IMAGINARY WHITTAKER MODULES FOR EXTENDED AFFINE LIE ALGEBRAS

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Abstract

We classify irreducible Whittaker modules for generalized Heisenberg Lie algebra $\mathfrak{t}$ and irreducible Whittaker modules for Lie algebra $\tilde{\mathfrak{t}}$ obtained by adjoining m degree derivations $d_1, d_2, \ldots, d_m$ to $\mathfrak{t}$. Using these results, we construct imaginary Whittaker modules for non-twisted extended affine Lie algebras and prove that the imaginary Whittaker modules of $\mathbb{Z}$-independent level are always irreducible.

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Introduction

In Block's classification [Bl] of all irreducible modules for the three-dimensional simple Lie algebra $\mathfrak{sl}_2$, they fall into two families: highest (lowest) weight modules and a family which are irreducible modules over a Borel subalgebra of $\mathfrak{sl}_2$ including Whittaker modules. This result illustrates the prominent role played by Whittaker modules.

The class of Whittaker modules for an arbitrary finite-dimensional complex semisimple Lie Algebra $\mathfrak{g}$ was defined by Kostant. Kostant defined and systematically studied in [Ko] Whittaker modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$. He showed that these modules with a fixed regular Whittaker function (Lie homomorphism) on a nilpotent radical are (up to isomorphism) in bijective correspondence with central characters of $U(\mathfrak{g})$. Specifically, irreducible Whittaker modules correspond to the maximal ideals of the center $Z(\mathfrak{g})$. In [Wa], N.Wallach gave new proofs of Kostant's results in the case that $\mathfrak{g}$ is the product of complex Lie algebras isomorphic to $\mathfrak{sl}_n$. E.McDowell [Mc], and D.Milicic and

W.Soegbel [MS] studied a category of modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ which includes the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ as well as those Whittaker modules where the Whittaker function on a nilpotent radical may be irregular (degenerate). The irreducible objects in this category are constructed by inducing over a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ from an irreducible Whittaker module or from a highest weight module for the reductive Levi factor of $\mathfrak{p}$ (when the Whittaker function is zero).

Naturally, the next important task is to study Whittaker modules over infinite-dimensional Lie algebras. Affine Lie algebras are the most extensively studied and most useful ones among infinite-dimensional Kac-Moody algebras. The integrable highest weight modules were the first class of representations over affine Kac-Moody algebras being extensively studied, see [Ka] for detailed discussion of results. In [Ch], Chari classified all irreducible integrable weight modules with finite-dimensional weight spaces over the untwisted affine Lie algebras. Chari and Pressley [CP1], then extended this classification to all affine Lie algebras. The results of [Ch] and [CP1] state that every irreducible integrable weight module with finite-dimensional weight spaces is either a highest weight module or a loop module. Very recently, a complete classification for all irreducible weight modules with finite-dimensional weight spaces over affine Lie algebras were obtained in [FT, DG]. As for irreducible weight modules with infinite-dimensional weight spaces and irreducible non-weight modules, the first examples were given by Chari and Pressley in [CP2] by taking the tensor product of some irreducible integrable highest weight modules and integrable loop modules over affine Lie algebras. Besides the irreducible modules constructed in [CP2], a class of irreducible weight modules over affine Lie algebras with infinite-dimensional weight spaces were constructed in [BBFK]. A complete classification for all irreducible (weight and non-weight) modules over affine Lie algebras with locally nilpotent action of the nilpotent radical were obtained in [MZ]. All irreducible modules over untwisted affine Lie algebras with locally finite action of the nilpotent radical were classified in [GZ].

A class of irreducible non-weight modules for untwisted affine Lie algebras from irreducible Whittaker modules over the subalgebra generated by imaginary root spaces were constructed in [Chr]. These modules are called imaginary Whittaker modules since they are different from the above Whittaker modules in nature.

Extended affine Lie algebras, first introduced by mathematical physicists [H-KT], are a higher-dimensional generalization of affine Kac-Moody Lie algebras. Roughly speaking, extended affine Lie algebras are complex Lie algebras characterized by a symmetric non-degenerate invariant bilinear form, a finite-dimensional ad-diagonalizable abelian subalgebra (i.e, a Cartan subalgebra), a discrete irreducible root system and ad-nilpotency of the root spaces attached to non-isotropic roots. It turns out the root systems of such Lie algebras are precisely the extended affine root systems introduced by Saito [Sa] in the study of elliptic singularities. Those Lie algebras and root systems have been further studied in [AABGP], [BGK] and [ABGP], and among others. Our purpose in this thesis to investigate the properties of imaginary Whittaker modules over non-twisted extended affine Lie algebras.

The organization of the thesis is as follows: Some basic definitions and notations are given in Chapter 1; in Chapter 2, we classify the irreducible Whittaker modules for generalized Heisenberg Lie algebras $\mathfrak{t}$; in Chapter 3, we classify the irreducible Whittaker modules for Lie algebras $\tilde{\mathfrak{t}}$ obtained by adjoining m degree derivations $d_1, d_2, \ldots, d_m$ to $\mathfrak{t}$; while in Chapter 4, we use our results from Chapter 3 to construct imaginary Whittaker modules for non-twisted extended affine Lie algebras and investigate their properties.

1 Preliminaries

A Lie algebra $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ with a product $[\cdot,\cdot]$, called Lie bracket, which is bilinear and satisfies two additional conditions:

1. $[x,x] = 0$ for all x in $\mathfrak{g}$,

2. $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in \mathfrak{g}$. (Jacobi identity)

For any algebra $\mathcal{A}$ we denote its center by $Z(\mathcal{A})$. Let n be a positive integer and let $\mathfrak{t}$ be a Lie algebra over $\mathbb{C}$ with the following properties:

1. $\mathfrak{t}$ has a one-dimensional center, $Z(\mathfrak{t}) = \mathbb{C}c$,

2. $\mathfrak{t}$ is $\mathbb{Z}$-graded, $\mathfrak{t} = \oplus_{i\in\mathbb{Z}}\mathfrak{t}_i$,

3. $dim_{\mathbb{C}}\mathfrak{t}_i = n$ for all $i \in \mathbb{Z}, i \neq 0$, and $\mathfrak{t}_0 = \mathbb{C}c$.

Set $\mathfrak{t}^+ = \oplus_{i>0}\mathfrak{t}_i$, $\mathfrak{t}^- = \oplus_{i<0}\mathfrak{t}_i$. We assume that there is a basis $\{x_{ri}\}_{1\leq r\leq n}$ of $\mathfrak{t}_i$ and a basis $\{y_{ri}\}_{1\leq r\leq n}$ of $\mathfrak{t}_{-i}, i \in \mathbb{Z}_{>0}$ such that

$$[c, x_{ri}] = [c, y_{ri}] = 0, \quad [x_{ri}, x_{sj}] = [y_{ri}, y_{sj}] = 0, \quad [x_{ri}, y_{sj}] = \delta_{rs}\delta_{ij}c$$

for all $1 \leq r, s \leq n, i \in \mathbb{Z}_{>0}$. It follows that $degree\, x_{ri} = degree\, x_{si} = i, degree\, y_{ri} = degree\, y_{si} = -i$ for all $1 \leq r, s \leq n, i \in \mathbb{Z}_{>0}$.

The algebra $\mathfrak{t}$ is an infinite-dimensional Heisenberg Lie algebra [Chr]. We extend the above definition to a generalized Heisenberg Lie algebra $\mathfrak{t}$ with three similar properties as infinite-dimensional Heisenberg Lie algebras:

1. $\mathfrak{t}$ has a m-dimensional center, $Z(\mathfrak{t}) = \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \ldots \oplus \mathbb{C}c_m$,

2. $\mathfrak{t}$ is $\mathbb{Z}^m$-graded, $\mathfrak{t} = \oplus_{\alpha \in \mathbb{Z}^m} \mathfrak{t}_\alpha$,

3. $dim_{\mathbb{C}} \mathfrak{t}_\alpha = n$ for all $\alpha \in \mathbb{Z}^m, \alpha \neq 0$, and $\mathfrak{t}_0 = \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \ldots \oplus \mathbb{C}c_m$,

for some positive integers m and n.

We can order the elements of $\mathbb{Z}^m$ lexicographically, that is, for $\alpha, \beta \in \mathbb{Z}^m, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$, $\alpha < \beta$ if and only if, for some $i = 1, 2, \ldots, m, \alpha_i < \beta_i$, and for all $j > i, \alpha_j = \beta_j$. Set $\mathbb{Z}^m + = \{\alpha \in \mathbb{Z}^m | \alpha < 0\}$, where we denote $0 = (0, 0, \ldots, 0)$. Set $\mathfrak{t}^+ = \oplus_{\alpha \in \mathbb{Z}^m +} \mathfrak{t}_\alpha, \mathfrak{t}^- = \oplus_{\alpha \in \mathbb{Z}^m +} \mathfrak{t}_{-\alpha}$. We assume that there is a basis $\{x_{r\alpha}\}_{1 \leq r \leq n}$ of $\mathfrak{t}_\alpha$ and a basis $\{y_{r\alpha}\}_{1 \leq r \leq n}$ of $\mathfrak{t}_{-\alpha}, \alpha \in \mathbb{Z}^m +$ such that

$$[c_i, x_{r\alpha}] = [c_i, y_{r\alpha}] = 0, \qquad [x_{r\alpha}, x_{s\beta}] = [y_{r\alpha}, y_{s\beta}] = 0,$$

$$[x_{r\alpha}, y_{s\beta}] = \delta_{rs} \delta_{\alpha\beta} (\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m)$$

for all $1 \leq r, s \leq n, \alpha, \beta \in \mathbb{Z}^m +$. It follows that $degree\, x_{r\alpha} = degree\, x_{s\alpha} = \alpha$, and $degree\, y_{r\alpha} = degree\, y_{s\alpha} = -\alpha$ for all $1 \leq r, s \leq n, \alpha \in \mathbb{Z}^m +$.

2 Whittaker modules for Heisenberg Lie algebras $\mathfrak{t}$

2.1 Whittaker modules for $\mathfrak{t}$

In this section we describe the irreducible Whittaker modules for $\mathfrak{t}$. All the results of this section are valid for generalized Heisenberg Lie subalgebras of any extended affine Lie algebras.

Definition 2.1 *Let $\eta : U(\mathfrak{t}^+) \to \mathbb{C}$ be an algebra homomorphism such that $\eta|_{\mathfrak{t}^+} \neq 0$, and let V be a $U(\mathfrak{t})$-module.*

1. A non-zero vector $v \in V$ is called a Whittaker vector of type η if $xv = \eta(x)v$ for all $x \in U(\mathfrak{t}^+)$
2. V is called a Whittaker module for $\mathfrak{t}$ if V contains a cyclic Whittaker vector v (i.e. $v \in V$ is a Whittaker vector and $V = U(\mathfrak{t})v$).

Notation 2.2 *Let V be a Whittaker module of type η for $\mathfrak{t}$ with cyclic Whittaker* *vector v. Let $\eta' : U(\mathfrak{t}^+) \to \mathbb{C}$ be an algebra homomorphism and assume that $x_{r\alpha}v = \eta'(x_{r\alpha})v$ for some $1 \leq r \leq n, \alpha \in \mathbb{Z}^m+$. Then $\eta(x_{r\alpha}) = \eta'(x_{r\alpha})$.*

Next we will construct Whittaker modules for $\mathfrak{t}$. Set $\mathfrak{b} = \mathfrak{t}^+ \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \ldots \oplus \mathbb{C}c_m$. Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ and let $\mathbb{C}_{\eta,\vec{a}} = \mathbb{C}\tilde{v}$ be a one-dimensional vector space viewed as a $\mathfrak{b}$-module by

$$c_i\tilde{v} = a_i\tilde{v}, \quad x\tilde{v} = \eta(x)\tilde{v} \tag{2.1}$$

for all $1 \leq i \leq m$ and $x \in U(\mathfrak{t}^+)$. Set

$$M_{\eta,\vec{a}} = U(\mathfrak{t}) \otimes_{U(\mathbf{b})} \mathbb{C}_{\eta,\vec{a}}, \quad v = 1 \otimes \tilde{v}. \tag{2.2}$$

Define an action of $U(\mathfrak{t})$ on $M_{\eta,\vec{a}}$ by left multiplication (on the first tensor factor). Note that $M_{\eta,\vec{a}} = U(\mathfrak{t})v$ and that $M_{\eta,\vec{a}}$ is a Whittake module for $\mathfrak{t}$.

Since $\mathbb{Z}^m+$ is totally ordered and enumerated as

$$(0, 0, \ldots, 0, 1) < (0, 0, \ldots, 0, 2) < \ldots,$$

we can denote that $k_i = (k_{i\alpha}, k_{i\beta}, \ldots)$, where $\alpha = (0, 0, \ldots, 0, 1), \beta = (0, 0, \ldots, 0, 2)$, for all $i = 1, 2, \ldots, n$. Let $\underline{k} = (k_1, k_2, \ldots, k_n)$ and only finitely many $k_{r\alpha}$ are non zero. Denote I be the set of all such $\underline{k}$. Then we can order the elements of I lexicograpically and denote this total order by $\leq$.

Let $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$ be an algebra homomorphism. For any $\underline{k} \in I$, since there are only finitely many $k_{r\alpha} \neq 0$, we may define:

1. $|\underline{k}| = \sum_{\substack{\alpha\in\mathbb{Z}^m+\\ 1\le r\le n}} k_{r\alpha}$,

2. $y^{\underline{k}} = \prod_{\substack{\alpha\in\mathbb{Z}^m+\\ 1\le r\le n}} y_{r\alpha}^{k_{r\alpha}}$,

3. $\underline{k}! = \prod_{\substack{\alpha\in\mathbb{Z}^m+\\ 1\le r\le n}} k_{r\alpha}!$,

4. $(x-\eta)^{\underline{k}} = \prod_{\substack{\alpha\in\mathbb{Z}^m+\\ 1\le r\le n}} (x_{r\alpha} - \eta(x_{r\alpha}))^{k_{r\alpha}}$,

5. $(y-\xi)^{\underline{k}} = \prod_{\substack{\alpha\in\mathbb{Z}^m+\\ 1\le r\le n}} (y_{r\alpha} - \xi(y_{r\alpha}))^{k_{r\alpha}}$.

Proposition 2.3 *Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ and assume $M_{\eta,\vec{a}}$ and v are as defined in Definition* 2.1. *Then the following hold:*

1. The set $\{y^{\underline{k}}v | \underline{k} \in I\}$ is a basis of $M_{\eta,\vec{a}}$ as a $\mathbb{C}$-vector space.

2. As a $U(\mathfrak{t}^-)$-module, $M_{\eta,\vec{a}}$ is isomorphic to $U(\mathfrak{t}^-)$.

3. $M_{\eta,\vec{a}}$ is free as a $U(\mathfrak{t}^-)$-module.

Proof.

1. Since $U(\mathfrak{t}) \cong U(\mathfrak{t}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$ by Poincaré-Birkoff-Witt theorem in section 17.3 [Hu], $U(\mathfrak{t})$ is a free right $U(\mathfrak{b})$-module with basis $\{y^{\underline{k}} | \underline{k} \in I\}$. Hence $M_{\eta,\vec{a}} = U(\mathfrak{t}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\eta,\vec{a}} \cong (U(\mathfrak{t}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\eta,\vec{a}} \cong U(\mathfrak{t}^-) \otimes_{\mathbb{C}} (U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\eta,\vec{a}}) \cong U(\mathfrak{t}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\eta,\vec{a}}$ is a $\mathbb{C}$-vector space with basis $\{y^{\underline{k}} | \underline{k} \in I\}$.

2. This is obvious from the proof of Proposition 2.3(1).

3. Since $U(\mathfrak{t}^-)$ is a domain, it follows that $M_{\eta,\vec{a}}$ is torsion-free as a $U(\mathfrak{t}^-)$-module. Hence $M_{\eta,\vec{a}}$ is free as a $U(\mathfrak{t}^-)$-module since $M_{\eta,\vec{a}}$ is cyclic as a $U(\mathfrak{t}^-)$-module.

$\square$

Lemma 2.4 *Let $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^m$ and $v \in M_{\eta,\vec{a}}$ be defined as in Definition* 2.1*, we have the following:*

1. if $\vec{a} = (a_1, a_2, \dots, a_n) \neq 0$, then

$$(x-\eta)^{\underline{k}} y^{\underline{k}} v = \{ \prod_{1 \le r \le n, \alpha \in \mathbb{Z}^m+} (\alpha_1 a_1 + \alpha_2 a_2 + \dots, + \alpha_n a_n)^{k_{r\alpha}} \} \underline{k}! v \tag{2.3}$$

   for any $\underline{k} \in I$.

2. if $\vec{a} = (a_1, a_2, \dots, a_n) \neq 0$ and $\underline{k}, \underline{l} \in I$ with $\underline{k} < \underline{l}$, then $(x-\eta)^{\underline{l}} y^{\underline{k}} v = 0$.

3. if $\vec{a} = (a_1, a_2, \dots, a_n) = 0$, then $x_{r\alpha} y^{\underline{k}} v = \eta(x_{r\alpha}) y^{\underline{k}} v$ for all $1 \le r \le n, \alpha \in \mathbb{Z}^m+, \underline{k} \in I$.

Proof.

1. Since $[x_{r\alpha}, x_{s\beta}] = [y_{r\alpha}, y_{s\beta}] = 0$ and $[x_{r\alpha}, y_{s\beta}] = \delta_{rs}\delta_{\alpha\beta}(\alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_m c_m)$, we have the following calculation:

$$\begin{aligned}
(x_{r\alpha} - \eta(x_{r\alpha}))y_{r\alpha} &= y_{r\alpha}(x_{r\alpha} - \eta(x_{r\alpha})) + \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_m c_m, \\
(x_{r\alpha} - \eta(x_{r\alpha}))y_{r\alpha}^2 &= y_{r\alpha}[y_{r\alpha}(x_{r\alpha} - \eta(x_{r\alpha})) + 2(\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_m c_m)], \\
(x_{r\alpha} - \eta(x_{r\alpha}))y_{r\alpha}^3 &= y_{r\alpha}^2[y_{r\alpha}(x_{r\alpha} - \eta(x_{r\alpha})) + 3(\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_m c_m)],
\end{aligned}$$

and by induction we may have

$$(x_{r\alpha}-\eta(x_{r\alpha}))y_{r\alpha}^{k_{r\alpha}} = y_{r\alpha}^{k_{r\alpha}-1}[y_{r\alpha}(x_{r\alpha}-\eta(x_{r\alpha}))+k_{r\alpha}(\alpha_1 c_1+\alpha_2 c_2+\cdots+\alpha_m c_m)].$$

Hence,

$$\begin{aligned}
(x_{r\alpha}-\eta(x_{r\alpha}))y_{r\alpha}^{k_{r\alpha}}v &= y_{r\alpha}^{k_{r\alpha}-1}k_{r\alpha}(\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_m a_m)v \\
(x_{r\alpha}-\eta(x_{r\alpha}))^{k_{r\alpha}}y_{r\alpha}^{k_{r\alpha}}v &= y_{r\alpha}^{k_{r\alpha}-1}k_{r\alpha}(\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_m a_m)v \\
&= k_{r\alpha}(\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_m a_m)(x_{r\alpha}-\eta(x_{r\alpha}))^{k_{r\alpha}-1}y_{r\alpha}^{k_{r\alpha}-1}v \\
&= k_{r\alpha}k_{r\alpha}-1(\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_m a_m)^2 \\
&\quad \cdot(x_{r\alpha}-\eta(x_{r\alpha}))^{k_{r\alpha}-2}y_{r\alpha}^{k_{r\alpha}-2}v \\
&= \ldots \\
&= k_{r\alpha}!(\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_m a_m)^{k_{r\alpha}}v.
\end{aligned}$$

Since $[x_{r\alpha}, x_{s\beta}] = [y_{r\alpha}, y_{s\beta}] = 0$, we have

$$(x-\eta)^{\underline{k}}y^{\underline{k}}v = \underline{k}!\Pi_{1\leq r\leq n,\alpha\in\mathbb{Z}^m+}(\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_m a_m)^{k_{r\alpha}}v$$

for any $\underline{k} \in I$.

2. $\underline{k} < \underline{l} \Rightarrow \exists 1 \le r \le n, \alpha \in \mathbb{Z}^m+$ such that $k_{r\alpha} < l_{r\alpha}$, so

$$\begin{aligned}(x_{r\alpha} - \eta(x_{r\alpha}))^{l_{r\alpha}} y_{r\alpha}^{k_{r\alpha}} v &= k_{r\alpha}!(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m)^{k_{r\alpha}} \\ &\quad *(x_{r\alpha} - \eta(x_{r\alpha}))^{l_{r\alpha} - k_{r\alpha}} v \\ &= 0 \\ \Rightarrow (x-\eta)^{\underline{l}} y^{\underline{k}} v &= 0.\end{aligned}$$

3. If $\vec{a} = (a_1, a_2, \ldots, a_m) = 0 \Rightarrow [x_{r\alpha}, y_{s\beta}] = 0$ for all $1 \le r, s \le n, \alpha, \beta \in \mathbb{Z}^m+ \Rightarrow$ $x_{r\alpha} y^{\underline{k}} v = \eta(x_{r\alpha}) y^{\underline{k}} v$ for all $1 \le r \le n, \alpha \in \mathbb{Z}^m+, \underline{k} \in I$.

□

2.2 Whittaker modules for $\mathfrak{t}$ with $a_1, a_2, \ldots, a_m$ $\mathbb{Z}$-independent

In this section, we classify all irreducible Whittaker modules for $\mathfrak{t}$ with $a_1, a_2, \ldots, a_m$ $\mathbb{Z}$-independent.

Proposition 2.5 *Let $\vec{a} = (a_1, a_2, \ldots, a_m)$ be $\mathbb{Z}$-independent, then $M_{\eta,\vec{a}}$ is irreducible as a $U(\mathfrak{t})$-module.*

Proof. Let N be a nonzero $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{a}}$ and let $0 \neq u \in N$. Then, u has a unique expression

$$u = \sum_{\underline{k}} \lambda_{\underline{k}} y^{\underline{k}} v,$$

where only finitely many $\lambda_{\underline{k}} \neq 0$. Let $\underline{l} = max\{\underline{k} \in I | \lambda_{\underline{k}} \neq 0\}$. If $\underline{l} = \underline{0}$, then $v \in N$ and so $N = M_{\eta,\vec{a}}$.

Assume that $\underline{l} \neq \underline{0}$, then

$$(x-\eta)^{\underline{l}} u = \{\prod_{1 \leq r \leq n, \alpha \in \mathbb{Z}^m+} (\alpha_1 a_1 + \alpha_2 a_2 + \ldots, +\alpha_n a_n)^{l_{r\alpha}}\} \underline{l}! \lambda_{\underline{l}} v \in N.$$

Since $\lambda_{\underline{l}} \neq 0$ and $\prod_{1 \leq r \leq n, \alpha \in \mathbb{Z}^m+} (\alpha_1 a_1 + \alpha_2 a_2 + \ldots, +\alpha_n a_n)^{l_{r\alpha}} \neq 0$, we have that $v \in N$, so $N = M_{\eta,\vec{a}}$ and $M_{\eta,\vec{a}}$ is irreducible as a $U(\mathfrak{t})$-module. $\square$

Proposition 2.6 *Let $\vec{a} = (a_1, a_2, \ldots, a_m)$ be $\mathbb{Z}$-independent, then $M_{\eta,\vec{a}}$ is the unique (up to isomorphism) irreducible Whittaker module of type η on which $c_1, c_2, \ldots, c_m$ acts on the Whittaker vector v by $a_1, a_2, \ldots, a_m$ respectively.*

Proof. Let M' be a Whittaker $\mathfrak{t}$-module of type η with cyclic Whittaker vector v' such that $c_1 v' = a_1 v', c_2 v' = a_2 v', \ldots, c_m v' = a_m v'$, then we only need to show that $M' \cong M_{\eta,\vec{a}}$. Let $\mathbb{C}_{\eta,\vec{a}}$ be defined the same as in Definition 2.1. Then the map

$$f : U(\mathfrak{t}) \otimes \mathbb{C}_{\eta,\mathbf{a}} \to M'$$

defined by

$$(u, rv) \mapsto ruv',$$

where $r \in \mathbb{C}, u \in U(\mathfrak{t})$, is bilinear. Moreover if $w \in U(\mathfrak{b})$, then

$$\begin{aligned} f(uw, rv) &= r(uw)v' \\ &= f(u, w(rv)). \end{aligned}$$

Hence there exists an induced linear map

$$f : M_{\eta,\vec{a}} = U(\mathfrak{t}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\eta,\vec{a}} \to M'$$

defined by

$$u \otimes rv \mapsto ruv',$$

which is a homomorphism of (left) $U(\mathfrak{t})$-modules, and it is obviously surjective as $M' = U(\mathfrak{t})v'$. Since $M_{\eta,\vec{a}}$ is irreducible, f is then one-to-one. Thus, $M' \cong M_{\eta,\vec{a}}$ as desired.

