k \\
&+ \int_0^1 \int_0^{u_i^{\xi_{c,i}/\xi_{c,k}}} u_i^{\frac{\xi_{c,k,(\overline{i,k})} + \alpha_{c,(i,k)}}{\xi_{c,k}}} u_k^{\frac{\xi_{c,i,(\overline{i,k})}}{\xi_{c,i}}} \left(u_i^{-\frac{1}{\xi_{c,k}}} + u_k^{-\frac{1}{\xi_{c,i}}} - 1 \right)^{-\gamma_{c,(i,k)}} du_k du_i \\
&= I_1(\boldsymbol{\xi}) + I_2(\boldsymbol{\xi}). \tag{4.4.9}
\end{aligned}$$

We further compute $I_1(\boldsymbol{\xi})$ whereas the other integral can be tackled in a similar fashion. By change of variables and evoking Equation (3.197(1)) in [Gradshteyn and Ryzhik \(2014\)](#), we obtain that

$$\begin{aligned}
I_1(\boldsymbol{\xi}) &= \xi_{c,i} \xi_{c,k} \int_{\mathbf{R}_+} (1+x)^{-\xi_{c,k} - \xi_{c,k,(\overline{i,k})} - 1} \int_{\mathbf{R}_+} (1+2x+y)^{-\gamma_{c,(i,k)}} (1+x+y)^{-\xi_{c,i} - \xi_{c,i,(\overline{i,k})} - \alpha_{c,(i,k)} - 1} dy dx \\
&= \frac{\xi_{c,k}}{2} \int_{\mathbf{R}_+} (1+2x)^{-\gamma_{c,(i,k)}} (1+x)^{-(b+1-\gamma_{c,(i,k)})} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; 2\xi_{c,i} + 1; \frac{x}{1+2x} \right) dx \\
&= \frac{\xi_{c,k}}{4} \int_0^1 (1-v)^{b-1} (1-v/2)^{-(b+1-\gamma_{c,(i,k)})} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; 2\xi_{c,i} + 1; v/2 \right) dv,
\end{aligned}$$

where $b = 2\xi_{c,i} + 2\xi_{c,k} - \xi_{c,(i,k)}$. Furthermore, note that as the ${}_2F_1$ hypergeometric function has the following integral representation for all $v \in \mathbf{R}$,

$$\frac{1}{2\xi_{c,i}} {}_2F_1 \left(\gamma_{c,(i,k)}, 1; 2\xi_{c,i} + 1; v/2 \right) = \int_0^1 (1-t)^{2\xi_{c,i}-1} \left(1 - \frac{v}{2}t \right)^{-\gamma_{c,(i,k)}} dt$$

(Equation 9.111 in loc. cit.), we obtain the following string of integrals

$$\begin{aligned}
I_1(\boldsymbol{\xi}) &= \frac{\xi_{c,i}\xi_{c,k}}{2} \int_0^1 (1-t)^{2\xi_{c,i}-1} \int_0^1 (1-v)^{b-1} \left(1 - \frac{v}{2}\right)^{-(b+1-\gamma_{c,(i,k)})} \left(1 - \frac{t}{2}v\right)^{-\gamma_{c,(i,k)}} dv dt \\
&\stackrel{(1)}{=} \frac{\xi_{c,i}\xi_{c,k}}{2b} \int_0^1 (1-t)^{2\xi_{c,i}-1} F_1\left(1, \gamma_{c,(i,k)}, b+1-\gamma_{c,(i,k)}, b+1; 1/2, t/2\right) dt \\
&\stackrel{(2)}{=} \frac{\xi_{c,i}\xi_{c,k}}{b} \int_0^1 (1-t)^{2\xi_{c,i}-1} {}_2F_1\left(1, \gamma_{c,(i,k)}; b+1; t-1\right) dt \\
&= \frac{\xi_{c,i}\xi_{c,k}}{b} \int_0^1 y^{2\xi_{c,i}-1} {}_2F_1\left(1, \gamma_{c,(i,k)}; b+1; -y\right) dy \\
&\stackrel{(3)}{=} \frac{\xi_{c,k}}{2b} {}_3F_2\left(2\xi_{c,i}, 1, \gamma_{c,(i,k)}; 2\xi_{c,i}+1, b+1; -1\right),
\end{aligned}$$

where F_1 is the bivariate hypergeometric function, and $\stackrel{(1)}{=}$, $\stackrel{(2)}{=}$, and $\stackrel{(3)}{=}$, hold by Equations (3.211), (9.182(1)) and (7.512(12)), respectively, in [Gradshteyn and Ryzhik \(2014\)](#). The expression for $I_2(\boldsymbol{\xi})$ is then by analogy

$$I_2(\boldsymbol{\xi}) = \frac{\xi_{c,i}}{2b} {}_3F_2\left(2\xi_{c,k}, 1, \gamma_{c,(i,k)}; 2\xi_{c,k}+1, b+1; -1\right).$$

We note in passing that the hypergeometric functions in $I_1(\boldsymbol{\xi})$ and $I_2(\boldsymbol{\xi})$ converge absolutely since $b+1-\gamma_{c,(i,k)} > 1$ for $1 \leq i \neq k \leq n$. This completes the proof. $\square$

Two immediate consequences are formulated next. We note in passing that while Corollary 4.4.1 confirms the findings in [Embrechts et al. \(2003\)](#), Corollary 4.4.2 is seemingly new.

Corollary 4.4.1. *Let $\gamma_{c,(i,k)} \equiv 0$ and leave the rest of the conditions in Theorem 4.4.1 unchanged. In this case, the Clayton MRF copula reduces to the Marshall-Olkin copula, succinctly $C_{\boldsymbol{\alpha}}$, with the measure of Spearman's rank correlation given by*

$$\rho_S(C_{\boldsymbol{\alpha}}) = \frac{3\alpha_{c,(i,k)}}{2\xi_{c,i} + 2\xi_{c,k} - \alpha_{c,(i,k)}}. \quad (4.4.10)$$

Proof. The assertion follows since ${}_3F_2(a, b, 0; c, d; z) \equiv 1$, for any real a, b, c, d, z . $\square$

Corollary 4.4.2. *Let $\alpha_{c,(i,k)} \equiv 0$ and leave the rest of the conditions in Theorem 4.4.1 unchanged. In this case, the Clayton MRF copula reduces to the class of non-exchangeable Archimedean copulas, succinctly C_γ , with the measure of Spearman's rank correlation given by*

$$\rho_S(C_\gamma) = 3 \left[{}_3F_2 \left(1, 1, \gamma_{c,(i,k)}; 2\xi_{c,i} + 1, 2\xi_{c,k} + 1; 1 \right) - 1 \right]. \quad (4.4.11)$$

Proof. First notice that according to Theorem 3.4.1. in [Furman and Su \(2015b\)](#), we have that

$$\begin{aligned} & \frac{1}{s-2} \left((s_2 - 1) {}_3F_2(s_1 - 1, 1, a; s_1, s - 1; -1) + (s_1 - 1) {}_3F_2(s_2 - 1, 1, a; s_2, s - 1; -1) \right) \\ &= {}_3F_2(a, 1, 1; s_1, s_2; 1), \end{aligned} \quad (4.4.12)$$

where a, b, c are all positive and such that $s_1 = a + b > 2$ and $s_2 = a + c > 2$. Put $\alpha_{c,(i,k)} \equiv 0$, $a = 2\xi_{c,(i,k)} = 2\gamma_{c,(i,k)}$, $b = 2\xi_{c,i,\overline{(i,k)}} + 1$ and $c = 2\xi_{c,k,\overline{(i,k)}} + 1$, then we have that $s_1 = \xi_{c,i} + 1$ and $s_2 = \xi_{c,k} + 1$, and the assertion follows using (4.4.1). $\square$

We next study the dependence of extreme default times, i.e., the tail dependence of the Clayton subclass of MRF copulas. It is well known that jointly dependent r.v.'s can become independent when the extreme outcomes are of interest ([Donnelly and Embrechts, 2010](#); [Furman et al., 2015a](#)), yet losses due to risk portfolios are as a rule driven by extremes therein. Interestingly, this was the class of MRF dependencies, that motivated us to revisit the classic approaches (for example, [Coles et al., 1999](#); [Joe, 1997](#); [Ledford and Tawn, 1996](#)) to measuring the extent of tail dependence in copulas and to propose an alternative method ([Furman et al., 2015b](#)). As the majority of the existing methods for quantifying tail dependence aim at random pairs, we specialize the discussion in the next section to the bivariate case, only. Some of our results in the sequel can be extended to the multivariate case with just a bit

of an effort, others are rather involved when explored in higher dimensions and can serve as great future research topics for a technically adept mathematician.

4.5 Tail dependence of the generalized Clayton copula

“The devil is in the tails” was the title of a paper by [Donnelly and Embrechts \(2010\)](#) who refuted the harsh criticism of mathematics ([Salmon, 2012](#)) in general, and of the Gaussian copula-based credit risk model of ([Li, 2000](#)) in particular. As a member of the unholy trinity ([Kousky and Cooke, 2009](#)), the notion of tail dependence is in the very center of this controversy. Speaking plainly, tail dependence is about the clustering of extreme events. In the context of default risk, such clustering is written formally as

$$\mathbf{P}[\tau_i \leq t, \tau_k \leq t] = 1 - \mathbf{P}[\tau_i > t] - \mathbf{P}[\tau_k > t] + \mathbf{P}[\tau_i > t, \tau_k > t],$$

for $1 \leq i \neq k \leq n$ and $t \uparrow t^*$, where t^* is the maximal survivorship time of the risk portfolio $\{i, k\}$. In view of the above and keeping in mind that the Clayton MRF copulas are in fact survival copulas, i.e., they couple d.d.f.’s, in what follows we restrict our attention to the copula $C_{\xi}(u, v) := \mathbf{P}[(U, V) \in R(u, v)]$, where the rectangle $R(u, v) := [0, u] \times [0, v]$ ‘shrinks’ along the diagonal $\{(u, v) \in [0, 1] : u = v\}$ or along a more intricate path $(\varphi(u), \psi(u))_{0 \leq u, v \leq 1}$, where $u \downarrow 0$ and φ and ψ are eligible functions.

Speaking generally, there exist a variety of ways to quantify the extent of tail dependence in bivariate random vectors with dependence structures given by copula functions $C : [0, 1]^2 \rightarrow [0, 1]$ ([Durante and Sempi, 2015](#); [Nelsen, 2006](#)). Arguably the most popular measure of lower tail dependence is nowadays attributed to [Joe \(1993\)](#) (also [Sibuya, 1959](#))

and given by

$$\lambda_L := \lambda_L(C) = \lim_{u \downarrow 0} \frac{C(u, u)}{u}. \quad (4.5.1)$$

Non-zero (more precisely $(0, 1]$) values of (4.5.1) suggest lower tail dependence in C . Just like other synthetic measures, λ_L is not always reliable as it sometimes underestimates the extent of lower tail dependence in copulas as the next example demonstrates.

Example 4.5.1. Consider the following copula (Nelsen, 2006, Example 3.3)

$$C_\theta(u, v) = \begin{cases} u & 0 \leq u \leq \theta v \leq \theta, \\ \theta v, & 0 \leq \theta v < u < 1 - (1 - \theta)v, \\ u + v - 1, & \theta \leq 1 - (1 - \theta)v \leq u \leq 1, \end{cases}$$

parametrized by $\theta \in [0, 1]$. This copula has two singularities, and it is fully commonotonic (fully counter-commonotonic) for $\theta = 1$ ($\theta = 0$). It is easy to see that

$$\lambda_L(C_\theta) = \lim_{u \downarrow 0} \frac{\theta u}{u} = \theta.$$

Let $\lambda_L^*(C_\theta)$ be the measure as in (4.5.1) but now along the path $(u\sqrt{\theta}, u/\sqrt{\theta})_{0 \leq u \leq 1}$ rather than the diagonal (u, u) . Clearly in this case

$$\lambda_L^*(C_\theta) = \lim_{u \downarrow 0} \frac{C(u\sqrt{\theta}, u/\sqrt{\theta})}{u} = \sqrt{\theta} > \theta = \lambda_L(C_\theta),$$

for non-trivial $\theta \in (0, 1)$.

