Probability 3: Discrete Random Variables

Module Outline

- Review: Bayes' Rule
- Random variables: Discrete and Continuous
- Probability distributions
- Expectation and Variance
- Common discrete distributions

Review: Conditional probability

- Conditional probability: $P(B|A)$
- General multiplication rule: $P(A \cap B) = P(B|A)P(A)$
- Total/Marginal probability:
  $P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$
  $= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + ... + P(A_n)P(B|A_n)$

Review: Bayes' Rule

- **Bayes' rule**:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

- Start with $P(B) \equiv$ **prior**,

  Updating due to new information: $P(B|A) \equiv$ **posterior**

new info.

Review: Bayes' Rule example

- Prob. (infection) $= 0.3$. Do a **Test**:

  $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- **Ex**: Use Bayes' rule to find $P(\text{infected} \mid +ive) =$

Review: Bayes' Rule example

$Pr(\text{not inf}) = 0.7$

- Prob. (infection) $= 0.3$. Do a **Test**:

  $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- **Ex**: Use Bayes' rule to find $P(\text{infected} \mid +ive) =$

- $P(\text{inf} \mid +ive) = \dfrac{P(+ \mid \text{inf})P(\text{inf})}{P(+ \mid \text{inf})P(\text{inf}) + P(+ \mid \text{not inf})P(\text{not inf})} =$

posterior

$$= \frac{0.7 \times 0.3}{0.7 \times 0.3 + 0.1 \times 0.7}$$

$$= \frac{0.21}{0.21 + 0.07} = \frac{.21}{.28} = \frac{21}{28}$$

$$= 3/4 = 0.75$$

Review contd.: A second test?

- <u>1*st* test</u>: +ive. **Does a second test**.

  Test: $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- <u>New prior</u>: $P(\text{inf} \mid \text{1st } +ive) =$ 0.75

Review contd.: A second test?

- 1*st* test: +ive. **Does a second test**.

  Test: $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- New prior: $P(\text{inf} \mid 1\text{st} + ive) =$ 0.75 $P(\text{not inf}) = 0.25$

- $P(\text{inf} \mid 2\text{nd} + ive) = \frac{P(+ \mid \text{inf})P(\text{inf})}{P(+ \mid \text{inf})P(\text{inf}) + P(+ \mid \text{not inf})P(\text{not inf})} =$

$$= \frac{0.7 \times 0.75}{0.7 \times 0.75 + 0.1 \times 0.25}$$

$$= 0.95$$

Review: Bayes' Rule extended

- **Bayes' rule**:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

- Suppose three possibilities for B : B_1, B_2, B_3

$$P(B_1|A) = ?$$

$$= \frac{P(B_1 \cap A)}{P(A)}$$

B_1 $\quad$ B_2 $\quad$ B_3

Review: Bayes' Rule extended

- **Bayes' rule**:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

- Suppose three possibilities for $B : B_1,\ B_2,\ B_3$
- Now, **Bayes' rule (extended)**:

$$\begin{aligned} P(B_1|A) &= \frac{P(A \cap B_1)}{P(A)} \\ &= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)} \end{aligned}$$

Example: Bayes' Rule extended

- 3 *types of borrowers*:

  Good diligent: $P(\textit{default} \mid \text{good dil}) = 0.1$,

  prior: $P(\text{good dil}) = 0.4$.

  Good negligient: $P(\textit{default} \mid \text{good neg}) = 0.4$,

  prior: $P(\text{good neg}) = 0.4$.

  Bad: $P(\textit{default} \mid \text{bad}) = 1$, prior: $P(\text{bad}) = 0.2$.

Example: Bayes' Rule extended

- 3 *types of borrowers*:

  Good diligent: $P(\mathit{default} \mid \text{good dil}) = 0.1$,

  prior: $P(\text{good dil}) = 0.4$.

  Good negligient: $P(\mathit{default} \mid \text{good neg}) = 0.4$,

  prior: $P(\text{good neg}) = 0.4$.

  Bad: $P(\mathit{default} \mid \text{bad}) = 1$, prior: $P(\text{bad}) = 0.2$.

- **Observe a default:** $P(\text{bad} \mid \text{default}) = ?$

Example: Types of borrowers

- **Good diligent**: $P(\mathit{def} \mid \text{good dil}) = 0.1$, $P(\text{good dil}) = 0.4$.

  Good negligient: $P(\mathit{def} \mid \text{good neg}) = 0.4$,

  $P(\text{good neg}) = 0.4$.

  Bad: $P(\mathit{def} \mid \text{bad}) = 1$, $P(\text{bad}) = 0.2$.

- $\boxed{\textbf{Observe a default:}}$ $P(\text{bad} \mid \text{default}) =$ df bad

Bayes' Rule

$$\frac{P(\cancel{\text{bad}}|\cancel{df})P(\text{bad})}{P(df|\text{bad})P(\text{bad}) + P(df|\text{g dil})P(\text{g dil}) + P(df|\text{g neg})P(\text{g neg})}$$

$$= \frac{1 \times 0.2}{1 \times 0.2 + 0.1 \times 0.4 + 0.4 \times 0.4}$$

$$= \frac{0.2}{0.2 + 0.04 + 0.16} = \frac{2}{4} = 0.5$$

Ex: $P(\text{good dil}|\text{default}) = ?$

ANS: 0.1

Random Variables

- *Random variable* $\widetilde{X}$: A variable taking on *numerical values* based on the outcome of some random event

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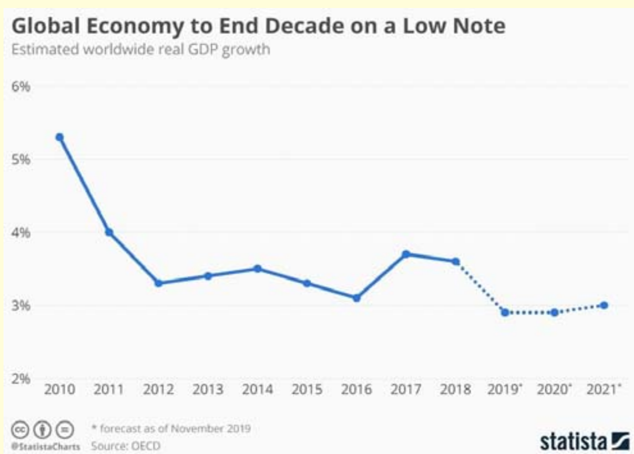
- Examples: **[Discrete]** No. of forest fires, crimes, spectators, children, minutes your phone lasts on a single charge......



Random Variables

- *Random variable* $\widetilde{X}$: A variable taking on *numerical values* based on the outcome of some random event
- Examples: **[Discrete]** No. of forest fires, crimes, spectators, children, minutes your phone lasts on a single charge......
- Examples: **[Continuous]** growth rate, crime rate, profits/losses, water level in a lake,.....



Random Variables

- *Random variable* $\widetilde{X}$: A variable taking on *numerical values* based on the outcome of some random event

- Examples: **[Derived r.v.]** Letter grade based on numerical scores,......

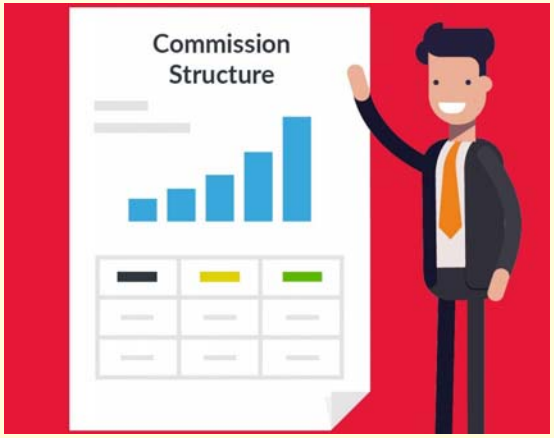
%	GPA	Letter grade
90-100%	4.0	A+
85-90%	3.9	A
80-84%	3.7	A-
77-89%	3.3	B+
73-76%	3.0	B
70-72%	2.7	B-
67-69%	2.3	C+
63-66%	2.0	C
60-62%	1.7	C-

Random Variables

- *Random variable* $\widetilde{X}$: A variable taking on *numerical values* based on the outcome of some random event

- Examples: **[Derived r.v.]** Letter grade based on numerical scores,......

- Examples: Profits based on sale numbers, commission based on sales, ...



Random Variables and Probabilities

- What are the probabilities of the different outcomes?

Random Variables and Probabilities

$\tilde{x}$ $\tilde{y}$

- What are the probabilities of the different outcomes?
- **Prob. mass function** $(p.m.f)$ $P(\widetilde{X} = x) \equiv p(x)$

particular value x

Random Variables and Probabilities

- What are the probabilities of the different outcomes?
- **Prob. mass function** $(p.m.f)$ $P(\widetilde{X} = x) \equiv p(x)$
- **Example 1**: A restaurant has 5 tables.

  $\widetilde{X}$ = number of tables occupied

0
1
2
3
4
5

Random Variables and Probabilities

- What are the probabilities of the different outcomes?
- **Prob. mass function** ($p.m.f$) $P(\widetilde{X} = x) \equiv p(x)$
- **Example 1**: A restaurant has 5 tables.

  $\widetilde{X}$ = number of tables occupied

-

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

$\rightarrow$ p.m.f.

