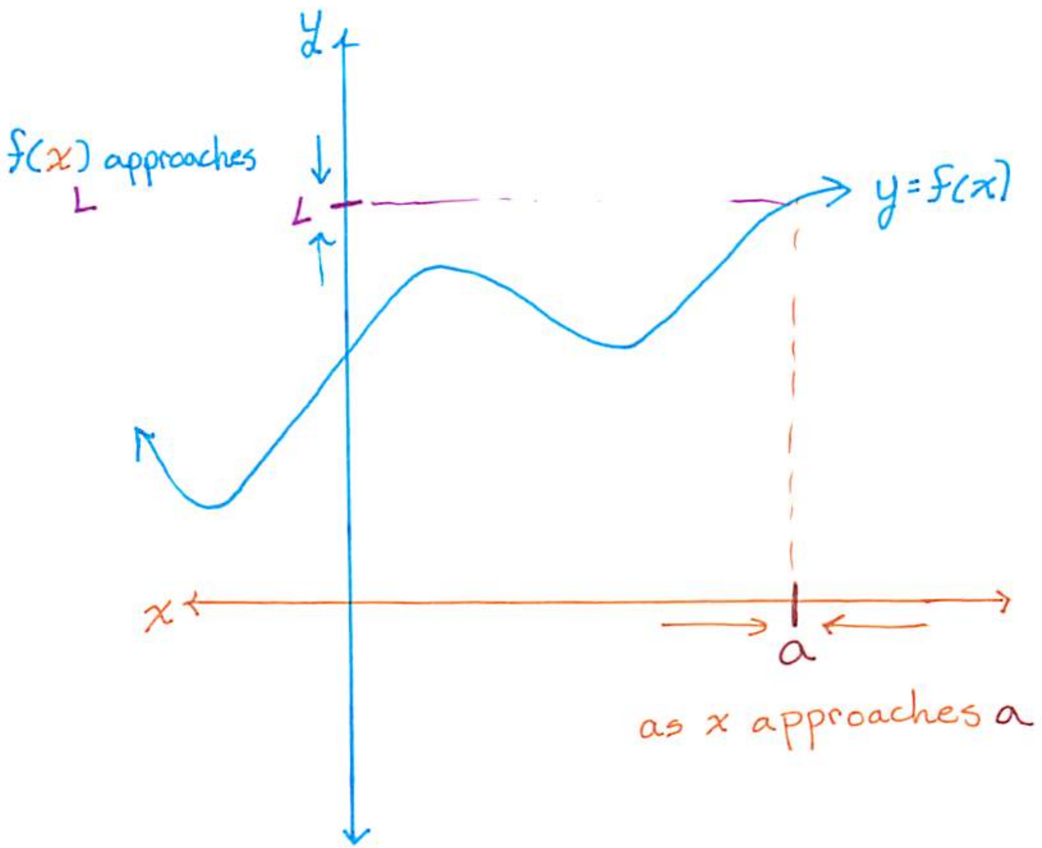
Meaning of $\lim_{x\to a} f(x) = L$

$\boxed{\lim_{x\to a} f(x) = L}$ means "can make $f(x)$ values as close as would like to L by making x close enough to a"

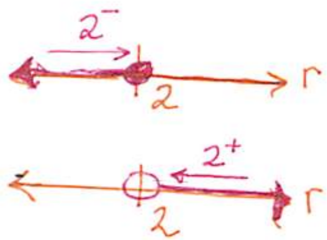


Left-Limit vs Right-Limit

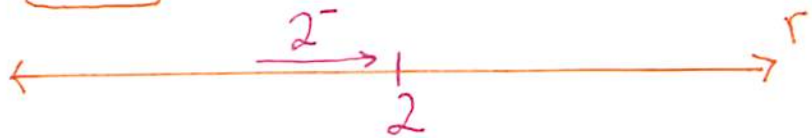
Sometimes a limit DNE (does not exist) because depends on direction of approach

SUPPOSE

$$g(r) = \begin{cases} r & \text{if } r \le 2 \\ -r & \text{if } r > 2 \end{cases}$$

