MAKING CONNECTIONS TO DEEPEN NEW TEACHERS' UNDERSTANDING OF ELEMENTARY MATHEMATICS

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ABSTRACT

This study investigates how Ontario newly graduated teachers who have obtained certification but have not begun teaching, can make connections within their existing elementary mathematical knowledge. Typically, beginning teachers have a fragmented understanding of elementary mathematics; for example, they know how to use standard algorithms but not why they work. New teachers often lack prior experiences constructing connections between algorithms and less formal images, like pictures or diagrams, for themselves. Twenty-five years of research has shown that working with a student's prior knowledge and connecting it with new material is the key to developing mathematics understanding. New teachers will need to support their future students in this recommended connection practice. However, asking teachers to support students in making mathematical connections seems like an unreasonable expectation when they have not had opportunities to connect their own understanding. In teacher education, researchers have uncovered what teachers do and don't know, but have yet to determine how to support teachers in making connections among their own fragmented mathematics ideas.

Researchers are also saying that knowing mathematics is not enough; teachers also need extensive experience using strategies and methods that help students connect their prior mathematical knowledge with new ideas. This case study took place in a course with 15 certified newly graduated teachers. It extends educational research by investigating how teachers can deepen their mathematical understanding using tools and strategies similar to the ones they can/will use in their future teaching. Simultaneously, the study also offered newly graduated teachers extensive experience with recommended teaching strategies/methods for elementary students. Pirie-Kieren Theory and met-befores were used to frame the case study analysis of the video data and show how newly graduated teachers deepened their existing mathematical understanding by experiencing the methods similar to those recommended for elementary classrooms.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1 SEEKING SOLUTIONS FOR ELEMENTARY MATHEMATICS

1.1. The Motivation for My Research

Mathematics education is increasingly important preparation for work and daily living, and the elementary classroom is the starting point, the place where students build their initial mathematical understandings and form their attitudes about math. Yet, despite over 25 years of relevant research and advancements in curricula, some jurisdictions, including Ontario and Canada overall, are seeing declining results in terms of elementary students' standardized test scores, the indicator that governments use to assess teaching efficacy. Between 2014 and 2018, Ontario's grade six math scores declined by 5 percentage points, and grade three scores dropped by 6 percent (Globe and Mail, 2018). This recent data reignited the periodic public and government concerns about elementary teachers' mathematics knowledge and teaching methods/strategies as well as overall teacher preparation.

This is not a new concern. The educational research community has long been concerned about elementary mathematics. To summarize the literature discussed later in chapter 2, researchers have investigated and agreed on how to support children in growing their mathematical understanding—in essence, by working on math tasks and connecting new ideas to what they already know (e.g. Stylianides and Stylianides, 2007). Researchers (e.g., Hiebert, 2013) have also found that recommended methods are not being used in most classrooms because many elementary teachers lack the necessary mathematics knowledge and experience with these tools—representations and strategies. Teachers tend to know math formulas, but not how to connect them with the concrete materials and pictures used to introduce math ideas to beginning learners. They usually know how to show and explain math rules to a class (Ball, 2001), but not how to orchestrate the real-time activities of young learners, individually and collectively working to grow their math understanding (e.g., Smith & Stein, 2011). Though most teachers have studied the recommended approaches, they have not actually experienced them in their own years of classroom learning, so they tend to teach the way they were taught and the vicious cycle continues to the next generation of elementary students (Ball, 1988).

The outstanding question is how can elementary teachers acquire the knowledge and experiences needed to effectively support their students in growing their understanding of mathematics. Teacher education researchers have focused on what teachers need to know and be able to do and developed related teaching resources. They have also discussed the importance of teachers having a deep understanding of mathematics (e.g. Adler, Hossain, Stevenson, Clarke, Archer, & Grantham, 2014). However, when investigating the effectiveness of current teacher education courses, researchers find that teachers are dissatisfied with their course work and are not implementing the practices they studied (Gainsburg, 2012). Recently a few researchers (e.g., McGowan, 2017) are calling teacher educators to use the teaching recommendations for school students in teacher education. This is where my research fits in.

1.2 Background of the Researcher

The belief that adult learning is different from child learning. In her review of three learning theories, McDonough (2013) found that the learning processes for adults and children are similar—notably both benefit from active engagement in the learning process and from making connections between prior knowledge and new knowledge. There were just two differences: adults are more self-motivated and they have more prior experience than children. The similarities between adult and child learning are also evident in research recommendations for teaching prospective teachers and elementary students as well as my own experience.

In teaching elementary students, and adult pre and in-service teachers, I have found that the learning process is similar for all age groups—using prior knowledge to do math tasks, and connecting multiple solutions to deepen understanding of concepts and how these relate to procedures. The difference is: when working within existing knowledge, adults often have a strong commitment to the elementary math rules and formulas they have been using for years. So using less formal strategies and making connections can be more challenging for them than for elementary students. Normally the adults I have worked with are also motivated to work through the connection process and satisfied by the results and insights they gain. All learners (adults and school children) benefit from starting with experiences with the concrete representations and then moving to more general representations.

This study about how prospective teachers can grow their mathematical understanding is inspired and informed by my 17 years of experience in elementary mathematics education, and, more recently, my 6 years as a teacher education instructor.
My mathematics teaching practice was initially developed in middle-school classrooms,

where I found that research-based, innovative teaching practices (e.g. Boaler, 2008) turned tedious math lessons into engaging activities.

Later, as I worked with in-service teachers as a mathematics coach and additional qualification course instructor, I gained an appreciation of the mathematics teaching challenges facing in-service teachers and learned how to use co-planning and co-teaching to support them in implementing research-based approaches. When I started teaching teacher candidates, I initially used a presentation format for my classes because I thought that was more appropriate for a university setting. But, when I noticed that I was boring my student teachers, I adapted my elementary school practices to support these adult learners in growing their understanding of elementary mathematics in learning environments similar to those they are expected to use with their future students (e.g., Liljedahl, 2016).

My case study draws on the research-informed teaching practices I have been using for over 10 years. As a practitioner doing research, it provided the opportunity to apply a rigorous research lens to my teaching practices.

1.3 The Research Questions

This study addresses concerns about elementary teachers' understanding of mathematics and how teachers can be supported to "transform and increase their understanding of mathematics" (Ball, 1990). It draws on the extensive research about learning and teaching mathematics, in particular: the benefits of teachers learning from new experiences (e.g., Adler, Ball, Krainer, Lin & Novotna, 2005; Mason, 2002; National Research Council, 2001), and how leveraging prior knowledge and making connections in the context of a course designed for teachers' professional learning can deepen mathematics understanding (Martin & Towers, 2016; McGowan, 2017).

My research investigated the elementary mathematics knowledge and learning processes of newly graduated teachers (NGTs) who have completed degree requirements, but have not yet begun teaching. The objective was to study: how they work on and use their prior knowledge and experience to connect mathematical concepts from Ontario's elementary curriculum, in particular:

- How can newly graduated teachers connect their mathematics ideas based on recommended strategies for teaching elementary students?
- How can the approaches recommended for teaching children mathematics be adapted to support newly graduated teachers in growing their existing understanding?
- How does a course that focuses on working with existing knowledge and connection-making support a growth in understanding for newly graduated teachers?

The objective of this investigation is to demonstrate that the research related to teaching elementary students is also applicable to teaching future teachers. In particular, to show how eliciting prior knowledge by doing elementary mathematics tasks and connecting existing knowledge can enhance teacher education by addressing the persistent cycle of teachers teaching the way they were taught.

CHAPTER 2: THE RESEARCH CONTEXT

This study focuses on how newly graduated teachers (NGTs) can connect their mathematical understandings so they are prepared for teaching their future elementary students. The related research falls into three categories: elementary students and mathematical understanding; recommended teaching practices to support elementary students in growing their understanding; teacher education - the knowledge and experiences NGTs bring and the types of experiences teachers needed to bridge research and practice. This chapter covers these areas and the theoretical framework used for the study.

2.1 Mathematical Understanding and School Students

Many educators and researchers use the word “connected” when talking about mathematics and mathematics understanding. However, a review of the literature indicates that few provide a definition of connections in mathematics education. Pirie and Kieren’s Theory (1994) includes the notion of connected by using the term *embedded* to describe how layers of mathematical understanding relate to each other. Wearne and Heibert (1988) describe the connecting process as the construction of links between individual symbols and familiar referents. Specifically, connecting “involves a gradual accumulation of rich meanings for written symbols as the symbols are connected with a variety of appropriate referents” (p. 380).

The work of teaching and learning mathematics involves connecting prior knowledge to new ideas as well as understanding the connections within mathematics itself. Many jurisdictions, including Ontario, have incorporated connecting into their education policies and curricula (e.g. Ontario Mathematics Curriculum, 2005). In what follows, I review the literature on mathematical understanding and its relationship to connection-making for students.

2.1.1 The Importance of Connections to Mathematical Understanding

The notion of understanding and how understanding relates to connections can be traced back to Skemp's (1976) seminal work about the two types of mathematics being taught in classrooms at that time: *instrumental,* knowing how, and *relational,* knowing why. Current researchers still agree that understanding involves both what to do and why to do it (e.g., Stylianides & Stylianides, 2007). Although an instrumental understanding offers the learner immediate rewards by quickly validating the use of rules to get correct answers, Skemp highlighted the importance of relational understanding because it is about how mathematical ideas are "inter-related" and supports students to remember mathematics as parts of a connected whole.

The idea of mathematical understanding as a connected whole has become a key concept in learning and teaching mathematics research. Curriculum documents and related materials view mathematics understanding as a process of making connections between and within mathematical ideas. The National Council of Teachers for Mathematics (NCTM) (2000), for example, called for students to "understand how mathematical ideas interconnect and build on one another to produce a coherent whole" (p. 64). What if a student's mathematical ideas are **not** connected: operations like addition with multiplication or multiplication with division, or representations like diagrams with formulas? If you don't understand the foundational concepts for procedures, like how borrowing rules in the subtraction can be understood as breaking apart blocks, then mathematics would be viewed as a subject of isolated topics or simply a list rules. Certainly this could limit the usefulness of mathematics in practical situations, and impede learning of more advanced concepts.

Skemp's (1976) discussion of instrumental and relational understanding builds on Bruner's (1966) earlier work described mathematical understanding in terms of three modes of representation: *enactive* or physical, *iconic* or pictorial, and *symbolic* or abstract. Later, many researchers asserted that the order of representations also matters. In their study of connecting decimals with concrete materials, Wearne and Hiebert (1988) found that prior instruction focusing on syntactic rules seemed to interfere with making sense of the concept. Their 4th, 5th and 6th grade participants had varying levels of previous experience with decimals: 4th graders had no experience, 5th graders had studied the notation, and 6th graders had studied decimal computations. The instructor first taught all students how to use blocks to represent numbers less than one, using a 1000 small block cube to represent one whole unit, a 100 block piece to represent 0.1, and a 10 block piece to represent 0.01. Then students engaged in activities to support them in recognizing relationships between the blocks and the symbolic notations. In the final step, the students did addition and subtraction questions with decimal numbers, like 2.3 + .62, first using blocks and later using symbols.

Wearne and Hiebert (1988) found that the students without previous experience who used blocks first more readily developed an understanding of one tenth being more than one one-hundredth. Or in other words, these scholars laid claims that starting by developing a conceptual understanding of decimals helped the inexperienced students make sense of syntactic rules more readily than those who had studied decimals previously.

Similarly, Cramer, Post and del Mas (2002) found that teaching resources that emphasized connecting multiple conceptual representations were more effective than a traditional textbook approach, when they compared test scores from sixty-six 4th and 5th grade classrooms studying fractions. In one group, students explored concepts with fractions like equivalence and addition by using fraction circles, chips, pictures, story problems, and written symbols. Students in this group were asked to make connections between different representations and encouraged to discuss mathematical ideas before attaching symbols to them. In the second group, students were provided with textbook illustrations related to fractions with the primary goal of developing competence at the symbolic level. Cramer et al. (2002) found that additional conceptual experiences helped students tackle fraction tasks successfully, even though these participants spent less time learning about the rules.

Using classroom-based research, Pirie (1988) raised concerns about how students' mathematical understanding was being described and split between knowing how and knowing why. Her findings about Katie, an 11 year old study participant who was learning to divide fractions, illustrated how a learner can start with symbolic materials, notice patterns, and then develop and use their own rules for an operation—rules which were the same as the standard algorithm. This particular student's understanding included both the steps for dividing fractions as well as why multiplying by the reciprocal, flipping the fraction, works. Katie made the connection between the initial pie-diagrams and her own rule for dividing fractions by actively working with a peer—talking, drawing diagrams, checking, arguing and trying out new ideas. Her growth of understanding was a continuum of concept formations with no split between knowing how and knowing why.

More recent research continues to consider mathematics as a connected whole with relational understanding including procedures within that whole. Carpenter, Levi, Franke and Zeringue (2005) focused on relational thinking as a way of approaching mathematics in its “entirety rather than as a process to be carried out step by step” (p.54). They argued that knowing how and when to use procedures requires the same flexibility as that involved in relational thinking, and made the case for relational thinking to be a way of investigating connections in the pursuit of learning mathematics with understanding. Molina, Castro and Ambrose (2005) considered relational thinking to be a type of connecting that includes both procedures and concepts; they defined relational thinking as an examination of “two or more mathematical ideas or objects alternatively looking for connections between them” (p. 3).

The research about the order of representations—moving from blocks/concrete objects and pictures to symbolic algorithms—provided the theoretical underpinnings for connecting new ideas with more familiar ones; however, it was often implemented in a step-wise, linear fashion (Cramer, 2003; Hiebert & Carpenter, 1992). The implicit assumption was that learners develop their understanding of mathematical concepts in this one-directional manner (from the concrete to the more abstract symbols). Lesh (1979), Hiebert and Carpenter (1992), and Pirie and Kieren (1994) proposed non-linear models to better reflect the complexity of mathematics connections and potential learning sequences, as discussed below.

Lesh’s (1979) translation model includes five modes of representation: manipulatives, pictures, real-life contexts, verbal symbols and written symbols. This model expanded the scope of representations and emphasized connections within and between them. Hiebert and Carpenter (1992) drew upon Lesh’s model to make the case for mathematical understanding as a web of linear and non-linear connections to things students already know. To illustrate this system, they used the image of a spider’s web as a visual metaphor for internal representation of intersecting lines. They proposed that mathematical ideas can be understood if they are a part of a network. This internal network is made up of mental representations, and understanding is based on the number and strength of the connections among these representations. In one study, for example, students developed a thorough understanding of subtraction when the algorithm was connected with other mental representations such as physical blocks (Hiebert & Carpenter, 1992).

Pirie and Kieren’s (1989; 1992; 1994a; 1994b) Theory for the Dynamical Growth of Mathematical Understanding uses a three-dimensional model layered like an onion to illustrate eight embedded layers of understanding. The idea of embedding means that to grow understanding in an outer layer, the learner must also understand the related earlier layers. Recognizing that sometimes embedding does not occur, Pirie and Kieren (1994) also include the notion of disjointed for understandings that are not connected. The layers in this theory support researchers and teachers in describing how an individual grows their understanding of a mathematics concept. Primitive Knowing, the first layer, is a starting place that includes everything a learner already knows, not just mathematics. Pirie-Kieren’s theory characterizes learning as constantly changing and moving forward and back in a non-linear fashion, so a learner can be observed growing their understanding from their earliest informal representations, or *images* using their term, through the highest layer of formal understanding. Using this theory, pathways can be traced through the eight layers of understanding and mapped onto a diagram to document how individuals develop their understanding.

2.1.2 Making Connections, the Pathway to Deeper Understanding

In the literature, deep understanding is associated with having connections among mathematical ideas. The Pirie-Kieren Theory posits that learners have existing images while also potentially creating new ones related to the topic, and that they move back and forth through layers of understanding deepening their understanding by connecting the less formal images with more formalised mathematics (Pirie & Kieren, 1994). As a result, learners develop a rich set of ideas related to a specific topic as they go back to what they already know embedding, or connecting, informal images with their formal understanding of mathematics. Without these connections, the learner's algorithms are disconnected from their less formal understandings.

In Pirie-Kieren Theory, deep understanding is not a destination, it is a process, so learners continue to deepen their understanding of a topic as they expand their repertoire of images, embedding them with their understanding of more formal representations. The National Research Council (NRC) (2001) says that when learners connect pieces of knowledge to deepen their understanding they can use their knowledge more productively. The NRC suggests that by connecting steps in the algorithm to how place value is being distributed, learners can develop a deep understanding of two-digit multiplication. For example, the algorithm 15 x 12 can be decomposed into 10 and 2 and rewritten like this: (2 x 15) + (10 x 15).

The importance of connected knowledge is reflected in elementary mathematics curriculum documents that make statements about making connections when learning mathematics for deep understanding. The Ontario Ministry of Education, for instance, states “the more connections students make, the deeper their understanding” (Ontario Ministry of Education, 2005, p. 16). The National Council of Teachers of Mathematics (NCTM) (2000) includes mathematical connections in their Standards of Practice: “When students can connect mathematical ideas, their understanding is deeper and more lasting” (p. 64). As the word “connection” suggests, connecting requires two or more images or representations related to a topic so a learner can develop an understanding of the relationships between or among them.

Comparing two solution strategies emphasizes connection making because it requires students to understand both methods and look for the relationships between them, helping them to deepen their understanding of a concept (Takahashi, 2008). Researchers (e.g. Star, Newton, Pollack, Kokka, Rittle-Johnson & Durkin, 2015) discuss the importance of making connections between multiple solutions with the purpose of noticing a solution that takes less time to complete. They associate connection making with being flexible. Flexibility, according to Star and his colleagues (2015), is “the ability to generate, use and evaluate multiple solution methods for given problems” (p.198). Flexibility in number sense develops from opportunities for learners to “tinker with numbers” (Fosnot & Dolk, 2001, p. 117).

Tinkering is working with numbers, for example, turning “unfriendly” problems into “friendly” ones or decomposing numbers to make calculations easier. An example of tinkering with numbers to develop flexibility is provided by Gray and Tall’s (1994) study that looked at how elementary students solved number problems. They found that high achieving students have flexibility, that is, when these students were presented with 19 + 7, they changed the problem into 20 + 6, deriving a new fact from known fact. Students preferred answering 20 + 6 because they could see the answer without regrouping, using friendly numbers in addition.

Boaler (2015) also associated number sense with flexibility and drew on Gray and Tall's work to advocate for ways to learn multiplication facts other than by memorizing them. When working with a multiplication question, like 18 x 5, the teacher may ask students to talk about different ways of solving it mentally, like 9 x 10 (half of 18 and double 5). What is important to note about Boaler's work is that in each example, deriving a different or second way of solving the question is rooted in what a learner already knows—their prior knowledge.

Making connections between a familiar solution strategy and a different method develops flexibility and supports fluency with numbers, a recommendation that has received widespread agreement among mathematics education researchers. Baroody (2006) describes fluency and basic math facts as "efficient, appropriate, and flexible application (p. 22). Similarly, Parish (2014) included flexibility in his description of mathematical fluency, saying that fluency is "knowing how a number can be composed and decomposed and using that information to be flexible and efficient with solving problems" (p. 232). The Common Core State Standards for Mathematics, a curriculum widely used in the United States, includes flexibility as one the tenets of fluency (CCSSI, 2010). For example, one student might solve the question 91-79 by counting backwards from 91 by ones to the answer 12 whereas a second student might count up from 79 to 91 by ones. By investigating connections between the two solution strategies, students can improve their flexibility—have more than one way of doing math and choosing the more efficient way.

2.1.3 Prior Knowledge, the Foundation for Connecting

The action of connecting prior knowledge with new ideas is deemed to be essential for learning with understanding in the mathematics classroom. Using prior knowledge to support future learning is well documented in the literature. Ausubel, Novak, and Hanesian (1968) said that the most important factor for learning and teaching is what the learner already knows. In the land-mark book *Making Sense: Teaching and learning mathematics understanding*, Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier and Human (1997) agreed on the following definition of mathematical understanding: “we understand something if we see how it is related or connected to other things we know” (p. 4). Stylianides and Stylianides (2007) used an expanded definition, “learning mathematics with understanding involves making connections among ideas; these connections are considered to facilitate the transfer of prior knowledge to novel situations” (p.106), and thus positioned prior knowledge and experiences as the starting point for further learning. The research about connecting prior knowledge has also been accepted by curriculum bodies. For example, the NCTM (2009) includes connection making in its definition of sense making, “developing understanding of a situation, context, or concept by connecting it with existing knowledge” (p. 43).

The term *met-before* was coined by McGowan and Tall (2010) to address the significance that past knowledge and experience have on connecting mathematical ideas. In calling attention to met-befores—“mental structure[s] that we have now as a result of experiences we have met before" (p.171)—they make the case for learning being personalized and dependent on prior experience. However, they also recognized that prior knowledge is not necessarily helpful; it may be either supportive or problematic for an individual student. Problematic met-befores impede learning new material due to gaps or misunderstandings in the learner's prior knowledge.

Continuing their research, McGowan and Tall (2013) emphasized the importance of met-befores in learning mathematics because making connections promotes deep understanding. They argued that new learning experiences that build on prior experiences are better remembered, however, when learning experiences are not connected with a student's prior knowledge, the material is either not learned or easily forgotten. McGowan and Tall (2013) also demonstrated that mathematical symbols with multiple meanings can become problematic for learning. They discussed this in terms of the minus sign, which can mean take away, negative numbers, or adding the inverse.

By analyzing interviews with undergraduate mathematics students, they found that learning about the minus sign in one context became problematic for learning mathematics in a different context. For example, -3 could signify either take away 3 or a negative integer 3. So while prior knowledge is always an important factor in further learning, it is not always helpful. It is sometimes an obstacle that requires investigation by the teacher and the learner.

2.2 Recommended Practices for Teaching School Students

2.2.1 Supporting Students in Connecting with Prior Knowledge

Educational research supports learning through connection-making. Supporting children in connecting with prior knowledge is a common classroom practice that is emphasized in literature. In Pirie and Kieren's Theory for the Dynamical Growth of Mathematical Understanding (1994), learners continuously build on their prior knowledge starting with their Primitive Knowing and existing images—mental or physical ideas—they may have related to the topic they are studying. More recently, McGowan (2017) recommended that teachers consider "the effects of existing knowledge—both positive and negative—that students have now as a result of experiences they have met before, at every level of development" (p. 24).

In her book on classroom talk, Cazden (2001) used the NRC's key finding, "[a]dults help children make connections between new situations and familiar ones" (p. 61) to highlight the teacher's role in scaffolding student learning. Several different teaching practices can be used to support students in accessing their prior knowledge. For example, Martin and Towers (2016) studied a high school teacher using a whole-class discussion to support his students in connecting their prior understanding of slope to the new topic of vectors. At the beginning of the lesson, the teacher asked his students to recall what they had previously learned about slope. The action of bringing students into conversations about what they already know helps to build understanding. Students may also connect with their prior knowledge by applying what they know to doing math tasks selected by the teacher.

Lampert (1991) drew attention to the ongoing action of making connections for both teachers and students in her description of non-traditional teaching and learning of mathematics.

In this view, teachers select tasks that connect with students' prior knowledge, and students connect previous understandings with the new material. To support students in making connections, for example, a teacher might select a subtraction task like 91 - 79. Students working on this task use their existing ideas about regrouping in addition to making connections about how regrouping is similar and different in subtraction.

Making predictions is another teaching strategy for eliciting students' prior knowledge and helping students connect to new material. In their study of middle school students, Kasmer and Kim (2012) found that making predictions provided all students with opportunities to evoke prior knowledge and use it to bridge to the new material. At the beginning of a lesson about linear and exponential relationships, the class was asked to predict which of two rates would be steeper if drawn as lines on a graph and then explain why. This study found that making explicit connections to their previous understanding of linear or exponential equations helped students make sense of the mathematics they were studying. Comparing the work at hand with their predictions gave students an opportunity to make connections among related concepts.

Wright (2014) also found predictions to be valuable teaching actions, when he studied a sample of students participating in the New Zealand Numeracy Development Project. While exploring tools such as a double number line, the teacher encouraged students working in small groups to think ahead about the results of proportional reasoning problems, like finding the value of a 40 percent discount applied to a $16 item. Using the Pirie-Kieren model, Wright (2014) found that an instructional sequence of folding back as well as folding forward helped students to develop understanding, when the students were asked anticipatory questions while using blocks and number lines.

Though using students' prior knowledge and connecting it with tasks and materials may seem to be a single teaching practice, it usually involves several teaching moves. Supporting students to recalling involves the deliberate action of gathering a learner's prior knowledge, similar to Pirie and Kieren's (1994) action of folding back to earlier understandings. As discussed, there are several ways to elicit students' prior knowledge. The teacher may, in fact, decide to draw on more than one representation or on multiple aspects of the learner's prior knowledge. Using this existing understanding to support further learning often requires the additional actions of comparing the recalled images to the new topic at hand and finding the connections among these images.

2.2.2 Comparing and Connecting Multiple Solution Strategies

When a teacher's practice includes having their students investigate more than one way of solving a mathematics question, they support connection making. Rittle-Johnson and Star (2007) investigated the value of comparing and contrasting multiple solutions and accessing students' prior knowledge. They studied the effect of multiple solution methods on students' conceptual and procedural knowledge of linear equations. Two tasks showing different solution methods were presented to two groups of grade 7 students, the compare group and the sequential group. Each solution showed a different method for solving an equation like $5 (y+1) = 3 (y+1) + 8$ with one involving more steps than the other. The compare group students were asked to compare the two methods side-by-side whereas the sequential group looked at two different methods one after the other and reflected on the methods without directions to compare.

Rittle-Johnson and Star (2007) found that students in the compare group made connections between the time it took for each solution to complete. In other words, the more efficient strategy. In post-tests these students were twice as likely to use more efficient methods. In addition, students in the compare group showed signs of making connections with solution strategies that were more efficient, thus developing their flexibility—using more than one way of doing the mathematics.

Lee (2016) investigated how grade 9 and first year university students studying mathematics made connections using *look-back strategies.* In particular, Lee (2016) investigated how the use of Look-back strategies influenced problem solving performance using multiple solution methods. She examined and categorized students' use of four look-back strategies: recall, review, check and compare, and found that students who spontaneously looked back more frequently performed better. Her review category builds upon George Polya's (1945) four phases of problem solving which are still in curricula, including Ontario's mathematics curriculum (2005): understanding the problem, formulating a plan, executing the plan and looking back. Though in Polya's work, "looking back" means reviewing completed work, rather than retrieving prior knowledge.

In Japan, mathematics lessons often include a dynamic, collaborative whole-class review process called *neriage*, or "polishing up ideas" that focuses on connecting mathematical ideas (Shimizu, 1999). At the onset of a lesson, students access their prior knowledge by developing their own solutions. Then the teacher selects a few solutions, and improves these solutions by leading whole-class discussions that focuses on comparing the different methods used, thinking deeply about their meanings, and highlighting particular parts of the solutions so students learn the goal of the mathematics lesson (Takahashi, 2008). Polishing supports learning by eliciting students' prior knowledge as well as allowing them to compare and make connections between mathematics solutions. For success in polishing, the teacher needs the mathematical understanding to choose tasks that challenge students and illustrate the goal of the lesson. In addition, the teacher is able to select solution strategies that can be connected without telling the students the answer.

Smith and Stein's (2011) provide an expanded list of recommended teaching practices for orchestrating classroom discussions that includes some approaches similar to polishing and focus on connecting. They describe five practices for teachers to use: anticipating, monitoring, selecting, sequencing and connecting. When *anticipating*, the teacher predicts several solution strategies to a lesson task that they expect their students might use. *Monitoring* occurs as the teacher circulates around the classroom, making observations about students' mathematical responses. In *selecting*, the teacher chooses a few samples of student work to share and discuss with the whole class. Then in *sequencing,* the teacher decides which selected solution to discuss first—usually this would be the solution that the majority of the class has done, because it represents what the class already knows and encourages connecting with a second, less familiar solution strategy.

