ON LAPLACE TRANSFORMS, GENERALIZED GAMMA CONVOLUTIONS, AND THEIR APPLICATIONS IN RISK AGGREGATION

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Abstract

This dissertation begins with two introductory chapters to provide some relevant background information: an introduction on the Laplace transform and an introduction on Generalized Gamma Convolutions (GGCs). The heart of this dissertation is the final three chapters comprised of three contributions to the literature.

In Chapter 3, we study the analytical properties of the Laplace transform of the log-normal distribution. Two integral expressions for the analytic continuation of the Laplace transform of the log-normal distribution are provided, one of which takes the form of a Mellin-Barnes integral. As a corollary, we obtain an integral expression for the characteristic function; we show that the integral expression derived by Leipnik in [45] is incorrect. We present two approximations for the Laplace transform of the log-normal distribution, both valid in $\mathbb{C} \setminus (-\infty, 0]$. In the last section, we discuss how one may use our results to compute the density of a sum of independent log-normal random variables.

In Chapter 4, we explore the topic of risk aggregation with moment matching approximations. We put forward a refined moment matching approximation (MMA) method for approximating the distributions of the sums of insurance risks. Our method approximates the distributions of interest to any desired precision, works equally well for light and heavy-tailed distributions, and is reasonably fast irrespective of the number of the involved summands.

In Chapter 5, we study the convergence of the Gaver-Stehfest algorithm. The Gaver-Stehfest algorithm is widely used for numerical inversion of Laplace transform. In this chapter we provide the first rigorous study of the rate of convergence of the Gaver-Stehfest algorithm. We prove that the Gaver-Stehfest approximations of order n converge exponentially fast if the target function is analytic in a neighbourhood of a point and they converge at a rate $o(n^{-k})$ if the target function is $(2k+3)$-times differentiable at a point.

I dedicate this work to my son Nathan

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Chapter 1

Introduction

The Laplace transform is a powerful mathematical tool used in various fields, including engineering, physics, and applied mathematics. It was named after the French mathematician Pierre-Simon Laplace, who developed the transform in the late 18th century.

The origins of the Laplace transform can be traced back to the work of mathematicians such as Leonhard Euler and Joseph Fourier. Euler used the notion of a generating function, which allowed the representation of a sequence of numbers as a power series. Fourier, on the other hand, discovered that functions can be represented as a sum of sinusoidal functions through the use of Fourier series.

Pierre-Simon Laplace built upon these ideas and developed the Laplace transform as a generalization of Fourier series. He published his findings in the treatise "Analytic Theory of Probability" in 1812. In this work, Laplace used the transform to solve problems in probability theory, specifically in the study of the central limit theorem and the law of large numbers.

The Laplace transform gained significant attention and popularity in the early 20th century due to its applications in engineering and physics. The transform allows the simplification of differential equations, making them more amenable to analysis and solution. Engineers and physicists realized its potential in solving a wide range of problems, including electrical circuits, control systems, vibrations, heat transfer, and fluid dynamics.

The Laplace transform operates on a function of a real variable and transforms it into a function of a complex variable. One of the key advantages of the Laplace transform is its ability to convert differential equations into algebraic equations. Differential equations that are difficult to solve directly often become simpler algebraic equations, and, once the solution is obtained, it can be transformed back to the original domain using the inverse Laplace transform.

Similarly, the Laplace transform can be used to convert a convolution of probability distributions into a product of Laplace transforms. Once the product is evaluated, the result can be transformed back into a probability distribution using the inverse Laplace transform.

The idea of convolution has a long history in mathematics, with roots in the works of Laplace, Fourier, and others. Convolution is an operation that combines two functions to produce a third function, which represents the integration of the product of the two original functions. In the field of probability, convolution corresponds to the distribution of a sum of two independent random variables and is extended to a sum of any number of independent random variables by way of iteration.

The gamma distribution is a flexible probability distribution that has properties allowing it to model a variety of real-world phenomena. The concept of generalized gamma convolutions (GGC's) emerged in the 20th century as researchers sought to extend the applicability of gamma distributions through convolutions. The foundations of GGC's can be traced back to the work of the Swedish mathematician Olof Thorin.

In the 1970's, Thorin published four papers ([61], [60], [62], and [59]) on the infinite divisibility of probability distributions. In 1977, Thorin published a paper, [61], titled "On the infinite divisibility of the lognormal distribution" in the Scandinavian Actuarial Journal. In this remarkable paper, Thorin introduced the concept of GGC's (using the term generalized Γ-convolutions) and used them to show that the log-normal distribution is infinitely divisible. Thorin's results laid the groundwork for the study of GGC's and marked an important step forward in the theory of infinite divisibility. In 1978, Thorin generalized his results to powers of gamma variables and extended the class of GGC's to include distributions on the whole real line.

Lennart Bondesson, another Swedish mathematician, further developed Thorin's ideas and made significant contributions to the theory of GGC's in the late 20th century. He published several influential works on this topic, including the book, [15], titled "Generalized Gamma Convolutions and Related Classes of Distributions and Densities", in 1992.

The class of GGC's has applications in various fields and contains many important distributions such as the gamma, inverse gamma, inverse-Gaussian, exponential, Lomax, log-normal, and Weibull. In actuarial science, they have been used for modeling insurance claim severities, lifetime distributions, and other phenomena with heavy-tailed behavior. In finance, these distributions have been employed to model asset returns and risk measures.

A common problem in mathematical modeling is to strike the right balance between complexity and simplicity: the mathematical model should be realistic enough to accurately describe the complicated natural phenomena, yet it should be simple enough and amenable to analysis that would (hopefully) lead to an insight. In actuarial science and finance, this problem appears when modeling various risks by random variables.

Quantitative risk management often begins with a set of random variables representing profit or loss, and models the aggregate financial position as a function of these random variables. For example, in the Individual Risk Model of actuarial science, the claims of an insurance company are modeled as a sum of the claims of many insured individuals. The ultimate objective is to utilize the model to accurately calculate probabilities concerning the value of this aggregate position or apply a measure of risk to this aggregate position.

Despite being an elementary procedure on a theoretical basis, calculating (or even approximating) the distribution of a sum of independent random variables can be troublesome. Indeed, the distribution of the sum is given by an integral (the convolution of the cumulative distribution functions) that in most cases does not have a known closed form. A number of numerical techniques have been developed for computing or approximating the aggregate distribution, including those which make use of the Laplace transform.

This dissertation explores the use of Laplace transforms and GGC's with applications to risk aggregation. In Chapter 1 and Chapter 2, we provide some relevant background information with an introduction on the Laplace transform and an introduction on GGC's, respectively.

In Chapter 3, we study the analytical properties of the Laplace transform of the log-normal distribution. Two integral expressions for the analytic continuation of the Laplace transform of the log-normal distribution are provided, one of which takes the form of a Mellin-Barnes integral. As a corollary, we obtain an integral expression for the characteristic function; we show that the integral expression derived by Leipnik in [45] is incorrect. We present two approximations for the Laplace transform of the log-normal distribution, both valid in $\mathbb{C} \setminus (-\infty, 0]$. In the last section, we discuss how one may use our results to compute the density of a sum of independent log-normal random variables.

In Chapter 4, we explore the topic of risk aggregation with moment matching approximations.

In the vast majority of situations, insurers are interested in the properties of the sums of the risks they are exposed to, rather than in the stand-alone risks per se. Unfortunately, the problem of formulating the probability distributions of the aforementioned sums is rather involved, and as a rule does not have an explicit solution. As a result, numerous methods to approximate the distributions of the sums have been proposed, with the moment matching approximations (MMAs) being arguably the most popular. We put forward a refined MMA method for approximating the distributions of the sums of insurance risks. The method approximates the distributions of interest to any desired precision, works equally well for light and heavy-tailed distributions, and is reasonably fast irrespective of the number of the involved summands.

In Chapter 5, we study the convergence of the Gaver-Stehfest algorithm. The Gaver-Stehfest algorithm is widely used for numerical inversion of Laplace transform. In this chapter we provide the first rigorous study of the rate of convergence of the Gaver-Stehfest algorithm. We prove that Gaver-Stehfest approximations converge exponentially fast if the target function is analytic in a neighbourhood of a point and they converge at a rate $o(n^{-k})$ if the target function is $(2k+3)$-times differentiable at a point.

1.1 Publication information

The contents of Chapters 4, 5, and 6 have been published. The results appearing in Chapter 5 represent joint work Edward Furman and Alexey Kuznetsov and the results appearing in Chapter 6 represent joint work with Alexey Kuznetsov. A modified version of: Chapter 4 has appeared in *Journal of Computational and Applied Mathematics* [48]; Chapter 5 has appeared in *Variance* [49]; Chapter 4 has appeared in the *IMA Journal of Numerical Analysis* [43].

Chapter 2

The Laplace transform

The Laplace transform is an integral transform named after the famous French scientist Pierre-Simon Laplace (1749-1827). This transform has a rich history and is widely employed across many mathematical disciplines. Among its numerous applications, it can be used to transform a convolution of two (or more) functions into a product of their respective Laplace transforms. This property is especially useful as a tool in the field of probability, as the distribution of a sum of independent random variables is the convolution of the underlying distributions.

In this chapter we review the basic properties of the Laplace-Stieltjes transform, discuss its inversion, and explore common computational methods.

2.1 The Laplace-Stieltjes transform

Assume the function $\alpha : (0, \infty) \to \mathbb{C}$ is of bounded variation in the interval $(0, R)$, for all $R > 0$, and has real and imaginary parts u and v respectively,

$$\alpha(t) = u(t) + \mathrm{i}v(t), \qquad t \in (0, \infty).$$

If z is a complex variable, it follows that the Stieltjes integral

$$\int_0^R e^{-zt} \mathrm{d}\alpha(t)$$

exists for every complex number $z = x + \mathrm{i}y$ and has value

$$\int_0^R e^{-xt}\cos(yt)\mathrm{d}u(t) - \int_0^R e^{-xt}\sin(yt)\mathrm{d}v(t) + \mathrm{i}\int_0^R e^{-xt}\sin(yt)\mathrm{d}u(t) + \mathrm{i}\int_0^R e^{-xt}\cos(yt)\mathrm{d}v(t)$$

We define the improper integral

$$\int_0^\infty e^{-zt}\mathrm{d}\alpha(t) = \lim_{R\to\infty}\int_0^R e^{-zt}\mathrm{d}\alpha(t), \tag{2.1}$$

and say the integral converges for the complex number z if the limit exists. If the limit does not exist, we say the integral diverges. When the integral (2.1) converges, it defines a function of the complex variable z:

Definition 1. The function

$$\mathcal{L}^*\{\alpha\}(z) := \int_0^\infty e^{-zt}\mathrm{d}\alpha(t) \tag{2.2}$$

is called the *Laplace-Stieltjes transform of the function* α.

The classic definition of the Laplace transform is included in the definition of the Laplace-Stieltjes transform. Indeed, if the function $\alpha(t)$ in Definition 1 is absolutely continuous on $(0,\infty)$, (2.2) may be written as a Lebesgue integral. In particular,

$$\int_0^\infty e^{-zt}\mathrm{d}\alpha(t) = \int_0^\infty e^{-zt}\alpha'(t)\mathrm{d}t,$$

and the integral takes the form of the classical Laplace transform:

Definition 2. The function

$$\mathcal{L}\{f\}(z) := \int_0^\infty e^{-zt}f(t)\mathrm{d}t \tag{2.3}$$

is called the*Laplace transform of the function* f.

In Chapter 6, the Laplace transform as defined by Definition 2 will form the basis of our study of the convergence of the Gaver-Stehfest algorithm. When working with this definition, it is common to use capital letters to refer to the Laplace transform of a function, e.g. $F(z) = \mathcal{L}\{f\}(z)$.

In Chapters 4 and 5, we explore the Laplace transform of the log-normal distribution and an algorithm for the aggregation of risk random variables, respectively. When we refer to the Laplace transform of a random variable (or probability distribution) we are referring to the following definition:

Definition 3. Let X be a random variable. The function

$$\mathcal{L}\{X\}(z) := \mathbb{E}\left[e^{-zX}\right] \tag{2.4}$$

is called the *Laplace transform of the random variable* X.

Since capital letters are often reserved for cumulative distribution functions, in this context we will refer to the Laplace transform of a random variable using the Greek letter ϕ, e.g. $\phi(z) = \mathcal{L}\{X\}(z)$.

Note that if X is a positive random variable with cumulative distribution function $F(x)$, $x \geq 0$, we have

$$\mathbb{E}\left[e^{-zX}\right] = \int_0^\infty e^{-zx}\mathrm{d}F(x)$$

and Definition 3 corresponds to Definition 1. Furthermore, if the positive random variable X has probability density function $f(x)$, $x > 0$, we have

$$\mathbb{E}\left[e^{-zX}\right] = \int_0^\infty e^{-zx}f(x)\mathrm{d}x$$

and Definition 3 corresponds to Definition 2. If the random variable X has support on the negative real line, equation (2.4) corresponds to the bilateral Laplace transform, which we will not cover in this work.

The domain of the Laplace-Stieltjes transform is determined by the convergence of the integral defined by (2.1). The following theorem provides a sufficient condition for the Laplace-Stieltjes transform to be defined on a half-plane:

Theorem 1 ([65]). *If*

$$\sup_{0 \leq R < \infty} \left| \int_0^R e^{-z_0 t}\mathrm{d}\alpha(t) \right| < \infty, \qquad z_0 = x_0 + \mathrm{i}y,$$

the integral defined by (2.1) *converges for all* $z \in \mathbb{C}$ *for which* $\mathrm{Re}(z) > x_0$.

In particular, we note that if the integral (2.1) converges for $z_0 = x_0 + \mathrm{i}y$, it converges for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > x_0$. Theorem 1 leads us to three possibilities for the domain of the Laplace-Stieltjes transform of the function α:

i) $\mathcal{L}^*\{\alpha\}(z)$ is not defined for any $z \in \mathbb{C}$.

ii) $\mathcal{L}^*\{\alpha\}(z)$ is defined for all $z \in \mathbb{C}$.

iii) There exists a number $x_c \in \mathbb{R}$ for which $\mathcal{L}^*\{\alpha\}(z)$ is defined for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > x_c$, and undefined for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) < x_c$.