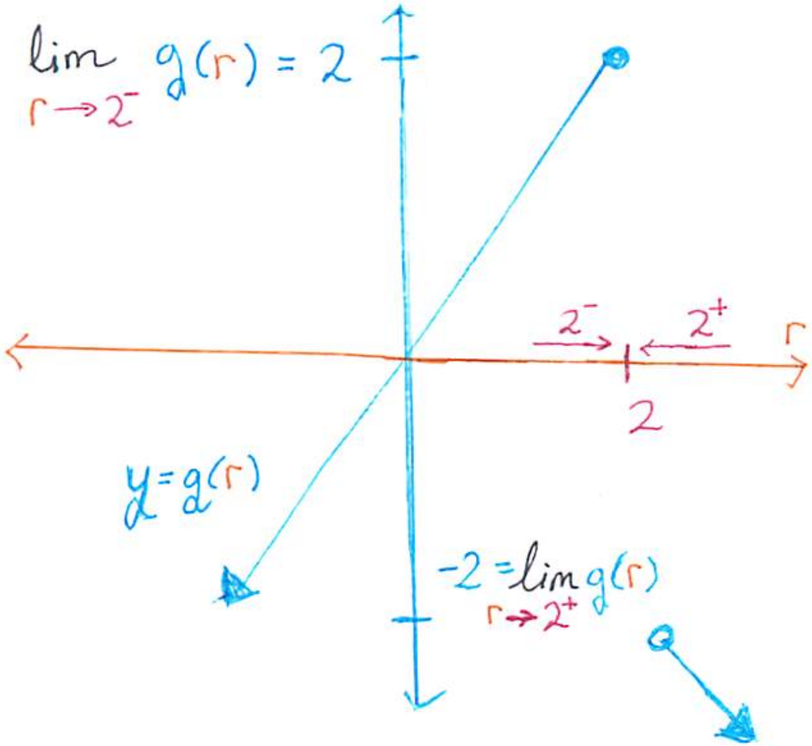


$\boxed{r \to 2^-}$



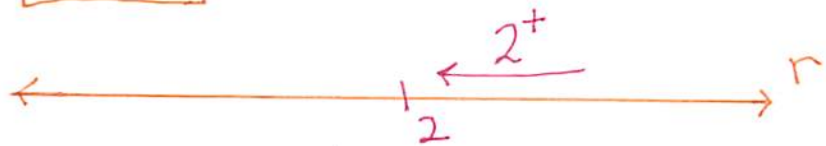
Look at values for $g(r)$ when r is really close to 2 but less than 2

$$\lim_{r \to 2^-} g(r) = 2$$



$$-2 = \lim_{r \to 2^+} g(r)$$

$\boxed{r \to 2^+}$



Look at values for $g(r)$ when r is really close to 2 but greater than 2

$\lim_{x \to a} f(x) = L$ has the same meaning as $\left\{ \begin{array}{l} \lim_{x \to a^-} f(x) = L \text{ AND} \\ \lim_{x \to a^+} f(x) = L \end{array} \right\}$

SO $\lim_{r \to 2} g(r)$ DNE

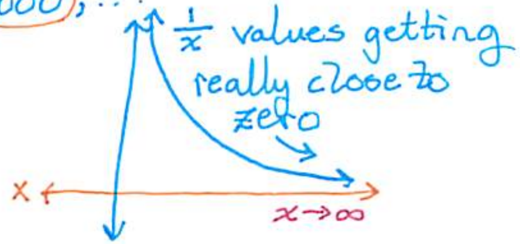
Limits of Quotient Functions

Example 1: $\boxed{\lim\limits_{x \to \infty} \frac{1}{x}}$

Check what happens as sub in larger & larger positive x values

$\frac{1}{1}, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$

$$\boxed{\lim_{x \to \infty} \frac{1}{x} = 0}$$

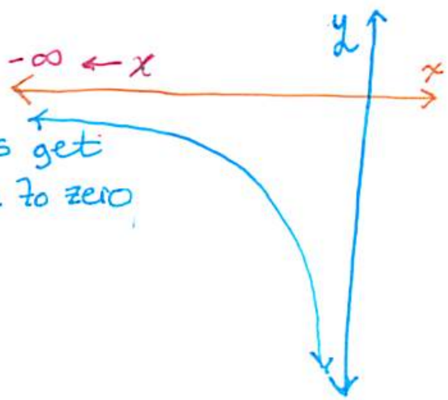


Example 2: $\lim\limits_{x \to -\infty} \frac{1}{x}$

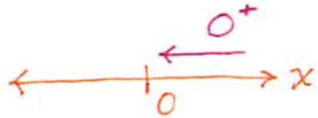
check what happens as substitute in larger and larger negative x values