In *connecting*, the teacher draws attention to the connections between the mathematical ideas in selected student samples to support all the students in making connections. For example, the teacher might allude to how two students' strategies are similar or different from each other (Hodge and Cobb, 2003) or ask students to identify what is similar or different. In an elementary classroom, this might sound like students and teacher talking about how borrowing, a rule in the subtraction algorithm, is similar and different to trading in one tens-block for ten ones-blocks. Based on these two different solution strategies, students could connect how breaking apart blocks is similar to borrowing a 10 in the algorithm. If teachers don't have their own experiences with connecting, they would typically not be able to lead their students through this type of exercise where they make connections themselves.

2.2.3 Teaching that Connects Students' Ideas and its Challenges

The recommended teaching practices embody a shift toward student engagement in activities that access their prior knowledge and promote their growth of understanding. In elementary classrooms, students are thinking and talking while they are doing math tasks, rather than listening to the teacher explain how it is done. In discussing multiple solutions, Smith and Stein (2011) distinguished between students building on each others' ideas and teachers presenting different ways to solve a problem. Boaler (2015) recommended that classroom discourse shift from teacher-to-student to students talking with each other (i.e., making connections amongst their ideas).

Warner (2008) made a similar point, and recommended teaching actions that encouraged student interactions, such as peer-to-peer explanations that foster students making connections between their prior knowledge and new mathematics understanding. Mathematics classrooms are no longer about the teacher at the front of the room reminding students what they already know, showing and telling them how to do mathematics, and thereby sharing their own preferred solution strategies. Teachers now need practices that elicit student ideas, encourage discussion and provoke further work by responding to students in the moment. Asking situationally appropriate questions is a versatile teaching action that supports student to make connections.

Prompts are well-timed interventions by a teacher that support students' thinking, often by revisiting their prior knowledge to connect it with what they are learning. By using prompts, teachers can help students get unstuck or direct attention to useful ways to make progress in the mathematics classroom (Mason, 2014). Taking a close look at teacher actions that support Pirie-Kieren's process of recursive growth, Martin (2008) categorized these intentional interventions as *explicit* and *unfocused* comments or questions that support learners in revisiting earlier understandings and connect their understanding. An explicit intervention tells the learner exactly what to do or look for, like find this number in that diagram, whereas an unfocused prompt is more general, like suggesting that the student try another method. Prompting is a non-judgemental way of meeting students where they are, while supporting further work and the progression of learning.

Schoenfeld (2014) incorporated many recommended teaching practices, including connections, into his notion of *powerful classrooms*—classrooms where students perform well on mathematics tests and problem solving. His rubric entitled *Teaching for Robust Understanding of Mathematics (TRU)* identifies five dimensions of powerful classrooms and the teacher actions and classroom activities that support achievement in each dimension, as summarized below:

- The Mathematics, classroom activities that support making connections between procedures, concepts and contexts.
- Cognitive Demand, the teacher engages students in productive struggles by providing hints or scaffolds.
- Access to Mathematical Content, the teacher maximizes mathematical participation among students.
- Agency, Authority, and Identity, the students explain their mathematical ideas and build on their peers' ideas.

- Use of Assessment, the teacher solicits student mathematical ideas and is responsive to their thinking.

In Schoenfeld's (2014) rubric, teachers support their students' learning by providing experiences in making connections, continuing to work based on a prompt, and talking about their own ideas as well as the ideas of their peers. Importantly, the teachers make classroom decisions that respond to a student's prior knowledge and experience and emphasize connections.

If all researchers' recommendations were taken together, what would an elementary classroom look like; what would the students be doing and what would their teachers be doing? The students would be busy working on math tasks and talking among themselves about what they are doing and thereby making connections. They would be bringing what they know and growing their understanding through making connections. The teacher would have prepared for this class by selecting the tasks and considering potential student responses that emphasize connection making. Then during the class, after initiating the classwork, the teacher would become a participant in the students' mathematics work, asking questions and providing prompts to encourage students to share their ideas and make connections. Respecting a learner's need to grow their own understanding, the teacher would use their knowledge of mathematics to provide insights into what students know and how they are progressing through connection making, rather than to provide answers or explain the topic to their students. This in-the-moment nature of teaching has been investigated by researchers in terms of developing the awareness, noticing and listening capabilities of teachers.

Working in the dynamic classroom environments that researchers recommend makes additional demands on teachers. Not only do they need to know elementary mathematics and how to teach this subject to young learners, now they need plan for and operate in real time with students engaging in activities that foster mathematical connections. Connection making, according to Mason (2002) is a "juxtaposition of the present with the past in some way" (p. 84), and this requires a teacher's *awareness.* Mason (1998) describes awareness as "a complex concept comprising both conscious and unconscious powers and sensitivities which enable people to act freshly and creatively in the moment" (p. 243). For Mason (1998), awareness-in-action is "sensitivities to certain situations which provoke and enable action" (p. 258), which includes the ability to see similarity and generality. In the context of teaching mathematics, awareness-in-action is needed to provide well-timed prompts to encourage further student work or look for connections. Mason's (1998) second level of awareness, awareness-in-discipline, is concerned with the "habits of thought, forms of fruitful questions and methods of resolution of those questions" (p. 260). For elementary mathematics teachers, this means using their mathematics knowledge in a situationally appropriate manner to foster student progress without providing direct instruction or telling the answer. Mason (1998) describes his third level of awareness, awareness-in-counsel, in terms of a "collective wisdom" that is "separated from immediate action" (p. 261). Awareness-in-council promotes a sense of community among learners, an ideal that teachers can foster using group work and a sense of shared purpose in their classrooms.

Awareness is a set of skills that teachers can learn though practice. To some extent, they can plan for it by using recommended sequencing, selecting relevant tasks, and anticipating student responses, as well as considering what unexpected ideas might emerge. Rowland, Thwaites and Jared (2015) describe unexpected events as contingent situations that are challenging to deal with because they unpredictable and, therefore, there is no pre-set way to deal with these events. Preparing teachers to act in-the-moment requires them to actually experience contingent situations, because no amount of planning can predict all responses or reactions from a classroom full of students all busy working on mathematics.

2.2.4 Implementing Recommended Teaching Practices

The above mentioned teaching recommendations have been described in elementary school policies and in Ontario's (2005) curriculum, including references to using multiple representations and using prior knowledge to connect mathematics ideas. In particular, the Ontario curriculum calls for students to make connections by "seeing the relationships among procedures and concepts," and recommends that "[e]ffective instructional approaches and learning activities draw on students' prior knowledge, capture their interest, and encourage meaningful practice" (p. 16). In addition, the NCTM (2000) has recommended that teachers "ask guiding questions" that refer to earlier lessons on the topic related to what they're studying to make connections. Important to note is that these questions are intended to make students "explicitly aware of the mathematical connections" (p. 64) within the topic. In an elementary classroom, for example, this can sound like the teacher asking students working on subtraction to notice the relationship between what they know about breaking apart blocks and what they're learning about regrouping in an algorithm.

Although some elementary teachers are implementing these recommended practices, many are still focusing on traditional approaches such as textbook work—reviewing a printed example, then practicing the same procedure using a list of similar questions. Perhaps this is not surprising because researchers have observed that these recommended teaching actions that focus on connections are challenging for teachers to implement, especially when they have not experienced the recommended practice themselves. Durkin, Star and Rittle-Johnson (2017), for example, found that teachers made little use of the curriculum resources that they developed to support comparing and connecting multiple solution methods—such as a which-is-better table showing two different solution methods for an algebra question.

Ideally, elementary teachers would have a deep understanding of the mathematics they are teaching, meaning they should have a connected understanding. Specifically, they would be conversant with all kinds of representations for mathematical ideas and understand the connections between them. This would support every facet of their role from anticipating what their students might do through classroom interactions like providing a prompt to elicit further work and draw connections. To build on students' existing knowledge, teachers need to be aware of multiple solution strategies and the many nuanced ways to support students in seeing relationships among strategies. In presenting two different solution strategies for subtracting, for example, the teacher could ask if they can find ideas from one solution in the other, thus building on each others' ideas and making connections. Teachers would also have experience learning and teaching in a dynamic classroom environment that calls on their mathematical understanding so they are in the moment with their students, following the flow of students' ideas and responding appropriately to support students to make connections.

In reality, elementary teachers typically have a fragmented knowledge of mathematics and little experience with recommended teaching methods based on their early learning experiences. Thirty years ago, Ball (1988) published the observation that traditional teaching and learning experiences were the norm in many elementary classrooms, because teachers tend to teach the way they were taught. Prospective teachers' experiences may have begun with blocks and diagrams and moved on to learning formulas, without understanding the foundational ideas and how they are connected (Ball, 1990; Ma, 1999). Though many prospective teachers can recall borrowing rules for subtraction, for example, they have either forgotten or lack experiences breaking apart blocks to illustrate how regrouping works.

Using Harel and Tall's (1991) term, teacher candidates' knowledge can be characterized as *disjunctive generalizations*—general rules that are not connected to concepts. Lacking a connected understanding of mathematics, they have difficulty supporting students in making connections. As Smith and Stein (2011) observed years later, connecting mathematics, a recommended teaching practice, is challenging for teachers.

This gap between educational research recommendations and teaching practices has persisted. A decade later, Ball (2001) found that, instead of students doing mathematics and connecting their ideas, they were following the teacher's instructions. Boaler (2008) also commented that most classroom teachers were using the methods they used in school, rather the methods that are recommended. Today educational research (e.g. Hiebert, 2013) and updated curricula are still not making it into classrooms because many elementary teachers are accustomed to showing their students how math is done, explaining the steps, and then asking students to practice doing similar tasks (Ball, 2001; Orpwood and Brown, 2015). As Boaler (2015) said, these traditional models "set [students] on the wrong path trying to memorize methods, instead of interacting with number flexibly" (p. 2). This approach, which McGowan (2017) says occurs far too often in classrooms, is often associated with showing "students how to use a rule to get the 'right' answer" (p. 20).

Lacking relevant experience, teachers are often not well prepared to implement the research recommending that teachers encourage their students to draw on what they already know and engage them in doing and talking about math tasks in the classroom. Activities that promote student thinking and learning are more effective than simply listening and following instructions (Boaler, 2008). Many classroom teachers are still falling back on the methods that they experienced in school, rather than the methods that are currently recommended (e.g. Boaler, 2008), and unwittingly perpetuating the teaching cycle of showing and telling students what and how to do mathematics. As a result, many elementary students are not learning mathematics with understanding, despite the years of research related to mathematics understanding, updated curricula and recommended teaching practices. We need to look at teacher education to investigate what might be causing this.

2.3 Preparation for Teaching Elementary Mathematics

2.3.1 What Teachers Need to Know

Much of the research related to teacher mathematics education has focused on what teachers need to know about mathematics to support students' learning or about how they need this but much more to be effective teachers. In terms of what teachers need to know, researchers have developed several frameworks: *Subject Matter Knowledge* (SMK) and *Pedagogical Content Knowledge* (PCK) (Shulman, 1986), *Mathematics Proficiency* (NRC, 2001) as well as *Mathematics Knowledge for Teaching* (MTK) (Ball, Hill & Bass, 2005) and the *Knowledge Quartet* (Rowland, Huckstep & Thwaits, 2005). In parallel, several other researchers have focused on connecting mathematics ideas, in particular, the connectionist orientation (Askew, Rhodes, Brown, William & Johnson, 1997) and profound understanding (Ma, 1999). These approaches have described the end results of teacher preparation programs—what teachers need to know and be able to do—rather than how prospective teachers can develop the required knowledge and skills.

Shulman (1986) categorized what teachers need to know as *subject matter knowledge* (SMK), the teacher's knowledge of mathematics content and concepts, and *pedagogical content knowledge* (PCK), the teacher's knowledge of how to make the subject understandable to others. Ma's work (1999) blurred the lines between SMK and PCK when she linked a teacher's connected knowledge with the recommended practice of supporting their students in making connections. For example, when a teacher knows that regrouping can be illustrated using blocks, their practice includes specific questions that prompt students to connect the action of breaking apart blocks to the formal rules of subtracting, thus their knowing and practice are intertwined.

Education researchers continue to cite Shulman's (1986) work as underpinning their theories. Deborah Ball, an eminent researcher in the field of elementary mathematics education, has maintained for decades that an elementary teacher's knowledge is both specific and special. In the dozens of papers she has authored and co-authored (e.g. Ball, Hill & Bass, 2005), Ball built on Shulman's (1986) ideas of PCK and SMK to provide mathematics educators with specific examples of the interplay between PCK and SMK. She said "knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably" (p. 21). For example, Ball and her colleagues (2005) argued that knowing how to perform the multiplication algorithm alone is not enough for teaching. When a teacher's mathematics understanding is limited to an algorithm, their lessons reflect a *show and tell* (Ball, 2001) model for teaching and learning. In this paradigm, the teacher looks for mathematics understanding as either correctly or incorrectly following steps that the teacher has outlined. Students' own mathematical ideas are not considered.

In collaboration with mathematics educators and mathematicians, Ball participated in creating a framework called *Mathematics Knowledge for Teachers* (MKT). This framework brought PCK and SMK together and served to identify the effects that teacher knowledge has on learning mathematics (Hill, Schilling & Ball, 2004). The framework focuses on "whether teachers' knowledge is procedural or conceptual, whether it is connected to big ideas or isolated into small bits" (Hill & Ball, 2005, p. 332). To make the connection between the teachers' knowledge and related teaching practices, MKT includes the duties associated with the "work of teaching," defined as:

> explaining terms and concepts to students, interpreting students' statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and effects of teachers' mathematical knowledge on student achievement providing students with examples of mathematical concepts, algorithms, or proofs. (Hill et al., 2005, p. 373)

Drawing upon the MKT framework to identify and classify mathematics connections made by prospective teachers, Eli et al. (2013) suggested that using MKT is a benefit for mathematics educators because it can support teachers to "access and unpack knowledge in a connected, effective manner" (p. 122). Unpacking in a connected manner requires drawing on, using and connecting with prior knowledge. Yet, the study does not refer to the use of a teacher's prior mathematics knowledge and experience.

Since the conception of MKT, Ball's work continues to address what teachers need to know and be able to do in order to teach effectively. To make the case for a teacher's content knowledge of mathematics being special, Ball, Thames, and Phelps (2008) once again pulled apart SMK and PCK, and illustrated this idea using an egg shape, which has since become known as *Ball's Egg*. This egg is divided into SMK and PCK, and then each part was further subdivided. SMK is split into three sub-categories—common, horizontal and specialized—with horizontal and specialized content focusing on teachers. Horizontal knowledge includes the ability to see connections between mathematical ideas that appear later in the curriculum, for instance, how equal groupings in the primary grades develop into multiplication and proportional relationships in later grades. Specialized content knowledge includes the ability to connect a topic, such as subtraction, from prior to future years (Ball et al., 2008). In practice, this would look like a grade two teacher connecting the previously learned concept of regrouping to make a 10 in addition, to regrouping through borrowing a 10 in subtraction.

Askew and his colleagues (1997) identified key factors that allowed teachers to enact recommendations for teaching numeracy in their classrooms. Their case study identified three types of orientations towards teaching; connectionist, transmission and discovery. They argue that a connectionist orientation is ideal because students in these classrooms made greater gains in numeracy compared with students who were in classrooms with teachers who held beliefs associated with the other two orientations.

Through classroom observations and interviews, Askew et al. (1997) identified five connectionist-oriented teachers based on their beliefs about what it means to be a numerate student, how students learn and how best to teach them. A connectionist-oriented teacher believes that to be numerate, learners have an "awareness of the links between different aspects of the mathematics curriculum," and that students "become numerate through purposeful interpersonal activity based on interactions with others" (p. 35). Lastly, and perhaps most pertinent to this study, is that a teacher with this orientation believes that the best way for students to explore solutions and choose efficient strategies is through dialogue with their teacher and peers (i.e., students making connections).

2.3.2 Mathematics for Teachers as Deep, Connected Understanding

Recently, the Ontario Ministry of Education funded an analysis on the state of mathematics education in Ontario that emphasized the importance of a deep, connected understanding for teachers (Orpwood & Brown, 2015). This investigation was motivated, in part, by college deans of technology concerns about the number of students who were unsuccessful in their chosen programs because of weaknesses in their mathematics knowledge. The resulting report *Closing The Numeracy Gap*, Orpwood and Brown (2015) reviewed research reports from agencies that test students in elementary, secondary and college levels and found that mathematics scores have been declining at all levels, painting a “gloomy picture" of mathematics education in Ontario. The report’s section on teacher education recommends a program of study that develops a teacher’s understanding beyond knowing how to do mathematics, saying that “teachers need to have a deep understanding of a mathematics concept and its connections to other concepts, to know the why behind the mathematics” (p. 16). Having a connected understanding allows teachers to help students who experience mathematics challenges. A teacher who understands the relationship between addition and subtraction, for example, can facilitate a student’s transition from using addition strategies to using comparable subtraction strategies.

Earlier, Ma (1999) made a similar recommendation after comparing elementary teacher knowledge in China and North America, and finding that North American teachers over emphasized procedures in classrooms due to limitations in their understanding of the underlying structure in mathematics. Showing and telling students about rules, like borrowing in the subtraction algorithm, dominated mathematics lessons. Ma (1999) recommended that teacher knowledge reflect a *profound understanding of fundamental mathematics*, including—connectedness, multiple perspectives, basic ideas and longitudinal coherence (Ma, 1999, p. 122). In terms of connectedness, a teacher who has a profound understanding of fundamental mathematics makes connections "between concepts and procedures", and "among different mathematical operations and sub-domains." Multiple perspectives, basic ideas and longitudinal coherence are "kinds of connections" that provide students with different methods to grow their understanding of mathematics (p. 123).

Davis and Simmt (2006) used multiplication to illustrate why teachers need a deep understanding to support their students in learning the multiple meanings of a concept, how meanings originate and evolve, and how they interconnect under different contexts. In their article, they provide detailed examples of multiple representation of multiplication. For example, primary students experience multiplication as equal groupings; and this meaning evolves in the Ontario curriculum for junior level students when they connect multiplication to area—rows and columns. As students continue to move through the curriculum, they are exposed to other meanings of multiplication like rate of change. This longitudinal coherence, as described by Ma (1999), involves connecting multiple representations within a concept.

When discussing mathematics for teaching, Adler and her colleagues (2014) said, "Deep subject knowledge is connected knowledge" that supports "flexible thinking and problem-solving or doing of mathematics" (p. 16). In their study, they interviewed 18 secondary teacher candidates who had taken a Mathematics Enhancement Course (MEC) designed to unpack the participants' previous understandings and to support them in a deepening their understandings of mathematics for teaching. When talking about their experiences, the teacher candidates discussed how reasoning, connectedness and developing a mathematics disposition were related to an in-depth understanding of mathematics.

Orpwood and Brown (2015) summarized the benefit of deep mathematical understanding for teachers, as follows:

> Teachers who are themselves strong mathematically will therefore make connections amongst mathematics concepts so that students gain a rich understanding of the landscape that each concept occupies. Making these connections transparent for students and making 'numeracy across the curriculum' a reality will help prepare students for the more complex mathematics that will come later. (p. 11)

2.3.3 Putting Knowledge into Practice

The National Research Council (NRC, 2001) characterized the work of mathematics teaching in its *mathematics proficiency* framework using a picture of a braided rope to illustrate how five strands are connected and work together. To describe what teaching and learning mathematics should look like in schools, the NRC reviewed the relevant research and endorsed a mathematics proficiency framework with five strands to support learning mathematics in schools: Conceptual understanding—comprehension of mathematical concepts, operations and relations; Procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently and appropriately; Strategic competence—ability to formulate, represent and solve mathematical problems; Adaptive reasoning—capacity for logical thought, reflection, explanation and justification; Productive disposition—habitual inclination to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001, p.116).

Mathematics Proficiency has been used by mathematics education researchers to study how the above mentioned strands were connected with respect to prospective teacher's knowledge. In Kaasila, Pehkonen and Hellinen (2010) studied pre-service teachers' conceptual understanding of division without using the algorithm and used the proficiency framework to report their findings. Their question was: if we know the fact that $498 \div 6 = 83$, how could $491 \div 6$ be determined? This asks teachers to derive a fact from a known fact. The researchers outlined the solutions that satisfied this connection using the following reasoning:

> If 6 is contained 83 times in 498, it is contained 81 times in 491. To 492 it would go 82 times but because 491 is one less, 6 goes into it only 81 times. Only one unit is missing for 6 to go 82 times to the dividend. Hence the remainder is 6 – 1. Result: 81 remainder 5. (p. 11)

In this explanation, the teacher connects what is already known with a reason for figuring out or deriving what is not known. In so doing, the teacher is using prior knowledge in the same way that students do to grow their understanding of mathematics. Less than half of the pre-service teachers were able to provide complete solutions. Kaasila, et al. (2010) recommended that pre-service teachers be provided with more experiences working with questions that require adaptive reasoning.

Hiebert, Morris and Glass (2003) drew on the NRC's mathematics proficiency as a framework for how school students can learn from experiences when their teachers experiment with lessons. They also proposed using it as a model for teaching prospective teachers that focuses on learning from experiences similar to those of a practicing teacher. Teacher candidates need to learn how to do the same work as in-service teachers: designing lessons with learning goals, monitoring their implementation, collecting feedback, and interpreting the feedback in order to revise and improve future lessons. Hiebert et al., (2003) recommend that teacher preparation programs focus "on helping students acquire the tools they will need to learn to teach rather than the finished competencies of effective teaching" (p. 202).

The Knowledge Quartet (KQ**)** framework was proposed by Rowland et al. (2005) to offer the mathematics researcher an option that relates SMK and PCK to classroom practices. The KQ consists of four dimensions—*foundation, transformation, connection and contingency*—that are used as a tools for making observations about a teacher's knowledge and how it comes into play in a classroom. To characterize KQ dimensions, Rowland and his colleagues analyzed videotaped lessons of a teacher candidates teaching lessons in their teaching placements.

Rowland and his colleagues' (2005; 2014) first domain foundation is a Knowledge category focusing on teachers' theoretical backgrounds, including their knowledge about mathematics and educational literature plus their related beliefs. The other three categories are described as knowledge-in-action. Transformation is the teacher's ability to transform their content knowledge into forms that support students' learning. Connection is a combination of understanding the coherence of mathematics plus skills in managing class discussions and sequencing mathematics topics. It includes connection-making as teaching actions, like anticipating decisions about sequencing, and making connections between procedures and concepts. Notably the well-accepted practice of using prior knowledge is not explicitly included in the Connection category, though it is implied perhaps in managing class discussions, sequencing topics, and connecting procedures and concepts. The final category Contingency deals with unpredictable nature of classroom teaching, like thinking on your feet and responding to children's ideas in the moment. A teacher might deviate from their lesson plan, for example, in order to respond to a student's idea because this idea provides learning opportunities.

Though the Knowledge Quartet succeeds in aligning elementary mathematics subject matter with related teaching actions, it focuses on what teachers need to know and be able to do, but not on how they can develop the knowledge and skills. This is a significant omission, given that the last three KQ categories are described as knowledge-in-action. As Mason and Davis (2013) demonstrated in their in-depth interviews with teachers, knowing that an action is needed, even knowing how to do the action, does not necessarily prepare a teacher for acting in the moment. As they said, "what matters is what actions come to the surface." (p.187) So based on teachers' mathematics experiences, in the moment they may explain how to do task, rather than use the prompts as they had planned to use. Whether teaching actions are planned or extemporaneous responses to contingencies, teaching in the moment is a skill teachers need for implementing recommended classroom activities that promote children's thinking, doing and expressing their mathematics ideas.

2.3.4 Concerns about Teacher Mathematics Education

The stage has been set for transforming teacher education—theories about mathematics understanding, recommendations for teaching, frameworks for teacher education and insights into the need for teachers to have deep understanding —plus curricula that incorporate key research recommendations. Yet progress in teacher education has been slow, and researchers continue to identify issues with teacher preparation for teaching mathematics and offer few solutions to how teachers can connect their prior mathematical knowledge.

Several approaches have been used to address teacher preparation, including higher level mathematics training for teachers and step-by-step methods for teachers to follow. In addition, teacher educators often ask prospective teachers to reflect upon how their past learning experiences may have differed from the ones they are studying. These approaches have not addressed the pressing issue of teachers' mathematical knowledge and experience at the level they are teaching.

Teacher education researchers have found that mathematics content courses are not effective for teacher preparation. In her early work, Ball (1990) debunked recommendations that elementary teachers participate in undergraduate mathematics courses to ensure subject matter knowledge. Other researchers have also observed that prospective teachers who have studied university level mathematics are not necessarily prepared for teaching mathematics when they arrive in the classroom (Hart and Swars, 2009; Beswick, 2012; Grossman, Hammerness and McDonald, 2009).

To investigate how mathematics courses are perceived by teacher candidates, Hart and Swars (2009) studied prospective teachers' experiences in a mathematics content course led by instructors without a background in education. In their interviews, the teacher candidates said the material was inappropriate because it lacked a focus on elementary concepts and how children might approach topics. In addition, the participants observed that the course leader's disposition made them feel as though they were not capable of doing mathematics. They also said that the pedagogical approaches in the course were poor—frequent use of instructor presentations with students expected to take notes. Consistent with NRC's (2001) recommendations for teacher preparation, Hart and Swars' (2009) interview findings reinforced that "successful experiences in learning mathematics content during teacher preparation should involve methods of inquiry and problem solving, as well as a presentation of content in such a way that students appreciate the applications of mathematics" (p.169).

Though researchers have recommended that teacher educators leverage prior knowledge when working with prospective mathematics teachers (Ball, 1990), few studies have focused on making mathematics connections in teacher education (Eli, Lee & Mohr-Schroeder, 2013). Instead, research has focused on the knowledge required for teaching and on teaching methods. Teacher educators typically give teacher candidates resources and have them plan lessons using these resources (e.g. Durkin, Star & Rittle Johnson, 2017). Such resources might tell the teacher candidates several solution strategies and exactly how they are connected.

Learning experiences that focus on telling prospective teachers what to do and how to do it potentially reinforce what they experienced as elementary school students, perpetuating the cycle of teaching the way they were taught. Thus elementary mathematics teachers may continue to teach using a *show and tell* (Ball, 2001) approach—the teacher shows and tells students what to do, then the students follow these instructions by doing what the teacher did, rather than using activities like the group work on tasks that elicit multiple students ideas recommended by Boaler (2008) for teaching mathematics.

Gainsburg (2012) investigated whether in-service teachers were teaching in ways that were recommended by their program of study, particularly their university course on reform-based methods for teaching mathematics. Of the nine reform-based teaching practices that Gainsburg (2012) surveyed, three involved connecting students' understanding to topics within and outside the domain of mathematics, as well as connecting to real-life contexts. Though the participating in-service teachers valued their educational experience, they found implementing the practices challenging due to their own limited experiences with conceptual (e.g. lesson plan templates) and practical tools (e.g. strategies for immediate use).

Gainsburg (2012) also discussed the "two worlds" that exist between the university setting and school classrooms. She said that bridging these two-worlds requires teacher preparation programs to change their methods by offering courses that focus on practical tools before discussing concepts of teaching and learning. Other researchers agree with her finding that mathematics courses for prospective teachers should be based on practical experiences (Grossman, Hammerness & McDonald, 2009; Hart & Swars, 2009; Liljedahl, Rösken & Rolka, 2006).

In addition, teacher education courses focusing on *tips and tricks* from the teacher educator's classroom practice (Loughran, 2005) or learning *about* instructional methods (Grossman et al., 2009) have failed to provide teacher candidates with experiences connecting their prior knowledge and experiences with new material. The teachers who participated in the Gainsburg (2012) study reported that they learned most by seeing and doing, but mainly by doing (i.e., learning within or through practical strategies). Based on these findings, she recommended that teacher preparation offer a variety of opportunities for pre-service teachers to use teaching practices and tools to increase the likelihood that these methods will be relevant to their future in-serve teaching. The number line, for example, is a practical tool that can be used to represent a variety of topics like counting, adding, subtracting and proportional relationships. Teacher candidates need to learn through and within their use of the number line.