On a different note, when limit (4.5.1) is zero, it is often useful to turn to the somewhat more delicate index of weak tail dependence $\chi_L \in [-1, 1]$ (Coles et al., 1999; Fischer and Klein, 2007) that is given by

$$\chi_L := \chi_L(C) = \lim_{u \downarrow 0} \frac{2 \log u}{\log C(u, u)} - 1, \quad (4.5.2)$$

and/or to the index of intermediate tail dependence $\kappa_L := \kappa_L(C) \in [1, 2]$ (Ledford and Tawn, 1996) that solves the equation

$$C(u, u) = \ell(u)u^{\kappa_L} \quad \text{when } u \downarrow 0, \quad (4.5.3)$$

assuming that we can find a slowly varying at 0 function $\ell(u)$. Unfortunately, as shown in Furman et al. (2015b) and Example 4.5.2 below, κ_L can be misleading in a manner that is entirely similar to that of Example 4.5.1.

Speaking generally, indices (4.5.1), (4.5.2) and (4.5.3) may underestimate the amount of tail dependence in copulas that are symmetric or asymmetric, with or without singularities (Furman et al., 2015b). The reason is that all the aforementioned indices of lower tail dependence rely entirely on the behavior of copulas along their main diagonal $(u, u)_{0 \leq u \leq 1}$. However, the tail dependence of copulas can be substantially stronger along the paths other than the main diagonal. This can be a serious disadvantage, as reported by, e.g., Schmid and Schmidt (2007), Zhang (2008), Li et al. (2014), and Furman et al. (2015b).

We next compute indices (4.5.1), (4.5.2) and (4.5.3) in the context of the Clayton subclass of MRF copulas. We recollect to this end, that similarly to the general MRF copulas, Clayton MRF copula functions admit default specifications with the exogenous r.f.'s having stochastically independent hitting times (idiosyncratic r.f.'s), fully comonotonic hitting times (systemic r.f.'s) and positively orthant dependent hitting times (general r.f.'s that are not in the aforementioned two groups).

Proposition 4.5.1. *In the context of the Clayton subclass of MRF dependencies, we have, for $1 \leq i \neq k \leq n$, that*

$$\lambda_L(C_{\xi}) = \begin{cases} 0, & \xi_{c,i,\overline{(i,k)}} \neq 0 \text{ or } \xi_{c,k,\overline{(i,k)}} \neq 0 \\ 2^{-\gamma_{c,(i,k)}}, & \xi_{c,i,\overline{(i,k)}} = \xi_{c,k,\overline{(i,k)}} = 0. \end{cases}.$$

Hence, the random default times $(\tau_i, \tau_k)'$ are lower tail dependent in the sense of (4.5.1) if the underlying default specification does not include idiosyncratic r.f.'s. Furthermore, the higher the contribution of the systemic r.f.'s is (higher values of $\alpha_{c,(i,k)}$ and thus, for fixed margins, lower values of $\gamma_{c,(i,k)}$), the more lower tail dependent the default times $(\tau_i, \tau_k)'$ are.

Proof. By (4.4.6), we have the limit

$$\lambda_L(C_\xi) = \lim_{u \downarrow 0} \frac{u^{\frac{\xi_{c,i}, \overline{(i,k)}}{\xi_{c,i}} + \frac{\xi_{c,k}, \overline{(i,k)}}{\xi_{c,k}} + \frac{\alpha_{c,(i,k)}}{\xi_{c,i} \wedge \xi_{c,k}}} \left(u^{-\frac{1}{\xi_{c,i}}} + u^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}}}{u}. \quad (4.5.4)$$

First consider the case when $\xi_{c,i}, \overline{(i,k)} \neq 0$ and/or $\xi_{c,k}, \overline{(i,k)} \neq 0$, and set without loss of generality $\xi_{c,i} < \xi_{c,k}$. Then the limit becomes

$$\begin{aligned} \lambda_L(C_\xi) &= \lim_{u \downarrow 0} \frac{u^{\frac{\xi_{c,i}, \overline{(i,k)}}{\xi_{c,i}} + \frac{\xi_{c,k}, \overline{(i,k)}}{\xi_{c,k}} + \frac{\alpha_{c,(i,k)}}{\xi_{c,i} \wedge \xi_{c,k}}} \left(u^{-\frac{1}{\xi_{c,i}}} + u^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}}}{u} \\ &= \lim_{u \downarrow 0} u^{-\frac{\gamma_{c,(i,k)}}{\xi_{c,i}} + \frac{\xi_{c,k}, \overline{(i,k)}}{\xi_{c,k}}} \left(u^{-\frac{1}{\xi_{c,i}}} + u^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}} \\ &= \lim_{u \downarrow 0} u^{\frac{\xi_{c,k}, \overline{(i,k)}}{\xi_{c,k}}} \left(1 + u^{-\frac{1}{\xi_{c,k}} + \frac{1}{\xi_{c,i}}} - u^{\frac{1}{\xi_{c,i}}} \right)^{-\gamma_{c,(i,k)}} = 0. \end{aligned}$$

In the other case, i.e., when both $\xi_{c,i}, \overline{(i,k)}$ and $\xi_{c,k}, \overline{(i,k)}$ are zero, we have that that $\xi_{c,i} = \xi_{c,k} = \gamma_{c,(i,k)} + \alpha_{c,(i,k)}$, where $1 \leq i \neq k \leq n$, hence limit (4.5.4) becomes

$$\begin{aligned} \lambda_L(C_\xi) &= \lim_{u \downarrow 0} \frac{u^{\frac{\alpha_{c,(i,k)}}{\xi_{c,i}}} \left(u^{-\frac{1}{\xi_{c,i}}} + u^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}}}{u} \\ &= \lim_{u \downarrow 0} u^{-\frac{\gamma_{c,(i,k)}}{\xi_{c,i}}} \left(2u^{-\frac{1}{\xi_{c,i}}} - 1 \right)^{-\gamma_{c,(i,k)}} \\ &= \lim_{u \downarrow 0} \left(2 - u^{\frac{1}{\xi_{c,i}}} \right)^{-\gamma_{c,(i,k)}} = 2^{-\gamma_{c,(i,k)}}. \end{aligned}$$

This completes the proof. □

Proposition 4.5.2. *Within the Clayton subclass of MRF dependencies, we have, for $1 \leq i \neq k \leq n$, that the index of weak lower tail dependence is given by*

$$\chi_L(C_{\xi}) = \frac{\xi_{c,(i,k)}}{\xi_{c,i} + \xi_{c,i,(i,k)}} \bigwedge \frac{\xi_{c,(i,k)}}{\xi_{c,k} + \xi_{c,k,(i,k)}}, \quad (4.5.5)$$

whereas the index of intermediate lower tail dependence is given by

$$\kappa_L(C_{\xi}) = 2 - \frac{\xi_{c,(i,k)}}{\xi_{c,i}} \bigwedge \frac{\xi_{c,(i,k)}}{\xi_{c,k}}. \quad (4.5.6)$$

Proof. We only need to prove (4.5.6), as the other formula follows from the relationship $\chi_L(C) = 2/\kappa_L(C) - 1$. Then, for $\xi_{c,i} < \xi_{c,k}$ and $1 \leq i \neq k \leq n$, we have, by (4.4.6) and for $u \in (0, 1)$, that

$$\begin{aligned} C_{\xi}(u, u) &= u^{\frac{\xi_{c,i,(i,k)} + \alpha_{c,(i,k)}}{\xi_{c,i}} + \frac{\xi_{c,k,(i,k)}}{\xi_{c,k}}} \left(u^{-\frac{1}{\xi_{c,i}}} + u^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}} \\ &= u^{\frac{\xi_{c,i,(i,k)} + \gamma_{c,(i,k)} + \alpha_{c,(i,k)}}{\xi_{c,i}} + \frac{\xi_{c,k,(i,k)}}{\xi_{c,k}}} \left(1 + u^{-\frac{1}{\xi_{c,k}} + \frac{1}{\xi_{c,i}}} - u^{\frac{1}{\xi_{c,i}}} \right)^{-\gamma_{c,(i,k)}} \\ &= u^{1 + \frac{\xi_{c,k,(i,k)}}{\xi_{c,k}}} \left(1 + u^{-\frac{1}{\xi_{c,k}} + \frac{1}{\xi_{c,i}}} - u^{\frac{1}{\xi_{c,i}}} \right)^{-\gamma_{c,(i,k)}}, \end{aligned}$$

which yields $\kappa_L(C_{\xi}) = 1 + \frac{\xi_{c,k,(i,k)}}{\xi_{c,k}}$. For $\xi_{c,i} \geq \xi_{c,k}$, we have by analogy that $\kappa_L(C_{\xi}) = 1 + \frac{\xi_{c,i,(i,k)}}{\xi_{c,i}}$, which establishes (4.5.6). This completes the proof. $\square$

We have already hinted that classic indices (4.5.1), (4.5.2) and (4.5.3) may underestimate the extent of tail dependence in copulas. In the next example, we elucidate this phenomenon in the context of the Clayton subclass of MRF dependencies.

Example 4.5.2. *Consider the bivariate Clayton MRF copula with $\gamma_{c,(i,k)} \equiv 0$ and $\xi_{c,i,(i,k)} \neq 0$ and $\xi_{c,k,(i,k)} \neq 0$. Then, by (4.4.6),*

$$C_{\alpha}(u_i, u_k) = u_i^{\frac{\xi_{c,i,(i,k)}}{\xi_{c,i}}} u_k^{\frac{\xi_{c,k,(i,k)}}{\xi_{c,k}}} \left(u_i^{\frac{\alpha_{c,(i,k)}}{\xi_{c,i}}} \wedge u_k^{\frac{\alpha_{c,(i,k)}}{\xi_{c,k}}} \right), \quad (u_i, u_k)' \in [0, 1]^2. \quad (4.5.7)$$

Appealing to Proposition 4.5.1 and 4.5.2, we readily have, for $1 \leq i \neq k \leq n$, that

$$\lambda_L(C_\alpha) = 0 \text{ and } \kappa_L(C_\alpha) = 2 - \frac{\alpha_{c,(i,k)}}{\xi_{c,i}} \bigwedge \frac{\alpha_{c,(i,k)}}{\xi_{c,k}}.$$

Then, denote by $\lambda_L^*(C_\alpha)$ and $\kappa_L^*(C_\alpha)$ two indices à la (4.5.1) and (4.5.3), respectively, but along an alternative than diagonal path, and let such path be the singularity path $(u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, u^{2\xi_{c,k}/(\xi_{c,i}+\xi_{c,k})})_{0 \leq u \leq 1}$. In this case, we readily obtain that

$$\lambda_L^*(C_\alpha) = 0 = \lambda_L(C_\alpha) \text{ and } \kappa_L^*(C_\alpha) = 2 - \frac{2\alpha_{c,(i,k)}}{\xi_{c,i} + \xi_{c,k}} \leq \kappa_L(C_\alpha),$$

where the equality holds only if $\xi_{c,i} = \xi_{c,k}$.

To summarize, the example shows that in the context of the Clayton MRF copulas, all classic indices of tail dependence may yield inappropriate (not maximal) measures of extreme default times' co-movements

If there existed a one word paradigm that could characterize the modern regulatory accords in financial risk management, then it would be 'prudence'. Indeed, regulators around the globe have been making tremendous efforts to convey the necessity of modelling the effects of 'low probability high severity risks' on the risk portfolios of insurance companies and banks. This was the vein of prudence, that motivated us to introduce the paths and indices of maximal tail dependence in copulas (Furman et al., 2015b).

Definition 4.5.1. A function $\varphi : [0, 1] \rightarrow [0, 1]$ is called admissible if it satisfies the following conditions:

(C1) $\varphi(u) \in [u^2, 1]$ for every $u \in [0, 1]$; and

(C2) $\varphi(u)$ and $u^2/\varphi(u)$ converge to 0 when $u \downarrow 0$.

Then the path $(\varphi(u), u^2/\varphi(u))_{0 \leq u \leq 1}$ is admissible whenever the function φ is admissible. Also, we denote by $\mathcal{A}$ the set of all admissible functions φ .