$\frac{1}{-1} = -1, \frac{1}{-10} = -\frac{1}{10}, \frac{1}{-100} = -\frac{1}{100}, \frac{1}{-1000} = -\frac{1}{1000}$

$$\boxed{\lim_{x \to -\infty} \frac{1}{x} = 0}$$



Example 3: $\boxed{\lim\limits_{x \to 0^+} \frac{1}{x}}$

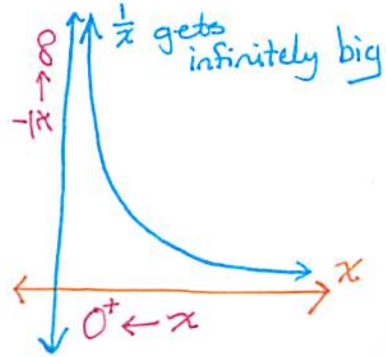


Check what happens as sub in smaller & smaller positive x values

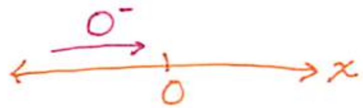
$\frac{1}{1/10} = 10, \frac{1}{1/100} = 100, \frac{1}{1/1000} = 1000 \ldots$

$\frac{1}{\text{very very small positive}}$

$$\boxed{\lim_{x \to 0^+} \frac{1}{x} = \infty}$$



Example 4: $\boxed{\lim\limits_{x \to 0^-} \frac{1}{x}}$

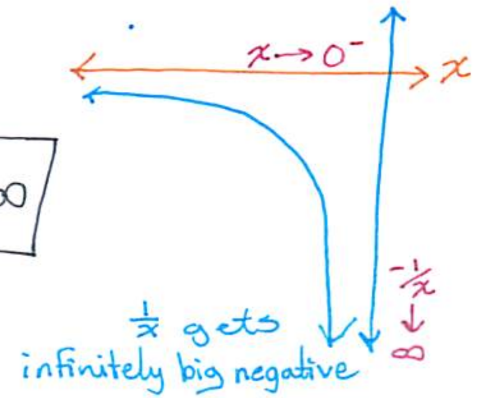


Check what happens as sub in smaller & smaller negative x values

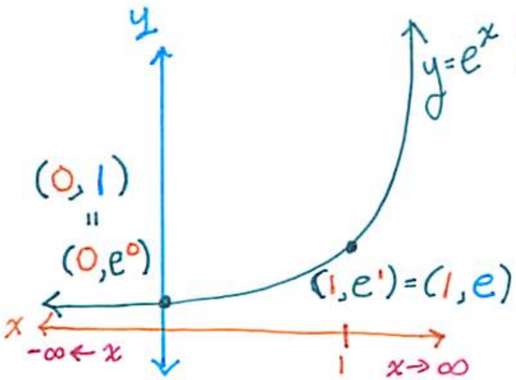
$\frac{1}{-1/10} = -10, \frac{1}{-1/100} = -100, \frac{1}{-1/1000} = -1000 \ldots$