The evidence is in—lecture-format teacher education courses are not working, even when their delivery of content is commendable. Learning about teaching is not helping teachers to enact the theories or teaching practices that they study, nor is it addressing deficits in teachers' mathematics understanding. Teacher candidates have said that they learn best by doing (ie., learning within and through the use of a strategy or tool). Absent the necessary doing experiences, teachers are not implementing what they have learned in their teacher education classes. To break the cycle of teaching mathematics the way teachers were originally taught (Ball, 1988), researchers have recognized the benefits of teachers learning from within new experiences (Adler, et al., 2005; Mason, 2002). Furthermore, researchers continue to make recommendations for universities to offer methods courses that explore tools and strategies that the teacher candidates will themselves will be using as teachers (Hart & Swars, 2009; Gainsburg, 2012; Liljedahl et al., 2006).

2.3.5 Evolving Practices within Teacher Education

Within the last decade, mathematics education for teachers is moving beyond experts at the front of the room showing and telling future teachers what and how to teach. Grossman and her colleagues (2009) proposed teacher education "move away from a curriculum focused on what teachers need to know" to a "curriculum organized around core practices" (p. 274) that focus on *pedagogies of enactment* so that teacher candidates learn a core set of practices, such as anticipating students' responses. Grossman and her colleagues go on to recommend preparing beginning teachers by having them rehearse in a teacher education classroom rather than actual classrooms "can help novices hone their practice and prepare them for when they will need to respond in the moment" (p. 279). Since Grossman and her colleagues proposed pedagogies of enactment, much has been written about rehearsals and how they can prepare novice teachers to learn within and from practice (Lampert, 2010).

Rehearsals have become a form of clinical practice to support what many researchers refer to as *ambitious teaching* (Lampert et al., 2013). Ambitious teaching is considered adaptive teaching because it focuses on listening and responding to students' mathematical ideas in the moment (with a focus on connections). To develop knowledge, skill and identities of ambitious teaching, teacher candidates require opportunities to practice or rehearse things they do not yet know how to do—first, in a setting where they can experiment with ideas and make mistakes, such as in a mathematics methods course; and second, in situations where they can experience corresponding changes between enactments and their consequences, such as in actual classrooms with students. Rehearsals engage novice teachers and teacher educators in exchanges that are primarily occurring moment by moment.

More recently, Kazemi, Ghousseini, Cunard, and Chan Turrou (2016) discuss how teacher educators can lead a rehearsal as a cycle of interactions with prospective teachers, making instructional decisions that are responsive to their students' contributions in the moment. Rehearsals provide novice teachers with repeated opportunities to investigate, reflect on, and enact future mathematics lesson through coached feedback. In rehearsal, small groups of novice teachers leads an instructional activity with their peers acting as students. While the teachers enact an activity they practice eliciting and responding to their colleagues' ideas, the teacher educator continuously interrupts to orients them to each others' thinking and the big ideas in mathematics. The dynamic results in the teacher educator at the centre of learning, acting as both a coach and student, commenting on students' thinking and subject matter, and instructional decision making.

In these types of classrooms, the teacher educator's role scaffolds the rehearsal by making a teaching moves like prompting to elicit further work, providing feedback, offering a students' perspective, and facilities a discussion of instructional decisions. To prepare for this class, the teacher educator purposefully selects instructional tasks, and anticipates how the novice teachers will enact these lessons with their colleagues so they can prepare themselves to make in-the-moment decisions. This work allows teacher educators to monitor "the types of instructional questions that novice teachers encounter as they try to put into practice their knowledge about mathematics, children's thinking, and teaching that they learn in the investigative domains of our learning cycle" (p. 28).

Rehearsing what teacher candidates might say and do in response to students' ideas was an objective of Zazkis, Liljedahl and Sinclair's (2009) proposal for *lesson plays,* as an alternative to traditional lesson planning. The authors asked pre-service teachers to imagine an account of what might occur when teaching a math lesson in a classroom and then write scripts for things they would say and do in this lesson. They found that this process helps teachers "develop a larger repertoire of possible actions and reactions" (p. 46). Although this activity has teacher candidates anticipate what they might say and do, it does not work on deepening their existing mathematics understanding, nor does it allow them to work with unexpected ideas that emerge in the moment.

Video is being increasingly used in teacher education to show good teaching, or bring elementary students' mathematical ideas into a university course. Yet research results suggest that this activity has produced only mild results in changing a teachers' beliefs about teaching and learning mathematics. Beswick and Muir (2013), for example, used video to allow pre-service teachers to compare a video-recorded teacher's practice with their own practicum experiences. Seeing the videos did not enable the prospective teachers to notice the difference between their experience and the practices demonstrated in the video and tended to confirm participants' existing beliefs. Similarly, Ineson, Voutsina, Fielding, Barber, and Rowland (2015) analyzed pre-service teachers' reflections about video-recordings of good mathematics lessons. Although teachers made connections between what they already knew and new ideas, their responses to the material were dominated by their beliefs about mathematics. Subsequent discussions with their peers, however, helped them to see other perspectives. Bartell, Webel, Bowen, Dyson (2013) also used video as a supplemental tool to show children's mathematical understanding and found that videos needed to be paired with other explorations of mathematical concepts, like working on math tasks that elicited their ideas about how and why the subtraction algorithm works.

Researchers in mathematics education continue to make recommendations for teacher educators to develop curriculum materials to translate their work into practice (Durkin, Star & Rittle Johnson, 2017; Stylianides & Stylianides, 2007). At best, these recommendations help teacher educators to develop innovative practices; however, recommendations can also overload teacher preparation courses with materials about lesson planning, instructional strategies and assessment (Eli, Schroder & Lee, 2011).

In these evolving practices for prospective teachers, much of the focus is about learning how to teach without any attention to providing experiences for prospective teacher to improve their own understanding of elementary mathematics. Even studies that focus on the important area of connection-making, like Eli, Schroeder and Lee's (2011) work on card sorting, are not considering participants' prior knowledge and experience.

There seems to be an implicit assumption that adults learn differently from children—they learn from listening and reading, rather than the activities recommended for children like working on tasks and talking about mathematics. However, in her review of three learning theories, McDonough (2013) found that the learning processes for adults and children are similar in that both groups of learners benefit from active engagement in the learning process and from making connections between prior knowledge and new knowledge. There were just two differences: adults are more self-motivated and they have more prior experience than children. The similarities between adult and child learning are also evident in research recommendations for teaching prospective teachers and elementary students.

Educational researchers have recommended that teachers have a connected understanding of mathematics so they are able to generate and use multiple solution strategies with ease. Yet activities in teacher preparation programs, like methods courses, offer teacher candidates scripted step-by-step curriculum or role-playing exercises (e.g. rehearsals) that do not address opportunities for them to work on and within their existing understandings of mathematics.

2.3.6 Curricula for Educating Elementary Mathematics Teachers

Elementary teacher education in mathematics is often based on the mathematics curriculum for elementary schools plus the relevant mathematics education research, with each instructor personalizing the focus of their courses. However, a few universities have developed a teacher education curriculum specifically for elementary mathematics. One example is Ball, Sleep, Borst and Bass' (2009) practice-based teacher education approach—a methods course that helps novices "do instruction, not just hear and talk about it" (p.459) through the lens of MKT. Ball and her colleagues have developed structured and detailed lesson plans for each class that use tasks to expose students to both content and pedagogy. These tasks are specifically designed to elicit "the kinds of thinking, reasoning, and communicating used in teaching" (p.462). For instance, one tightly-scripted lesson plan for a division of fractions task, 1 3/4 ÷ 1/2, included an hour and a half of detailed teaching actions along with related notes and cautions.

Hiebert and his colleagues' (2007) curriculum structure also uses elementary mathematics tasks to elicit mathematics thinking similar to that of students in elementary classrooms. This model includes the following three phases: a) specify the critical learning goals for prospective teachers; b) collect test scores and use this evidence of students' learning to drive task revisions; and c) gather and store knowledge in a shared product. When Berk and Hiebert (2009) used a mathematics lesson to illustrate this model, they argued that the initial task should elicit prospective teachers' misconceptions about a topic so they could actively confront and resolve the misconception.

Internationally, Japan completely revolutionized the way they teach and learn mathematics country-wide by drawing on North American research. To learn about reform-based teaching and learning practices, in-service teachers in Japan participate in an on-going professional development model called *lesson study*. In lesson study, small groups of teachers go through a process that requires them to act as researchers. Initially, the group co-plans a lesson. Then the lesson is used in one of the group members' classrooms while other members observe and take notes—this part is viewed as the experiment. Based on the group's thoughts about how the lesson unfolded, the lesson is either revised or used in its original form by a second member in a different classroom and so on until all members have used the lesson. These lessons are eventually documented and shared in a public report (Fernandez, 2002). This model for professional learning has transformed the way that mathematics is taught and learned in Japan. Lesson study has been brought to the United States, but cultural differences have made it challenging to produce the same results (Fernandez, 2002). In particular, opportunities for teacher collaboration in North American schools are less common than in Japanese schools. In his list of reasons for why there has been limited uptake, Fernandez identified gaps in teachers' mathematics content knowledge as the key factor.

2.3.7 Applying Mathematical Understanding Research to Teacher Education

So how can teacher educators ensure that teacher candidates know the mathematics for the levels they are teaching and how can they be supported to enact research? Teachers, like their students, benefit from actively engaging in learning experiences that build on their prior knowledge and experiences. The decades of mathematics education research, plus my own teaching experience support the need for transforming professional development for teachers. Teachers, like their students, need to work on mathematics tasks, express their mathematics ideas, and be exposed to others' mathematical ideas so that they too can make connections.

This study targets this question directly by offering an example of how teacher candidates can deepen their mathematics understanding while experiencing the type of teaching practices recommended by researchers. It builds on the strong themes in the mathematical understanding literature—using prior knowledge and experiences, and connection-making to deepen understanding—and extends the emerging teaching practices that focus on teacher candidates doing mathematics and teaching actions, rather than simply being told about them.

Mathematics education literature (e.g. McGowan, 2017; Martin & Towers, 2016) recommends that educators should start with what their school students know and engage them in doing and discussing mathematics. This would have teacher educators beginning their lessons by eliciting what their students already know about a topic, then engaging them in doing mathematics through a task. However, research evidence shows that teacher candidates often have a procedural bias and a fragmented understanding of the concepts (McGowan, 2017), and that these traditional instruction methods are not effective for learning mathematics for teaching (Boaler, 2008).

Teacher candidates are students that need to develop a deep understanding of mathematics. But many researchers have focused on what teachers should know and do in the classroom, rather than how they can develop the required knowledge and skills. New teacher education practices are more active, enabling teacher candidates to try out teaching actions, and to see and discuss real classroom activities (e.g. Kazemi et al., 2016). However, their effectiveness is hampered by the candidates' previous experiences with the subject of mathematics (McGowan, 2017) and with the teaching models they experienced in school. In addition, developing a teacher education curriculum is a worthy goal that could promote quality and consistency, but we need to ensure that these efforts address the issues of teacher candidates' prior knowledge and experiences, before we institutionalize these programs.

Teacher candidates may be learning about the curriculum and desirable teaching practices and yet be unable to use these practices in their future classrooms. The important practice of connecting students' prior knowledge with new material is difficult for teachers who lack experiences with mathematical connections from their own schooling (Adler, et al., 2005). Given the lack of connecting experiences, many teachers may also have gaps in their mathematics knowledge at the level they are teaching—their met-befores may be disconnected

McGowan's (2017) recent recommendations are promising because she is working on the key issue of the teacher candidates' prior knowledge and how to deepen their understanding of mathematics by making connections. She observed that teacher candidates bring a "deeply ingrained procedural orientation to mathematics" (p. 23) for getting answers, and that this propensity is valued above all. To break from this, she recommends that teachers identify their own met-befores and examine how they impact their teaching. She suggests that a course is necessary to develop a deeper understanding of arithmetic and algebra with "tasks designed to change the procedural orientation and superficial, fragmented knowledge" (p. 36). These tasks would bring a leaners met-befores to the surface so misconceptions could be uncovered. Then the teacher educator would use instructional strategies like generating multiple solution strategies and making connections between them to "overcome and transform students' problematic met-befores" (p. 28).

This study provides an example of the type of course that McGowan (2017) has recommended—it has carefully selected tasks that when enacted allow teachers to bring and work on their met-befores by generating and connecting between multiple methods using the tools and strategies they will be expected to use in their classroom. To be more effective, teacher education can use techniques similar to those that teacher candidates will find in their curriculum and recommended practices for teaching their future students (e.g., Martin & Towers, 2016).

CHAPTER 3 THEORETICAL FRAMEWORKS

The purpose of my research is to study how teachers can deepen their understanding of elementary mathematics by connecting their existing mathematical ideas. I identified two theories that are well-suited to this investigation. To emphasize and describe the domain of prior knowledge, I selected McGowan and Tall's (2010) concept of met-befores. To analyze and describe the newly graduated teachers' learning processes in detail, I used Pirie-Kieren's (1994) Theory for the Dynamical Growth of Mathematical Understanding. Both Pirie-Kieren theory and the concept of met-befores focus on prior knowledge, are rooted in connection making and talk about learning being an individual or personal process. Previous studies on connection-making have used resources like those discussed earlier in the Literature Review, for example, the Knowledge Quartet, mathematics proficiency and Mathematics Knowledge for Teaching (MKT). However, these approaches are not appropriate because they focus on what teachers need to know, rather than how they can acquire this understanding.

To learn fundamental mathematics concepts, an elementary student constructs physical and mental representations by engaging in mathematics tasks. In light of this, much of an elementary teacher's work is associated with supporting their students' developing understanding by anticipating the physical and mental representations the student will construct, and then encouraging them to connect their representations. The Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding provides a model that can support this work because the theory encourages learners to construct a rich set of physical and mental representations called images to use in making connections (Pirie & Kieren, 1989; 1992; 1994a;1994b). Emphasizing the importance of relationships, Pirie and Kieren (1994) say "it is the interconnecting of these images that leads to the level of property noticing" (p. 178). In other words, making connections is essential for learners to notice important mathematical concepts.

3.1 Prior Knowledge and the Concept of Met-befores

This study focuses on newly graduated teachers' existing knowledge of elementary mathematics and connecting their fragmented knowledge, not learning new topics in mathematics, the more typical application of these theories. The study's aim was to work on and with their prior knowledge or met-befores to connect existing knowledge. Pirie-Kieren theory provides a lens for how learners access their prior knowledge and revisit it as they grow their understanding. As Cavey (2002) observed in the Pirie-Kieren context, prospective teachers often have outer layer understandings, like rules and formulas, that can limit their growth in understanding school mathematics, if they do not see the value in revisiting their earlier mathematical ideas. Similarly, using McGowan and Tall's (2010) concept of met-befores is particularly relevant to new teachers working on elementary mathematics. As learners, they have many years of experience using arithmetic formulas in school and their daily lives, but likely little or no recall of the abstract representations or concepts in elementary mathematics.

Met-befores affect a student's experience and attitudes toward math. Though learners may bring similar mathematics ideas, they may have opposing views towards these ideas. What might appear to be problematic or supportive to one NGT might be the opposite for another. For instance, connecting the action of breaking apart blocks to the subtraction rules of regrouping may appear straight forward to some learners and puzzling to others. Attitudes can also be affected; making mistakes in math class can elicit feelings of shame from one learner and indifference from another.

Because this study was concerned with what participants did with their met-befores, I also considered how McGowan and Tall (2010) have further characterized met-befores as either supportive or problematic. Supportive met-befores “enhance the chance of making sense of new ideas” (p.171) whereas problematic met-befores can “impede learning and can frustrate the learner in making sense of new ideas” (p. 171). However, this study was considering existing knowledge, not learning the “new ideas” as stated in McGowan and Tall’s (2010) definitions for the types of met-befores learners encounter. Its aim was to work on and with a variety of met-befores to connect existing knowledge, rather than to bridge to new ideas. In this context, met-befores were used to characterize how prior knowledge related to solving mathematics tasks—what mathematics did learners recall and use.

More recent research (e.g. Martin & Towers, 2016) has pointed to the ways that teachers can remind learners of met-befores for the purpose of uncovering potential problems in learning new material. In this paradigm, the teacher identifies the met-before and encourages students to use it in their work. However, McGowan (2017) recommends that learners identify their own met-befores, a more useful approach for this study participants who may have not previously worked together and bring a variety of past knowledge and experience. Both met-before researchers and Pirie-Kieren stress that learning is an individual process. Pirie-Kieren (1992) have articulated their beliefs about the personal nature of learning:

- Progress towards learning goals may not be achieved by all students nor in the expected way by all students,
- There are multiple paths to a similar mathematical understanding,
- Different people can understand the same thing differently, and

- There are different levels of understanding but there is no final endpoint to understanding a given topic. (Beswick and Muir, 2012, p. 30)

3.2 Significance of Pirie-Kieren Theory to this Study

Of the various analytical frameworks that have been used in studies about connections, the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding offered the most suitable frame for this study. It is well established among mathematics educators and uses prior knowledge as a significant resource for learning. In addition, it has been specifically used in teacher education (e.g., Cavey and Berenson, 2005). Therefore, it was appropriate for analyzing how individual NGT elementary teachers deepened their understanding of elementary-level mathematics.

The Pirie-Kieren Theory describes a nested model that resembles the cross-section of an onion consisting of eight layers—Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventizing— where "image" can be physical representations, like pictures, as well as mental ideas (Figure 1). It provides a framework for describing how an individual grows their understanding of mathematics concepts by moving forward and backward through these layers, in what Pirie-Kieren Theory describes as a non-linear pathway. This study used the first five of the eight Pirie-Kieren layers of understanding to analyze where participants started and how they moved through the layers while working on elementary mathematics tasks.

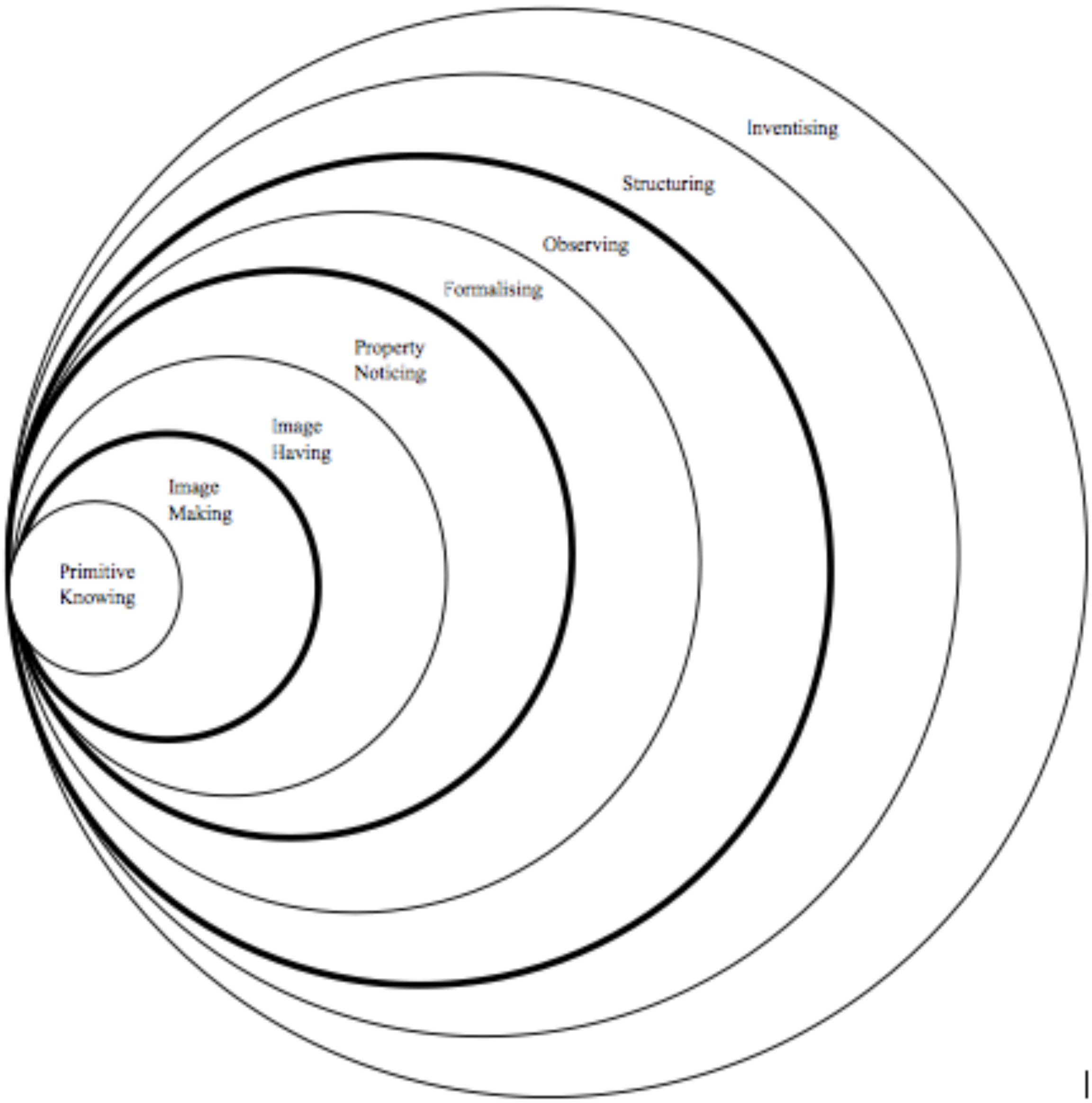


Figure 1. Pirie-Kieren Dynamical Theory for Growth of Mathematical Understanding

Primitive Knowing, the first layer, is the learner's starting point. Martin and Towers (2016) made a clear distinction related to this layer in Pirie-Kieren Theory: Primitive Knowing "is observed to be everything that a learner knows (and can do) except the knowledge about the particular concept that is being considered by the observer" (p. 90, emphasis added). For example, an observer studying a learner's pathway of understanding related to subtraction would include everything a learner says and does unrelated to the topic of subtraction, including drawing dots or number lines.

In the next two layers, Image Having and Image Making, images include multiple modes of expression; ideas, mental imagery, or pictorial representations that learners have or make. In Image Having, the learner has a mental construct without having to do any activities to create or recreate it. This includes their starting images about the concept being considered. In subtraction, for example, learners might have an image of difference without having to draw a number line showing the difference between two numbers. Image Making occurs when learners develop representations that apply their previous knowings to make distinctions and use them in new ways. Learners move between Image Making and Image Having while devising plans to solve problems; therefore, the boundary between these two layers is permeable and arguably unnecessary—what Pirie-Kieren call a "don't need" boundary.

The nested Pirie-Kieren model also includes some distinct boundaries that rely on learners to make connections to previous layers of understanding. The primary one for this study is between Image Having and Property Noticing, the point where learners use images they make and have to notice relationships and discuss their workings in the context of the task. Property

Noticing occurs when learners connect, combine or manipulate images and recognize general features of the mathematics concept (Pirie & Kieren, 1994).

Formalising, the first formal layer of understanding, embeds the previously four layers and is marked by an awareness that a method works for all relevant examples. Algorithms, for example, are a generalized method that can be applied to a set of similar problems, like arithmetic operations. Characteristics of formalising include the learner's ability to explain and justify the method. Pirie and Kieren (1994) refer to formalising when learners are "capable of enunciating and appreciating a formal mathematics definition or algorithm" (p. 171).

3.3 Observing and Analyzing the Growth of Mathematics Understanding

Pirie-Kieren Theory provides an observational approach for analyzing mathematics learning. It enables researchers and teachers to categorize an individual learner's actions and expressions and then trace their progress as they work through a mathematics topic and build their understanding. A key feature of Pirie-Kieren Theory is that learners of mathematics may start at any layer and move to other layers when faced with tasks.

Folding back happens when learners use outer layer knowings to inform inner layer activities, and maybe provoked by invocative interventions. These interventions can be used to call upon learners "to describe a cognitive shift to an inner level of understanding" (Pirie & Martin, 2000, p. 130), supporting the learners' growth at both the inner and outer layers of understanding (Pirie & Kieren, 1994). Pirie-Kieren Theory also recognizes moving forward through the layers when learners use knowings from inner layers to inform outer layer activities. This movement can be prompted by a *provocative intervention* (Pirie & Kieren, 1994). Back and forth action reflects the dynamic learning process that involves a continual folding back to inner layers to thicken understanding. Pirie-Kieren also describes a third type of intervention, *validating* where a teacher or student confirm existing understanding. Folding back and forth results in non-linear learning pathways that can be mapped using the Pirie-Kieren model to trace through the eight layers of mathematical understanding, as shown above in Figure 1. This non-linear growth distinguishes the Pirie-Kieren theoretical model from other perspectives. It is particularly appropriate for studying newly graduated teachers who tend to bring outer layer formalised understandings and then fold back to connect these with be less formal images, like diagrams or blocks.

Martin (2008) furthered the concept of interventions by developing an analytic model that teachers and researchers can use. This framework distinguishes among the sources, like teachers or peers, of intentional teaching and learning interventions that in turn can be either unfocused or explicit in nature. These unfocused and explicit prompts can provoke students to fold back to previous knowings for the purpose of working with and using prior knowledge in mathematics lessons (Martin, 2008). An explicit prompt "is one whose form suggests or even states to the learner which concept, image or existing understanding, in the opinion of the person making the intervention, the learner should fold back to and access" (Martin, 2008, p. 72).

On the other hand, unfocused prompting is characterized as, "the learner is stimulated to consider earlier understandings but without being directed precisely where to fold back and exactly which mathematical idea and existing understanding to work with" (Martin, 2008, p. 72). Unfocused prompts are less directive and allow learners to choose the option they fold back to. In either case, learners fold backwards and forwards to use their previous understandings to connect with the mathematics they are learning.

Pirie and Martin (2000) also identified a specific type of folding back called *collecting*. In collecting, the learner may retrieve previous knowledge from their memory, or consult their notes or a reference book. Collecting is distinctly different from other folding back actions because it does not provoke changes to the learner's earlier understandings.

Deep understanding emerges from revisiting previous understandings. Pirie and Kieren (1994) say that individuals learn by "folding back again and again to enable them to build broader, but also more sophisticated or deeper understanding" (p. 173). They see folding to other layers as re-experiencing the same material in a different light, because the previous material is influenced by the inner or outer layers the learners are experiencing, offering learners opportunities "to re-remember and re-construct new understanding" (p. 188). This is consistent with Piaget's (1973) theory that all knowledge grows by reformulation where prior knowledge is not replaced, but integrated into a coherent whole. Pirie-Kieren Theory see the actions and expressions associated with revisiting and connecting inner layer understandings with more formalised mathematics as deepening or *thickening* of mathematical understanding. They draw on a metaphor of folding paper in half to illustrate that folding back creates a thicker understanding. Connecting existing images to thicken understandings at formal layers was the goal of this study.

Evidence of folding back and thickening can be seen in Martin and Towers (2016) study that investigated the purposeful action of folding grade 12 students' understanding back to previous knowledge in order to create a problem for learning new material. In doing so, Martin and Towers (2016) found that growing a deep and connected understanding can emerge from revisiting met-befores, particularly working on and with existing mathematics knowledge that present problems for leaners. For example, using an addition strategies that change questions from 19 + 7 to 20 + 6 by taking one away and adding it to the second number is helpful for quick mental calculations, but present problems to learners when they apply this approach to subtraction: $19 - 7 \neq 20 - 6$. Working on how this met-before addition strategy can be applied to a strategy for solving a subtraction question constitutes the work involved in using with existing met-befores.

Folding back or forward is particularly useful when learners' images are not connected, what Pirie and Kieren call *disjointed*. Harel and Tall's (1991) use a similar term, disjunctive generalizations, when they describe what happens when learners existing schemas are not build up without connecting the less formal and more formal mathematics. They warn educators that when students' generalizations are disjunctive, they typically have to learn more procedures than are needed to solve the general class of problems. Pirie and Kieren (1994) elaborate on disjointed understandings in their discussion of the necessity for embedding less formal understandings in formalising ones.