The number x_c is called the *abscissa of convergence* and the line $x = x_c$ is called the *axis of convergence* (this notation extends to the first two scenarios by defining $x_c = \infty$ and $x_c = -\infty$, respectively).

It is a relatively simple exercise to construct examples for each of the possible domains. We leave it to the reader to verify that

$$\alpha(t) = \int_0^t e^{e^x} \mathrm{d}x,$$

and

$$\alpha(t) = \int_0^t e^{-e^x} \mathrm{d}x,$$

result in Laplace transforms with domains i) and ii), respectively. To illustrate the most interesting case, domain iii), we consider a positive random variable X with cumulative distribution function $F(x)$, $x \geq 0$. The Laplace transform of X is given by

$$\phi(z) = \int_0^\infty e^{-zx} \mathrm{d}F(x),$$

in accordance with Definition 3. Since X is a random variable, we have $\phi(0) = 1$ and we know that $\phi(z)$ is defined on $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. In this example, the abscissa of convergence may be less than zero, however we require further information about the function $F(x)$ to make this determination. For example, if X is exponentially distributed with cdf $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, the abscissa of convergence is $x_c = -\lambda$. However, if X is log-normally distributed with cdf

$$F(x) = \int_0^x \frac{1}{\sqrt{2\pi\sigma^2}t} e^{-\frac{(\ln(t)-\mu)^2}{2\sigma^2}} \mathrm{d}t, \qquad \mu \in \mathbb{R}, \sigma > 0,$$

the abscissa of convergence is $x_c = 0$.

An important property of Laplace-Stieltjes transforms is that they are analytic in the half-plane on which they are defined. Furthermore, if the determining function is monotonic, the Laplace-Stieltjes transform has a singularity at the point where the axis of convergence intersects the real line.

Theorem 2 ([65])**.**

i) If $\mathcal{L}^*\{\alpha\}(z)$ *converges for* $\mathrm{Re}(z) > x$*, then* $\mathcal{L}^*\{\alpha\}(z)$ *is analytic for* $\mathrm{Re}(z) > x$*.*

ii) If $\alpha(t)$ is a monotonic function, $\mathcal{L}^*\{\alpha\}(z)$ has a singularity at the point $z = x_c$.

Theorem 2 ensures that the Laplace transform of a positive random variable is (at least) analytic on the right half-plane. Moreover, since the cdf $F(x)$ is monotonic, the Laplace transform has a singularity at the point $z = x_c$.

The uniqueness of a Laplace transform's determining function is a distinctive property which makes the Laplace transform a practical tool. For example, this feature allows us show that two random variables have the same distribution if they have the same Laplace transform. The following theorem provides the precise statement for uniqueness; note that the function $\alpha(t)$ is *normalized* if $\alpha(0+) = 0$ and

$$\alpha(t) = \frac{\alpha(t+) + \alpha(t-)}{2}, \qquad t > 0.$$

Theorem 3 ([65])**.**

i) If $\alpha_1(t)$ and $\alpha_2(t)$ are two normalized functions which satisfy $\mathcal{L}^*\{\alpha_1\}(z) = \mathcal{L}^*\{\alpha_2\}(z)$ for all z in some common region of convergence, then $\alpha_1(t) = \alpha_2(t)$, $t \in [0, \infty)$.

ii) If $f_1(t)$ and $f_2(t)$ are two functions which satisfy $\mathcal{L}\{f_1\}(z) = \mathcal{L}\{f_2\}(z)$ for all z in some common region of convergence, then $f_1(t) = f_2(t)$ almost everywhere in $[0, \infty)$.

The last property we need to mention is that the Laplace transform of the convolution of two functions is equal to the product of their transforms:

$$\mathcal{L}\{f_1 * f_2\}(z) = \int_0^\infty e^{-zx}(f_1 * f_2)(x)\mathrm{d}x = \mathcal{L}\{f_1\}(z) \cdot \mathcal{L}\{f_2\}(z) \tag{2.5}$$

In the context of probability, this means that the Laplace transform of the sum of two independent random variables is equal to the product of their transforms:

$$\mathcal{L}\{X_1 + X_2\}(z) = \mathbb{E}\left[e^{-z(X_1+X_2)}\right] = \mathcal{L}\{X_1\}(z) \cdot \mathcal{L}\{X_2\}(z) \tag{2.6}$$

2.2 The inverse Laplace transform

In this section, we provide the formula for inversion of the Laplace (-Stieltjes) transform commonly known as the *Bromwich* integral or the inverse Laplace transform. We begin with the classical Laplace transform:

Theorem 4 ([65]). *Assume that the function $f(t)$ belongs to $L_1((0,R),\mathrm{d}t)$, for each $R>0$, and the Laplace transform*

$$\mathcal{L}\{f\}(z)=\int_0^\infty e^{-zt}f(t)\mathrm{d}t,$$

is absolutely convergent on the line $\mathrm{Re}(z)=c$. If $f(t)$ is of bounded variation in a neighbourhood of t when $t\geq 0$, then

$$\lim_{T\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\mathcal{L}\{f\}(z)e^{zt}\mathrm{d}z=\begin{cases}0 & t<0\\ \frac{f(0+)}{2} & t=0\\ \frac{f(t+)+f(t-)}{2} & t>0\end{cases}\tag{2.7}$$

Note that it is not necessary for the integral

$$\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathcal{L}\{f\}(z)e^{zt}\mathrm{d}z\tag{2.8}$$

to exist. Consider the function $f(t)=1$, which satisfies the assumptions of Theorem 4. For each $c>0$,

$$\mathcal{L}\{f\}(z)=\int_0^\infty e^{-zt}\mathrm{d}t=z^{-1},\qquad \mathrm{Re}(z)>c,$$

but the integral

$$\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{e^{zt}}{z}\mathrm{d}z,\qquad t\geq 0,$$

diverges and only its principal value exists. The inversion formula for the more general Laplace-Stieltjes transform is similar:

Theorem 5 ([65]). *Assume that the function $\alpha(t)$ is a normalized function that belongs to $L_1((0,R),\mathrm{d}t)$, for each $R>0$, and the Laplace-Stieltjes Transform*

$$\mathcal{L}^*\{\alpha\}(z)=\int_0^\infty e^{-zt}\mathrm{d}\alpha(t),$$

has abscissa of convergence x_c. If $c>0$ and $c>x_c$, then

$$\lim_{T\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\mathcal{L}^*\{\alpha\}(z)\frac{e^{zt}}{z}\mathrm{d}z=\begin{cases}0 & t<0\\ \frac{\alpha(0+)}{2} & t=0\\ \alpha(t) & t>0\end{cases}\tag{2.9}$$

Note that Theorem 5 requires the contour of integration to be in the right-half plane and to the right of the axis of convergence.

To illustrate the utility of these inversion theorems, we consider a scenario that we will encounter in Chapter 5. Suppose that we have the Laplace transform of a positive, continuous random variable X, but the cdf, F, is unknown. Since we are certain that F is a normalized function that belongs to $\mathrm{L}_1((0,R),\mathrm{d}x)$, for each $R>0$, and the Laplace transform of X

$$\phi(z)=\int_0^\infty e^{-zx}\mathrm{d}F(x),$$

has abscissa of convergence $x_c \leq 0$, Theorem 5 guarantees that

$$F(x)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\phi(z)\frac{e^{zx}}{z}\mathrm{d}z, \qquad x>0, \tag{2.10}$$

for any $c>0$. Moreover, if we know that X is absolutely continuous, then X has a pdf $f \in \mathrm{L}(0,\infty)$ and the Laplace transform

$$\phi(z)=\int_0^\infty e^{-zx}f(x)\mathrm{d}x,$$

is absolutely convergent on the line $\mathrm{Re}(z)=c$, for any $c>0$. In this case, Theorem 4 guarantees that

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\phi(z)e^{zx}\mathrm{d}z, \qquad x>0. \tag{2.11}$$

2.3 Inversion methods

Methods for computing the inverse Laplace transform of a function fall into two categories: analytic methods and numerical methods. As the transform is defined by integration, it is no surprise that we must rely on numerical methods for the majority of functions. Nonetheless, we begin with an analytic example as it is instructive to see how one may compute the Laplace transform using analytic techniques.

We illustrate analytic inversion of a simple function using classic techniques from complex analysis. Consider an absolutely continuous random variable with gamma distribution, $X \sim \mathit{Gamma}(\alpha,\beta)$, where $\alpha\in\mathbb{N}$ and $\beta>0$. The Laplace transform of X is given by

$$\phi(z)=\beta^\alpha(z+\beta)^{-\alpha}.$$

Invoking Theorem 4, we can recover the pdf, $f(x)$, by computing

$$f(x)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\phi(z)e^{zx}\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\beta^\alpha e^{zx}}{(z+\beta)^\alpha}\mathrm{d}z, \qquad x>0. \tag{2.12}$$

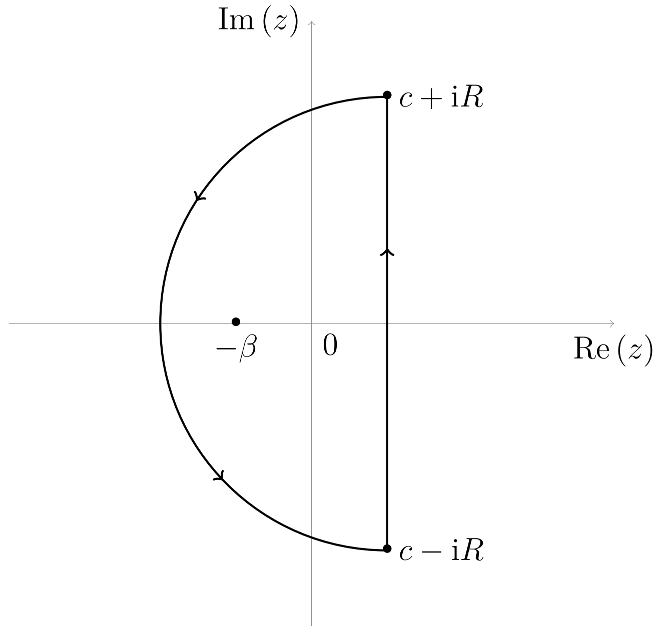


Figure 2.1: The contour γ_R

Recognizing that the integrand of (2.12) is a meromorphic function with a single pole at $z = -\beta$, we can deform the contour of integration to the left of the origin. Indeed, when $\alpha > 1$ one may show that

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{C_\epsilon(-\beta)} \frac{\beta^\alpha e^{zx}}{(z+\beta)^\alpha} \mathrm{d}z, \qquad x > 0, \tag{2.13}$$

where the contour $C_\epsilon(-\beta)$ is the circle of radius $\epsilon > 0$, centered at $-\beta$, with positive (counter-clockwise) orientation. To show this, one would show that

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \phi(z) e^{zx} \mathrm{d}z = \lim_{R\to\infty} \int_{\gamma_R} \phi(z) e^{zx} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{C_\epsilon(-\beta)} \phi(z) e^{zx} \mathrm{d}z,$$

where γ_R is the semi-circle of radius R depicted in Figure 2.1. Invoking Cauchy's integral theorem we obtain the pdf of X,

$$f(x) = \frac{1}{(\alpha-1)!} \left[\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}z^{\alpha-1}} \beta^\alpha e^{zx} \right]_{z=-\beta} = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x > 0.$$

If the parameter α was not an integer, we would have a branch point at $z = -\beta$ rather than a pole, and the argument above would not be valid. In this situation, we would deform the

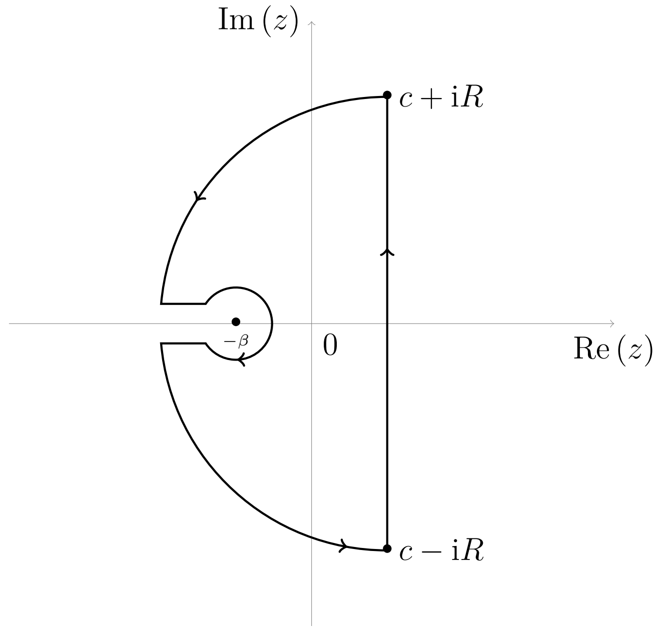


Figure 2.2: The contour $\tilde{\gamma}_R$

contour of integration to the left of the origin avoiding the branch cut $(-\infty, -\beta)$ using a keyhole contour, as in Figure 2.2. In this case, one may show that

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \phi(z) e^{zx} \mathrm{d}z = \int_H \phi(z) e^{zx} \mathrm{d}z,$$

where H is a Hankel contour, depicted in Figure 2.3. As the Hankel contour converges to the line segment $(-\infty, -\beta)$ from above and below, one may show that

$$\begin{aligned} f(x) &= -\frac{1}{\pi} \int_\beta^\infty \mathrm{Im}\left[\phi(-t + i \cdot 0)\right] e^{-tx} \mathrm{d}t \\ &= \beta^\alpha e^{-\beta x} \frac{\sin(\alpha\pi)}{\pi} (-1)^{-\alpha} \int_\beta^\infty (-t+\beta)^{-\alpha} e^{(-t+\beta)x} \mathrm{d}t \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x > 0. \end{aligned}$$

Analytic inversion methods often fail because the functions we wish to invert are simply too complicated. We commonly encounter functions for which integration does not yield a known closed form or functions for which repeated differentiation becomes unreasonable. In these cases, the pragmatic approach is to apply a numerical method for inversion.