A number of observations are instrumental to clarify the definition. First, condition (C1) makes sure that both $\varphi(u) \in [0, 1]$ and $u^2/\varphi(u) \in [0, 1]$, whereas condition (C2) is motivated by the fact that we are interested in the behavior of the copula C near the lower-left vertex of its domain of definition. Second, it is clear that the function $\varphi_0(u) = u$, $u \in [0, 1]$ is admissible and yields the main diagonal $(u, u)_{0 \leq u \leq 1}$. Last but not least, for the independence copula, it holds that $C^\perp(\varphi(u), u^2/\varphi(u)) = u^2$, $u \in [0, 1]^2$, which is path-independent as expected, thus warranting the choice $\psi(u) = u^2/\varphi(u)$, $u \in [0, 1]$.

In order to determine the strongest extreme co-movements of risks for any copula C , we search for functions $\varphi \in \mathcal{A}$ that maximize the probability

$$\Pi_\varphi(u) = C(\varphi(u), u^2/\varphi(u)), \quad u \in (0, 1)$$

or, equivalently, the function

$$d_\varphi(C, C^\perp)(u) = C(\varphi(u), u^2/\varphi(u)) - C^\perp(\varphi(u), u^2/\varphi(u)), \quad u \in (0, 1),$$

which is non-negative for Positively Quadrant Dependent (PQD) ([Lehmann, 1966](#)) copulas C . Then an admissible function $\varphi^* \in \mathcal{A}$ is called a *function of maximal dependence* if

$$\Pi_{\varphi^*}(u) = \max_{\varphi \in \mathcal{A}} \Pi_\varphi(u) \tag{4.5.8}$$

for all $u \in (0, 1)$. The corresponding admissible path $(\varphi^*(u), u^2/\varphi^*(u))_{0 \leq u \leq 1}$ is called a *path of maximal dependence*. Generally speaking, the path φ^* is not unique, but for each such path the value of Π_{φ^*} is the same. In what follows, we use the notation $\Pi^*(u)$ instead of $\Pi_{\varphi^*}(u)$.

Prudent variants of measures (4.5.1), (4.5.2) and (4.5.3) are then introduced as

$$\lambda_L^* := \lambda_L^*(C) = \lim_{u \downarrow 0} \frac{\Pi^*(u)}{u}, \text{ instead of } \lambda_L(C) = \lim_{u \downarrow 0} \frac{C(u, u)}{u} \quad (4.5.9)$$

and

$$\chi_L^* := \chi_L^*(C) = \lim_{u \downarrow 0} \frac{2 \log u}{\log \Pi^*(u)} - 1, \text{ instead of } \chi_L = \lim_{u \downarrow 0} \frac{2 \log u}{\log C(u, u)} - 1, \quad (4.5.10)$$

subject to the existence of the limits, and also

$$\Pi^*(u) = \ell^*(u)u^{\kappa_L^*}, \quad u \downarrow 0, \text{ as opposed to } \Pi(u) = \ell(u)u^{\kappa_L}, \quad u \downarrow 0, \quad (4.5.11)$$

assuming that there exist slowly varying at zero functions $\ell^*(u)$ and $\ell(u)$ (Ledford and Tawn, 1996).

A useful technique for deriving function(s) of maximal dependence, and thus in turn of the corresponding indices, consists of three steps:

- (S1) search for critical points of the function $x \mapsto C(x, u^2/x)$ over the interval $[u^2, 1]$ and for each $u \in [0, 1]$;
- (S2) check which of the solution(s) is/are global maximum/maxima; and
- (S3) verify that the function $u \mapsto \varphi^*(u)$ is admissible.

We next formulate and prove the main result of this section. Figure 4.1 visualize some of the notions in it.

Theorem 4.5.1. *Consider the Clayton subclass of MRF copulas, thwn we have, for $1 \leq i \neq k \leq n$, that*

- *the index of maximal strong lower tail dependence is given by*

$$\lambda_L^*(C_\xi) = \lambda_L(C_\xi);$$

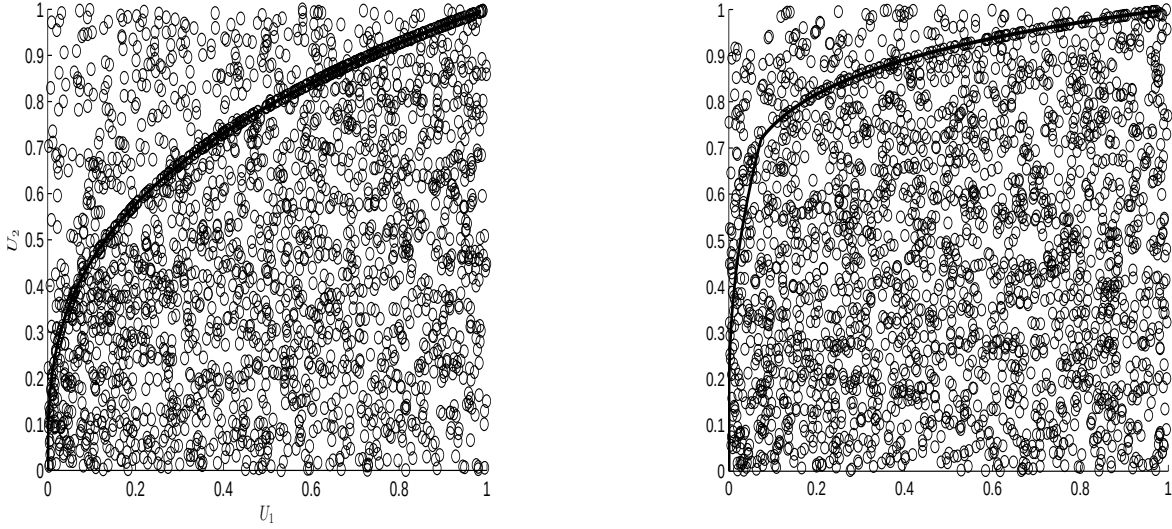


Figure 4.1: Scatterplots of the Clayton MRF copula for $\xi_{c,i,\overline{(i,k)}} = 3$, $\xi_{c,k,\overline{(i,k)}} = 0.3$, $\gamma_{c,(i,k)} = 0.5$, $\alpha_{c,(i,k)} = 0.6$ (left panel) and $\xi_{c,i,\overline{(i,k)}} = 10$, $\xi_{c,k,\overline{(i,k)}} = 0.3$, $\gamma_{c,(i,k)} = 0.5$, $\alpha_{c,(i,k)} = 0.6$ (right panel) with the paths of maximal dependence superimposed on both panels.

- the index of maximal weak lower tail dependence is given by

$$\chi_L^*(C_{\xi}) = \frac{\xi_{c,(i,k)}}{\xi_{c,i,\overline{(i,k)}} + \xi_{c,(i,k)} + \xi_{c,k,\overline{(i,k)}}};$$

- the index of maximal intermediate lower tail dependence is given by

$$\kappa_L^*(C_{\xi}) = 2 \left(1 - \frac{\xi_{c,(i,k)}}{\xi_{c,i,\overline{(i,k)}} + 2\xi_{c,(i,k)} + \xi_{c,k,\overline{(i,k)}}} \right).$$

Proof. Assume without loss of generality that $\xi_{c,i} > \xi_{c,k}$, $1 \leq i \neq k \leq n$, which is the case when the singularity curve of the Clayton MRF copula lies in the upper left section of its domain of definition. Also, let, for $1 \leq i \neq k \leq n$,

$$\delta_i := \frac{\xi_{c,i,\overline{(i,k)}} + \alpha_{c,(i,k)}}{\xi_{c,i}} - \frac{\xi_{c,k,\overline{(i,k)}}}{\xi_{c,k}} \text{ and } \delta_k := \frac{\xi_{c,i,\overline{(i,k)}}}{\xi_{c,i}} - \frac{\xi_{c,k,\overline{(i,k)}} + \alpha_{c,(i,k)}}{\xi_{c,k}},$$

then, by (4.4.6), we have that

$$C_{\xi}(x, u^2/x) \tag{4.5.12}$$

$$= \begin{cases} x^{\delta_i} u^{2(1-\xi_{c,(i,k)}/\xi_{c,k})} \left(x^{-\frac{1}{\xi_{c,i}}} + \left(\frac{u^2}{x} \right)^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}}, & x \leq u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})} \\ x^{\delta_k} u^{2(1-\gamma_{c,(i,k)}/\xi_{c,k})} \left(x^{-\frac{1}{\xi_{c,i}}} + \left(\frac{u^2}{x} \right)^{-\frac{1}{\xi_{c,k}}} - 1 \right)^{-\gamma_{c,(i,k)}}, & x > u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})} \end{cases},$$

and we are interested in the behaviour of (4.5.12) on the interval $[u^2, 1]$, which is to this end split into two intervals $[u^2, u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})})$ and $[u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, 1]$ with $u \in (0, 1)$.

For $x \in [u^2, u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})})$, we first note that

$$\frac{\partial}{\partial x} \ln C_{\xi}(x, u^2/x) = 0$$

if and only if

$$\zeta_i(x) := \left(\delta_i + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} \right) x^{-1/\xi_{c,i}} + \left(\delta_i - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \right) \left(\left(\frac{u^2}{x} \right)^{-1/\xi_{c,k}} - 1 \right) - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} = 0$$

or, equivalently, if and only if

$$\eta_i(x) := \zeta_i(x) x^{-\frac{1}{\xi_{c,k}}} = 0. \tag{4.5.13}$$

Equation (4.5.13) does not have solutions for $x \in (u^2, u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})})$ since $\eta_i(x)$ is non-increasing therein

$$\begin{aligned} \eta'_i(x) &= x^{-\frac{1}{\xi_{c,i}} - \frac{1}{\xi_{c,k}} - 1} \left(- \left(\delta_i + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} \right) \left(\frac{1}{\xi_{c,i}} + \frac{1}{\xi_{c,k}} \right) + \frac{\delta_i}{\xi_{c,k}} x^{\frac{1}{\xi_{c,i}}} \right) \\ &\leq -x^{-\frac{1}{\xi_{c,i}} - \frac{1}{\xi_{c,k}} - 1} \frac{\delta_i}{\xi_{c,i}} \leq 0 \end{aligned}$$

and such that

$$\begin{aligned} \eta_i(u^2) &= u^{-\frac{1}{\xi_{c,k}}} \left(\left(\delta_i + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} \right) u^{-2/\xi_{c,i}} - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \right) > 0; \\ \eta_i(u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}) &= u^{-\frac{2\xi_{c,i}}{\xi_{c,k}(\xi_{c,i}+\xi_{c,k})}} \left(\left(2\delta_i + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \right) u^{-\frac{2}{\xi_{c,i}+\xi_{c,k}}} - \delta_i \right) \\ &= u^{-\frac{2\xi_{c,i}}{\xi_{c,k}(\xi_{c,i}+\xi_{c,k})}} \left(\left(1 + \delta_i - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \right) u^{-\frac{2}{\xi_{c,i}+\xi_{c,k}}} - \delta_i \right) > 0. \end{aligned}$$

Hence, we conclude that $x \mapsto C_{\xi}(x, u^2/x)$ is strictly increasing on $(u^2, u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})})$ and cannot attain its maximum or maxima there.