$\frac{1}{\text{very very small negative}}$

$$\boxed{\lim_{x \to 0^-} \frac{1}{x} = -\infty}$$

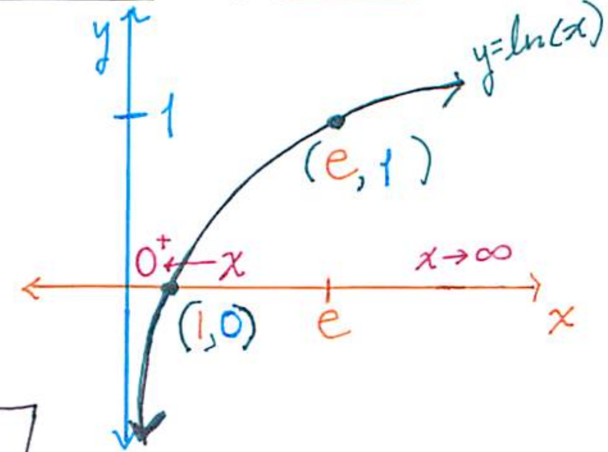


Limits of the Exponential Function & Natural Logarithm



Key Fact

$x = e^y \longleftrightarrow y = \ln(x)$



$$\boxed{\lim_{x\to\infty} e^x = \infty}$$

(outputs get infinitely big positive/
graph goes infinitely high as move to the right)

$$\boxed{\lim_{x\to-\infty} e^x = 0}$$

Why?: $e^{-x} = \frac{1}{e^x}$ so

if input for x really big negative values,
get $\frac{1}{\text{v.v.b.p.}} \to 0$

$$\boxed{\lim_{x\to\infty} \ln(x) = \infty}$$

Why?: $y = \ln\left(\begin{matrix}\text{very very}\\ \text{big positive}\end{matrix}\right) \longleftrightarrow \begin{matrix}\text{very very}\\ \text{big positive}\end{matrix} = e^y$

so y must get very big, i.e. $\to \infty$

$$\boxed{\lim_{x\to0^+} \ln(x) = -\infty}$$

Why?: To get e^y close to zero,
need $e^y = \frac{1}{\text{v.v.s}}$

so $y \to -\infty$

Taking Limits is a "Linear Operation"!

SUPPOSE $\left\{ \begin{array}{l} c \text{ is a constant AND} \\ \lim_{x \to a} f(x) \text{ AND } \lim_{x \to a} g(x) \text{ both exist} \end{array} \right\}$

THEN

I Summation Rule:

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

II Subtraction Rule:

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

III Constant Multiple Rule:

$$\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x)$$

Example: $\left\{ \begin{array}{l} f(x) = e^x \\ g(x) = \frac{1}{x} \\ a = \infty \\ c = \pi \end{array} \right\}$

SO $\left\{ \begin{array}{l} \lim_{x \to a} f(x) = \lim_{x \to -\infty} e^x = 0 \\ \lim_{x \to a} g(x) = \lim_{x \to -\infty} \frac{1}{x} = 0 \end{array} \right\}$

I $\lim_{x \to -\infty} \left[e^x + \frac{1}{x}\right] = \overbrace{\lim_{x \to -\infty} e^x}^{0} + \overbrace{\lim_{x \to -\infty} \frac{1}{x}}^{0} = 0 + 0 = \underline{\underline{0}}$

II $\lim_{x \to -\infty} \left[e^x - \frac{1}{x}\right] = \overbrace{\lim_{x \to -\infty} e^x}^{0} - \overbrace{\lim_{x \to -\infty} \frac{1}{x}}^{0} = 0 - 0 = \underline{\underline{0}}$

III $\lim_{x \to -\infty} [\pi e^x] = \pi \underbrace{\lim_{x \to -\infty} e^x}_{0} = \pi \cdot 0 = \underline{\underline{0}}$

Some More Useful Limit Properties : Roots

SUPPOSE $\begin{cases} n \text{ is a positive integers} \\ \text{AND} \\ a>0 \text{ if } n \text{ is even} \end{cases}$

$$\text{(X)} \quad \lim_{x\to a}\left(\sqrt[n]{x}\right) = \sqrt[n]{a}$$

Recall How to Think About ln(x)