The most direct method is to apply a numerical quadrature method to the integral

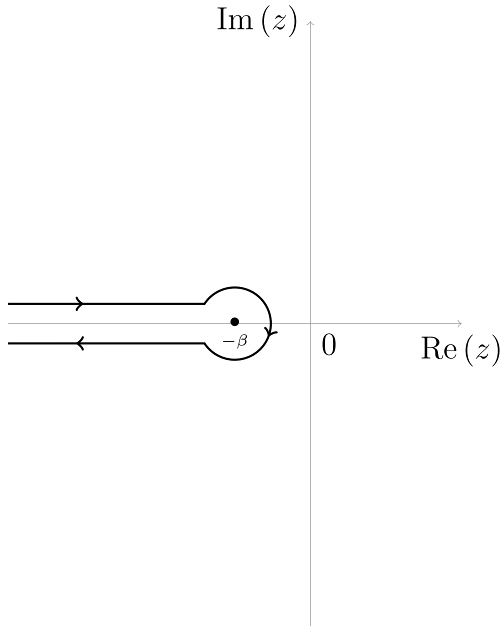


Figure 2.3: The contour H.

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathcal{L}^*\{\alpha\}(z)\frac{e^{zx}}{z}\mathrm{d}z,$$

or the integral

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathcal{L}\{\alpha\}(z)e^{zx}\mathrm{d}z.$$

This approach has several disadvantages:

- this is not an all-purpose method and we must tailor our methodology to suit the function under consideration.
- the method requires values of the Laplace transform at complex numbers.
- the argument of the exponential term has positive real part.

We can often avoid the last disadvantage by transforming the contour of integration. If the integrand can be analytically continued to the left of the origin, it may be possible to shift the contour so that the exponential term has negative real part. Moreover, if the integrand has sufficient decay in this half-plane as $|z| \to \infty$, we may integrate over the negative real line as in the preceding example.

If we are dealing with a specific function and we can calculate values of the Laplace transform in the complex plane, this method can be simple to implement and yield excellent results. However, we often need an all-purpose method that is accurate, applicable to a wide range of functions, and does not require the values of the Laplace tansform at complex numbers.

For the remainder of this section, we discuss several popular algorithms for recovering the target function

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(z) e^{zx} \mathrm{d}z, \tag{2.14}$$

from its Laplace transform

$$F(z) = \int_0^\infty e^{-zx} f(x) \mathrm{d}x. \tag{2.15}$$

2.3.1 Filon's Method

We can express the Bromwich integral 2.14 in terms of the cosine transformation as follows

$$\frac{2e^{cx}}{\pi} \int_0^\infty \mathrm{Re}\left[F(c+\mathrm{i}u)\right] \cos(ux) \mathrm{d}u, \qquad x > 0. \tag{2.16}$$

Since the integral 2.16 is an oscillatory integral, applying a numerical method directly can be challenging. For example, a numerical method such as the trapezoid rule would require an increasingly finer mesh for large values of x because the period of the function $\cos(ux)$ decreases as x increases.

Filon's method applies to oscillatory integrals of the form

$$\int_a^b g(u) cos(ux) \mathrm{d}u. \tag{2.17}$$

By approximating the function $g(u)$ with second order Lagrange interpolating polynomials, the method avoids the discretization problem since the product of a polynomial and the cosine function can be integrated explicitly.

The method is implemented by first creating a mesh consisting of $2N+1$ points, $u_k = a + k(b-a)/2N$, $k = 0, 1, \ldots, 2N$, and dividing the integral in 2.17 into N sub-integrals as follows,

$$\int_a^b g(u)\cos(ux)\mathrm{d}u = \sum_{k=0}^{N-1} \int_{u_{2k}}^{u_{2k+2}} g(u)\cos(ux)\mathrm{d}u$$

Next, we approximate the function $g(u)$ on each sub-interval $[u_{2k}, u_{2k+2}]$ with a second order Lagrange interpolating polynomial using the points (u_{2j}, g_{2j}), (u_{2j+1}, g_{2j+1}), and (u_{2j+2}, g_{2j+2}), where $g_j := g(u_j)$. With $g(u) \approx c_0^{(j)} + c_1^{(j)} u + c_2^{(j)} u^2$ on $[u_{2j}, u_{2j+2}]$, we have

$$\int_a^b g(u)\cos(ux)\mathrm{d}u \approx \sum_{k=0}^{N-1} \int_{u_{2k}}^{u_{2k+2}} \left(c_0^{(j)} + c_1^{(j)} u + c_2^{(j)} u^2\right)\cos(ux)\mathrm{d}u$$

and the expression on the right-hand side is computed explicitly.

2.3.2 The Euler algorithm

With change of variable $u \mapsto v/x$ and the change $c \mapsto c/x$, the Bromwich integral 2.14 can be written as follows

$$f(x) = \frac{e^c}{2\pi x}\int_{\mathbb{R}} F\left(\frac{c + \mathrm{i}v}{x}\right) e^{\mathrm{i}v}\mathrm{d}v. \tag{2.18}$$

The Euler algorithm approximates the integral in 2.18 by applying the trapezoid rule combined with the Euler acceleration method to improve the convergence rate. The approximation is given by

$$f^E(x; M) = \frac{10^{\frac{M}{3}}}{x}\sum_{n=0}^{2M}(-1)^n a_n \operatorname{Re}\left[F\left(\left(\ln\left(10^{\frac{M}{3}}\right) + \pi \mathrm{i} n\right) x^{-1}\right)\right], \tag{2.19}$$

where the coefficients a_n are defined by

$$\begin{aligned} a_0 &= \frac{1}{2}, \\ a_n &= 1, \qquad \text{for } 1 \le n \le M, \\ a_{2M} &= 2^{-M}, \\ a_{2M-k} &= a_{2M-k+1} + 2^{-M}\binom{M}{n}, \qquad \text{for } 1 \le n \le M. \end{aligned}$$

For the Euler algorithm, it is recommended to set $M = \lceil 1.7j \rceil$ if j significant digits are required, and then set the system precision to M [42].

2.3.3 The fixed Talbot algorithm

As with the Euler algorithm, the fixed Talbot algorithm starts with the Bromwich integral but then the contour of integration is transformed so that $\operatorname{Re}(z) \to -\infty$ on this contour. The benefit of this transformation is that the integrand converges to zero much faster, however, the method requires that the function can be analytically continued to the complex plane minus the cut $(-\infty, 0]$. In the case that the Laplace transform corresponds to a GGC random variable, this condition is satisfied.

The fixed Talbot approximation is given by

$$f^T(x; M) = \frac{1}{x} \sum_{n=0}^{M-1} \operatorname{Im}\left[a_n F\left(b_n x^{-1}\right)\right], \tag{2.20}$$

where the coefficients a_n and b_n are given by

$$\begin{aligned}
b_0 &= \frac{2M}{5}, \\
b_n &= \frac{2\pi n}{5}\left(\cot\left(\frac{\pi n}{M}\right) + \mathrm{i}\right), \qquad \text{for } 1 \leq n < M, \\
a_0 &= \frac{\mathrm{i}}{5} e^{b_0}, \\
a_n &= \left(b_n - \frac{5}{2M}|b_n|^2\right)\frac{e^{b_n}}{n\pi}, \qquad \text{for } 1 \leq n < M.
\end{aligned}$$

For the fixed Talbot algorithm, it is also recommended to set $M = \lceil 1.7j \rceil$ if j significant digits are required, and then set the system precision to M [42].

2.3.4 Expansion using generalized Laguerre polynomials

Several numerical methods adopt the following methodology:

1. Assume the target function $f(x)$ (Laplace transform $F(z)$) has, or can be approximated by, some form of expansion. The Laplace transform (inverse Laplace transform) of the expansion is known.

2. The known Laplace transform $F(z)$ is evaluated on a mesh of points to determine the coefficients of the expansion.

In their survey and comparison of methods, [23] suggest the expansion of $f(x)$ using exponential functions seldom yields results with a high accuracy. On the other hand, expanding the target function using generalized Laguerre polynomials resulted in an all-purpose method that produces exceptional results for a wide range of functions. The approximate expansion of $f(x)$ is

$$f(x) \approx x^{\alpha} e^{-cx} \sum_{k=0}^{N} a_k \frac{k!}{(\alpha + k)!} L_k^{\alpha}(t/T), \tag{2.21}$$

where α, c, and T are parameters and the generalized Laguerre polynomials are calculated recursively using the relations

$$\begin{aligned} L_0^\alpha(x) &= 1 \\ L_1^\alpha(x) &= 1 + \alpha - x \\ kL_k^\alpha(x) &= (2k + \alpha - 1 - x)L_{k-1}^\alpha(x) - (k - 1 + \alpha)L_{k-2}^\alpha(x) \end{aligned}$$

The expansion (2.21) has Laplace transform

$$F(z) \approx \sum_{k=0}^{N} a_k (z + c - 1)^k (z + c)^{-(k+\alpha+1)} \tag{2.22}$$

Introducing the new variable $w = (z + c - 1)/(z + c)$, we have

$$G(w) := (z + c)^{\alpha+1} F(z) = \sum_{k=0}^{N} a_k w^k. \tag{2.23}$$

Setting $w = e^{\mathrm{i}\theta}$, the coefficients a_k, $k = 0, \ldots, N$, are calculated using trigonometric interpolation.

2.3.5 Piessens' Method

Using a similar approach, Piessens' method assumes the Laplace transform F can be written as an expansion of Jacobi polynomials. Employing a special case of the Jacobi polynomials, the Chebyshev polynomials, we have

$$F(z) \approx x^{-\alpha-1} \sum_{k=0}^{N} a_k T_k(1 - bz^{-1}). \tag{2.24}$$

Inverting the series term-by-term, the target function takes the form

$$f(x) \approx \frac{x^\alpha}{\alpha!} \sum_{k=0}^{N} a_k \phi_k(bx/2), \tag{2.25}$$

where ϕ_k is a polynomial of degree k. In 1990, Cope [20] showed that the method converges exponentially fast when the Laplace transform has the form $F(z) = z^{-\gamma} G(z)$ with $\gamma > 0$ and G analytic at ∞.

2.3.6 The Gaver-Stehfest algorithm

The Gaver-Stehfest algorithm is a popular inversion algorithm that was first developed by Gaver [31] in 1966 and subsequently improved by Stehfest [54, 55] in 1970. The algorithm establishes a sequence of approximations which converge to the generating function of the Laplace transform. The great advantage of the Gaver-Stehfest approximations is that they only require the Laplace transform to be evaluated on the positive real line. The downside of this algorithm is that the method requires the use of high-precision arithmetic; a trade-off that is easily accommodated with modern day computers.

Suppose $f : (0, \infty) \mapsto \mathbb{R}$ is a locally integrable function such that its Laplace transform

$$F(z) := \int_0^\infty e^{-zx} f(x) \mathrm{d}x \tag{2.26}$$

is finite for all $z > 0$. The n^{th} Gaver-Stehfest approximation at the point $x > 0$ is given by

$$f_n(x) := \ln(2)x^{-1} \sum_{k=1}^{2n} a_k(n) F\left(k \ln(2)x^{-1}\right), \quad n \geq 1,\ x > 0, \tag{2.27}$$

where

$$a_k(n) := \frac{(-1)^{n+k}}{n!} \sum_{j=[(k+1)/2]}^{\min(k,n)} j^{n+1} \binom{n}{j} \binom{2j}{j} \binom{j}{k-j}, \quad 1 \leq k \leq 2n.$$

In Chapter 6, we prove that Gaver-Stehfest approximations converge exponentially fast if the generating function is analytic in a neighbourhood of a point and they converge at a rate $o(n^{-k})$ if the generating function is $(2k + 3)$-times differentiable at a point.

Chapter 3

Generalized Gamma Convolutions

A probability distribution is infinitely divisible if it can be shown to have the same distribution as a sum of an arbitrary number of independent and identically distributed random variables. Since an insurance claim is often viewed as a sum of a number of partial claims, an actuary should confirm that a probability distribution is infinitely divisible before using it to model an aggregate claim.

In 1977, Olof Thorin, an actuarial mathematician, introduced the class of probability distributions called Generalized Gamma Convolutions (GGCs), and he used them to show that the log-normal distribution was infinitely divisible. The infinite divisibility of the log-normal distribution was on open question for many years, and Olof Thorin's work paved the way for the rich collection of results that followed.

It turns out that several important probability distributions used in the field of actuarial science are GGCs. In this chapter, we provide the definition of a GGC and review several characterizations.

3.1 Definition of a GGC

Recall that the Gamma distribution is an absolutely continuous random variable, $X \sim Gamma(\alpha, \beta)$, with positive parameters α, and β and probability density function given by

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x > 0.$$

The Laplace transform of the gamma distribution, X, is given by

$$\phi(z) = \left(\frac{\beta}{z+\beta}\right)^{\alpha}$$

Let $S_n = X_1 + X_2 + \ldots + X_n$ be a convolution (i.e., sum) of n independent gamma random variables. The Laplace transform of S_n is given by

$$\begin{aligned}\phi_{S_n}(z) &= \phi_{X_1} \cdot \phi_{X_2} \cdot \ldots \cdot \phi_{X_n} \\ &= \prod_{i=1}^{n} \beta_i^{\alpha_i}(z+\beta_i)^{-\alpha_i} \\ &= \exp\left[\sum_{i=1}^{n} \alpha_i \ln\left(\frac{\beta_i}{z+\beta_i}\right)\right]\end{aligned}$$

Considering the limiting distributions as we take $n \to \infty$, we are led to the following definition of a GGC. In this chapter, we use the moment-generating function rather than the Laplace transform since this is common practice in the literature. Recall that the moment-generating function satisfies $M(s) = \phi(-s)$.

Definition 4 ([15])**.** A GGC is a probability distribution F on $[0, \infty)$ with moment-generating function (mgf) of the form

$$M(s) = \exp\left[as + \int_0^{\infty} \ln\left(\frac{t}{t-s}\right) U(\mathrm{d}t)\right], \qquad s \leq 0 \text{ (or } s \in \mathbb{C} \setminus (0,\infty)),$$

where $a \geq 0$ and $U(\mathrm{d}t)$ is a nonnegative measure, called the Thorin measure, on $(0, \infty)$ satisfying

$$\int_{(0,1]} |\ln t| U(\mathrm{d}t) < \infty, \text{ and } \int_{(1,\infty)} t^{-1} U(\mathrm{d}t) < \infty,$$

This class of distributions is often denoted by $\mathscr{T}$, in honour of Olof Thorin. The $\mathscr{T}$-class is a rich class of distributions that is closed with respect to convolutions and weak limits.