In Pirie-Kieren terms, "disjointed" describes mathematical understanding that is thin or disconnected. For instance, knowing what to do could be disjointed from why it works. To reconcile disjointed images, learners need to build their understanding by making connections between their images and more formalized understandings. As Pirie and Kieren (1994) have observed, ideally formalising grows from the learner's "personal mathematical structures and unfolds from less formal image-based mathematical understandings" (p. 43). When students learn piecemeal rules, their understanding may not grow until they work on connecting disjointed images for themselves. For example, when learners join a rule like: *whatever you do to one side* *of the equal sign you have to do to the other* with images of "balance" where the one side of the equal sign needs to be "the same as" the other side, understanding of that particular concept thickens. Disjointed mathematical understandings may emerge as problematic met-befores that impede a learners ability to make sense of why the above rule makes sense. By working on these met-befores learners can make connections that support the progression of their learning.

The embedded nature of Pirie-Kieren Theory supports a connected understanding. Pirie-Kieren's model of nested layers of understanding emphasizes connectedness by showing each layer as building upon all of the previous layers. Connection-making takes place when learners see two or more images and notice the relationship between them. This is particularly evident as they move into Pirie-Kieren's bounded layers of understanding. For example, Property Noticing is characterized by learners examining and reflecting upon images and making connections between them. Learners bring their prior knowings and experiences, construct multiple images and connect these images by demonstrating actions and expressions, such as "doing, and reviewing, seeing, and saying, predicting and recording" (Pirie & Kieren, 1994, p. 175).

Taken together, met-befores and Pirie-Kieren Theory provided a lens to discuss all the previous knowledge and experiences that learners bring when working on a task. Pirie-Kieren Theory supports an analysis of an individual's growth in mathematical understanding, and the concept of met-befores provides a breakdown for how specific previously learned mathematics ideas affect future learning. Using both enables a detailed analysis of how learners use their existing mathematical ideas to connect and deepen their understanding.

CHAPTER 4 STUDYING NEW TEACHERS CONNECTING MATHEMATICS

4.1 Methodology & Methods

Consistent with Pirie-Kieren Theory's premise and my experience, mathematics is a subject that is learned through actions and expressions. Thus this study's methodology and methods were approaches that elicited participants' prior knowledge and experiences and provided opportunities for participants to connect their existing ideas. This section provides a detailed description of the study design: the overall methodology, the details of the course used in the case study, and the data collection and analysis methods.

4.1.1 Qualitative Case Study Methodology

My research was investigating how newly graduated teachers (NGTs) can make connections to deepen their understanding of elementary mathematics. To observe this, I needed to create a setting where the NGTs were motivated to use what they knew and had the opportunity to make connections. As the researcher and NGTs' teacher, I had to go beyond observing and taking field notes to creating the learning environment needed to elicit the real-time interactions—in the moment responses—and learning I wanted to study. I designed this environment based on research recommendations I use in my teaching practice. I knew how to use mathematics tasks in a classroom to reveal where elementary teachers' mathematics knowledge is fragmented. Teachers usually do not know that their understanding is disjointed unless they are provoked and supported to use their knowledge in problem solving.

I selected a qualitative case study methodology to investigate how newly graduated teachers (NGTs) can use their prior knowledge to connect mathematical concepts and deepen their understanding because no other methodology supports this kind of research. This methodology is aligned with the holistic inquiry model that investigates a contemporary phenomenon within its natural setting (Stake, 1995). In this study, the phenomenon was the learning activities of the NGTs within the setting of a course. Recently, other proponents (e.g., Bruce, et al. 2010; Case & Light, 2011) have also found that case study methods are meaningful when considering questions about specific applications that enhance teaching and learning. More specifically, my research fits well with the Stake's *instrumental case study* approach because it is used to provide a general understanding of learning activities and permits researchers to gain an insider's perspective of an issue (Stake 1995). In addition, the instrumental case study aims to accomplish insight from a group of individuals who experienced a phenomenon, exactly what was needed to revisit mathematics with the purpose of connecting prior knowledge.

As the course designer and instructor, I was a practitioner doing research. This work is not new to me — since 2008 I have been working in roles that directly support pre-service and in-service teachers to develop their practices in mathematics education. Based on these experiences, I know what beginning and veteran teachers typically bring to professional learning courses on mathematics, including several common disjointed understandings. This work allowed me to do what Lampert (2000) described as doing research from the inside. Using a case study aligns with her argument in favour of teachers being active participants in understanding the problems of practice. By applying theoretical rigour to my teaching method, I expected to learn more about how my practice works and uncover possible improvements.

Mason (2002) has said that researching from the inside requires the researcher to be aware of the processes involved in exploring new or familiar topics and specifically how these occur in learning mathematics. He explained that being sensitive to others' experiences is very difficult, especially because inside researchers can fall into the trap posed by knowing more than their students. Mason (2002) recommended that you resist sharing your own experiences and making assumptions about what should be accomplished. These suggestions align well with the methodology for a course that focuses on newly graduated teachers working on their existing knowledge. Following Mason's advice, I needed to be open to the knowledge and experiences that newly graduated teachers bring, and the work they do to make connections for themselves.

Mason (2002) also discussed the "fuzzy areas" of doing research from the inside when the researcher is torn between experimenting on people and acting as a practitioner, striving to improve practice. To achieve a balance between being an instructor and a researcher, he emphasizes the importance of being attentive and being able to consider your options in the moment, saying, "if you feel caught between intervening and not intervening, between making a direct and specific suggestion or trying an indirect prompt, between telling and asking, then you can enter fully into the moment" (p. 222). The course was structured so that I and the NGTs doing co-teaching entered into the moment with the participants' mathematical ideas, ready with our anticipated responses, and also open to unexplored ideas.

4.1.2 Research Design: The Additional Qualification Course

The setting for the case study was a ten-day Additional Qualification (AQ) elementary mathematics course specifically designed for new graduates of the teacher education program at a University in Toronto, Ontario. The 15 participating NGTs had completed all the requirements of the Bachelor of Education program and were waiting to officially graduate. They had yet to begin their career and gain in-service teaching status. These NGTs had registered and paid for the

AQ course because they wanted to learn more about teaching mathematics; therefore, they were assumed to be more motivated and less anxious than typical elementary NGTs learning mathematics.

The AQ course used elementary mathematics tasks to elicit solution strategies that reflected the NGTs' prior knowledge, an effective approach based on my teaching experience. As Cavey (2002) said, "there is much to be learned about the effectiveness of using teaching tasks in adding depth, breadth, and thoroughness (Ma, 1999) to prospective teachers' mathematically specific knowledge for teaching" (p. 1086). During the course, participants used several solution strategies to make connections within their existing mathematics knowledge, while learning about related educational theories. They also reflected on what they were learning and how this might affect their future teaching. It was a thinking classroom, according to Liljedahl (2016):

> a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion. (p. 362)

The course offered the NGT participants an opportunity to think and learn together about making connections in mathematics in multiple roles: as students solving elementary-level tasks, as co-teachers facilitating the learning process, and as themselves, new teachers studying related theories. Learning, in this way, was a collection of practices that resulted in the NGTs "living in, and hence learning from" (Mason, 2002, p. 44) their experiences during the AQ course.

The course format and content were similar to the other additional qualification courses that I offer for beginning teachers and in-service teachers, but required additional documentation to support the study (i.e. capturing images of the blackboard before, during and after each task)

The AQ course format and content. The first nine course days of the 10-day course followed a similar pattern. The mornings included learning or discussing educational theory, working on the day's task, documenting multiple images and making connections between their images. Time was also provided for NGTs to record some of the documented images into their journals and record a connection by looking back at their work and video-recording a description of this connection. On day ten, the NGTs observed a grade 5 class working on one of the tasks that they investigated earlier in the AQ course.

To begin each course task, NGTs were randomly assigned a partner and blackboard space to encourage interacting with their peers, expressing their understanding, and investigating multiple solution methods. Daily randomized groupings were used because of the influence this strategy has on collaborating in earnest and decreasing anxieties about groupings. In particular Liljedahl (2016) found that randomized groups eliminated social barriers, increased the sharing of knowledge between students, decreased the reliance on the teacher for answers, improved engagement in the task and overall enthusiasm for mathematics.

The NGTs worked in groups standing at blackboards—a vertical, non-permanent work space that Liljedahl (2016) found promoted learners' eagerness to take risks in recording ideas, increased discussion and persistence on the task at hand. While working on tasks, NGTs generated multiple solution strategies by recording preferred mathematics solutions as well as any they did not initially anticipate but documented as a result of seeing their peers solution strategies. This daily group practice facilitated what Mason (2002) calls professional noticing "which serve to support both 'picking up ideas' and 'trying them out for ourselves'" (p. 30). In addition to documenting multiple solutions on the board, NGTs documented connections between their solutions, and relevant properties that explained how the concepts worked.

Polishing, a method used in Japan (Shimizu, 1999), was used regularly to revise work and promote discussions with other course participants. On most days, time was allocated to polishing solutions—erasing most, or all, of the board work and rewriting or revising their solutions to present NGTs' mathematics ideas clearly to peers who had not participated in their initial work. The experience ranged from erasing all of their own work and recording a polished solution of their ideas, to reviewing another groups' work and polishing it by erasing most of it and using one idea as a starting point to create a polished solution.

The NGTs also documented their work in personal notebooks, sometimes copying a portion of their mathematical ideas off the blackboard and sometimes working through another example. These personal records were not intended for collaborative work, though NGTs freely shared them with others in their groups.

Course afternoons were devoted to NGTs looking back at their day's documentation, adding more notes, and video-recording the details of connections that occurred for them during the day. In addition, they worked on assigned reading and sometimes participated in planning for the next day. Pirie-Kieren Theory (1994) was included in the course readings since it was the primary analysis framework for studying the NGT's growth of understanding during the AQ course. It provided the language and insights for interpreting course events and supporting NGTs' ability to translate their experiences into future teaching skills.

Course instruction and co-teaching. As the instructor, I focused on planning course activities, selecting tasks and anticipating the solution strategies that the NGTs might bring and use to solve the tasks. I engaged in the learning process by listening to NGTs' responses and encouraging them to record their work on the blackboards. I circulated around the room asking questions about their ideas to elicit further work, more images or clarifications. When I looked at their work on the board and simply read aloud what I saw, many NGTs explained their mathematical ideas in greater detail. As I responded in the moment with prompts and suggestions about looking at others' solutions, the NGTs noticed how solutions were connected or not connected. I enacted the practices recommended by Smith and Stein (2011). Beginning on day two, this role was shared with the NGTs through co-teaching.

Many researchers have found co-teaching to be an effective method for professional development. In co-teaching, two teachers work side-be-side, with the lead teacher sometimes described as a coach during a lesson (West & Cameron, 2013). The coach is not an extra teacher working with a small group of students. Instead the teacher and coaching teacher work together to learn from and with each other. West and Cameron (2013) said that this work resembles athletic coaching because the coach models specific teaching moves "so the teacher can see what the move looks and sounds like in real time" (p. 14). Co-teaching offers teachers a chance to slow down the learning process and really listen to what students are saying.

Mathematics educators have been using this method as a model for coaching in-service teachers (e.g. West & Cameron, 2013) and for mentoring teacher candidates in their field placements (Yopp, Ellis, Bonsangue, Duarte, & Meza, 2014). Yopp et al.'s (2014) study looked at using co-teaching for teacher preparation as a model for practicum placements where the co-teachers are the teacher candidate and a mentor teacher. They reported that co-teaching models offer better support than traditional models, like attending workshops, for preparing mathematics teachers to help plan and enact lessons aligned with recommended practices that enable students to learn through reasoning and sense making. Though the participants learned about co-teaching through a workshop model, they experienced the work of co-teaching during their practicum placement with their mentor teachers.

Co-teaching can look very different in classrooms depending on the roles for the participating teachers. The table below describes the seven co-teaching strategies outlined by Yopp and his colleagues (2014).

Co-Teaching Strategy	**Description**
One Teach, One Observe	One teacher assumes instructional responsibility, and the other conducts agreed-upon targeted observations of the students or the co-teacher.
One Teach, One Assist	One teacher assumes primary instructional responsibility, and the other provides assistance.
Team	Both teachers have instructional responsibility and are actively involved in the lesson.
Parallel	Each teacher teaches half the class. The lesson and materials are the same.
Station	Students rotate to different stations where different instruction and activities are occurring. Each co-teacher leads a station, and there may be independent stations or student-led stations as well.
Supplemental	One teacher works with students who are at grade level, and the other works with a group of students who have already met the standard to offer enrichment or with students who need remediation.
Alternative	Both teachers address the same objectives but use different approaches, based on the needs of the learners.

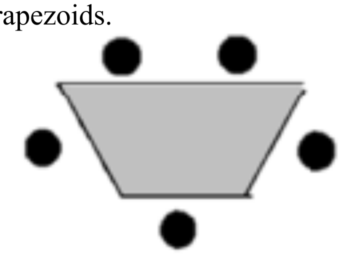
Table 1: Co-Teaching Strategies

Co-teaching approach used in this study was based on the *Team* strategy where the instructional responsibilities are shared among the co-teachers. I used this approach because it engaged the NGTs in the lesson so they could practice listening and responding in-the-moment to their colleagues' mathematical ideas. NGTs were expected to sign up for three co-planning and co-teaching sessions. At the end of each day, a group of five NGTs co-planned the following day's task, including anticipating colleagues' thinking, connecting ideas, and considering questions that would prompt participants to make connections.

At the beginning of most tasks, I led a group warm-up exercise using an informal representation method, like blocks or a number line. I posed a question to the group and then used this method to illustrate the ideas that they offered. For example, on the third day, prior to exploring addition strategies, I elicited everything they knew about the number 12, and recorded some of their ideas on a ten frame: 1 ten and 2 ones were represented by filling one full ten frame with counters and a second ten frame with two counters.

As NGTs worked on course tasks, the NGT co-planners transitioned into their role of co-teaching. The small group of co-teachers and I made observations about their mathematics work based on how it compared with our anticipated solutions. We also asked questions to uncover their thinking, encouraged them to communicate their work on the blackboard, and prompted them to construct or review different images so they could connect their own ideas with other solutions. The tasks were drawn from several topics: counting, patterning, equality, and the operational sense for addition, multiplication, subtraction, division and fractions. They were all sourced from scholarly articles or mathematics education resources for teachers. The course tasks and their rationale are listed by course-day in Table 1.

Table 2: Tasks for the 10-Day AQ Course Case Study

Day	Task	Purpose and Source
1	Constructing a fractions kit Find as many quantities or combinations of quantities that make exactly 3/4 Make 2/3 in many ways	• Used to introduce Pirie-Kieren theory and connect their experience with Pirie and Kieren's 1994 paper • From Pirie and Kieren's (1994) paper discussing student Theresa's growth of understanding
2	How do you see the dots? Use these visualizations to determine the number of dots around two and three trapezoids. 	• Used to introduce concepts associated with subitizing, counting and number patterns • Inspired by Boaler's (2016) work on seeing as understanding
3	3 butterflies landed on a bush. Then, 4 more butterflies landed. Later, 8 more butterflies joined them on the bush. How many butterflies are on the bush altogether? Show your work.	• Used as the transition from counting to addition strategies • From the Ontario Ministry of Education's (2011) description of Bansho, a method for making connections among symbolic and numeric representations

Day	Task	Purpose and Source
4	13 + 6 in base 7	• Used to elicit prior knowledge of counting, quantities and number relationships in base 10 to solve an addition question in base 7 • From Zazkis (1999) recommendation for pre-service teachers, because it develops skills that are fundamental for the teaching of mathematics
5	How many ways can you calculate: 91 minus 79?	• Used because it would elicit multiple solution strategies to investigate place value and regrouping concepts • From Ma's (1999) book *Knowing and Teaching Elementary Mathematics*
6	You have 6 m of a ribbon. Each bow requires 5/6 of a metre. How many bows can you make? How much is left over?	• Used because depending on solution strategy— algorithm versus counting— two different numerical answers will be reached. • From Pirie's (1988) work that illustrated student Kate's actions and expressions for dividing fractions
7	A dirt field is being sodded with rolls of grass. Each roll of sod grass is 1 metre by 1 metre. After 3 hours, the landscaping crew sodded 12 metres by 15 metres. What is the area of the field after 3 hours?	• Used to connect multiplication with area and the distributive property • From Hyde's (2009) book because he described how this task can be used
8	Use your preferred or an easy mental math strategy to solve the questions you identified as more challenging, on your Mad Minutes worksheet	• Used to emphasize a variety of mental math strategies, and the preferences we have for some over others • From (Husband and Rapke, 2015) about adapting traditional Mad Minutes into an inquiry-oriented experience, and Tall and Grey's (1994) work that recommends students derive facts from known facts

Day	Task	Purpose and Source
9	4 friends buy 36 cookies for \$12. Each person contributes money: \$2 Helen, \$3 Isabelle, \$4 Donna, \$3 Michelle. How many cookies does each person get in proportion to the money they paid?	• Used to elicit multiple strategies for multiplication and division • Idea inspired from paying attention to proportional reasoning (2011) Ontario Ministry of Education
10	Field trip Observed a grade 5 class while they worked on the ribbons and bows task (number 6 above).	The field trip was planned for the purpose of observing students doing mathematics that had been done in the course.

4.1.3 Study Data Collection and Analysis

Selecting video for data collection. I chose video as a data collection method because of the compelling evidence video offers educators about a learners' experience. Studies that use video as a methodological tool have made significant contributions to the mathematics education literature. For example, the large-scale Third International Mathematics and Science Study (TIMSS) videos of mathematics lessons in grade 8 classrooms provided powerful evidence of the diverse ways mathematics is experienced in Germany, the United States and Japan (Hiebert & Stigler, 2000). The video analysis revealed specific nuances and difference in mathematics lessons that deepened viewer's understanding— an overhead projector was used as a tool in the US to provide students with step-by-step instructions. On the other hand, in Japan, blackboards were used to document a collection of students' mathematical ideas at the beginning, middle and end of a lesson. Video provides the viewer with a visualization of how learning mathematics can be experienced differently depending on where you live.

Video for collecting data aligned well with my case study methodology. Case studies seek to understand a phenomena, and video documents the sights and sounds of a particular phenomenon (Powell et al., 2003). In this study, the phenomenon of connecting was evident in the participants' actions and expressions so using video allowed me to review the data multiple times, gaining new perspective with each viewing (Pirie, 1996). Multiple viewings of the video data strengthened my analysis because I can review and consider the specific actions and expressions the NGTs used multiple times. Using video data alongside students' written work was essential for documenting the events of the course because, as Pirie (1996) said, it provided "the least intrusive, yet most inclusive, way of studying a phenomenon" (p. 4). This is especially important because of my dual role as researcher and instructor.

Video provided a detailed record of the whole picture and allowed the researcher to continue to revisit scenes at a later date, freeing the instructor to interact with learners while they are working on tasks. In her discussion on researching from the inside, Lampert (2000) says that "video makes it possible to have a running image of the teacher-student-subject interaction" (p. 68). As the course instructor, the video captured my teacher-student-subject actions and expressions that supported connection-making.

On the other hand, using video to collect data can also create issues. A benefit for using video is that it captures everything, but this can also be problematic. Researchers (e.g. Martin, 1999) say that when researchers use video they run the risk of producing too much data, an overwhelming amount for the researcher to analyze. Powell and his colleagues (2003) say that video can also capture participants in compromising positions, and therefore, create ethical problems for the researcher about permission and access. However, Powell et al. (2003) suggest that the researcher address these ethical concerns by showing participants the video, giving them another opportunity to provide consent.

Data collection. The video recordings and NGT journals provided both real-time and daily-summary data for the study. The videos captured the live interactions at the blackboard as NGTs recalled, used and discussed the mathematics that they had met before. The recordings also captured the actions and expressions involved in constructing and connecting solutions, including what they did, saw and said. I also captured photos of the blackboards at different stages of their work each day, including before and after blackboards were erased for a polishing exercise. By the end of the course, there were approximately 75 hours of recorded live interactions among the NGTs, based on approximately 2.5 hours of live video from three different video cameras each day.

Further evidence of connecting occurrences was documented in NGTs' look-back videos at the end of each day and in their journals. The NGTs produced a total of 135 short videos that recorded daily occurrences and described a connection they had made. I transcribed all 135 videos and used their comments about connections to identify the mathematics they had met before. Then I located these occurrences in the real-time videos and transcribed the related interactions.

The NGTs' daily journals recorded their preferred solution and an unanticipated one that emerged during each investigation, plus as many connections between the solution strategies as possible. The NGTs choices about what work to record in their individual journals provided additional insights and details about what they noticed and valued as important. Their recorded images also provided further evidence of what they did with their met-befores.

Data analysis. As I viewed each video of NGTs working, I cross-referenced the video-recorded action with their journals and the photos I had periodically taken of their blackboard work. The video data, journal documentation, and photographs provided a complete record of NGT actions and expressions during the course, as well as their personal interpretations of these activities. I used this detailed and overlapping data for identifying and analyzing how NGTs used their prior knowledge, made connections, and noticed properties. Then I wrote a narrative about what occurred for each NGT and mapped their growth of understanding on a Pirie-Kieren diagram.

The data were analyzed using the Pirie-Kieren Theory's (1994) embedded layers of understanding and McGowan and Tall's (2010) concept of met-befores. Actions and expressions were used to characterize the understandings that the NGTs brought and worked on with their colleagues. For example, actions and expressions of doing, reviewing, seeing and saying can be attributed to a Pirie-Kieren layer of understanding, whereas applying and justifying can be analyzed through a different layer of understanding. In terms of met-befores, I analyzed NGTs' actions and expressions for evidence of their prior knowledge and experience as supportive and/or problematic.

The analysis of the video data was adapted from Powell, Francisco, and Maher's (2003) analytical model for studying mathematical met-befores used to make connections. Their seven phases are: viewing the video data attentively, describing the video data, identifying critical events, transcribing, coding, constructing a storyline and composing a narrative. A researcher may go back and forth through these phases. I used these phases as I studied the video data:

- Viewing the video data attentively and transcribing. I started viewing the NGT's look-back videos—their perspectives on how and what connections occurred during each day of the course. To begin, I transcribed the 135 look-back videos that the 15 NGTs generated at the end of nine course days.
- Identifying met-befores. I looked for the specific words and phrases in the video transcriptions that NGTs used when they identified their met-befores related to course tasks, in particular: "I know that," "Because I know," "I automatically started with," "I went straight for," "I knew that I had to," "I remembered," "I immediately."

- Viewing the live-video data attentively. Based on specific look-back videos that were identified as showing evidence of met-befores, I viewed the live-video to see when and how a met-before emerged.
- Describing the video data. I reviewed the video, paying close attention to the occasions when the NGTs used or referred to images, recording notes about the NGTs' actions and expressions and making sure not to interpret them at that point.
- Identifying critical events. I identified critical events when NGTs combined or manipulated images, and noted when they discussed important properties or explained how the mathematics worked. I cross-referenced these moments from the live-video with the NGTs' written work in their journals. Critical events occurred when the NGTs found two different answers for a task, or what they referred to as a "mismatch."
- Transcribing. I transcribed the live-video related to the critical events.
- Constructing a storyline and composing a narrative. Using the data from multiple sources— look-back videos, live videos, and written work—I constructed a narrative that described the NGTs' actions and expressions associated with their growth of mathematical understanding.

4.1.4 Assumptions about Prior Mathematics Experience

When working on course tasks, the NGTs' starting point was their prior mathematics experience, their met-befores. In Pirie-Kieren Theory (1994), the authors acknowledge that it is not possible to know a learner's actual experience so it is necessary to use reasonable assumptions. Given the NGTs' ages and educational background, I made four assumptions about their knowledge and experience related to elementary mathematics—their Primitive Knowing in Pirie-Kieren terms:

1. NGTs had a working knowledge of arithmetic operations—addition, subtraction, multiplication, and division—using whole numbers and fractions.
2. They knew algorithms that supported these operations.
3. They had experiences using physical methods like counting, diagrams and tables that could be used to illustrate the operations.
4. They also had experience using these operations to solve questions involving physical quantities and their associated units.

These assumptions were based on NGTs' likely experiences in school, as a student and a teacher candidate, and in using elementary mathematics operations in their daily lives. Five of the NGTs had participated in one of my methods courses for elementary mathematics in their B.Ed program. So, although I had previously taught some of the NGTs, I made no assumptions about the depth of their understandings, that is, whether their informal and formal images were connected.

In addition, when selecting the AQ course tasks and in daily course planning, I made some assumptions about what past knowledge might emerge, and potentially be worked on by the NGTs. However, there was no way to anticipate all the NGTs' previous knowings that could be supportive or problematic.

4.2 Detailed Analyses of Selected Tasks and Newly Graduated Teachers

The following section describes and analyzes how selected NGTs investigated two of the tasks from the AQ course: subtraction and division of a whole number by a fraction. I selected the subtraction task because it yielded the richest data— almost half of the transcriptions of the NGTs' look-back videos went onto a second page. In terms of the live-video, the division of fractions task resulted in the longest investigations—NGTs continued to discuss this task well into their lunch break. These two tasks provoked the most work and discussion, disagreements and surprises. The subtraction and division by a fraction tasks also covered the range of elementary mathematics in the course and provoked more than one solution strategy from NGTs.

The tasks in the AQ course were activities designed to elicit the NGTs' prior knowledge and provide opportunities for connection-making. Each analysis section begins with a description of the task. This is followed by a detailed narrative and analysis of how two or three NGTs worked on the task, using Pirie-Kieren Theory to interpret and describe their actions and expressions, and map their growth of understanding using Pirie-Kieren's layers of understanding. Each task analysis concludes with a discussion of the met-befores that NGTs used as they worked through that particular task.

The participants used as examples are identified by their pseudonyms—Deb, Hillary, Nellie, and Pat.

4.2.1 The Subtraction Task

"**How many ways can you calculate 91 - 79?"** Before this task was read aloud by one of the co-teachers, I did a 20-minute "minds on" exercise where I asked NGTs: "what's more 21 or 14?" I asked them to use a hundreds chart to convince their peers that one number was more than the other. My intention was to get NGTs to use the hundreds chart as a tool to communicate and compare 21 and 14. The NGTs worked on their own and shared their resulting hundreds chart with the person beside them. Then we walked around and looked at everyones' chart and voted on which solution was most convincing. Most NGTs selected a solution that had highlighted one full row of 10 in yellow with one partial row of 4 ones in yellow, compared with two full rows of green with 1 box in green. This solution convinced the group that the student knew that 21 had two tens, whereas 14 only had one ten.

Figure 2. Comparing 21 and 14 on a hundreds chart

Then the day's co-teachers randomized the group and read the subtraction task to the AQ class: "How many ways can you calculate 91-79?" The NGTs began working on this task while the co-teachers circulated around the room. Participants spent more than an hour generating strategies, comparing solutions, and documenting them in their journals.

Newly graduated teachers selected for analysis. I selected the work of two NGTs to analyze in detail: Deb because her approach was unique and Hillary because her method was more typical. Deb's was unique because her work illustrated connection-making across topics in mathematics—her knowledge of addition and algebra. Although the task required Deb to subtract in as many ways as she could, the traditional algorithm did not surface during her investigation, even when she discovered an error. In contrast, Hillary remained within the topic of subtraction. She began with the algorithm for subtraction, which she described as "very traditional," and investigated the rules she knew using blocks. As Cavey (2002) observed—prospective teachers often have a formalized understanding and need to work on less formal images. In Pirie-Kieren terms, Hillary, like many prospective teachers, had a thin understanding of subtraction.

Deb's growth of understanding for subtraction. On this particular day, Deb was partnered with two of her colleagues, Pat and Emma. Netta, a co-teacher, was closely monitoring this group and returned several times to listen and ask questions. Within the few minutes it took for me to set up and begin the live recording, Deb had already recorded an idea on the blackboard as follows:

91 - 79 = 90 - 80
= 10

The live video began with Deb drawing a hundreds chart on the chalkboard below this idea. Pat, Emma and Deb were using the chart to show 91 and 79 and the numbers that make up the difference between these numbers. They had blocks on the floor, and all three of them moved between sitting on the floor using blocks and standing up to record their ideas on the blackboard. Deb and Emma were both counting on their fingers. About 10 minutes into the live video, Pat used Deb's idea with smaller numbers and recorded them on the board, like this:

21 - 9 = 20 -10
= 10

Confused by the results of this strategy, Pat turned to Emma and Deb and spoke:

Pat:	It's 12, right.
Deb:	Yeah.
Pat:	How come it's 12?
Deb:	How can we make that work? *(pointing to 91 - 79 = 90 - 80 on the board)*

Primitive Knowing. Deb's idea to change the numbers was part of her Primitive Knowing based on a friendly number strategy that she knew worked in addition. For Deb, this friendly numbers strategy involved changing numbers for the purpose of performing operations. She demonstrated this later in her journal entry with an example of how friendly numbers work in addition: 19 plus 5 can be changed to 20 plus 4 by adding one to 19 and taking away 1 from 5.