$p(2) = 0.4$

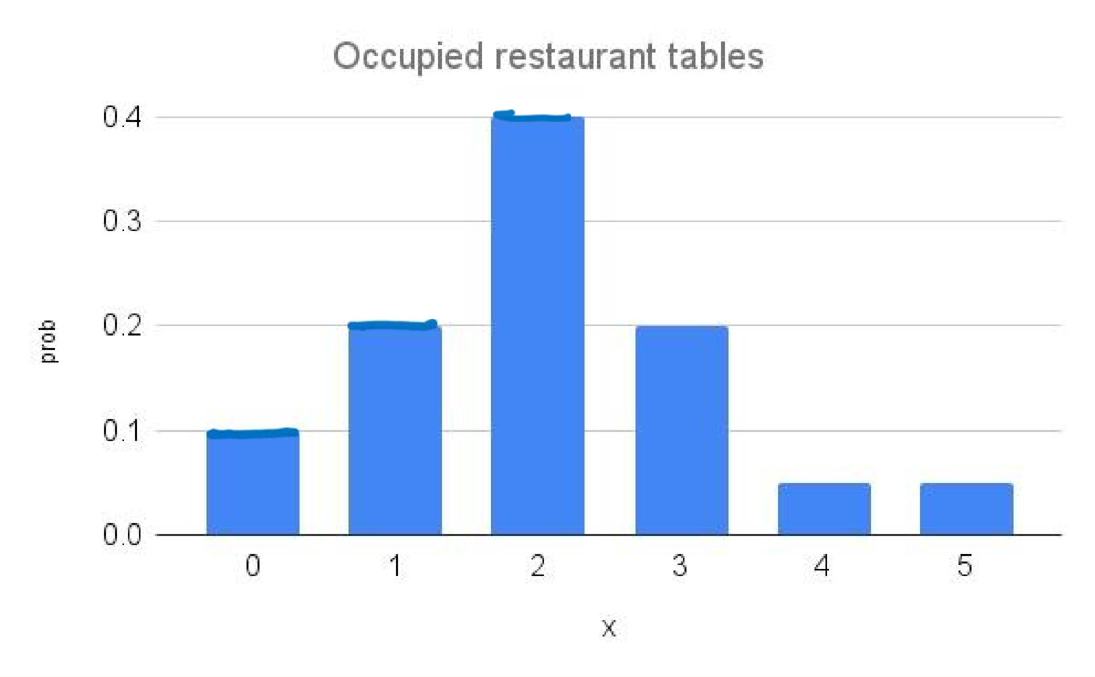
$0 \leq p(x) \leq 1$

$\sum_x p(x) = 1$

Example 1

- $\widetilde{X}$ = number of tables occupied

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05



Random Variables and Probabilities: Example

$6+5$
$5+6$

- **Example 2**: 2 dice rolled. $\widetilde{X}$ = sum of their outcome. $6+6$

$\widetilde{X}$	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$2: 1+1$

$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$4: 2+2 \rightarrow \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
$3+1 \rightarrow$
$1+3 \rightarrow$

$6: 3+3$
$5+1$
$1+5$
$2+4$
$4+2$

$3: \begin{cases} 1+2 \rightarrow \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \\ 2+1 \end{cases}$

$5: 4+1$
$1+4$
$2+3$
$3+2$

$7: 6+1, 1+6$
$5+2, 2+5$
$3+4, 4+3$

Example 2

- 2 dice rolled. $\widetilde{X}$ = sum of their outcome.

$\widetilde{X}$	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\Sigma = 1$

Example 1 contd.

- **Example 1**: A restaurant has 5 tables.

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

- *Profit per table* = \$50; *fixed costs* = \$100

Example 1 contd.

- **Example 1**: A restaurant has 5 tables.

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

- *Profit per table* = \$50; *fixed costs* = \$100
- $\widetilde{Z}$ = total profit for the restaurant

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0·1	0·2	0·4	0·2	0·05	0·05

Example 1 contd.

- $\widetilde{X}$ = number of tables occupied, $\widetilde{Z}$ = total profit

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

-

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

- *Other probs.*: Prob. that restaurant is occupied? $1 - p(0)$

$$= 1 - 0.1$$
$$= 0.9$$

Example 1 contd.

- $\widetilde{X}$ = number of tables occupied, $\widetilde{Z}$ = total profit

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

-

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

- *Other probs.*: Prob. that restaurant is occupied?
- Prob. that restaurant makes +*ive* profits?

$0\cdot 2 + 0\cdot 05 + 0\cdot 05 = 0\cdot 3$

Clicker question 1

- $\widetilde{X}$ = number of tables occupied

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

Q: What is prob. that an odd number of tables are occupied?

(a) 0.1 **(b)** 0.2

(c) 0.4 **(d)** 0.45

Solution to Clicker question 1

- $\widetilde{X}$ = number of tables occupied

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

Q: What is prob. that an odd number of tables are occupied?

(a) 0.1 **(b)** 0.2

(c) 0.4 **(d)** 0.45

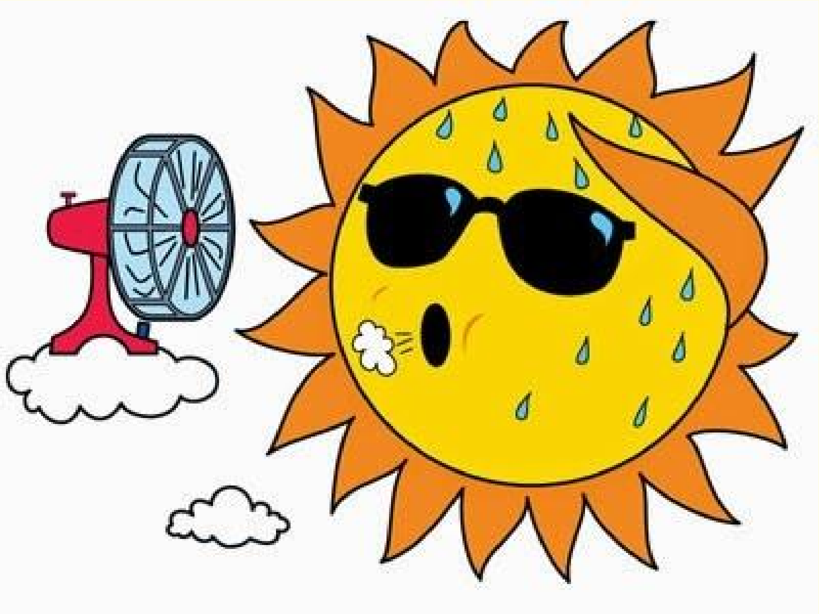
$0.2 + 0.2 + 0.05$

$= 0.45$

Example 2

- An ice-cream parlour's sales depends on the temp $\tilde{t}$

$\tilde{t}$	< 20	$20 - 25$	$25 - 30$	$30 - 35$	$35 - 40$	$40+$
$p(t)$	0.15	0.15	0.2	0.2	0.25	0.05



Example 2

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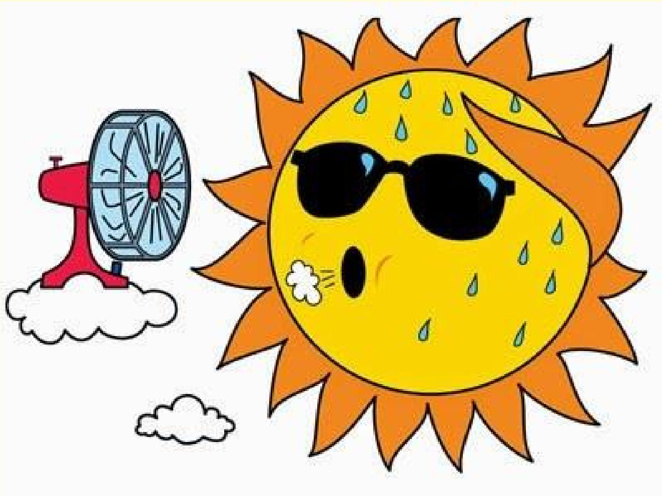
$\widetilde{t}$	< 20	$20 - 25$	$25 - 30$	$30 - 35$	$35 - 40$	$40+$
$p(t)$	0.15	0.15	0.2	0.2	0.25	0.05

- If $\widetilde{t} < 20$, sales $\widetilde{s} = 0$; $20 \leq \widetilde{t} < 30$, sales $\widetilde{s} = 30$; $30 \leq \widetilde{t} < 40$, sales $\widetilde{s} = 60$; $40 \leq \widetilde{t}$, sales $\widetilde{s} = 100$



Example 2

- An ice-cream parlour's sales depends on the temp. $\widetilde{t}$

$\widetilde{t}$	< 20	$20-25$	$25-30$	$30-35$	$35-40$	$40+$
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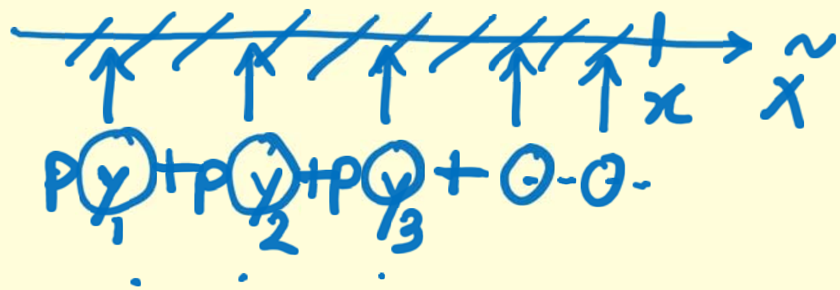
$\widetilde{s}$	0	30	60	100
$p(s)$	0·15	0·35	0·45	0·05

Cumulative distribution function

- **Cumulative distribution function** ($c.d.f$): Prob. that $\widetilde{X}$ takes values less than or equal to x :

$$F(x) = P(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$$

$F(x)$



Cumulative distribution function

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$$F(x) = P(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$$

- **Example 1**:

$\widetilde{X}$	0	1	2	3	4	5
$p.m.f.$	0.1	0.2	0.4	0.2	0.05	0.05
$c.d.f.$	0.1	0.3	0.7	0.9	0.95	1

$\rightarrow F(5)$

$F(3)$ $F(4)$

$F(0) = P(\tilde{X} \leq 0)$ $F(1) = P(\tilde{X} \leq 1)$

0 or 1

C.d.f. for Example 1 contd.

Profits

$\widetilde{Z}$	-100	-50	0	50	100	150	
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05	$\rightarrow$ pmf
cdf $F(z)$	0.1	0.3	0.7	0.9	0.95	1	

$$F(100) = \Pr(\widetilde{Z} \leq 100) = 0.95$$

C.d.f. for Example 1 contd.