Theorem 1 (Closure theorem [15])**.** *If $F_n \in \mathscr{T}$, $n = 1, 2, \ldots, \infty$, and $F_n \to F$ weakly (with F non-defective), then $F \in \mathscr{T}$.*

In fact, the $\mathscr{T}$-class is equal to the class of weak limits of finite convolutions of gamma distributions. For the remainder of this section, we list several characterizations of GGC's that are based on the moment-generating function or the density of the distribution.

3.2 Moment-generating function characterizations of GGC's

The first theorem provides a characterization of GGC's in terms of the Lévy measure and shows that every GGC is infinitely divisible. The moment generating function of an infinitly divisible distribution on $[0, \infty)$ can be represented as

$$M(s) = \exp\left[as + \int_0^\infty (e^{st} - 1)L(\mathrm{d}t)\right],$$

where $a \geq 0$ and the Lévy measure, L, is non-negative and satisfies $\int_0^\infty \min(1, t)L(\mathrm{d}t) < \infty$. Note that a function, f, is completely monotone if $(-1)^n f^n(x) \geq 0$, $n \in \mathbb{N}_0$.

Theorem 2 ([15])**.** *A probability distribution on $[0, \infty)$ is GGC if and only if it is infinitely divisible and the Lévy measure has a density l such that $xl(x)$, $x > 0$, is completely monotone.*

We consider the stable distribution with moment generating function $M(s) = \exp[-(-s)^\alpha]$, $0 < \alpha < 1$, as an illustrative example. Writing

$$\frac{M'(s)}{M(s)} = \alpha(-s)^{\alpha-1} = a + \int_0^\infty e^{st} tl(t)\mathrm{d}t,$$

we see that $a = 0$ and $tl(t) = \frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha}$ is completely monotone. Thus, Theorem 2 ensures that the stable distribution is a GGC.

A function of a complex variable, ψ, is called a Pick function, denoted by $\psi \in \mathcal{P}$, if it is analytic in the upper half-plane, $\mathrm{Im}\,(z) > 0$, and has a non-negative imaginary part in the upper half-plane. Furthermore, if ψ is also continuous up to and on the real interval (a, b), it is denoted by $\psi \in \mathcal{P}(a, b)$.

If M is a moment-generating function of a GGC, then

$$\frac{M'(z)}{M(z)} = a + \int_0^\infty \frac{1}{t - z} U(\mathrm{d}t),$$

and so $M'(z)/M(z) = \psi \in \mathcal{P}(-\infty, 0)$. Thus, the following theorem provides a characterization of GGC's in terms of Pick functions.

Theorem 3 ([15])**.** *A probability distribution on $[0, \infty)$ is GGC if and only if its moment-generating function, M, is analytic and non-zero in $\mathbb{C} \setminus [0, \infty)$ and $\mathrm{Im}\,[M'(z)/M(z)] \geq 0$ for all $\mathrm{Im}\,(z) > 0$.*

Returning to our illustrative example, the stable distribution has moment-generating function $M(s) = \exp\left[-(-s)^{\alpha}\right]$ and it follows from Theorem 3 that it is a GGC since $M'(s)/M(s)$ is a Pick function.

The following theorem is referred to as the inversion theorem. Thorin used techniques based off the inversion theorem to prove that the log-normal distribution is a GGC.

Theorem 4 ([15]). *Let M be the moment-generating function of a probability distribution F on $[0, \infty)$ such that*

a) M is analytic and zero-free in $\mathbb{C} \setminus [0, \infty)$, and continuously differentiable up to and on the cut with non-zero boundary values.

b) $M'(z)/M(z) \to a$ uniformly as $|z| \to \infty$, $z \in \mathbb{C} \setminus [0, \infty)$

c) $\arg\left(M(z)\right)$ is increasing for $z > 0$, or, equivalently, $\operatorname{Im}\left[M'(z)/M(z)\right] \geq 0$ for $z > 0$.

Then F is a GGC with left extremity a. Furthermore, the Thorin measure, U, has density $u(t) = \frac{1}{\pi} \cdot \operatorname{Im}\left[M'(z)/M(z)\right]$ and satisfies $\int_{(0,z]} U(\mathrm{d}t) = \frac{1}{\pi} \cdot \arg\left[M(z)\right]$, $z > 0$.

The following theorem provides a useful test to determine if a moment-generating function corresponds to a GGC.

Theorem 5 ([15]). *A probability distribution on $[0, \infty)$ is GGC if and only if its moment-generating function, M, is analytic in $\mathbb{C} \setminus [0, \infty)$ and $\operatorname{Im}\left[M'(z) \cdot \overline{M(z)}\right] \geq 0$ for all $\operatorname{Im}(z) > 0$.*

3.3 Densities of GGC's

Many distributions do not have a moment-generating function (e.g., Pareto distribution) or have a moment-generating function that is difficult to work with (e.g., log-normal distribution). In these cases, the theorems of the previous section are not easily employed to verify whether a distribution is a GGC. In this section, we consider two characterizations of GGC's in terms of distribution's density.

Theorem 6 ([15]). *If $g(t)$ is log-concave on $(0, \infty)$ and the random variable X has a probability density function of the form*

$$f(x) = \int_0^{\infty} t e^{-xt} g(t) \mathrm{d}t, \tag{3.1}$$

then $X \in \mathscr{T}$.

Theorem 6 can be used to show that the Pareto distribution is a GGC. The Pareto distribution is a positive distribution with parameters $\alpha > 0$ and $\beta > 0$ and probability density function

$$f(x) = \frac{\alpha\beta^\alpha}{(x+\beta)^{\alpha+1}}, \qquad x > 0,$$

which can be written in the form

$$f(x) = \int_0^\infty t e^{-xt} g(t) \mathrm{d}t, \tag{3.2}$$

with

$$g(t) = \frac{1}{\Gamma(\alpha)} \beta^\alpha t^\alpha e^{-\beta t}. \tag{3.3}$$

Since $g'(t)/g(t) = \alpha/t - \beta$ is decreasing, $g(t)$ is log-concave and Theorem 6 shows that the Pareto distribution is a GGC.

The second theorem we consider shows that a distribution with a hyperbolically completely monotone density is a GGC.

Definition 5. A non-negative function, f, on $(0, \infty)$ is hyperbolically completely monotone (HCM) if, for each $u > 0$, $f(uv) \cdot f(u/v)$ is completely monotone as a function of $w = v + v^{-1}$.

Theorem 7 ([15])**.** *If the random variable X has a probability density function, f, on $(0, \infty)$ such that f is HCM, then $X \in \mathscr{T}$.*

Theorem 7 can be used to show that the log-normal distribution is a GGC. Since the general density is obtained by change of scale, we consider the random variable $X \sim \mathrm{LN}(0, \sigma^2)$, with probability density function given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\ln x)^2}{2\sigma^2}\right], \qquad x > 0.$$

A function of the form $h \circ g$ is completely monotone when h is completely monotone and g is non-negative with a completely monotone derivative. Thus, with

$$f(uv)f(u/v) = \frac{1}{2\pi\sigma^2 u^2} \exp\left[-\frac{(\ln u)^2}{\sigma^2} - \frac{(\ln v)^2}{\sigma^2}\right],$$

it suffices to show that $(\ln v)^2$ has completely monotone derivative with respect to $w = v+v^{-1}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}w}(\ln v)^2 = \frac{\ln v - \ln(v^{-1})}{v - v^{-1}},$$

and

$$\ln v = \int_{-\infty}^{0} \left(\frac{1}{t-v} - \frac{1}{t-1} \right) \mathrm{d}t,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}w} (\ln v)^2 = \int_{-\infty}^{0} \frac{1}{1+t^2-tw} \mathrm{d}t,$$

which is completely monotone with respect to w. Thus, Theorem 7 ensures that the log-normal distribution is a GGC.

Chapter 4

The Laplace transform of the log-normal distribution

In this chapter we study the analytical properties of the Laplace transform of the log-normal distribution. Two integral expressions for the analytic continuation of the Laplace transform of the log-normal distribution are provided, one of which takes the form of a Mellin-Barnes integral. As a corollary, we obtain an integral expression for the characteristic function; we show that the integral expression derived by Leipnik in [45] is incorrect. We present two approximations for the Laplace transform of the log-normal distribution, both valid in $\mathbb{C} \setminus (-\infty, 0]$. In the last section, we discuss how one may use our results to compute the density of a sum of independent log-normal random variables.

4.1 Introduction

A positive random variable X is said to have a log-normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, written $X \sim \mathrm{LN}(\mu, \sigma^2)$, if it has probability density function given by

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right], \qquad x > 0.$$

The log-normal distribution has a wide range of applications in the natural sciences and fields like finance, actuarial science, economics and engineering. Integral transforms, such as the Laplace and Fourier transforms, of the log-normal distribution have received considerable attention in the literature for several decades. The Laplace transform of X, henceforth denoted by ϕ, is defined by

$$\phi(z;\mu,\sigma) := \mathbb{E}\left[e^{-zX}\right] = \int_0^\infty e^{-zx} f(x;\mu,\sigma)\mathrm{d}x, \qquad \mathrm{Re}\,(z) \geq 0. \tag{4.1}$$

The characteristic function of X, henceforth denoted by φ, is the restriction of ϕ to the imaginary axis:

$$\varphi(t;\mu,\sigma) := \mathbb{E}\left[e^{\mathrm{i}tX}\right] = \phi(-\mathrm{i}t;\mu,\sigma), \;\; t \in \mathbb{R}. \tag{4.2}$$

Since these integral transforms have no known closed form, there has been substantial effort to put forth viable approximation methods (see [6] for a thorough overview and numerical comparison of several methods). Some authors, such as Barouch and Kaufman [11], Barakat [10], Holgate [36], and Leipnik [45], have proposed series representations for (4.2). Others, including Gubner [33] and Tellambura and Senaratne [58], have proposed numerical integration methods for computing (4.1). Gubner's numerical integration procedure reduces oscillations of the integrand by deforming the contour of integration. Tellambura and Senaratne improved upon Gubner's method by deriving the steepest-descent contour and by providing two, related, closed-form contours.

More recently, Asmussen et al. [6] used a modified version of Laplace's method to derive an asymptotically equivalent, closed-form approximation for (4.1). Moreover, Asmussen et al. [6] constructed a Monte Carlo estimator and, based on this framework, Laub et al. [44] generalized the approach to approximate the Laplace transform of a finite sum of dependent log-normals.

There are several disadvantages with existing methods in the literature. Examples include:

- The majority of methods are only valid, at most, for arguments in the right half plane, $\{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq 0\}$. As a result, one must exclude some efficient paths of integration when performing an inversion of the Laplace transform.

- It appears that there are no convergent series representations in the literature that are valid on the entire domain of analyticity. Since ϕ is not analytic at the origin, the Taylor series representation centered at any point will have finite radius of convergence. For example, the formal Taylor series of ϕ, centered at the origin, is given by

$$\sum_{n=0}^{\infty} \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2}. \tag{4.3}$$

It is easy to see that the series (4.3) diverges for all $z \neq 0$.

- In 1991, Leipnik [45] presented the following expression for the characteristic function: Let $X \sim \mathrm{LN}(0, \sigma^2)$, then, for $t > 0$ and $0 < k < 1$, the characteristic function is given by

$$\varphi(t; 0, \sigma) \stackrel{?}{=} \frac{1}{2\pi} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \sin(\pi s) \Gamma(s) e^{-(\ln t + \mathrm{i}\frac{\pi}{2})s + \frac{\sigma^2}{2}s^2} \mathrm{d}s. \tag{4.4}$$

  It has been reported that the right-hand side of (4.4), and the subsequent series for φ derived in [45], are unreliable in numerical computations (see [27], and [6]). We claim that the result is incorrect. To see that (4.4) is incorrect, observe that the integrand is entire and that one may take $k \in \mathbb{R}$. After shifting the contour of integration to the left of the origin (taking $k < 0$), it is easy to see that the expression in (4.4) is $O(t^{|k|})$, as $t \to 0$. Hence, the expression converges to 0 as $t \to 0$, violating the fact that the characteristic function must converge to 1 as $t \to 0$.

  Leipnik obtains (4.4) by first deriving a functional differential equation, and then solving it using a method due to de Bruijn. In this method, the differential equation is transformed into a forward difference-differential equation and an ansatz solution is posed. It appears that Leipnik imposed an inconvenient condition on the ansatz; specifically, in equation (25) of [45], he imposed the condition $S(z-1) = -S(z)$ when he could have taken $S(z-1) = S(z)$. As a result, Leipnik searched for an anti-periodic solution for S and ultimately obtained $S(z) = \sin \pi z$ rather than $S(z) = 1$.

In this paper, we explore the analytic continuation of the Laplace transform of the log-normal distribution and present new, efficient, series *approximations* of ϕ that are valid on $\mathbb{C} \setminus (-\infty, 0]$. In Sections 4.2 and 4.3, we provide two (integral) expressions of the analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$, one of which takes the form of a Mellin-Barnes integral. As a corollary, we obtain an integral expression for the characteristic function φ (this expression is stated by Dufresne in [27] without proof). In the third section of this paper, we exploit the Mellin-Barnes integral expression and use knowledge of the gamma function to derive series approximations for ϕ that are valid for arguments in $\mathbb{C} \setminus (-\infty, 0]$.

The first approximation we present in Section 4.4 is a convergent series for which the error term is uniformly bounded on $\mathbb{C} \setminus (-\infty, 0]$ by a constant that can be made arbitrarily small (by choice of a parameter). Furthermore, the approximation is asymptotic to ϕ as the magnitude of the argument decreases to zero. The second approximation we present is a sum which improves as the parameter $\sigma \to \infty$. The terms of the series/sum are composed of expressions involving error functions and/or Hermite polynomials. The approximations are used to compute ϕ for several real arguments and the results compared to the values obtained by way of numerical integration.