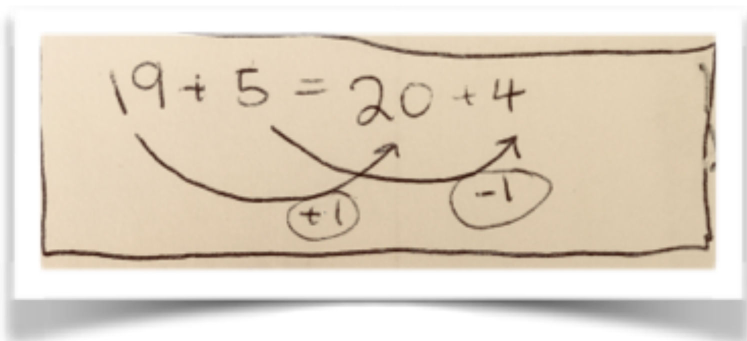


Figure 3. Deb's Primitive Knowing for Addition

Image Making. In making her first image for the subtraction task, Deb used her friendly number strategy by subtracting 1 from 91 and adding 1 to 79 to make 90 - 80. As she recalled in her look-back video, "so when we first came to this question I automatically wanted to make these numbers friendlier to use, so I changed them." The live-video opened with her constructing her second image, a hundreds chart to identify the difference between 91 and 79 through counting. However, Deb and Emma's focus quickly shifted back to friendly numbers when Pat used this strategy with smaller numbers and also got an incorrect answer and said, “How come it’s 12?” This provoked Deb to point at her original image on the blackboard, and respond: “How can we make that work?” Then Deb started working with Pat and Emma on the 10 = 12 problem.

Deb was puzzled by the two different answers, as evidenced in her journal entry below and in her look-back video, when she said, “I needed to add plus 2, but I couldn’t figure out why because to get from 91 to 90 I subtracted 1.” Deb circled the two +1’s, drew an arrow to the other side of the equal sign, and circled another +2. She wondered why she needed to add +2.

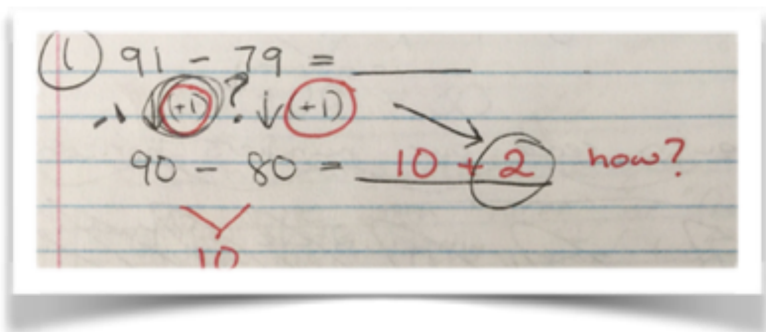


Figure 4. Deb’s Image Making: Where is the plus two?

Deb’s friendly number strategy that resulted in 10 did not match the correct answer 12 that she produced by using the counting and balancing ideas she retrieved from her Primitive Knowing. At this point, her image of making numbers friendly was not connected to how this could influence the results in subtraction. In Pirie-Kieren terms, this is disjointed because Deb’s use of friendly numbers was not linked to her Primitive Knowing about counting and difference.

Following Deb’s question to her group about how to make this work, she stood up and began using hand gestures and explaining the missing 2. Deb, Pat, Emma and Netta were all facing the blackboard, and Deb was leading the conversation. As she was talking to her peers at the blackboard, she recalled a rule for solving equations and said, "so whatever we do to this side, we have to do to this side," as she pointed to each side of the equal sign. Then the following conversation between the group members occurred:

Deb: So I'm minusing one here and adding one here *(pointing to the left side of the equation)*.

Netta: What number are you wanting to change to make it easier to work with?

Deb: I want to make this a 90 and I want to make this an 80 *(on the chalkboard, Deb records 90 above 91 and the 80 above 79)*. So I'm taking one away from the big number and adding one to the small number, and whatever the answer is, I have to add 2 because I made the taking away number bigger by one and I made the initial number smaller by one … so that's two numbers *(circles the -1 and +1 on the board)*.

Pat: So why isn't it *(inaudible)*. Wait, I don't know.

Deb: (*explaining to Pat*) I make the bigger number *(gestures with her hands far apart)* smaller by one *(gestures hands closer together)*, and I made the smaller number bigger by one *(continues hand gestures)*. So that's two numbers.

Pat: So wouldn't that just cancel each other out.

Deb: No. *(repeats)* I make the bigger number smaller by one *(gestures with hands again)* so you have to add one to the answer. Then I took the smaller number bigger by one, so that's another one I need to add. Let's show the two numbers and I think I can show you better. *(Deb sits down to work with the blocks.)*

In this short dialogue between Deb's group, Deb demonstrated an understanding of the missing 2 through her expressions and actions. With her hands, she gestured distances increasing and decreasing as a result of the values she shifted using her friendly number strategy. Deb was invested in working more on this idea and convincing her peer Pat how this made sense when she said, “Let’s show the two numbers (referring to the blocks) and I think I can show you better.” Following Deb’s lead, the group moved to the floor again to work with the blocks.

The blocks were one of the methods that Deb’s team used to investigate the missing 2. As Deb said later when she talked about the subtraction task, “It took a couple of tools to get to the number line which really showed me what I was doing ….” A co-teacher’s prompt had interrupted their work and led them to construct a number line that eventually helped them to represent the missing 2 on the board.

Folding back to Primitive Knowing. When Pat came to the same incorrect solution and was also puzzled, it provoked Deb to stand up and start talking about what she already knew. Deb drew upon her prior knowings of the equal sign, when she wrote her friendly numbers on the blackboard as 91 - 79 = 90 - 80. Initially she expected the difference to be the same. The equal sign as a notation for balance was part of Deb’s prior knowing of equality.

Later in her look-back video, Deb confirmed that the difference between 91 and 79 was not 10. She said, “at first I thought that balanced out, but I knew that the answer wasn’t 10.” Deb’s sense of the subtraction equation being about balance is part of her Primitive Knowing. To determine the difference between 79 and 91 is 12, Deb used her prior knowings of counting and a hundreds chart. Deb moved back into her Primitive Knowing and counted, rather than moving forward to retrieve the algorithm for subtraction from Pirie-Kieren’s formalising layer.

Deb’s initial images captured the two different answers reached on each side of the equal sign. Her friendly number strategy that worked in addition subtracted one from the 91 and added one to the 79 (-1, +1) producing 90 - 80. The incorrect answer 10 did not match the +2 required to balance her equation.

Image Making. In her look-back video, Deb explained that Netta, the co-teacher, prompted them to use a different representation by asking, "Is there a tool you could use to check?" Deb recalled, "…so I went to the number line."

Deb and her group constructed the following number line on the blackboard to learn that the 2 came from +1 on each end. Her number line showed the difference of 10 between 80 and 90 and two +1's, giving a total difference of 12. Using this diagram, she saw the numerical relationships and why her friendly number strategy was not giving the expected answer.

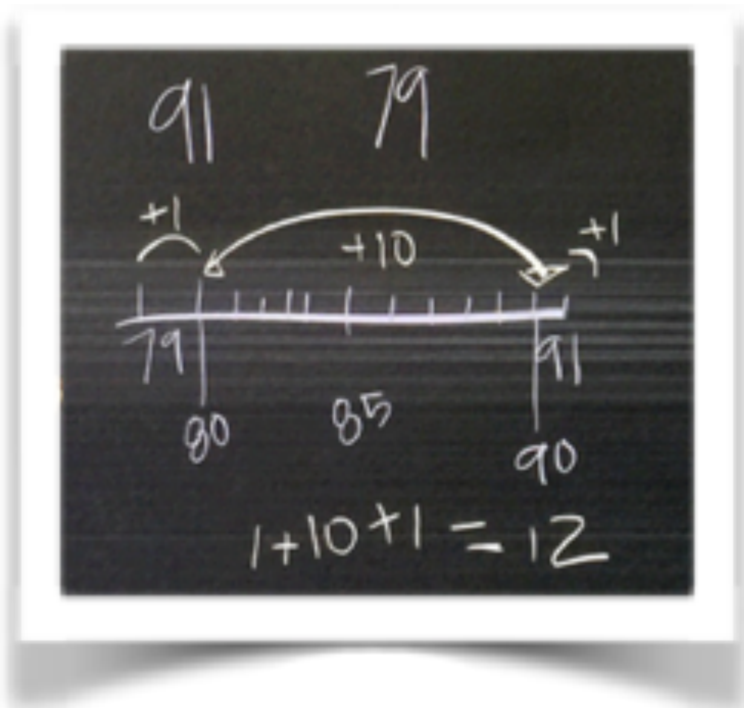


Figure 5. Deb's Image Making: the Number Line

Property Noticing. In her look-back video, Deb started by first talking about why her original image didn't work, "I had changed the number in both directions and therefore had changed the difference." She demonstrated how difference works by holding her hands 7 units apart and clasping a paper number line between two fingers at 21 cm and two fingers at 14 cm.

Then Deb called on a peer to pull the number line all the way to 0 cm and 7 cm. She referred to this experience as an “aha moment” that “explained the idea of constant difference.” By explaining how this method works with the numbers in her one metre paper number line, Deb's understanding of subtraction grew and demonstrated the property of constant difference in subtraction.

In her journal entry shown below, Deb documented a number line using the numbers from the warm-up exercise: 21 - 14 = 20 - 13. This illustration displayed how the difference remained the same even though it was shifted by one to the left. Deb had combined images—the number line with shifting the entire difference by one. Annotating the mathematics like this and stating that this showed “the difference between the two numbers,” whereas her original image “had changed the difference,” is evidence that Deb was noticing how her friendly number strategy moved in the same direction.

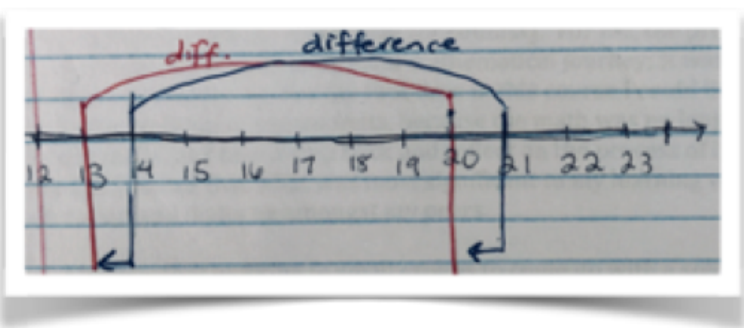


Figure 6. Deb’s Property Noticing: Constant Difference

Formalising. Deb recorded evidence of formalising this concept in her journal. Returning to the task 91 - 79, Deb equated this to 92 - 80 and explained the balance between the two sides of the equal sign as a mental shift “in her head” as well as on the physical paper number line and realized that she was “not counting anymore.” By translating her use of the number line into a new equation and seeing a relationship, she thickened her understanding of the difference between operating in subtraction, rather than operating in addition.

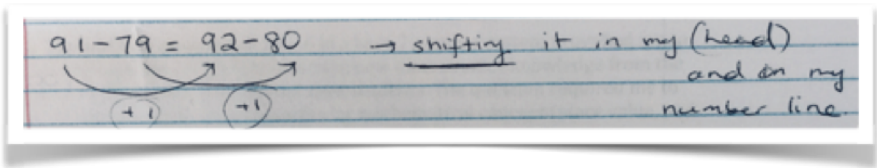


Figure 7. Deb's Formalising: Constant Difference

Mapping Deb's growth of understanding of subtraction. In summary, Deb started solving this subtraction task using an image based on a friendly number method typically used in addition, and realized that the result was not correct based on folding back to counting on her fingers and her knowledge of the equal sign. Then she made a new representation of the task, and ultimately noticed "constant difference," a relevant property for subtraction, and why her initial method didn't work, thus deepening her understanding of subtraction. Her movements through the Pirie-Kieren layers of understanding are mapped below, beginning with Primitive Knowing and ending in Formalising.

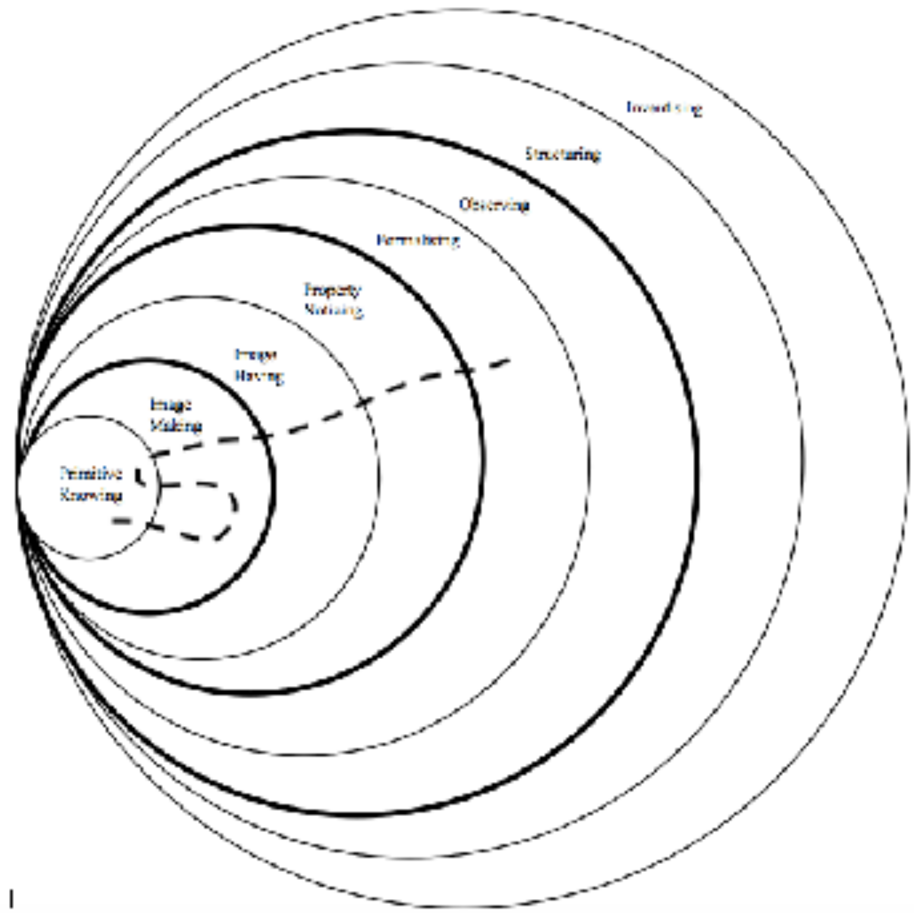


Figure 8. Mapping of Deb's Growth of Understanding for subtraction

Hillary's growth of understanding for subtraction. Most NGTs used the subtraction algorithm that included rules of borrowing to regroup a ten and then subtracting the numbers as their initial approach. This was especially true for Hillary who journaled about "immediately" using the subtraction algorithm.

On the opposite end of the blackboard, during the same investigation used earlier to describe Deb's growth of understanding, Hillary had recorded the algorithm on the board before I started the second video recorder. Her partner Lina was working beside her, drawing a hundreds chart. After two minutes, neither of them had said much to each other.

Formalising. Hillary walked her partner Lina through the algorithm, writing it on the board and explaining the steps involved in how it worked: “You bring a ten over. I can’t minus 9 so you borrow from the 9 to 8, and you put 10 here 10 +1 and then 11 - 9 is 2.” Hillary was bringing forth her formalised understanding of subtraction.

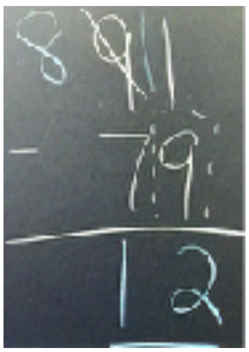


Figure 9. Hillary’s Formalized Understanding of Subtraction

Later in her look-back video, Hillary said, "I used base ten blocks to show my understanding, however, at first I used an algorithm.” Then she recited her algorithm rhythmically, “so I said to myself, 1 cannot minus 9 because smaller numbers cannot subtract bigger numbers, so I had to borrow a 10 from the 9 and bring it over the 1 to make it 11.”

Below the algorithm, Hillary drew a second image using blocks: 9 tens and 1 one, and below this 7 tens and 9 ones. She subtracted by crossing out all the 7 tens and 9 ones and 7 of the 9 tens. and crossed out 8 of them. She recorded the leftover blocks—one ten and two ones—beside the first equal sign, then the numeral 12 beside the second equal sign.

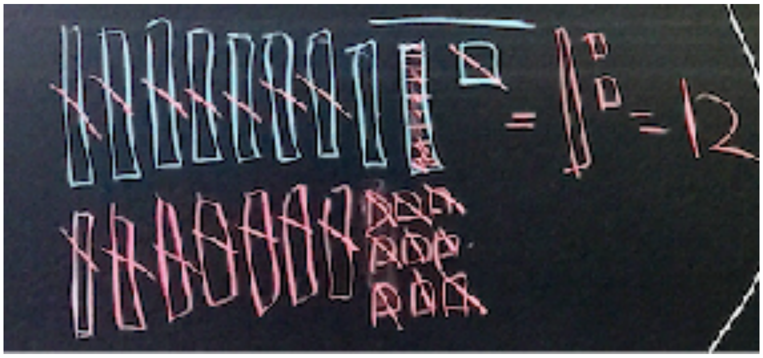


Figure 10. Hillary's Image of Blocks

About four minutes into the investigation, I approached Hillary, and we had the following conversation about the two solutions she had constructed on the board:

Marc: This is 11, right? (*pointing to the 11 in the algorithm*)
Hillary: Yeah.
Marc: Do you see 11 there (p*ointing in a circular motion around the picture of blocks)*? Have you connected these two?
Hillary: No, I did them separately.
Marc: You did them separately.
Hillary: Yeah.
Marc: So I want to know where… so right here you made that 11 and you subtracted 9. So I want you to find the 11 subtract 9 and there's a difference of 2. Find it in here (*pointing to the blocks diagram*).

To encourage Hillary to explore the connection between her algorithm and her blocks solution, I prompted her to locate the numbers from the algorithm in the blocks. I walked away, and Lina and Hillary began talking and working on this together.

A few minutes later, I returned to ask Hillary if she found the 11. She replied, "I didn't" and explained to me how she solved her blocks solution. I realized that she had approached it from the tens place value:

Marc: Ahhh, I know what we are doing. We are starting here with these numbers (*pointing to the tens column*). But when you did this algorithm, you started here (*pointing to the ones columns*). So I want you to think about starting going this way (*pointing to the left from the ones column*) with these blocks *(points to the blocks solution*).

Hillary: Okay.

Marc: (*walks away*)

Image Making. On the chalkboard, Hillary drew a third solution strategy by using a blocks diagram a second time. She recorded 9 tens and a one to represent the 91, then she crossed out the one, divided one of her 10's into 10 sections, and crossed out 8 of them. Then she checked by counting by ones. Lina was beside her, and they were discussing what Hillary was recording on the board as she was doing it. After making this third image, she compared it to the algorithm and explained to Lina that borrowing in the algorithm and breaking apart a ten using the blocks are different. She elaborated:

> But in base 10 blocks, you don't have to borrow from a 10 over to a one, because you can just break it apart like that (pointing to the drawings of blocks). This algorithm makes it so that you have to figure out the rules or you can't do it.

Figure 11. Hillary's Image of Using blocks in a Different Way

This comparison between the blocks solution and the algorithm became more clear about 15 minutes into this investigation when I approached Hillary to ask for the third time if she found the 11. Hillary and I discussed this, as follows:

Hillary: We didn't because I figured out that with these blocks you don't need to because you can break it apart.

Marc: Okay, so breaking it apart. What do you mean—breaking it apart?

Hillary: (*inaudible*) … so you can just take what you need.

Marc: Oh, yeah, yeah … help me understand.

Hillary: Because in this algorithm we've been taught that a smaller number can't be subtracted by a bigger number.

Marc: Right.

Hillary: Yeah.

Marc: So you can't subtract the bigger number (*pointing to the 1 and the 9 in the ones column*).

Hillary: So you have to borrow from the 10s in order for us to subtract.

Marc: Right.

Hillary: When I laid it out like this (*pointing to the blocks solution*), you don't have to technically borrow, you just have to break it apart.

Marc: What do you mean—break it apart?

Hillary: A ten.

Marc: Oh, you're breaking apart a 10?

Hillary: Yeah.

Marc: So you're not breaking apart a 10 here (*pointing to algorithm*).

Hillary: Here I feel like we are borrowing.

Marc: So is borrowing and breaking apart different?

Hillary: Yeah.

Marc: Oh, okay.

Hillary: That's why I can't find the 11 in here, because we are borrowing a 10 from here and 10 plus one is 11.

At this point in the discussion, Hillary's understanding of the algorithm was disjointed from her block diagram. In other words, the rules for borrowing in the algorithm and the actions of breaking apart a 10 using blocks were viewed as two different ideas. This was evident when she said to Lina that you have to know the (borrowing) rule or you can't do it, and that borrowing does not exist in the blocks diagram. Further evidence of her understanding being disjointed occurred when I asked her if borrowing and breaking apart were different, and she responded, "Yes."

Following the above conversation between myself and Hillary, I prompted her to use physical blocks. Hillary sat down at a table and began by bringing 9 tens and 1 one block in front of her. She immediately started taking away 10's, and I interrupted her, saying "Oh, no, no …. so now you see how you started with the left side?" Hillary laughed and said, "Oh, yeah, the ones first."

Hillary handed me a 10 block and said she needed to exchange this for 10 ones. Lina, who's observing Hillary said "Ohhhh," and I asked her to say what she was thinking. She explained to Hillary where the 11 and the 8 were in the blocks while pointing at the algorithm on the board and the blocks on the table in front of Hillary. Then the following dialogue unfolded between Hillary and me while she was using the blocks:

Marc: If you have the 91 and then the first thing you do is borrow. In the algorithm, you're taught to borrow a what?

Hillary: A ten.

Marc: So borrow the ten.

Hillary: Here (*slides the ten block over*).

Marc: And then what do you have?

Hillary: I have 11.

Marc: You have 11.

Hillary: Yeah. I have to take 9 from the 11.
Marc: And now and how are you doing to do that?
Hillary: I have to trade in.
Marc: Ahhh (*pause*). What's another word for trade in?
Hillary: Borrowing.
Marc: What's another word for borrow?
Hillary: Uhh.
Marc: What are you doing to this ten (*hold up 10 block*)?
Hillary: Breaking it apart.

Property Noticing. Hillary's trading the 10 block in for 10 one blocks helped her to connect her ideas of borrowing and breaking apart. While manipulating blocks, Hillary referred to trading in a ten block as "borrowing," then following a prompt, asking what she was doing with the ten block, she replied "breaking it apart." Hillary used the words borrowing and breaking apart interchangeably to describe regrouping, traditionally known as borrowing. Breaking it apart and borrowing were no longer different actions; Hillary used these two terms for the same action.

Formalising. Later in her look-back video, Hillary recalled the moment she found the 11 generated by her algorithm method in her blocks solution, she said, "I couldn't find the 11 until I realized that I have to separate the 11 from the 91 on its own to figure out 11 and then subtract 79 from that block." In other words, she decomposed the 91 into 80 + 11. She went on to say, "I realized I regrouped the 10 into units [ones] in order to subtract." Hillary's realization occurred when she sat down at the table and started using the physical blocks. By connecting her rule for borrowing in the algorithm and the action of breaking apart the physical blocks, Hillary's understanding grew to include the image of breaking apart and regrouped units in the subtraction algorithm.

Hillary reported that she found the algorithm “confusing” until she compared it to the blocks solution. This confusion plus her initial recitation of the formula reflected Hillary’s initially thin formal understanding of subtraction. Constructing her blocks images and looking for her algorithm numbers in those images, as she said, “helped me understand how units are distributed within the standard algorithm.” Hillary’s understanding of subtraction is thicker now that she can connect her formal “borrowing” methods to informal images.

To conclude our work with subtraction, the NGTs, including the co-teachers, were relocated to a different work space to work on another task. Reviewing images that were not their own, each group was instructed to erase 75 percent of the work that was in front of them. They had to make sense of the work and keep one idea in the work that they wanted to polish; that is, teachers had to identify about 25 percent of the work and use this idea to clearly communicate a mathematics solution. Two colleagues drew on Hillary and Lina’s work to create the following “polished” solution that clearly communicated their idea.

During this polishing exercise, the two NGT’s assigned to Hillary and Lina’s blackboard space went through a similar process of finding the 11 in the blocks solution that was described above. Excited to share their connection with me, one NGT in the group announced that she found the 11. Ally, the other NGT, added the following comment:“we are trained to think in numbers, not in the visual. So now we can see …. how they relate to the visual and the physical manipulation of pulling them away.” Ally’s connection between the algorithm and the blocks is based on seeing numbers represented visually, and the action of pulling blocks away.

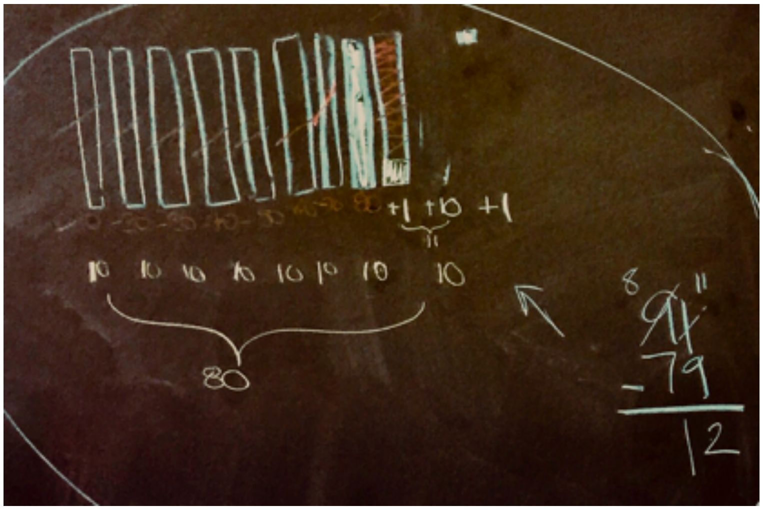


Figure 12. Hillary’s Fully Formalized Understanding of Subtraction

Mapping Hillary’s growth of understanding of subtraction. In summary, the second subtraction example shows how Hillary thickened her understanding of the subtraction algorithm by folding back and working to connect her algorithm with an image of blocks. Her U-shaped pathway through Pirie and Kieren layers is diagrammed below, beginning and ending in the Formalising layer of understanding.