$p(25) = 0$

$Pr(\tilde{z} \le 200) = 1$

$\tilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05
$F(z)$	0.1	0.3	0.7	0.9	0.95	1

cdf

$Pr(\tilde{z} \le -200) = 0$

- What is $F(22)$? $F(124)$? $F(200)$? $F(-200)$?

$F(22) = Pr[\tilde{z} \le 22]$

$= p(-100) + p(-50) + p(0)$

$= 0.7$

$F(50) = Pr(\tilde{z} \le 50)$

$= 0.9$

Graph of c.d.f. for Example 1 contd.

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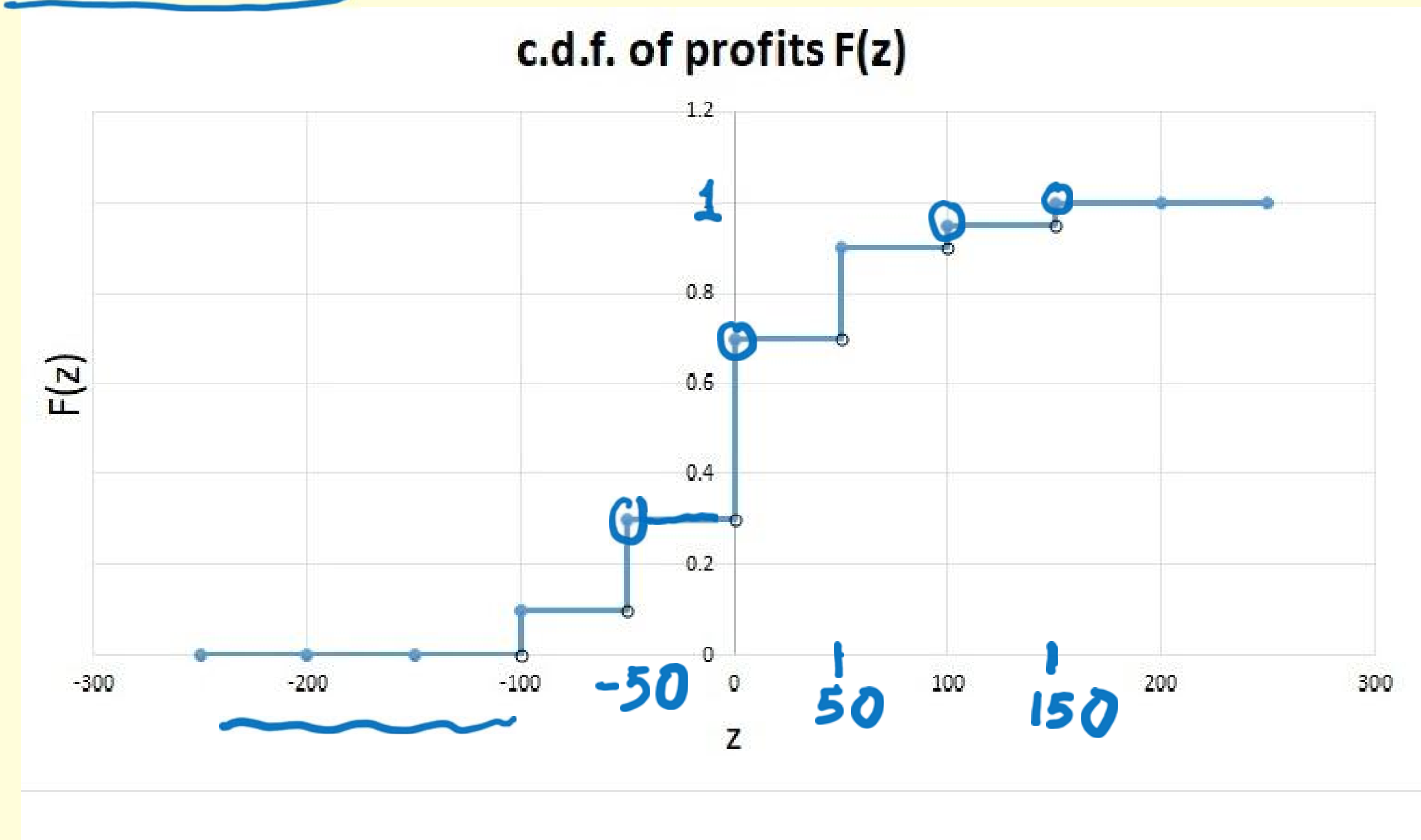
$\tilde{Z}$	-100	-50	0	50	100	150
$F(z)$	0.1	0.3	0.7	0.9	0.95	1

- Graph: step-function



Cumulative distribution function: Properties

- $0 \le F(x) \le 1$

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- $0 \leq F(x) \leq 1$
- $F(x)$ is non-decreasing

Cumulative distribution function: Properties

- $0 \leq F(x) \leq 1$
- $F(x)$ is non-decreasing
- If maximum possible value of $\widetilde{X}$ is x^{max} :

  $F(x) = 1$ for $x \geq x^{\text{max}}$.

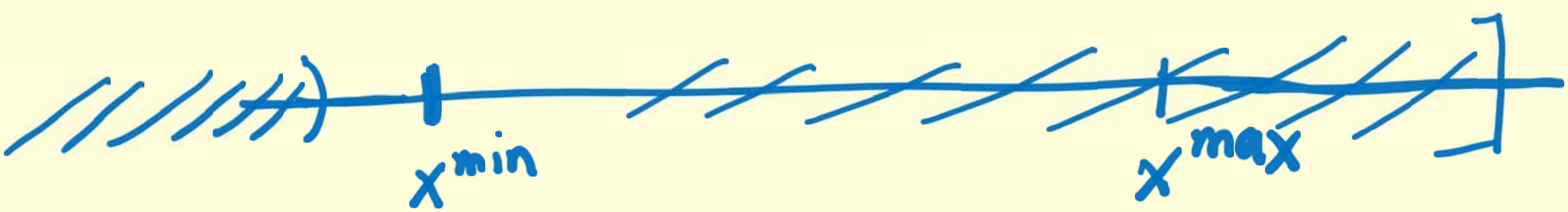
$Pr(\tilde{X} \leq x^{max}) = 1$

Cumulative distribution function: Properties

- $0 \leq F(x) \leq 1$
- $F(x)$ is non-decreasing
- If maximum possible value of $\widetilde{X}$ is $x^{\max}$:

  $F(x) = 1$ for $x \geq x^{\max}$.
- If minimum possible value of $\widetilde{X}$ is $x^{\min}$:

  $F(x) = 0$ for $x < x^{\min}$.



Clicker question 2

- A random variable $\widetilde{X}$ has the following p.m.f.:

$\widetilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2

Q: What is $F(60)$?

(a) 0.25 **(b)** 0.45

(c) 0.65 **(d)** 1

Solution to Clicker question 2

- A random variable $\widetilde{X}$ has the following p.m.f.:

$\widetilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2

$Pr(\widetilde{X} \leq 60) = 1$

Q: What is $F(60)$?

(a) 0.25

(b) 0.45

(c) 0.65

(d) 1

C.d.f. for Clicker question

- *Clicker question*: start with p.m.f.; derive c.d.f.:

$\tilde{X}$	10	20	30	40	50	60	
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2	pmf
$F(x)$	0.4	0.6	0.7	0.75	0.8	1	cdf ?

$F(x) = 0$ for $x < 10$

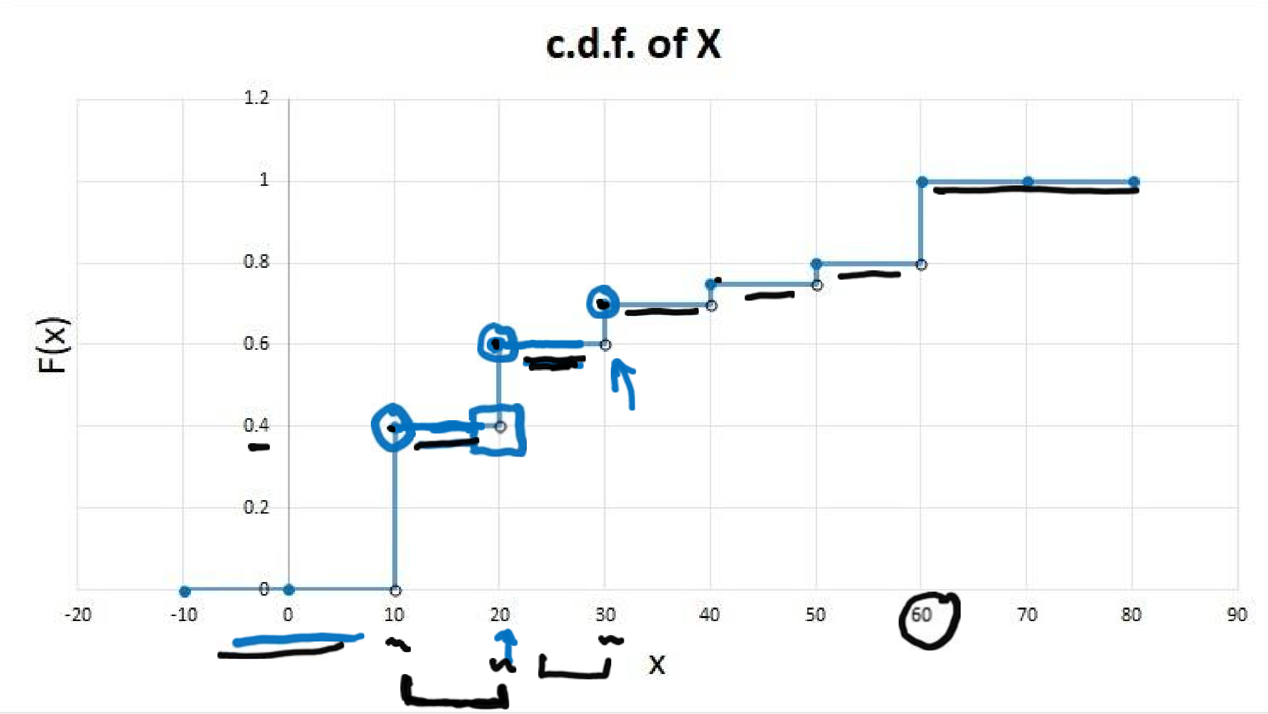
$F(x) = 1$ for $x \geq 60$

$$F(15) = Pr[\tilde{X} \leq 15]$$
$$= 0.4$$
$$F(32) = Pr[\tilde{X} \leq 32] = 0.7$$

C.d.f. for Clicker question

$\tilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2
$F(x)$	0.4	0.6	0.7	0.75	0.8	1

- Graph: step-function



$$Pr(\tilde{X} \le 15) = Pr(\tilde{X} \le 10)$$

Deriving p.m.f from c.d.f.