Let us now turn to $x \in [u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, 1]$. We note that

$$\frac{\partial}{\partial x} \log(C_{\xi}(x, u^2/x)) = 0$$

if and only if

$$\zeta_k(x) := \left(\delta_k + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} \right) (x^{-1/\xi_{c,i}} - 1) + \left(\delta_k - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \right) \left(\frac{u^2}{x} \right)^{-1/\xi_{c,k}} + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} = 0$$

if and only if

$$\eta_k(x) := \zeta_k(x) x^{\frac{1}{\xi_{c,i}}} = 0. \quad (4.5.14)$$

Equation (4.5.14) may have at most one solution for $x \in (u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, 1)$ and $u \in (0, 1)$, as

$$\begin{aligned} \eta_k(1) &= - \left(\frac{\gamma_{c,(i,k)} + \alpha_{c,(i,k)}}{\xi_{c,i}} \right) u^{-2/\xi_{c,k}} + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} < 0; \\ \eta'_k(x) &= x^{\frac{1}{\xi_{c,i}}-1} \left(\left(\delta_k - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \right) \left(\frac{1}{\xi_{c,i}} + \frac{1}{\xi_{c,k}} \right) \left(\frac{u^2}{x} \right)^{-\frac{1}{\xi_{c,k}}} - \frac{\delta_k}{\xi_{c,k}} \right) \\ &= x^{\frac{1}{\xi_{c,i}}-1} \left(\left(-\frac{\xi_{c,(i,k)}}{\xi_{c,i}} \right) \left(\frac{1}{\xi_{c,i}} + \frac{1}{\xi_{c,k}} \right) \left(\frac{u^2}{x} \right)^{-\frac{1}{\xi_{c,k}}} - \frac{\delta_k}{\xi_{c,k}} \right) \leq 0; \end{aligned}$$

and the sign of

$$\begin{aligned} \eta_k(u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}) &= \left(\delta_k - \frac{\alpha_{c,(i,k)}}{\xi_{c,i}} \right) - \delta_k u^{\frac{2}{\xi_{c,i}+\xi_{c,k}}} \\ &= \left(\frac{\xi_{c,i,(i,k)} - \alpha_{c,(i,k)}}{\xi_{c,i}} - \frac{\xi_{c,k,(i,k)} + \alpha_{c,(i,k)}}{\xi_{c,k}} \right) - \delta_k u^{\frac{2}{\xi_{c,i}+\xi_{c,k}}} \\ &= \left(\frac{\xi_{c,i,(i,k)}\xi_{c,k} - \xi_{c,k,(i,k)}\xi_{c,i} - \alpha_{c,(i,k)}(\xi_{c,i} + \xi_{c,k})}{\xi_{c,i}\xi_{c,k}} \right) - \delta_k u^{\frac{2}{\xi_{c,i}+\xi_{c,k}}} \end{aligned}$$

is unknown. Consequently, we have that the function $x \mapsto C_{\xi}(x, u^2/x)$ may or may not achieve its maximum on the interval $(u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, 1)$, and there may be one such maximum, only.

To summarize, there are two possibilities:

(A.) (Figure 4.2, left panel) - the function $x \mapsto C_{\xi}(x, u^2/x)$ is strictly increasing on $(u^2, u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})})$ and strictly decreasing on $(u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, 1)$. Therefore its maximum is achieved at $x = u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}$, the function of maximum dependence is $\varphi^*(u) = u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}$ and the path of maximal dependence is $(u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, u^{2\xi_{c,k}/(\xi_{c,i}+\xi_{c,k})})$, where $u \in [0, 1]$. Also, the indices λ_L^* , κ_L^* and χ_L^* follow, respectively, from (4.5.9), (4.5.10) and (4.5.11).

(B.) (Figure 4.2, right panel) - the function $x \mapsto C_{\xi}(x, u^2/x)$ has its maximum in $(u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})}, 1)$. In this case, we cannot formulate the function of maximal tail dependence explicitly, and so the path of maximal tail dependence is unknown. Nevertheless, the indices of maximal tail dependence can be written in a closed form. The only non-trivial index is the index of maximal intermediate lower tail dependence. In this respect, we know that the function of maximal dependence exists, is unique and satisfies the equation $\zeta_k(x) = 0$, or equivalently

$$x = u^{2\xi_{c,i}/(\xi_{c,i}+\xi_{c,k})} r(x), \quad (4.5.15)$$

where

$$r(x) = \left(\left(\delta_k - \frac{\gamma_{c,(i,k)}}{\xi_{c,k}} \right) / \left(\delta_k x^{\frac{1}{\xi_{c,i}}} - \left(\delta_k + \frac{\gamma_{c,(i,k)}}{\xi_{c,i}} \right) \right) \right)^{-\frac{\xi_{c,i}\xi_{c,k}}{\xi_{c,i}+\xi_{c,k}}}.$$

Then the substitution of (4.5.15) into (4.5.12) yields

$$C_{\xi}(\varphi^*(u), u^2/\varphi^*(u)) = u^{1+(\xi_{c,i,\overline{(i,k)}}+\xi_{c,k,\overline{(i,k)}})/(\xi_{c,i}+\xi_{c,k})} s(u),$$

where the function $s(u)$ is such that $\lim_{u \downarrow 0} s(u) = \text{const}(\in \mathbf{R}_+)$. The index of maximal weak lower tail dependence is obtained from the relationship $\xi_L^* = 2/\kappa_L^* - 1$. Finally, the index of maximal strong lower tail dependence is non-zero if and only if $\kappa_L^*(C_{\xi}) \equiv 1$, which in the context of the Clayton MRF copulas implies exchangeability as in this case $\xi_{c,i,\overline{(i,k)}} \equiv 0$ and $\xi_{c,k,\overline{(i,k)}} \equiv 0$. This completes the proof. $\square$

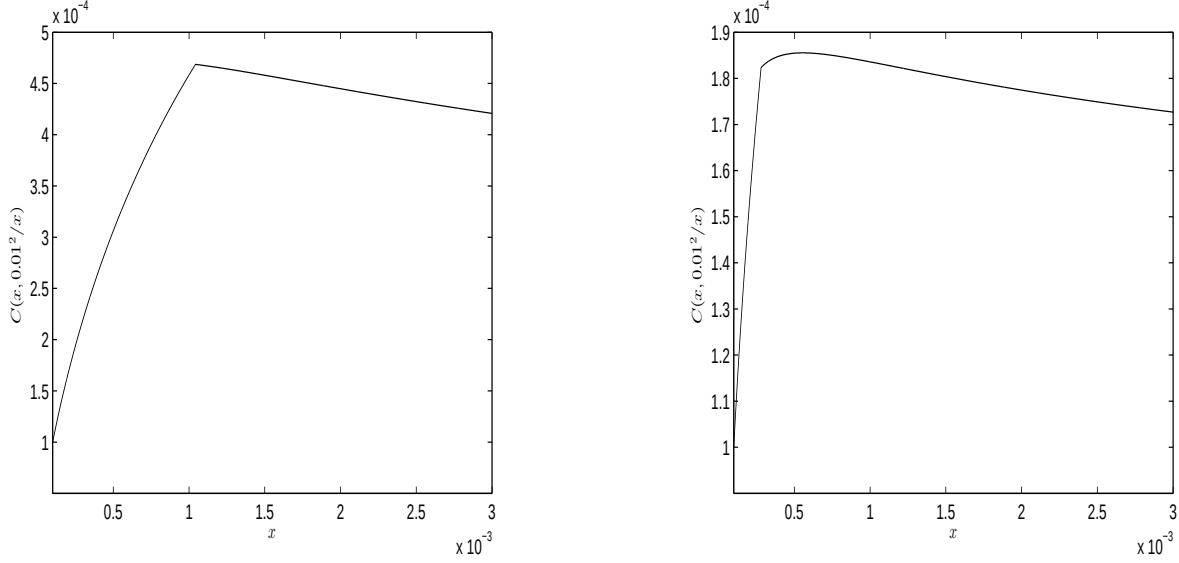


Figure 4.2: The function $C(x, 0.01^2/x)$ for $\xi_{c,i,\overline{(i,k)}} = 3$, $\xi_{c,k,\overline{(i,k)}} = 0.3$, $\gamma_{c,(i,k)} = 0.5$, $\alpha_{c,(i,k)} = 0.6$ (left panel) and $\xi_{c,i,\overline{(i,k)}} = 10$, $\xi_{c,k,\overline{(i,k)}} = 0.3$, $\gamma_{c,(i,k)} = 0.5$, $\alpha_{c,(i,k)} = 0.6$ (right panel).

4.6 Conclusions

Copulas have become an important element of best practice ERM, displacing in many contexts other more conservative approaches to modelling stochastic dependence. Choosing the ‘right’ copula is however not an easy call, and the temptation to make the decision in favour of a tractable rather than a meaningful copula is high. The use of the Gaussian copula to price the collateralized debt obligations (Li, 2000) is an illuminating example.

A ‘good’ copula should conform to a problem at hand, be asymmetric and exhibit some extent of tail dependence (Staudt, 2010). The MRF copulas that we have introduced and studied in this paper are exactly such. Namely, they admit meaningful in the context of the ERM stochastic representations, arise from a number of default risk specifications with stochastic default barrier and are in general not symmetric in their domain of definition

as well as in the corresponding tail behaviour. The MRF copulas cover the full range of non-negative dependence and are yet quite tractable analytically.

Among immediate applications, the MRF copulas generalize the CreditRisk⁺ framework by augmenting systemic risk factors having fully comonotonic hitting times, thus making the overall default times more positively orthant dependent. As the CreditRisk⁺ method has been criticized for underestimating the clustering of real world defaults, we believe that the MRF copulas may serve as a reasonable substitution. Last but not least, as the notion of systemic risk is of fundamental importance in the new era of prudent risk management, we think that the MRF copulas may be of interest for modelling general dependent (insurance) risks, well beyond the context of credit risk.

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Chapter 5

Paths and indices of maximal tail dependence

Motivated by the asymmetric dependencies studied in Chapters 2 to 4 , the following paper introduces a new notion of Maximal Dependence Paths and then employs it to measure tail dependence in copulas. This chapter has been published in ASTIN Bulletin with Edward Furman and Ričardas Zitikis.

PATHS AND INDICES OF MAXIMAL TAIL DEPENDENCE

Abstract. We demonstrate both analytically and numerically that the existing methods for measuring tail dependence in copulas may sometimes underestimate the extent of extreme co-movements of dependent risks and, therefore, may not always comply with the new paradigm of prudent risk management. This phenomenon holds in the context of both symmetric and asymmetric copulas with and without singularities. As a remedy, we introduce a notion of *paths of maximal (tail) dependence* and utilize the notion to propose several new indices of tail dependence. The suggested new indices are conservative, conform with the basic concepts of modern quantitative risk management, and are capable of differentiating between distinct risky positions in situations when the existing indices fail to do so.

Keywords and phrases: multivariate distribution; copula; tail dependence; maximal dependence; fatal shock; multivariate Pareto; enterprise risk management.

5.1 Introduction

Current regulatory frameworks require enhanced techniques for measuring and managing extremal risks of financial enterprises. In particular, this may involve

- 1) analyses of marginal risks and dependence structures between them (cf., e.g., [McNeil et al., 2005](#); [Jaworski et al., 2010](#); [Jaworski et al., 2013](#); [Durante et al., 2015](#); and references therein); and
- 2) behaviour of aggregated risks, such as sums (cf., e.g., [Asimit and Badescu, 2010](#); [Asimit et al., 2011](#); [Embrechts et al., 2013](#); [Embrechts et al., 2014](#); [Puccetti and Rüschendorf, 2014](#); and references therein).

In many cases, both routes are feasible and complementary to each other. However, there are situations when only one of them is desirable and thus being pursued. We refer to, for example, [Churchill and Matul \(2012\)](#), [Biener and Eling \(2012\)](#), and references therein, for situations in micro-insurance where aggregation is not feasible.

Dependence is quite often modelled using copulas, which have become a well established mathematical tool in actuarial and financial research and practice (cf., e.g., [Frees et al., 1996](#); [Frees and Wang, 2006](#); [McNeil et al., 2005](#); [Embrechts, 2009](#); [Frees, 2009](#); [Jaworski et al., 2010](#); [Jaworski et al., 2013](#); and references therein), as well as in many other areas such as economics (cf., [Patton, 2012](#)), medicine (cf., [Nikoloulopoulos and Karlis, 2008](#)), reliability engineering and life sciences (cf., e.g., [Balakrishnan and Lai, 2009](#)), and so on. Briefly, a bivariate function $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a copula if it is grounded, two-increasing, and have uniform marginals (cf., [Nelsen, 2006](#)).

When quantifying co-movements of extreme risks, the behaviour of C around the upper-right and lower-left vertices of the domain of definition $[0, 1] \times [0, 1]$ gives rise to indices of upper and lower tail dependence. While these indices are concerned with phenomena in opposing risk-tails, they can be formally unified by shifting attention from the underlying copula to the corresponding survival copula. For example, given the index of lower-tail dependence $\lambda_L := \lambda_L(C)$, which is

$$\lambda_L = \lim_{u \downarrow 0} \frac{C(u, u)}{u}, \quad (5.1.1)$$

the corresponding index of upper-tail dependence $\lambda_U := \lambda_U(C)$ is

$$\lambda_U = \lim_{u \downarrow 0} \frac{1 - 2(1 - u) + C(1 - u, 1 - u)}{u} = \lim_{u \downarrow 0} \frac{\widehat{C}(u, u)}{u},$$

where $\widehat{C}$ is the survival copula of C ; hence, $\lambda_U(C) = \lambda_L(\widehat{C})$. (Throughout the paper we use ‘:=’ instead of the customary equality sign ‘=’ when we want to emphasize that equality is ‘*by definition.*’)

Similar arguments hold for other indices of tail dependence, such as the index of weak lower-tail dependence $\chi_L := \chi_L(C)$ given by

$$\chi_L = \lim_{u \downarrow 0} \frac{2 \log u}{\log C(u, u)} - 1, \quad (5.1.2)$$

and its upper variant $\chi_U := \chi_U(C) = \chi_L(\widehat{C})$ (cf., [Coles et al., 1999](#); [Heffernan, 2000](#); [Fischer and Klein, 2007](#); and references therein). In view of these notes, we restrict our following considerations to the behaviour of copulas near the lower-left vertex of their domain of definition.