$\ln(x) = y \longleftrightarrow e^y = x$

$\ln(1) = 0 \longleftrightarrow e^0 = 1$

$\ln(e) = 1 \longleftrightarrow e^1 = e$

$n=4$
$a=\pi$

Example: $\lim_{x\to\pi^2} x^{1/4} = \lim_{x\to\pi^2} \sqrt[4]{x} = \sqrt[4]{\pi^2}$

$$= (\pi^2)^{1/4} = \pi^{(2\cdot 1/4)} = \pi^{1/2} = \underline{\underline{\sqrt{\pi}}}$$

$$\text{(XI)} \quad \lim_{x\to a}\left(\sqrt[n]{f(x)}\right) = \sqrt[n]{\lim_{x\to a} f(x)}$$

where $\begin{cases} n \text{ is a positive integer} \\ \text{AND} \\ \lim_{x\to a} f(x) > 0 \text{ if } n \text{ is even} \end{cases}$

Example: $\lim_{x\to e} \sqrt[2]{\ln(x)} = \sqrt[2]{\lim_{x\to e} \ln(x)}$

$$= \sqrt{\ln(e)}$$

$$= \sqrt{1}$$

$$= \underline{\underline{1}}$$

Some More Useful Limit Properties : Product, Quotient & Power Rules

IV Product Rule:

$$\lim_{x\to a}\left[f(x)\,g(x)\right] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

SUPPOSE $\left\{\begin{array}{l} f(x)=x \\ g(x)=e^{x} \\ a=0 \end{array}\right\}$

SO: $\lim_{x\to a} f(x)=0$; $\lim_{x\to a} g(x)=1$

IV $\lim_{x\to 0}\left[x e^{x}\right] = \lim_{x\to 0} x \cdot \lim_{x\to 0} e^{x} = 0\cdot 1 = \underline{\underline{0}}$

V Quotient Rule:

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

V $\lim_{x\to 0}\frac{x}{e^{x}} = \dfrac{\lim_{x\to 0} x}{\lim_{x\to 0} e^{x}} = \frac{0}{1} = \underline{\underline{0}}$

VI Power Rule:

positive integer

$$\lim_{x\to a}\left(\left[f(x)\right]^{n}\right) = \left(\lim_{x\to a} f(x)\right)^{n}$$

SUPPOSE $\left\{\begin{array}{l} f(x)=x \\ n=3 \\ a=2 \end{array}\right\}$, So $\lim_{x\to a} f(x) = \lim_{x\to 2} x = 2$

VI $\lim_{x\to 2}\left(x^{3}\right) = \left(\lim_{x\to 2} x\right)^{3} = (2)^{3} = \underline{\underline{8}}$

We used in the power rule example:

VIII $\lim_{x\to a} x = a$

And power rule example shows how to combine VI and VIII to: $\lim_{x\to a} x^{n} = a^{n}$

positive integer

Example of Applying Limit Properties

Example: $\displaystyle \lim_{x \to \pi} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$

IV

$$= \frac{\displaystyle\lim_{x \to \pi} \left[x^3 + 4x^2 - 3\right]}{\displaystyle\lim_{x \to \pi} \left[x^2 + 5\right]}$$

I AND II

$$= \frac{\displaystyle\lim_{x \to \pi} x^3 + \lim_{x \to \pi} 4x^2 - \lim_{x \to \pi} 3}{\displaystyle\lim_{x \to \pi} x^2 + \lim_{x \to \pi} 5}$$

III

$$= \frac{\displaystyle\lim_{x \to \pi} x^3 + 4\lim_{x \to \pi} x^2 - \lim_{x \to \pi} 3}{\displaystyle\lim_{x \to \pi} x^2 + \lim_{x \to \pi} 5}$$

IV AND IX

$$= \frac{(\pi)^3 + 4(\pi)^2 - 3}{(\pi)^2 + 5}$$

Important Observation:

Could get same answer by "plugging" π for x in

$$\frac{x^3 + 4x^2 - 3}{x^2 + 5}$$

Will be "Direct Substitution Property"!

Direct Substitution Property for Limits of Polynomials and Rational Functions

Last Time:

$$\lim_{x\to\pi} \frac{x^3+4x^2-3}{x^2+5} = \frac{(\pi)^3+4(\pi)^2-3}{(\pi)^2+5}$$

i.e. applying limit laws or "plugging in" π gave exactly the same thing!