□

Corollary 2.7 *Let $\vec{a} = (a_1, a_2, \ldots, a_m)$ be $\mathbb{Z}$-independent. Let M' be a Whittaker $\mathfrak{t}$-module of type η with cyclic Whittaker vector v' such that $c_i v' = a_i v'$ for all $1 \leq i \leq m$. Then $M' \cong M_{\eta,\vec{a}}$.*

Proposition 2.8 *Let $\vec{a} = (a_1, a_2, \ldots, a_m)$ be $\mathbb{Z}$-independent. Then the space of Whittaker vectors (of type η) for $M_{\eta,\vec{a}}$ is one-dimensional.*

Proof. Let $\eta' : U(\mathfrak{t}) \to \mathbb{C}$ be an algebra homomorphism. Suppose that $w \in M_{\eta,\vec{a}}$ is a Whittaker vector of type η'. We show that $\eta = \eta'$ and that $w \in \mathbb{C}v$. By Proposition 2.3(1), w has a unique expression

$$w = \sum_{\underline{k}} \lambda_{\underline{k}} y^{\underline{k}} v,$$

where only finitely many $\lambda_{\underline{k}} \neq 0$. We may assume that $\lambda_{\underline{k}} \neq 0$ for some $\underline{k} \neq \underline{0}$, otherwise we would have $w \in \mathbb{C}v$ and the proof is done. Let $\underline{0} \neq \underline{l} = max\{\underline{k} | \lambda_{\underline{k}} \neq 0\}$. By Lemma 2.4(1), we have

$$(x-\eta)^{\underline{l}} w = \lambda_{\underline{l}} \underline{l}! \Pi_{1 \leq r \leq n, \alpha \in \mathbb{Z}^m} (\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m)^{l_{r\alpha}} v.$$

Since $\mathfrak{t}^+$ is abelian and w is a Whittaker vector of type η',

$$\begin{aligned} (x_{r\alpha} - \eta'(x_{r\alpha}))(x-\eta)^{\underline{l}} w &= (x-\eta)^{\underline{l}} (x_{r\alpha} - \eta'(x_{r\alpha})) w \\ &= 0 \end{aligned}$$

for all $1 \leq r \leq n, \alpha \in \mathbb{Z}^m +$. Thus

$$\begin{aligned} (x_{r\alpha} - \eta'(x_{r\alpha}))v &= (\lambda_{\underline{l}} \underline{l}! \Pi_{1 \leq r \leq n, \alpha \in \mathbb{Z}^m +} (\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m)^{l_{r\alpha}})^{-1} \\ &\quad * (x_{r\alpha} - \eta'(x_{r\alpha}))(x-\eta)^{\underline{l}} w \\ &= 0 \end{aligned}$$

for all $1 \leq r \leq n, \alpha \in \mathbb{Z}^m +$. Which is to say $\eta'(x_{r\alpha}) = \eta(x_{r\alpha})$ for all $1 \leq r \leq n, \alpha \in \mathbb{Z}^m +$. so we have $\eta = \eta'$. This implies that

$$(x-\eta)^{\underline{l}} w = 0 \Rightarrow \lambda_{\underline{k}} = 0,$$

which is a contradiction to our choice of $\underline{l}$. Therefore, $w = \lambda v$ for some $\lambda \in \mathbb{C}$ as desired.

$\square$

Proposition 2.9 *Let $\vec{a} = (a_1, a_2, \ldots, a_m)$ be $\mathbb{Z}$-independent. Then $M_{\eta,\vec{a}} \cong M_{\eta',\vec{a}'}$ as $U(\mathfrak{t})$-modules if and only if $\eta = \eta'$ and $\vec{a} = \vec{a}'$.*

Proof. We only need to prove that if $M_{\eta,\vec{a}} \cong M_{\eta',\vec{a}'}$, then $\eta = \eta'$ and $\vec{a} = \vec{a}'$, because the other direction is obviuos. Since $M_{\eta,\vec{a}} \cong M_{\eta',\vec{a}'}$, let $f : M_{\eta,\vec{a}} \to M_{\eta',\vec{a}'}$ be an isomorphism of $U(\mathfrak{t})$-modules and choose $v \in M_{\eta,\vec{a}}$ as a Whittaker vector. Then $a_i' f(v) = c_i f(v) = f(c_i v) = f(a_i v) = a_i v$ for $i = 1, 2, \ldots, m$. So, $a_i' = a_i$ for $i = 1, 2, \ldots, m$ and $\vec{a} = \vec{a}'$. Moreover,

$$\begin{aligned}(u - \eta(u))f(v) &= f((u - \eta(u))v) \\ &= f(0) \\ &= 0\end{aligned}$$

for all $u \in U(\mathfrak{t}^+)$, which implies that $f(v)$ is a Whittaker vector of type η in $M_{\eta',\vec{a}'}$. By Proposition 2.8 , it follows that $\eta = \eta'$. $\square$

2.3 Whittaker modules for $\mathfrak{t}$ with $a_1, a_2, \ldots, a_m$ $\mathbb{Z}$-dependent

In this chapter, we assume that $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m \neq 0$ and $a_1, a_2, \ldots, a_m$ are $\mathbb{Z}$-dependent. Let $\Omega = \{\underline{k} \in I|$ there exists at least one entry $k_{r\alpha} \neq 0$ such that $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m = 0\}$. For any $\underline{k} \in I$, denote $[\underline{k}]_{r,\alpha}$ the same as $\underline{k} \in I$ except that, if $k_{r\alpha} \neq 0$ for $\underline{k}$, then the $(r, \alpha)^{th}$ position is $k_{r\alpha} - 1$ instead of $k_{r\alpha}$.

Proposition 2.10 *Let $\vec{a} = (a_1, a_2, \ldots, a_m)$ be $\mathbb{Z}$-dependent. Then $N_\eta = span_{\mathbb{C}}\{y^{\underline{k}}v|\underline{k} \in \Omega\}$ is a maximal submodule of $M_{\eta,\vec{a}}$.*

Proof. First we show that N_η is a proper submodule of $M_{\eta,\vec{a}}$. For any $w \in N_\eta$, w has a unique expression

$$w = \sum_{\underline{k}\in\Omega} \lambda_{\underline{k}} y^{\underline{k}} v,$$

where only finitely many $\lambda_{\underline{k}} \in \Omega$ are not zero.

1. For any $r = 1, 2, \ldots, m, \alpha \in \mathbb{Z}^m+$. If $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$, then

$$x_{r\alpha} w = \sum_{\underline{k}\in\Omega} \lambda_{\underline{k}} \eta(x_{r\alpha}) y^{\underline{k}} v \in N_\eta.$$

If $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m \neq 0$, then we can rewrite w as

$$w = \sum_{\underline{k}\in\Omega, k_{r\alpha}>0} \lambda_{\underline{k}} y^{\underline{k}} v + \sum_{\underline{k}\in\Omega, k_{r\alpha}=0} \lambda_{\underline{k}} y^{\underline{k}} v,$$

and we have

$$\begin{aligned}
x_{r\alpha}w &= \sum_{\underline{k}\in\Omega, k_{r\alpha}>0} \lambda_{\underline{k}} y^{\underline{k}} x_{r\alpha}v + \sum_{\underline{k}\in\Omega, k_{r\alpha}=0} \lambda_{\underline{k}} y^{\underline{k}} x_{r\alpha}v \\
&\quad + \sum_{\underline{k}\in\Omega, k_{r\alpha}>0} \lambda_{\underline{k}} k_{r\alpha} y^{[\underline{k}]_{r,\alpha}} (\alpha_1 c_1 + \ldots + \alpha_m c_m)v \\
&= \sum_{\underline{k}\in\Omega} \lambda_{\underline{k}} \eta(x_{r\alpha}) y^{\underline{k}} v + \sum_{\underline{k}\in\Omega, k_{r\alpha}>0} \lambda_{\underline{k}} k_{r\alpha} y^{[\underline{k}]_{r,\alpha}} (\alpha_1 a_1 + \ldots + \alpha_m a_m)v.
\end{aligned}$$

Since $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m \neq 0$, it must be $[\underline{k}]_{r,\alpha} \in \Omega$ given that $\underline{k} \in \Omega$. Thus $x_{r\alpha}w \in N_\eta$. So, for any $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$, we have $x_{r\alpha}w \in N_\eta$, which shows that N_η is stable under $U(\mathfrak{t}^+)$.

2. For any $\underline{k}' \in I$, $y^{\underline{k}'}w = \sum_{\underline{k}\in\Omega} \lambda_{\underline{k}} y^{\underline{k}'} y^{\underline{k}} v = \sum_{\underline{k}\in\Omega} \lambda_{\underline{k}} y^{\underline{k}+\underline{k}'} v \in N_\eta$.

The above implies that N_η is stable under $U(\mathfrak{t})$ and $N_\eta \neq M_{\eta,\vec{a}}$, so N_η is a proper submodule of $M_{\eta,\vec{a}}$. Consider $V = \{y^{\underline{k}}v | \underline{k} \in I\backslash\Omega\}$. It is easy to see that V is a $\mathbb{C}$-basis of $M_{\eta,\vec{a}}/N_\eta$. Next we will show that $M_{\eta,\vec{a}}/N_\eta$ is irreducible as a $U(\mathfrak{t})$-module. Similar as the proof of Proposition 2.5, let K be a $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{a}}/N_\eta$. Then for any $0 \neq w \in K$, w has a unique expression

$$w = \sum_{\underline{k}\in I\backslash\Omega} \lambda_{\underline{k}} y^{\underline{k}} v,$$

where only finitely many $\underline{k} \in I\backslash\Omega$ are not zero. Let $\underline{l} = max\{\underline{k} \in I\backslash\Omega | \lambda_{\underline{k}} \neq 0\}$. If $\underline{l} = \underline{0}$, then $v \in K$ and so $K = M_{\eta,\vec{a}}/N_\eta$. Assume that $\underline{l} \neq \underline{0}$. Then

$(x-\eta)^{\underline{l}}w = \{\prod_{1\leq r\leq n,\alpha\in\mathbb{Z}^m+}(\alpha_1 a_1+\alpha_2 a_2+\ldots,+\alpha_m a_m)^{l_{r\alpha}}\}\underline{l}!\lambda_{\underline{l}}v \in N$. Since $\lambda_{\underline{l}} \neq 0$ and $\prod_{1\leq r\leq n,\alpha\in\mathbb{Z}^m+}(\alpha_1 a_1+\alpha_2 a_2+\ldots,+\alpha_n a_n)^{l_{r\alpha}} \neq 0$, this implies that $v \in K$, and so $K = M_{\eta,\vec{a}}/N_\eta$ and thus $M_{\eta,\vec{a}}/N_\eta$ is irreducible as a $U(\mathfrak{t})$-module. So, N_η is a maximal submodule of $M_{\eta,\vec{a}}$.

□

For every $1 \leq r \leq n$ and $\alpha \in \mathbb{Z}^m+$, let $\underline{e}_{r,\alpha}$ be the element of Ω which has 1 in the $(r,\alpha)^{th}$ position and zeros elsewhere.

Proposition 2.11 *$N_\eta^{(r,\alpha)} = span_{\mathbb{C}}\{y^{\underline{k}}v|\underline{k}\in\Omega, \underline{k}\neq\underline{e}_{r,\alpha}\}$ is a maximal $U(\mathfrak{t})$-submodule of N_η for every $\underline{e}_{r,\alpha}\in\Omega$.*

Proof. First we show that $N_\eta^{(r,\alpha)}$ is a proper submodule of N_η. For any $w \in N_\eta^{(r,\alpha)}$, w has a unique expression

$$w = \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}} \lambda_{\underline{k}} y^{\underline{k}} v,$$

where $\lambda_{\underline{k}} \neq 0$ for only finitely many $\underline{k} \in \Omega \setminus \underline{e}_{r,\alpha}$.

Obviously, $N_\eta^{(r,\alpha)}$ is stable under $U(\mathfrak{t}^-)$ since for any $\underline{k'} \in I$, we have

$$y^{\underline{k'}}w = \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}} \lambda_{\underline{k}} y^{\underline{k}+\underline{k'}} v \in \tilde{N}_\eta.$$

For any $i = 1, 2, \ldots, m$,

$$c_i w = \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}} \lambda_{\underline{k}} y^{\underline{k}} c_i v$$

$$= \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}} \lambda_{\underline{k}} a_i y^{\underline{k}} v \in N_\eta^{(r,\alpha)}.$$

So, $N_\eta^{(r,\alpha)}$ is stable under $\mathbb{C}c_1 \oplus \mathbb{C}c_2 \ldots \mathbb{C} \oplus c_m$.

Now we claim that $\widetilde{N}_\eta^{(r,\alpha)}$ is also stable under $U(\mathfrak{t}^+)$. By induction we have

$$x_{r\alpha} y_{s\beta}^k = y_{s\beta}^k x_{r\alpha} + k\delta_{r,s}\delta_{\alpha,\beta} y_{s,\beta}^{k-1}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m),$$

where $1 \leq r, s \leq n, \alpha, \beta \in \mathbb{Z}^m +, k \in \mathbb{Z}_{\geq 0}$.

For any $r = 1, 2, \ldots, n$, and $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{Z}+$, if $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m = 0$, then

$$x_{r\alpha} w = \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}} \lambda_{\underline{k}} y^{\underline{k}} x_{r\alpha} v = \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}} \lambda_{\underline{k}} \eta(x_{r\alpha}) y^{\underline{k}} v \in N_\eta^{(r,\alpha)}.$$

If $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m \neq 0$, denote $[\underline{k}]_{r\alpha}$ the same as $\underline{k}$ except that, if $k_{r\alpha} > 0$, the element at $(r, \alpha)^{th}$ position is $k_{r\alpha} - 1$ instead of $k_{r\alpha}$. Then, we can rewrite w as

$$w = \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}, k_{r\alpha}>0} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} v + \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}, k_{r\alpha}=0} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} v.$$

So we have

$$\begin{aligned} x_{r\alpha} w &= \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}, k_{r\alpha}>0} \lambda_{\underline{k},\underline{p}} y^{[\underline{k}]_{r\alpha}} (\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m) v \\ &\quad + \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}, k_{r\alpha}>0} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} v + \sum_{\underline{k}\in\Omega\setminus\underline{e}_{r,\alpha}, k_{r\alpha}=0} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} v. \end{aligned}$$

Since $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m \neq 0$ and $\underline{k} \in \Omega \setminus \underline{e}_{r,\alpha}$, we have $[\underline{k}]_{r\alpha} \in \Omega \setminus \underline{e}_{r,\alpha}$ and

$x_{r\alpha}w \in N_\eta^{(r,\alpha)}$.

For any $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$, we have $x_{r\alpha}w \in N_\eta^{(r,\alpha)}$, so $N_\eta^{(r,\alpha)}$ is stable under $U(\mathfrak{t}^+)$. Thus, $N_\eta^{(r,\alpha)}$ is a proper submodule of N_η.

Moreover, $N_\eta/N_\eta^{(r,\alpha)} = span_{\mathbb{C}}\{y^{\underline{e}_{r,\alpha}}v\}$, which is a one-dimensional $\mathbb{C}$-vector space, so $N_\eta^{(r,\alpha)}$ is a maximal $U(\mathfrak{t})$-submodule of N_η. $\square$

Proposition 2.12 *Every maximal $U(\mathfrak{t})$-submodule of N_η is of the form $N_\eta^{(r,\alpha)}$ for some $\underline{e}_{r,\alpha} \in \Omega$.*

Proof. Assume that there exists a maximal submodule M of N_η such that $M \neq N_\eta^{(r,\alpha)}$ for all $1 \leq r \leq n$ and $\alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Then by the maximality of M and $N_\eta^{(r,\alpha)}$ in N_η, we have $M + N_\eta^{(r,\alpha)} = N_\eta$. So, $(M + N_\eta^{(r,\alpha)})/M \cong N_\eta^{(r,\alpha)}/M \cap N_\eta^{(r,\alpha)}$ and it follows that $N_\eta/M \cong N_\eta^{(r,\alpha)}/M \cap N_\eta^{(r,\alpha)}$. Since $N_\eta^{(r,\alpha)}$ is not irreducible, we have $M \cap N_\eta^{(r,\alpha)} \neq 0$. Let $N_{r,\alpha} = span_{\mathbb{C}}\{y_{r,\alpha}v\}$. Note that $N_\eta^{(r,\alpha)} \cap N_{r,\alpha} = 0$, hence $(M \cap N_\eta^{(r,\alpha)}) \cap (M \cap N_{r,\alpha}) = 0$. Thus, as vector spaces, $(M \cap N_\eta^{(r,\alpha)}) + (M \cap N_{r,\alpha}) = (M \cap N_\eta^{(r,\alpha)}) \oplus (M \cap N_{r,\alpha})$. Since $N_\eta/M \cong N_\eta^{(r,\alpha)}/M \cap N_\eta^{(r,\alpha)}$, $N_\eta/N_\eta^{(r,\alpha)} \cong M/M \cap N_\eta^{(r,\alpha)}$ is irreducible and we must have

$$M = (M \cap N_\eta^{(r,\alpha)}) \oplus (M \cap N_{r,\alpha}).$$

Suppose that $M \cap N_{r,\alpha} \neq 0$ for all $1 \leq r \leq n$ and $\alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$, then $w = y_{r\alpha} v \in M$ for all $1 \leq r \leq n$ and $\alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Since $\{y_{r\alpha} v | 1 \leq r \leq n, \alpha \in \mathbb{Z}^m+, \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0\}$ generates N_η, we get that $N_\eta \in M$, which can not happen because we assumed that M is a maximal submodule of N_η. So, $M \cap N_{r,\alpha} = 0$ for some $1 \leq r \leq n$ and $\alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Then we get $M = M \cap N_\eta^{(r,\alpha)}$ and by the maximality of M we have $M = N_\eta^{(r,\alpha)}$. But this is a contradiction as we assumed that $M \neq N_\eta^{(r,\alpha)}$ for all $1 \leq r \leq n$ and $\alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. We conclude that $M = N_\eta^{(r,\alpha)}$ for some $1 \leq r \leq n$ and $\alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. □

Proposition 2.13 *The space of Whittaker vectors (of type η) for M_η / N_η is one-dimensional.*

Proof. Let $w \neq 0$ be a Whittaker vector for M_η / N_η, then $(x - \eta)^{\underline{k}} w \in N_\eta$ for all $\underline{k} \in I$. We can write w as

$$w = \sum_{\underline{k} \in I \setminus \Omega} \lambda_{\underline{k}} y^{\underline{k}} v + N_\eta,$$

where only finitely many $\lambda_{\underline{k}}$ are not zero. Let $\underline{l} = max\{\underline{k} \in I \setminus \Omega, \lambda_{\underline{k}} \neq 0\}$. If $\underline{l} = \underline{0}$, then $w = \lambda v + N_\eta$ for some nonzero $\lambda \in \mathbb{C}$. Assume that $\underline{l} \neq \underline{0}$, then we can see that $(x - \eta)^{\underline{l}} w = \{\prod_{1 \leq r \leq n, \alpha \in \mathbb{Z}^m+} (\alpha_1 a_1 + \alpha_2 a_2 + \ldots, +\alpha_n a_n)^{l_{r\alpha}}\} \underline{l}! v + N_\eta$. Since $\underline{l} \notin \Omega$,, we have $\alpha_1 a_1 + \alpha_2 a_2 + \ldots, +\alpha_n a_n \neq 0$ for every $l_{r\alpha} > 0$. But this is a contradiction because $(x-\eta)^{\underline{l}} w \in N_\eta$. Thus, we have $w = \lambda v + N_\eta$ for some $\lambda \in \mathbb{C}$, which implies that the space of Whittaker vectors (of type η) for M_η / N_η is one-dimensional. $\square$

Theorem 2.14 *N_η is the unique maximal submodule of $M_{\eta,\vec{a}}$.*

Proof. Let K be a maximal $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{a}}$ and suppose that $K \neq N_\eta$. Then $K \cap N_\eta$ is a maximal $U(\mathfrak{t})$-submodule of N_η. Since $K + N_\eta = M_{\eta,\vec{a}}$, so $N_\eta/(K \cap N_\eta) \cong M_{\eta,\vec{a}}/K$ and then we must have $K \cap N_\eta = N_\eta^{(r,\alpha)}$ for some $\underline{e}_{r,\alpha} \in \Omega$. Hence $N_\eta^{(r,\alpha)} \subseteq K$. Since $K/(K \cap N_\eta) \cong M_{\eta,\vec{a}}/N_\eta$ and $M_{\eta,\vec{a}}/N_\eta$ has a Whittaker vector, there exists $w \in K, w \notin N_\eta$ such that $w + (K \cap N_\eta)$ is a Whittaker vector in $K/(K \cap N_\eta)$. Thus, by Proposition 2.13, we may assume that $w = v + \sum_{\underline{k} \in \Omega} \lambda_{\underline{k}} y^{\underline{k}} v$ after by multiplying a scalar. Then $0 \neq y_{r\alpha} w = y_{r\alpha} v + \sum_{\underline{k} \in \Omega} \lambda_{\underline{k}} y_{r\alpha} y^{\underline{k}} v \in K \cap N_\eta = N_\eta^{(r,\alpha)}$. Since $\sum_{\underline{k} \in \Omega} \lambda_{\underline{k}} y_{r\alpha} y^{\underline{k}} v \in N_\eta^{(r,\alpha)}$, we get $y_{r\alpha} v \in N_\eta^{(r,\alpha)}$, which is a contradiction with the defnition of $N_\eta^{(r,\alpha)}$. Hence $K = N_\eta$ and we get that N_η is the unique maximal submodule of $M_{\eta,\vec{a}}$. $\square$

2.4 Whittaker modules for $\mathfrak{t}$ with $a_1 = a_2 = \cdots = a_m = 0$

In this chapter we will investigate the maximal $U(\mathfrak{t})$-submodules for $M_{\eta,\vec{a}}$ with $a_1 = a_2 = \cdots = a_m = 0$. We denote $M_{\eta,\vec{a}}$ as $M_{\eta,\vec{0}}$.