Instead of comparing $C(u, u)$ and $u \in (0, 1)$ as in limits (5.1.1) and (5.1.2), we can more generally explore the asymptotic behaviour of $C(u, u)$ in the form

$$C(u, u) = \ell(u)u^{\kappa_L} \quad \text{when } u \downarrow 0, \quad (5.1.3)$$

which defines the index $\kappa_L := \kappa_L(C) \in [1, \infty)$ of lower-tail dependence, assuming of course that equation (5.1.3) holds for a slowly varying at 0 function $\ell(u)$ (cf., [Ledford and Tawn, 1996](#)). We note that smaller values of κ_L correspond to stronger interdependences, thus motivating the search for the smallest possible analogues of κ_L , and this is our main goal in the present paper.

The rest of the paper is organized as follows. In Section 5.2 we describe the main framework for our ideas, including the class of admissible dependence paths and its subclass of maximal dependence paths. In Section 5.3 we present a numerical example that shows that the existing indices of tail dependence may underestimate the extent of extreme co-movements of dependent risks. In Section 5.4 we explore several families of copulas for which paths of maximal dependence can be derived in closed form, whereas somewhat more intricate examples are given in Section 5.6. In Section 5.5, we further extend our general framework by introducing new tail orderings of distinct copulas. Section 5.7 concludes the paper.

5.2 Paths of maximal dependence and related indices

In what follows we work with bivariate copulas $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ only, because they convey the main idea of the present paper in a simple and illuminating way. With the notation $R(u, v) = [0, u] \times [0, v]$, the copula-value $C(u, v)$ is the probability that the bivariate random vector (U, V) with uniform marginals U and V falls into the rectangle $R(u, v)$, that is,

$$C(u, v) = \mathbf{P}[(U, V) \in R(u, v)]. \quad (5.2.1)$$

The class of all rectangles $R(u, v)$ contains the subclass of all squares $R(u, u)$, and these are the ones that have traditionally been used for measuring the strength of tail dependence. Namely, the classical indices of lower-tail dependence are based on the behaviour of probability (5.2.1) when the rectangle $R(u, v)$ shrinks along the diagonal

$$\{(u, v) \in [0, 1]^2 : u = v\} \quad (5.2.2)$$

when $u \downarrow 0$. (Throughout the paper, we work only with diagonal (5.2.2) and thus call it *the* diagonal.) In the context of the present paper, the diagonal is a path of (tail) dependence, but there are of course many other possible paths and we shall next describe them.

In order to make the problem well posed, some restrictions on the class of possible paths must be imposed. First, we observe that for the independence copula, it is natural to require that every path would reflect the same degree of tail dependence, that is, would have the same probability (5.2.1). For two functions $\varphi, \psi : [0, 1] \rightarrow [0, 1]$, this implies that every path $(\varphi(u), \psi(u))_{0 \leq u \leq 1}$ has to be necessarily of the form $(\varphi(u), u^2/\varphi(u))_{0 \leq u \leq 1}$, that is, we must have $\psi(u) = u^2/\varphi(u)$. Certainly, both $\varphi(u)$ and $u^2/\varphi(u)$ must be in the interval $[0, 1]$, and thus $\varphi(u)$ must be in the interval $[u^2, 1]$, which justifies the first property of the following definition.

Definition 5.2.1. We call a function $\varphi : [0, 1] \rightarrow [0, 1]$ admissible if it satisfies the following two properties:

- (1) $\varphi(u) \in [u^2, 1]$ for every $u \in [0, 1]$; and
- (2) both $\varphi(u)$ and $u^2/\varphi(u)$ converge to 0 when $u \downarrow 0$.

We call the path $(\varphi(u), u^2/\varphi(u))_{0 \leq u \leq 1}$ admissible whenever the function φ is admissible. Finally, we use $\mathcal{A}$ to denote the set of all admissible functions φ .

The second property of Definition 5.2.1 is related to the fact that we are interested in the behaviour of copulas near the lower-left vertex of their domain of definition.

Motivated by the idea of determining the strongest extreme co-movements of risks, among all admissible functions $\varphi \in \mathcal{A}$, we search for those that maximize the probability

$$\Pi_{\varphi}(u) = C(\varphi(u), u^2/\varphi(u)) \quad (5.2.3)$$

or, equivalently, the distance function

$$d_{\varphi}(C, C^{\perp})(u) = C(\varphi(u), u^2/\varphi(u)) - C^{\perp}(\varphi(u), u^2/\varphi(u)), \quad (5.2.4)$$

where $C^{\perp}$ is the independence copula, i.e., $C^{\perp}(u, v) = uv$ for all $0 \leq u, v \leq 1$. Obviously, the function $\varphi_0(u) = u$ is admissible and yields the representation of the diagonal path that serves as a building block for the classical indices of lower-tail dependence. However, this path may not maximize probability (5.2.3), as we illustrate in following Example 5.2.1. In fact, in view of (5.2.4), the classical index κ_L may serve as neither a maximal nor minimal measure of tail dependence of the copula C .

Example 5.2.1. Consider the Marshall-Olkin copula

$$C_{a,b}(u, v) = \min(u^{1-a}v, uv^{1-b}) \quad \text{for } 0 \leq u, v \leq 1, \quad (5.2.5)$$

where $a, b \in [0, 1]$ are parameters. We check that $C_{a,b}(u, u) = \min(u^{2-a}, u^{2-b})$ and thus

$$\Pi_{\varphi_0}(u) = u^{\kappa_L} \quad \text{with } \kappa_L = 2 - \min\{a, b\}.$$

Next we choose the admissible function $\varphi_1(u) = u^{2b/(a+b)}$. See Figure 5.1 for the corresponding path, which we later show to be the (only) path of maximal dependence. We check that

$$\Pi_{\varphi_1}(u) = u^{\kappa_L^*} \quad \text{with } \kappa_L^* = 2 - \frac{2ab}{a+b}. \quad (5.2.6)$$

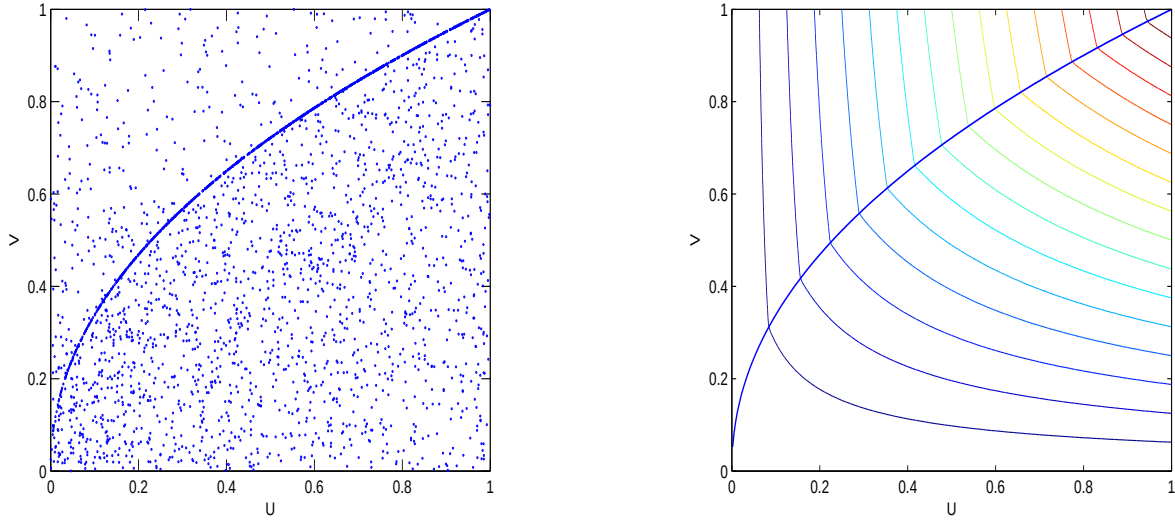


Figure 5.1: Simulated Marshall-Olkin copula (left) and its contours (right) when $a = 0.3529$ and $b = 0.75$, with the path of maximal dependence $(\varphi_1(u), u^2/\varphi_1(u))_{0 \leq u \leq 1}$ superimposed on the right-hand panel.

Clearly, $\kappa_L^* \leq \kappa_L$ with the equality holding only when $a = b$, that is, when the Marshall-Olkin copula is symmetric.

This example motivates the following definition of a subclass of admissible functions.

Definition 5.2.2. Given a copula C , an admissible function $\varphi^* \in \mathcal{A}$ is called a function of maximal dependence if

$$\Pi_{\varphi^*}(u) = \max_{\varphi \in \mathcal{A}} \Pi_{\varphi}(u) \quad \text{for all } u \in [0, 1].$$

We conveniently use the simpler notation $\Pi^*(u)$ instead of $\Pi_{\varphi^*}(u)$, but may enhance it to $\Pi^*(u \mid C)$ when a need arises to emphasize the dependence of $\Pi^*(u)$ on C . Also, we refer to $(\varphi^*(u), u^2/\varphi^*(u))_{0 \leq u \leq 1}$ as a path of maximal dependence.

A useful technique for deriving function(s) and thus path(s) of maximal dependence –

and we have frequently used it in our explorations – is based on:

- searching for critical points of the function $x \mapsto C(x, u^2/x)$ over the interval $[u^2, 1]$, and for each $u \in [0, 1]$, and then
- checking which of the solutions is/are global maximum/maxima.

Accomplishing these tasks may sometimes be relatively easy (Section 5.4), sometimes challenging (Furman et al., 2015), and in some cases obtaining closed-form solutions may not even be possible (Section 5.6). Sometimes, especially when formulas for conditional copulas are readily available, it is useful to recall that partial derivatives of copulas are conditional copulas, and thus the task of determining the set of critical points becomes equivalent to finding all the solutions in $x \in [u^2, 1]$ to the equation

$$xC_{2|1}\left(\frac{u^2}{x} \mid x\right) = \frac{u^2}{x}C_{1|2}\left(x \mid \frac{u^2}{x}\right). \quad (5.2.7)$$

For example, equation (5.2.7) has played a pivotal role when handling the Gaussian copula by Furman et al. (2015).

Functions of maximal dependence may or may not be continuous, and this is related to uniqueness of the path of maximal dependence. To demonstrate this fact, we start with the following theorem.