Example of:

Direct Substitution Property (DSP):

IF $\left\{ \begin{array}{l} f \text{ is a polynomial or rational function AND} \\ a \text{ is in the domain of } f \end{array} \right\}$

THEN

$$\lim_{x\to a} f(x) = f(a)$$

These 2 properties must hold (check them) to apply "Then statement"

Can assume this holds as long as in a situation where the 2 properties hold

In other words:

Can plug a into such functions (for x), as did with π

Direct Substitution Property Works Even if Must "Cancel"

Example: $\lim_{x \to -3} \frac{x^2+6x+9}{x^2+3x}$

Can we apply DSP?

No: Subbing -3 into denominator $\rightsquigarrow 0$

Can't divide by $0 \rightsquigarrow -3$ is not in the domain

Try factoring & cancelling 1st, before applying DSP:

$$\lim_{x \to -3} \frac{x^2+6x+9}{x^2+3x} = \lim_{x \to -3} \frac{(x+3)^2}{x(x+3)}$$

$$= \lim_{x \to -3} \frac{(x+3)\boxed{(x+3)}}{x\boxed{(x+3)}} \quad =1$$

$$= \lim_{x \to -3} \frac{x+3}{x}$$

Now we're ready!

$$\overset{\text{DSP}}{=} \frac{(-3)+3}{(-3)} = \underline{\underline{0}}$$

Formality of Example

IF $f(x) = g(x)$ when $x \neq a$

THEN $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$

(if the limit exists)

Q: Why was it ok to factor, cancel, & plug in?

A: Don't care what happens at $x = -3$

$\lim_{x\to a} f(x)$ when $f(x)$ not defined at $x=a$: Part 1

Q: How do we compute $\lim_{x\to a} f(x)$ when $f(x)$ is not defined at $x=a$?

Short Answer:
Ignore what happens at $x=a$!

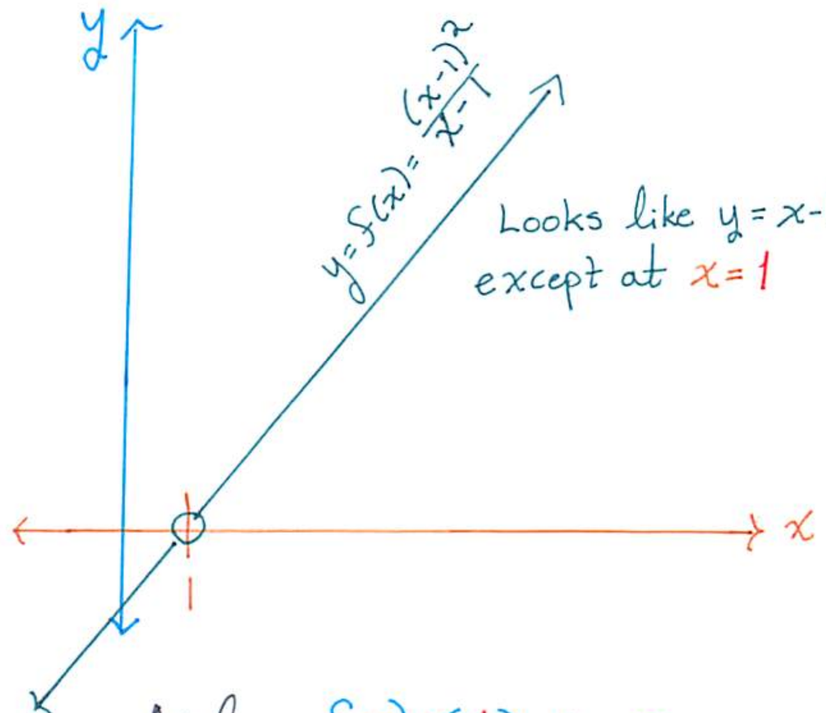
Answer by Example:

Example 1: $f(x)=\dfrac{(x-1)^2}{(x-1)}$, $\lim_{x\to 1} f(x) = ?$

This is $\begin{cases} x-1 & \text{where } x\neq 1 \\ \text{undefined} & \text{where } x=1 \end{cases}$