In Section 4.5, we discuss how one may use the analytic continuation of ϕ to compute the density of a sum of independent log-normals via Laplace inversion. By deforming the contour of the Bromwich integral to a Hankel contour, we obtain a real integral with an integrand which decays exponentially. The result is an integral which is easily evaluated numerically.

4.2 The analytic continuation of the Laplace transform of the log-normal distribution

The integral definition of the function ϕ, given by (4.1), is finite when $\mathrm{Re}\,(z) \geq 0$ and it is well known that it is analytic in the right half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$. It will be convenient for us to express this function as

$$\phi(z; \mu, \sigma) = C(\mu, \sigma) \int_0^\infty \frac{1}{x} \exp\left[-zx - \frac{1}{2\sigma^2}(\ln x)^2 + \frac{\mu}{\sigma^2} \ln x\right] \mathrm{d}x, \tag{4.5}$$

where $C(\mu, \sigma) := (2\pi\sigma^2)^{-1/2} \exp\left(-\mu^2/2\sigma^2\right)$. Noting that the integral in (4.5) is finite for all $\mu \in \mathbb{C}$, we define

$$\Phi(z, w; \sigma) := C(w, \sigma) \int_0^\infty \frac{1}{x} \exp\left[-zx - \frac{1}{2\sigma^2}(\ln x)^2 + \frac{w}{\sigma^2} \ln x\right] \mathrm{d}x, \qquad (z, w) \in \overline{\mathbb{C}^+} \times \mathbb{C}. \tag{4.6}$$

Since $\Phi(z, \mu; \sigma) = \phi(z; \mu, \sigma)$, the function Φ is an extension of ϕ. Here $\overline{\mathbb{C}^+}$ denotes the closure of $\mathbb{C}^+$, and throughout this paper we take the logarithm to be complex with the principal branch. The main result of this section is given in the following theorem. It provides us with an expression for $\phi(z; \mu, \sigma)$ which is analytic on $\mathbb{C} \setminus (-\infty, 0]$.

Theorem 6. *Let $\sigma > 0$ and let Φ be defined by (4.6). Then the Laplace transform of $X \sim \mathrm{LN}(\mu, \sigma^2)$ is analytically continued to $\mathbb{C} \setminus (-\infty, 0]$ by the equation*

$$\phi(z; \mu, \sigma) = \Phi(1, \mu + \ln z; \sigma). \tag{4.7}$$

Proof. Fix $\sigma > 0$ and let $F(z, w, x) = x^{-1} \exp\left[-zx - (\ln x)^2/2\sigma^2 + w \ln x/\sigma^2\right]$ so that

$$\Phi(z, w; \sigma) = C(w, \sigma) \int_0^\infty F(z, w, x) \mathrm{d}x.$$

The function $C(\cdot,\sigma)$ is entire, and, for each $z \in \mathbb{C}^+$, $F(z,\cdot\,,\cdot)$ is continuous on $\mathbb{C}\times(0,\infty)$, and, for each pair $(z,x) \in \mathbb{C}^+ \times (0,\infty)$, $F(z,\cdot\,,x)$ is entire. Thus, for each $n \in \mathbb{N}$, and for each $z \in \mathbb{C}^+$, the function $\Phi_n(z,\cdot\,;\sigma)$ defined by

$$\Phi_n(z,w;\sigma) := C(w,\sigma)\int_{\frac{1}{n}}^{n} F(z,w,x)\mathrm{d}x, \qquad w \in \mathbb{C}$$

is entire. Since $\Phi_n(z,\cdot\,;\sigma) \to \Phi(z,\cdot\,;\sigma)$ uniformly on compact subsets of $\mathbb{C}$, the function $\Phi(z,\cdot\,;\sigma)$ is entire. To prove the theorem, we make the formal substitution $ct = x$ in (4.6) which yields

$$\begin{aligned}
\Phi(z,w;\sigma) &= C(w,\sigma)\int_0^\infty \frac{1}{ct}\exp\left[-zct - \frac{1}{2\sigma^2}(\ln ct)^2 + \frac{w}{\sigma^2}\ln ct\right]c\mathrm{d}t \\
&= C(w,\sigma)\exp\left(-\frac{1}{2\sigma^2}(\ln c)^2 + \frac{w}{\sigma^2}\ln c\right) \\
&\qquad\qquad \times \int_0^\infty \frac{1}{t}\exp\left[-zct - \frac{1}{2\sigma^2}(\ln t)^2 + \frac{(w-\ln c)}{\sigma^2}\ln t\right]\mathrm{d}t \\
&= \Phi(cz, w - \ln c;\sigma), \qquad (4.8)
\end{aligned}$$

where (4.8) holds provided $cz \in \mathbb{C}^+$. Setting $c = 1/z$ we obtain

$$\Phi(z,w;\sigma) = \Phi(1, w + \ln z;\sigma). \qquad (4.9)$$

Therefore, since $\Phi(z,\cdot\,;\sigma)$ is an entire function for each $z \in \mathbb{C}^+$, setting $w = \mu$ yields an analytic continuation for the Laplace transform of X defined by (4.1). □

4.3 The analytic continuation of ϕ as a Mellin-Barnes integral

In this section we derive an alternate expression for ϕ, the Laplace transform of $X \sim \mathrm{LN}(\mu,\sigma^2)$, in the form of a Mellin-Barnes integral. As a consequence, we obtain the corresponding expression for the characteristic function of the log-normal random variable X. As far as we are aware, there is no other explicit proof of the result in the literature. For convenience, we will often write $f(x)$ and $\phi(z)$ instead of $f(x;\mu,\sigma)$ and $\phi(z;\mu,\sigma)$ with the understanding that μ and σ are the parameters of X.

Theorem 7. *Let $X \sim \mathrm{LN}(\mu, \sigma^2)$ and $k > 0$. Then the Laplace transform of X has integral expression*

$$\phi(z) = \frac{1}{2\pi i} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \Gamma(s) e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s, \qquad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{4.10}$$

Proof. The Mellin transform of ϕ, denoted by $M[\phi; \cdot\,]$, is defined by

$$M[\phi; s] = \int_0^\infty z^{s-1} \phi(z) \mathrm{d}z, \qquad s = k + \mathrm{i}t.$$

Using the definition of the Laplace transform, Fubini's theorem, and the fact that

$$\int_0^\infty z^{s-1} e^{-zx} \mathrm{d}z = x^{-s}\Gamma(s), \qquad \mathrm{Re}\,(s) > 0,$$

we have

$$\begin{aligned} M[\phi; s] &= \int_0^\infty z^{s-1} \left(\int_0^\infty e^{-zx} f(x) \mathrm{d}x \right) \mathrm{d}z \\ &= \int_0^\infty f(x) \left(\int_0^\infty z^{s-1} e^{-zx} \mathrm{d}z \right) \mathrm{d}x \\ &= \Gamma(s) \int_0^\infty x^{-s} f(x) \mathrm{d}x \\ &= \Gamma(s) e^{-\mu s + \frac{\sigma^2}{2}s^2}, \qquad \mathrm{Re}(s) > 0. \end{aligned}$$

This also shows that $z^{k-1}\phi(z) \in L^1(0,\infty)$ for $k > 0$. Furthermore, ϕ is continuous on $(0,\infty)$ and so, by Mellin's inversion formula ([63] Pg.46, Theorem 28),

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} z^{-s} M[\phi; s] \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \Gamma(s) e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s, \qquad z \in (0, \infty).$$

We can extend this function to take arguments in $\mathbb{C} \setminus (-\infty, 0]$, and, in fact, it is analytic on this set. Therefore, our new expression for ϕ must agree with the analytic continuation given in Section 4.2 by the uniqueness of analytic continuation. ∎

Since $\varphi(t) = \phi(-it)$, we have the following corollary to Theorem 7

Corollary 1. *Let $X \sim \mathrm{LN}(\mu, \sigma^2)$ and $k > 0$. Then the characteristic function of X has integral expression*

$$\varphi(t) = \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \Gamma(s) e^{-(\mu+\ln|t| - \mathrm{sgn}\,(t)\mathrm{i}\frac{\pi}{2})s+\frac{\sigma^2}{2}s^2} \mathrm{d}s, \qquad t \in \mathbb{R} \setminus \{0\}. \tag{4.11}$$

4.4 Series approximations and numerical computation of ϕ

In the first subsection we present series approximations which may be used to compute ϕ on $\mathbb{C} \setminus (-\infty, 0]$. In the second subsection, we present numerical results using the series approximations and compare the error using numerical integration as a benchmark.

4.4.1 Series approximations

The following theorem introduces a convergent series that approximates ϕ with an error that can be made arbitrarily small. Note that the approximation bears some resemblance to the results of Barouch and Kaufman [11] who investigated series approximations of the characteristic function.

Theorem 8. *Let $X \sim \mathrm{LN}(\mu, \sigma^2)$ and $\alpha \geq 1$. Then the Laplace transform of X has expression*

$$\phi(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \operatorname{erfc}\left(\frac{\mu + \ln(z/\alpha) + \sigma^2 n}{\sqrt{2}\sigma}\right) + O(e^{-\alpha}), \qquad z \in \mathbb{C} \setminus (-\infty, 0]. \quad (4.12)$$

Furthermore,

$$\phi(z) \sim \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \operatorname{erfc}\left(\frac{\mu + \ln(z/\alpha) + \sigma^2 n}{\sqrt{2}\sigma}\right), \ \text{as } z \to 0. \qquad (4.13)$$

The function erfc *is the complimentary error function.*

Proof. Let $\alpha \geq 1$. For $\operatorname{Re}(s) > 0$, we may write $\Gamma(s) = \gamma(s, \alpha) + \Gamma(s, \alpha)$ where $\gamma(\cdot, \alpha)$ and $\Gamma(\cdot, \alpha)$ are the upper and lower incomplete gamma functions defined by

$$\gamma(s, \alpha) := \int_0^{\alpha} t^{s-1} e^{-t} \mathrm{d}t, \ \text{and } \Gamma(s, \alpha) := \int_{\alpha}^{\infty} t^{s-1} e^{-t} \mathrm{d}t.$$

Substituting this sum into (4.10), and replacing $\gamma(\cdot, \alpha)$ with the power series expansion

$$\gamma(s, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\alpha^{s+n}}{(s+n)},$$

we obtain

$$\begin{aligned}\phi(z) = &\sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \frac{1}{(s+n)} e^{[\ln \alpha - (\mu + \ln z)]s + \frac{\sigma^2}{2} s^2} \mathrm{d}s \\ &+ \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \Gamma(s, \alpha) e^{-(\mu + \ln z)s + \frac{\sigma^2}{2} s^2} \mathrm{d}s,\end{aligned}$$

where $k > 0$ (when necessary). The interchange of summation and integration in the first term is justified by Fubini's theorem and the fact that the integral can be bounded by a Gaussian integral, independent of n. To complete the proof, we need to:

i) compute $\frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \frac{1}{(s+n)} e^{[\ln\alpha - (\mu + \ln z)]s + \frac{\sigma^2}{2}s^2} \mathrm{d}s, \qquad n \in \mathbb{N} \cup \{0\}$, and

ii) bound $\left| \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \Gamma(s,\alpha) e^{-(\mu+\ln z)s + \frac{\sigma^2}{2}s^2} \mathrm{d}s \right|$.

We compute the integral in i) using differentiation with respect to a parameter. Letting

$$F_n(w) = \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \frac{1}{(s+n)} e^{w(s+n) + \frac{\sigma^2}{2}s^2} \mathrm{d}s,$$

we have

$$F_n'(w) = \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} e^{w(s+n) + \frac{\sigma^2}{2}s^2} \mathrm{d}s = \frac{1}{\sqrt{2\pi}\sigma} e^{wn - \frac{w^2}{2\sigma^2}},$$

and so, for $w \in \mathbb{R}$,

$$F_n(w) = F_n(-\infty) + \int_{-\infty}^{w} F_n'(y)\mathrm{d}y = 0 + \frac{e^{\frac{\sigma^2}{2}n^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{w} e^{-\frac{(y-\sigma^2 n)^2}{2\sigma^2}} \mathrm{d}y = \frac{e^{\frac{\sigma^2}{2}n^2}}{2} \operatorname{erfc}\left(\frac{-w + \sigma^2 n}{\sqrt{2}\sigma} \right).$$

It can be shown that the interchange of integration and differentiation is justified, and $F_n(-\infty) = 0$ using the dominated convergence theorem. Since the complementary error function is entire, the result extends to arguments in $\mathbb{C}$ by analytic continuation. Therefore

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \frac{1}{(s+n)} e^{[\ln\alpha - (\mu+\ln z)]s + \frac{\sigma^2}{2}s^2} \mathrm{d}s &= e^{-[\ln\alpha - (\mu+\ln z)]n} F_n\left(\ln\alpha - (\mu + \ln z)\right) \\ &= \alpha^{-n} z^n e^{\mu n + \frac{\sigma^2}{2}n^2} \cdot \frac{1}{2} \operatorname{erfc}\left(\frac{\mu + \ln(z/\alpha) + \sigma^2 n}{\sqrt{2}\sigma} \right) \end{aligned}$$

To bound the integral in ii), observe that the integrand is entire and we may choose any $k \in \mathbb{R}$ for the vertical contour. Choosing $k \leq 1$, we have $|\Gamma(s,\alpha)| \leq e^{-\alpha}$ for $s = k + \mathrm{i}t, \;\; t \in \mathbb{R}$, and so

$$\begin{aligned} \left| \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \Gamma(s,\alpha) e^{-(\mu+\ln z)s + \frac{\sigma^2}{2}s^2} \mathrm{d}s \right| &\leq \frac{e^{-\alpha}}{2\pi} \int_{-\infty}^{\infty} e^{-(\mu + \ln|z|)k + \operatorname{Arg}(z)t + \frac{\sigma^2}{2}(k^2 - t^2)} \mathrm{d}t \\ &\leq \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{\pi^2}{2\sigma^2} + \frac{\sigma^2}{2}k^2 - \alpha - k(\mu + \ln|z|)}. \end{aligned}$$

We set $k = 0$ to obtain the error term in (4.12) and we choose k to be negative to show (4.13). $\square$

The following theorem presents an approximation of ϕ that improves as the parameter σ increases.