Notation 2.15 *Let $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$ be an algebra homomorphism, and let J_ξ be the ideal in $U(\mathfrak{t}^-)$ generated by $y_{r\alpha} - \xi(y_{r\alpha})$ for all $1 \le r \le n, \alpha \in \mathbb{Z}^m+$.*

Lemma 2.16 *Let $M^{(\xi)}_{\eta,\vec{0}} = J_\xi v$ in $M_{\eta,\vec{0}}$. Then $M^{(\xi)}_{\eta,\vec{0}}$ is a maximal $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{0}}$.*

Proof. Since J_ξ is an ideal of $U(\mathfrak{t}^-)$, it follows that $M^{(\xi)}_{\eta,\vec{0}}$ is stable under $U(\mathfrak{t}^-)$. By Lemma 2.4(3), $M^{(\xi)}_{\eta,\vec{0}}$ is stable under $U(\mathfrak{t}^+)$, and it is obviously stable under $\mathfrak{t}_0$. Hence, $M^{(\xi)}_{\eta,\vec{0}}$ is a $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{0}}$ and is proper because $v \notin M^{(\xi)}_{\eta,\vec{0}}$. Since $M^{(\xi)}_{\eta,\vec{0}} = span_{\mathbb{C}}\{(y-\xi)^{\underline{k}} v | \underline{k} \in I, \underline{k} \neq 0\}$ and the set $span_{\mathbb{C}}\{(y-\xi)^{\underline{k}} v | \underline{k} \in I\}$ is a $\mathbb{C}$-basis of $M_{\eta,\vec{0}}$, we get that $M_{\eta,\vec{0}}/M^{(\xi)}_{\eta,\vec{0}} = \mathbb{C}v$. So, $M^{(\xi)}_{\eta,\vec{0}}$ is a maximal $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{0}}$. $\square$

Lemma 2.17 *Every maximal ideal of $U(\mathfrak{t}^-)$ is of the form J_ξ for some algebra homomorphism $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$.*

Proof. Let M be a maximal ideal of $U(\mathfrak{t}^-)$, then $U(\mathfrak{t}^-)/M$ is a field extension of $\mathbb{C}$. Since every proper field extension of $\mathbb{C}$ must contain a copy of $\mathbb{C}(z)$, where

z is algebraically independent over $\mathbb{C}$, hence it must have uncountable dimension. Since $dim_{\mathbb{C}} U(\mathfrak{t}^-)/M$ is countable, $U(\mathfrak{t}^-)/M$ is not a proper field extension and $U(\mathfrak{t}^-)/M = \mathbb{C}$. So, for every $1 \leq r \leq n, \alpha \in \mathbb{Z}^m+$, there exists $\xi_{r\alpha} \in \mathbb{C}$ such that $y_{r\alpha} = \xi_{r\alpha} + M \Rightarrow y_{r\alpha} - \xi_{r\alpha} \in M$. Let $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$ be the algebra homomorphism defined by $\xi(y_{r\alpha}) = \xi_{r\alpha}$ for all $1 \leq r \leq n, \alpha \in \mathbb{Z}^m+$. Then $J_\xi \subset M$, and by the maximality of J_ξ, we have $M = J_\xi$. □

Set $P = U(\mathfrak{t}^-)$. By the PBW theorem, we may view P as a polynomial ring in the variables $y_{r\alpha}, 1 \leq r \leq n, \alpha \in \mathbb{Z}^m+$. For any $u \in P$, define the action of $U(\mathfrak{t})$ on u by: $y_{r\alpha}$ acts on u as multiplication by $y_{r\alpha}$, $x_{r\alpha}u = \eta(x_{r\alpha})u$ and $c_1 u = c_2 u = \cdots = c_m u = 0$.

Lemma 2.18 *Every maximal $U(\mathfrak{t})$-submodule of P has the form J_ξ for some algebra homomorphism $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$.*

Proof. Let K be a maximal $U(\mathfrak{t})$-submodule of P. Then K is a proper $U(\mathfrak{t})$-submodule of P with the action of $U(\mathfrak{t}^-)$ defined above. Clearly, K is an ideal of P. Hence K must be contained in some maximal ideal of $P = U(\mathfrak{t}^-)$. By Lemma 2.16, $K \subset J_\xi$ for some algebra homomorphism $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$. However, J_ξ is a $U(\mathfrak{t})$-submodule of P, so it is stable under the action of $U(\mathfrak{t}^+)$ and $c_1, c_2, \ldots, c_m$ defined above. Hence $K = J_\xi$ by the maximality of K as a $U(\mathfrak{t})$-submodule of P. □

Theorem 2.19 *Every maximal $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{0}}$ has the form $M_{\eta,\vec{0}}^{(\xi)}$ for some algebra homomorphism $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$.*

Proof. Define $f : P \to M_{\eta,\vec{0}}$ by $u \mapsto uv$ for all $u \in P$. As in Proposition 2.3(2), we know that f is an isomorphism of (left) $U(\mathfrak{t}^-)$-modules, where the action of $U(\mathfrak{t}^-)$ on P is by left multiplication. It is easy to see that f is actually an isomorphism of (left) $U(\mathfrak{t})$-modules. Let M be a maximal $U(\mathfrak{t})$-submodule of $M_{\eta,\vec{0}}$. Then $f^{-1}(M)$ is a maximal $U(\mathfrak{t})$-submodule of P. By Lemma 2.18, it follows that $f^{-1}(M) = J_\xi$ for some algebra homomorphism $\xi : U(\mathfrak{t}^-) \to \mathbb{C}$. So $M = (J_\xi) = J_\xi v = M_{\eta,\vec{0}}^{(\xi)}$ as desired. $\square$

2.5 The center of $U(\mathfrak{t})$ and annihilator ideals

In this section, we describe the center of the enveloping algebra $U(\mathfrak{t})$. Then we show how the annihilator in $U(\mathfrak{t})$ of an irreducible Whittaker module for $\mathfrak{t}$ of $\mathbb{Z}$-independent levels is generated. Let $Z = Z(U(\mathfrak{t}))$ be the center of the enveloping algebra $U(\mathfrak{t})$ of $\mathfrak{t}$.

Proposition 2.20 $Z = \mathbb{C}[c_1, c_2, \ldots, c_m]$.

Proof. Since it is obvious that $\mathbb{C}[c_1, c_2, \ldots, c_m] \subseteq Z$, we only need to prove $Z \subseteq \mathbb{C}[c_1, c_2, \ldots, c_m]$. Let $u = \sum \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}} \in Z$, where $c^{\underline{b}} = c_1^{b_1} c_2^{b_2} \ldots c_m^{b_m}$ and only finitely many non-zero $\lambda_{\underline{k},\underline{l},\underline{b}}$ occur in the sum. Assume that there exists $\underline{m} \in I, \underline{m} \neq \underline{0}$, such that $\lambda_{\underline{k},\underline{m},\underline{b}} \neq 0$ for some $\underline{k} \in I, \underline{b} \in \mathbb{Z}^m, b_1, b_2, \ldots, b_m \geq 0$. Let $\alpha \in \mathbb{Z}^m+, 1 \leq r \leq n$ be such that $m_{r\alpha} \neq 0$. Then the set

$$I_{r,\alpha} = \{(\underline{k}, \underline{l}, \underline{b}) | \lambda_{\underline{k},\underline{l},\underline{b}} \neq 0 \ for\ some\ \underline{k}, \underline{l} \in I, \underline{b} \in \mathbb{Z}^m \ with\ l_{r\alpha} \neq 0\}$$

is non-empty and we can write

$$u = \sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}} + \sum_{(\underline{k},\underline{l},\underline{b}) \notin I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}}.$$

Now for any $\underline{k} \in I, 1 \leq s \leq n, \beta \in \mathbb{Z}^m+$, let $\underline{k}^{(s,\beta)}$ be defined as: $k_{r\alpha}^{(s,\beta)} = k_{r\alpha}$ if $(r, \alpha) \neq (s, \beta)$ and $k_{s\beta}^{(s,\beta)} = k_{s\beta} - 1$. Note that if $\underline{k}, \underline{l} \in I$ and $\underline{k}^{(s,\beta)} = \underline{l}^{(s,\beta)}$ for some $1 \leq s \leq n, \beta \in \mathbb{Z}^m+$, then $\underline{k} = \underline{l}$. Since

$$[x_{r\alpha}, y_{s\beta}] = \delta_{rs}\delta_{\alpha\beta}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m),$$

we have

$$\begin{aligned}
x_{r\alpha}^{l_{r\alpha}} y_{r\alpha} &= l_{r\alpha} x_{r\alpha}^{l_{r\alpha}-1}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m) + y_{r\alpha} x_{r\alpha}^{l_{r\alpha}}, \\
u y_{r\alpha} &= y_{r\alpha} \sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}} + y_{r\alpha} \sum_{(\underline{k},\underline{l},\underline{b}) \notin I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}} \\
&\quad + \sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} l_{r\alpha} y^{\underline{k}} x^{\underline{l}^{(r,\alpha)}} c^{\underline{b}} (\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m).
\end{aligned}$$

Since $uy_{r\alpha} = y_{r\alpha}u$, it follows that

$$y_{r\alpha} \sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}} + y_{r\alpha} \sum_{(\underline{k},\underline{l},\underline{b}) \notin I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}}$$

$$= y_{r\alpha} \sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}} + y_{r\alpha} \sum_{(\underline{k},\underline{l},\underline{b}) \notin I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} y^{\underline{k}} x^{\underline{l}} c^{\underline{b}}$$

$$+ \sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} l_{r\alpha} y^{\underline{k}} x^{\underline{l}^{(r,\alpha)}} c^{\underline{b}} (\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m).$$

This implies

$$\sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} l_{r\alpha} y^{\underline{k}} x^{\underline{l}^{(r,\alpha)}} c^{\underline{b}} (\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_n c_n) = 0.$$

We have

$$\sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} \alpha_i l_{r\alpha} y^{\underline{k}} x^{\underline{l}^{(r,\alpha)}} c^{\underline{b}} c_i = 0,$$

for every $1 \leq i \leq m$. Since $\alpha \in \mathbb{Z}^m+$, there exists at least one $1 \leq j \leq m$ such that $\alpha_j \neq 0$. So we have

$$\sum_{(\underline{k},\underline{l},\underline{b}) \in I_{r,\alpha}} \lambda_{\underline{k},\underline{l},\underline{b}} l_{r\alpha} y^{\underline{k}} x^{\underline{l}^{(r,\alpha)}} c^{\underline{b}} = 0.$$

Note that if $(\underline{k'}, \underline{l'}^{(r,\alpha)}, \underline{b'}) = (\underline{k}, \underline{l}^{(r,\alpha)}, \underline{b})$ in the above sum, then $\underline{k'} = \underline{k}, \underline{l'}^{(r,\alpha)} = \underline{l}^{(r,\alpha)}, \underline{b'} = \underline{b}$. So $\lambda_{\underline{k},\underline{l},\underline{b}} l_{r\alpha} y^{\underline{k}} x^{\underline{l}^{(r,\alpha)}} c^{\underline{b}} = 0$ for all $(\underline{k}, \underline{l}, \underline{b}) \in I_{r,\alpha}$, which implies $\lambda_{\underline{k},\underline{l},\underline{b}} = 0$ for all $(\underline{k}, \underline{l}, \underline{b}) \in I_{r,\alpha}$ and this is a contradiction. Hence such $\underline{m}$ does not exist and u can be written as $u = \sum_{\underline{k},\underline{b}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} c^{\underline{b}} \in Z$. Now, assume that there exists $\underline{k} \in I, \underline{k} \neq \underline{0}$, such that $\lambda_{\underline{k},\underline{b}} \neq 0$ for some $\underline{b} \in \mathbb{Z}^m, b_1, b_2, \ldots, b_m \geq 0$. Let $\alpha \in \mathbb{Z}^m+, 1 \leq r \leq n$ be such that $k_{r\alpha} \neq 0$. Then the set

$$J_{r,\alpha} = \{(\underline{k}, \underline{b}) | \lambda_{\underline{k},\underline{b}} \neq 0 \ for\ some\ \underline{k} \in I, \underline{b} \in \mathbb{Z}^m\ with\ k_{r\alpha} \neq 0\}$$

is non-empty and we can write

$$u = \sum_{(\underline{k},\underline{b}) \in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} c^{\underline{b}} + \sum_{(\underline{k},\underline{b}) \notin J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} c^{\underline{b}}.$$

we have

$$\begin{aligned} x_{r\alpha} y_{r\alpha}^{k_{r\alpha}} &= k_{r\alpha} y_{r\alpha}^{k_{r\alpha}-1}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m) + y_{r\alpha}^{k_{r\alpha}} x_{r\alpha}, \\ x_{r\alpha} u &= \sum_{(\underline{k},\underline{b}) \in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} x_{r\alpha} c^{\underline{b}} + \sum_{(\underline{k},\underline{b}) \notin J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} x_{r\alpha} c^{\underline{b}} \\ &+ \sum_{(\underline{k},\underline{b}) \in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} k_{r\alpha} y^{\underline{k}^{(r,\alpha)}} c^{\underline{b}}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m). \end{aligned}$$

Since $x_{r\alpha} u = u x_{r\alpha}$, it follows that

$$\begin{aligned} &\sum_{(\underline{k},\underline{b}) \in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} x_{r\alpha} c^{\underline{b}} + \sum_{(\underline{k},\underline{b}) \notin J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} x_{r\alpha} c^{\underline{b}} \\ &+ \sum_{(\underline{k},\underline{b}) \in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} k_{r\alpha} y^{\underline{k}^{(r,\alpha)}} c^{\underline{b}}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m) \\ &= \sum_{(\underline{k},\underline{b}) \in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} x_{r\alpha} c^{\underline{b}} + \sum_{(\underline{k},\underline{b}) \notin J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} y^{\underline{k}} x_{r\alpha} c^{\underline{b}}. \end{aligned}$$

This implies

$$\sum_{(\underline{k},\underline{b}) \in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} k_{r\alpha} y^{\underline{k}^{(r,\alpha)}} c^{\underline{b}}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m) = 0.$$

We have

$$\sum_{(\underline{k},\underline{b})\in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}}\alpha_i k_{r\alpha} y^{\underline{k}^{(r,\alpha)}} c^{\underline{b}} c_i = 0,$$

for every $1 \leq i \leq m$. Since $\alpha \in \mathbb{Z}^m+$, there exists at least one $1 \leq j \leq m$ such that $\alpha_j \neq 0$. So we have

$$\sum_{(\underline{k},\underline{b})\in J_{r,\alpha}} \lambda_{\underline{k},\underline{b}} k_{r\alpha} y^{\underline{k}^{(r,\alpha)}} c^{\underline{b}} = 0.$$

So, $\lambda_{\underline{k},\underline{b}} k_{r\alpha} y^{\underline{k}^{(r,\alpha)}} c^{\underline{b}}$ for all $(\underline{k},\underline{b}) \in J_{r,\alpha}$, which implies $\lambda_{\underline{k},\underline{b}} = 0$ for all $(\underline{k},\underline{b}) \in J_{r,\alpha}$ and this is a contradiction. Hence such $\underline{k}$ does not exist and u can be written as $u = \sum_{\underline{b}\in\mathbb{Z}^m} \lambda_{\underline{b}} c^{\underline{b}} \in \mathbb{C}[c_1, c_2, \ldots, c_m]$. $\square$

Now, for any $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$, let $Z_{\vec{a}}$ be the ideal in Z generated by $c_1 - a_1, c_2 - a_2, \ldots, c_m - a_m$. We will show that the annihilator ideal in $U(\mathfrak{t})$ of an irreducible Whittaker module for $\mathfrak{t}$ with $a_1, a_2, \ldots, a_m$ $\mathbb{Z}$-independent is generated by $Z_{\vec{a}}$. In the setting of Whittaker modules for finite dimensional complex semisimple Lie algebra $\mathfrak{g}$, Kostant showed that the annihilator in the enveloping algebra $U(\mathfrak{g})$ of an irreducible Whittaker module for $\mathfrak{g}$ is centrally generated [Kos]. In [On], M.Ondrus showed that the annihilator of any Whittaker module for the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is centrally generated. In [Chr], Christodoulopoulou showed that the annihilator ideal in $U(\mathfrak{t})$ of an irreducible Whittaker module for $\mathfrak{t}$ is centrally generated when $m = 1$ and $a_1 \neq 0$.

Proposition 2.21 *If* $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ *is* $\mathbb{Z}$*-independent, then* $Ann_{U(\mathfrak{t})} M_{\eta,\vec{a}} = U(\mathfrak{t}) Z_{\vec{a}}$.

Proof. It is obvious that $U(\mathfrak{t}) Z_{\vec{a}} \subset Ann_{U(\mathfrak{t})} M_{\eta,\vec{a}}$, we only need to show that for any $u \in Ann_{U(\mathfrak{t})} M_{\eta,\vec{a}}$, we have $u \in U(\mathfrak{t}) Z_{\vec{a}}$. By the PBW theorem, u can be written as

$$\sum_{\underline{l},\underline{k} \in I, \underline{b} \in \mathbb{Z}^m} \lambda_{\underline{l},\underline{k},\underline{b}} y^{\underline{l}} (x - \eta)^{\underline{k}} (c - \vec{a})^{\underline{b}},$$

where $(c - \vec{a})^{\underline{b}} = \prod_{i=1}^{i=m} (c_i - a_i)^{b_i}$ and there are only finitely many nonzero terms in the sum. If $b_1^2 + b_2^2 + \ldots + b_m^2 > 0$ and $\underline{l}, \underline{k} \in I$, we have $y^{\underline{l}} (x - \eta)^{\underline{k}} (c - \vec{a})^{\underline{b}} \in Ann_{U(\mathfrak{t})} M_{\eta,\vec{a}}$. We may assume that

$$\sum_{\underline{l},\underline{k} \in I} \lambda_{\underline{l},\underline{k}} y^{\underline{l}} (x - \eta)^{\underline{k}}.$$

For the Whittaker vector v, since $uv = 0$, we get that $\lambda_{\underline{l},\underline{0}} = 0$ for all $\underline{l}$ by Proposition 2.3(1). Since $u \neq 0$, we may assume that there exist $\underline{l}, \underline{k} \in I$, $\underline{k} \neq 0$ such that $\lambda_{\underline{l},\underline{k}} \neq 0$. Let $\underline{k'} = min\{\underline{k} \in I | \lambda_{\underline{l},\underline{k}} \neq 0 \ for \ some \ \underline{l} \in I\}$ and $\underline{k'} \neq 0$. Then by Lemma 2.4, we have

$$0 = u y^{\underline{k'}} v = \sum_{\underline{l} \in I} \lambda_{\underline{l},\underline{k'}} \underline{k'}! \{ \prod_{r,\alpha} (\alpha_1 a_1 + \ldots + \alpha_m a_m)^{k'_{r\alpha}} \} y^{\underline{l}} v.$$

Since $a_1, a_2, \ldots, a_m$ are $\mathbb{Z}$-independent, $\prod_{r,\alpha} (\alpha_1 a_1 + \ldots + \alpha_m a_m)^{k'_{r\alpha}} \neq 0$. So we have $\lambda_{\underline{l},\underline{k'}} = 0$ for all such $\underline{l}$ and this is a contradiction by our choice of $\underline{k'}$. Thus, $u \in U(\mathfrak{t}) Z_{\vec{a}}$ as desired. $\square$

3 Whittaker modules for $\tilde{\mathfrak{t}}$

3.1 Extending $\mathfrak{t}$ by m derivations

Let $\mathfrak{t}$ be the Heisenberg algebra defined in Chapter 2. Set $\tilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \ldots \oplus \mathbb{C}d_m$, and extend the Lie bracket on $\mathfrak{t}$ to $\tilde{\mathfrak{t}}$ by

$$[c_i, d_j] = 0, [d_i, x_{r\alpha}] = \alpha_i x_{r\alpha}, [d_i, y_{r\alpha}] = -\alpha_i y_{r\alpha}, [d_i, d_j] = 0,$$

for all $1 \leq i, j \leq m, 1 \leq r \leq n, \alpha \in \mathbb{Z}^m+$.

Set $\tilde{\mathfrak{t}}^+ = \mathfrak{t}^+ = \bigoplus_{\alpha \in \mathbb{Z}^m+} \mathfrak{t}_\alpha, \tilde{\mathfrak{t}}^- = \mathfrak{t}^- = \bigoplus_{\alpha \in \mathbb{Z}^m+} \mathfrak{t}_\alpha$ and $\tilde{\mathfrak{t}}_0 = \mathfrak{t}_0 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \ldots \oplus \mathbb{C}d_m$.

Definition 3.1 *Let $\eta : U(\tilde{\mathfrak{t}}^+) \to \mathbb{C}$ be an algebra homomorphism such that $\eta|_{\tilde{\mathfrak{t}}^+} \neq 0$, and let V be a $U(\tilde{\mathfrak{t}})$-module.*

1. A non-zero vector $v \in V$ is called a Whittaker vector of type η if $xv = \eta(x)v$ for all $x \in U(\tilde{\mathfrak{t}}^+)$.

2. V is called a Whittaker module for $\tilde{\mathfrak{t}}$ if V contains a cyclic Whittaker vector v (i.e. $v \in V$ is a Whittaker vector and $V = U(\tilde{\mathfrak{t}})v$).

Next we will construct Whittaker modules for $\tilde{\mathfrak{t}}$. Set $\tilde{\mathfrak{b}} = \mathfrak{t}^+ \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \ldots \oplus \mathbb{C}c_m$. Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ and let $\mathbb{C}_{\eta,\vec{a}} = \mathbb{C}\tilde{v}$ be a one-dimensional vector space viewed as a $\tilde{\mathfrak{b}}$-module by

$$c_i\tilde{v} = a_i\tilde{v}, \quad x\tilde{v} = \eta(x)\tilde{v},$$

for all $1 \leq i \leq m$ and $x \in U(\tilde{\mathfrak{t}}^+)$. Set

$$\widetilde{M}_{\eta,\vec{a}} = U(\tilde{\mathfrak{t}}) \otimes_{u(\tilde{b})} \mathbb{C}_{\eta,\vec{a}}, \quad v = 1 \otimes \tilde{v}.$$

Define an action of $U(\tilde{\mathfrak{t}})$ on $\widetilde{M}_{\eta,\vec{a}}$ by left multiplication (on the first tensor factor). Note that $\widetilde{M}_{\eta,\vec{a}} = U(\tilde{\mathfrak{t}})v$ and that $\widetilde{M}_{\eta,\vec{a}}$ is a Whittaker module for $\tilde{\mathfrak{t}}$.

Proposition 3.2 *Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$, and $d^{\underline{p}} = d_1^{p_1} d_2^{p_2} \ldots d_m^{p_m}$, where $\underline{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{Z}^m_{\geq 0}$. Then we have*

1. *The set $\{y^{\underline{k}} d^{\underline{p}} | \underline{k} \in I, \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$ is a basis of $\widetilde{M}_{\eta,\vec{a}}$ as a $\mathbb{C}$-vector space.*

2. *As a $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$-module, $\widetilde{M}_{\eta,\vec{a}}$ is isomorphic to $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \ldots \oplus \mathbb{C}d_m)$.*

3. *$\widetilde{M}_{\eta,\vec{a}}$ is free as a $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \ldots \oplus \mathbb{C}d_m)$-module.*

Proof.

1. Since $U(\tilde{\mathfrak{t}}) \cong U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{b}})$ by PBW theorem, $U(\tilde{\mathfrak{t}})$ is a free right $U(\tilde{\mathfrak{b}})$-module with basis of $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$. And since $\{y^{\underline{k}}d^{\underline{p}} | \underline{k} \in I, \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$, we have $\widetilde{M}_{\eta,\vec{a}} = U(\tilde{\mathfrak{t}}) \otimes_{U(\tilde{\mathfrak{b}})} \mathbb{C}_{\eta,\vec{a}} \cong (U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m) \otimes_{\mathbb{C}} U(\tilde{\mathfrak{b}})) \otimes_{U(\tilde{\mathfrak{b}})} \mathbb{C}_{\eta,\vec{a}} \cong U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m) \otimes_{\mathbb{C}} (U(\tilde{\mathfrak{b}}) \otimes_{U(\tilde{\mathfrak{b}})} \mathbb{C}_{\eta,\vec{a}}) \cong U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m) \otimes_{\mathbb{C}} \mathbb{C}_{\eta,\vec{a}}$ is a $\mathbb{C}$-vector space with basis $\{y^{\underline{k}}d^{\underline{p}} | \underline{k} \in I, \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$.

2. This is obvious from the proof of Proposition 3.2(1).

3. Since $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$ is a domain, it follows that $\widetilde{M}_{\eta,\vec{a}}$ is torsion-free as a $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$-module. Hence $\widetilde{M}_{\eta,\vec{a}}$ is free as a $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$-module since $\widetilde{M}_{\eta,\vec{a}}$ is cyclic as a $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$-module.

□

Proposition 3.3 *Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ be $\mathbb{Z}$-independent and $M_{\eta,\vec{a}}$ be the irreducible Whittaker $U(\mathfrak{t})$ module (of type η) constructed in Chapter 2. Then $M_{\eta,\vec{a}}$ is isomorphic to a proper $U(\mathfrak{t})$-submodule of $\widetilde{M}_{\eta,\vec{a}}$.*

Proof. In $\widetilde{M}_{\eta,\vec{a}}$, set $V = U(\mathfrak{t})v$. By Corollary 2.7, $V \cong M_{\eta,\vec{a}}$ and V is a proper subspace of $\widetilde{M}_{\eta,\vec{a}}$ by Propositions 2.3(1) and 3.2(1). □

For any $k \in \mathbb{Z}_{>0}, 1 \leq i \leq k \in \mathbb{Z}$, let $(k)_i = k(k-1)(k-2)\dots(k-i+1)$ be the falling factorial. Set $(k)_i = 0$ if $i < 0$ or $i > k$, and $(k)_0 = 1$.