- *Clicker question*: start with p.m.f.; derive c.d.f.:

$\widetilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2
$F(x)$						

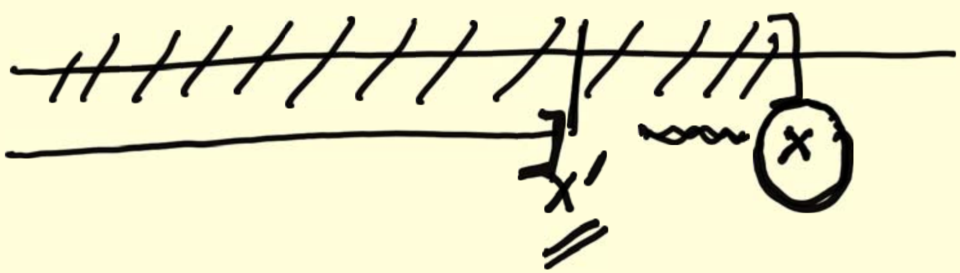
- **Opposite:** Start with $c.d.f.\ F(x)$; how to get $p.m.f.\ p(x)$?

Deriving p.m.f from c.d.f.

- *Clicker question*: start with p.m.f.; derive c.d.f.:

$\widetilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2
$F(x)$						

- **Opposite:** Start with $c.d.f.$ $F(x)$; how to get $p.m.f.$ $p(x)$?
- $F(x) = P(\widetilde{X} \leq x) = \sum_{y:\, y \leq x} p(y)$



Deriving p.m.f from c.d.f.

- *Clicker question*: start with p.m.f.; derive c.d.f.:

$\widetilde{X}$	10	20	30	40	50	60
~~$p(x)$~~	~~0.4~~	~~0.2~~	~~0.1~~	~~0.05~~	~~0.05~~	~~0.2~~
→ $F(x)$	0.4	0.6	0.7	0.75		

?

- **Opposite:** Start with $c.d.f.$ $F(x)$; how to get $p.m.f.$ $p(x)$?
- $F(x) = P(\widetilde{X} \leq x) = \sum_{y:\, y \leq x} p(y)$
- $\Rightarrow$ p.m.f. $=$ difference between adjacent c.d.f.s

Deriving p.m.f from c.d.f: Examples

- *Start with c.d.f.:*

$\tilde{X}$	10	20	30	40	50	60
$F(x)$	0.4	0.6	0.7	0.75	0.8	1
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2

Deriving p.m.f from c.d.f: Examples

- *Start with c.d.f.*:

$\widetilde{X}$	10	20	30	40	50	60
$F(x)$	0.4	0.6	0.7	0.75	0.8	1
$p(x)$						

- **Example 1 contd.**: derive *pmf* from *cdf*

$\widetilde{Z}$	-100	-50	0	50	100	150
$F(z)$	0.1	0.3	0.7	0.9	0.95	1
$p(z)$				0.2	0.05	0.05

Clicker question 2

- A random variable $\widetilde{X}$ has the following c.d.f.:

$\widetilde{X}$	1	2	3	4	5	6
$F(x)$	0.1	0.2	0.4	0.5	0.75	1

Q: What is $p(6)$?

(a) 0.1 **(b)** 0.2

(c) 0.25 **(d)** 0.75

Solution to Clicker question 2

- A random variable $\widetilde{X}$ has the following c.d.f.:

$\widetilde{X}$	1	2	3	4	5	6
$F(x)$	0.1	0.2	0.4	0.5	0.75	1

$1 - 0.75 = 0.25$

Q: What is $p(6)$?

(a) 0.1 **(b)** 0.2

(c) 0.25 **(d)** 0.75

Expected value

- **Example 1** (contd.) $\widetilde{Z}$ = profit

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

Expected profits?

0.1: -100
0.2: -50
0.4: 0
0.2: 50
0.05: 100
0.05: 150

$$
\begin{aligned}
& -100 \times 0.1 = -10 \\
+ & -50 \times 0.2 = -10 \\
+ & \ 0 \times 0.4 = 0 \\
+ & \ 50 \times 0.2 = 10 \\
+ & \ 100 \times 0.05 = 5 \\
+ & \ 150 \times 0.05 = 7.5 \\
\hline
& \quad 2.5
\end{aligned}
$$

Expected value

- **Example 1** (contd.) $\widetilde{Z}$ = profit

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

Expected profits?

- **Expected value** (*mean*) of $\widetilde{X}$:

$$E\widetilde{X} = \mu_X = \sum_{\text{all } x} xp(x)$$

mu

Expected value

- **Example 1** (contd.) $\widetilde{Z}$ = profit

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

Expected profits?

- **Expected value** (*mean*) of $\widetilde{X}$:

$$E\widetilde{X} = \mu_X = \sum_{\text{all } x} xp(x)$$

$\neq$

- ! *Not the same* as average
only when $p(x) = \frac{1}{N}$

$$\frac{1}{N}\sum x = \sum \frac{1}{N} x$$

Expected value

- **Example 1** (contd.) $\widetilde{Z}$ = profit

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

  Expected profits?

- **Expected value** (*mean*) of $\widetilde{X}$:

$$\rightarrow E\widetilde{X} = \mu_X = \sum_{\text{all } x} x p(x)$$

- ! *Not the same* as average
  only when $p(x) = \frac{1}{N}$
- **Weighted average**

Example 3

- Diameter $\widetilde{d}$ of tree-trunks in a forest:

$\widetilde{d}$ (in m)	1	2	3	4
$p(d)$	0.3	0.4	0.2	0.1

Example 3

- Diameter $\widetilde{d}$ of tree-trunks in a forest:

$\widetilde{d}$ (in m)	1	2	3	4
$p(d)$	0.3	0.4	0.2	0.1

- Mean diameter $\mu_d =$ $1\times 0.3 + 2\times 0.4 + 3\times 0.2 + 4\times 0.1$

$= 0.3 + 0.8 + 0.6 + 0.4$

$= 2.1\ m$

Example 3

- Diameter $\widetilde{d}$ of tree-trunks in a forest:

$\widetilde{d}$ (in m)	1	2	3	4
$p(d)$	0.3	0.4	0.2	0.1

- Mean diameter $\mu_d =$

- **Expected value** of a function of $\widetilde{X}$:

$$Eg(\widetilde{X}) = \sum_{\text{all } x} g(x)p(x)$$

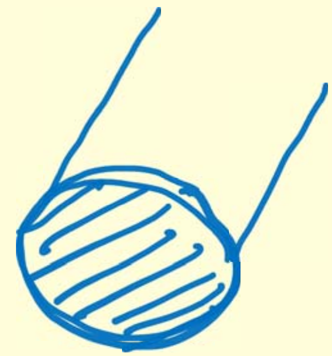
Example 3 contd.

- Diameter $\tilde{d}$ of tree-trunks in a forest;

  Lumber yield y from a tree of diameter d : $10\pi d^2$

  $g(d)$

  Expected lumber yield?



Example 3 contd.

- Diameter $\widetilde{d}$ of tree-trunks in a forest;

  Lumber yield y from a tree of diameter d : $10\pi d^2$

  Expected lumber yield?

-

$\widetilde{d}$	1	2	3	4
$p(d)$	0.3	0.4	0.2	0.1
$\widetilde{y} = 10\pi\widetilde{d}^2$	10π	40π	90π	160π

$$E\widetilde{y} = 10\pi \times 0.3 = 3\pi$$

Mean yield $\mu_y =$

$$+ 40\pi \times 0.4 = 16\pi$$
$$+ 90\pi \times 0.2 = 18\pi$$
$$+ 160\pi \times 0.1 = 16\pi$$

$$53\pi \doteq 53 \times \frac{22}{7}$$
$$= 166.57$$

Properties of Expected value

- $E(\text{constant}) = \text{constant}$

$E(3) = 3$

$E(\pi) = \pi$

Properties of Expected value

- $E(\text{constant}) = \text{constant}$
- **Expected value** (*mean*) of $\widetilde{X}$:

$$E\widetilde{X} = \mu_X = \sum_{\text{all } x} \underline{xp(x)}$$

Properties of Expected value

- $E(\text{constant}) = \text{constant}$
- **Expected value** (*mean*) of $\widetilde{X}$:

$$E\widetilde{X} = \mu_X = \sum_{\text{all } x} xp(x)$$

- $E(a + b\widetilde{X}) = a + bE\widetilde{X}$

$$E b\widetilde{X} = \sum bxp(x)$$
$$= b\sum xp(x)$$
$$= bE\widetilde{X}$$

Properties of Expected value

- $E(\text{constant}) = \text{constant}$
- **Expected value** (*mean*) of $\widetilde{X}$:

$$E\widetilde{X} = \mu_X = \sum_{\text{all } x} xp(x)$$

- $E(a + b\widetilde{X}) = a + bE\widetilde{X}$
- $E(b\widetilde{X} + c\widetilde{Y}) = bE(\widetilde{X}) + cE(\widetilde{Y})$

Example 4

- **Example 4:** Cake sales in two locations:

$\tilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1

$\tilde{s}_2$	0	20	50	100
$p(s_2)$	0.1	0.2	0.5	0.2

Expected sales: $E\tilde{s}_1 = 13$, $E\tilde{s}_2 = 0 + 4 + 25 + 20$

$0 + 6 + 4 + 3$

$= 49$

Example 4

- **Example 4:** Cake sales in two locations:

$\tilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1

$\tilde{s}_2$	0	20	50	100
$p(s_2)$	0.1	0.2	0.5	0.2

Expected sales: $E\tilde{s}_1 = 13$, $E\tilde{s}_2 = 49$

- Fixed costs = \$25 for location 1, \$100 for location 2.