Will use $x-1$

Ignore! Don't care what happens at $x=1$



Looks like $y=x-1$, except at $x=1$

A: $\lim_{x\to 1} f(x) = (1)-1 = \underline{\underline{0}}$

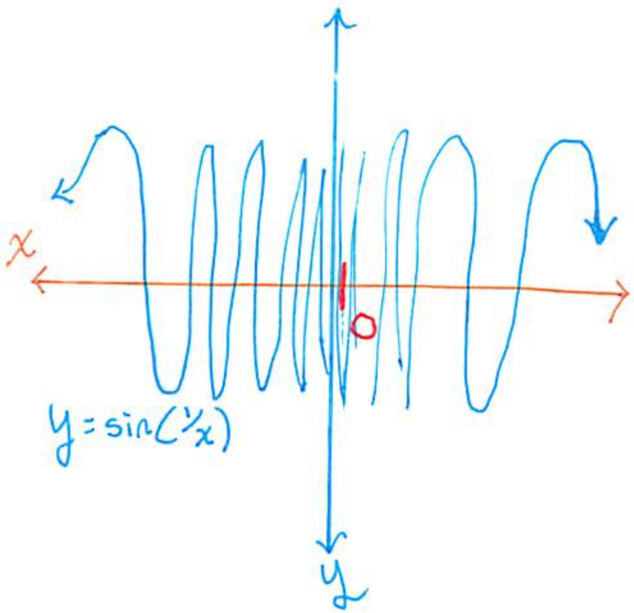
$\lim_{x \to a} f(x)$ When $f(x)$ is Not Defined at $x=a$: Part 2

Moral of Last Time: Ignore what happens at $x=a$!

This Time: Sometimes the limit $D^{\text{oes}}N^{\text{ot}}E^{\text{xist}}$

Example 2: $f(x) = \sin\left(\frac{1}{x}\right)$

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = ?$$



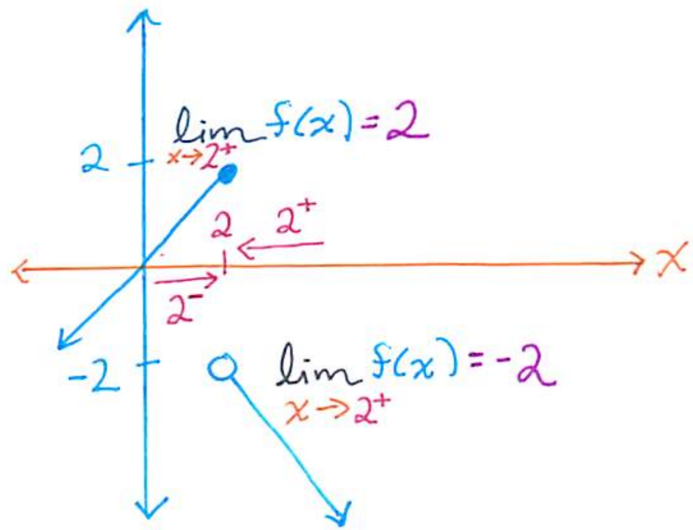
Since it is impossible to get close enough to $x=0$ to make $f(x)$-values all really close to the same value

Example 3 (Can depend on direction approach $x=a$ from)

$$f(x) = \begin{cases} x & \text{if } x \leq 2 \\ -x & \text{if } x > 2 \end{cases}$$

$$\lim_{x \to 2} f(x) = ?$$

A: DNE since $\overbrace{\lim_{x \to 2^-} f(x)}^{2} \neq \overbrace{\lim_{x \to 2^+} f(x)}^{-2}$



$$\boxed{\text{Careful Please! } \infty - \infty \text{ Could Be Anything!}}$$

Examples $\left[\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x) = \infty - \infty \right]$

① $f(x) = x + 2$, $g(x) = x$

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$

so $\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to\infty} [(x+2) - x] = \lim_{x\to\infty} (2) = \boxed{2}$

② $f(x) = 2x$, $g(x) = x$

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$

so $\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to\infty} [(2x) - (x)] = \lim_{x\to\infty} (x) = \boxed{\infty}$

③ $f(x) = x$, $g(x) = 2x$

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$

so $\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to\infty} [(x) - (2x)] = \lim_{x\to\infty} (-2x) = \boxed{-\infty}$