Lemma 3.4 *Let $1 \leq r, s \leq n, \alpha, \beta \in \mathbb{Z}^m+, \alpha \neq \beta, q, e \in \mathbb{Z}_{\geq 0}, \underline{p} \in \mathbb{Z}^m_{\geq 0}, C^j_q = q!/j!(q-j)!$, then we have*

1. $(x_{r\alpha} - \eta(x_{r\alpha}))^q d^{\underline{p}} = \sum_{j=0}^{j=q} C^j_q (-1)^{q-j} \eta(x_{r\alpha})^{q-j} \prod_{i=0}^{i=m} (d_i - j\alpha_i)^{p_i} x^j_{r\alpha}$.

2. $(x_{r\alpha} - \eta(x_{r\alpha}))^q y^e_{r\alpha} = \sum_{j=0}^{min(e,q)} C^j_q (e)_j y^{e-j}_{r\alpha} (\alpha_1 c_1 + \cdots + \alpha_m c_m)^j (x_{r\alpha} - \eta(x_{r\alpha}))^{q-j}$.

3. $(x_{r\alpha} - \eta(x_r\alpha))^q y^{q'}_{s\beta} = y^{q'}_{s\beta} (x_{r\alpha} - \eta(x_r\alpha))^q$.

Proof.

1. For any $1 \leq i \leq m, 1 \leq r \leq n, e, q \in \mathbb{Z}_{\geq 0}, \underline{p} = (p_1, p_2, \dots, p_m) \in \mathbb{Z}^m_{\geq 0}, \alpha \in \mathbb{Z}^m+$, by induction, we have:

$$\begin{aligned} [d_i, x_{r\alpha}] &= \alpha_i x_{r\alpha}, \\ x_{r\alpha} d_i &= (d_i - \alpha_i) x_{r\alpha}, \\ x^l_{r\alpha} d_i &= (d_i - l\alpha_i) x^l_{r\alpha}, \\ x^l_{r\alpha} d^{p_i}_i &= (d_i - l\alpha_i)^{p_i} x^l_{r\alpha}. \end{aligned} \tag{3.1}$$

So, by induction we have

$$\begin{aligned}
(x_{r\alpha}-\eta(x_{r\alpha}))^q d_1^{p_1} &= [\sum_{j=0}^{j=q}(-1)^{q-j}C_q^j\eta(x_{r\alpha})^{q-j}{x_{r\alpha}}^j]d_1^{p_1}\\
&= \sum_{j=0}^{j=q}(-1)^{q-j}C_q^j\eta(x_{r\alpha})^{q-j}(d_1-j\alpha_1)^{p_1}x_{r\alpha}^j,\\
(x_{r\alpha}-\eta(x_{r\alpha}))^q d_1^{p_1}d_2^{p_2} &= [\sum_{j=0}^{j=q}(-1)^{q-j}C_q^j\eta(x_{r\alpha})^{q-j}(d_1-j\alpha_1)^{p_1}x_{r\alpha}^j]d_2^{p_2}\\
&= \sum_{j=0}^{j=q}(-1)^{q-j}C_q^j\eta(x_{r\alpha})^{q-j}(d_1-j\alpha_1)^{p_1}(d_2-j\alpha_2)^{p_2}x_{r\alpha}^j,\\
(x_{r\alpha}-\eta(x_{r\alpha}))^q d^{\underline{p}} &= \sum_{j=0}^{j=q}(-1)^{q-j}C_q^j\eta(x_{r\alpha})^{q-j}[\prod_{i-0}^{i=m}(d_i-j\alpha_i)^{p_i}]x_{r\alpha}^j.
\end{aligned}$$

2. $[x_{r\alpha}, y_{s\beta}] = \delta_{r,s}\delta_{\alpha,\beta}(\alpha_1c_1+\cdots+\alpha_mc_m)$ implies that

$$\begin{aligned}
(x_{r\alpha}-\eta(x_{r\alpha}))y_{r\alpha} &= y_{r\alpha}(x_{r\alpha}-\eta(x_{r\alpha}))+(\alpha_1c_1+\cdots+\alpha_mc_m),\\
(x_{r\alpha}-\eta(x_{r\alpha}))y_{r\alpha}^2 &= y_{r\alpha}^2(x_{r\alpha}-\eta(x_{r\alpha}))+2(\alpha_1c_1+\cdots+\alpha_mc_m)y_{r\alpha}.
\end{aligned}$$

By induction on e, we can show that

$$(x_{r\alpha}-\eta(x_{r\alpha}))y_{r\alpha}^e = y_{r\alpha}^e(x_{r\alpha}-\eta(x_{r\alpha}))+e(\alpha_1c_1+\cdots+\alpha_mc_m)y_{r\alpha}^{e-1},$$

which proves (2) for $q=1, e\geq 1$. Now for all $q\leq e$, suppose that (2) is true for $1, 2, \ldots, q-1$. Then we have

$$
\begin{aligned}
(x_{r\alpha} - \eta(x_{r\alpha}))^{q-1} y_{r\alpha}^{e} &= \sum_{j=0}^{q-1} C_{q-1}^{j} (e)_j y_{r\alpha}^{e-j} (\alpha_1 c_1 + \cdots + \alpha_m c_m)^j \\
&\quad * (x_{r\alpha} - \eta(x_{r\alpha}))^{q-1-j}, \\
(x_{r\alpha} - \eta(x_{r\alpha}))^{q} y_{r\alpha}^{e} &= \sum_{j=0}^{q-1} C_{q-1}^{j} (e)_j (x_{r\alpha} - \eta(x_{r\alpha})) y_{r\alpha}^{e-j} \\
&\quad * (\alpha_1 c_1 + \cdots + \alpha_m c_m)^j (x_{r\alpha} - \eta(x_{r\alpha}))^{q-1-j} \\
&= \sum_{j=0}^{q-1} C_{q-1}^{j} (e)_j (y_{r\alpha}^{e-j} (x_{r\alpha} - \eta(x_{r\alpha})) + (e-j)(\alpha_1 c_1 + \ldots \\
&\quad + \alpha_m c_m) y_{r\alpha}^{e-j-1}) (\alpha_1 c_1 + \cdots + \alpha_m c_m)^j (x_{r\alpha} - \eta(x_{r\alpha}))^{q-1-j} \\
&= \sum_{j=0}^{q-1} C_{q-1}^{j} (e)_j y_{r\alpha}^{e-j} (\alpha_1 c_1 + \cdots + \alpha_m c_m)^j (x_{r\alpha} - \eta(x_{r\alpha}))^{q-j} \\
&\quad + \sum_{j=0}^{q-1} C_{q-1}^{j} (e)_j (e-j) (\alpha_1 c_1 + \cdots + \alpha_m c_m)^{j+1} y_{r\alpha}^{e-j-1} \\
&= \sum_{j=0}^{q} C_{q-1}^{j} (e)_j y_{r\alpha}^{e-j} (\alpha_1 c_1 + \cdots + \alpha_m c_m)^j (x_{r\alpha} - \eta(x_{r\alpha}))^{q-j}
\end{aligned}
$$

$$+\sum_{j=0}^{q} C_{q-1}^{j-1}(e)_{j-1}(e-j+1)(\alpha_1 c_1+\cdots+\alpha_m c_m)^j y_{r\alpha}^{e-j}$$

$$+\sum_{j=0}^{q} C_q^j(e)_j y_{r\alpha}^{e-j}(\alpha_1 c_1+\cdots+\alpha_m c_m)^j(x_{r\alpha}-\eta(x_{r\alpha}))^{q-j}.$$

Since that $C_{q-1}^q=0, C_{q-1}^{-1}=0$ and $C_{q-1}^j+C_{q-1}^{j-1}=C_q^j$, (2) is true for all $q\leq e$.

Now, for $q>e$,

$$\begin{aligned}(x_{r\alpha}-\eta(x_{r\alpha}))^q y_{r\alpha}^e &= (x_{r\alpha}-\eta(x_{r\alpha}))^{q-e}(x_{r\alpha}-\eta(x_{r\alpha}))^e y_{r\alpha}^e \\ &= (x_{r\alpha}-\eta(x_{r\alpha}))^{q-e}\sum_{j=0}^{e} C_e^j(e)_j y_{r\alpha}^{e-j} \\ &\quad *(\alpha_1 c_1+\cdots+\alpha_m c_m)^j(x_{r\alpha}-\eta(x_{r\alpha}))^{e-j}.\end{aligned}$$

So by induction, we have that

$$(x_{r\alpha}-\eta(x_{r\alpha}))^q y_{r\alpha}^e=\sum_{j=0}^{e} C_q^j(e)_j y_{r\alpha}^{e-j}(\alpha_1 c_1+\cdots+\alpha_m c_m)^j(x_{r\alpha}-\eta(x_{r\alpha}))^{q-j}.$$

All the above show that

$$(x_{r\alpha}-\eta(x_{r\alpha}))^q y_{r\alpha}^e=\sum_{j=0}^{min(e,q)} C_q^j(e)_j y_{r\alpha}^{e-j}(\alpha_1 c_1+\cdots+\alpha_m c_m)^j(x_{r\alpha}-\eta(x_{r\alpha}))^{q-j}.$$

3. The relation $[x_{r\alpha}, y_{s\beta}]=0$ for $\alpha\neq\beta$ implies $(x_{r\alpha}-\eta(x_r\alpha))^q y_{s\beta}^{q'}=y_{s\beta}^{q'}(x_{r\alpha}-\eta(x_r\alpha))^q$.

$\square$

Next, we will discuss some standard facts for further use. For any $m, k \in \mathbb{Z}_{\geq 0}$, let

$$\Delta^m(x^k) = \sum_{j=0}^{m} (-1)^{m-j} C_m^j (x+j)^k \tag{3.2}$$

be the m-th forward difference of the monomial x^k. When $m = 1$, we will omit the superscript and just write Δ. Let

$$\sigma(k, m) = \Delta^m(x^k)|_{x=0} = \sum_{j=0}^{m} (-1)^{m-j} C_m^j j^k. \tag{3.3}$$

$\sigma(k, m)$ is sometimes referred to as the ordered Stirling number and is equal to the number of set compositions of $\{1, 2, \ldots, k\}$ of length m. If $0 \leq m \leq k$, then $\frac{1}{m!}\sigma(k, m)$ is the Stirling number of the second kind. It is easy to see that $\sigma(k, 1) = 1$ and $\sigma(k, k) = k!$ for all $k \geq 1$. Note that $\Delta(x^k)$ is a polynomial in x of degree $k-1$ for every $k > 1$. By induction on m, we can show that $\Delta^m(x^k)$ is a polynomial in x of degree at most $k - m$ for every $1 \leq m \leq k$. Hence $\Delta^k(x^k)$ is constant for all x, and in fact $\Delta^k(x^k) = k!$ for all $k \geq 0$, since $\Delta^k(x^k) = \sigma(k, k) = k!$ for all $k \geq 0$. From this, it follows that $\Delta^m(x^k) = 0$ if $0 \leq k < m$. As $\sigma(k, m) = \Delta^m(x^k)$, we get that $\sigma(k, m) = 0$ if $0 \leq k < m$.

Lemma 3.5 *Assume that $\widetilde{M}_{\eta,\vec{a}}$ and v are defined as in Definition* 3.1. *Let $1 \leq i \leq m, 1 \leq r, s \leq n, q \in \mathbb{Z}_{\geq 0}, \underline{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{Z}_{\geq 0}^m, \alpha \in \mathbb{Z}^m+$. Then*

1. $(x_{r\alpha}-\eta(x_{r\alpha}))^q d^{\underline{p}}v=(-1)^q(\prod_{i=1}^{m}\alpha_i^{p_m})q!\eta(x_{r\alpha})^q v$ *if* $q=p_1+p_2+\cdots+p_m$.

2. $(x_{r\alpha}-\eta(x_{r\alpha}))^q d^{\underline{p}}v=0$ *if* $q>p_1+p_2+\cdots+p_m$.

3. *If* $\vec{a}=(a_1,a_2,\ldots,a_m)\neq 0$, *then*

$$
\begin{aligned}
&(x_{r\alpha}-\eta(x_{r\alpha}))^{|\underline{p}|+s}y_{r\alpha}^s d^{\underline{p}}v\\
&=(-1)^{|\underline{p}|}(\prod_{i=1}^{m}\alpha_i^{p_m})(|\underline{p}|+s)!(\alpha_1c_1+\cdots+\alpha_mc_m)^s\eta(x_{r\alpha})^{|\underline{p}|}v,
\end{aligned}
$$

and $(x_{r\alpha}-\eta(x_{r\alpha}))^{q+s}y_{r\alpha}^s d^{\underline{p}}v=0$ *if* $|p|+s<q$.

Proof.

1.

$$
\begin{aligned}
(x_{r\alpha}-\eta(x_{r\alpha}))^q d^{\underline{p}}v &= \sum_{j=0}^{j=q}(-1)^{q-j}C_q^j\eta(x_{r\alpha})^{q-j}[\prod_{i-0}^{i=m}(d_i-j\alpha_i)^{p_i}]x_{r\alpha}^j v\\
&= \eta(x_{r\alpha})^q\sum_{j=0}^{j=q}(-1)^{q-j}C_q^j[\prod_{i-0}^{i=m}(d_i-j\alpha_i)^{p_i}]v.
\end{aligned}
$$

For the convenience of typesetting, we denote $\underline{i}=(i_1,i_2,\ldots,i_m)\in\mathbb{Z}^m$ and set $A=\{\underline{i}\mid 0\leq i_1\leq p_1,0\leq i_2\leq p_2,\ldots,0\leq i_m\leq p_m\}$. Since

$$
\begin{aligned}
\prod_{i-0}^{i=m}(d_i-j\alpha_i)^{p_i} &= \sum_{\underline{i}\in A}(-1)^{i_1+i_2+\cdots+i_m}C_{p_1}^{i_1}C_{p_2}^{i_2}\cdots C_{p_m}^{i_m}\\
&\qquad *\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}d_1^{p_1-i_1}d_2^{p_2-i_2}\cdots d_m^{p_m-i_m}j^{i_1+i_2+\cdots+i_m}.
\end{aligned}
$$

So, by the fact that $\sigma(k,k)=k!$ and $\sigma(k,m)=0$ for all $0\le k<m$, we have

$$(x_{r\alpha}-\eta(x_{r\alpha}))^q d^{\underline{p}}v$$

$$=\eta(x_{r\alpha})^q\sum_{j=0}^{j=q}(-1)^{q-j}C_q^j(\sum_{\underline{i}\in A}(-1)^{i_1+i_2+\cdots+i_m}C_{p_1}^{i_1}C_{p_2}^{i_2}\cdots C_{p_m}^{i_m}$$

$$*\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}d_1^{p_1-i_1}d_2^{p_2-i_2}\cdots d_m^{p_m-i_m}j^{i_1+i_2+\cdots+i_m})v$$

$$=\eta(x_{r\alpha})^q(\sum_{\underline{i}\in A}(-1)^{i_1+i_2+\cdots+i_m}C_{p_1}^{i_1}C_{p_2}^{i_2}\cdots C_{p_m}^{i_m}\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}$$

$$*d_1^{p_1-i_1}d_2^{p_2-i_2}\cdots d_m^{p_m-i_m}\sigma(i_1+i_2+\cdots+i_m,q))v$$

$$=(-1)^q\eta(x_{r\alpha})^q q!\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m}v.$$

2. This part is obvious from the proof of Lemma 3.5(1).

3. It follows from Lemma 3.4(2) that

$$(x_{r\alpha}-\eta(x_{r\alpha}))^{|\underline{p}|+s}y_{r\alpha}^s d^{\underline{p}}v$$
$$=\sum_{j=0}^{s}C_{|\underline{p}|+s}^j(s)_j y_{r\alpha}^{s-j}(\alpha_1c_1+\cdots+\alpha_mc_m)^j(x_{r\alpha}-\eta(x_{r\alpha}))^{|\underline{p}|+s-j}d^{\underline{p}}v.$$

By Lemma 3.5(2), we have that

$$(x_{r\alpha} - \eta(x_{r\alpha}))^{|\underline{p}|+s-j} d^{\underline{p}} v = 0,$$

for all $j = 0, 1, 2, \ldots, s-1$ and

$$(x_{r\alpha} - \eta(x_{r\alpha}))^{|\underline{p}|} d^{\underline{p}} v = (-1)^{|\underline{p}|} (\prod_{i=1}^{m} \alpha_i^{p_m}) |\underline{p}|! \eta(x_{r\alpha})^{|\underline{p}|} v.$$

Hence,

$$(x_{r\alpha} - \eta(x_{r\alpha}))^{|\underline{p}|+s} y_{r\alpha}^s d^{\underline{p}} v$$

$$= C_{|\underline{p}|+s}^s s! (\alpha_1 c_1 + \cdots + \alpha_m c_m)^s (-1)^{|\underline{p}|} (\prod_{i=1}^{m} \alpha_i^{p_m}) |\underline{p}|! \eta(x_{r\alpha})^{|\underline{p}|} v$$

$$= (-1)^{|\underline{p}|} (s + |\underline{p}|)! (\prod_{i=1}^{m} \alpha_i^{p_m}) (\alpha_1 c_1 + \cdots + \alpha_m c_m)^s \eta(x_{r\alpha})^{|\underline{p}|} v$$

as desired. This implies that $(x_{r\alpha} - \eta(x_{r\alpha}))^{q+s} y_{r\alpha}^s d^{\underline{p}} v = 0$ if $|p| + s < q$.

$\square$

For any $\underline{k} \in I$, let $||i\underline{k}|| = \sum_{1 \le r \le n, \alpha \in \mathbb{Z}^m+} \alpha_i k_{r\alpha}$.

Lemma 3.6 *Let* $\underline{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{Z}_{\ge 0}^m, \underline{k} \in I$. *Then*

1. $x^{\underline{k}} d^{\underline{p}} = (\prod_{i=0}^{m} (d_i - ||i\underline{k}||)^{p_i}) x^{\underline{k}}$.

2. $y^{\underline{k}} d^{\underline{p}} = (\prod_{i=0}^{m} (d_i + ||i\underline{k}||)^{p_i}) y^{\underline{k}}$.

3. $d^{\underline{p}} x^{\underline{k}} = x^{\underline{k}} (\prod_{i=0}^{m} (d_i + ||i\underline{k}||)^{p_i})$.

4. $d^{\underline{p}} y^{\underline{k}} = y^{\underline{k}} (\prod_{i=0}^{m} (d_i - ||i\underline{k}||)^{p_i})$.

Proof.

1. By equation 3.1 we have

$$\begin{aligned} x_{r\alpha}^{k_{r\alpha}} d_i &= (d_i - k_{r\alpha}\alpha_i) x_{r\alpha}^{k_{r\alpha}} \\ \Rightarrow x^{\underline{k}} d_i &= (d_i - ||i\underline{k}||) x^{\underline{k}} \\ \Rightarrow x^{\underline{k}} d_i^{p_i} &= (d_i - ||i\underline{k}||)^{p_i} x^{\underline{k}} \\ \Rightarrow x^{\underline{k}} d^{\underline{p}} &= (\prod_{i=0}^{m} (d_i - ||i\underline{k}||)^{p_i}) x^{\underline{k}}. \end{aligned}$$

2. By equation 3.1 we have

$$\begin{aligned} y_{r\alpha}^{k_{r\alpha}} d_i &= (d_i + k_{r\alpha}\alpha_i) y_{r\alpha}^{k_{r\alpha}} \\ \Rightarrow y^{\underline{k}} d_i &= (d_i + ||i\underline{k}||) y^{\underline{k}} \\ \Rightarrow y^{\underline{k}} d_i^{p_i} &= (d_i + ||i\underline{k}||)^{p_i} y^{\underline{k}} \\ \Rightarrow y^{\underline{k}} d^{\underline{p}} &= (\prod_{i=0}^{m} (d_i + ||i\underline{k}||)^{p_i}) y^{\underline{k}}. \end{aligned}$$

3. By induction, we have

$$\begin{aligned} d_i x_{r\alpha}^{k_{r\alpha}} &= x_{r\alpha}^{k_{r\alpha}}(d_i + k_{r\alpha}\alpha_i) \\ \Rightarrow d_i x^{\underline{k}} &= x^{\underline{k}}(d_i + ||i\underline{k}||) \\ \Rightarrow d_i^{p_i} x^{\underline{k}} &= x^{\underline{k}}(d_i + ||i\underline{k}||)^{p_i} \\ \Rightarrow d^{\underline{p}} x^{\underline{k}} &= x^{\underline{k}}(\prod_{i=0}^{m}(d_i + ||i\underline{k}||)^{p_i}). \end{aligned}$$

4. By induction, we have

$$\begin{aligned} d_i y_{r\alpha}^{k_{r\alpha}} &= y_{r\alpha}^{k_{r\alpha}}(d_i - k_{r\alpha}\alpha_i) \\ \Rightarrow d_i y^{\underline{k}} &= y^{\underline{k}}(d_i - ||i\underline{k}||) \\ \Rightarrow d_i^{p_i} y^{\underline{k}} &= y^{\underline{k}}(d_i - ||i\underline{k}||)^{p_i} \\ \Rightarrow d^{\underline{p}} y^{\underline{k}} &= y^{\underline{k}}(\prod_{i=0}^{m}(d_i - ||i\underline{k}||)^{p_i}). \end{aligned}$$

□

3.2 Whittaker modules for $\tilde{\mathfrak{t}}$ with $a_1, a_2, \ldots, a_m$ $\mathbb{Z}$-independent

Definition 3.7 *Let $\eta : U(\tilde{\mathfrak{t}}^+) \to \mathbb{C}$ be an algebra homomorphism and Γ be the collection of all η such that: if given $\alpha \in \mathbb{Z}^m+$ with $\alpha_1\alpha_2\ldots\alpha_m \neq 0$, for each $1 \leq i \leq m$, we can fix all α_j, for $j = 1, 2, \ldots, i-1, i+1, \ldots, m$, and still have infinitely many α_i such that $\eta|_{\tilde{\mathfrak{t}}_\alpha} \neq 0$.*

From this chapter to the end of the article, if not specifically noticed, we assume that $\eta \in \Gamma$.

Proposition 3.8 *Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ be $\mathbb{Z}$-independent. If $\eta \in \Gamma$, then $\widetilde{M}_{\eta,\vec{a}}$ is irreducible as a $U(\tilde{\mathfrak{t}})$-module.*

Proof. Let K be a non-zero $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{M}_{\eta,\vec{a}}$. Since $\widetilde{M}_{\eta,\vec{a}}=U(\tilde{\mathfrak{t}})v$ and $U(\mathfrak{t})v$ is irreducible as $U(\mathfrak{t})$-module, we only need to show that $K \cap U(\mathfrak{t})v \neq 0$. Let $0 \neq w \in K$ and w has a unique expression

$$w = \sum_{\underline{k},\underline{p}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v,$$

where $\lambda_{\underline{k},\underline{p}} \neq 0$ for only finitely many $\underline{k} \in I, \underline{p} \in \mathbb{Z}^m_{\geq 0}$. Let $l = max\{|\underline{p}| = p_1 + p_2 + \ldots + p_m | \lambda_{\underline{k},\underline{p}} \neq 0$ for some $\underline{k} \in I\}$. If $l = 0$, then $w \in U(\mathfrak{t})v$ and so $K \cap U(\mathfrak{t})v \neq 0$. Now, consider the case $l > 0$, we will show that there exists $u \in U(\tilde{\mathfrak{t}})$ such that $0 \neq uw \in K \cap U(\mathfrak{t})v$. Since $\eta \in \Gamma$, there must exist $1 \leq r \leq n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1\alpha_2 \ldots \alpha_m \neq 0$, $\eta(x_{r\alpha}) \neq 0$ and $k_{r\alpha} = 0$ for all $\underline{k}$ with $\lambda_{\underline{k},\underline{p}} \neq 0$ for some $\underline{p}$. Then

$$(x_{r\alpha} - \eta(x_{r\alpha}))^l w = (x_{r\alpha} - \eta(x_{r\alpha}))^l \sum_{\underline{k},\underline{p}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v.$$

By Lemma 3.5, we have

$$\begin{aligned}
(x_{r\alpha} - \eta(x_{r\alpha}))^l w &= \sum_{\underline{k},|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} (x_{r\alpha} - \eta(x_{r\alpha}))^l d^{\underline{p}} v \\
&= \sum_{\underline{k},|\underline{p}|=l} (-1)^l \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m} \eta(x_{r\alpha})^l l! y^{\underline{k}} v
\end{aligned}$$

$$= \sum_{\underline{k}} (-1)^l l! \eta(x_{r\alpha})^l (\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m}) y^{\underline{k}} v.$$

If $\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m} \neq 0$ for some $\underline{k}$ with $\lambda_{\underline{k},|\underline{p}|=l} \neq 0$, then $0 \neq (x_{\gamma\alpha} - \eta(x_{\gamma\alpha}))^l w \in K \cap U(\mathfrak{t})v$ and the proof is done. If $\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m} = 0$ for all $\underline{k}$ with $\lambda_{\underline{k},|\underline{p}|=l} \neq 0$. Since $\alpha_1 \alpha_2 \ldots \alpha_m \neq 0$, for fixed $\underline{k}$, we have $\underline{p}' \neq \underline{p}$ and $|\underline{p}'| = |\underline{p}| = l$. Since $\underline{p}' \neq \underline{p}$, there must exsits $1 \leq j \leq m$ such that $p_j \neq {p'}_j$. Consider

$$\begin{aligned} & \sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m} \\ = & \sum_{|\underline{p}|=l} (\lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \ldots \alpha_{j-1}^{p_{j-1}} \alpha_{j+1}^{p_{j+1}} \cdots \alpha_m^{p_m}) \alpha_j^{p_j} \end{aligned}$$

as a finite term polynomial $f(\alpha_j)$ for α_j.