Profit per ice-cream: Location 1: \$1; Location 2: \$1.50.

Exp. profits $= E(1\tilde{s}_1 - 25 + 1{\cdot}50\tilde{s}_2 - 100)$

$= E(\tilde{s}_1 + 1{\cdot}5\,\tilde{s}_2 - 125)$

$= E\tilde{s}_1 + 1{\cdot}5\,E\tilde{s}_2 - 125 = 13 + 1{\cdot}5 \times 49 - 125$

$= -38{\cdot}50$

Example 4

- **Example 4:** Cake sales in two locations:

$\tilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1

$\tilde{s}_2$	0	20	50	100
$p(s_2)$	0.1	0.2	0.5	0.2

  Expected sales: $E\tilde{s}_1 =$, $E\tilde{s}_2 =$

- Fixed costs = \$25 for location 1, \$100 for location 2.

  Profit per ice-cream: Location 1: \$1; Location 2: \$1.50.

- Expected profits = −\$38.50

Clicker question 4

- A random variable $\widetilde{Y}$ has expected value of \$20.

  Q: What is $E(40 + 5\widetilde{Y})$?

  (a) 20 **(b)** 40

  (c) 100 **(d)** 140

Solution to Clicker question 4

- A random variable $\widetilde{Y}$ has expected value of \$20.

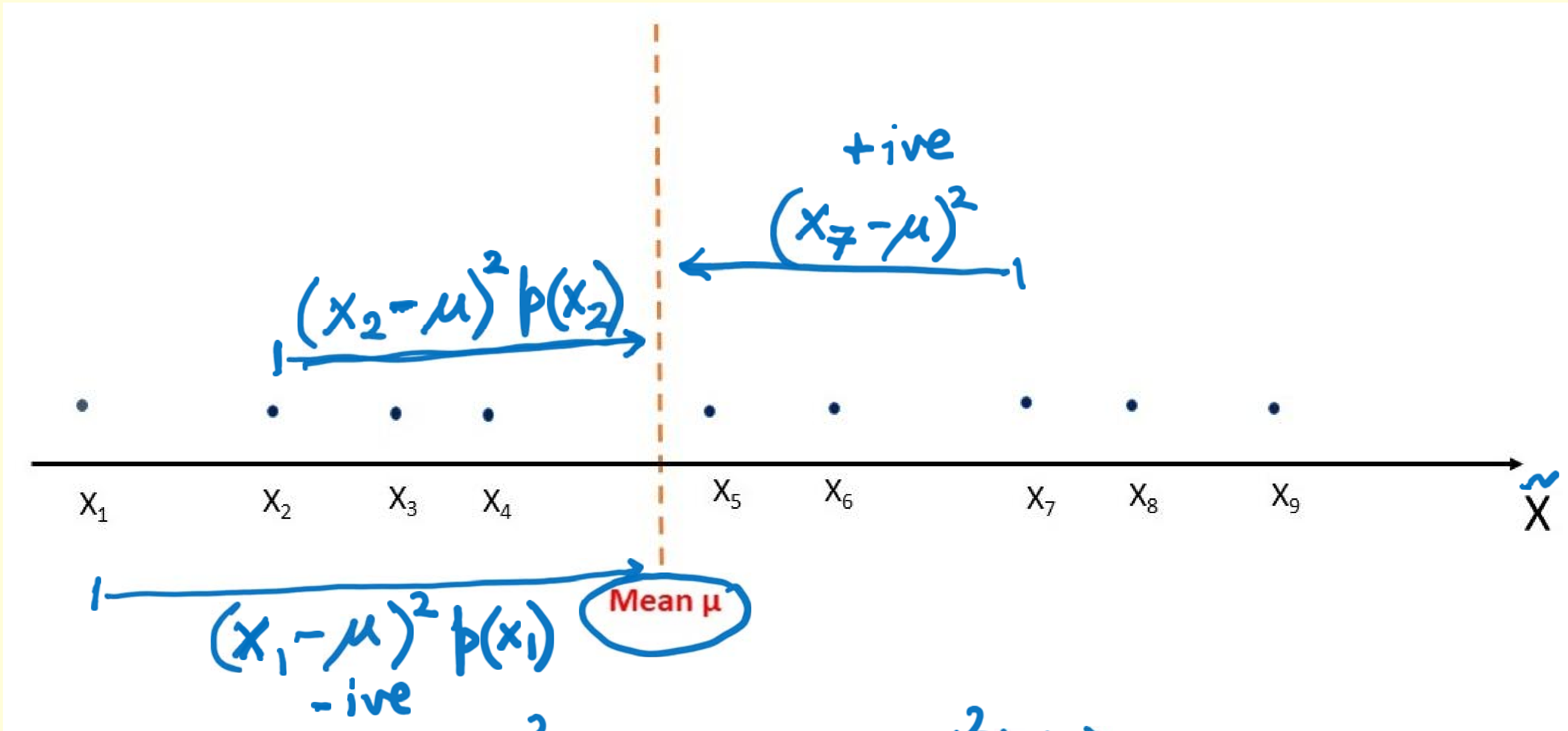
  Q: What is $E(40 + 5\widetilde{Y})$? $= 40 + 5E\widetilde{Y}$

  $= 40 + 5 \times 20 = 140$

(a) 20 **(b)** 40

(c) 100 **(d)** 140

Variance: Measure of variability



$$+ (x_2-\mu)^2 p(x_2) + (x_3-\mu)^2 p(x_3) + \cdots$$
$$= Var(\tilde{X})$$

Variance: Measure of variability

- $$Var(\tilde{X}) = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2] =$

sigma

Variance: Measure of variability

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2]$

Variance: Measure of variability

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2]$
- $\sigma_X^2 = \sum_{\text{all } x} (x - \mu_X)^2 p(x)$

Variance: Measure of variability

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2]$
- $\sigma_X^2 = \sum_{\text{all } x} (x - \mu_X)^2 p(x)$
- $\underbrace{\textbf{Standard deviation of}}_{\text{s.d.}}$ $\widetilde{X} = \sqrt{Variance} = \sqrt{\underset{\sim}{\sigma_X^2}} = \underset{\sim}{\sigma_X}$

Variance

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X-\mu_X)^2] = \sum_{\text{all } x} (x-\mu_X)^2 p(x)$

- Cake sales in location 1:

$\widetilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1
$(\widetilde{s}_1 - \mu_S)^2$				

$E\widetilde{s} = \mu_S = 0 + 6 + 4 + 3 = 13$

Variance

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2] = \sum_{\text{all } x} (x - \mu_X)^2 p(x)$

- Cake sales in location 1:

$\widetilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1
$(\widetilde{s}_1 - \mu_S)^2$	13^2	3^2	7^2	17^2

$E\widetilde{s} = \mu_S = 13$

- Variance of $\widetilde{s} = \sigma_s^2 = 13^2 \times 0.1 + 3^2 \times 0.6 + 7^2 \times 0.2 + 17^2 \times 0.1 = 61$

Variance

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2] = \sum_{\text{all } x} (x - \mu_X)^2 p(x)$

- Cake sales in location 1:

$\widetilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1
$(\widetilde{s}_1 - \mu_S)^2$				

$E\widetilde{s} = \mu_S =$

- Variance of $\widetilde{s} = \sigma_s^2 =$ 61

- Standard deviation of $\widetilde{s} = \sigma_s = \sqrt{Variance} =$ $\sqrt{61} = 7.8$

Variance: an alternate (easier) formula

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2] =$

$$= E[X^2 + \mu_X^2 - 2X\mu_X]$$

$$= E[X^2] + \mu_X^2 - E[2X\mu_X]$$

$$= E[X^2] + \mu_X^2 - 2\mu_X E[X]$$

$$E[X] \rightarrow \mu_X$$

$$= E[X^2] + \mu_X^2 - 2\mu_X^2$$

$$Var(\widetilde{X}) = E[X^2] - \mu_X^2 \leftarrow$$

Variance: an alternate (easier) formula

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2] =$
- $E(X - \mu_X)^2 = E(X^2) - (\overset{EX}{\mu_X})^2 = E(X^2) - (EX)^2$

$$\boxed{\sigma_X^2 = E(X^2) - (EX)^2}$$

Variance: Example 4

- **Variance of** $\widetilde{X} = \sigma_X^2 = E(X^2) - (EX)^2$

Variance: Example 4

- **Variance of** $\widetilde{X} = \sigma_X^2 = \boxed{E(X^2) - (EX)^2}$

- Cake sales in location 1:

$\widetilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1

$E\widetilde{s} = \mu_S = 13$, $E\widetilde{s}^2 = 0 \times 0.1 + 100 \times 0.6 + 400 \times 0.2 + 900 \times 0.1 = 230$

Variance of $\widetilde{s} = \sigma_s^2 = Es^2 - (Es)^2$

$= 230 - (13)^2$

$= 230 - 169 = 61$

Common Discrete Distributions 1: Bernoulli

- A single trial is conducted:
  outcome: *Success* $(\widetilde{Z} = 1)$ or *Failure* $(\widetilde{Z} = 0)$

Common Discrete Distributions 1: Bernoulli

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  outcome: *Success* $(\widetilde{Z} = 1)$ or *Failure* $(\widetilde{Z} = 0)$

$\widetilde{Z}$	1	0
$p(z)$	p	$1 - p$

Common Discrete Distributions 1: Bernoulli

- A single trial is conducted:

  outcome: *Success* $(\widetilde{Z} = 1)$ or *Failure* $(\widetilde{Z} = 0)$

$\widetilde{Z}$	1	0
$p(z)$	p	$1-p$

- Mean $= p$ ✓

  Variance $= p(1-p)$ ✓

$$E\widetilde{Z} = p$$

$$Var = E\widetilde{Z}^2 - (E\widetilde{Z})^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$s.d. = \sqrt{p(1-p)}$$

z^2	1	0
	p	$(1-p)$

Common Discrete Distributions 1: Bernoulli

- A single trial is conducted:

  outcome: *Success* $(\widetilde{Z} = 1)$ or *Failure* $(\widetilde{Z} = 0)$

-

$\widetilde{Z}$	1	0
$p(z)$	p	$1 - p$

- Mean $= p$

  Variance $= p(1 - p)$

- $\widetilde{Z}$ is an **indicator variable**. Examples?