Theorem 9. *Let $X \sim \mathrm{LN}(\mu, \sigma^2)$, and $M, N \in \mathbb{N}$. Then the Laplace transform of X has expression*

$$\begin{aligned}\phi(z) = &\sum_{n=0}^{N} \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \operatorname{erfc}\left(\frac{\mu + \ln z + \sigma^2 n}{\sqrt{2}\sigma}\right) \\ &+ \sum_{m=0}^{M} \frac{(-1)^m a_m}{\sqrt{2\pi}\sigma^{m+1}} e^{-\frac{(\mu+\ln z)^2}{2\sigma^2}} H_m\left(-\frac{(\mu + \ln z)}{\sigma}\right) + O(\sigma^{-M-2}).\end{aligned} \tag{4.14}$$

The function erfc *is the complimentary error function, H_m is the m^{th} Hermite polynomial defined by*

$$H_m(x) := (-1)^m e^{\frac{x^2}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m} e^{-\frac{x^2}{2}},$$

and the coefficients a_m are defined by

$$a_m = \frac{\Gamma^{(m+1)}(1)}{(m+1)!} + (-1)^{m+1} \cdot \sum_{j=1}^{N} \frac{(-1)^j}{j!} \frac{1}{j^{m+1}}.$$

Proof. Let $N \in \mathbb{N}$. Recall that the function Γ has a simple pole at $s = -n, \ n = 0, 1, 2, \ldots$, with residue $\operatorname{Res}(\Gamma, -n) = (-1)^n/n!$. We may remove the first $N+1$ poles of Γ by writing

$$\Gamma(s) - \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)}$$

to obtain a function, denoted γ_N, that is holomorphic on $\{s \in \mathbb{C} : |s| < N+1\}$. Thus, we write

$$\Gamma(s) = \gamma_N(s) + \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)},$$

where, for $|s| < N+1$, we may write

$$\gamma_N(s) = \sum_{m=0}^{\infty} a_m s^m,$$

with $a_m = \gamma_N^{(m)}(0)/m!$. Note that as $\sigma \to \infty$, the mass of the integrand in the integral (4.10) is increasingly supported on the set $\{k + \mathrm{i}t \in \mathbb{C} : t \in (-N-1, N+1)\}$. Thus, we choose

$M \in \mathbb{N}$ and write

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \left(\gamma_N(s) + \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)} \right) e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s$$
$$= \sum_{n=0}^{N} \frac{(-1)^n}{n!} \cdot \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \frac{1}{(s+n)} e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s$$
$$+ \sum_{m=0}^{M} a_m \cdot \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} s^m e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s + R_M(z), \tag{4.15}$$

where,

$$R_M(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left(\gamma_N(s) - \sum_{m=0}^{M} a_m s^m \right) e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s.$$

To complete the proof we need to:

i) compute $\frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} \frac{1}{(s+n)} e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s, \quad n \in \{0, 1, \ldots, N\}$,

ii) compute $a_m, \;\; m \in \{0, 1, \ldots, M\}$,

iii) compute $\frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} s^m e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s, \quad m \in \{0, 1, \ldots, M\}$, and

iv) bound $|R_M(z)|$

The integral in i) was computed in the proof of Theorem 3. To determine ii), we need to compute $\gamma_N^{(m)}(0)$. Note that, for $|s| < 1$, we may write

$$\Gamma(s) - \frac{1}{s} = \sum_{j=0}^{\infty} b_j s^j,$$

where $b_j = \Gamma^{(j+1)}(1)/(j+1)!$. So, for $|s| < 1$, we may write

$$\gamma_N(s) = \sum_{j=0}^{\infty} b_j s^j - \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)}.$$

Differentiating, we have

$$\gamma_N^{(m)}(s) = \sum_{j=m}^{\infty} (j)_m b_j s^{j-m} + (-1)^{m+1} m! \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)^{m+1}}$$

and

$$\gamma_N^{(m)}(0) = m! b_m + (-1)^{(m+1)} m! \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{n^{m+1}}.$$

Therefore,

$$a_m = \frac{\gamma_N^{(m)}(0)}{m!} = b_m + (-1)^{(m+1)} \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{n^{m+1}}.$$

To compute iii), we let

$$G(w) = \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} e^{ws+\frac{\sigma^2}{2}s^2} \mathrm{d}s = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{w^2}{2\sigma^2}}.$$

Then

$$G^{(m)}(w) = \frac{\mathrm{d}^m}{\mathrm{d}w^m} \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{w^2}{2\sigma^2}} \right) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \frac{(-1)^m}{\sigma^m} e^{-\frac{w^2}{2\sigma^2}} H_m\left(\frac{w}{\sigma}\right),$$

and therefore,

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{k-\mathrm{i}\infty}^{k+\mathrm{i}\infty} s^m e^{-(\mu+\ln z)s+\frac{\sigma^2}{2}s^2} \mathrm{d}s &= G^{(m)}\left(-(\mu+\ln z)\right) \\ &= \frac{(-1)^m}{\sqrt{2\pi}\sigma^{m+1}} e^{-\frac{(\mu+\ln z)^2}{2\sigma^2}} H_m\left(\frac{-(\mu+\ln z)}{\sigma}\right), \end{aligned}$$

where, again, it can be shown that the interchange of integration and differentiation is justified. Finally, to show iv), we note that

$$\left|\gamma_N(s) - \sum_{m=0}^{M} a_m s^m\right| \le C|s|^{M+1}, \qquad s \in \mathrm{i}\mathbb{R},$$

for some $C > 0$. To see this, observe that, for $|s| < N$, we have

$$\left|\gamma_N(s) - \sum_{m=0}^{M} a_m s^m\right| = \left|\sum_{m=M+1}^{\infty} a_m s^m\right| = \left|\sum_{m=0}^{\infty} a_{M+1+m} s^m\right| \cdot |s|^{M+1} \le C_1|s|^{M+1}.$$

We also have

$$|\gamma_N(s)| = \left|\Gamma(s) - \frac{1}{s} - \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)}\right| \le \left|\Gamma(s) - \frac{1}{s}\right| + \left|\sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)}\right| \le C_2, \qquad s \in \mathrm{i}\mathbb{R},$$

so that, for $|s| \ge N$, we have

$$\left|\gamma_N(s) - \sum_{m=0}^{M} a_m s^m\right| \le |\gamma_N(s)| + \left|\sum_{m=0}^{M} a_m s^m\right| \le C_2 + C_3|s|^M \le C_4|s|^{M+1}.$$

Thus,

$$|R_M(z)| \le \frac{C}{2\pi} \int_{-\infty}^{\infty} |t|^{M+1} e^{\mathrm{Arg}\,(z)t - \frac{\sigma^2}{2}t^2}\, dt = \frac{C}{2\pi} \int_{-\infty}^{\infty} \left|\frac{x}{\sigma}\right|^{M+1} e^{\frac{\mathrm{Arg}\,(z)}{\sigma}x - \frac{x^2}{2}}\, \frac{1}{\sigma} dx \le C'\sigma^{-M-2}.$$

□

4.4.2 Numerical example

We can compute $\phi(z)$, $z \in \mathbb{C} \setminus (-\infty, 0]$, via numerical integration using either of the relations (4.7) or (4.10) given by Theorems 6 or 7, respectively. If we choose to use the former, then we need to compute $\Phi(1, \mu + \ln z; \sigma)$ as defined by (4.6). For simplicity, we will discuss the computation of $\Phi(1, a + \mathrm{i}b; \sigma)$, for $a, b \in \mathbb{R}$.

Making the substitution $x \mapsto e^x$ we have

$$\Phi(1, a + \mathrm{i}b; \sigma) = C(a + \mathrm{i}b, \sigma) \int_{-\infty}^{\infty} \exp\left[-e^x - \frac{x^2}{2\sigma^2} + \frac{(a + \mathrm{i}b)x}{\sigma^2}\right] \mathrm{d}x.$$

We can write this integral in the form

$$\int_{-\infty}^{\infty} g(x) e^{\mathrm{i}tx} \mathrm{d}x, \tag{4.16}$$

where

$$g(x) = \exp\left[-e^x - \frac{x^2}{2\sigma^2} + \frac{ax}{\sigma^2}\right], \text{ and } t = \frac{b}{\sigma^2}.$$

The integral (4.16) can be computed numerically using Filon's quadrature method [29]. First, we determine an interval, $[x_0, x_{2N}]$, which supports most of the integrand's mass and create a mesh consisting of $2N + 1$ points, x_j, $j = 0, \ldots, 2N$. The integral is then written as a sum of N integrals over $[x_{2j}, x_{2j+2}]$, $j = 0, \ldots, N - 1$:

$$\int_{-\infty}^{\infty} g(x) e^{\mathrm{i}tx} \mathrm{d}x \approx \int_{x_0}^{x_{2N}} g(x) e^{\mathrm{i}tx} \mathrm{d}x = \sum_{j=0}^{N-1} \int_{x_{2j}}^{x_{2j+2}} g(x) e^{\mathrm{i}tx} \mathrm{d}x.$$

On each subinterval $[x_{2j}, x_{2j+2}]$, we approximate $g(x)$ with a second order Lagrange interpolating polynomial using the data points $(x_{2j}, g_{2j}), (x_{2j+1}, g_{2j+1})$, and (x_{2j+2}, g_{2j+2}), where $g_j := g(x_j)$. Thus, with $g(x) \approx c_0^{(j)} + c_1^{(j)} x + c_2^{(j)} x^2$ on $[x_{2j}, x_{2j+2}]$, we have

$$\int_{-\infty}^{\infty} g(x) e^{\mathrm{i}tx} \mathrm{d}x \approx \sum_{j=0}^{N-1} \int_{x_{2j}}^{x_{2j+2}} \left(c_0^{(j)} + c_1^{(j)} x + c_2^{(j)} x^2\right) e^{\mathrm{i}tx} \mathrm{d}x. \tag{4.17}$$

The integrals on the right hand side of (4.17) can be computed explicitly. With an appropriate interval of integration, $[x_0, x_{2N}]$, and N sufficiently large, an accurate approximation of the integral (4.16) is obtained.

Theorem 8 was used to numerically compute $\phi(z)$ for several real values of z; Table 4.1 shows the results corresponding to $\sigma = 0.0625, 0.25, 0.75$, and 1 with $\mu = 0$. In each case, the expression in (4.12) was truncated to 41 terms and evaluated using $\alpha = 10$. Table 4.2

Table 4.1: The function ϕ computed using (4.12), truncated to 41 terms, with $\alpha = 10$.

	$\sigma = 0.0625$	$\sigma = 0.25$	$\sigma = 0.75$	$\sigma = 1$
z	$\phi(z)$	$\phi(z)$	$\phi(z)$	$\phi(z)$
0.5	0.60624	0.60196	0.57541	0.56171
1	0.36788	0.36804	0.37469	0.38176
1.5	0.22346	0.22825	0.26086	0.2807
2	0.13586	0.14342	0.18984	0.21631
3	0.050369	0.058656	0.10995	0.14025
5	0.0070017	0.011065	0.045898	0.072028
10	3.9289e-05	0.00028124	0.0096044	0.022991

displays the absolute difference (labeled AD) between $\phi(z)$ computed using (4.12) and the value of $\phi(z)$ computed by way of numerical integration.

Theorem 9 was used to numerically compute $\phi(z)$ for several real values of z; Table 4.3 shows the results corresponding to $\sigma = 1, 1.5, 2$, and 2.5 with $\mu = 0$. In each case, (4.14) was used with $N = 5$, and $M = 10$. Table 4.4 displays the absolute difference (labeled AD) between $\phi(z)$ computed using (4.14) and the value of $\phi(z)$ computed by way of numerical integration.

Table 4.2: absolute difference between $\phi(z)$ computed using (4.12) and $\phi(z)$ computed using numerical integration.

	$\sigma = 0.0625$	$\sigma = 0.25$	$\sigma = 0.75$	$\sigma = 1$
z	AD	AD	AD	AD
0.5	6.572520e-14	6.661338e-14	5.155796e-10	1.478849e-08
1	1.110223e-16	4.013456e-14	1.456569e-08	9.738506e-08
1.5	4.440892e-16	2.525757e-14	6.936278e-08	2.349823e-07
2	2.775558e-17	1.468270e-14	1.760354e-07	3.975210e-07
3	1.942890e-16	2.212219e-11	5.105122e-07	7.251790e-07
5	1.756408e-15	6.980607e-08	1.292328e-06	1.225385e-06
10	1.450124e-05	6.055084e-06	2.185399e-06	1.648996e-06

Table 4.3: The function ϕ computed using (4.14) with $N = 5$, and $M = 10$.

	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$	$\sigma = 2.5$
z	$\phi(z)$	$\phi(z)$	$\phi(z)$	$\phi(z)$
0.5	0.56169	0.54186	0.53012	0.523
1	0.38175	0.39772	0.41216	0.42396
1.5	0.28073	0.31674	0.34538	0.36751
2	0.21634	0.26336	0.30039	0.32893
3	0.14024	0.19613	0.24163	0.27744
5	0.072008	0.12725	0.17708	0.21855
10	0.023002	0.062944	0.10844	0.15117

Table 4.4: absolute difference between $\phi(z)$ computed using (4.14) and $\phi(z)$ computed using numerical integration.

	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$	$\sigma = 2.5$
z	AD	AD	AD	AD
0.5	3.503349e-05	6.444704e-07	2.668369e-08	3.269640e-09
1	1.716174e-05	3.009667e-08	1.325727e-09	9.603728e-10
1.5	1.196279e-04	8.780255e-07	2.567335e-08	1.195540e-09
2	1.371616e-04	1.352950e-06	4.380613e-08	2.832041e-09
3	6.300887e-05	1.180623e-06	5.771800e-08	4.875454e-09
5	2.828704e-04	7.533040e-07	4.059893e-08	6.225437e-09
10	4.467548e-04	3.262143e-06	4.479331e-08	4.615517e-09

4.5 The density of a sum of independent log-normal random variables

We have introduced two integral expressions which analytically continue the Laplace transform of $X \sim \mathrm{LN}(\mu, \sigma^2)$ to $\mathbb{C} \setminus (-\infty, 0]$. We have also provided series approximations which may be used for numerical computations. In the last section of this paper, we consider an application which utilizes the analytic continuation of the Laplace transform of X.