Since $\eta \in \Gamma$,we may keep all $\alpha_i, i = 1, 2, \ldots, m, i \neq j$ fixed and have infinitely many α_j such that $\eta(x_{r,\alpha}) \neq 0$. $f(\alpha_j) = 0$ has only finite solutions in $\mathbb{Z}$, so we may choose $\alpha_j \in \mathbb{Z}$ such that $f(\alpha_j) \neq 0$. Then for this $\alpha \in \mathbb{Z}_{\geq 0}^m$,

$$\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m} \neq 0.$$

Since $\{y^{\underline{k}} | \underline{k} \in I\}$ is the $\mathbb{C}$ basis for $U(\mathfrak{t})v$, we have that

$$0 \neq \sum_{\underline{k}} (-1)^l l! \eta(x_{r\alpha})^l (\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m}) y^{\underline{k}} v \in K \cap U(\mathfrak{t})v$$

and the proof is done. $\square$

Proposition 3.9 *Let $\vec{a} = (a_1, a_2, \ldots, \alpha_m) \in \mathbb{C}^m$ be $\mathbb{Z}$-independent. If $\eta \in \Gamma$, then the space of Whittaker vectors for $\widetilde{M}_{\eta,\vec{a}}$ is one dimensional.*

Proof. Let $\eta' : U(\tilde{\mathfrak{t}}) \to \mathbb{C}$ be an algebra homomorphism. Suppose that $w \in \widetilde{M}_{\eta,\vec{a}}$ is a Whittaker vector of type η'. We show that $\eta = \eta'$ and that $w \in \mathbb{C}v$. By Proposition 3.2(1), w has a unique expression

$$w = \sum_{\underline{k},\underline{p}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v,$$

where only finitely many $\lambda_{\underline{k},\underline{p}} \neq 0$. Let $l = max\{|\underline{p}| = p_1 + p_2 + \cdots + p_m | \lambda_{\underline{k},\underline{p}} \neq 0$ for some $\underline{k} \in I\}$. If $l = 0$, then $w \in U(\mathfrak{t})v$, hence $w \in \mathbb{C}v$ by Proposition 2.5. Suppose that $l > 0$. We will show that this lead to a contradiction. By our assumption on η, we may choose $\alpha \in \mathbb{Z}^m+, 1 \leq r \leq n$ such that $\eta(x_{r\alpha}) \neq 0$ and $k_{r\alpha} = 0$ for all $\underline{k}$ such that $\lambda_{\underline{k},\underline{p}} \neq 0$ for some $\underline{p}$. By Lemma 3.4(3) and 3.5(1), we have that

$$\begin{aligned}
(x_{r\alpha} - \eta(x_{r\alpha}))^l w &= \sum_{\underline{k},|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} (x_{r\alpha} - \eta(x_{r\alpha}))^l d^{\underline{p}} v \\
&= \sum_{\underline{k},|\underline{p}|=l} (-1)^l \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m} \eta(x_{r\alpha})^l l! y^{\underline{k}} v \\
&= \sum_{\underline{k}} (-1)^l l! \eta(x_{r\alpha})^l (\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m}) y^{\underline{k}} v.
\end{aligned}$$

Let $ker(\eta)$ be the kernel of η in $U(\tilde{\mathfrak{t}}^+)$. We claim that there exist $0 \neq u_+ \in ker(\eta)$ such that $u_+ w = v$. Let $\underline{q} = max\{\underline{k} | \lambda_{\underline{k},|\underline{p}|=l} \neq 0\}$ (with respect to the lexicographic order in I). If $\underline{q} = \underline{0}$, then by the formula above, we get

$$(x_{r\alpha} - \eta(x_{r\alpha}))^l w = (-1)^l l! \eta(x_{r\alpha})^l (\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m}) v \in \mathbb{C}v.$$

Thus, the claim holds in this case with $u_+ = (x_{r\alpha} - \eta(x_{r\alpha}))^l$. Suppose that $\underline{q} \neq \underline{0}$, then by the formula above and Lemma 2.4(1) we have

$$\begin{aligned} &(x-\eta)^{\underline{m}}(x_{r\alpha} - \eta(x_{r\alpha}))^l w \\ &= (-1)^l l! \eta(x_{r\alpha})^l (\sum_{|\underline{p}|=l} \lambda_{\underline{k},\underline{p}} \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_m^{p_m}) v \end{aligned}$$

and this is an element of

$$\mathbb{C}\{\prod_{1 \le r \le n, \alpha \in \mathbb{Z}^m+} (\alpha_1 a_1 + \ldots, +\alpha_m a_m)^{k_{r\alpha}}\} \underline{m}! v.$$

Multiplying $(x-\eta)^{\underline{m}}(x_{r\alpha} - \eta(x_{r\alpha}))^l$ by an appropriate scalar, we get an element $u_+ \in U(\tilde{\mathfrak{t}}^+)$ such that $u_+ w = v$. This proves the claim. Since $U(\tilde{\mathfrak{t}}^+)$ is abelian and w is a Whittaker vector of type η', we have

$$(x_{s\beta} - \eta'(x_{s\beta}))v = (x_{s\beta} - \eta'(x_{s\beta}))u_+ w = u_+(x_{s\beta} - \eta'(x_{s\beta}))w = 0$$

for all $1 \le s \le n, \beta \in \mathbb{Z}^m+$. Therefore $\eta = \eta'$. Since $u_+ \in ker(\eta)$, this implies $v = u_+ w = \eta(u_+) w = 0$, which is a contradiction. $\square$

Proposition 3.10 *Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ be $\mathbb{Z}$-independent. If $\eta \in \Gamma$, and M' is a Whittaker $\tilde{\mathfrak{t}}$-module of type η with cyclic Whittaker vector v' such that $c_1 v' = a_1 v', c_2 v' = a_2 v', \ldots, c_m v' = a_m v'$, then $M' \cong \widetilde{M_{\eta,\vec{a}}}$ and so M' is irreducible.*

Proof. Let $\mathbb{C}_{\eta,\vec{a}} = \mathbb{C}v$. Then the map

$$f : U(\tilde{\mathfrak{t}}) \otimes \mathbb{C}_{\eta,\vec{a}} \to M',$$

defined by $(u, rv) \mapsto ruv'$ for $r \in \mathbb{C}, u \in U(\tilde{\mathfrak{t}})$, is bilinear. Moreover if $w \in U(\tilde{\mathfrak{b}})$, then

$$f(uw, rv) = r(uw)v' = f(u, w(rv)).$$

Hence there exists an induced linear map

$$f : \widetilde{M}_{\eta,\vec{a}} = U(\tilde{\mathfrak{t}}) \otimes_{U(\tilde{\mathfrak{b}})} \mathbb{C}_{\eta,\vec{a}} \to M',$$

defined by $u \otimes rv \mapsto ruv'$, which is a homomorphism of (left) $U(\tilde{\mathfrak{t}})$-modules, and it is obviously surjective as $M' = U(\tilde{\mathfrak{t}})v'$. Since $\widetilde{M}_{\eta,\vec{a}}$ is irreducible, f is then one-to-one. Thus, $M' \cong \widetilde{M}_{\eta,\vec{a}}$ as desired. □

Corollary 3.11 *Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ be $\mathbb{Z}$-independent. If $\eta \in \Gamma$, then $\widetilde{M}_{\eta,\vec{a}}$ is the unique (up to isomorphism) irreducible Whittaker $\tilde{\mathfrak{t}}$-module of type η on which c_i acts on the Whittaker vector v by a_i for $i = 1, 2, \ldots, m$.*

Proposition 3.12 *Let $\eta' : U(\tilde{\mathfrak{t}}^+) \to \mathbb{C}$ be a nonzero algebra homomorphism and $\eta \in \Gamma$. Let $\vec{a}, \vec{a}' \in \mathbb{C}^m$ and both $\mathbb{Z}$-independent. Then $\widetilde{M}_{\eta,\vec{a}} \cong \widetilde{M}_{\eta'\vec{a}'}$ as $U(\tilde{\mathfrak{t}})$-modules if and only if $\eta = \eta'$ and $\vec{a} = \vec{a}'$.*

Proof. This follows from the proof of Proposition 3.9. $\square$

Now we describe a filtration of $\widetilde{M}_{\eta,\vec{a}}$ by $U(\mathfrak{t})$ modules. For $s = 0, 1, 2, 3, \ldots$, let

$$\widetilde{M}^{(s)}_{\eta,\vec{a}} = span_{\mathbb{C}}\{y^{\underline{k}} d^{\underline{p}} v | \underline{k} \in I, |\underline{p}| \leq s\}.$$

Note that $\widetilde{M}^{(0)}_{\eta,\vec{a}} = span_{\mathbb{C}}\{y^{\underline{k}} v | \underline{k} \in I\} \cong M_{\eta,\vec{a}}$ and that $\widetilde{M}^{(s)}_{\eta,\vec{a}}$ is a $U(\mathfrak{t})$-module for each s by Lemma 3.4.

Proposition 3.13 *The sequence*

$$\widetilde{M}^{(0)}_{\eta,\vec{a}} \subsetneqq \widetilde{M}^{(1)}_{\eta,\vec{a}} \subsetneqq \cdots \subsetneqq \widetilde{M}^{(s)}_{\eta,\vec{a}} \subsetneqq \ldots$$

is a filtration of $\widetilde{M}_{\eta,\vec{a}}$ *by* $U(\mathfrak{t})$*-modules. Moreover, if* $a_1, a_2, \ldots, a_m$ *are* $\mathbb{Z}$*-independent, then* $\widetilde{M}^{(s)}_{\eta,\vec{a}}/\widetilde{M}^{(s-1)}_{\eta,\vec{a}}$ *is an irreducible Whittaker* $U(\mathfrak{t})$*-module.*

Proof. Since $\widetilde{M}^{(s)}_{\eta,\vec{a}}$ is stable under $U(\mathfrak{t})$ for all $s = 0, 1, 2, \ldots$, the sequence is a filtration by $U(\mathfrak{t})$-modules. Since $\widetilde{M}^{(s)}_{\eta,\vec{a}}/\widetilde{M}^{(s-1)}_{\eta,\vec{a}} \cong M_{\eta,\vec{a}}$ as $U(\mathfrak{t})$-modules, we have $\widetilde{M}^{(s)}_{\eta,\vec{a}}/\widetilde{M}^{(s-1)}_{\eta,\vec{a}}$ irreducible as a whittaker $U(\mathfrak{t})$-module. $\square$

3.3 Whittaker modules for $\tilde{\mathfrak{t}}$ with $a_1, a_2, \ldots, a_m$ $\mathbb{Z}$-dependent

Proposition 3.14 *Let* $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m \neq 0$ *and* $a_1, a_2, \ldots, a_m$ *be* $\mathbb{Z}$*-dependent,* $\eta \in \Gamma$*. Then* $\widetilde{N}_\eta = span_{\mathbb{C}}\{y^{\underline{k}} d^{\underline{p}} v | \underline{k} \in \Omega\}$ *is a maximal submodule of*

$\widetilde{M}_{\eta,\vec{a}}$.

Proof. First we show that $\widetilde{N}_\eta$ is a proper submodule of $\widetilde{M}_{\eta,\vec{a}}$. For any $w \in \widetilde{N}_\eta$, w has a unique expression

$$w = \sum_{\underline{k}\in\Omega,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v,$$

where $\lambda_{\underline{k},\underline{p}} \neq 0$ for only finitely many $\underline{k} \in \Omega, \underline{p} \in \mathbb{Z}^m_{\geq 0}$.

Obviously, $\widetilde{N}_\eta$ is stable under $\tilde{\mathfrak{t}}^-$ since for any $\underline{k'} \in I$, we have

$$y^{\underline{k'}} w = \sum_{\underline{k}\in\Omega,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}+\underline{k'}} d^{\underline{p}} v \in \widetilde{N}_\eta.$$

For any $i = 1, 2, \ldots, m$,

$$c_i w = \sum_{\underline{k}\in\Omega,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} c_i v$$

$$= \sum_{\underline{k}\in\Omega,\underline{p}\in\mathbb{Z}^m_{\geq 0}} a_i \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v \in \widetilde{N}_\eta.$$

So $\widetilde{N}_\eta$ is stable under $\mathbb{C}c_1 \oplus \mathbb{C}c_2 \ldots \oplus \mathbb{C}c_m$. Now, for any $\underline{p'} \in \mathbb{Z}^m_{\geq 0}$, by Lemma 3.6 we have

$$d^{\underline{p'}} w = \sum_{\underline{k}\in\Omega,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} (\prod_{i=0}^{i=m} (d_i - ||i\underline{k}||^{p'_i})) d^{\underline{p}} v \in \widetilde{N}_\eta.$$

So $\widetilde{N}_\eta$ is stable under $\mathbb{C}d_1 \oplus \mathbb{C}d_2 \ldots \oplus \mathbb{C}d_m$.

Now we claim that $\widetilde{N}_\eta$ is also stable under $\tilde{\mathfrak{t}}^+$. For any $r = 1, 2, \ldots, n, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{Z}^m +$. If $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m = 0$, by induction we have

$$x_{r\alpha} y^k_{s\beta} = y^k_{s\beta} x_{r\alpha} + k\delta_{r,s}\delta_{\alpha,\beta} y^{k-1}_{s,\beta} (\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m).$$

Denote $[\underline{k}]_{r\alpha}$ the same as $\underline{k}$ except that, if $k_{r\alpha} > 0$, the element at $(r,\alpha)^{th}$ position is $k_{r\alpha} - 1$ instead of $k_{r\alpha}$. Then, we can rewrite w as

$$w = \sum_{\underline{k}\in\Omega, k_{r\alpha}\neq 0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v + \sum_{\underline{k}\in\Omega, k_{r\alpha}=0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v.$$

SO,

$$\begin{aligned}
x_{r\alpha} w &= \sum_{\underline{k}\in\Omega, k_{r\alpha}\neq 0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} k_{r\alpha}\lambda_{\underline{k},\underline{p}} y^{[\underline{k}]_{r\alpha}} d^{\underline{p}} (\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m) v \\
&\quad + \sum_{\underline{k}\in\Omega, k_{r\alpha}\neq 0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v + \sum_{\underline{k}\in\Omega, k_{r\alpha}=0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v \\
&= \sum_{\underline{k}\in\Omega, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v \\
&= \sum_{\underline{k}\in\Omega, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \eta(x_{r\alpha}) \lambda_{\underline{k},\underline{p}} y^{\underline{k}} \Big(\prod_{i=1}^{i=m} (d_i - \alpha_i)^{p_i}\Big) v \in \widetilde{N}_\eta.
\end{aligned}$$

If $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m \neq 0$. Then for any $\underline{k} \in \Omega$, we have $[\underline{k}]_{r\alpha} \in \Omega$, so

$$\begin{aligned}
x_{r\alpha} w &= \sum_{\underline{k}\in\Omega, k_{r\alpha}\neq 0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} k_{r\alpha}\lambda_{\underline{k},\underline{p}} y^{[\underline{k}]_{r\alpha}} d^{\underline{p}} (\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m) v \\
&\quad + \sum_{\underline{k}\in\Omega, k_{r\alpha}\neq 0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v + \sum_{\underline{k}\in\Omega, k_{r\alpha}=0, \underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v
\end{aligned}$$

$$= \sum_{\underline{k}\in\Omega,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v + \sum_{\underline{k}\in\Omega,k_{r\alpha}\neq 0,\underline{p}\in\mathbb{Z}^m_{\geq 0}} k_{r\alpha}\lambda_{\underline{k},\underline{p}} y^{[\underline{k}]_{r\alpha}} d^{\underline{p}}(\alpha_1 a_1 + \ldots + \alpha_m a_m)v$$

$$= \sum_{\underline{k}\in\Omega,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \eta(x_{r\alpha})\lambda_{\underline{k},\underline{p}} y^{\underline{k}} (\prod_{i=1}^{i=m}(d_i - \alpha_i)^{p_i})v$$

$$+ \sum_{\underline{k}\in\Omega,k_{r\alpha}\neq 0,\underline{p}\in\mathbb{Z}^m_{\geq 0}} k_{r\alpha}\lambda_{\underline{k},\underline{p}} y^{[\underline{k}]_{r\alpha}} d^{\underline{p}}(\alpha_1 a_1 + \ldots + \alpha_m a_m)v \in \widetilde{N}_\eta.$$

Since for any $r = 1, 2, \ldots, n, \alpha =\in \mathbb{Z}^m+$, we have $x_{r\alpha} w \in \widetilde{N}_\eta$, so $\widetilde{N}_\eta$ is stable under $\tilde{\mathfrak{t}}^+$. Thus, $\widetilde{N}_\eta$ is a proper submodule of $\widetilde{M}_{\eta,\vec{a}}$.

Now consider $\widetilde{M}_\eta/\widetilde{N}_\eta \cong span_{\mathbb{C}}\{y^{\underline{k}} d^{\underline{p}} v | \underline{k} \in I, \underline{k} \notin \Omega, \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$. By Proposition 3.8, $\widetilde{M}_\eta/\widetilde{N}_\eta$ is irreducible as a $U(\tilde{\mathfrak{t}})$-module. Thus $\widetilde{N}_\eta$ is a maximal submodule of $\widetilde{M}_{\eta,\vec{a}}$.

□

For $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$, let $\underline{e}_{r,\alpha}$ be the element of I which has 1 in the $(r,\alpha)^{th}$ position and zeros everywhere else. Denote $\Omega_{r,\alpha} = \Omega \setminus \underline{e}_{r,\alpha}$.

Lemma 3.15 *Let* $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m \neq 0$ *be* $\mathbb{Z}$*-dependent,* $\eta \in \Gamma$*. Then* $\widetilde{N}^{(r,\alpha)}_\eta = span_{\mathbb{C}}\{y^{\underline{k}} d^{\underline{p}} v | \underline{k} \in \Omega_{r,\alpha}, \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$ *is a maximal* $U(\tilde{\mathfrak{t}})$*-module of* $\widetilde{N}_\eta$*.*

Proof. First we show that $\widetilde{N}^{(r,\alpha)}_\eta$ is a proper submodule of $\widetilde{N}_\eta$. For any $w \in \widetilde{N}^{(r,\alpha)}_\eta$,

w has a unique expression

$$w = \sum_{\underline{k} \in \Omega_{r,\alpha}, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v,$$

where $\lambda_{\underline{k},\underline{p}} \neq 0$ for only finitely many $\underline{k} \in \Omega_{r,\alpha}, \underline{p} \in \mathbb{Z}^m_{\geq 0}$.

For any $\underline{0} \neq \underline{k'} \in I$, we have

$$y^{\underline{k'}} w = \sum_{\underline{k} \in \Omega_{r,\alpha}, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}+\underline{k'}} d^{\underline{p}} v.$$

Suppose that $y^{\underline{k'}} w \notin \widetilde{N}^{(r,\alpha)}_\eta$, since $w \in \widetilde{N}_\eta$, we have $y^{\underline{k'}} w \in \tilde{N}_\eta$. Then there must exist $\underline{k} \in \Omega_{r,\alpha}, \underline{p} \in \mathbb{Z}^m_{\geq 0}, \lambda_{\underline{k},\underline{p}}$, such that $y^{\underline{k}+\underline{k'}} d^{\underline{p}} v \in \widetilde{N}_\eta \setminus \widetilde{N}^{(r,\alpha)}_\eta$, which implies that $\underline{k} + \underline{k'} = \underline{e}_{r,\alpha}$. So, $\underline{k} = \underline{e}_{r,\alpha}$ or $\underline{k'} = \underline{e}_{r,\alpha}$. If $\underline{k} = \underline{e}_{r,\alpha}$, then $\underline{k'} = \underline{0}$ is a contradiction. If $\underline{k'} = \underline{e}_{r,\alpha}$, then $\underline{k} = \underline{0}$, but $\underline{0} \notin \Omega_{r,\alpha}$ and this is a contradiction. So $y^{\underline{k'}} w \in \widetilde{N}^{(r,\alpha)}_\eta$ and this shows that $\widetilde{N}^{(r,\alpha)}_\eta$ is stable under $\tilde{\mathfrak{t}}^-$. Similar to Proposition 3.14, $\widetilde{N}^{(r,\alpha)}_\eta$ is stable under $\mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \ldots \oplus \mathbb{C}d_m$ and $\mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \ldots \oplus \mathbb{C}c_m$.

Now we claim that $\widetilde{N}_\eta$ is also stable under $\tilde{\mathfrak{t}}^+$. For any $s = 1, 2, \ldots, m, \beta \in \mathbb{Z}^m +$, by Lemma 3.6, we have

$$\begin{aligned} x_{s\beta} w &= \sum_{\underline{k} \in \Omega_{r,\alpha}, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \eta(x_{s\beta}) \lambda_{\underline{k},\underline{p}} y^{\underline{k}} (\prod_{i=1}^{i=m} (d_i - \beta_i)^{p_i}) v \\ &\quad + \sum_{\underline{k} \in \Omega_{r,\alpha}, k_{s\beta} \neq 0, \underline{p} \in \mathbb{Z}^m_{\geq 0}} k_{s\beta} \lambda_{\underline{k},\underline{p}} y^{[\underline{k}]_{s\beta}} d^{\underline{p}} (\beta_1 a_1 + \ldots + \beta_m a_m) v. \end{aligned}$$

Assume that $x_{s\beta}w \notin \widetilde{N}_\eta^{(r,\alpha)}$, then it must be that

$$\sum_{\underline{k}\in\Omega_{r,\alpha},k_{s\beta}\neq 0,\underline{p}\in\mathbb{Z}_{\geq 0}^m} k_{s\beta}\lambda_{\underline{k},\underline{p}}y^{[\underline{k}]_{s\beta}}d^{\underline{p}}(\beta_1a_1+\beta_2a_2+\ldots+\beta_ma_m)v \neq 0.$$

So, $\beta_1a_1+\beta_2a_2+\ldots+\beta_ma_m \neq 0$ and this implies $[\underline{k}]_{s\beta} \neq \underline{e}_{r,\alpha}$ given that $\underline{k} \in \Omega_{r,\alpha}$.

Thus, $x_{s\beta}w \in \widetilde{N}_\eta^{(r,\alpha)}$ and this is a contradiction with our assumption. Since for any $s = 1,2,\ldots,n, \beta \in \mathbb{Z}^m+$, we have $x_{s\beta}w \in \widetilde{N}_\eta^{(r,\alpha)}$, so $\widetilde{N}_\eta^{(r,\alpha)}$ is stable under $\tilde{\mathfrak{t}}^+$. Thus, $\widetilde{N}_\eta^{(r,\alpha)}$ is a proper submodule of $\widetilde{N}_\eta$.