$\widetilde{Z} = 1 \rightarrow$ like

$= 0 \rightarrow$ don't like

Bernoulli Distribution: Examples

- **Bernoulli distribution**: any 2 outcome event

Bernoulli Distribution: Examples

- **Bernoulli distribution**: any 2 outcome event
- e.g. *pick a random person*:

  is he/she wearing an orange shirt?

  is university-educated?

$\tilde{Z} = 1$ orange

$= 0$ anything else

Bernoulli Distribution: Examples

- **Bernoulli distribution**: any 2 outcome event
- e.g. *pick a random person*:

  is he/she wearing an orange shirt?

  is university-eduacted?
- e.g. *pick a random tree in a forest*:

  is it taller than 20m?

  is it diseased?

$$\tilde{z} = 1 \quad \text{if } h \geq 20$$
$$= 0 \quad \text{o/wise}$$

Common Discrete Distributions 2: Binomial distribution

- Suppose n independent and identical trials are conducted:

  Each trial can either be *Success* (with prob. p) or *Failure*

Common Discrete Distributions 2: Binomial distribution

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- $\widetilde{X}$ = total number of successes (out of n)

Common Discrete Distributions 2: Binomial distribution

- Suppose n independent and identical trials are conducted: Each trial can either be *Success* (with prob. p) or *Failure*
- $\widetilde{X}$ = total number of successes (out of n)
- $\widetilde{X}$ follows $Binomial(n, p)$ distribution

Common Discrete Distributions 2: Binomial distribution

- Suppose n independent and identical trials are conducted:

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- $\widetilde{X}$ = total number of successes (out of n)
- $\widetilde{X}$ follows $Binomial(n, p)$ distribution
- Examples:

Binomial distribution

- $\widetilde{X}$ follows *Binomial*(n, p) distribution: Probabilities?

$$\rightarrow \tilde{X} = \{0, 1, 2, \ldots\ldots, n\} \equiv \text{sample space}$$

$$P(\tilde{X} = x)$$

Binomial distribution

- $\widetilde{X}$ follows $Binomial(n, p)$ distribution: Probabilities?
- For $x \in \{0, 1, 2, ...n\}$,

$$Prob(\widetilde{X} = x) = \boxed{\binom{n}{x} p^x (1-p)^{n-x}}$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

and $x! = x(x-1)(x-2).......1.$

$$\frac{n!}{x!(n-x)!}$$

Binomial distribution: Example

- **Example 5**: Pick 5 people at random.

  Prob. of anyone being university-educated = 0.3

  Q: Prob. that exactly 3 in this sample are univ-educated?

$$P(\tilde{X} = 3)$$



Binomial distribution: Example

- **Example 5**: Pick 5 people at random.

  Prob. of anyone being university-educated $= 0.3$

  Q: Prob. that exactly 3 in this sample are univ-educated?

- Binomial: $n = 5,\ p = 0.3$

Binomial distribution: Example

- **Example 5**: Pick 5 people at random.

  Prob. of anyone being university-educated $= 0.3$

  Q: Prob. that exactly 3 in this sample are univ-educated?

- Binomial: $n = 5$, $p = 0.3$ $\quad \binom{n}{x} p^x (1-p)^{n-x}$

- *Prob*$(\widetilde{X} = 3) = \binom{5}{3}(0.3)^3(1-0.3)^{5-3} = 10 \times 0.027 \times 0.49$

  $= 0.132$

  $\binom{5}{3} = \frac{5!}{3!2!} = \dfrac{5.4.\cancel{3.2.1}}{\cancel{3.2.1}.2.1} = \dfrac{20}{2} = 10$

Binomial distribution

- <u>Set-up</u>: n *independent* trials, each with 2 possible outcomes

  identical trials: same prob. of success $= p$

$\widetilde{X}$ = total number of successes (out of n)

Binomial distribution

- Set-up: n *independent* trials, each with 2 possible outcomes

  identical trials: same prob. of success $= p$

  $\widetilde{X}$ = total number of successes (out of n)

- $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$

Binomial distribution

- Set-up: n *independent* trials, each with 2 possible outcomes

  identical trials: same prob. of success $= p$

  $\widetilde{X}$ = total number of successes (out of n)

- $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$

- **Mean** $= np$

  Variance $= np(1-p)$

  s.d. $= \sqrt{np(1-p)}$

Bernoulli : $\mu = p$

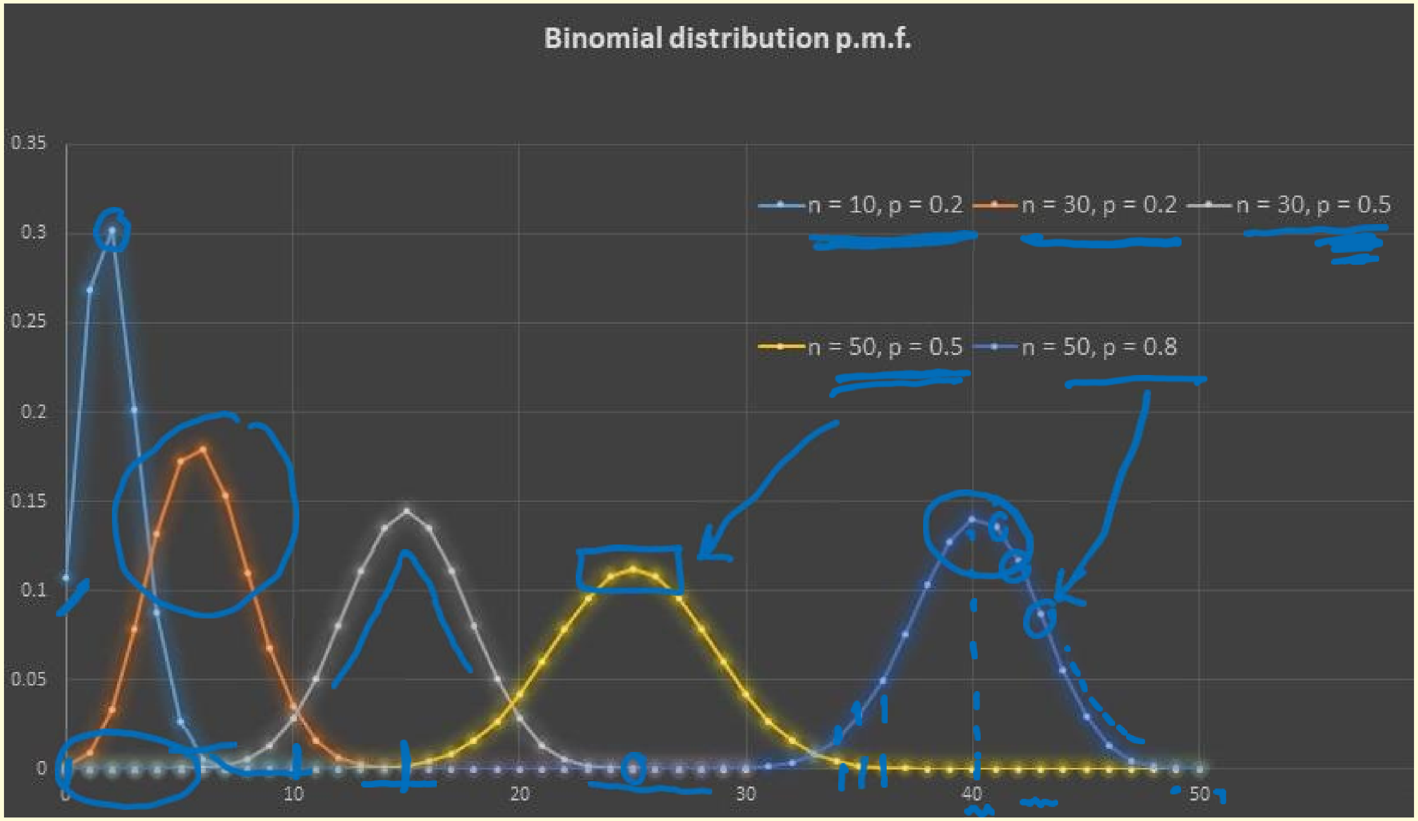
Var $= p(1-p)$

Binomial distribution: Shape of p.m.f.

- **p.m.f.**: $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$. How do they look?

Binomial distribution: Shape of p.m.f.