Theorem 5.2.1. *If a copula has only one function of maximal dependence, then the function is continuous.*

Proof. We prove by contradiction. Hence, assume that there is a point u_0 of discontinuity of the function φ^* of maximal dependence. This implies that there is a sequence $u_n \rightarrow u_0$ such that $\varphi^*(u_n) \rightarrow x_0 \neq \varphi^*(u_0)$. By the continuity of the function $(v, w) \mapsto C(v, w)$ (cf.,

e.g., [Embrechts et al., 2003](#); [Nelsen, 2006](#)), we have

$$C(\varphi^*(u_n), u_n^2/\varphi^*(u_n)) \rightarrow C(x_0, u_0^2/x_0). \quad (5.2.8)$$

By the uniqueness of the function of maximal dependence, x_0 cannot maximize the function $x \mapsto C(x, u_0^2/x)$ because otherwise x_0 would be equal to $\varphi^*(u_0)$. On the other hand, we have

$$C(\varphi^*(u_n), u_n^2/\varphi^*(u_n)) = \max_{x \in [u_n^2, 1]} C(x, u_n^2/x) \rightarrow \max_{x \in [u_0^2, 1]} C(x, u_0^2/x), \quad (5.2.9)$$

where the limit holds due to the following reasons. When $u_n < u_0$, then

$$\begin{aligned} \max_{x \in [u_n^2, 1]} C(x, u_n^2/x) &= \max \left\{ \max_{x \in [u_n^2, u_0^2]} C(x, u_n^2/x), \max_{x \in [u_0^2, 1]} C(x, u_n^2/x) \right\} \\ &\rightarrow \max \left\{ u_0^2, \max_{x \in [u_0^2, 1]} C(x, u_0^2/x) \right\} \\ &= \max_{x \in [u_0^2, 1]} C(x, u_0^2/x), \end{aligned}$$

where the right-most equation holds due to $u_0^2 = C(x, u_0^2/x)$ for $x = 1$. When $u_n > u_0$, then

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left| \max_{x \in [u_n^2, 1]} C(x, u_n^2/x) - \max \left\{ \max_{x \in [u_0^2, u_n^2]} C(x, u_0^2/x), \max_{x \in [u_n^2, 1]} C(x, u_0^2/x) \right\} \right| \\ &= \lim_{n \rightarrow \infty} \left| \max_{x \in [u_n^2, 1]} C(x, u_n^2/x) - \max \left\{ u_0^2, \max_{x \in [u_n^2, 1]} C(x, u_0^2/x) \right\} \right| \\ &= \lim_{n \rightarrow \infty} \left| \max_{x \in [u_n^2, 1]} C(x, u_n^2/x) - \max_{x \in [u_n^2, 1]} C(x, u_0^2/x) \right| \\ &\leq \lim_{n \rightarrow \infty} \max_{x \in [u_n^2, 1]} |C(x, u_n^2/x) - C(x, u_0^2/x)| = 0 \end{aligned}$$

because of the uniform continuity of the function $(v, w) \mapsto C(v, w)$ (cf., e.g., [Embrechts et al., 2003](#); [Nelsen, 2006](#)). This completes the proof of limit (5.2.9). Combining statements (5.2.8) and (5.2.9), we get the equation

$$\max_{x \in [u_0^2, 1]} C(x, u_0^2/x) = C(x_0, u_0^2/x_0), \quad (5.2.10)$$

which implies that x_0 must be equal to $\varphi^*(u_0)$ because of the uniqueness of the maximum. We have arrived at a contradiction, which establishes the continuity of φ^* and completes the proof of Theorem 5.2.1. $\square$

For copulas with at least two paths of maximal dependence, there are automatically infinitely many paths of maximal dependence, as has been very rightly noted by one of the referees of the current paper: given two functions φ_1^* and φ_2^* of maximal dependence, their combination $\varphi_A^*(u) := \varphi_1^*(u)\mathbf{1}_A(u) + \varphi_2^*(u)\mathbf{1}_{A^c}(u)$ is a function of maximal dependence for any $A \subset [0, 1]$, where $\mathbf{1}_A$ and $\mathbf{1}_{A^c}$ are the indicator functions of the set A and its complement A^c , respectively. Of course, functions such as φ_A^* are perfectly legitimate and the probability of maximal dependence will be the same irrespective of the chosen path: $\Pi_{\varphi_A^*} = \Pi_{\varphi_1^*} = \Pi_{\varphi_2^*}$. Consequently, the new indices of tail dependence that we introduce next do not depend on the chosen path of maximal dependence.

We now use Definition 5.2.2 to introduce three conservative variants of the classical indices λ_L , χ_L and κ_L . Namely, assuming that the limits below exist, we set

$$\lambda_L^* := \lambda_L^*(C) = \lim_{u \downarrow 0} \frac{\Pi^*(u)}{u}, \quad (5.2.11)$$

$$\chi_L^* := \chi_L^*(C) = \lim_{u \downarrow 0} \frac{2 \log u}{\log \Pi^*(u)} - 1,$$

and let $\kappa_L^* := \kappa_L^*(C)$ be such that

$$\Pi^*(u) = \ell^*(u)u^{\kappa_L^*} \quad \text{when } u \downarrow 0 \quad (5.2.12)$$

for a slowly varying at 0 function $\ell^*(u)$, assuming that such a function exists. In our illustrative examples below, we concentrate on calculating κ_L^* only.

We conclude this section by noting that this paper is not the first attempt to abandon the diagonal section of copulas when measuring tail dependence. For example, in their research of

large claims reinsurance, [Asimit and Jones \(2008\)](#) rely on the asymptotic behaviour of copulas along non-diagonal paths. [Asimit and Badescu \(2010\)](#) rely on the extreme behaviour of copulas along a variety of paths when exploring a time dependent risk model with dependent inter-claim times and claim amounts. [Joe et al. \(2010\)](#) introduce the tail dependence function $b(w_1, w_2; C) = \lim_{u \downarrow 0} C(uw_1, uw_2)/u$ for $w_1 > 0$ and $w_2 > 0$. We refer to [Asimit et al. \(2011\)](#), [Weng and Zhang \(2012\)](#), and [Li and Wu \(2013\)](#) for further developments on the topic. [Hua and Joe \(2014\)](#) use the excess-of-loss economic pricing functional to propose and study a measure of tail dependence that does not rely on the diagonal dependence path. There are also several other related works but none of them – due to different research goals – aim at maximal-dependence paths and, in turn, at indices of maximal dependence.

5.3 Numerical illustration

Here we present a numerical example that questions the decisive role of the diagonal path in measuring tail dependence in copulas. We nevertheless stress at the outset that our discussion is not a criticism of the role of the diagonal when investigating copulas – it does play a pivotal role in the analysis of a variety of other aspects as elucidated by, e.g., [Durante et al. \(2015\)](#) and references therein.

Let X and Y be two random variables, which we assume to follow the same Pareto-II (also known as Lomax) distribution. Its decumulative distribution and probability density functions are given by, respectively,

$$\bar{F}(x) = \left(\frac{x - \mu}{\sigma} + 1 \right)^{-\alpha} \quad \text{and} \quad f(x) = \frac{\alpha}{\sigma} \left(\frac{x - \mu}{\sigma} + 1 \right)^{-(\alpha+1)} \quad (5.3.1)$$

for all $x \geq \mu \in (-\infty, \infty)$. Throughout this section, we set $\mu = 0$ and $\sigma = 1$ for simplicity, and set $\alpha = 4$ to ensure the finiteness of all quantities that we consider.

Let the dependence structure between X and Y be given by the Marshall-Olkin copula; see equation (5.2.5). Hence, the joint cumulative distribution function of X and Y is equal to $C_{a,b}(F(x), F(y)), 0 \leq x, y < \infty$ (for applications of this model to insurance, we refer to, e.g., Asimit et al., 2010). Following Embrechts et al. (2003), we set $a = 0.3529$. For several values of $b \in [0, 1]$, we calculate

- (i) Kendall's $\tau = \tau(C_{a,b})$ index of dependence
- (ii) classical index $\kappa_L = \kappa_L(C_{a,b})$ of lower-tail dependence given by equation (5.1.3)
- (iii) newly suggested index $\kappa_L^* = \kappa_L^*(C_{a,b})$ defined by equations (5.2.6) and (5.2.12)
- (iv) value-at-risk $\text{VaR}_q[Z] = \inf\{x \in \mathbf{R} : F_Z(x) \geq q\}$ for $Z = X + Y$
- (v) conditional tail expectation $\text{CTE}_q[Z] = \mathbf{E}[Z | Z > \text{VaR}_q[Z]]$
- (vi) modified tail variance

$$\text{MTVar}_q[Z] = \text{CTE}_q[Z] + \frac{1}{\text{CTE}_q[Z]} \mathbf{Var}[Z | Z > \text{VaR}_q[Z]]$$

where $q \in (0, 1)$. When conducting our numerical calculations, we set q to 0.990 and 0.995 in all of the aforementioned weighted risk measures (cf., e.g., Furman and Zitikis, 2010; and references therein). We have summarized the results in Table 5.1.

Noting that smaller values of κ_L^* mean stronger tail dependence, it is illuminating to observe from Table 5.1 that the smaller the values of κ_L^* are, the larger the values of VaR, CTE and MTVar are, whereas the classical index κ_L does not change.

Table 5.1: Quantities (i)–(vi) for the Marshall-Olkin copula with Pareto-II marginals.

Parameters		Indices of dependence			Risk measures		
q	b	τ	κ_L	κ_L^*	$\text{VaR}_q[Z]$	$\text{CTE}_q[Z]$	$\text{MTVar}_q[Z]$
0.9900	0.7500	0.3158	1.6471	1.5200	3.4621	4.8599	5.5808
	0.5000	0.2609	1.6471	1.5862	3.4095	4.7606	5.4691
	0.3529	0.2143	1.6471	1.6471	3.3612	4.6926	5.3951
0.9950	0.7500	0.3158	1.6471	1.5200	4.2925	5.8976	6.7004
	0.5000	0.2609	1.6471	1.5862	4.2114	5.7782	6.5552
	0.3529	0.2143	1.6471	1.6471	4.1460	5.6801	6.4268

5.4 Examples

We start with several families of copulas for which paths of maximal dependence are derivable in closed form and with moderate amount of effort.

5.4.1 Marshall-Olkin copula

Recall that the Marshall-Olkin copula is defined by formula (5.2.5). Next, for every $u \in [0, 1]$, the function $x \mapsto C_{a,b}(x, u^2/x)$ defined on the interval $[u^2, 1]$ is equal to $u^{2(1-b)}x^b$ for all $x \leq x_0 = u^{2b/(a+b)}$ and u^2/x^a for all $x \geq x_0$. Hence, the unique maximum of the function is achieved at the point $x = x_0$, and thus the function of maximal dependence of the Marshall-Olkin copula is unique and given by

$$\varphi^*(u) = u^{2b/(a+b)}. \quad (5.4.1)$$

Consequently, the maximal probability is

$$\Pi^*(u) = u^{2-2ab/(a+b)}, \quad (5.4.2)$$

and thus the lower-tail index of maximal dependence is

$$\kappa_L^* = 2 - \frac{2ab}{a+b}. \quad (5.4.3)$$

5.4.2 Mixture of Marshall-Olkin copulas

We see from formula (5.4.1) that the path of maximal dependence is diagonal if and only if $a = b$, and thus if and only if the Marshall-Olkin copula is symmetric. We next show that this fact cannot be generalized to arbitrary symmetric copulas. Namely, there are symmetric copulas whose paths of maximal dependence are not diagonal. To show this, we use the 0.5/0.5 mixture of two mirrored (around the diagonal) Marshall-Olkin copulas, that is,

$$C(u, v) = \frac{1}{2} \left(C_{a,b}(u, v) + C_{b,a}(u, v) \right) \quad \text{for } 0 \leq u, v \leq 1, \quad (5.4.4)$$

with some $a \neq b$. The copula $C(u, v)$ is symmetric. Tedious calculations which will be given later on show that there are two paths of maximal dependence (Figure 5.2) given by the functions

$$\varphi_1^*(u) = u^{2b/(a+b)} \quad \text{and} \quad \varphi_2^*(u) = u^{2a/(a+b)}. \quad (5.4.5)$$

Hence, none of the paths of maximal dependence coincides with the diagonal because $a \neq b$, and for any of the two paths, the maximal probability is

$$\begin{aligned} \Pi^*(u) &= \frac{1}{2} u^{2-2ab/(a+b)} \left(1 + u^{2(ab-\min\{a,b\}^2)/(a+b)} \right) \\ &= \frac{1}{2} u^{2-2ab/(a+b)} (1 + o(1)) \quad \text{when } u \downarrow 0. \end{aligned} \quad (5.4.6)$$

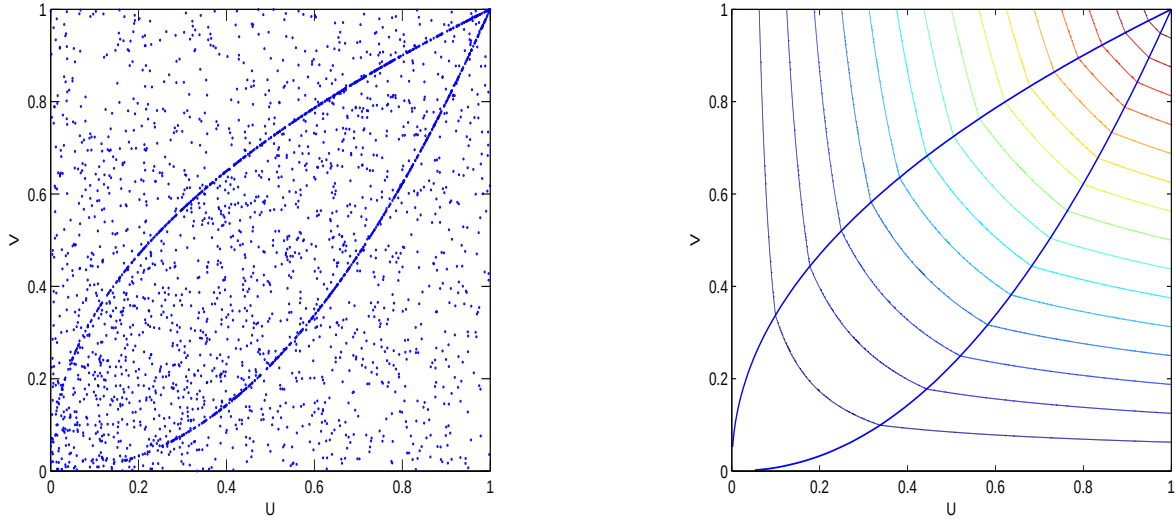


Figure 5.2: Simulated mixture of two Marshall-Olkin copulas (left) and its contours (right) when $a = 0.3529$ and $b = 0.75$, with the two paths of maximal dependence $(\varphi_k^*(u), u^2/\varphi_k^*(u))_{0 \leq u \leq 1}$, $k = 1, 2$, superimposed on the right-hand panel.