Now consider $\widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)} \cong span_{\mathbb{C}}\{y_{r\alpha}d^{\underline{p}}v|\underline{p} \in \mathbb{Z}_{\geq 0}^m\}$. Let A be a proper $U(\tilde{\mathfrak{t}})$-submodule of $span_{\mathbb{C}}\{y_{r\alpha}d^{\underline{p}}v|\underline{p} \in \mathbb{Z}_{\geq 0}^m\}$ and $0 \neq u \in A$. Then u has an unique expression

$$u = \sum_{\underline{p}\in\mathbb{Z}_{\geq 0}^m} \lambda_{\underline{p}}y_{r\alpha}d^{\underline{p}}v.$$

From Proposition 3.8, we have for some $s = 1,2,\ldots,n, \beta \in \mathbb{Z}^m+$,

$$(x_{s\beta}-\eta(x_{s\beta}))^l u = \lambda y_{r\alpha}v,$$

where $l = max\{|\underline{p}|\}$ and λ is a nonzero constant. Now, for any $\underline{p} \in \mathbb{Z}_{\geq 0}^m$,

$$\lambda^{-1}\prod_{i=1}^{i=m}(d_i+\alpha_i)^{p_i}(x_{s\beta}-\eta(x_{s\beta}))^l u = y_{r\alpha}d^{\underline{p}}v.$$

Thus, u generates $\widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)}$ and $A = \widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)}$. So $\widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)}$ is irreducible as a $U(\tilde{\mathfrak{t}})$-module and all the above proved that $\widetilde{N}_\eta^{(r,\alpha)}$ is a maximal proper $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{N}_\eta$. $\square$

Proposition 3.16 *Every maximal $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{N}_\eta$ is of the form $\widetilde{N}_\eta^{(r,\alpha)}$ for some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$.*

Proof. By Lemma 3.15, $\widetilde{N}_\eta^{(r,\alpha)}$ is a maximal $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{N}_\eta$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Assume that there exists a maximal submodule M of $\widetilde{N}_\eta$ such that $M \neq \widetilde{N}_\eta^{(r,\alpha)}$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Let $\widetilde{N}_{r,\alpha} = span_{\mathbb{C}}\{y_{r\alpha} d^{\underline{p}} v | \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$, then $\widetilde{N}_\eta = \widetilde{N}_{r,\alpha} \oplus \widetilde{N}_\eta^{(r,\alpha)}$ and we have

$$M = (M \cap \widetilde{N}_{r,\alpha}) \oplus (M \cap \widetilde{N}_\eta^{(r,\alpha)}).$$

Suppose that $M \cap \widetilde{N}_{r,\alpha} \neq 0$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Then for any $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$, we have

$$0 \neq u = \sum_{\underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{p}} y_{r\alpha} d^{\underline{p}} v \in M.$$

From the proof of Lemma 3.15, we have that $span_{\mathbb{C}}\{y_{r\alpha} d^{\underline{p}} v | \underline{p} \in \mathbb{Z}^m_{\geq 0}\} \in M$. Since $\{y_{r\alpha} d^{\underline{p}} v | \underline{p} \in \mathbb{Z}^m_{\geq 0}, r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+, \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0\}$ generates $\widetilde{N}_\eta$, we have that $\widetilde{N}_\eta \subset M$, which can not happen because we assumed that M is a proper maximal submodule of $\widetilde{N}_\eta$. So, $M \cap \widetilde{N}_{r,\alpha} \neq 0$ for some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Then we have $M = M \cap \widetilde{N}_\eta^{(r,\alpha)}$ and by the maximality of M we have $M = \widetilde{N}_\eta^{(r,\alpha)}$. But this is a contradiction as we assumed that $M \neq \widetilde{N}_\eta^{(r,\alpha)}$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Thus, we conclude that $M = \widetilde{N}_\eta^{(r,\alpha)}$ for some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. □

Proposition 3.17 *The space of Whittaker vectors (of type η) for $\widetilde{M_{\eta,\vec{a}}}/\widetilde{N}_\eta$ is one-dimensional.*

Proof.

Let $w \neq 0$ be a Whittaker module for $\widetilde{M_{\eta,\vec{a}}}/\widetilde{N}_\eta$, then $(x - \eta)^{\underline{k}} w \in \widetilde{N}_\eta$ for all $\underline{k} \in I$. We can write

$$w = \sum_{\underline{k} \in I \setminus \Omega, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v + \widetilde{N}_\eta,$$

where only finitely many $\lambda_{\underline{k},\underline{p}} \neq 0$. Let $l = max\{|\underline{p}| | \lambda_{\underline{k},\underline{p}} \neq 0\}$. If $l = 0$, then by Proposition 2.13, we have that $w = \lambda v + \widetilde{N}_\eta$ for some $\lambda \in \mathbb{C}$. If $l > 0$, then by the proof of Proposition 3.8, there are some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$, such that

$$(x_{r\alpha} - \eta(x_{r\alpha}))^l w = \sum_{\underline{k} \in I \setminus \Omega} \lambda_{\underline{k}} y^{\underline{k}} v + \widetilde{N}_\eta,$$

where there is at least one $\underline{k}$ such that $\lambda_{\underline{k}} \neq 0$ and this is the same as the case that $l = 0$. So we always have $w = \lambda v + \widetilde{N}_\eta$ for some $\lambda \in \mathbb{C}$.

□

Theorem 3.18 *$\widetilde{N}_\eta$ is the unique maximal submodule of $\widetilde{M_{\eta,\vec{a}}}$.*

Proof. Let K be a maximal $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{M_{\eta,\vec{a}}}$ and suppose that $K \neq \widetilde{N}_\eta$. Then $K \cap \widetilde{N}_\eta$ is a maximal $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{N}_\eta$. By Proposition 3.16, we have $K \cap \widetilde{N}_\eta = \widetilde{N}_\eta^{(r,\alpha)}$ for some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$ such that $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m = 0$. Hence $\widetilde{N}_\eta^{(r,\alpha)} \subset K$. Since $K/(K \cap \widetilde{N}_\eta) \cong \widetilde{M_{\eta,\vec{a}}}/\widetilde{N}_\eta$ and $\widetilde{M_{\eta,\vec{a}}}/\widetilde{N}_\eta$ has a Whittaker vector, there exists $w \in K, w \notin \widetilde{N}_\eta$, such that $w + (K \cap \widetilde{N}_\eta)$ is a Whittaker vector in $K/(K \cap \widetilde{N}_\eta)$. Thus, by Proposition 3.9, we may assume that

$$w = v + \sum_{\underline{k} \in \Omega, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v$$

after by multiplying a scalar. Then $0 \neq y_{r\alpha} w = y_{r\alpha} v + \sum_{\underline{k} \in \Omega, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y_{r\alpha} y^{\underline{k}} d^{\underline{p}} v \in K \cap \widetilde{N}_\eta = \widetilde{N}_\eta^{(r,\alpha)}$. Since $\sum_{\underline{k} \in \Omega, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y_{r\alpha} y^{\underline{k}} d^{\underline{p}} v \in \widetilde{N}_\eta^{(r,\alpha)}$, we have $y_{r\alpha} v \in \widetilde{N}_\eta^{(r,\alpha)}$, which is a contradiction with the definition of $\widetilde{N}_\eta^{(r,\alpha)}$. Hence $K = \widetilde{N}_\eta$ and $\widetilde{N}_\eta$ is the unique maximal submodule of $\widetilde{M_{\eta,\vec{a}}}$. □

3.4 Whittaker modules for $\tilde{\mathfrak{t}}$ with $a_1 = a_2 = \cdots = a_m = 0$

Proposition 3.19 *$\widetilde{N}_\eta = span_{\mathbb{C}}\{y^{\underline{k}} d^{\underline{p}} v | \underline{k} \neq \underline{0}, \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$ is a maximal submodule of $\widetilde{M_{\eta,\vec{0}}}$.*

Proof. First we show that $\widetilde{N}_\eta$ is a proper submodule of $\widetilde{M_{\eta,\vec{0}}}$. For any $w \in \widetilde{N}_\eta$, w

has a unique expression

$$w = \sum_{\underline{k}\neq\underline{0},\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v,$$

where $\lambda_{\underline{k},\underline{p}} \neq 0$ for only finitely many $\underline{k} \neq \underline{0}, \underline{p} \in \mathbb{Z}^m_{\geq 0}$. Obviously, $\widetilde{N}_\eta$ is stable under $\tilde{\mathfrak{t}}^-$ since for any $\underline{k'} \in I$, we have

$$y^{\underline{k'}} w = \sum_{\underline{k}\neq\underline{0},\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}+\underline{k'}} d^{\underline{p}} v \in \widetilde{N}_\eta.$$

For any $i = 1, 2, \ldots, m$,

$$\begin{aligned} c_i w &= \sum_{\underline{k}\neq\underline{0},\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} c_i v \\ &= \sum_{\underline{k}\neq\underline{0},\underline{p}\in\mathbb{Z}^m_{\geq 0}} a_i \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v \in \widetilde{N}_\eta. \end{aligned}$$

So $\widetilde{N}_\eta$ is stable under $\mathbb{C}c_1 \oplus \mathbb{C}c_2 \ldots \oplus \mathbb{C}c_m$. Now, for any $\underline{p'} \in \mathbb{Z}^m_{\geq 0}$, by Lemma 3.6, we have

$$d^{\underline{p'}} w = \sum_{\underline{k}\neq\underline{0},\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} (\prod_{i=0}^{i=m} (d_i - ||i\underline{k}||^{p'_i})) d^{\underline{p}} v \in \widetilde{N}_\eta.$$

So $\widetilde{N}_\eta$ is stable under $\mathbb{C}d_1 \oplus \mathbb{C}d_2 \ldots \oplus \mathbb{C}d_m$.

Now we claim that $\widetilde{N}_\eta$ is also stable under $\tilde{\mathfrak{t}}^+$. We can rewrite w as

$$w = \sum_{\underline{k}\neq\underline{0},k_{r\alpha}\neq 0,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v + \sum_{\underline{k}\neq\underline{0},k_{r\alpha}=0,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v.$$

So we have

$$\begin{aligned} x_{r\alpha}w &= \sum_{\underline{k}\neq\underline{0},k_{r\alpha}=0,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v + \sum_{\underline{k}\neq\underline{0},k_{r\alpha}\neq 0,\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v \\ &\quad + \sum_{\underline{k}\neq\underline{0},k_{r\alpha}\neq 0,\underline{p}\in\mathbb{Z}^m_{\geq 0}} k_{r\alpha}\lambda_{\underline{k},\underline{p}} y^{[\underline{k}]_{r\alpha}} d^{\underline{p}}(\alpha_1 a_1 + \ldots + \alpha_m a_m)v \\ &= \sum_{\underline{k}\neq\underline{0},\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} x_{r\alpha} d^{\underline{p}} v \\ &= \sum_{\underline{k}\neq\underline{0},\underline{p}\in\mathbb{Z}^m_{\geq 0}} \eta(x_{r\alpha})\lambda_{\underline{k},\underline{p}} y^{\underline{k}} (\prod_{i=1}^{i=m}(d_i-\alpha_i)^{p_i})v \in \widetilde{N}_\eta. \end{aligned}$$

Since for any $r = 1, 2, \ldots, n, \alpha =\in \mathbb{Z}^m+$, we have $x_{r\alpha}w \in \widetilde{N}_\eta$, so $\widetilde{N}_\eta$ is stable under $\tilde{\mathfrak{t}}^+$. Thus, $\widetilde{N}_\eta$ is a proper submodule of $\widetilde{M}_{\eta,\vec{0}}$.

Now consider $\widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta \cong span_{\mathbb{C}}\{d^{\underline{p}}v|\underline{p} \in \mathbb{Z}^m_{\geq 0}\}$. For any $0 \neq w \in span_{\mathbb{C}}\{d^{\underline{p}}v|\underline{p} \in \mathbb{Z}^m_{\geq 0}\}$, w has an unique expression

$$w = \sum_{\underline{p}\in\mathbb{Z}^m_{\geq 0}} \lambda_{\underline{p}} d^{\underline{p}} v,$$

where only finitely many $\lambda_{\underline{p}} \neq 0$. Let $l = max\{|\underline{p}||\lambda_{\underline{p}} \neq 0\}$. If $l = 0$, then $w = \lambda v$ for some nonzero constant $\lambda \in \mathbb{C}$. If $l > 0$, then from the proof of Proposition 3.8, there is some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}+$ such that

$$(x_{r\alpha} - \eta(x_{r\alpha}))w = \lambda v,$$

for some nonzero constant $\lambda \in \mathbb{C}$. We always have the fact w generates $\widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta$ and so $\widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta$ is irreducible as a $U(\tilde{\mathfrak{t}})$-module. Thus $\widetilde{N}_\eta$ is a maximal submodule of $\widetilde{M}_{\eta,\vec{0}}$. $\square$

Lemma 3.20 *$\widetilde{N}_\eta^{(r,\alpha)} = span_{\mathbb{C}}\{y^{\underline{k}} d^{\underline{p}} v | \underline{k} \in I \setminus \{\underline{0}, \underline{e}_{r,\alpha}\}, \underline{p} \in \mathbb{Z}_{\geq 0}^m\}$ is a maximal $U(\tilde{\mathfrak{t}})$-module of $\widetilde{N}_\eta$.*

Proof. Since $c_1, c_2, \ldots, c_m$ acts by zero on v, $\widetilde{N}_\eta^{(r,\alpha)}$ is stable under $U(\tilde{\mathfrak{t}})$. Thus, $\widetilde{N}_\eta^{(r,\alpha)}$ is a proper submodule of $\widetilde{N}_\eta$.

Now consider $\widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)} \cong span_{\mathbb{C}}\{y_{r\alpha} d^{\underline{p}} v | \underline{p} \in \mathbb{Z}_{\geq 0}^m\}$. Let A be a proper $U(\tilde{\mathfrak{t}})$-submodule of $span_{\mathbb{C}}\{y_{r\alpha} d^{\underline{p}} v | \underline{p} \in \mathbb{Z}_{\geq 0}^m\}$ and $u \in A$. Then u has an unique expression

$$u = \sum_{\underline{p} \in \mathbb{Z}_{\geq 0}^m} \lambda_{\underline{p}} y_{r\alpha} d^{\underline{p}} v,$$

where only finitely many $\lambda_{\underline{p}} \neq 0$ for $\underline{p} \in \mathbb{Z}_{\geq 0}^m$. From Proposition 3.8, we have for some $s = 1, 2, \ldots, n, \beta \in \mathbb{Z}^m+$,

$$(x_{s\beta} - \eta(x_{s\beta}))^l u = \lambda y_{r\alpha} v,$$

where $l = max\{|\underline{p}|\}$ and λ is a nonzero constant. Now, for any $\underline{p} \in \mathbb{Z}_{\geq 0}^m$,

$$\lambda^{-1} \prod_{i=1}^{i=m} (d_i + \alpha_i)^{p_i} (x_{s\beta} - \eta(x_{s\beta}))^l u = y_{r\alpha} d^{\underline{p}} v.$$

Thus, u generates $\widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)}$ and $A = \widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)}$. So $\widetilde{N}_\eta/\widetilde{N}_\eta^{(r,\alpha)}$ is irreducible as an $U(\tilde{\mathfrak{t}})$-module and all the above proved that $\widetilde{N}_\eta^{(r,\alpha)}$ is a maximal proper $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{N}_\eta$. $\square$

Remark 3.21 *It is easy to see that* $N = span_{\mathbb{C}}\{y^{\underline{k}}d^{\underline{p}}v | \underline{k} \in I \setminus \{\underline{0}, \underline{e}_{r,\alpha}, \underline{e}_{s,\beta}\}, \underline{p} \in \mathbb{Z}^m_{\geq 0}\} = \widetilde{N}_\eta^{(r,\alpha)} \cap \widetilde{N}_\eta^{(s,\beta)}$ *for* $(r,\alpha) \neq (s,\beta)$ *is a proper* $U(\tilde{\mathfrak{t}})$*-submodule of* $\widetilde{N}_\eta^{(r,\alpha)}$*, so* $\widetilde{N}_\eta^{(r,\alpha)}$ *is not irreducible.*

Proposition 3.22 *Every maximal* $U(\tilde{\mathfrak{t}})$*-submodule of* $\widetilde{N}_\eta$ *is of the form* $\widetilde{N}_\eta^{(r,\alpha)}$ *for some* $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$.

Proof. By Lemma 3.20, $\widetilde{N}_\eta^{(r,\alpha)}$ is a maximal $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{N}_\eta$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$. Assume that there exists a maximal submodule M of $\widetilde{N}_\eta$ such that $M \neq \widetilde{N}_\eta^{(r,\alpha)}$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$. Let $\widetilde{N}_{r,\alpha} = span_{\mathbb{C}}\{y_{r\alpha}d^{\underline{p}}v | \underline{p} \in \mathbb{Z}^m_{\geq 0}\}$, then $\widetilde{N}_\eta = \widetilde{N}_{r,\alpha} \oplus \widetilde{N}_\eta^{(r,\alpha)}$ and we have

$$M = (M \cap \widetilde{N}_{r,\alpha}) \oplus (M \cap \widetilde{N}_\eta^{(r,\alpha)}).$$

Suppose that $M \cap \widetilde{N}_{r,\alpha} \neq 0$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$. Then for any $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$, we have

$$0 \neq u = \sum_{\underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{p}} y_{r\alpha} d^{\underline{p}} v \in M.$$

From the proof of Lemma 3.20, we have that $span_{\mathbb{C}}\{y_{r\alpha}d^{\underline{p}}v|\underline{p} \in \mathbb{Z}^m_{\geq 0}\} \in M$. Since $\{y_{r\alpha}d^{\underline{p}}v|\underline{p} \in \mathbb{Z}^m_{\geq 0}, r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+\}$ generates $\widetilde{N}_\eta$, we have that $\widetilde{N}_\eta \subset M$, which can not happen because we assumed that M is a proper maximal submodule of $\widetilde{N}_\eta$. So, $M \cap \widetilde{N}_{r,\alpha} \neq 0$ for some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$. Then we have $M = M \cap \widetilde{N}^{(r,\alpha)}_\eta$ and by the maximality of M we have $M = \widetilde{N}^{(r,\alpha)}_\eta$. But this is a contradiction as we assumed that $M \neq \widetilde{N}^{(r,\alpha)}_\eta$ for all $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$. Thus, we conclude that $M = \widetilde{N}^{(r,\alpha)}_\eta$ for some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$.

□

Proposition 3.23 *The space of Whittaker vectors (of type η) for $\widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta$ is one-dimensional.*

Proof.

Let $w \neq 0$ be a Whittaker module for $\widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta$, then $(x-\eta)^{\underline{k}}w \in \widetilde{N}_\eta$ for all $\underline{k} \in I$. We can write

$$w = \sum_{\underline{k} \neq \underline{0}, \underline{e}_{r,\alpha}, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v + \widetilde{N}_\eta,$$

where only finitely many $\lambda_{\underline{k},\underline{p}} \neq 0$. Let $l = max\{|\underline{p}||\lambda_{\underline{k},\underline{p}} \neq 0\}$. If $l = 0$, then by Proposition 2.13, we have that $w = \lambda v + \widetilde{N}_\eta$ for some $\lambda \in \mathbb{C}$. If $l > 0$, then by the proof of Proposition 3.8, there are some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$, such that

$$(x_{r\alpha} - \eta(x_{r\alpha}))^l w = \sum_{\underline{k} \neq \underline{0}, \underline{e}_{r,\alpha}} \lambda_{\underline{k}} y^{\underline{k}} v + \widetilde{N}_\eta,$$

where there is at least one $\underline{k}$ such that $\lambda_{\underline{k}} \neq 0$ and this is the same as the case that $l = 0$. We always have $w = \lambda v + \widetilde{N}_\eta$ for some $\lambda \in \mathbb{C}$. Thus, the space of Whittaker vectors (of type η) for $\widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta$ is one-dimensional.

□

Theorem 3.24 *$\widetilde{N}_\eta$ is the unique maximal submodule of $\widetilde{M}_{\eta,\vec{0}}$.*

Proof. Let K be a maximal $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{M}_{\eta,\vec{0}}$ and suppose that $K \neq \widetilde{N}_\eta$. Then $K \cap \widetilde{N}_\eta$ is a maximal $U(\tilde{\mathfrak{t}})$-submodule of $\widetilde{N}_\eta$. By Proposition 3.22, we have $K \cap \widetilde{N}_\eta = \widetilde{N}_\eta^{(r,\alpha)}$ for some $r = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^m+$. Hence $\widetilde{N}_\eta^{(r,\alpha)} \subset K$. Since $K/(K \cap \widetilde{N}_\eta) \cong \widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta$ and $\widetilde{M}_{\eta,\vec{0}}/\widetilde{N}_\eta$ has a Whittaker vector, there exists $w \in K, w \notin \widetilde{N}_\eta$, such that $w + (K \cap \widetilde{N}_\eta)$ is a Whittaker vector in $K/(K \cap \widetilde{N}_\eta)$. Thus, by Proposition 3.9, we may assume that

$$w = v + \sum_{\underline{k} \neq \underline{0}, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y^{\underline{k}} d^{\underline{p}} v$$

after by multiplying a scalar. Then $0 \neq y_{r\alpha} w = y_{r\alpha} v + \sum_{\underline{k} \neq \underline{0}, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y_{r\alpha} y^{\underline{k}} d^{\underline{p}} v \in K \cap \widetilde{N}_\eta = \widetilde{N}_\eta^{(r,\alpha)}$. Since $\sum_{\underline{k} \neq \underline{0}, \underline{p} \in \mathbb{Z}^m_{\geq 0}} \lambda_{\underline{k},\underline{p}} y_{r\alpha} y^{\underline{k}} d^{\underline{p}} v \in \widetilde{N}_\eta^{(r,\alpha)}$, we have $y_{r\alpha} v \in \widetilde{N}_\eta^{(r,\alpha)}$, which is a contradiction with the definition of $\widetilde{N}_\eta^{(r,\alpha)}$. Hence $K = \widetilde{N}_\eta$ and $\widetilde{N}_\eta$ is the unique maximal submodule of $\widetilde{M}_{\eta,\vec{0}}$. □

4 Imaginary Whittaker modules for non-twisted extended affine Lie algebras

4.1 Imaginary Whittaker modules

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank n over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Δ the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, $\{\varphi_1, \varphi_2, ..., \varphi_n\}$ a set of simple roots for Δ. Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\varphi\in\Delta} \mathfrak{g}_\varphi$. Set $\mathfrak{n}^\pm = \bigoplus_{\varphi\in\Delta^+} \mathfrak{g}_{\pm\varphi}$, where Δ^+ is the set of positive roots cooresponding to Δ. Denote L as the Laurent polynomial ring generated by m commutative variables $t_1, t_2, \ldots, t_m$, which is $L = \mathbb{C}[t_1^{\pm1}, t_2^{\pm1}, ..., t_m^{\pm1}]$. For $\alpha \in \mathbb{Z}^m$, we denote $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_m^{\alpha_m}$ in L. Let $\bar{\mathfrak{g}}$ be the non-twisted extended affine Lie algebra associated with $\mathfrak{g}$, then

$$\bar{\mathfrak{g}} = (\mathfrak{g} \otimes L) \oplus \mathbb{C}c_1 \oplus ... \oplus \mathbb{C}c_m \oplus \mathbb{C}d_1 \oplus ... \oplus \mathbb{C}d_m.$$

The Lie bracket is given by

1. $[c_i, \bar{\mathfrak{g}}] = 0$, for all $i = 1, 2, ..., m$,

2. $[d_i, d_j] = 0$, for all $i, j = 1, 2, ..., m$,

3. $[d_i, x \otimes t^\alpha] = \alpha_i x \otimes t^\alpha$, for all $\alpha \in \mathbb{Z}^m, x \in \mathfrak{g}, i = 1, 2, ..., m$,

4. $[x \otimes t^\alpha, y \otimes t^\beta] = [x, y] \otimes t^{\alpha+\beta} + \delta_{\alpha+\beta,0} K(x, y)(\alpha_1 c_1 + \alpha_2 c_2 + ... + \alpha_m c_m)$, for all $\alpha, \beta \in \mathbb{Z}^m, x, y \in \mathfrak{g}$, where K is the Killing form on $\mathfrak{g}$.