- **p.m.f.**: $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$. How do they look?
- Depends on n and p : **parameters**

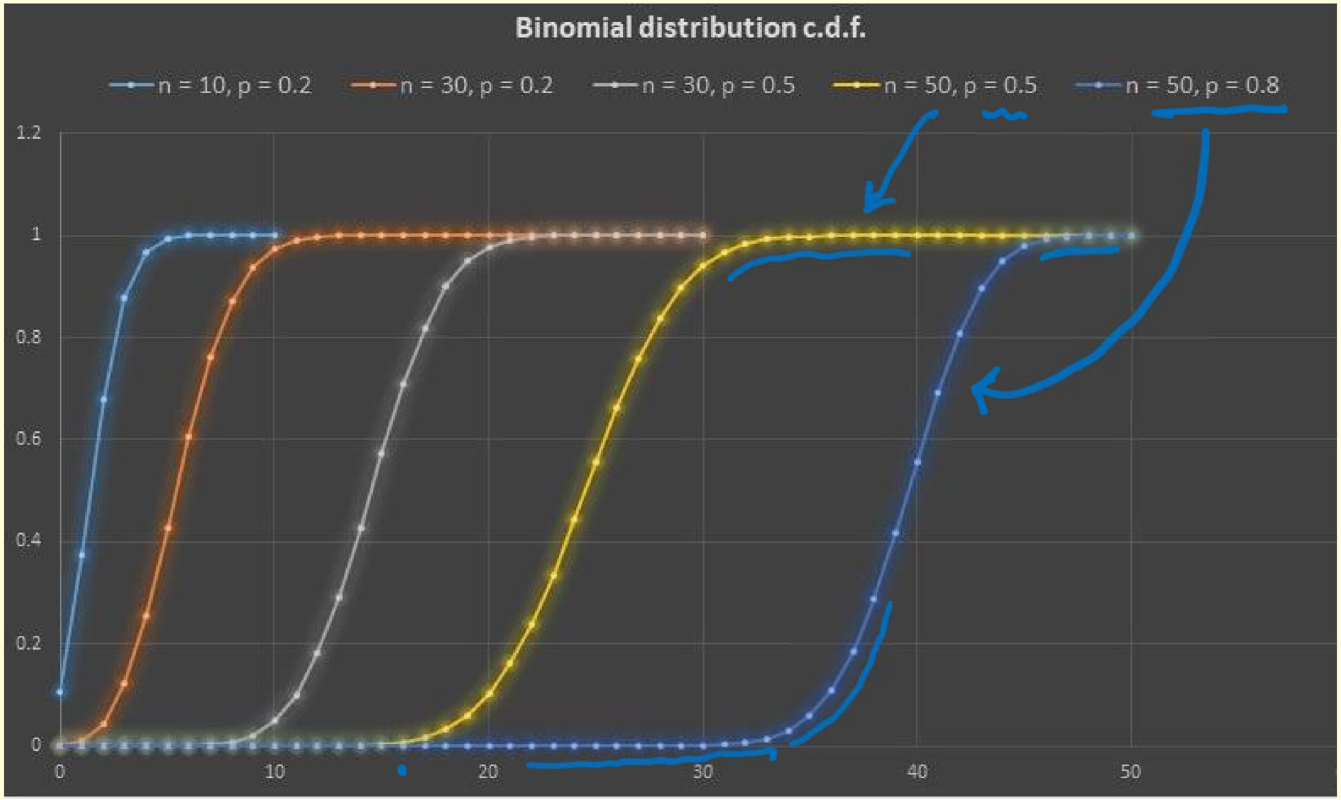


Binomial distribution: Shape of c.d.f.

- **c.d.f.**: $Prob(\widetilde{X} \le x) = \underbrace{\sum_{y:\, y \le x}}_{0,1,2,\cdots,x} p(y) = \sum_{y:\, y \le x} \binom{n}{y} p^y (1-p)^{n-y}$.

Binomial distribution: Shape of c.d.f.

- **c.d.f.**: $Prob(\widetilde{X} \leq x) = \sum_{y:\, y \leq x} p(y) = \sum_{y:\, y \leq x} \binom{n}{y} p^y (1-p)^{n-y}$.

- 

Binomial distribution: Example

- **Example 6**: Voter turnout rate in 2021 fed. election = 60%. $p = 0.6$

  Pick 8 people at random. $n = 8$

  Q: Prob. that exactly 4 in this sample voted?

  $P(\tilde{X} = 4) = ?$

Binomial distribution: Example

- **Example 6**: Voter turnout rate in 2021 fed. election $= 60\%$.

  Pick 8 people at random.

  Q: Prob. that exactly 4 in this sample voted?

- Binomial: $n = 8$, $p = 0.6$

Binomial distribution: Example

- **Example 6**: Voter turnout rate in 2021 fed. election = 60%.

  Pick 8 people at random.

  Q: Prob. that exactly 4 in this sample voted?

$(0.6)^4 \leftarrow 4$ voted

$(1-0.6)^4 \leftarrow 4$ didn't

- Binomial: $n = 8$, $p = 0.6$
- $Prob(\widetilde{X} = 4) = \binom{8}{4}(0.6)^4(1-0.6)^{8-4} =$ 0.23

  ? p

$\binom{8}{4} = \frac{8!}{4!4!} =$

$$= \frac{\overset{2}{\cancel{8}} \times 7 \times \cancel{6} \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{4 \times 3 \times 2 \times 1}} = 70$$

1111 0000

1011 1000

1100 1100

Binomial distribution: Example

- **Example 6**: Voter turnout rate in 2021 fed. election $= 60\%$.

  Pick 8 people at random.

  Q: Prob. that exactly 4 in this sample voted?

- Binomial: $n = 8$, $p = 0.6$

- $Prob(\widetilde{X} = 4) = \binom{8}{4}(0.6)^4(1-0.6)^{8-4} =$

  $\binom{8}{4} = \frac{8!}{4!4!} =$

- Prob. that 4 or fewer voted $=$

  $P(\tilde{X}=0) + P(\tilde{X}=1) + \cdots + P(\tilde{X}=4)$

  $= F(4)$

Binomial distribution: Example

- **Example 6**: Voter turnout rate in 2021 fed. election = 60%.

  Pick 8 people at random.

  Q: Prob. that exactly 4 in this sample voted?

- Binomial: $n = 8$, $p = 0.6$

- $Prob(\widetilde{X} = 4) = \binom{8}{4}(0.6)^4(1-0.6)^{8-4} =$

  $\binom{8}{4} = \frac{8!}{4!4!} =$

- Prob. that 4 or fewer voted =

- Mean number who voted $= np =$ $8 \times 0.6 = 4.8$

  $Variance = np(1-p) =$ $8 \times 0.6 \times 0.4 = 1.92$

Clicker question 5

- The probability that any given e-mail is spam is 0.25. Suppose you receive 80 e-mails in a day.

  Q: What is the mean number of spam e-mails that you will receive in a day?

  (a) 20 **(b)** 25

  (c) 40 **(d)** 80

Soluton to Clicker question 5

- The probability that any given e-mail is spam is 0.25. Suppose you receive 80 e-mails in a day. $n = 80$ $p = 0.25$

  Q: What is the mean number of spam e-mails that you will receive in a day?

  $np = 80 \times 0.25$
  $= 20$

  (a) 20 **(b)** 25

  (c) 40 **(d)** 80

Common Discrete Distributions 3: Poisson distribution

- $\widetilde{X}$ = number of times an event occurs within a given time interval/area

$$\tilde{X} = 0, 1, 2, 3, 4, \ldots\ldots\ldots \infty$$

$$P(\tilde{X} = x) = ?$$

Common Discrete Distributions 3: Poisson distribution

- $\widetilde{X} =$ number of times an event occurs within a given time interval/area
- Possible values for $\widetilde{X} = 0, 1, 2, 3, 4, \ldots\ldots\ldots$

Common Discrete Distributions 3: Poisson distribution

- $\widetilde{X}$ = number of times an event occurs within a given time interval/area
- Possible values for $\widetilde{X} = 0, 1, 2, 3, 4, \ldots\ldots\ldots$
- For $x \in \{0, 1, 2, 3, \ldots., \infty\}$,

$$\boxed{Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}}$$

p.m.f. of Poisson dist.

$x! \to x.(x-1)\ldots\ldots.1$

where $e \approx \underline{2.71828}$

Common Discrete Distributions 3: Poisson distribution

- $\widetilde{X}$ = number of times an event occurs within a given time interval/area
- Possible values for $\widetilde{X} = 0, 1, 2, 3, 4, \ldots\ldots\ldots$
- For $x \in \{0, 1, 2, 3, \ldots., \infty\}$,

$$Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where $e \approx 2.71828$

- Only one *parameter*: λ = mean number of times the event occurs within the given time interval/area

Poisson distribution: Example

- **Example 7**: On average, Pam receives 5 e-mails in an hour.

  Q: Prob. that she will receive 6 e-mails in the next hour?

$P(\underset{\sim}{X} = 6)$

$\lambda = 5$

Poisson distribution: Example

- **Example 7**: On average, Pam receives 5 e-mails in an hour.

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- **Example 7**: On average, Pam receives 5 e-mails in an hour.

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- Poisson: $Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $\longrightarrow$ $e^{-5}5^6/6!$

- Here: $\lambda = 5$; $Prob(\widetilde{X} = 6) = \frac{e^{-6}5^6}{6!} =$ $0 \cdot 146$

Poisson distribution: Example

- **Example 7**: On average, Pam receives 5 e-mails in an hour.

  Q: Prob. that she will receive 6 e-mails in the next hour?

- Poisson: $Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $\leftarrow x = 0$

- Here: $\lambda = 5$; $Prob(\widetilde{X} = 6) = \frac{e^{-6}5^6}{6!} =$

- Prob. that she gets no e-mail: $Prob(\underline{\widetilde{X} = 0}) = \frac{e^{-5}5^0}{0!} = 0.0067$

$0! = 1$

$6! = 6 \times 5 \times 2 \times \cdots \times 1$

Poisson distribution: Example

- **Example 7**: On average, Pam receives 5 e-mails in an hour.

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- Prob. that she gets no e-mail: $Prob(\widetilde{X} = 0) = \frac{e^{-5}5^0}{0!} =$

- Prob. she gets 6 e-mails or less: F(6)

  $Prob(\widetilde{X} = 0) + Prob(\widetilde{X} = 1) + ... + Prob(\widetilde{X} = 6)$ = 0·762

Poisson distribution: Examples

- On average, 10 crimes in a city per day $\lambda = 10$

  Q: Prob. that there will be 8 crimes tomorrow? $Pr(\tilde{X} = 8)$?

Poisson distribution: Examples

memory-less

- On average, 10 crimes in a city per day

  Q: Prob. that there will be 8 crimes tomorrow?

- Number of website visitors in an hour, number of births in a day, bankruptcies in a month,..........

Poisson distribution: Examples

- On average, 10 crimes in a city per day

  Q: Prob. that there will be 8 crimes tomorrow?