Hence, the lower-tail index of maximal dependence is

$$\kappa_L^* = 2 - \frac{2ab}{a+b}, \quad (5.4.7)$$

which is the same as in equation (5.4.3) for the ordinary Marshall-Olkin copula.

Proof. Without loss of generality, let $a < b$. Then $x_{a,b} = u^{2b/(a+b)}$ is smaller than $x_{b,a} = u^{2a/(a+b)}$, and so the two functions $x \mapsto C_{a,b}(x, u^2/x)$ and $x \mapsto C_{b,a}(x, u^2/x)$ are increasing on the interval $(u^2, x_{a,b})$ and decreasing on $(x_{b,a}, 1)$. Hence, the maximum of the function $x \mapsto C(x, u^2/x)$ can only be achieved on the interval $[x_{a,b}, x_{b,a}]$, where we have the formula

$$C(x, u^2/x) = \frac{1}{2} \left(u^{2(1-a)} x^a + u^2/x^a \right).$$

We split the interval $[x_{a,b}, x_{b,a}]$ into two subintervals: $[x_{a,b}, u]$ where the function $x \mapsto C(x, u^2/x)$ is decreasing, and $[u, x_{b,a}]$ where the function is increasing. From this we conclude

that the function $x \mapsto C(x, u^2/x)$ can achieve its maximum only at $x_{a,b}$ and/or $x_{b,a}$, and it attains the same value at both end-points. Consequently, the two end-points are maxima and thus define two functions of maximal dependence, which are φ_1^* and φ_2^* given by formulas (5.4.5). For any of these two functions, φ_k^* , we have

$$C(\varphi_k^*(u), u^2/\varphi_k^*(u)) = \frac{1}{2} \left(u^{2-2(a \wedge b)^2/(a+b)} + u^{2-2ab/(a+b)} \right).$$

With $\kappa = 2 - 2ab/(a+b)$ we check that

$$\frac{\Pi^*(u)}{u^\kappa} = \frac{C(\varphi_k^*(u), u^2/\varphi_k^*(u))}{u^\kappa} \rightarrow \frac{1}{2}$$

when $u \downarrow 0$. Consequently, the lower-tail index of maximal dependence κ_L^* is equal to κ .

This concludes the proof of equations (5.4.5) and (5.4.7). $\square$

5.4.3 Farlie-Gumbel-Morgenstern copula

We saw in the previous subsection that the symmetry of a copula does not necessarily imply that the diagonal is a path of maximal dependence. Of course, some symmetric copulas do have diagonal paths of maximal dependence. One of such examples is the already discussed Marshall-Olkin copula when $a = b$. Another example is the Farlie-Gumbel-Morgenstern (FGM) copula

$$C(u, v) = uv(1 + \alpha(1-u)(1-v)) \quad \text{for } 0 \leq u, v \leq 1,$$

with parameter $\alpha \in [-1, 1]$. Our choice to include the FGM copula among illustrative examples has been motivated by the fact that within the context of the present paper, it sometimes matters whether a copula is or is not positively quadrant dependent (PQD) (Nelsen, 2006). Specifically for the FGM copula, when $\alpha < 0$, then the copula is negatively

quadrant dependent (NQD) (Nelsen, 2006); when $\alpha > 0$, then it is PQD; and when $\alpha = 0$, then it is the independence copula. We shall see the role of the sign of α in our following considerations.

To find the path of maximal dependence for the FGM copula, for every $u \in [0, 1]$, we first search for those $x \in [u^2, 1]$ that solve the equation $(\partial/\partial x)C(x, u^2/x) = 0$. Since

$$\begin{aligned}\frac{\partial C(x, u^2/x)}{\partial x} &= \frac{\partial}{\partial x} u^2 (1 + \alpha(1 - u^2/x)(1 - x)) \\ &= u^2 \alpha (-1 + u^2/x^2),\end{aligned}$$

we have $(\partial/\partial x)C(x, u^2/x) = 0$ if and only if $x = u$. Therefore, when $\alpha > 0$, then the diagonal is the unique path of maximal dependence, that is, $\varphi^*(u) = u$. In this case, the maximal probability is $\Pi^*(u) = u^2(1 + \alpha(1 - u)^2)$, and thus the lower-tail index of maximal dependence is

$$\kappa_L^* = 2.$$

As to the case $\alpha \leq 0$, we first note that when $\alpha = 0$, then every admissible path is a path of maximal dependence. When $\alpha < 0$, then $C(x, u^2/x)$ reaches its maximal value at either u^2 or 1 or both, but neither of the functions $\varphi(u) = u^2$ and $\varphi(u) = 1$ is admissible because they fail to satisfy property (2) of Definition 5.2.1.

5.5 Lower-tail dependence comparisons

We now look at the index λ_L^* (cf. equation (5.2.11)) from a different angle. To this end we first note that for the Fréchet upper bound copula $C^\top(u, v) = \min\{u, v\}$, $0 \leq u, v \leq 1$, the path of maximal dependence is diagonal. Consequently, $\Pi^*(u \mid C^\top) = u$ and so, given any

copula C , the index $\lambda_L^* := \lambda_L^*(C)$ can be rewritten as

$$\lambda_L^*(C) = \lim_{u \downarrow 0} \frac{\Pi^*(u \mid C)}{\Pi^*(u \mid C^\top)}.$$

This suggests that, given two copulas C_1 and C_2 , we can compare their lower-tail dependencies using the index

$$\lambda_L^*(C_1, C_2) = \lim_{u \downarrow 0} \frac{\Pi^*(u \mid C_1)}{\Pi^*(u \mid C_2)}.$$

Obviously, $\lambda_L^*(C) = \lambda_L^*(C, C^\top)$.

Definition 5.5.1. *The copula C_1 is said to be more lower-tail maximally-dependent (LTMD) than the copula C_2 if $\lambda_L^*(C_1, C_2) > 1$; and less LTMD if $\lambda_L^*(C_1, C_2) < 1$.*

An analogous interpretation can be made about the index $\chi_L^* := \chi_L^*(C)$, but now using the independence copula $C^\perp(u, v) = uv, 0 \leq u, v \leq 1$, for which every admissible path is a path of maximal dependence. Hence,

$$\chi_L^*(C) = \lim_{u \downarrow 0} \frac{\log \Pi^*(u \mid C^\perp)}{\log \Pi^*(u \mid C)} - 1.$$

We generalize this index to the case of two arbitrary copulas C_1 and C_2 by introducing the index

$$\chi_L^*(C_1, C_2) = \lim_{u \downarrow 0} \frac{\log \Pi^*(u \mid C_2)}{\log \Pi^*(u \mid C_1)} - 1.$$

Obviously, $\chi_L^*(C) = \chi_L^*(C, C^\perp)$.

Definition 5.5.2. *The copula C_1 is said to be more weakly lower-tail maximally-dependent (WLTMD) than the copula C_2 if $\chi_L^*(C_1, C_2) > 0$; and less WLTMD if $\chi_L^*(C_1, C_2) < 0$.*

We can now compare all the specific copulas considered in this paper, as well as other ones. Without getting too specific at the moment, assume that C_1 and C_2 are such that,

for some constants $c^*(C_k)$ and $\kappa_L^*(C_k)$, $k = 1, 2$, the asymptotic formulas $\Pi^*(u \mid C_1) = c^*(C_1)u^{\kappa_L^*(C_1)}(1 + o(u))$ and $\Pi^*(u \mid C_2) = c^*(C_2)u^{\kappa_L^*(C_2)}(1 + o(u))$ hold when $u \downarrow 0$. Two cases follow:

- When $\kappa_L^*(C_1) = \kappa_L^*(C_2)$, then

$$\lambda_L^*(C_1, C_2) = \frac{c^*(C_1)}{c^*(C_2)}.$$

To illustrate, we note that the classical Marshall-Olkin copula, which we denote here by C_{MO} , and mixture (5.4.4) of two Marshall-Olkin copulas, which we denote here by C_{MIXMO} , have the same κ_L^* -indices. Equations (5.4.2) and (5.4.6) imply that $\lambda_L^*(C_{\text{MO}}, C_{\text{MIXMO}}) = 2$, which we interpret as saying that C_{MO} is more LTMD than C_{MIXMO} whenever $a \neq b$. When $a = b$, then we of course have $\lambda_L^*(C_{\text{MO}}, C_{\text{MIXMO}}) = 1$ because $C_{\text{MIXMO}} = C_{\text{MO}}$ in this case.

- When $\kappa_L^*(C_1) \neq \kappa_L^*(C_2)$, then only the index $\chi_L^*(C_1, C_2)$ is of interest, which can be expressed by the formula

$$\chi_L^*(C_1, C_2) = \frac{\kappa_L^*(C_2)}{\kappa_L^*(C_1)} - 1.$$

In particular, we have $\chi_L^*(C) = \chi_L^*(C, C^\perp) = 2/\kappa_L^*(C) - 1$.

5.6 Further examples

For all the copulas hitherto, we have derived closed-form expressions for their paths of maximal dependence. This may not always be the case as we illustrate with following examples; yet, we shall be able to derive closed-form expressions for the index κ_L^* .

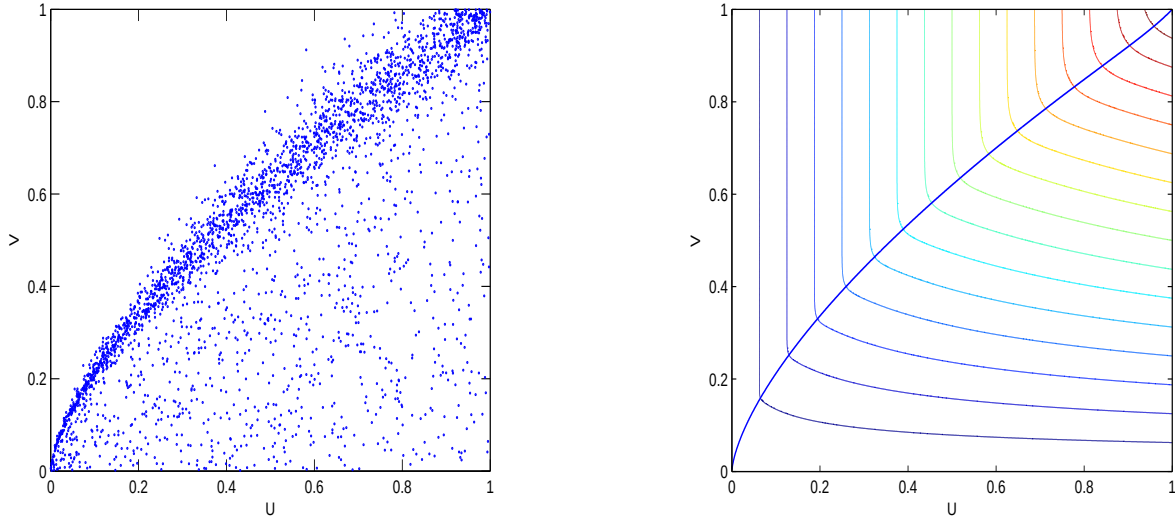


Figure 5.3: Simulated generalized Clayton copula (left) and its contours (right) when $\gamma_0 = 0.04$ and $\gamma_1 = 0.02$, with the path of maximal dependence $(\varphi^*(u), u^2/\varphi^*(u))_{0 \leq u \leq 1}$ superimposed on the right-hand panel.