Let $\{\theta_1, \theta_2, ..., \theta_n\}$ be an orthonomal basis of $\mathfrak{h}$ such that $K(\theta_i, \theta_j) = \delta_{i,j}$. Set $x_{r\alpha} = \theta_r \otimes t^\alpha, y_{r\alpha} = \theta_r \otimes t^{-\alpha}$ for $r = 1, 2, ...n, \alpha \in \mathbb{Z}^m+$. Let $\mathfrak{t} = \oplus_{\alpha \in \mathbb{Z}^m} \mathfrak{t}_\alpha$, where

$$\begin{cases} \mathfrak{t}_\alpha = \mathfrak{h} \otimes t^\alpha, & \alpha \neq 0, \\ \mathfrak{t}_\alpha = \mathbb{C}c_1 \oplus ... \oplus \mathbb{C}c_m, & \alpha = 0. \end{cases} \tag{4.1}$$

Thus $\mathfrak{t}$ is a generalized Heisenberg subalgebra of $\bar{\mathfrak{g}}$, $\{x_{r\alpha}\}_{1 \leq r \leq n}$ is a basis of $\mathfrak{t}_\alpha$, $\{y_{r\alpha}\}_{1 \leq r \leq n}$ is a basis of $\mathfrak{t}_{-\alpha}$ for all $\alpha \in \mathbb{Z}^m+$, such that

$$\begin{aligned} [c_i, x_{r\alpha}] &= [c_i, y_{r\alpha}] = 0, \\ [x_{r\alpha} x_{s\beta}] &= [y_{r\alpha}, y_{s\beta}] = 0, \\ [x_{r\alpha}, y_{s\beta}] &= \delta_{rs}\delta_{\alpha\beta}(\alpha_1 c_1 + \alpha_2 c_2 + \ldots + \alpha_m c_m). \end{aligned}$$

for all $1 \leq r, s \leq n$, $1 \leq i \leq m$, $\alpha, \beta \in \mathbb{Z}^m+$.

Set $\mathfrak{t}^\pm = \oplus_{\alpha \in \mathbb{Z}^m+} \mathfrak{t}_{\pm\alpha}$, $\tilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{C}d_1 \oplus ... \oplus \mathbb{C}d_m$. The subalgebras $\mathfrak{t}, \tilde{\mathfrak{t}}$ motivated the definitions in the previous chapters, and so we may apply all the results on Whittaker modules to $\mathfrak{t}$ and $\bar{\mathfrak{t}}$ from chapters 2 and chapter 3.

Now, let $\bar{\mathfrak{n}}^{\pm} = \mathfrak{n}^{\pm} \otimes L$, then the extended affine Lie algebra $\bar{\mathfrak{g}}$ has the following decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{n}}^{-} \oplus (\tilde{\mathfrak{t}} \oplus \mathfrak{h}) \oplus \bar{\mathfrak{n}}^{+}.$$

The subalgebra $\mathfrak{p} = (\tilde{\mathfrak{t}} \oplus \mathfrak{h}) \oplus \bar{\mathfrak{n}}^{+}$ is a parabolic subalgebra of $\bar{\mathfrak{g}}$. Moreover, $[\tilde{\mathfrak{t}}, \mathfrak{h}] = 0$ and $\bar{\mathfrak{n}}^{+}$ is an ideal of $\mathfrak{p}$.

Assume that $\lambda \in (\mathfrak{h} \oplus \mathbb{C}c_1 \oplus ... \oplus \mathbb{C}c_m)^*$ and $\eta \in \Gamma$. Let $\bar{L}_{\eta,\lambda}$ be the unique (up to isomorphism) irreducible Whittaker $\tilde{\mathfrak{t}}$-module of type η and levels $\lambda(c_1), \lambda(c_2), \ldots, \lambda(c_m)$. Denote $\vec{a} = (\lambda(c_1), \lambda(c_2), \ldots, \lambda(c_m))$, then we have:

1. $\bar{L}_{\eta,\lambda} = \widetilde{M}_{\eta,\vec{a}}$, if $\lambda(c_1), \lambda(c_2), ..., \lambda(c_m)$ are $\mathbb{Z}$-independent,

2. $\bar{L}_{\eta,\lambda} = \widetilde{M}_{\eta,\vec{a}}/\widetilde{N}_{\eta}$, if $\lambda(c_1), \lambda(c_2), ..., \lambda(c_m)$ are $\mathbb{Z}$-dependent,

3. $\bar{L}_{\eta,\lambda} = \widetilde{M}_{\eta,\vec{a}}/\widetilde{N}_{\eta}$, if $\lambda(c_1) = \lambda(c_2) = ... = \lambda(c_m) = 0$.

Let $\tilde{v} \in \bar{L}_{\eta,\lambda}$ be a Whittaker vector of type η. Define a $U(\mathfrak{p})$-module structure on $\bar{L}_{\eta,\lambda}$ by letting

1. $hw = \lambda(h)w$ for all $h \in \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m, w \in \bar{L}_{\eta,\lambda}$,

2. $\bar{\mathfrak{n}}^{+}w = 0$ for all $w \in \bar{L}_{\eta,\lambda}$.

Set

$$V_{\eta,\lambda} = U(\bar{\mathfrak{g}}) \otimes_{U(\mathfrak{p})} \bar{L}_{\eta,\lambda}, v = 1 \otimes \bar{v}.$$

Define an action of $U(\bar{\mathfrak{g}})$ on $V_{\eta,\lambda}$ by multiplication on the left on the $U(\bar{\mathfrak{g}})$ factor. We will say that $V_{\eta,\lambda}$ is an imaginary Whittaker module of type (η, λ) for $\bar{\mathfrak{g}}$.

Let Q^+ be be the non-negative integral linear span of $\varphi_1, \varphi_2, ..., \varphi_n$ and extend an element $\mu \in (\mathfrak{h})^*$ to an element of $(\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m)^*$ by letting $\mu(c_1) = \mu(c_2) = ... = \mu(c_m) = 0$. For $\phi \in Q^+$, set

$$U(\bar{\mathfrak{n}}^-)^{-\phi} = \{u \in U(\bar{\mathfrak{n}}^-) | [h, u] = -\phi(h)u, h \in \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m\}.$$

For $\mu \in (\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m)^*$, set

$$V^{\mu}_{\eta,\lambda} = \{w \in V_{\eta,\lambda} | hw = \mu(h)w, h \in \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m\}.$$

Proposition 4.1

1. As $U(\bar{\mathfrak{n}}^-)$-modules, $V_{\eta,\lambda} \cong U(\bar{\mathfrak{n}}^-) \otimes_{\mathbb{C}} \bar{L}_{\eta,\lambda}$. Moreover, $V_{\eta,\lambda}$ is free as a $U(\bar{\mathfrak{n}}^-)$-module.

2. The map $w \to 1 \otimes w$ defines a $\mathfrak{p}$-isomorphism of $\bar{L}_{\eta,\lambda}$ onto the $\mathfrak{p}$-submodule $U(\mathfrak{p})v$ of $V_{\eta,\lambda}$.

3. $V_{\eta,\lambda} = \oplus_{\phi \in Q^+} V^{\lambda-\phi}_{\eta,\lambda}$, and $V^{\lambda-\phi}_{\eta,\lambda} \cong U(\bar{\mathfrak{n}}^-)^{-\phi} \otimes_{\mathbb{C}} \bar{L}_{\eta,\lambda}$ as modules for $\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m$. In particular, $V^{\lambda}_{\eta,\lambda} \cong \bar{L}_{\eta,\lambda}$.

Proof.

1. Since $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}^- \oplus \mathfrak{p}$, the PBW theorem implies that $U(\bar{\mathfrak{g}}) \cong U(\bar{\mathfrak{n}}^-) \otimes_{\mathbb{C}} U(\mathfrak{p})$. So $V_{\eta,\lambda} = U(\bar{\mathfrak{g}}) \otimes_{U(\mathfrak{p})} \bar{L}_{\eta,\lambda} \cong U(\bar{\mathfrak{n}}^-) \otimes_{\mathbb{C}} \bar{L}_{\eta,\lambda}$ as vector space over $\mathbb{C}$. Thus the map $f : U(\bar{\mathfrak{n}}^-) \otimes_{\mathbb{C}} \bar{L}_{\eta,\lambda} \to V_{\eta,\lambda}$ defined by $(u, w) \mapsto uw$ is an isomorphism of $U(\bar{\mathfrak{n}}^-)$-modules. It follows by Corollary 5.13 [Hun] that $V_{\eta,\lambda}$ is free as a $U(\bar{\mathfrak{n}}^-)$-module.

2. This part is evident from the definitions.

3. First, claim that $U(\bar{\mathfrak{n}}^-) = \oplus_{\phi \in Q^+} U(\bar{\mathfrak{n}}^-)^{-\phi}$. For every $(u, w) \in U(\bar{\mathfrak{n}}^-)^{-\phi} \otimes_{\mathbb{C}} \bar{L}_{\eta,\lambda}$, since $u \in U(\bar{\mathfrak{n}}^-)^{-\phi}, w \in \bar{L}_{\eta,\lambda}$, we have $[h, u] = -\phi(h)u \Leftrightarrow hu - uh = -\phi(h)u \Leftrightarrow huw - uhw = -\phi(h)uw \Leftrightarrow huw - u\lambda(h)w = -\phi(h)uw \Leftrightarrow h(uw) = (\lambda - \phi)(h)uw \Leftrightarrow uw \in V_{\eta,\lambda}^{\lambda-\phi}$. So the isomorphism f in (1) is an isomorphism between $U(\bar{\mathfrak{n}}^-)^{-\phi} \otimes_{\mathbb{C}} \bar{L}_{\eta,\lambda}$ and $V_{\eta,\lambda}^{\lambda-\phi}$ for every $\phi \in Q^+$. In particular, if $\phi = 0$, then $U(\bar{\mathfrak{n}}^-) = \mathbb{C}$ and $V_{\eta,\lambda}^{\lambda} \cong \bar{L}_{\eta,\lambda}$.

□

Proposition 4.2 *Every $U(\bar{\mathfrak{g}})$-submodule M of $V_{\eta,\lambda}$ has a decomposition $M = \oplus_{\phi \in Q^+} M \cap V_{\eta,\lambda}^{\lambda-\phi}$ into weight spaces relative to $\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m$.*

Proof. Since $V_{\eta,\lambda} = \oplus_{\phi \in Q^+} V_{\eta,\lambda}^{\lambda-\phi} \Rightarrow M = \oplus_{\phi \in Q^+} M \cap V_{\eta,\lambda}^{\lambda-\phi}$. □

Proposition 4.3 *Assume* $\lambda, \lambda' \in (\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m)^*$, *Let* $\eta' : U(\bar{\mathfrak{t}}^+) \to \mathbb{C}$ *be a algebra homomorphism,* $\lambda'(c_1), \lambda'(c_2), \ldots, \lambda'(c_m)$ *are* $\mathbb{Z}$*-independent and* $\eta' \in \Gamma$. *Then* $V_{\eta,\lambda} \cong V_{\eta',\lambda'}$ *as* $U(\mathfrak{g})$*-modules if and only if* $\eta = \eta'$ *and* $\lambda = \lambda'$.

Proof. We only need to prove that if $V_{\eta,\lambda} \cong V_{\eta',\lambda'}$, then $\eta = \eta'$ and $\lambda = \lambda'$ because the other direction is obvious. Let $f : V_{\eta,\lambda} \to V_{\eta',\lambda'}$ be an isomorphism of $U(\mathfrak{g})$ modules. Let $D(\lambda)$ (resp $D(\lambda')$) be the set of weights of $V_{\eta,\lambda}$ (resp. $V_{\eta',\lambda'}$) for the action of $\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m$, then $\lambda \in D(\lambda')$. Hence there exists $\phi \in Q^+$ such that $\lambda = \lambda' - \phi$. Similarly, $\lambda' = \lambda - \phi'$ for some $\phi' \in Q^+$, which implies that $\phi = -\phi'$. Thus, $\phi = \phi' = 0$ since $\phi, \phi' \in Q^+$. Therefore $\lambda = \lambda'$ and f restricted on $V_{\eta,\lambda}^{\lambda}$ is an isomorphism of $U(\tilde{\mathfrak{t}})$-modules from $V_{\eta,\lambda}^{\lambda}$ to $V_{\eta',\lambda}^{\lambda}$. Consequently, $\bar{L}_{\eta,\lambda} \cong \bar{L}_{\eta',\lambda}$. Choose $v \in \bar{L}_{\eta,\lambda}$ as a Whittaker vector, then

$$(u - \eta(u))f(v) = f((u - \eta(u))v) = f(0) = 0$$

for all $u \in U(\tilde{\mathfrak{t}}^+)$, which implies that $f(v)$ is a Whittaker vector of type η in $\bar{L}_{\eta',\lambda}$. By Proposition 3.9, it follows that $\eta = \eta'$. □

4.2 An irreducibility criterion

For the rest of this section, we will focus on imaginary Whittaker modules with $\mathbb{Z}$-independent level for extended affine Lie algebra $\bar{\mathfrak{g}}$ and show that they irreducible.

Fix $\eta \in \Gamma$, let $\mathfrak{m} = \bar{\mathfrak{n}}^- \oplus \tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus ... \oplus \mathbb{C}d_m$. Note that $\bar{\mathfrak{n}}^-$ is an ideal in $\mathfrak{m}$.

Proposition 4.4 *Let $\lambda \in (\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus ... \oplus \mathbb{C}c_m)^*, \lambda(c_1), \lambda(c_2), \ldots, \lambda(c_m)$ be $\mathbb{Z}$-independent, then $V_{\eta,\lambda}$ is torsionfree as left $U(\mathfrak{m})$-module.*

Proof. Denote $\vec{a} = (\lambda(c_1), \lambda(c_2), \ldots, \lambda(c_m))$. Since $\lambda(c_1), \lambda(c_2), \ldots, \lambda(c_m)$ are $\mathbb{Z}$-independent, we have $\bar{L}_{\eta,\lambda} = \widetilde{M}_{\eta,\vec{a}}$. Let $\{\omega_s\}_{s\in S}$ be a $\mathbb{C}$-basis of $U(\bar{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$, then $\{\omega_s\}_{s\in S}$ is also a $\mathbb{C}$-basis of $\bar{L}_{\eta,\lambda}$. By the PBW theorem $U(\mathfrak{m}) \cong U(\bar{\mathfrak{n}}^-) \otimes_{\mathbb{C}} U(\bar{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$. Hence $U(\mathfrak{m})$ is a free left $U(\bar{\mathfrak{n}}^-)$-module with basis $\{\omega_s\}_{s\in S}$. Moreover, by Proposition 4.1, $\{\omega_s v\}_{s\in S}$ is a basis of $V_{\eta,\lambda}$ as a free $U(\bar{\mathfrak{n}}^-)$-module. The map $f : V_{\eta,\lambda} \to U(\mathfrak{m})$ defined by $u \otimes wv \mapsto uw, u \in U(\bar{\mathfrak{n}}^-), w \in U(\bar{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$ is obviously surjective. Let $u = \sum_s y_s\omega_s v \in V_{\eta,\lambda}$, where $y_s \in U(\bar{\mathfrak{n}}^-)$. Then $f(u) = \sum_s y_s\omega_s = 0$ would imply that $y_s = 0$ for all s, so $u = 0$. Hence f is an isomorphism of vector space over $\mathbb{C}$. Suppose that $y \in \mathfrak{m}, u \in U(\bar{\mathfrak{n}}^-), w \in U(\bar{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$. Since $\bar{\mathfrak{n}}^-$ is an ideal in $\mathfrak{m}$, we have $[y, u] \in u(\bar{\mathfrak{n}}^-)$. Therefore $f([y, u] \otimes w) = [y, u]w$. Moreover, since $\mathfrak{m} = \bar{\mathfrak{n}}^- \oplus \tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m$, there must exist unique $u_1 \in \bar{\mathfrak{n}}^-$ and $u_2 \in \tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m$ such that $y = u_1 + u_2$. Hence $f(y(u \otimes wv)) = f(yu \otimes wv) = f(uy \otimes wv) + f([y, u] \otimes wv) = f(uu_1 \otimes wv) + f(uu_2 \otimes wv) + [y, u]w = uu_1w + f(u \otimes u_2wv) + [y, u]w = uu_1w + uu_2w + [y, u]w = uyw + [y, u]w = y(uw)$. Hence f is an isomorphism of $U(\mathfrak{m})$-modules. Since $U(\mathfrak{m})$ is a domain, it follows that $V_{\eta,\lambda}$ is torsion-free as a $U(\mathfrak{m})$-module. $\square$

We begin by establishing some notation. For any $\mu = \sum_{i=1}^{n} k_i\varphi_i \in Q^+$, let $ht(\mu) = \sum_{i=1}^{n} k_i$. If $\gamma, \omega \in \Delta^+, \gamma = \sum_{i=1}^{n} \kappa_i\varphi_i, \omega = \sum_{i=1}^{n} \nu_i\varphi_i$, then we define $\gamma \leq \omega$ if and only if $(\kappa_1, \kappa_2, \ldots, \kappa_n,) \leq (\nu_1, \nu_2, \ldots, \nu_n,)$ in the lexicographic order. Thus, $\leq$ is a total order on Q^+ which satisfies the following property: if $\gamma, \omega \in \Delta^+, \gamma \leq \omega$ and $\omega - \gamma \in \Delta$, then $\omega - \gamma \in \Delta^+$. Fix a Chevalley basis $\{e_\gamma | \gamma \in \Delta\} \cup \{h_i | 1 \leq i \leq n\}$ for $\mathfrak{g}$. For $\gamma \in \Delta, \alpha \in \mathbb{Z}^m+$, we define element $e_{\gamma+\alpha}$ as follows

$$e_{\gamma+\alpha} = e_\gamma \otimes t^\alpha.$$

Since $\bar{\mathfrak{n}}^- = \mathfrak{n}^- \otimes L$, the set

$$B = \{e_{-\gamma+\alpha} | \gamma \in \Delta^+, \alpha \in \mathbb{Z}^m+\}$$

is a basis of $\bar{\mathfrak{n}}^-$.

If $\gamma, \omega \in \Delta^+, \alpha, \beta \in \mathbb{Z}^m+$, define $e_{-\gamma+\alpha} < e_{-\omega+\beta}$ if $\gamma < \omega$ or $\gamma = \omega$ and $\alpha \leq \beta$. Then $\leq$ is a total order on B. Let $l = |\Delta^+|$ and let $\gamma_1 < \gamma_2 < \cdots < \gamma_l$ be an ordered listing of the roots in Δ^+ using the total order above. For $1 \leq i \leq l$, set

$$E_i^{\kappa_i} = \prod_{\alpha \in \mathbb{Z}^m+} e_{-\gamma_i+\alpha}^{\kappa_i(\alpha)},$$

where $\kappa_i : \mathbb{Z}^m+ \to \mathbb{Z}_{\geq 0}$ has only finite support. Set

$$E^{\underline{\kappa}} = E_1^{\kappa_1} E_2^{\kappa_2} \cdots E_l^{\kappa_l}.$$

Then by the PBW theorem, the set

$$A = \{E^{\underline{\kappa}} | \underline{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_l), \kappa_i : \mathbb{Z}^m + \to \mathbb{Z}_{\geq 0}\}$$

is a basis for $U(\bar{\mathfrak{n}}^-)$. For any $\underline{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_l)$ and any i, set

$$N_{\underline{\kappa},i} = \{\alpha \in \mathbb{Z}^m + | \kappa_i(\alpha) \neq 0\}.$$

Since $\kappa_i : \mathbb{Z}^m \to \mathbb{Z}_{\geq 0}$ has only finite support, $N_{\underline{\kappa},i}$ is a finite set for every i. Given $\underline{\kappa} \neq 0$, denote $N_{\underline{\kappa}} = N_{\underline{\kappa},i}$ with i minimum so that $N_{\underline{\kappa},i}$ not empty. Suppose $E^{\underline{\kappa}} \in A$, and $N_{\underline{\kappa}} = N_{\underline{\kappa},i}$, for $\alpha \in N_{\underline{\kappa}}$, let $(E^{\underline{\kappa}})_{[\alpha]}$ be the same as $E^{\underline{\kappa}}$ but with power $e_{-\gamma_i+\alpha}^{\kappa_i(\alpha)-1}$. By the definitions, it is easy to very the following:

Lemma 4.5 *1. if $\alpha, \alpha' \in N_{\underline{\kappa}}, \alpha \neq \alpha'$, then*

$$(E^{\underline{\kappa}})_{[\alpha]} \neq (E^{\underline{\kappa}})_{[\alpha']}.$$

2. Assume $\underline{\kappa} \neq \underline{\kappa}', N_{\underline{\kappa}} = N_{\underline{\kappa},i}, N_{\underline{\kappa}'} = N_{\underline{\kappa}',i}$. If $\alpha \in N_{\underline{\kappa}} \cap N_{\underline{\kappa}'}$, then

$$(E^{\underline{\kappa}})_{[\alpha]} \neq (E^{\underline{\kappa}'})_{[\alpha]}.$$

Lemma 4.6 *Let $x, y \in \mathfrak{g}, u_1, u_2, \ldots, u_n \in U(\mathfrak{g}), k \in \mathbb{Z}_{\geq 0}$. Then*

1. $[y, u_1 \cdots u_n] = \sum_{i=1}^n u_1 \cdots u_{i-1}[y, u_i]u_{i+1} \cdots u_n$.

2. $[y, x^k] = \sum_{i=1}^n x^{k-i}[y, x]x^{i-1} = kx^{k-1}[y, x] + \sum_{i=2}^k x^{k-i}[[y, x], x^{i-1}]$.

Proof. Since $u_1[y,u_2]+[y,u_1]u_2 = u_1(yu_2-u_2y)+(yu_1-u_1y)u_2 = yu_1u_2-u_1u_2y = [y,u_1u_2]$, by induction on n we have

$$\begin{aligned}
[y, u_1\cdots u_n] &= u_1\cdots u_{n-1}[y,u_n] + [y, u_1\cdots u_{n-1}]u_n \\
&= u_1\cdots u_{n-1}[y,u_n] + \sum_{i=1}^{n-1} u_1\cdots u_{i-1}[y,u_i]u_{i+1}\cdots u_n \\
&= \sum_{i=1}^{n} u_1\cdots u_{i-1}[y,u_i]u_{i+1}\cdots u_n.
\end{aligned}$$

The second equation is just a special case of the first one. □

Lemma 4.7 *Assume $1 \neq E^{\underline{\kappa}} \in A$, Let $\beta \in \mathbb{Z}^m+$ such that $\alpha < \beta$ for all $\alpha \in N_{\underline{\kappa}} = N_{\underline{\kappa},i}$. Let y be a non-zero element of $\mathfrak{g}_{\gamma_i} \otimes t^{-\beta} \subset \bar{\mathfrak{n}}^-$, then there exists $u \in U(\bar{\mathfrak{n}}^-)$ such that*

$$[y, E^{\underline{\kappa}}] = u + \sum_{\alpha\in N_{\underline{\kappa}}} \kappa_i(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_i+\alpha}]. \tag{4.2}$$

Moreover,

$$\sum_{\alpha\in N_{\underline{\kappa}}} \underline{\kappa}_i(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_i+\alpha}] \neq 0. \tag{4.3}$$

Proof. Note that $\bar{\mathfrak{g}}_{-\gamma_i-\beta} = \mathfrak{g}_{-\gamma_i} \otimes t^{-\beta}$ and $\bar{\mathfrak{g}}_{-\gamma_i+\alpha} = \mathfrak{g}_{-\gamma_i} \otimes t^{\alpha}$ for every $\alpha \in \mathbb{Z}^m+$. $[y, e_{-\gamma_i+\alpha}] = b[e_{\gamma_i-\beta}, e_{-\gamma_i+\alpha}] = b[e_{\gamma_i}\otimes t^{\beta}, e_{-\gamma_i}\otimes t^{\alpha}] = b[e_{\gamma_i}, e_{-\gamma_i}]\otimes t^{\alpha-\beta}$ for some $0 \neq b \in \mathbb{C}$. Since $[e_{\gamma_i}, e_{-\gamma_i}] = h_{\gamma_i} \neq 0$, $\beta > \alpha \Rightarrow t^{\alpha-\beta} \neq 0$, we have $[y, e_{-\gamma_i+\alpha}] \neq 0$. Moreover, $[y, e_{-\gamma_i+\alpha}] = bh_{\gamma_i}\otimes t^{\alpha-\beta} \Rightarrow [y, e_{-\gamma_i+\alpha}] \in \mathfrak{t}_{\alpha-\beta} \subset \bar{\mathfrak{t}}^-$ for all $\alpha \in N_{\underline{\kappa}}$. Since $\alpha \in N_{\underline{\kappa}} = N_{\underline{\kappa},i}$, we have $\kappa_j = 0$ for all $1 \leq j \leq i-1$. Thus, we may write

$E^{\underline{\kappa}} = E_i^{\kappa_i} E_{i+1}^{\kappa_{i+1}} \cdots E_l^{\kappa_l}$, by Lemma 4.6,

$$[y, E^{\underline{\kappa}}] = [y, E^{\kappa_i}]E^{\kappa_{i+1}} \cdots E^{\kappa_l} + E^{\kappa_i}[y, E^{\kappa_{i+1}} \cdots E^{\kappa_l}].$$

Since $\gamma_i < \gamma_j$ for all $i < j$, so, if $i < j$ and $\gamma_i - \gamma_j \in \Delta$ then $\gamma_i - \gamma_j \in \Delta^-$. Then by Lemma 4.2, $[y, E^{\kappa_{i+1}} \cdots E^{\kappa_l}] \in U(\bar{\mathfrak{n}}^-)$ because $[y, e_{-\gamma_j+\alpha}] = [e_{\gamma_i}, e_{-\gamma_j}] \otimes t^{\alpha-\beta}$ is equal to 0 if $\gamma_i - \gamma_j \notin \Delta$, or equal to $be_{\gamma_i-\gamma_j} \otimes t^{\alpha-\beta} \in U(\bar{\mathfrak{n}}^-)$ if $\gamma_i - \gamma_j \in \Delta$.