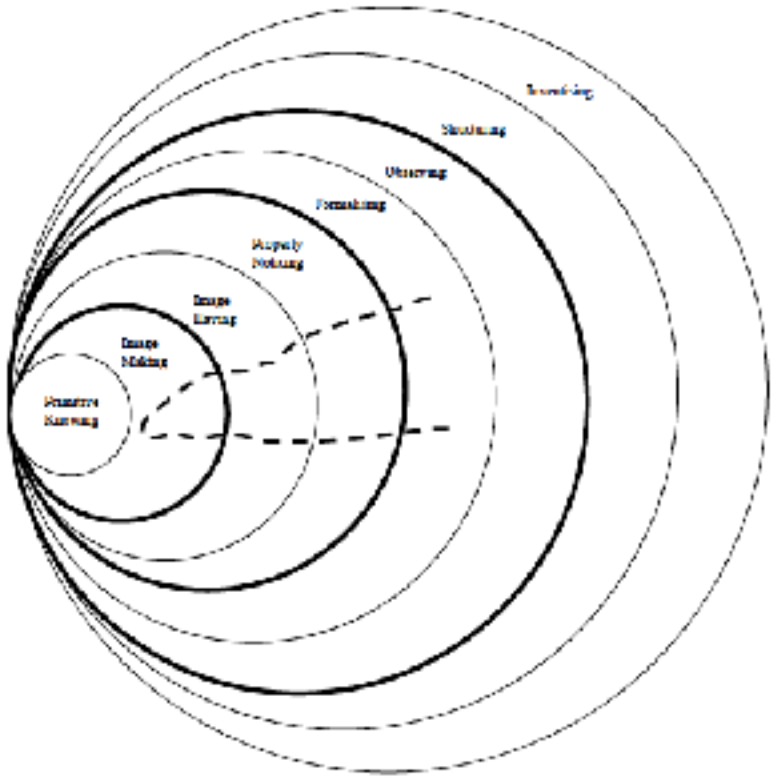


Figure 13. Mapping of Hillary's Understanding using the Pirie-Kieren Model

Intentional interventions. Deb and Hillary referenced specific prompts that were used by me and the co-teachers to support their growth in understanding. In both the live-video and her look-back video, Deb referred to a co-teacher prompting her to consider what her idea might look like on different tool. Hillary's look-back video discussed both the action of showing her idea using the physical blocks that occurred as a result of a the prompt asking her to find the 11 from her algorithm in a drawing of blocks on the board. These two prompts are categorized in Table 2 below as examples of explicit and unfocused intentional interventions that supported further work. For Deb and Hillary, the prompts acted to sustain their work at Image Making.

Table 3: Examples of Subtraction Interventions

Unfocused intervention	Explicit intervention
What would this look like on a tool? (Deb)	Where do you see the 11 and 8 in the diagram of blocks? (Hillary) Where's 11? Do you see the 11 in here? (pointing to the diagram of blocks on the blackboard)? (Marc)

Deb's being prompted to work with a different tool is an example of an unfocused intervention because the prompt asked her to think about what tool she could use without specifying the exact the mathematical idea that was being investigated. On the other hand, Hillary's recollection of being prompted to show the 11 using the physical blocks was an explicit intervention because I prompted her to use a specific tool— the physical blocks.

Subtraction met-befores for Deb and Hillary. In analyzing the data, I identified Deb's and Hillary's initial mathematics met-befores by how they described their starting point for the subtraction task as either "automatic" for Deb or "immediate" for Hillary. These first reactions showed that these methods were rooted in their past experience.

Before the live-video recorder was on, Deb had recorded her idea on the blackboard that brought forth a friendly number strategy she met-before in addition. A mismatch between her initial answer 10, and the correct result of 12 emerged when she used her counting and equal sign met-befores. Based on a prompt that focused her attention to use a different representation, Deb used another met-before, the number line, to support further work and enhanced her understanding of subtraction. Though applying her friendly number strategy met-before to subtraction was a problem that initially slowed her work, it allowed Deb to make sense of what constant difference looks like in subtraction.

Hillary's initial met-before, the subtraction algorithm, easily gave the correct answer, so I asked her to solve the task using blocks in the same direction —starting from the ones digits—as she had used with the algorithm. Then I asked her to connect these two solutions by finding the 11 used in her algorithm in her blocks solution. After Hillary spent more than 20 minutes trying to find the 11, I prompted her to use the physical blocks and act out the sequence. This work, plus my earlier prompt to start from the left, helped her to make the connection. Hillary's process — drawing blocks that mirrored her met-before algorithm and then actually using the physical blocks to perform the actions— connected the "borrowing" rules she recalled in her algorithm to a less formal solution strategy using blocks.

4.2.2 Division of Fractions Task

"You have 6 meters of ribbon. Each bow requires 5/6 of a meter. How many bows can you make? How much is left over?" This task investigated the inverse relationship between division and multiplication, and how the units, metres of ribbon and bows, were involved. To focus their thinking on fractions at the beginning of the task, I asked everyone to count aloud by two fractions: one half and one quarter. The class counted aloud while I drew jumps on the number lines on the board as shown below: One half, two halves, three halves, four halves; then one quarter, two quarters, three quarters, four quarters, five quarters, and so on, always saying the units aloud.

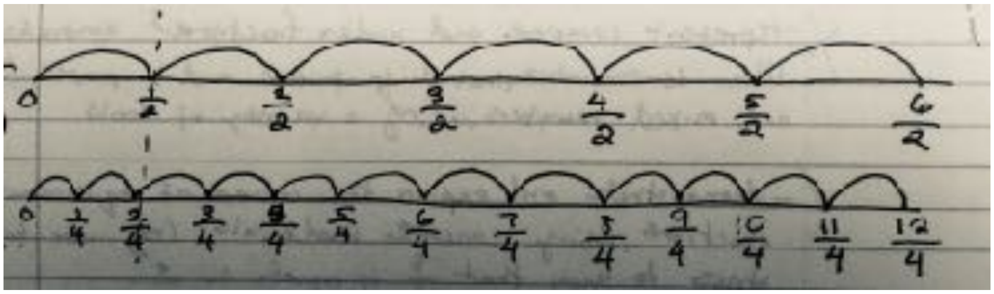


Figure 14. Initial Exercise: Counting by Fractions on a Number Line

Newly graduated teachers used two methods for this task: some began by drawing a number line and others by using the division algorithm for fractions. When two different answers emerged: 7 and 1/5 and 7 and 1/6, they all worked to understand and resolve how two different answers could make sense.

I selected Nellie's work for analysis because she often initiated tasks using algorithmic approaches, and this was no exception. In many cases Nellie illustrated how and when to use the rules she knew, but when asked questions about why her rules worked, Nellie was unable to answer. Her desire to find out why her algorithm worked, and the work she did during this task is a good example of connecting one's own understanding. The second example NGT Pat was selected because she was the only NGT who used an array to check her understanding and learn how units are effected by division.

Nellie's growth of understanding of division of fractions. On this particular day, the co-teacher Deb was assigned to observe Nellie and her partner Ally instead of walking around the room. The live video was positioned on the side of the blackboard so the video was capturing a side view of the group. The video began with Deb reading the problem to Nellie and Ally. Ally was the only one in view, standing at the blackboard with chalk in hand. Nellie, who was positioned behind Ally, was out of view of the video recorder for the first two minutes.

Initially, Ally and Nellie debated about what Deb had read aloud. Nellie had heard 5/6 of a bow and Ally 5/6 of ribbon. Based on Liljedahl's (2016) strategies for building a thinking classroom, I had instructed co-teachers to read the problem aloud only once to increase the dialogue about the task. Nellie and Ally discussed what they heard Deb say:

> Nellie: 5/6 of a bow.
> Ally: No, she said 5/6 of a ribbon, didn't she?
> Ally: Can you read it again? Are you allowed to read it? Can I read it? Can I look at it?
> Deb: *(gestures no)*
> Ally: No.
> Nellie: She said 5/6 of one metre.
> Ally: No. She said each ribbon is 6 metres.
> Nellie: Yeah and a bow takes 5/6 of a metre, I think.
> Ally: No. 5/6 of a ribbon.
> Nellie: *(pointing to the group beside them)*"they got 5/6 of a metre," *(pointing to another group on the blackboard)*, "so do they. 5/6 of a metre" *(pointing to another group)*.
> Ally: I coulda swore she said ribbon.
> Deb: *(reads the problem again in low voice)*
> Ally: Oh, a metre, dammit *(laughing)*.

This discrepancy in what they each heard—5/6 of a metre or 5/6 of ribbon—was followed by Nellie convincing Ally that it was, in fact, a 5/6 of a metre based on other groups' work. This illustrates the back and forth dialogue between Nellie and Ally, and how they, at times, competed for their own ideas to be heard. Once Nellie and Ally agreed on what the 5/6 meant, Nellie moved in closer to the board and positioned herself next to Ally. Ally asked her how she wanted to start solving the problem. With her arms far apart and just as Nellie was about to say number line, Ally interrupted and said, "I think a number line." They both laughed. This type of dialogue—talking over each other or interrupting one another continued throughout the hour they were working on this task, making the dialogue transcription challenging. Yet, throughout the transcription, there were several moments of laughter and places where they were building on each others' ideas as illustrated in the narrative below.

Nellie watched her peer Ally draw a number-line to represent one metre and then construct five jumps with each labelled in increments of 1/6th:

Ally: *(recoding the count for each jump)* So that's one bow.
Nellie: We have six metres.
Nellie: So now we need to make a number line that's even that goes to 6 metres, and if we make our jumps, we can even them out—do you get what I'm saying? We know this is how much it takes for one *(pointing to the number line on the board)*.
Ally: Yeah, so theoretically in my mind
Nellie: Or we could just multiply this *(pointing to the one line of 5/6)*.
Ally: Yeah, cause theoretically in my mind what's going to happen is we would end up with 6 of these *(pointing to the line)* we are going to have 6 lines.
Nellie: Yeah.
Ally: We are going to have a second one that goes here *(starts drawing second line under first line)*.
Nellie: Right, one for every single metre.
Ally: *(drawing line)*Third.
Nellie: Could we just do this *(pointing to the one)* times 6?
Ally: Forth, fifth, sixth.
Deb: *(to Nellie)* Why don't you do that? *(referring to times 6)*
Nellie: Okay, Okay, you're right each one of these is a metre, which is what I was -
Ally: So this is our second bow, our third bow, our forth bow, our fifth bow, sixth bow *(pointing to her number lines)*.
Nellie: Right, okay.
Ally: But then we also have
Nellie: Then you have to add each of them up.

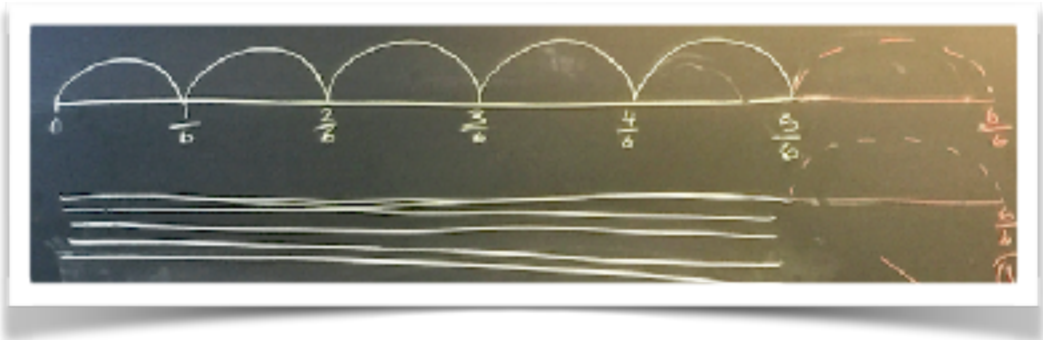


Figure 15. Counting by Fractions on a Number Line

Primitive Knowing. On two occasions during this exchange about the number lines, Nellie expressed her desire to multiply. And Deb, the co-teacher, encouraged this by saying, “why don’t you do that?” Nellie and Ally brought forth what they knew about counting fractions—the parts (numerator increases) while the whole (denominator) remains the same. Nellie also brought a multiplicative sense to this task by making copies of the number line and with her desire to stop counting and multiply by 6.

Image Having. Ally drew the first number line image—1 metre of ribbon repeated 6 times—with Nellie’s agreement so Nellie had an image that Ally created. The 6 number lines that were each divided into 1/6ths of a metre provoked Nellie to ask Deb if the problem required the 6 metres to be cut into 6 one-metre pieces. Deb replied, “that wasn’t part of the problem.” Nellie and Ally discussed this:

Nellie: Just that we had 6 metres of ribbon, it could have been all one long strip.
Ally: And if it had of been all one long strip, we all have this leftover (pointing to the dotted jumps on the board for each number line).
Nellie: Right *(counts the leftover jumps)*—One, two, three, four, five, six, and then five. You would have one, two, three
Ally: *(interrupts)* So we
Nellie: Four, five.
Ally: Would have 8 and 1/6 leftover.

Nellie: So we would actually have 1/6 leftover. Because we still have to figure that out—how much is leftover?

Nellie and Ally changed their thinking from considering the ribbon was cut into 6 one metre pieces to "one long strip" of ribbon. By putting their two sentences together, we can see how they were building on each others' ideas: "Just that we had six metres of ribbon, it could have been all one long strip. And if it had of been all one long strip, we all have this leftover."

At about the same time, Nellie and Ally both agreed that 1/6 was leftover, and although they had reached a different number of bows—Ally says 8 and Nellie says 5—neither of them mentioned this discrepancy. In part, because I don't think they heard each other say the number of bows.

Nellie, who had mentioned multiplying to Ally and was encouraged by co-teacher Deb to pursue this approach, eventually wrote 6 x 5/6 on the board. Then Deb brought her attention back to this image on the board by saying, "could you use this?"

Nellie: This will work for this because realistically I wouldn't have to draw all these. I could have just done this *(6 x 5/6 on board)*.
Deb: Does that get you to the
Nellie: Cut into 1 metre strips.
Deb: Okay, there's still a question that you have to answer.
Nellie: Yes. I know.

They were working separately, not talking for a full minute, and then Nellie recorded the following solution:

5/6 x 6
= 5/6 x 6/1
= 30/6
= 5

At some point later, Nellie drew an arrow from the 5 and recorded "5m of ribbon used." Ally continued drawing one large "super big" number line representation. Later in her journal, Nellie recorded a copy of Ally's solution of counting by 5/6's on a number line:

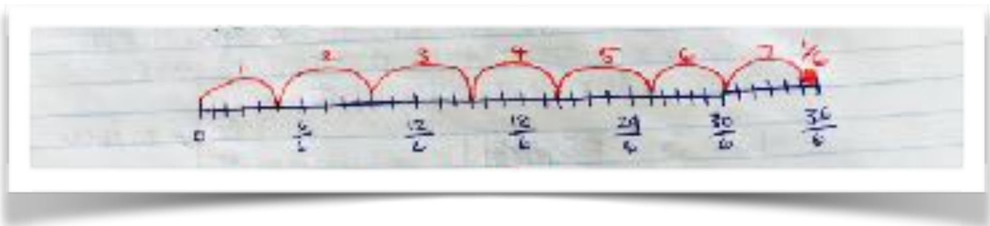


Figure 16. Nellie Counting by Fractions on a Number Line

The group started talking again when Nellie turned to Deb and asked her if what she wrote on the board made sense to her. When they started talking, I dropped by to view their work for the first time.

Marc: So this says 5/6 x 6. So this says 6 groups of 5/6. Can you show me your 6 groups of 5/6?
Ally: 1, 2, 3, 4, 5, 6 *(pointing to the board and counting number lines)*.
Marc: Ahhh
Nellie: We drew it all out so we knew it was going to be the same.
Marc: Where's your 30?
Nellie: Right here *(pointing to the number 30 in her algorithm)*.
Marc: No, but where's your 30 in your picture?
Nellie: Well, then we'd have to draw all the jumps.
Marc: Hang on—is it 1, 2, 3, 4, 5, 6 *(pointing to the number line and counting the jumps)*?
Ally: Hang on—6 one-metre strips of ribbon.
Nellie: It's the 1, 2, 3, 4, 5 *(counting the jumps on one number line)* six times.
Marc: Oh, I see and that gets you the thirty?
N & A: Yes
Marc: And if you divide them by 6, what happens?
Ally: See that doesn't work. You get 5 bows, but we have 6 bows.
Nellie: No, but not if they are cut here.

Ally: We'd still have 1, 2, 3, 4, 5, 6 bows.
(pause)
Nellie: So what happened to my equation?
Ally: *(laughs)*
Nellie: That doesn't make sense.
Ally: No, it doesn't.
Marc: Okay, I'm going to come back *(walking away).*
Ally: Please come back. *(with the intention give us some time, and come back later)*

While reviewing their work, I prompted them to connect their solutions by asking them to locate a number from Nellie's image on Ally's diagram. Together we found the 30 that was in Nellie's algorithm and on Ally's number line. When I questioned them about dividing the 30 by 6 in the algorithm, five bows didn't make sense to Ally. Ally convinced Nellie that 5 did not make sense since her diagram shows 6 bows. At this point, the algorithm that Nellie used for multiplying fractions did not make sense of the 6 number lines they had previously counted. In Pirie-Kieren terms, Nellie's algorithm for multiplying fractions was disjointed from the solution reached using 6 one-metre number lines. Nellie wondered what occurred in the algorithm, saying, "so what happened to my equation?" When I walked away, Nellie said, "I don't understand why it doesn't work." She spent the next 10 seconds reviewing the board. Ally interrupted her with an idea:

Ally: Maybe this will help *(She starts drawing a small bow to represent each 5/6 metre piece of ribbon above each jump)* One bow, two bows, three bows, four bows, five bows, six bows, seven bows, 1/6 leftover.
Deb: So your answer is?
Nellie: 7 bows
Ally: 7 bows
Deb; And?
Ally: 1/6 leftover.

At this point they had an agreement, 7 bows and 1/6 leftover. Deb who had been observing from the periphery came in closer to them and the following discussion took place:

Deb: How might you use division as a solution?
Ally: Can we figure out what went wrong with the multiplication first?
Nellie: Yeah, this is really bothering me. *(Nelly proceeds to record the algorithm for the division of fractions on the board.)*
Deb: I'm just wondering
Ally: *(interrupting)* so hold up
Deb: I'm just wondering if this might help you *(pointing to the divisions algorithm)*.

Although Nellie and Ally wanted to pursue why their multiplication algorithm produced the 5, Nellie complied with Deb's request, and recorded the algorithm and solution for dividing by fractions on the board as follows:

6 ÷ 5/6
= 6/1 x 6/5
= 36/5
= 7 1/5

Nellie and Ally still had not made sense of the 5, so it remained on the board beside the two possible answers: the 7 and 1/6 found by counting by 5/6 on a number line, and 7 and 1/5 reached when Nellie used the algorithm for dividing fractions based on Deb's request. Deb, who was prepared for this moment from our previous days co-planning session, capitalized on the two different answers by pointing to each of them on the blackboard and asking, "why do you think that is?" and continued:

Deb: OK, so what rule did you use to uh go from here to here *(pointing to the flipping of the fraction)* is that a rule that

Nellie: *(Interrupting)* Because to divide fractions, you take the second one, and you flip it and multiply, cause you have to do the same thing here.

Deb: OK, continue.

Nellie: But when you're dividing and multiplying *(pause)* your denominators should be the same only when you're adding and subtracting.

Formalising. Nellie had rules for how to multiply and divide fractions. Her comment about "denominators should be the same only when you're adding and subtracting" suggested she had an awareness that adding and subtracting have a different set of rules to follow than multiplying and dividing. Indeed, Nellie brought a more formalized approach at the onset of this task by noting that multiplication is a faster route to finding the answer. As Nellie put it, "because realistically [she] wouldn't have to draw all these," referring to the 6 lines that Ally drew. Later when Deb prompted her to use the division algorithm, Nellie performed the algorithm for multiplying fractions without any hesitation. This preference was also evident in her look-back video from the next day that focused on multiplication in a different context, when she said, "I notice that I automatically seem to go back to the algorithm."

Although she did not begin with a formalized approach in the live-video, Nellie used the algorithm for multiplying and dividing fractions on the blackboard, and referred to starting with the algorithm in both her journal and look-back video. Nellie said, "For me, I automatically started with the algorithm." She referred to the action involved in the algorithm as "flipping and multiplying" and applied this method by dividing the 6 metres by the 5/6 metre size of a bow. Nellie was using her division algorithm from her formalising layer of understanding. She was going to divide the one unit by 6 to find out how many groups of five there were.

Disjointed solutions. Nellie was puzzled when her result "did not match" Ally's initial drawing, as she explained in her look-back video, "I got 7 and 1/5, and her drawing said 7 and 1/6, and we could not figure out why 1/6 and 1/5 were different." In Pirie-Kieren terms, the two solutions were disjointed because Nellie's algorithmic thinking included numbers only, without considering how the units—ribbons and bows—were involved.

The two different answers provoked the group to work further on this task. As Nellie said later in her look-back video, "I had to figure out why." The discussion among the three of them continued:

Deb: What's the one sixth? *(pointing to the 1/6th on the board)*
Nellie: The 1/6th of the next ribbon cause we separated our ribbon into 6. But that's assuming that 6 was the whole because now 6 is no longer—I mean one is the whole cause we used 6 over 6 as our ending. But this is 36 out of 6 like this is 6 whole metres together but then we jumped by 5. So really this would be one out of the next 5 jumps
Deb: And what's the jump?
Nellie: One ribbon
Ally: No, the jump is 5/6
Nellie: No, the length of a bow
Deb: And so let's see, what is this one sixth *(pointing to the board)*?
Ally: OK, let's just do the next jump.
Deb: What is this—the 1/6th? The next jump? Can you show me?

In Nellie's explanation, she began to refer to the "whole" and then they discussed the units of bows and the ribbon in relationship to the whole. After this discussion, Ally continued working on the number line to show the next jump after 35. They counted by ones to 40 and recorded this on the number line by extending it. Deb told Nellie that she also did not understand the meaning of the 5 that resulted from their multiplication algorithm. Indicating that she was equally invested in their work. Following the next jump on the number line, Deb asked them to show that, then realized they had, she said "I guess you did." The three of them broke out in laughter. Deb persisted with her question—what does the 1/6th mean?

Deb: What does this one 1/6 mean? If that, what the 1/5 means. What does that 1/6 mean?
Nellie: There's 1/6 of the metre left.
Ally: No, there's 1/6 of a whole.
Deb: Am I allowed to say it?
Nellie: If I'm asking you, it's different. I'm telling you that that's 1/6 metre of ribbon leftover.
Deb: *(Nods)* OK, yeah.
Ally: So of the 6 metres of ribbon *(recording on the board)*
Nellie: Which matches up here, because I said one metre which is 1/6?
Ally: Yeah, but we got 5.
Nellie: I don't know why.

In this dialogue, Nellie mentioned the significance of the prompt about the meaning of the 1/6th had on her understanding. Deb was bursting at this point, saying, "am I allowed to say it?" to a fellow co-teacher. Nellie overheard her and responded, "If I'm asking you, it's different. I'm telling you that that's 1/6 metre of ribbon left over." Nellie was looking for confirmation here, and Deb confirmed by nodding and saying yes. This illustrates a challenge associated with co-teaching—having a variety of prompts to use instead of falling into telling and explaining.

Folding back to Image Making. In her look-back video, Nellie recalled that the prompt,"what does the 1/6 represent?" caused her to consider the two different units (ribbons and bows), specifically to see that "the ribbon….being split into parts of 6, but our bows were represented in pieces of 5" of the six pieces of a metre of ribbon. Then Nellie folded back to construct a new image—a new number line that counted by 1/5's seven times and identified the leftover as 1/5.

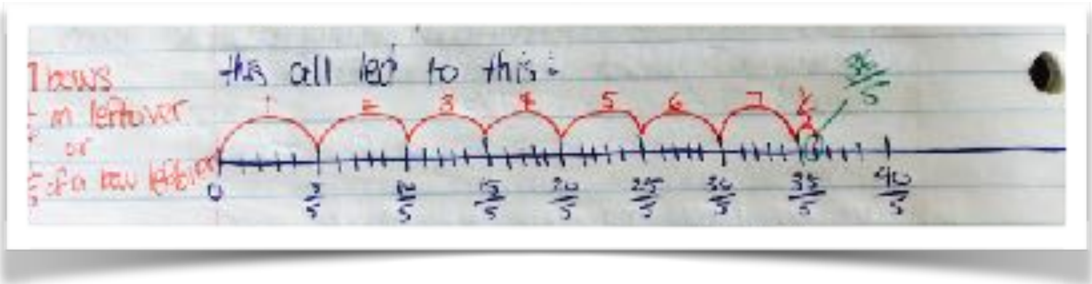


Figure 17. Nellie's Image Making: Counting by Fractions on a Number Line

This new image was different than her previous number lines because it distinguished between the two units, ribbon and bows. To the left of the number line, she recorded, "7 bows 1/6 m leftover or 1/5 of a bow leftover."

Property Noticing. By combining the initial number line—counting by sixths to get 7 and 1/6 leftover—with results of her algorithm, Nellie constructed the new number line image above and described this connection in her look-back video as, "the 1/5 represented a piece of the bow, rather than the 1/6 representing a piece of a metre that would have been the leftover." In the context of ribbons and bows, Nellie had explained the reason for the mismatch between the two answers by recognizing the two units involved—ribbons and bows.

In her journal, Nellie recorded the following two fractions, 5/6 and 6/5, and posed the question, "how are they related?" She circled the 6 in 5/6 and indicated that this was a "reference point for metres," then she drew a box around the 5 in 6/5's and referred to this as the "reference point for bows."

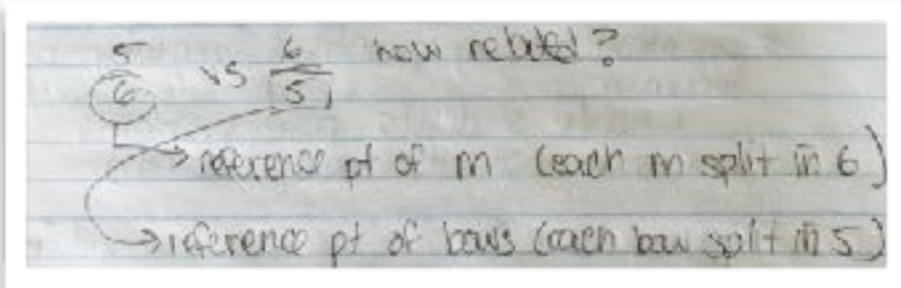


Figure 18. Nellie's Property Noticing: Reference Points

Formalising. Nellie used her general term "reference points" to describe the change in units, by associating it with the flipping action in the algorithm, "because you flip the fraction to multiply when you're dividing…your denominator changes and that's when your unit changes." Here Nellie was making the case for why her algorithm produced a different answer than counting by sixths on a number line had produced. Nellie explained and justified her notion of reference points further in her look-back video, "Whatever your reference point is will lead you to the answer…so in both cases the answers are correct, they are just looking at different aspects of the same question." Nellie's understanding was no longer tied to the specific ribbons and bows task. She had a thicker understanding of her algorithm for dividing by fractions because she was aware and explained the concept using reference points.

Mapping Nellie's growth of understanding of division of fractions. In summary, Nellie started by working on this division by fractions task with her partner Ally and found an answer using Ally's number line, though she actually wanted to use her algorithm for dividing by fractions. Later she found that her calculated answer was different than the initial number line answer. By drawing a new number line image to investigate this mismatch, she grew her understanding of how the algorithm worked. Nellie's movement through the Pirie-Kieren layers is diagrammed below, beginning in the Formalising layer of understanding and moving back and forth before ending in the Formalising layer of understanding.

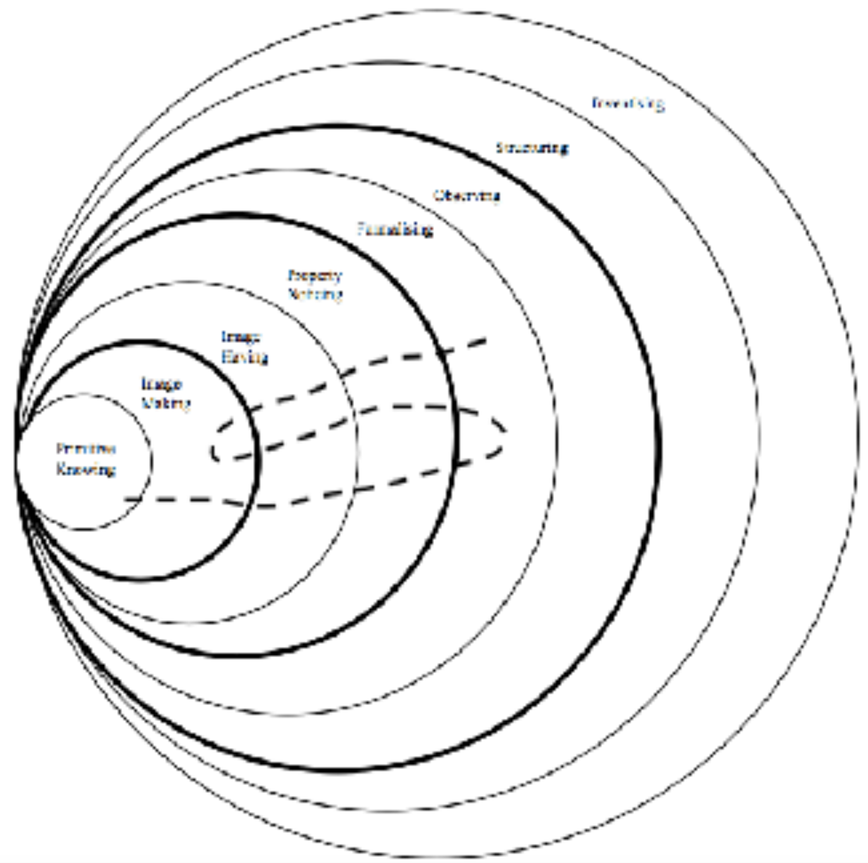


Figure 19. Mapping of Nellie's Understanding using the Pirie-Kieren Model

Pat's growth of understanding of dividing of fractions. Two board spaces down from Nellie's group, Pat worked on this task with her partner Moira. Mary, a co-teacher, was assigned to work with them. While Mary read the problem aloud, Pat was taking notes on the board. Pat was aware of my presence, as I walked by their group, and she said:

> Pat: Number line, number line, number line.
> Marc: *(laughing)*
> Moira: We've been prompted to use that number line.
> Marc: You don't have to, though. But it might help.