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- $\boxed{\begin{array}{l}\textbf{Mean} = \lambda \\ \textbf{Variance} = \lambda\end{array}}$

Poisson distribution: Shape of p.m.f.

- $Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}$. How do they look? Depends on λ

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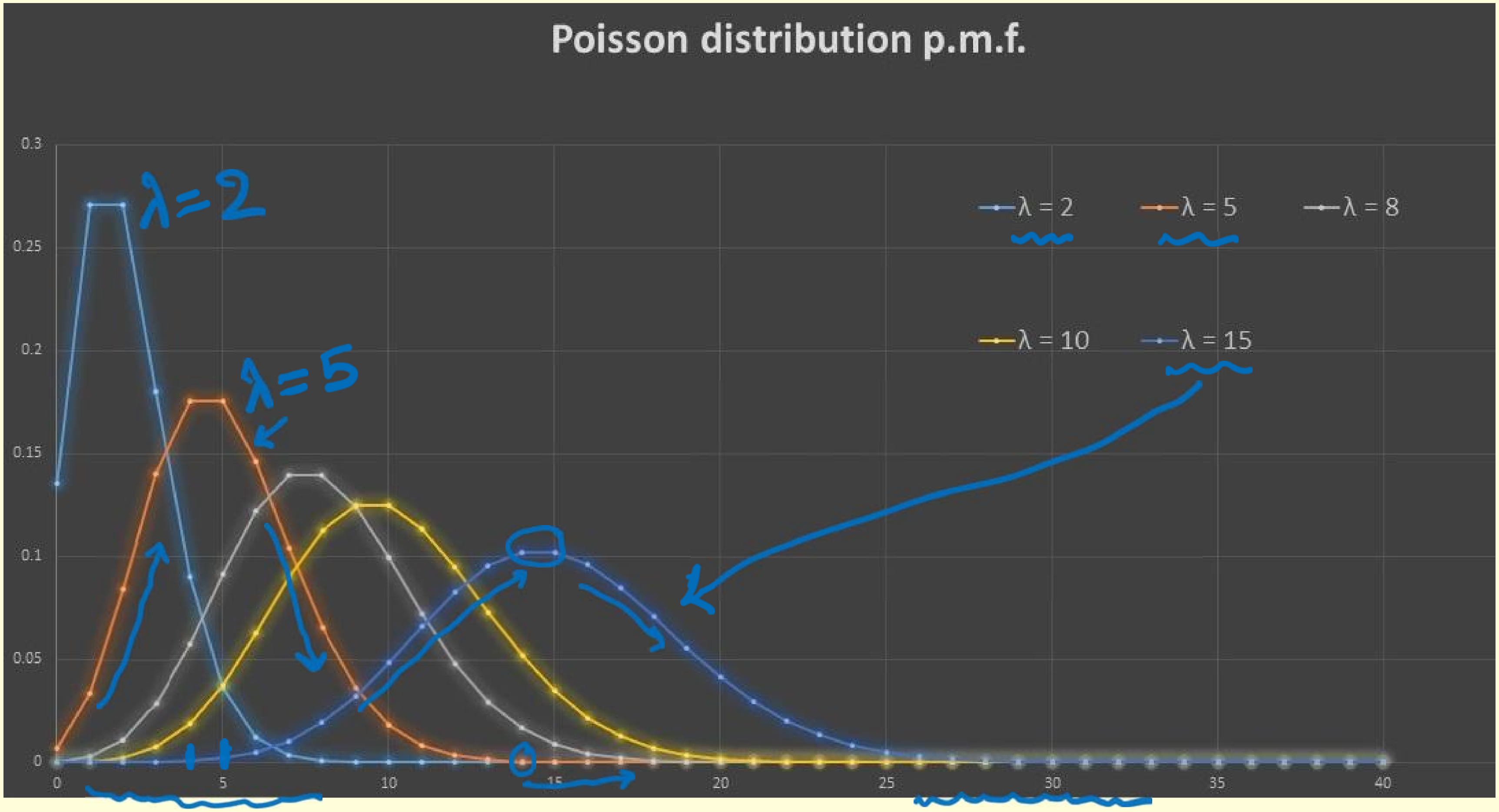
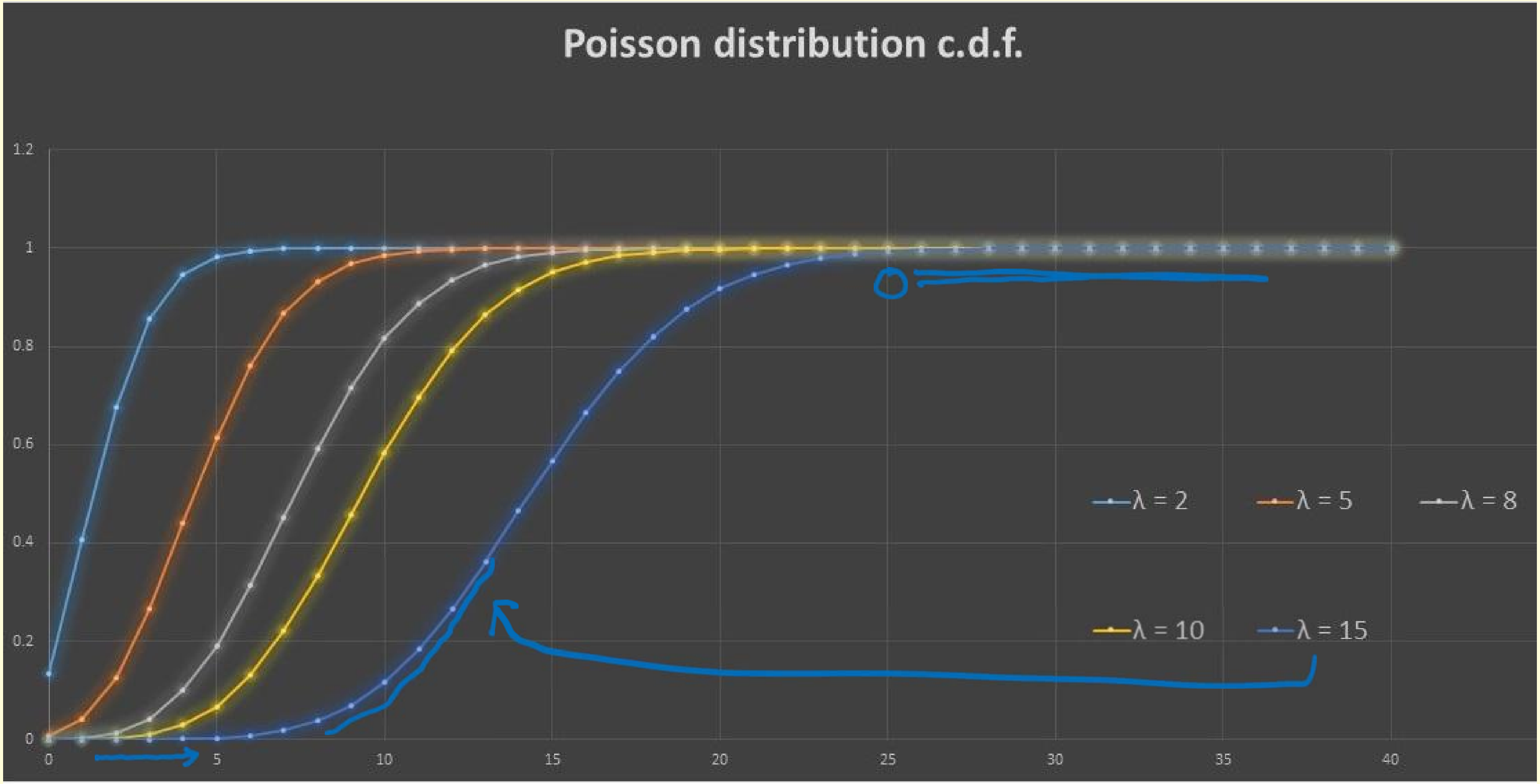
- 

Poisson distribution: Shape of c.d.f.



Excel: Computing Expectation and Variance

- **Column A**: possible x **Column B**: corresponding prob. $p(x)$
- **Column C**: $xp(x)$ $E\widetilde{X} = \sum_{\text{all } x} xp(x) = \text{SUM(C2: C11)}$
- Variance: $\sigma_X^2 = E[(X - \mu_X)^2] = E(X^2) - (EX)^2$
- **Column D**: $x^2p(x)$ $E(X^2) = \sum_{\text{all } x} x^2p(x) = \text{SUM(D2: D11)}$
- Variance $= D12 - (C12)$^2 : **Cell** $E12$

Excel: Binomial and Poisson Distributions

- *Binomial*: $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$

  Poisson: $Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}$
- **Column C**: possible $x \in \{0, 1, 2, 3, \ldots\ldots..)$ generate using $C4 = C3 + 1$ etc.
- <u>*Binomial pmf*</u>: **Column D**: $=$ BINOM.DIST(x, n, p, *FALSE*)
- <u>*Binomial cdf*</u>: **Column H**: $=$ BINOM.DIST(x, n, p, *TRUE*)

  In *Google Sheets*: BINOMDIST
- <u>*Poisson pmf*</u>: **Column D**: $=$ POISSON.DIST(x, n, p, *FALSE*)
- <u>*Poisson cdf*</u>: **Column H**: $=$ POISSON.DIST(x, n, p, *TRUE*)

  In *Google Sheets*: POISSON.DIST

Module recap

- Random variables: Discrete
- Probability mass function (p.m.f.) $p(x)$ $= P(\tilde{X} = x)$

  Cumulative distribution function (c.d.f.):

  $$Prob(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$$

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Module recap

- Random variables: Discrete
- Probability mass function (p.m.f.) $p(x)$

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  $$Prob(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$$
- Expectation, Variance (<u>next module</u>: Correlation)
- Common discrete distributions:

  Bernoulli (single trial)

  Binomial (number of successes in n trials)

  Poisson (number of occurrences in a time period/area)