Now we compute $[y, E^{\kappa_i}]$,

$$\begin{aligned} [y, E^{\kappa_i}] &= [y, \prod_{\alpha \in \ N_{\underline{\kappa}}} e_{-\gamma_i+\alpha}^{\kappa_i(\alpha)}] \\ &= \sum_{\alpha \in \ N_{\underline{\kappa}}} \cdots e_{-\gamma_i+\alpha-}^{\kappa_i(\alpha-)}[y, e_{-\gamma_i+\alpha}^{\kappa_i(\alpha)}]e_{-\gamma_i+\alpha+}^{\kappa_i(\alpha+)} \cdots , \end{aligned}$$

where $\alpha-$ $(\alpha+)$ is the element in $\mathbb{Z}^m+$ most close to α but are smaller (greater) than α with lexicographic order.

$$\begin{aligned} [y, e_{-\gamma_i+\alpha}^{\kappa_i(\alpha)}] &= \kappa_i(\alpha)(e_{-\gamma_i+\alpha})^{\kappa_i(\alpha)-1}[y, e_{-\gamma_i+\alpha}] \\ &\quad + \sum_{j=2}^{\kappa_i(\alpha)} (e_{-\gamma_i+\alpha})^{\kappa_i(\alpha)-j}[[y, e_{-\gamma_i+\alpha}], (e_{-\gamma_i+\alpha})^{j-1}]. \end{aligned}$$

Since $[y, e_{-\gamma_i+\alpha}] \in \tilde{\mathfrak{t}}^-$ for all $\alpha \in N_{\underline{\kappa}}$, we have

$$u' = \sum_{j=2}^{\kappa_i(\alpha)} (e_{-\gamma_i+\alpha})^{\kappa_i(\alpha)-j}[[y, e_{-\gamma_i+\alpha}], (e_{-\gamma_i+\alpha})^{j-1}] \in U(\bar{\mathfrak{n}}^-).$$

So,

$$[y, E^{\kappa_i}] \;=\; \sum_{\alpha \in \; N_{\underline{\kappa}}} \cdots e_{-\gamma_i+\alpha-}^{\kappa_i(\alpha-)} \{u' + \kappa_i(\alpha)(e_{-\gamma_i+\alpha})^{\kappa_i(\alpha)-1}$$

$$*[y, e_{-\gamma_i+\alpha}]\} e_{-\gamma_i+\alpha+}^{\kappa_i(\alpha+)} \cdots .$$

Again, since $[y, e_{-\gamma_i+\alpha}] \in \tilde{\mathfrak{t}}^-$ for all $\alpha \in N_{\underline{\kappa}}$, we can move $[y, e_{-\gamma_i+\alpha}]$ to the right side at the expense of commutators live in $U(\bar{\mathfrak{n}}^-)$, denoted as u''. So,

$$[y, E^{\kappa_i}] \;=\; \{ \sum_{\alpha \in \; N_{\underline{\kappa}}} \cdots e_{-\gamma_i+\alpha-}^{\kappa_i(\alpha-)} (\kappa_i(\alpha)(e_{-\gamma_i+\alpha})^{\kappa_i(\alpha)-1}) e_{-\gamma_i+\alpha+}^{\kappa_i(\alpha+)} \cdots \}$$

$$*[y, e_{-\gamma_i+\alpha}] + u'' + \sum_{\alpha \in \; N_{\underline{\kappa}}} \cdots e_{-\gamma_i+\alpha-}^{\kappa_i(\alpha-)} u' e_{-\gamma_i+\alpha+}^{\kappa_i(\alpha+)} \cdots$$

$$= \sum_{\alpha \in \; N_{\underline{\kappa}}} \kappa_i(\alpha)(E^{\kappa_i})_{[\alpha]}[y, e_{-\gamma_i+\alpha}] + u''',$$

for some $u''' \in U(\bar{\mathfrak{n}}^-)$. Thus, we have

$$[y, E^{\underline{\kappa}}] \;=\; [y, E^{\kappa_i}]E^{\kappa_{i+1}} \cdots E^{\kappa_l} + E^{\kappa_i}[y, E^{\kappa_{i+1}} \cdots E^{\kappa_l}]$$

$$= \{ \sum_{\alpha \in \; N_{\underline{\kappa}}} \kappa_i(\alpha)(E^{\kappa_i})_{[\alpha]}[y, e_{-\gamma_i+\alpha}] + u''' \} E^{\kappa_{i+1}} \cdots E^{\kappa_l}$$

$$+ E^{\kappa_i}[y, E^{\kappa_{i+1}} \cdots E^{\kappa_l}]$$

$$= \quad u + \sum_{\alpha \in N_{\underline{\kappa}}} \kappa_i(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_i+\alpha}]$$

for some $u \in U(\bar{\mathfrak{n}}^-)$.

Suppose that

$$\sum_{\alpha \in N_{\underline{\kappa}}} \kappa_i(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_i+\alpha}] = 0.$$

Since the elements of $\{(E^{\underline{\kappa}})_{[\alpha]} | \alpha \in N_{\underline{\kappa}}\}$ are linearly independent by Lemma 4.5, and by the PBW theorem, A is a basis of $U(\mathfrak{m})$ as a free right $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \oplus \cdots \oplus \mathbb{C}d_m)$-module. So $[y, e_{-\gamma_i+\alpha}] = 0$ for every $\alpha \in N_{\underline{\kappa}}$, which is not true. Hence

$$\sum_{\alpha \in N_{\underline{\kappa}}} \kappa_i(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_i+\alpha}] \neq 0.$$

$\square$

Recall that $\underline{k} = (k_{1\alpha}, k_{1\beta}, \dots, k_{2\alpha}, k_{2\beta}, \dots, k_{n\alpha}, k_{n\beta}, \dots) = (k_{r\alpha})_{1\leq r\leq n, \alpha\in\mathbb{Z}^m+}$. For any $\underline{k} \in I$, let $\underline{k}^\top = (k_{1\alpha}, k_{2\alpha}, \dots, k_{n\alpha}, k_{1\beta}, k_{2\beta}, \dots, k_{n\beta}, \dots)$. Let $I^\top = \{\underline{k}^\top | \underline{k} \in I\}$. We order the elements in $I^\top$ in the reverse lexicographic order. For any $y^{\underline{k}}, y^{\underline{l}}, d^{\underline{p}}, d^{\underline{q}}$, where $\underline{k}, \underline{l} \in I, \underline{p}, \underline{q} \in \mathbb{Z}^m_{\geq 0}$, we define $y^{\underline{k}} d^{\underline{p}} \leq y^{\underline{l}} d^{\underline{q}}$ if $\underline{k}^\top < \underline{l}^\top$ (in the reverse lexicographic order) or $\underline{k} = \underline{l}$ and $|\underline{p}| \leq |\underline{q}|$.

Theorem 4.8 *Let* $\lambda \in (\mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \cdots \oplus \mathbb{C}c_m)^*, \lambda(c_1), \lambda(c_2), \dots, \lambda(c_m)$ *be* $\mathbb{Z}$*-independent and* $\eta \in \Gamma$. *Then* $V_{\eta,\lambda}$ *is irreducible as a* $U(\bar{\mathfrak{g}})$*-module.*

Proof. Let K be a non-zero $U(\bar{\mathfrak{g}})$-submodule of $V_{\eta,\lambda}$, we will show that $K = V_{\eta,\lambda}$. It suffices to show that $K \cap \bar{L}_{\eta,\lambda} v \neq 0$ because$\bar{L}_{\eta,\lambda} v = U(\bar{\mathfrak{t}})v$ is irreducible as a $U(\bar{\mathfrak{t}})$-module and $V_{\eta,\lambda} = U(\bar{\mathfrak{g}})v$. By Proposition 4.1(3), it follows that $K \cap V^{\lambda-\mu}_{\eta,\lambda} \neq 0$ for some $\mu \in Q^+$. Assume that $0 \neq w \in K \cap V^{\lambda-\mu}_{\eta,\lambda}$. We claim that there exists $u \in U(\bar{\mathfrak{g}})$ such that $0 \neq uw \in \bar{L}_{\eta,\lambda} v$. We will proceed by induction on $ht(\mu)$. If $\mu = 0$, then we are done since $V^{\lambda}_{\eta,\lambda} = \bar{L}_{\eta,\lambda} v$. Suppose that the claim is true for all $\mu' \in Q^+$ with $ht(\mu') < ht(\mu)$. By Proposition 3.2(1) and Proposition 4.1(1), w has a unique expression

$$w = \sum_{q=1}^{k} (\sum_{\underline{\kappa}} \lambda_{\underline{\kappa},q} E^{\underline{\kappa}}) w_q d^{\underline{p_q}} v, \tag{4.4}$$

where $k \in \mathbb{Z}_{>0}, E^{\underline{\kappa}} \in A, \lambda_{\underline{\kappa},q} \in \mathbb{C}$, and for each q, only finitely many $\lambda_{\underline{\kappa},q} \in \mathbb{C} \neq 0$. $w_q \in \{y^{\underline{k}} | \underline{k} \in I\}$, $\underline{p_q} \in \mathbb{Z}^m_{\geq 0}$ and $w_q d^{\underline{p_q}} \neq w_{q'} d^{\underline{p_{q'}}}$ if $q \neq q'$.

Since $w \in V^{\lambda-\mu}_{\eta,\lambda} = U(\bar{\mathfrak{n}}^-)^{-u} \otimes_{\mathbb{C}} \bar{L}_{\eta,\lambda}$, for each $\underline{\kappa}$ such that $\lambda_{\underline{\kappa},q} \neq 0$ for some q, we have

$$[h, E^{\underline{\kappa}}] = -\mu(h)E^{\underline{\kappa}} \tag{4.5}$$

for all $h \in \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \cdots \oplus \mathbb{C}c_m$. We claim that

$$\mu = \sum_{i=1}^{l} \sum_{\alpha \in \mathbb{Z}^m +} \kappa_i(\alpha)\gamma_i. \tag{4.6}$$

$$\begin{aligned}
[h, e_{-\gamma_i+\alpha}e_{-\gamma_j+\beta}] &= [h, e_{-\gamma_i} \otimes t^{\alpha} e_{-\gamma_j} \otimes t^{\beta}] \\
&= [h \otimes 1, e_{-\gamma_i} \otimes t^{\alpha}]e_{-\gamma_j} \otimes t^{\beta} \\
&\quad + e_{-\gamma_i} \otimes t^{\alpha}[h \otimes 1, e_{-\gamma_j} \otimes t^{\beta}] \\
&= [h, e_{-\gamma_i}] \otimes t^{\alpha} e_{-\gamma_j} \otimes t^{\beta} + e_{-\gamma_i} \otimes t^{\alpha}[h, e_{-\gamma_j}] \otimes t^{\beta} \\
&= -\gamma_i(h)e_{-\gamma_i+\alpha}e_{-\gamma_j+\beta} - \gamma_j(h)e_{-\gamma_i+\alpha}e_{-\gamma_j+\beta} \\
&= -(\gamma_i + \gamma_j)(h)e_{-\gamma_i+\alpha}e_{-\gamma_j+\beta},
\end{aligned}$$

$$\begin{aligned}
\Rightarrow [h, E^{\underline{\kappa}}] &= [h, \prod_{i=1}^{l} \prod_{\alpha \in \mathbb{Z}^m +} e_{-\gamma_i+\alpha}^{\kappa_i(\alpha)}] \\
&= (-\sum_{i=1}^{l} \sum_{\alpha \in \mathbb{Z}^m +} \kappa_i(\alpha)\gamma_i)(h)E^{\underline{\kappa}}, \\
\Rightarrow \mu &= \sum_{i=1}^{l} \sum_{\alpha \in \mathbb{Z}^m +} \kappa_i(\alpha)\gamma_i.
\end{aligned}$$

For each q, redefine Ω as,

$$\Omega = \{\underline{\kappa} | \lambda_{\underline{\kappa},q} \neq 0\},$$

and denote $i_q = min\{j | N_{\underline{\kappa}} = N_{\underline{\kappa},j}, \underline{\kappa} \in \Omega_q\}$. Without loss of generality, we may assume that

$$i_1 = \cdots = i_j < i_{j+1} \leq \cdots \leq i_k.$$

Then we may write

$$w = \sum_{q=1}^{j} (\sum_{\underline{\kappa} \in \Omega_q} \lambda_{\underline{\kappa},q} E^{\underline{\kappa}}) w_q d^{\underline{p_q}} v + \sum_{q=j+1}^{k} (\sum_{\underline{\kappa} \in \Omega_q} \lambda_{\underline{\kappa},q} E^{\underline{\kappa}}) w_q d^{\underline{p_q}} v.$$

Let

$$N = \{\alpha | \alpha \in N_{\underline{\kappa}}, \underline{\kappa} \in \Omega_q, q = 1, 2, \ldots, k\}.$$

Recall that $\{y_{r\alpha} = \theta_r \otimes t^{-\alpha}\}_{1 \leq r \leq n}$ is a basis of $\mathfrak{t}_{-\alpha} for \alpha \in \mathbb{Z}^m+$. To avoid misunderstandings, we will write $y_{r,\alpha}$ for $y_{r\alpha}$. Let $\beta \in \mathbb{Z}^m+$ such that $\alpha < \beta, w_q < y_{r,\beta-\alpha}$ for all q, r and all $\alpha \in N$. Let $y = e_{\gamma_{i_1} - \beta}$, since $y \in \bar{\mathfrak{n}}^+$,

$$y w_q d^{\underline{p_q}} v = 0$$

for all $1 \leq q \leq k$. As $[y, e_{-\gamma_{i_1}+\alpha}] = [e_{\gamma_{i_1}}, e_{-\gamma_{i_1}}] \otimes t^{\alpha-\beta}$, we have $[y, e_{-\gamma_{i_1}+\alpha}] \neq 0$ because $[e_{\gamma_{i_1}}, e_{-\gamma_{i_1}}] \neq 0$. Moreover, if $\alpha \in N$, then

$$[y, e_{-\gamma_{i_1}+\alpha}] \in \mathfrak{t}_{\alpha-\beta} = \tilde{\mathfrak{t}}_{\alpha-\beta} \subset \tilde{\mathfrak{t}}^-,$$

since $\alpha < \beta$ for all $\alpha \in N$. Thus, for every $\alpha \in N$, there exist values $\nu_{r,\alpha} \in \mathbb{C}, 1 \leq r \leq n$, with at least one $\nu_{r,\alpha} \neq 0$ such that

$$[y, e_{-\gamma_{i_1}+\alpha}] = \sum_{r=1}^{n} \nu_{r,\alpha} y_{r,\beta-\alpha}$$

and this expression is unique. If $i_q = i_1 = \cdots = i_j$, then by Lemma 4.7, for all $\underline{\kappa} \in \Omega_q$ there exists $u_{\underline{\kappa},q} \in U(\bar{\mathfrak{n}}^-)$ such that

$$[y, E^{\underline{\kappa}}] = u_{\underline{\kappa},q} + \sum_{\alpha \in N_{\underline{\kappa}}} \kappa_{i_1}(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_{i_1}+\alpha}],$$

where $\sum_{\alpha \in N_{\underline{\kappa}}} \kappa_{i_1}(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_{i_1}+\alpha}] \neq 0$. So,

$$\begin{aligned}
yw &= \sum_{q=1}^{j}(\sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q} y E^{\underline{\kappa}}) w_q d^{\underline{p_q}} v + \sum_{q=j+1}^{k}(\sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q} y E^{\underline{\kappa}}) w_q d^{\underline{p_q}} v \\
&= \sum_{q=1}^{j}(\sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q}([y, E^{\underline{\kappa}}] - E^{\underline{\kappa}} y)) w_q d^{\underline{p_q}} v \\
&\quad + \sum_{q=j+1}^{k}(\sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q}([y, E^{\underline{\kappa}}] - E^{\underline{\kappa}} y)) w_q d^{\underline{p_q}} v \\
&= \sum_{q=1}^{j}(\sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q}[y, E^{\underline{\kappa}}]) w_q d^{\underline{p_q}} v + \sum_{q=j+1}^{k}(\sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q}[y, E^{\underline{\kappa}}]) w_q d^{\underline{p_q}} v \\
&= \sum_{q=1}^{j} \sum_{\underline{\kappa}\in\Omega_q} \sum_{\alpha\in N_{\underline{\kappa}}} \lambda_{\underline{\kappa},q} \kappa_{i_1}(\alpha)(E^{\underline{\kappa}})_{[\alpha]}[y, e_{-\gamma_{i_1}+\alpha}] w_q d^{\underline{p_q}} v \\
&\quad + \sum_{q=j+1}^{k} \sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q} u_{\underline{\kappa},q} w_q d^{\underline{p_q}} v + \sum_{q=j+1}^{k} \sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q}[y, E^{\underline{\kappa}}] w_q d^{\underline{p_q}} v
\end{aligned}$$

$$= \sum_{q=1}^{j} \sum_{\underline{\kappa}\in\Omega_q} \sum_{\alpha\in N_{\underline{\kappa}}} \sum_{r=1}^{n} \lambda_{\underline{\kappa},q} \kappa_{i_1}(\alpha)\nu_{r,\alpha}(E^{\underline{\kappa}})_{[\alpha]} y_{r,\beta-\alpha} w_q d^{\underline{p_q}} v$$

$$+ \sum_{q=j+1}^{k} \sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q} u_{\underline{\kappa},q} w_q d^{\underline{p_q}} v + \sum_{q=j+1}^{k} \sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q} [y, E^{\underline{\kappa}}] w_q d^{\underline{p_q}} v.$$

We claim that $yw \neq 0$. Suppose that $yw = 0$. Let

$$f : V_{\eta,\lambda} \to U(\mathfrak{m}),$$

defined by $u \otimes wv \mapsto uw, u \in U(\bar{\mathfrak{n}}^-), w \in U(\bar{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \cdots \oplus \mathbb{C}d_m)$. Then we have

$$0 = f(yw) = \sum_{q=1}^{j} \sum_{\underline{\kappa}\in\Omega_q} \sum_{\alpha\in N_{\underline{\kappa}}} \sum_{r=1}^{n} \lambda_{\underline{\kappa},q} \kappa_{i_1}(\alpha)\nu_{r,\alpha}(E^{\underline{\kappa}})_{[\alpha]} y_{r,\beta-\alpha} w_q d^{\underline{p_q}}$$

$$+ \sum_{q=j+1}^{k} \sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q} u_{\underline{\kappa},q} w_q d^{\underline{p_q}} + \sum_{q=j+1}^{k} \sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q} [y, E^{\underline{\kappa}}] w_q d^{\underline{p_q}}.$$

Since $w_{q'} < y_{r,\beta-\alpha}$ for all $q', r, , \alpha \in N$, it follows that

$$w_{q'} < w_q y_{r,\beta-\alpha}$$

for all q, q', so

$$w_{q'} d^{\underline{p_q}'} < w_q y_{r,\beta-\alpha} d^{\underline{p_q}}$$

for all $q, q', \alpha \in N$. As $N_{\underline{\kappa}} \subseteq N$, for all $\underline{\kappa} \in \Omega_q$ and all q, so

$$\sum_{q=1}^{j} \sum_{\underline{\kappa} \in \Omega_q} \sum_{\alpha \in N_{\underline{\kappa}}} \sum_{r=1}^{n} \lambda_{\underline{\kappa},q} \kappa_{i_1}(\alpha) \nu_{r,\alpha} (E^{\underline{\kappa}})_{[\alpha]} y_{r,\beta-\alpha} w_q d^{\underline{p_q}} = 0.$$

Let $\delta = min\{\alpha | \alpha \in N_{\underline{\kappa}}, \underline{\kappa} \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_j\}$. Suppose that $1 \leq r \leq n$ is maximal such that $\nu_{r,\delta} \neq 0$. Assume $\alpha \in N_{\underline{\kappa}}, \underline{\kappa} \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_j$, and $\alpha \neq \delta$. Then

$$y_{s,\beta-\alpha} < y_{r,\beta-\delta}$$

for all $1 \leq r, s \leq n$, since $\beta - \alpha < \beta - \delta$. Moreover, if $s < r$, then

$$y_{s,\beta-\delta} < y_{r,\beta-\delta}.$$

Hence

$$w_{q'} y_{s,\beta-\alpha} d^{\underline{p_q}'} < w_q y_{r,\beta-\delta} d^{\underline{p_q}}, 1 \leq s, r \leq n$$

for all $q, q', \alpha \in N_{\underline{\kappa}}$ and $\alpha \neq \delta, \underline{\kappa} \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_j$. Also

$$w_{q'} y_{s,\beta-\delta} d^{\underline{p_q}'} < w_q y_{r,\beta-\delta} d^{\underline{p_q}}, 1 \leq s < r \leq n$$

for all q, q'. Hence

$$\begin{aligned} \sum_{q=1}^{j} \sum_{\underline{\kappa} \in \Omega_q, \delta \in N_{\underline{\kappa}}} \lambda_{\underline{\kappa},q} \kappa_{i_1}(\delta) \nu_{r,\delta} (E^{\underline{\kappa}})_{[\delta]} y_{r,\beta-\delta} w_q d^{\underline{p_q}} &= 0 \\ \Rightarrow \sum_{q=1}^{j} \sum_{\underline{\kappa} \in \Omega_q, \delta \in N_{\underline{\kappa}}} \lambda_{\underline{\kappa},q} \kappa_{i_1}(\delta) \nu_{r,\delta} (E^{\underline{\kappa}})_{[\delta]} w_q y_{r,\beta-\delta} d^{\underline{p_q}} &= 0, \end{aligned}$$

since $y_{r,\beta-\delta}w_q = w_q y_{r,\beta-\delta}$. Let $1 \leq q \leq j$ such that $\delta \in N_{\underline{\kappa}}$ for some $\underline{\kappa} \in \Omega_q$. Since $w_{q'}y_{r,\beta-\delta}d^{\underline{p_{q}}'} \neq w_q y_{r,\beta-\delta}d^{\underline{p_q}}$ if $q \neq q'$, and $U(\mathfrak{m})$ is free as a right $U(\tilde{\mathfrak{t}}^- \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \cdots \oplus \mathbb{C}d_m)$-module, it must be

$$\sum_{\underline{\kappa}\in\Omega_q} \lambda_{\underline{\kappa},q}(E^{\underline{\kappa}})_{[\delta]} = 0.$$

Since the elements $E^{\underline{\kappa}}, \underline{\kappa} \in \Omega_q$ are linearly independent, and δ is fixed, so $(E^{\underline{\kappa}})_{[\delta]}, \underline{\kappa} \in \Omega_q$ must also be linearly independent. Then we have $\lambda_{\underline{\kappa},q} = 0$ for all $\underline{\kappa} \in \Omega_q$, which is a contradiction. This proves that $yw \neq 0$.

Since $\mu - \gamma_{i_1} \in Q^+, 0 \neq yw \in V_{\eta,\lambda}^{\lambda-(\mu-\gamma_{i_1})}$ and $ht(\mu - \gamma_{i_1}) < ht(\mu)$, by the inductive hypothesis there exists $u \in U(\bar{\mathfrak{g}})$ such that $0 \neq u(yw) = (uy)w \in \bar{L}_{\eta,\lambda}$, hence $K \cap \bar{L}_{\eta,\lambda} \neq 0$ as desired. $\square$

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