At this stage of the course, the NGTs were feeling comfortable with me and their peers. They were aware that "befores," our warm-up exercises, were ways to scaffold ideas and offered students tools to represent thinking. Pat's saying number line three times spoke to the comfort we had as a group. Since I had used a number line in our warm-up exercise, she took me up on my offer and started with it, although later in her work she said that the number line is not her preference.

Image Making: Pat's initial work. Pat started by dividing the units (1 metre) of a number line into 6 parts and then counting by fives and circling groups of fives to count the number of bows. Within 40 minutes of working on the task, Pat and Moira's board space was full of different solution strategies for dividing 6 metres of ribbon into 5/6 metre bows and determining the remainder. On the top left of Figure 20, they had circled 5 out of 6 dashes on a number line, and counted bows on the line to find 7 bows with 1/6 left over ribbon. Towards the bottom right of the figure, they used the division algorithm to find their second solution, 7 and 1/5 of a bow.

Image Making: checking her work. When Pat noticed the two different answers to the same question, 7 1/6 and 7 1/5, she decided to check her work. In Pirie-Kieren terms, these two images were disjointed: the formal algorithm that determined leftover bows was not connected to the leftover ribbon determined by the less formal counting on a number line.

As Pat explained in the live and look-back videos, she constructed an array figure on the middle right side of her board space. She said, “I tried to use the number line…and I wanted to check my work, and I found a different way where I actually used circles.” Pat drew her image of a 6 x 6 array of circled rows (6 metres) and columns (1/6th of a metre), and then grouped them into fives (5/6 metre in a bow) with one remaining circled dot labeled 1/6 metre. Figure 19 shows their board space:

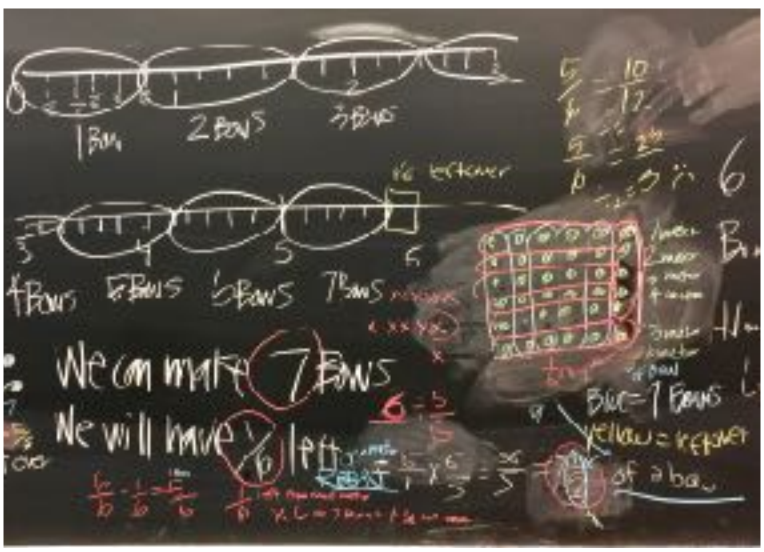


Figure 20. Image Making: Pat’s Initial Work

After the NGTs had been working on this task for more than 45 minutes, I got the attention of the whole class to find out how much more time was needed before sharing their solutions with their colleagues. The following exchange unfolded between me and the class, with the video was positioned on me so there was no way of identifying which NGTs actually spoke. In the transcription below, I used the plural form “NGTs” when multiple voices responded simultaneously and “NGT" when it was just one NGT responding.

Marc: I want to know right now. If we were to go and look at *(pause)* if someone were to look at your solution right now.
NGTs: *(laughter)*
Marc: Would they figure out
NGTs: *(interrupting)* No.
Marc: where you started?
NGTs: *(interrupting)* Probably not.
Marc: And where you came to? And my other question is: could they make sense of what you did *(pause)* Is that possible?
NGTs: No. I don’t think so.
Marc: No?
NGT: Probably not.
Marc: Is all of your thinking on the board?
NGTs: Pretty much. Yeah.
Marc: I want to know—do you want to *(pause)* I want to give you a choice between two things. I want to know if you want to polish your own solution
NGT: Yes.
Marc: Or if you’d like to polish someone else’s solution?
NGTs: Our own. Our own. Our own.
Marc: Okay. So before we polish our own then, let’s look at other peoples’ and then go back and polish our own. So I want you to look at what other people did, make sense of what other people did. Actually I’m only going to say this—make connections between your solution and the others. And then I’m going to give you an opportunity to go back and polish your own. Go ahead.

When I checked in to see if they were ready to share their work, many NGTs laughed at my suggestion that someone else might be able to make sense of their many representations on the board. They wanted to spend more time on their own work, cleaning it up and making it clearer for others to review. I took photos of their work, while they walked around to view each others’ work prior to polishing their own.

As I walked to each board taking photos, I saw the dots and circles solution for the first time. At first, I was unaware that Pat was the author, and I began talking to a few NGTs about it. Then Pat revealed that she had this solution to check her work.

Marc: This is an interesting representation that I didn't anticipate.
Pat: That *(pointing to the solution)*? I was checking what we had initially done.
Marc: Well, it's interesting because it started out as a set, right. And then she *(Pat)* groups it. Groups these sets. I'm still thinking about it as a set model. And I'm thinking that you like sets
Pat: I do. Yeah.

She went on to tell me that the number line was not something that she "probably would have started off" doing. Later in her journal, Pat identified her array solution with circles as her "preferred solution" and wrote, "what was interesting is the fact that I can calculate to see 7 bows can be made and 1/6 of ribbon is left, yet could not see [what] the 1/5 left represented." At that point, her two different answers had yet to be connected. Later in her look-back video, Pat said that she initially wanted to use the number line to solve the problem, but found this experience "confusing."

Image Making. Later, I asked the NGTs to polish their work by erasing it and rewriting or redrawing their preferred solution: "spend 20 minutes and we will have our polished solution … that you would look at from a grade six student … like really beautiful solution that is communicated with—I'm going to use language from our curriculum—-a high degree of effectiveness."

After this instruction was given, Pat and her partner Moira approached the board and began erasing it. As they worked on redrawing her solution, Pat drew circles and turned to Moira to ask her opinion about the direction for grouping the circles—horizontal or vertical. Moira recommended not changing the orientation because people had already seen it and liked it. After Pat drew the circles, she started talking and recording her words. I was standing off to the side and interjected questions as they talked and recorded their polished solution:

Moira: 7 1/6th of a meter leftover.
Pat: *(saying and recording)* 7 bows… 1/6th of a metre of *(pause)* ribbon,
Marc: Can I ask a question? This here, this is out of how many? This blue here is what?
Moira: 1/6
Marc: 1/6th 1, 2, 3, 4, 5, 6 *(counting the circles)*. So that's one sixth leftover, right? And if you put that into one bow *(pointing to the leftover)*.
Pat: Oh, 1/5th of a bow.
Moira: Yeah, see we had it! We didn't know why it worked.
Pat: Why it works, yes.

I asked them, "and if you put that into one bow?" Moira responded, confirming to Pat that they had already reached this conclusion. They continued polishing by adding an 8th bow with 1/5th ribbon missing. Their polished solution is shown below in Figure 20, to the right of Pat's original preferred solution. This new image includes the 1/5 of a bow that was not present in her initial image. In her journal, she later wrote about not seeing the 1/5 originally.

During the polishing exercise, Pat grew her understanding by reworking her original array image and connecting the two different remainders and their units. In her polished solution in the righthand diagram, Pat modified the circles and grouping in her previous image to make a new image. She included another vertical grouping of 4 circles with one missing circle at the bottom as a space for the piece of ribbon needed to complete a bow. Her polished solution made the distinction between the ungrouped circle —1/6 of a metre of leftover ribbon—and how it represented 1/5 of a bow. In her journal, Pat said she was "able to compare both the 1/6 and the 1/5m visually."

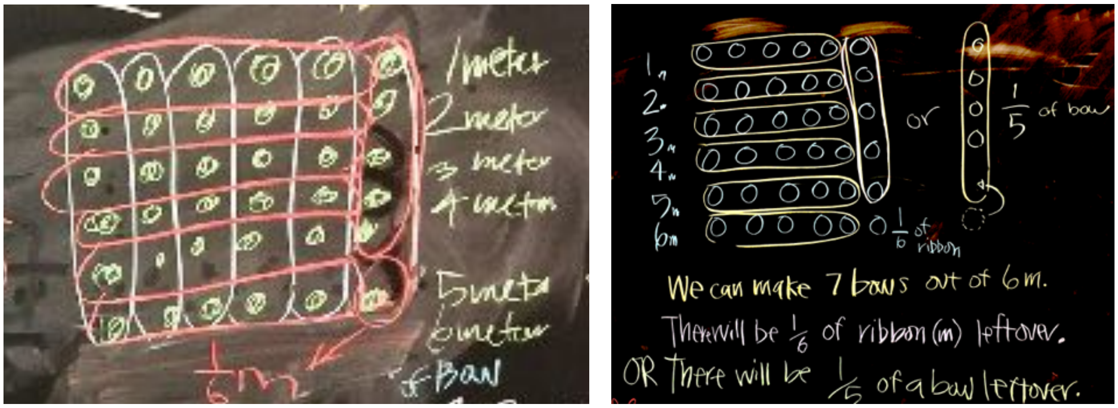


Figure 21. Image Making: Pat's Polished Work

Property Noticing. While polishing her solution, Pat responded, "Oh, 1/5th of a bow" when prompted to consider how much of a bow she could make with 1/6 of a ribbon. And this expression is followed up with an addition to her drawing that connected the 1/6 of leftover ribbon with the 1/5 of another bow. This showed that Pat noticed the relationship between these two numbers and their corresponding units. Later in her journal, she expressed her understanding of the two units—bows and ribbons—by stating "the 1/5 represents the leftover ribbon that makes 1 bow."

Mapping Pat's growth of understanding of division of fractions. In summary, Pat and Moira initially constructed a number line, and counted and grouped five 1/6s on this representation. They also used the algorithm for the division of fractions. To check her work, Pat later drew an array and counted the number of bows. Following a prompt to polish her work, she noticed how the two units were related and reworked her array to include both units. Her movements through the Pirie-Kieren layers of understanding are diagrammed below, beginning and ending in Image Making.

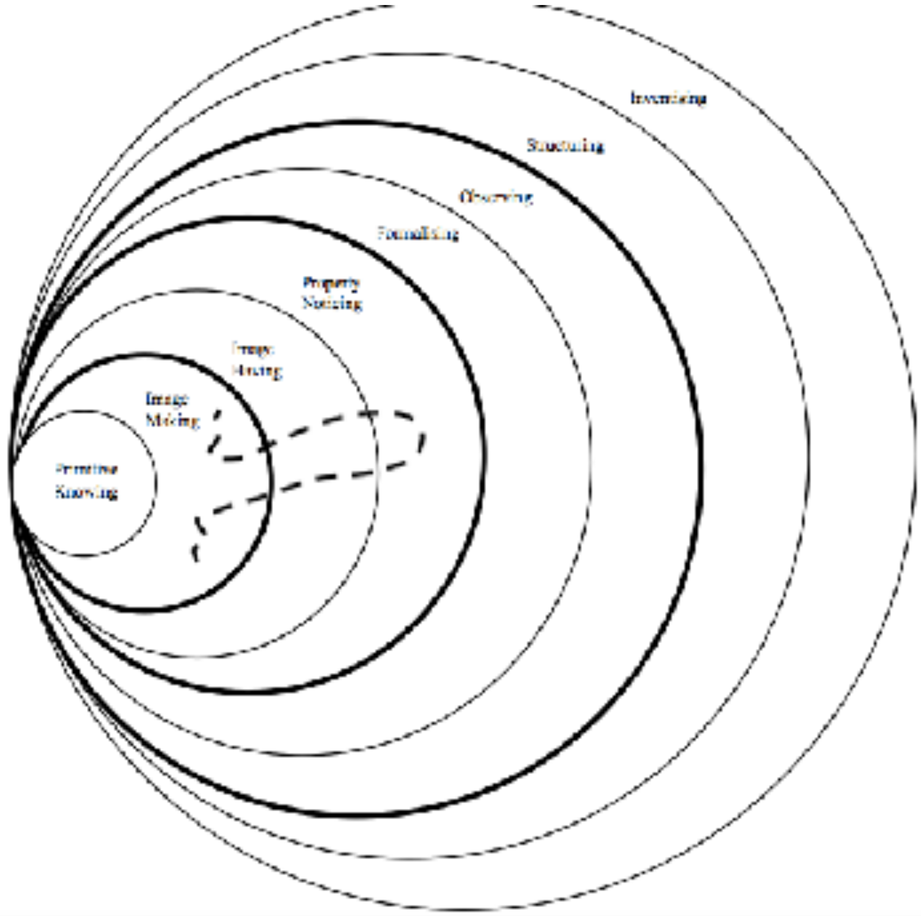


Figure 22. Mapping of Pat's Understanding using the Pirie-Kieren Model

Intentional interventions. Deb used several prompts (e.g. How might you use division as a solution?) in her interactions with Ally and Nellie. In her look-back video, Nellie recalled the specific prompt about the meaning of the 1/6th that she felt supported her growth in understanding. The one that Nellie recalled was different from prompt recorded in the live-video in which Deb focused her attention on the wholes. This prompt was an explicit intervention because it supported further work on differentiating between the two wholes, allowing Nellie to thicken her understanding of how units are being manipulated by the operations in her algorithm.

Pat did not refer to any specific prompts that supported her growth in understanding, yet from the live-video my prompt, “and if you put that into one bow,” elicited work that resulted in an additional drawing in her polished solution. This prompt was explicit in nature because it provoked Pat to recall the unit of bows and modify her polished array drawing to include how much of a bow could be made using the leftover ribbon.

Table 4: Examples of Interventions During the Division Task

Unfocused intervention	Explicit intervention
	What does this one 1/6 mean? (Nellie) How might you use division as a solution? (Nellie) And if you put that into one bow (Pat)

Division of fraction met-befores for Nellie and Pat. Nellie and Pat drew on the elementary mathematics that they had met before as they worked on this task, starting with counting by parts of a whole and grouping units of bows on number-lines. In both cases, the linear representation of a number line was not the preferred solution strategy. Nellie’s automatic response to use a division algorithm was evidence of her having met this rule before. When she calculated, however, her answer was different than her number-line answer. Pat had a similar experience; when she looked at her group’s number-line and algorithm answers, they were different.

Nellie and Pat used different met-befores to resolve this puzzle. Nellie drew her own version of a number-line that reflected the results of her algorithm. Connecting these two images, she saw how the different units accounted for the different answers resulting from her division algorithm. Pat, on the other hand, wanted to check the answer using dots in an array, an approach that she had met before and preferred. She grouped the dots to resolve the task, and modified it when she erased and redrew it during polishing. Both Nellie and Pat spent more than an hour working with their met-befores in the dividing fractions task. Although this could be characterized as impeding their progress, this work was necessary for them to make sense of how the two answers were related to the two different units— metres of ribbon and bows.

CHAPTER 5 STUDY FINDINGS

This case study investigated how newly graduated teachers elicited their prior knowledge and experience and used this to connect mathematical concepts from Ontario's elementary curriculum. It continues the long line of research related to learning mathematics, but focusing on future teachers, rather than students. Chapter 5 describes the findings from the case study. The first three sections, 5.1 through 5.3, answer the research questions about how this growth occurred and provide details about the NGTs' experiences and observations related to the connection-making process.

The two-part premise for this study was that elementary teachers lack an adequate understanding of mathematics for teaching and that their existing knowledge can be improved using teaching strategies similar to those recommended for elementary students. There is ample evidence (Adler et al., 2014; Ball, 1990; Ma, 1999; Smith & Stein, 2011) that a connected mathematics understanding is important for teachers, and yet prospective elementary teachers tend to have a fragmented understanding of the mathematics they will be teaching (McGowan, 2017). In this study, NGTs were working with and on their existing knowledge of elementary mathematics—work that is arguably essential for learning mathematics for teaching, yet rarely investigated by researchers.

Educational researchers and educators tend to focus on prior knowledge as the foundation for more advanced material, selecting earlier related concepts to support this progression. This method works well when learners are working forward through levels of a subject, but needs to be adapted for working within existing knowledge. This study demonstrated how current learning theories and teaching methods for school students can be adapted to support newly graduated teachers in growing their existing understanding of elementary mathematics. Analyzing the extensive data from video recordings and NGT journals yielded answers to the case study's research questions as discussed in the following three sections 5.1, 5.2 and 5.3.

5.1 How can Newly Graduated Teachers Connect their Mathematics Ideas Based on Recommended Strategies for Teaching Elementary Students?

When analyzing the data, a pattern emerged in how connections occurred: first, generating multiple images, then comparing images and identifying any disconnects, and ultimately resolving the disconnects to make new connections. These disconnects were examples of "disjointed" understandings as described by Pirie-Kieren Theory (1994a; 1994b). The connection-making process was iterative, that is, finding a disconnect between two images might lead to creating a third image or using another existing image. The three parts of this connection process are described below.

5.1.1 Generating Multiple Images based on Met-Befores

When NGTs were presented with an elementary mathematics task in the AQ course, they naturally called upon their prior knowledge and experience, their met-befores. Though they were free to start with any solution, some began with an informal approach using the method that was used in the warm-up exercise for the task. Others started by automatically recalling and using an algorithm, like Hillary in subtraction and Nellie in division.

The NGTs' starting images were based on applying the elementary mathematics that they had met before to the course tasks. These images were neither supportive nor problematic according to McGowan and Tall's (2010) definitions. They were helpful simply because they reflected the NGTs current understandings—what was being investigated and potentially connected. For example, Deb's friendly number strategy, useful in addition, ultimately provoked the thinking that helped develop her understanding of subtraction.

Some initial met-befores were considered typical while others were less common. Typical met-befores were related to the elementary mathematics operation being studied whereas less common met-befores brought in mathematics from outside this operation. In the subtraction task, for example, Deb's use of a familiar addition strategy resembled how elementary students might bring their met-befores to new learning experiences about another mathematics operation. On the other hand, Hillary and Nellie's process for subtraction and division of fractions started with their met-before algorithms, which was later compared and connected to a less formal image, illustrating a typical pathway that prospective teachers might follow in teacher education courses.

Producing one initial image was not enough; two or more images were needed to investigate connections within the NGTs' existing mathematical understanding. The AQ course was designed to ensure that NGTs worked with multiple images. It elicited multiple solution strategies, based on: instructions to use more than one approach, prompts by the instructor and co-teachers to try another method, and discussions among NGTs as they compared solutions. The prompts were explicit or unfocused interventions (Martin, 2008) that encouraged NGTs to fold back to a less formal image or fold forward to the relevant algorithm. The NGTs did not typically fold back on their own, because they had little experience with informal images, and the ones they did have were not connected to their formal rules. Prompting ensured that they drew upon several of their met-befores to create multiple images related to the task. In division, for example, Pat generated images of counting using number lines, dot arrays and the algorithm for dividing fractions.

The key was that the NGTs applied their met-befores to specific tasks and put them into action to see how well they worked. This created task-related images for NGTs to work with while investigating connections.

5.1.2 Comparing Images and Identifying Disconnects

Once the NGTs had developed multiple images, they folded back and forth as they identified and resolved disconnects among images or formula results—disjoints in Pirie-Kieren terms or problematic met-befores using McGowan and Tall's (2010) concept. Comparing multiple solutions revealed the presence or absence of connections among images. For the two analyzed tasks, subtraction and division by a fraction, none of the 4 NGTs brought a connected understanding that allowed them to immediately bridge their mathematical solutions to the tasks.

Several situations emerged as NGTs worked through course tasks. Sometimes their images gave two different answers—mismatches that clearly revealed a disconnect. In subtraction, Deb found two answers, 10 and 12. In this case, she realized that the answer was not reasonable and did not match counting on her fingers. In Nellie's case, a group of NGTs checked out each others' solutions on the blackboard and wondered how a number-line image could give a different answer from the calculation beside it. Occasionally, a NGT produced two solutions that gave the same answer based on results from an algorithm and the less formal diagram beside it, without seeing the connection between them, like Hillary's difficulty finding the 11 in subtraction. Prompts from co-teachers and the instructor encouraged the NGTs to do further work.

These disconnects were not problems; they were the basis for learning and connecting. Working with NGTs' met-befores to identify their disconnects involved a tremendous amount of time and teaching strategies, since the goal was to support them in their learning, rather than tell them the answers. Though the NGTs often found these disconnects puzzling, and sometimes frustrating, they were motivated to keep working and find the answer.

5.1.3 Resolving the Disconnects to Form New Connections

The connection-making process was dynamic. When the NGTs could not resolve disconnects among images, they created yet another image by reviewing a peer's work, using another method to cross-check their solution, or working through a prompt from a colleague or the instructor. For example, multiple instructor prompts supported Hillary to grow her understanding of borrowing in subtraction by first using a block diagram and then physical blocks. Often this further work was revealing because they found another image by folding back or forward to resolve the disconnect.

Sometimes the connection emerged later when NGTs checked their work, or at the end of the day when they were asked to polish their work by erasing the board then redoing their or another NGT's solution. For example, it was only after Pat redrew her array diagram that she reconciled the two answers to the division question. Other NGTs commented that her polished solution also helped them to "make sense" of the remainder in the division by a fraction task. Polishing allowed NGTs more time to work on the task, plus the motivation to make it easier for others to read and understand. As they rewrote or redrew their best solution, they adjusted it to make clearer connections.

Resolving a disconnect was a “now I see it” moment that involved an “interconnecting” of several images. It was satisfying for the NGTs. In some cases, they explained the problem with their earlier solution, discussed their new approach, or generalized by “enunciating and appreciating” (Pirie & Kieren, 1994, p. 171) how an algorithm worked, for example, Nellie’s “reference points” in the division by a fraction task.

5.2 How can Approaches Recommended for Teaching Children Mathematics be Adapted to Support Newly Graduated Teachers’ Growth of Understanding?

The AQ course was designed as a classroom where participants worked on and within elementary mathematics tasks that promoted thinking. Though the 4 NGTs were working on what some might call easy arithmetic, they were fully engaged and challenged by applying their met-befores to generate and connect solutions for elementary mathematics tasks. This study demonstrated how NGTs can grow their existing understanding. By working on elementary mathematics tasks in a supportive setting, they connected their prior knowledge, their met-befores, to deepen their understanding of the related concepts. Consistent with the similarities between how adults and children learn (eg. McDonough, 2013) and my experience in teacher education, the adult NGTs grew their understanding using practices similar to those they will apply with the children who will be their future students.

5.2.1 Course Features that Supported Connection-Making

Four course features were especially important for achieving the course’s connection-making objectives: selecting tasks, eliciting prior knowledge to make connections, working in groups and prompting. The course tasks were selected based on a combination of research findings and my experience. They elicited solution strategies that were expected to be challenging, like borrowing in subtraction where the rule is often memorized without understanding why the rule works. In their look-back videos for each task, many NGTs spoke about their confusion—the state of not understanding how one solution is connected to another.

The tasks were specifically selected to elicit algorithmic met-befores—prior knowledge and experiences that were based in following a set of rules to achieve the answer. They reflected elementary mathematics content and could be used in an elementary classroom with elementary students; however, because this group was likely to bring preferences for algorithms, the tasks were designed to elicit more than one solution—the task either asked for more than one way (e.g. the subtraction task), or the newly graduated teachers were prompted to look at an alternative solution (e.g. the division of fractions task).

The approach for eliciting prior knowledge was similar to the ways that prior knowledge is elicited in elementary classrooms. The NGTs were expected to talk about what they knew and record theses initial ideas on the board—the more ideas the better, as seen in Figure 20: Image Making: Pat's Initial Work. As this work unfolded, they began to make connections by investigating similarities and differences, a practice similar to the way school students make connections. Evidence of their process is outlined above in 4.1.

Working in randomized groups promoted talking and generating a variety of mathematical ideas, as NGTs developed and compared their images. Daily changes in groups maximized the exposure of NGTs to other participants' ideas and thus contributed to developing and discussing multiple solutions. It also ensured that that group dynamics were constantly in flux so NGTs did not fall into one type of role or behaviour. For example, students who tended to dominate the conversation or chalk for recording ideas sometimes had to compete with fellow colleagues who had similar dispositions. For example, Nellie and Ally had similar behaviours while working in a group—-they both held onto the chalk, recorded their own ideas and often talked over each other. Randomizing groups can be enacted in an elementary classroom with similar results.

Prompting was a critical, in-the-moment teaching intervention that provided guidance without telling the answers. It was used at every stage of the connection process to support thinking: generating multiple solution strategies, noticing disconnects, and finding connections between solution strategies. Telling the newly graduated teachers the answers would have stopped their thinking about and working on the task (Hiebert et al., 1997). Sometimes many prompts were needed, especially when NGTs saw two solutions as being separate ways of solving a task and needed to do a detailed comparison between them or create a third image to support their understanding.

Prompts—focused and unfocused—that supported connections were similar to the types of prompts that could be used with elementary students in classrooms. An unfocused prompt, for example, directed Deb to use a different tool supported her to grow her understanding of how her friendly number strategy in addition works in subtraction. On the other hand, a focused prompt that asked Hillary to find the 11 from by her algorithm in her blocks solution enabled her to thicken her understanding of subtraction. Both types of prompts can be used with students in classrooms; however, it would likely take a lot longer for newly graduated teachers to work through these prompts because their met-befores were so “deeply ingrained” (McGowan, 2017, p. 23).

5.2.2 Adapting Theoretical Frameworks to Existing Knowledge

Both Pirie-Kieren Theory and the concept of met-befores emphasize using prior knowledge in the context of learning mathematics. This study shifted the focus to making connections within NGTs' existing knowledge. Finding and fixing gaps in existing understanding meant looking for *disjointed* images in Pirie-Kieren terms or *problematic* met-befores as defined by McGowan and Tall (2010) and working to make connections that deepens understanding.

The primary theory used for analyzing study data, Pirie-Kieren's model for growth of understanding, assumes mathematics learners move forward and back through the layers of understanding as they gain a more sophisticated understanding of the subject matter. They start with their Primitive Knowing plus the images, or mental representations, they have about the topic. Then as learners progress, they create new images and revisit their earlier images about the topic, folding back and forth connecting images as they deepen their understanding.

The adult NGTs in this study had more prior knowledge related to the course tasks than a learner who is investigating an elementary math topic for the first time; they had several relevant images and already knew the algorithm at Pirie-Kieren's formalized layer. Therefore, I categorized all NGTs' prior knowledge as Primitive Knowing and reserved Image Having for situations where they worked with a colleague's image that they had not created themselves. NGT's algorithms did not embed their less formal images. In Pirie-Kieren terms, their formulas were thin and disjointed, or in the language of teacher education researchers, their knowledge was fragmented—exactly what this study expected to find. Rather than progressing through

Pirie-Kieren layers, NGTs were creating multiple images and folding back and forth with the goal of making connections within their existing knowledge.