5.6.1 Generalized Clayton copula

The generalized Clayton copula is given by the formula

$$C(u, v) = u^{\gamma_1/\tilde{\gamma}_1} (u^{-1/\tilde{\gamma}_1} + v^{-1/\gamma_0} - 1)^{-\gamma_0} \quad \text{for } 0 \leq u, v \leq 1, \quad (5.6.1)$$

where $\gamma_0 > 0$ and $\gamma_1 \geq 0$ are parameters, and $\tilde{\gamma}_1 = \gamma_0 + \gamma_1$. Tedious computations, which we will give later on, show that the function of maximal dependence φ^* (Figure 5.3) is unique and satisfies the equation

$$\zeta(\varphi^*(u)) = 0 \quad (5.6.2)$$

for every $0 \leq u \leq 1$, where

$$\zeta(x) = x^{-1/\gamma_0} (x^{-1/\tilde{\gamma}_1} - (\gamma_1/\tilde{\gamma}_1)) - (1 - (\gamma_1/\tilde{\gamma}_1)) u^{-2/\gamma_0}$$

for all $x \in [u^2, 1]$. Further tedious calculations show that the lower-tail index of maximal dependence is

$$\kappa_L^* = 1 + \frac{\gamma_1}{\gamma_1 + 2\gamma_0}. \quad (5.6.3)$$

Proof. We start with the equation

$$C(x, u^2/x) = x^{\gamma_1/\tilde{\gamma}_1} (x^{-1/\tilde{\gamma}_1} + u^{-2/\gamma_0} x^{1/\gamma_0} - 1)^{-\gamma_0}. \quad (5.6.4)$$

Finding the maximum of $C(x, u^2/x)$ with respect to $x \in [u^2, 1]$ is the same as finding the maximum of its logarithm $\log C(x, u^2/x)$, which is easier in this particular case. We have

$$\frac{\partial}{\partial x} \log(C(x, u^2/x)) = \frac{x^{-1/\tilde{\gamma}_1} + ((\gamma_1/\tilde{\gamma}_1) - 1)(u^2/x)^{-1/\gamma_0} - (\gamma_1/\tilde{\gamma}_1)}{x(x^{-1/\tilde{\gamma}_1} + (u^2/x)^{-1/\gamma_0} - 1)}.$$

The denominator $x(x^{-1/\tilde{\gamma}_1} + (u^2/x)^{-1/\gamma_0} - 1)$ is positive for all $x \in [u^2, 1]$, and thus we need to find those $x \in [u^2, 1]$ that make the numerator equal to 0. This is equivalent to solving the equation $\zeta(x) = 0$. The solution to this equation is unique because

$$\zeta(u^2) = u^{-2/\gamma_0} (u^{-2/\tilde{\gamma}_1} - 1) > 0,$$

$$\zeta(1) = (1 - (\gamma_1/\tilde{\gamma}_1)) - (1 - (\gamma_1/\tilde{\gamma}_1)) u^{-2/\tilde{\gamma}_1} < 0,$$

and, for all $x \in [u^2, 1]$,

$$\zeta'(x) = -\frac{2}{\tilde{\gamma}_1} x^{-1/\tilde{\gamma}_1 - 1/\gamma_0 - 1} - \frac{\gamma_1}{\tilde{\gamma}_1 \gamma_0} (x^{-1/\tilde{\gamma}_1 - 1/\gamma_0 - 1} - x^{-1/\gamma_0 - 1}) < 0.$$

We cannot derive a closed form solution to the equation $\zeta(x) = 0$, but we already know that the solution $x = \varphi^*(u) \in [u^2, 1]$ exists and is unique. Furthermore, the solution satisfies the equation

$$x^{-1/\tilde{\gamma}_1 - 1/\gamma_0} (1 - (\gamma_1/\tilde{\gamma}_1) x^{1/\tilde{\gamma}_1}) = (1 - (\gamma_1/\tilde{\gamma}_1)) u^{-2/\gamma_0}. \quad (5.6.5)$$

We also have that $1 - (\gamma_1/\tilde{\gamma}_1)x^{1/\tilde{\gamma}_1} \in [1 - (\gamma_1/\tilde{\gamma}_1), 1]$. From these facts we conclude that $x = \varphi^*(u) \downarrow 0$ when $u \downarrow 0$.

Due to the lack of closed-form expression for the function of maximal dependence, we cannot obtain a closed-form expression for the maximal probability $\Pi^*(u)$ either. Nevertheless, we can obtain a closed-form expression for κ_L^* . Starting with equation (5.6.5), we arrive at the following one:

$$x = \left(u^{2/\gamma_0} (1 - (\gamma_1/\tilde{\gamma}_1)x^{1/\tilde{\gamma}_1}) / (1 - (\gamma_1/\tilde{\gamma}_1)) \right)^{\tilde{\gamma}_1\gamma_0/(\tilde{\gamma}_1+\gamma_0)}. \quad (5.6.6)$$

Denote $r(x) = (1 - (\gamma_1/\tilde{\gamma}_1)x^{1/\tilde{\gamma}_1}) / (1 - (\gamma_1/\tilde{\gamma}_1))$. Replacing all the x 's on the right-hand side of equation (5.6.4) by the right-hand side of equation (5.6.6), we obtain

$$\begin{aligned} C(x, u^2/x) &= \{u^{2/\gamma_0} r(x)\}^{\tilde{\gamma}_1\gamma_0/(\tilde{\gamma}_1+\gamma_0)} \\ &= u^{\frac{2\gamma_1}{\tilde{\gamma}_1+\gamma_0}} r(x)^{\frac{\gamma_0\gamma_1}{\tilde{\gamma}_1+\gamma_0}} \left(r(x)^{-\frac{\gamma_0}{\tilde{\gamma}_1+\gamma_0}} u^{-\frac{2}{\tilde{\gamma}_1+\gamma_0}} + r(x)^{\frac{\tilde{\gamma}_1/\gamma_0}{\tilde{\gamma}_1+\gamma_0}} u^{-\frac{2}{\tilde{\gamma}_1+\gamma_0}} - 1 \right)^{-\gamma_0}. \end{aligned}$$

Consequently, with $\kappa = 1 + \gamma_1/(\tilde{\gamma}_1 + \gamma_0)$ we have

$$\frac{C(x, u^2/x)}{u^\kappa} = r(x)^{\frac{\gamma_0\gamma_1}{\tilde{\gamma}_1+\gamma_0}} \left(r(x)^{-\frac{\gamma_0}{\tilde{\gamma}_1+\gamma_0}} + r(x)^{\frac{\tilde{\gamma}_1/\gamma_0}{\tilde{\gamma}_1+\gamma_0}} - u^{\frac{2}{\tilde{\gamma}_1+\gamma_0}} \right)^{-\gamma_0} \rightarrow c \in (0, \infty)$$

when $u \downarrow 0$. This proves that the lower-tail index of maximal dependence κ_L^* is equal to κ .

The proof of equations (5.6.2) and (5.6.3) is finished. $\square$

5.6.2 Archimedean copula

Another example illustrating the difficulty of deriving closed-form expressions for paths of maximal dependence is provided by the Archimedean copula

$$C(u, v) = \psi^{-1}(\psi(u) + \psi(v)) \quad \text{for } 0 \leq u, v \leq 1, \quad (5.6.7)$$

where $\psi : [0, 1] \rightarrow [0, \infty]$ is such that $\psi(1) = 0$, $\psi'(u) < 0$, and $\psi''(u) > 0$ for all $0 < u < 1$. Assume that the Archimedean generator ψ is strict, that is, $\psi(u) \rightarrow \infty$ when $u \downarrow 0$, as only in this case can we have a path of maximal dependence. For every $u \in [0, 1]$, to determine those $x \in [u^2, 1]$ for which the equation $(\partial/\partial x)C(x, u^2/x) = 0$ holds, we notice that when the function $x\psi'(x)$ is increasing on $[u^2, 1]$, which we assume from now on, then

$$x\psi'(x) - (u^2/x)\psi'(u^2/x) = \begin{cases} < 0 & \text{when } x < u^2/x, \\ > 0 & \text{when } x > u^2/x. \end{cases}$$

Thus, $\psi(x) + \psi(u^2/x) \geq 2\psi(u)$ for all $x \in [u^2, 1]$ or, equivalently,

$$\psi^{-1}(\psi(x) + \psi(u^2/x)) \leq \psi^{-1}(2\psi(u)).$$

Therefore, when the function $x\psi'(x)$ is increasing on $[u^2, 1]$, then the diagonal is a path of maximal dependence. In this case, the Archimedean copula is PQD. We observe that all PQD Archimedean copulas documented by [Nelsen \(2006\)](#) have increasing functions $x\psi'(x)$.

5.7 Conclusions

We have demonstrated that the classical indices of tail dependence, which are based on the behaviour of copulas along the diagonal path, do not generally capture the maximal degree of tail dependence. For this reason, we have proposed conservative indices of tail dependence that hinge on the new notion of *paths of maximal dependence* that we have introduced herein. We have used a number of specific copulas, as well as a numerical example, to elucidate relevant main ideas both analytically and numerically. Our approach to assessing the tail dependence in copulas conforms with the new paradigm of prudence in modern quantitative risk management (cf. [OSFI, 2014](#)).

A number of challenging problems remain in this area. For example, statistical inference for the indices would be of interest, and we believe that tools developed in the area of M-estimators and related weak convergence theorems would be helpful (cf., e.g., [van der Vaart and Wellner, 1996](#); [van de Geer, 2000](#); and references therein). Another interesting problem would be to explore uniqueness and other properties of the path(s) of maximal dependence, and we think that results discussed by [Dharmadhikari and Joag-dev \(1988\)](#) would be particularly helpful.

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Chapter 6

Conclusions

The latest Solvency II directives require that insurers recognize the interdependences within and among different liability classes. This new paradigm arguably brings to an end the indisputable role of the assumption of independence that has shaped both research and applications in actuarial science and general quantitative risk management in the 20th century. The choice of an appropriate probability dependence model is however not an easy call. Indeed, while there exists only one way to describe stochastic independence, the forms of stochastic dependence are infinite.

In this dissertation we have introduced a new class of Multiple Risk Factor dependence structures. On the one hand, these structures emerge as an extension of the popular CreditRisk⁺ approach, and as such they formally describe default risk portfolios exposed to an arbitrary number of fatal risk factors with conditionally exponential hitting times that can be independent, positively dependent and fully comonotonic. On the other hand, the MRF structures can be viewed as a quite general family of multivariate probability distributions with Pareto of the 2nd kind univariate margins, and in this capacity, they can model risk portfolios of (insurance) losses with heavy-tailed and positively dependent risk components.

Departing to an extent from the classical distributional approach, we have used the doubly stochastic Poisson processes to explore the class of copula functions that underlie the MRF structures. It has been noted that a ‘good’ copula should conform to a problem at hand, be asymmetric and exhibit some extent of tail dependence ([Staudt, 2010](#)). The MRF copulas that we have introduced and studied in this dissertation are exactly such. Namely, they admit meaningful in the context of the ERM stochastic representations, arise from a number of default risk specifications with stochastic default barrier and are in general not symmetric in their domain of definition as well as in the corresponding tail behaviour. The MRF copulas cover the full range of non-negative dependence and are yet quite tractable analytically.

Last but not least, motivated by the asymmetric structure of the MRF dependencies, we have demonstrated in this dissertation that the classical indices of tail dependence which are based on the behaviour of copulas along the main diagonal, do not generally capture the maximal degree of tail dependence. For this reason, we have proposed conservative indices of tail dependence that hinge on the new notion of paths of maximal dependence that we have introduced herein. We have used a number of specific copulas, as well as a numerical example, to elucidate relevant main ideas both analytically and numerically. The proposed approach to assessing the tail dependence in copulas conforms to the new paradigm of prudence in modern quantitative risk management ([OSFI, 2014](#)).

Future research topics are numerous and include but are not limited to: statistical inference for the indices of maximal tail dependence as well as the application of the paths of maximal dependence to formulating new measures of global dependence, constructing new families of copulas and developing new classes of bivariate risk measures.

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