In this section we discuss a method to numerically compute the density of a sum of independent log-normal random variables. In this procedure, we obtain the density function by inverting the Laplace transform of the sum. Using the analytic continuation of the Laplace transform, we may deform the contour of the Bromwich integral into a Hankel contour and obtain an integral for which the integrand decays exponentially.

Proposition 1. *Let $X_j \sim \mathrm{LN}(\mu_j, \sigma_j^2)$, $j = 1, \ldots, n$, be independent and $X = \sum_{j=1}^n X_j$. Then X has density*

$$f_X(x) = -\frac{1}{\pi} \int_0^\infty \mathrm{Im}\left[\phi(-t + \mathrm{i} \cdot 0)\right] e^{-tx} \mathrm{d}t, \qquad x > 0. \tag{4.18}$$

Here ϕ and f_X denote the Laplace transform and probability density function of X, respectively, and $\phi(-t + \mathrm{i} \cdot 0) = \lim_{\epsilon \to 0^+} \phi(-t + \mathrm{i}\epsilon)$.

We will use the following lemma in the proof of Proposition 1

Lemma 1. *Let ϕ_j denote the Laplace transform of $X_j \sim \mathrm{LN}(\mu_j, \sigma_j^2)$. For every $k > 0$, $\phi_j(z) = O_k(|z|^{-k})$ as $|z| \to \infty$, $z \in \mathbb{C} \setminus (-\infty, 0]$. Consequently, for every $k > 0$, $\phi(z) = O_k(|z|^{-k})$ as $|z| \to \infty$, $z \in \mathbb{C} \setminus (-\infty, 0]$.*

Proof of Lemma 1. Let $k > 0$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. By Theorem 7, we have

$$\phi_j(z) = \frac{1}{2\pi} \int_{-\infty}^\infty z^{-(k+\mathrm{i}t)} \Gamma(k + \mathrm{i}t) e^{-\mu_j (k+\mathrm{i}t) + \frac{\sigma_j^2}{2}(k+\mathrm{i}t)^2} \mathrm{d}t,$$

and thus,

$$|\phi_j(z)| \leq \frac{1}{2\pi} \int_{-\infty}^\infty |z|^{-k} e^{\pi |t|} \Gamma(k) e^{-\mu_j k + \frac{\sigma_j^2}{2}(k^2 - t^2)} \mathrm{d}t = M_{k,j} |z|^{-k}.$$

To prove the second part of the lemma, let $k > 0$ and take $r = k/n$. Then, by the independence of the X_j's and the first part of the lemma,

$$|\phi(z)| = \prod_{j=1}^n |\phi_j(z)| \leq \prod_{j=1}^n M_{r,j} |z|^{-r} = M_k |z|^{-k}$$

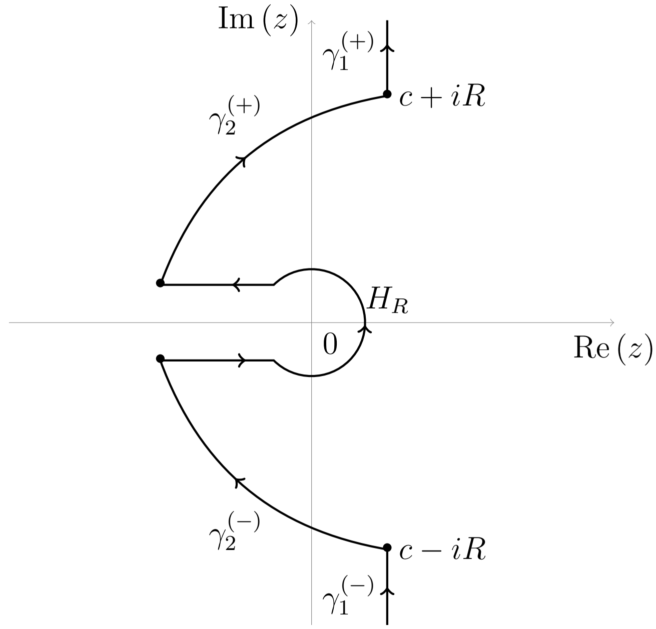


Figure 4.1: The contour $\Gamma_R = \gamma_1^{(-)} + \gamma_2^{(-)} + H_R + \gamma_2^{(+)} + \gamma_1^{(+)}$

where, $M_k = \prod_{j=1}^{n} M_{r,j}$. $\square$

Proof of Proposition 5. The density function of X, obtained by the inverse Laplace transform, is given by the Bromwich integral

$$f_X(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \phi(z) e^{zx} \mathrm{d}z, \tag{4.19}$$

for any $c > 0$. Using the analytic continuation of ϕ, the integrand of (4.19) is analytic on $\mathbb{C} \setminus (-\infty, 0]$ and we can deform the contour to the contour $\Gamma_R = \gamma_1^{(-)} + \gamma_2^{(-)} + H_R + \gamma_2^{(+)} + \gamma_1^{(+)}$, shown in Figure 4.1, for any $R > 0$. The density function is now given by

$$f_X(x) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_R} \phi(z) e^{zx} \mathrm{d}z. \tag{4.20}$$

We will show that the contributions of the contours $\gamma_1^{(+)}$ and $\gamma_2^{(+)}$ go to zero as $R \to 0$. Similarly, the contributions of $\gamma_1^{(-)}$ and $\gamma_2^{(-)}$ go to zero and as a result $\Gamma_R \to H$, as $R \to \infty$, where H is the Hankel contour in Figure 4.2. We first consider the contour $\gamma_1^{(+)}$ parametrized by $z(t) = c + \mathrm{i}t$, $t \in [R, \infty)$. For every $x > 0$, we have

$$\begin{aligned}\left|\frac{1}{2\pi\mathrm{i}}\int_{\gamma_1^{(+)}}\phi(z)e^{zx}\mathrm{d}z\right| &\leq \frac{1}{2\pi}\int_R^\infty \left|\phi(c+\mathrm{i}t)e^{(c+\mathrm{i}t)x}\right|\mathrm{d}t \\ &\leq \frac{1}{2\pi}e^{cx}M_2\int_R^\infty |c+\mathrm{i}t|^{-2}\mathrm{d}t \\ &= \frac{1}{2\pi}e^{cx}M_2\int_R^\infty \frac{1}{c^2+t^2}\mathrm{d}t \longrightarrow 0, \text{ as } R \to \infty,\end{aligned}$$

where we have used Lemma 1 with $k = 2$. Next we consider the contour $-\gamma_2^{(+)}$ parametrized by $z(t) = c + Re^{\mathrm{i}t}$, $t \in [\frac{\pi}{2}, \pi - \theta_R]$, where $\theta_R \to 0$ as $R \to \infty$. Since

$$|z(t)| = \left|c + Re^{\mathrm{i}t}\right| \geq R - c = \left(1 - \frac{c}{R}\right)R,$$

there exists $R' > 0$ such that $|z(t)| \geq \frac{1}{2}R$ when $R > R'$. So for every $x > 0$, and $R > R'$, we have

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\gamma_2^{(+)}}\varphi(z)e^{zx}\ dz\right| &\leq \max_{z\in-\gamma_2^{(+)}}\left\{|\varphi(z)|e^{\mathrm{Re}\,(z)x}\right\}\cdot\frac{\pi}{2}R \\ &\leq \max_{t\in[\frac{\pi}{2},\pi-\theta_R]}\left\{M_k|z(t)|^{-k}e^{(c+R\cos t)x}\right\}\cdot\frac{\pi}{2}R \\ &\leq M_k\left(\frac{1}{2}R\right)^{-k}e^{(c+R\cos(\frac{\pi}{2}))x}\cdot\frac{\pi}{2}R \\ &= \pi 2^{k-1}M_k e^{cx}R^{-k+1} \longrightarrow 0, \text{ as } R \to \infty,\end{aligned}$$

when we use Lemma 1 with any $k > 1$. Thus, taking $R \to \infty$, we have

$$f_X(x) = \frac{1}{2\pi\mathrm{i}}\int_H \phi(z)e^{zx}\mathrm{d}z. \tag{4.21}$$

The contour H is defined $\forall\delta > 0$ and $\forall\epsilon \in (0, \delta)$ and it is clear that $\theta_\epsilon \to 0$ as $\epsilon \to 0$. Rewriting (4.21) as

$$f_X(x) = \frac{1}{2\pi\mathrm{i}}\left\{\left(\int_{h_1} + \int_{h_2} + \int_{h_3}\right)\phi(z)e^{zx}\mathrm{d}z\right\},$$

where,

$$\begin{aligned}\int_{h_1}\phi(z)e^{zx}\mathrm{d}z &= \int_{\delta\cos\theta_\epsilon}^\infty \phi(-t-\mathrm{i}\epsilon)e^{(-t-\mathrm{i}\epsilon)x}\mathrm{d}t, \\ \int_{h_2}\phi(z)e^{zx}\mathrm{d}z &= \mathrm{i}\delta\int_{-\pi+\theta_\epsilon}^{\pi-\theta_\epsilon}\phi(\delta e^{\mathrm{i}t})e^{\delta e^{\mathrm{i}t}x+\mathrm{i}t}\mathrm{d}t, \text{ and} \\ \int_{h_3}\phi(z)e^{zx}\mathrm{d}z &= -\int_{\delta\cos\theta_\varepsilon}^\infty \varphi(-t+\mathrm{i}\epsilon)e^{(-t+\mathrm{i}\epsilon)x}\mathrm{d}t,\end{aligned}$$

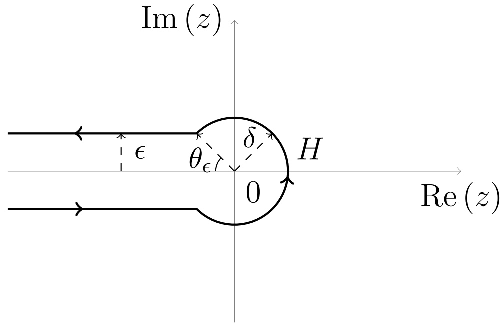


Figure 4.2: The Hankel contour H

and using the fact that $\overline{\phi(z)} = \phi(\overline{z})$, we have

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi\mathrm{i}}\left\{\mathrm{i}\delta\int_{-\pi+\theta_\epsilon}^{\pi-\theta_\epsilon}\phi(\delta e^{\mathrm{i}t})e^{\delta e^{\mathrm{i}t}x+\mathrm{i}t}\mathrm{d}t - 2\mathrm{i}\,\mathrm{Im}\left[\int_{\delta\cos\theta_\epsilon}^{\infty}\phi(-t+\mathrm{i}\epsilon)e^{(-t+\mathrm{i}\epsilon)x}\mathrm{d}t\right]\right\} \\
&\longrightarrow -\frac{1}{\pi}\,\mathrm{Im}\left[\int_0^\infty \phi(-t+\mathrm{i}\cdot 0)e^{-tx}\mathrm{d}t\right], \text{ as } \epsilon,\delta\to 0 \\
&= -\frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\phi(-t+\mathrm{i}\cdot 0)\right]e^{-tx}\mathrm{d}t.
\end{aligned}
$$

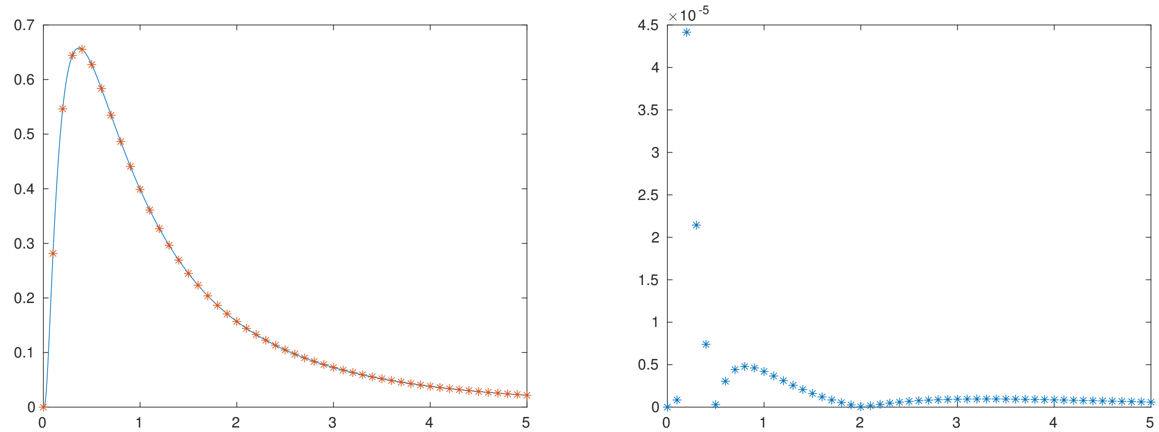
The interchange of the limit and integration can be justified by dominated convergence. □

To utilize Proposition 1, one must first compute $\phi_j(-t+\mathrm{i}\cdot 0)$, $j = 1,\ldots,n$. This can be performed using the methods from Section 4.4, or any alternative method (for example, see [16]). Since the random variables, X_j, $j = 1,\ldots,n$, are independent we have

$$\phi(-t+\mathrm{i}\cdot 0) = \prod_{j=1}^{n}\phi_j(-t+\mathrm{i}\cdot 0).$$

We can compute the integral in (4.18) in a similar fashion to the numerical integration method of Section 4.4.2.

To illustrate the method, we computed the Laplace transform of $X \sim \mathrm{LN}(0,1)$ using the Theorem 8 and used the inversion formula of Proposition 1 to obtain the density. Figure 4.3a shows a plot with both the closed form of f_X and our approximation. Figure 4.3b shows the relative error of the approximation.



(a) Actual density and our approximation.

(b) Relative error

Figure 4.3: Approximating the density of $X \sim \mathrm{LN}(0, 1)$

4.6 Conclusion

We have presented two derivations of the analytic continuation of the Laplace transform of the log-normal distribution, which we denote by ϕ. Since the Mellin transform of ϕ has closed form, we used the Mellin inversion formula to express ϕ in the form of a Mellin-Barnes integral. As a consequence, we obtained the corresponding expression for the characteristic function of the log-normal distribution. This expression is slightly different from the expression derived by Leipnik in [45]; we claim his expression is incorrect.