In this context, *folding forward* was particularly significant. Starting with an informal solution and then folding forward allowed NGTs to work with concepts first, like school students would, and then reconcile and connect this with the formulas they automatically used. Making connections between informal and formal representations enabled some NGTs to formalise their understanding of an elementary mathematics concept.

Both Pirie and Kieren and met-before theorists support actively working to build learners' understandings. In Pirie-Kieren theory (1994), "mathematical topics need to be worked on at the image making level before one can begin to look for an appropriate formalization or structure" (p. 188). Martin and Towers (2016) recommend that problematic met-befores "be worked on and with to enable a connected, deeper understanding" (p. 96). In the case study, the NGTs selected their own solution strategies based on their prior knowledge. Their experiences illustrate how met-befores that delay progress or cause confusion for the student can elicit more work and so are often helpful in working with existing ideas, rather than being problematic (Martin & Towers, 2016; McGowan, 2017). In essence, students learn mathematics by doing mathematics—a widely accepted maxim in elementary education that also applies to prospective teachers learning mathematics for teaching (Pirie & Kieren, 1994).

5.3 How does a Course Focusing on Working with Existing Knowledge and Connection-Making Support a Growth in Understanding for Newly Graduated Teachers?

During the AQ course, the NGTs made connections among mathematical ideas and thickened their understanding of elementary mathematics. For example, what does borrowing really mean in subtraction, and how do units of measure affect numerical answers in division by a fraction? Though *deep understanding* is not a well defined term, there is wide spread agreement within the educational community that connected knowledge is associated with deep understanding (Adler, et al., 2014; Ontario Mathematics Curriculum, 2005; NCTM, 2000).

5.3.1 Newly Graduated Teachers' Growth of Mathematical Understanding

When NGTs started working on course tasks, their rules and algorithms were not formalized because these images were disconnected or disjointed in Pirie-Kieren terms. NGTs did not embed the NGTs' less formal understandings. NGT's experiences learning rules and algorithms appeared to have been "an add-on to previous informal mathematics activity or understanding" (Pirie & Kieren, 1994, p. 43).

By working through course tasks, NGTs grew in their understanding as they applied their met-befores to the course tasks, and moved back and forth making connections between their original images and those that they were making. They noticed either important properties related to the mathematics task or provided reasons for why their methods worked. Hillary and Nellie, in particular, were then able to enunciate an appreciation for the algorithm. In other words, they could explain and justify their methods to their colleagues—a key characteristic of formalising understanding in Pirie-Kieren Theory.

At the end of the course, the NGTs had a deeper of understanding of the elementary mathematics they would be teaching. The next time they approached similar tasks, they would likely draw upon their reworked set of met-befores, and generate more connected solution strategies. In theoretical terms, the NGTs' pre-course knowledge—met-befores and Primitive Knowing—changed as they made new connections. This renewal of prior knowledge is consistent with both theories, and in fact, is expected in Pirie-Kieren theory. In my case study, however, this was the goal, not a side benefit of working on an unfamiliar concept.

Using Wright's (2014) term, the *endgame* for teachers learning elementary mathematics for teaching is different from the endgame for students learning an unfamiliar mathematics topic. For students, the endgame involves using materials and multiple images so "the development of abstract concepts that can be thought of without reference to actions and images" (Wright, 2014, p. 107). For teachers, the endgame involves less sophisticated representations connected to existing algorithms and rules so they can refer to these actions and multiple images in their teaching.

5.3.2 Newly Graduated Teachers' Understanding of the Connection Process

While engaging in the connection process to improve their understandings, NGTs were also learning about how this process works. As students, they reflected on and documented their experiences throughout the AQ course. As co-teachers, they participated in anticipating possible solutions for tasks and developing prompts that they might use to support their colleagues' connection-making.

In-the-moment prompting was challenging for some co-teachers. When their prompts did not get the expected response, they could not readily follow an unexpected path or quickly develop an alternative prompt—understandable because they were applying new skills to support their peers' work on a confusing task.

As they progressed through the 10-day AQ course, the NGTs' journals and look-back videos included more and more references to what "connections" mean. In my course-instructor role, I allowed participants to develop their own understanding of connections in a mathematics context by not providing a definition of the word, leaving it open to interpretations—like connecting to the real world or connecting to past and future teaching. Though some NGTs initially asked for clarification when asked to document a connection, later most NGTs understood how connecting applied to their work, and identified and documented connections between their mathematics images related to the course tasks.

Some NGTs also commented about making connections in their future teaching. One NGT noticed how the orientation of images affects seeing connections: "I think if we did a side-by-side comparison, our students wouldn't be able to make that connection … there needs to be some type of symmetry or mirroring … the way we were traditionally taught I wouldn't have made that connection." Another NGT talked about making notes: "Annotating over the course of the day, I've just had a growing feeling that annotating really really is something that my students can benefit from … You can't record it without understanding it. It's a great way for students to make connections between what they are doing and what someone else is doing." Reflecting on what she had learned, another participant said, "We need to get into our students' heads … think like them and put our experiences aside … and be in this primitive knowing, same level as them, trying to understand what they're trying to understand."

This case study offers answers to long-standing questions about how teachers can connect their own mathematics understanding and learn how to support their students in making connections. Teacher educators looking for new research-based practices can apply these methods to their work with teacher candidates, and researchers may use it as a springboard for further classroom-based research in teacher education.

CHAPTER 6 DISCUSSION

This chapter discusses the merits of working on existing knowledge as demonstrated in this case study, along with using approaches recommended for teaching elementary school students. It also includes broad recommendations to support implementing these kinds of experiences in teacher education.

6.1 Learning Experiences Connect Fragmented Existing Knowledge

Education researchers have found that prospective teachers often have a fragmented understanding of elementary mathematics (Adler et al, 2014; Ball, 1990; McGowan; 2017). In addition, their past classroom experiences affect their teaching practices more than what they read hear or talk about later in teacher education classes (Ball, 1988; 2001; McGowan, 2017).

This case study described how NGTs can grow their existing understanding of elementary mathematics by doing mathematics tasks that elicited their prior knowledge, and then working to make connections among their fragmented mathematical ideas. The AQ course was based on research recommendations and teaching practices for young learners, and adapted to teacher education by incorporating co-teaching, reading Pirie-Kieren Theory and recording reflections through look-back videos. The course structure and design provided an example of a learning environment that elicited NGTs' met-befores and fostered work to make connections, as described in Chapter 4. It also provided the kind of learning and teaching experiences necessary to set NGTs on a teaching pathway that is less likely to include a show-and-tell (Ball, 2001) approach to teaching.

This course focused on newly graduated teachers growing their mathematical understanding by working on math tasks to elicit their existing knowledge while simultaneously giving them experience in recommended teaching practices. This experience in teacher education is markedly different from the more common experiences like highly structured lessons (e.g. Ball, et al., 2009). that focus on either information about teaching or prescriptions for teaching, like rehearsal techniques (e.g. Kazemi et al. 2016)—experiences that do not specifically address a teacher's mathematics understanding, or the importance of making connections for themselves.

Educating prospective teachers, like other forms of teaching and learning, can be understood in three parts: what do the students know, how do they need to learn to become teachers, and what teaching approaches are most effective for supporting this learning. This case study investigated a teaching approach that mirrored recommendations for teaching elementary school mathematics.

6.1.1 Starting with What Newly Graduated Teachers Know

Using what students know as a starting point is an accepted learning principle that is commonly used in elementary classrooms and other learning settings. Their mathematics met-befores and what we might call their *teaching met-befores* were the starting points for NGTs in the AQ course education. In Pirie Kieren terms, this is called Primitive Knowing—using what students know as a starting point. In this study, course instruction or planned activities were dependent upon the newly graduated teachers' Primitive Knowing. Building on a students' starting place means that students need to express mathematical images while working on math tasks. For course instructors, this means selecting math tasks, orchestrating opportunities for sharing and discussion, listening and responding to existing ideas though prompts.

Elementary mathematics knowledge. In this case study, doing elementary mathematics tasks revealed information about the NGTs' mathematical understandings that neither the NGTs themselves nor their instructors could know in advance. Working with their met-befores to solve tasks revealed that the NGTs' existing knowledge was fragmented, as researchers have found in other settings, and making connections among solution methods was challenging. The NGT case study participants found that even seemingly simple operations, like subtraction, required an hour or more of work: uncovering existing knowledge through a task, expressing their math ideas, seeing others' mathematics ideas, and revising or doing further work based on an intervention (e.g., an explicit prompt).

Problem solving tendencies. The NGTs' solutions and related discussions also revealed some problem solving tendencies that could affect their future teaching. When solving tasks in context, like the ribbon and bows task, they focused on the numbers, often multiplying or dividing two numbers and then wondering what the answer meant in terms of the context. This was particularly evident when they used algorithms. As Sowder, Philipp, Armstrong and Schappelle (1998) have recognized, when teachers own understanding of algorithms—in particular, division of fractions—is "not yet robust" (p. 88), they tend to confuse their students.

Image preferences. NGT's solution strategies and comments also revealed image preferences that could be a problem for future teaching. The number line, for example, was a tool that was preferred by some and disliked by others. In subtraction, Deb said that representing her idea on a number line "was the only way [she] could identify that [she] had changed the number in both directions." Other NGTs said that using a number line allowed them to "see" their mental activities. However, Pat referred to the number line as being confusing, and other NGTs agreed with her saying that it was "utterly confusing" and not helpful in making sense of ideas. The number line is referenced 31 times in the Ontario Curriculum (2005) with specific recommendations for use with topics, such as counting by fractions (p. 67).

Image preferences develop naturally for learners. When they have been exposed to a variety of tools and strategies, they may self-select one that works best for them. On the other hand, preferences can be so powerful that once students have a favourite image it is hard to change. This is evident in Pirie-Kieren's (1994) observations about a student who pursued a familiar strategy over others. Yet the opposite can be said about the students in this course. While Deb was working on the division of fractions, for example, she said that she was stuck "because the number line was an unpreferred tool for me and I couldn't make sense of it." Her preference not to use a number line in this context could be viewed as a dismissal of its use, but we know that this is not true. Deb also referred to using the number line in order for her to see the missing two when she pursued her friendly number strategy. In this case, Deb is showing her flexible use of representations, and how in some contexts they can be very powerful and, in other situations, not helpful.

During the AQ course, some NGTs gained insights about using a variety of tools. One NGT realized that supporting future students may involve "helping kids go towards a certain tool," though she selected another tool to make her own connection. Another NGT saw how using different tools deepened her understanding, saying I was asked "to use the number line and the hundreds chart, and I realized that it was another way of coming to my solutions."

6.1.2 Future Teachers Learning about Connections

Previous research has investigated the kinds of knowledge and practices needed for effective elementary mathematics teaching, in brief—a connected knowledge and a connecting practice (Ma, 1999; Ball et al., 2005). What has been less clear are how these recommendations can be translated into a teacher education program. Teachers, like their students, need to learn about practices first by experiencing them, and then by linking these experiences to theoretical perspectives (Gainsburg, 2012).

Connecting existing knowledge by doing math tasks. In this study, the NGTs developed multiple solutions for carefully selected elementary tasks, identified areas where they could not see how the solutions were related and worked to find the connections between them. This process is analyzed in detail in chapter 4 and summarized in the Study Findings, chapter 5.

Having connections to less formal images serves two purposes tor teachers. It deepens their own mathematics understanding and provides a variety of connected images that they can use to support their students. Ma (1999) suggested that subject matter knowledge and pedagogical content knowledge are not separate for mathematics teachers; the what and the how of learning are intermingled in the classroom and in the minds of learners. When teacher candidates experience the value of multiple images and how to connect them, they are better prepared to address the needs of their future students and support them in making connections, what Smith and Stein (2011) refer to as the most challenging practice for teachers.

Practicing related teaching activities. The course was planned like elementary teachers or teacher education instructors might plan their courses, not scripted like a step-by-step curriculum or role-playing exercise. In preparing for each lesson, a group of NGT co-teachers anticipated multiple solution strategies, made connections between them, and considered questions that would prompt their peers to make connections. Then in real-time, the co-teachers observed their colleagues while they worked on tasks and intentionally intervened with prompting questions to support their colleagues' connection-making.

Teacher education classrooms, like the AQ course in this case study, provide valuable information about the state of teachers' existing knowledge and preferences, in addition to supporting them in connecting and deepening their understanding of mathematics for teaching. Seeing these results for teacher candidates, a teacher educator might ask: What tasks would support learning how to integrate physical contexts, like units of measure, into solution strategies? What experiences would demonstrate the usefulness of the number line and other tools? Or more comprehensively, how could these approaches be integrated into my regular classes?

6.2. The Dilemma in Elementary Mathematics Education

Over 20 years of educational research has done little to change the status quo in teacher education. Prior research seems to focus on what teachers know or do not know, but doesn't look at how they can work with and on their existing understanding—through approaches similar to the way students learn. This is suggested in the literature but barely examined. Breaking the cycle of teachers teaching the way they were taught requires a massive shift in how we educate future teachers in Ontario. This section covers the factors that impede change in teacher education. It reflects both research findings and my 20 years of teaching prospective teachers, in-service teachers and elementary students.

6.2.1 Established Teaching Approaches

Teacher educators are working within an established educational system that is not conducive to change, particularly when the instructors themselves are products of that system with no elementary classroom experience. In a university setting, both the instructors and teacher candidates can unwittingly perpetuate the status quo (Hart & Swars, 2009). Teacher candidates recall procedures for elementary math operations, but they don't know what they are missing. Based on their past learning experiences, they expect to listen and take notes. Their instructors are faced with over 40 students in the classroom, each with a different understanding of elementary mathematics. The classrooms are lecture halls, and as an instructor, you might have Powerpoint slides that you revise and use every year. As a teacher educator, it is natural to default to teaching what you know your students don't know and would like to learn, like lesson planning or evaluation. Delivery modes such as these are simply the way things are done because of the physical environment and the experiences of both instructors and students.

Research related to prospective teachers. As reviewed in chapter 2, teacher education researchers say that teachers need a deep, connected mathematics understanding. They recommend more problem solving, discussions (including explaining and justifying), thinking and reflecting, seeing and especially doing; and less presentations of theory when students have few practical experiences to draw from (Gainsburg, 2012). Plus there is a trend towards pedagogies of enactment (Grossman et al., 2009) such as rehearsals and lesson plays.

Teacher education is caught in the knowledge trap of higher learning, when teaching and learning are a complex sets of activities that involves both knowledge and in the moment experience working with students and using teaching practices. Learning about (i.e., talking about) how school students learn mathematics does not help a student do math. Elementary mathematics is an applied system that students learn by using in the context of the physical world and then generalize so it can be used in abstract form (Harel & Tall, 1991; Pirie & Kieren, 1994).

Learning about teaching does not help teacher candidates actually teach their young students. Certainly teaching ideas and tools can be helpful, but using these in the real-time dynamics of a classroom requires actual experience. Through many teacher educators may recognize this, it is risky to try approaches where you start with a students' existing knowledge because the dynamics of teaching and learning change (Milewski, & Strickland, 2016).

Teaching newly graduated teachers to work on and with their existing knowledge requires a skill set similar to how Kazemi and her colleagues (2016) describe the teacher educator's role in rehearsal— the teacher educator orients newly graduated teachers to each others' thinking, prompting to elicit further work and respond to mathematics ideas that emerge in the moment. The main difference between Kazemi et al's (2016) approach and the one outlined here is that the mathematics being worked on is rooted in newly graduated teachers' existing mathematical ideas and the kinds of connections they are making.

6.2.2 Misconceptions Affecting Mathematics Teaching

Two common assumptions about elementary mathematics and adult learning can also hamper implementing learning in teacher education.

The myth that elementary math is easy. In the AQ case study, NGTs found that working with their met-befores to solve tasks and make connections among solution methods was challenging—-belying common assumptions that elementary mathematics is simple arithmetic and that formulas are enough. As Klinger (2011) said:

> While concepts of number, number representation, and arithmetic operations are certainly fundamental, there is nothing basic (in the sense of 'simple' or 'obvious') about them – it has taken humankind millennia to invent/discover these concepts and to formalize them in terms of definitions, rules, procedures, and algorithms. (p. 13)

Most people can rely on formulas as a quick and effective method for doing calculations, leaving early solution strategies behind once they are no longer needed. As Susan Pirie observed in 1988, the "inability to reproduce the 'doing strategies' from 'first principles' does not indicate a diminished understanding, nor is it necessary or even desirable when problem solving" (p. 5). Elementary mathematics teachers, however, need an understanding of many tools and strategies, how to apply them flexibly to math tasks, and how to connect solutions—from the primitive to the formal—so they can guide learners to develop their understanding of mathematics.

6.3 Teaching for Understanding in Elementary Teacher Education

This section discusses how the case study's findings could be used to support an example of implementing research recommendations for teachers in teacher education. First, I propose that the AQ case study and the roles and relationships implicit in its methodology can serve as an example for how research recommendations can be implemented in teacher education. Then, I discuss how we might develop a description of what is included in elementary mathematics for teaching. I conclude with a proposed list of teaching principles that would support the kinds of teaching and learning experiences that allow NGTs to work with and on their existing knowledge that could unify teacher education without codifying the details of the program.

6.3.1 A Model for Teaching Elementary Mathematics

The course described in this dissertation offers the mathematics education community an example of what course-work focusing on connecting teacher candidates' existing mathematics knowledge looks like in practice. It demonstrated how newly graduated teachers (NGTs) can use their prior knowledge and experiences to connect their understanding of elementary mathematics, while also studying related theories and practicing co-teaching. Despite concerns about teachers' fragmented mathematics understanding and general agreement about the value of connection-making and using prior knowledge, researchers have been asking how teachers can acquire the needed knowledge and practices (Kazemi et al., 2016; Lampert, 2010; McGowan, 2017). This study offers a number of ways in which this might be accomplished as presented below.

Roles and relationships. The details of the AQ course are described in the study methodology and discussed its findings, but it is also useful to consider the roles and underlying relationships for a course that connects existing understanding.

- Each student starts with their own understanding and solution preferences and progresses in their own way.
- The instructor's role is to provide a safe, non-judgemental environment where participants do math in several informal and formal ways, and find and resolve disconnected ideas. The more disconnects the better, they should be celebrated and viewed as learning opportunities. No showing or telling, just prompting.
- Instructors and their adult students operate like peers, even though they have different knowledge and experiences, and different roles in the classroom.

Publicly sharing what they do not know feels risky for many teacher candidates. This stems from their background experiences that focused on getting the correct answer and their concern that they might be wrong. An open, safe learning environment allows students to be comfortable sharing their ideas even when they have difficulty solving elementary tasks. Safe environments support prospective teachers to take risks and overcome their reluctance to do mathematics.

Newly graduated teachers who feel that the instructor is confidence in their abilities persevere and ultimately grow their mathematical understanding. For example, when Hillary did not find the 11 initially, I did not give up on her; I knew she could connect these two representations. The study video transcriptions include many instances of me walking away from the NGTs. This purposeful teaching action supported learning in two ways: it provided NGTs with think time and time to talk with a peer, and it also allowed me to avoid over instructing, or telling the answers. In addition, the transcription shows me really listening and making sense of their ideas (e.g. saying, "Ahha"). This is teaching move elevated the NGT's status by signifying that their mathematical ideas were interesting and valuable for the collective. It also emphasized that doing mathematics is about generating ideas, not merely learning about math rules. For example, by listening carefully to Hillary's subtraction explanation, I realized her blocks method was solved in the opposite order from her algorithm. This enabled me to understand the work that needed to be done next: doing a solution using blocks starting from the same side as her algorithm. At another point in the course, I engaged the class by allowing them to decide what action they preferred to do—polish their own solution or polish a colleague's work.

Experiencing a safe learning environment can inform prospective teachers' future teaching, replacing the default show-and-tell methods from their past classroom experiences. The dialogue within the case study analysis illustrates the kind of relaxed working relationships that the NGTs achieved using this approach. For example, Deb's insistence on working with her colleague Pat to show how her friendly number strategy worked. Plus the several times in the video where Nellie, Ally and Deb were laughing, enjoying the process of co-teaching or working on the course tasks. Also, Pat's reference to the number line, and teasing me about focusing on it prior to working on the task. These examples illustrate the pleasures of working with and on existing knowledge in a safe environment.

6.3.2 Elementary Mathematics for Teaching

Teacher educators and teacher candidates would benefit from having a target for developing mathematical understanding. What mathematics are teacher candidates expected to learn by the end of their program, and what experiences to we want for them? Without a description of what mathematical understanding is needed at the elementary level, teachers and teacher educators may have a variety of opinions about what content is appropriate in methods courses.

My work with case-study participants and the related research and analysis suggest that Pirie-Kieren Theory could be used to define what deeper understanding means for elementary math teachers. The works of Pirie-Kieren are based on the goal of deepening understanding and the process of folding back to revisit previous understandings to achieve this (Pirie & Kieren, 1994). Other researchers describe deep understanding as connected knowledge (Adler et al., 2014). In Ma's (1999) discussion of Profound Understanding in Fundamental Mathematics, she defined the mathematics needed for teaching in elementary schools as a deep knowledge of mathematics at the level that it is being taught.

Based on this case study and these descriptions of deep understanding, we can develop a more detailed description of deep understanding for teachers. Pirie-Kieren (1994) describes their theory's formalising layer of understanding as embedding all the previous, informal layers of understanding. For teachers, this would be formalized understanding at the levels they are teaching. Then using Adler and her colleagues' (2014) description of deep understanding as connected knowledge, rather than the word "embedding," the description becomes: *Deep understanding for teachers is connected, formalized understanding of mathematics at the levels they are teaching.*

To be useful, this description needs to be more specific. *The mathematics at the level they are teaching* is described in the Ontario elementary curriculum. To be *formalised*, the algorithm must be connected to the less formal representations within the curriculum. So subtraction includes the algorithm connected to other images, like counting, the number line and blocks. Plus an operation like subtraction does not stand alone. How is it connected to addition, for example? Is the minus sign used to describe negative numbers at the level that is being taught?

6.3.3 Teaching Principles, a Curriculum Alternative for Teacher Education

Currently Ontario's teacher educators are working without a curriculum, except the elementary curriculum that their teacher candidates will use with their future students. This allows flexibility for individual institutions and teacher educators to tailor their courses, but presents system-wide challenges for Ontario, as recently discussed in a ministry-funded report and major newspaper (Orpwood, & Brown, 2015; The Globe and Mail, August, 2018).

As described earlier in section 2.2.5, some published teacher education curricula can be viewed as cautionary examples. Through commendable collaborative work, teacher educators have produced detailed lesson plans to use when teaching pre-service teachers. However, there is a risk associated with institutionalizing the status quo or current best practices. Are these step-by-step programs flexible enough to respond to students' needs, emerging research, changing contexts for elementary teaching, or the individual creativity of the instructors themselves? Detailed syllabi for courses can be constraining. Teaching principles is an approach that allows teacher educators to be more responsive in working with a prospective teachers in growing their existing knowledge.

Teaching principles. Agreeing to a set of teaching principles would support teacher educators in implementing research-based recommendations for teachers teaching students, without prescribing the specifics of lessons. The list of teaching principles below informed this study, and could be beneficial for elementary mathematics educators to consider as they develop programs of study for prospective teachers:

- Move from the practical to conceptual, for example, doing mathematics before introducing frameworks for teaching like lesson plan templates (Gainsburg, 2012).
- Start with and revisit what learners know from past experience or have more recently learned (McGowan, 2017);
- Select and work with elementary mathematics tasks designed to grow understanding and practice using a variety of tools and solution strategies (Kazemi et al., 2016); and
- Promote engagement by randomizing groups to work on math tasks, expressing mathematical ideas, and exposure to others' mathematical ideas (Liljedahl, 2016).

This list reflects research recommendations as well as my teaching experience. A consensus on teaching principles would harmonize elementary mathematics teaching and thus create a better learning experience for teacher candidates. This opportunity could be extended to other subjects in the elementary program to broaden its benefits, because, after all, teacher educators specialize in teaching one subject, but elementary teachers teach them all. These teaching principles together with a more detailed, connected description of elementary mathematics, and the roles and relationships described in section 6.3.1 are a foundation for implementing a course that offers participants opportunities to work on and with their existing mathematics understanding.

CHAPTER 7 CONCLUSION

This case study illustrates how teacher educators can elicit what teacher candidates already know and use it as a resource for teaching and learning elementary mathematics. It can serve as a reference point for educators looking for research-based, interactive experiences to support their students' learning. The study's theoretical framework, methodology and findings may also inspire and support future research. The limitations of the AQ case study are discussed in section 7.1, opportunities for future research are suggested in section 7.2, and my personal reflections on this work are included in section 7.3.

7.1 Limitations of this Study

This case study is one example of a self-selected group working with an experienced instructor. For this type of course, the instructor needs an in depth understanding of elementary mathematics. I have been using interactive teaching methods in my classes for over 10 years, supporting teachers in sharing what they do and do not know. For participants to identify and work with their met-befores, they must be willing to uncover their own disjointed understandings. A typical group of elementary teacher candidates would likely include several candidates with math anxiety who are more reluctant to take risks, especially if their solutions are incorrect. More work needs to be done to demonstrate the methodology's general applicability and effectiveness.

7.2 Opportunities for Future Research

7.2.1 Replicating the Case Study

This case study could be replicated with more typical groups in a variety of course settings using the same or an adapted methodology. Fruitful possibilities include: other additional qualification courses for pre-service or in-service teachers, or a series of classes within an existing teacher education course. To maintain the research focus on using prior knowledge and connection-making, further studies would ideally engage teachers in a supportive, interactive setting that provides learning and co-teaching opportunities. The essential elements are: selecting tasks, anticipating solutions, developing prompting strategies, incorporating group work and practices that focus on making connections. To adapt these approaches to other groups or settings, the teacher educator could use different course tasks that maintain a range of solution strategies at the level they are teaching. These studies might also include a follow up component to assess the impact of the experience on participants' future teaching.

7.2.2 Investigating the Impact on Future Teaching

This research was based on the premise that, if newly graduated teachers have experiences connecting their own understanding, then they are more likely to support their future students to do the same. Follow up studies could investigate how participants translate connecting experiences into their teaching practice, using a variety of followup techniques, for example, surveying of participants' perceptions or observing participants' classroom teaching.

7.2.3 Detailing Elementary Mathematics for Teacher Education

The proposed description for deep understanding—connected, formalizing understanding at the levels they are teaching—could be expanded by mapping curriculum concepts at the level teachers are teaching as discussed in section 6.3.2. This exercise would reveal how well the curriculum is connected—mathematics for teaching might be organized differently than what currently exists (e.g., geometry, number sense, patterning, etc.). Using methods like Davis' (2008) concept study, it could be done by teacher educators to support their instruction, or by teacher candidates or in-service teachers as a learning exercise. For example, can an elementary teacher who teaches the concept of rate trace this back and forth through curriculum guides for the purpose of connecting less sophisticated strategies with the formalising layer of understanding?

7.2.4 Further Studies of Connecting Existing Mathematics Knowledge

In my literature review, I found no research that specifically investigated deepening existing knowledge, except Pirie-Kieren theory (1994) where existing knowledge continuously grows as a result of participating in learning experiences. Yet it is critical at all educational levels and for many related subjects with a high mathematics component. In practice, we have been relying on review lessons, make-up courses or home study—usually without making direct use of what the learner knows and engaging them in deepening their own understanding. Using what prospective teachers bring to strengthen their elementary mathematics understanding is definitely worthy of more research attention.

7.3 The Teacher Educator's Reprise

At this point, it is enlightening to consider the 15 newly graduated teachers who participated in the AQ course case study. They were not beginners; they had completed all requirements for their Bachelor of Education degree, including 72 hours of mathematics methods courses, and hence were called newly graduated teachers. Several students had some experience with problem solving in classrooms because they had taken one of the courses I taught. They volunteered to participate in this additional qualification course and may never take another mathematics course, unless they choose to do so. How well prepared are they to start teaching math to children next September?

These 15 new graduates may become excellent math teachers through years of teaching experience and personal study or mentoring. This case study is an example of how we can create experiences that accelerate learning for prospective teachers, and an invitation for teacher educators and researchers to continue exploring how to deepen the elementary mathematics understanding of our future teachers.

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