Using the Mellin-Barnes expression for ϕ, we obtained two approximations which may be used in numerical computations. The error of the first approximation (see Theorem 8) can be made arbitrarily small and the approximation is asymptotic to ϕ as the magnitude of the argument goes to zero. The second approximation (see Theorem 9) improves as the parameter σ goes to infinity. Both approximations were shown to provide accurate results, however, we note that computation can be difficult if too many terms of the series employed.

In the last section, we showed how one may use the analytic continuation of the Laplace transform of a sum of independent log-normals to compute the density, via Laplace inversion. By deforming the vertical contour of the Bromwich integral to a Hankel contour, one may obtain a real integral for which the integrand decays exponentially. The result is an integral that can be computed numerically with ease.

The analytic continuation of the Laplace transform of the log-normal distribution has other applications. In 1977, Olof Thorin showed that the log-normal distribution is a Generalized Gamma Convolution (GGC) (see [61]). A GGC is a probability distribution F on $[0,\infty)$ with moment-generating function (mgf) of the form

$$M(s) = \exp\left[as + \int_0^\infty \ln\left(\frac{t}{t-s}\right)U(\mathrm{d}t)\right], \qquad s \leq 0 \text{ (or } s \in \mathbb{C} \setminus (0,\infty)),$$

where $a \geq 0$ and $U(\mathrm{d}t)$ is a nonnegative measure, called the Thorin measure, on $(0,\infty)$ satisfying

$$\int_{(0,1]} |\ln t| U(\mathrm{d}t) < \infty, \text{ and } \int_{(1,\infty)} t^{-1} U(\mathrm{d}t) < \infty,$$

([15], pg. 29). As Bondesson discusses in [15] and [16], one may compute the density of the Thorin measure using the analytic continuation of the Laplace transform of the log-normal distribution. The density, denoted here by U, can be computed using the formula

$$U(t) = \frac{1}{\pi}\mathrm{Im}\left[\frac{\phi'(-t+\mathrm{i}\cdot 0)}{\phi(-t+\mathrm{i}\cdot 0)}\right],$$

where $\phi(-t+\mathrm{i}\cdot 0) = \lim_{\epsilon\to 0^+}\phi(-t+\mathrm{i}\epsilon)$ (equivalently, as in Bondesson's derivation, one may approach the negative real line from below and multiply the result by -1).

Chapter 5

Risk aggregation: A general approach via the class of Generalized Gamma Convolutions

Risk aggregation is virtually everywhere in insurance applications. Indeed, in the vast majority of situations insurers are interested in the properties of the sums of the risks they are exposed to, rather than in the stand-alone risks per se. Unfortunately, the problem of formulating the probability distributions of the aforementioned sums is rather involved, and as a rule does not have an explicit solution. As a result, numerous methods to approximate the distributions of the sums have been proposed, with the moment matching approximations (MMAs) being arguably the most popular. The arsenal of existing MMAs is quite impressive and contains such very simple methods as the normal and shifted-gamma approximations that, respectively, match the first two and three moments, only, as well as such much more intricate methods as the one based on the mixed Erlang distributions [H.Cossette, D. Landriault, E. Marceau, and K. Moutanabbir (2016). Moment-based approximation with mixed Erlang distributions. *Variance* **10**(1), 166 – 182]. Note however that in practice the sums of insurance risks can have numerous and just a few summands; in the latter case the normal approximation is very questionable. Also, in practice the distributions of the stand alone risks can be light-tailed or heavy-tailed; in the latter case moments of higher orders (e.g., ≥ 2) may not exist, and so the approximation based on mixed Erlang distributions is of limited usefulness.

In this chapter we put forward a refined MMA method for approximating the distributions of the sums of insurance risks. Our method approximates the distributions of interest to any desired precision, works equally well for light and heavy-tailed distributions, and is reasonably fast irrespective of the number of the involved summands.

5.1 Introduction

Risk aggregation is of fundamental importance for insurance. This is because risk aggregation is in fact a precursor of risk pooling, a principle that is seen by some as the insurance's reason d'etre. To see how crucial risk aggregation is for risk pooling, consider a group of $n \in \mathbf{N}$ individuals (also, business lines, risk components in a portfolio of risks, etc), where each one of $i \in \{1, \ldots, n\}$ faces a risk represented by the random variable (RV) X_i, then a sharing rule $Y(X_1, \ldots, X_n)$ is called risk pooling if it is a function of the aggregate risk $X_1 + \cdots + X_n$, only, that is $Y(X_1, \ldots, X_n) = Y(X_1 + \cdots + X_n)$ [e.g., 1, for details]. Hence, it is clear that to reap the benefits of risk pooling, insurers must study and understand risk aggregation thoroughly. From now and on, $X_1, \ldots, X_n$ stand for insurance risk RVs, and $X_1 + \cdots + X_n := S_n$ denotes the associated aggregate risk.

Specific examples of risk aggregation are naturally abundant in all areas of insurance business. For classical applications, one has to go no further than the renowned individual [e.g., 24] and collective [e.g., 32] risk models. Specifically, in the case of the individual risk models (IRMs), for a fixed n as hitherto and independent but not identically distributed risk RVs $X_1, \ldots, X_n$, the aggregate risk is as before denoted by S_n. Also, in the case of the collective risk models (CRMs), the number of risks, denoted by N, is assumed random and independent of identically distributed and mutually independent (IID) RVs $X_1, \ldots, X_N$; the aggregate risk is then denoted by $S_N := X_1 + \cdots + X_N$.

In order to comprehend the implications of risk aggregation, and merely comply with the norms of the risk informed decision making, insurers are concerned with the stochastic properties of the RVs S_n and S_N, that is with the corresponding cumulative distribution functions (CDFs). Whether the IRM or CRM are considered, it is often tempting to approximate these CDFs with the use of the Lindeberg – Lévy central limit theorem (CLT). To this end, arguments like: (i) the number of risks is *large*, (ii) the risks are *not too* heterogeneous, and (iii) the distributions of the risk RVs are *not too* skewed are quite common. Unfortunately, none of the above must be true in reality. Furthermore, statements (ii) and

(iii) are often violated as insurance risk RVs due to distinct risk sources can be very unalike, and, as a rule, positively skewed. Another problem with the aforementioned variant of the CLT is that there are situations (not rare) where the risk RVs of interest have infinite second moments [e.g., 53, and references therein]. We refer to Brockett [18] and references therein for some interesting examples of how the CLT is misused in insurance applications.

The standard CLT-based approximation of the aggregate risk's CDF, a.k.a. the normal approximation (NA), can be considered a moment matching approximation (MMA) that hinges on the first two moments only. A generalization of NA that incorporates skewness is the so-called normal-power (NP) approximation [e.g., 51]. An alternative approach to count for skewness that is of great popularity among practising actuaries is the shifted-gamma approximation (SGA), which aims to match the first three moments [34]. Clearly, the choice of how many moments to match is somewhat ad-hoc. Thus more general approximations that aim at matching an arbitrary number of moments have been proposed [e.g., 22, for a recent reference].

Admittedly, MMAs, both the ones mentioned above and others alike, are convenient and intuitive to convey to upper management, yet, rather problematic. For instance, even the two moments-based NA method requires finite second moments, and it is inapplicable otherwise. The approach of Cossette et al. [22] achieves better accuracy at the price of requiring the finiteness of higher order moments. In addition, in the latter case, the method is often rather computationally intensive.

In this paper, we put forward a new efficient method to approximate the CDFs of the aggregate risk RVs S_n and S_N. Our approach approximates the CDFs of interest to any desired precision, works equally well for light and heavy-tailed CDFs, and is reasonably fast irrespective of the number of summands. We organize the rest of this paper as follows: Section 5.2 provides a high-level overview of our approach, and Sections 5.3, 5.4, and 5.5 explain in detail its three main pillars, which are, respectively, the class of Padé approximations, the family of Generalized Gamma Convolutions, and the Gaver-Stehfest algorithm. The theory we propose is then elucidated by a variety of practical examples borrowed from Bahnemann [8]. More specifically, in Section 5.6 we demonstrate the effectiveness of the new MMA in the context of stand-alone risk RVs first, and then in the context of aggregate risk RVs with and without policy modifications.

5.2 Brief description of the method

In order to outline the essence of our proposed technique in the simplest possible manner, in this section we consider the framework of the IRM, however the ideas are equally applicable in the context of the CRM (Section 5.6 of this paper). Let $X_1, \ldots, X_n$ be positive and mutually independent RVs with arbitrary corresponding CDFs $F_1, \ldots, F_n$. Also denote by $\phi_i(z) := \mathbb{E}[\exp(-zX_i)]$ the Laplace transform (LT) of the RV $X_i,\ i = 1, \ldots, n$. Our goal is to approximate the CDF F of the aggregate RV $S = X_1 + \cdots + X_n$.

Let $\phi(z) := \mathbb{E}[\exp(-zS)]$ (also, $\mathcal{L}(z)$) denote the LT of the aggregate risk RV S, then we readily have

$$\phi(z) = \prod_{i=1}^{n} \phi_i(z) \tag{5.1}$$

and

$$F(x) = \mathcal{L}^{-1}\left\{\frac{\phi(z)}{z}\right\}(x),\ x \geq 0. \tag{5.2}$$

Hence, by combining (5.1) and (5.2), we are able to obtain the desired approximation of the CDF F given that there exist reliable methods to: (i) approximate each LT $\phi_i(z)$, and (ii) invert LTs. Next in Sections 5.3 - 5.5, we demonstrate that (i) and (ii) can be achieved very successfully. Namely, in Section 5.3, we approximate the Laplace transform ϕ_i with the help of the Laplace transform of certain gamma convolutions, and we utilize the machinery of Padé approximations to determine the involved shape and rate parameters. In Section 5.4, we reintroduce the family of Generalized Gamma Convolutions, and, finally, in Section 5.5, we briefly explain the Gaver-Stehfest method to invert Laplace transforms. The proposed MMA may at the first glance seem complex, however, when implemented, it is utterly user-friendly and requires minimal human involvement.

5.3 Padé approximations

To start off, we note that (5.1) requires to approximate the LT of the RV X_i (the CDF of the RV X_i is then established via (5.2)). We accomplish this with the help of m-fold convolutions of Gamma-distributed RVs, succinctly $\Gamma_i \sim \mathit{Gamma}(\alpha_i, \beta_i),\ i = 1, \ldots, m$. In other words, we seek to choose the parameters $\{\alpha_i\}_{1\leq i\leq m}$ and $\{\beta_i\}_{1\leq i\leq m}$, so that the CDF of the approximant of order m RV $\tilde{X}_{i,(m)} := \sum_{i=1}^{m} \Gamma_i$ is *close* in an appropriate sense to the CDF of the RV X_i $i = 1, \ldots, n$. In the rest of the paper, we omit the subscript (m) whenever the order of the approximation is fixed. Also, for the sake of the expositional simplicity of the discussion in the present section, and sometimes thereafter, we omit the subscript i and write X instead of X_i, and ϕ instead of ϕ_i.

In terms of LTs, we seek to choose the parameters $\{\alpha_i\}_{1\le i\le m}$ and $\{\beta_i\}_{1\le i\le m}$, so that the approximant LT

$$\tilde{\phi}(z) := \mathbb{E}\left[\exp\left(-z\sum_{i=1}^{m}\Gamma_i\right)\right] = \prod_{i=1}^{m}(1+z/\beta_i)^{-\alpha_i}$$

is *close* to the function $\phi(z) = \mathbb{E}[\exp(-zX)]$. Our method is essentially an MMA. Note that the moments of X can be computed as $\mathbb{E}[X^k] = (-1)^n\phi^{(k)}(0)$, thus matching the first m moments of $\tilde{X}$ and X is equivalent to matching the derivatives (of order $k = 1, 2, \ldots, m$) of $\tilde{\phi}$ and ϕ at $z = 0$. However, our approach allows for a number of important improvements.

Note 1. A significant innovation of our method is that we match the derivatives of $\tilde{\phi}$ and ϕ not at $z = 0$ but at some point $z = z^* > 0$. This is crucial if we want our technique to be applicable to risk RVs with heavy tails, for which the moments (and thus the derivatives $\phi^{(k)}(0)$) may fail to exist. If the risk RV X has exponentially light tails, we may as well choose $z^* = 0$, but in general z^* must be strictly positive.

Note 2. Another distinguishing feature of our approach is that $\tilde{\phi}$ converges to ϕ uniformly and exponentially fast, and in particular the approximant RV $\tilde{X}$ converges in distribution to the RV X.

Note 3. Our method requires $2m$ parameters (unique up to permutation) to match the first $2m$ moments of the RVs $\tilde{X}$ and X. As at least $2m$ parameters are required to complete this task, the method we put forward herein is optimal in this sense.

5.3.1 Approximation of order two

In order to explain the main ideas behind our algorithm, let us consider a simple case where $m = 2$ and $z^* = 1$. In this case the problem reduces to the following one: we want to find four positive numbers α_1, α_2, β_1 and β_2 such that the derivatives of the LT ϕ at $z = 1$ coincide with derivatives of the approximant LT

$$\tilde{\phi}_{(2)}(z) = (1+z/\beta_1)^{-\alpha_1}(1+z/\beta_2)^{-\alpha_2}$$

also at point $z = 1$. In order to compute the four constants α_1, α_2, β_1 and β_2 we need to have at least four equations, so now we need to solve the following system

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}(1 + z/\beta_1)^{-\alpha_1}(1 + z/\beta_2)^{-\alpha_2}\Big|_{z=1} = \phi^{(k)}(1), \quad k \in \{1, 2, 3, 4\}. \tag{5.3}$$

If we were to compute the derivatives in the left-hand side of (5.3) and then to simplify the resulting equations, we would have obtained a fairly complicated system of four nonlinear equations in α_1, α_2, β_1 and β_2. However, analysing these equations theoretically would not be feasible due to their complexity and even solving them numerically would be a major problem.

In order to avoid the complexity, we take logarithms before differentiating. Thus, instead of matching the derivatives $\phi^{(k)}$ and $\tilde{\phi}^{(k)}_{(2)}$ for $k = 1,\ 2,\ 3,\ 4$, we match the derivatives of $\ln(\phi(z))$ and $\ln(\tilde{\phi}_{(2)}(z))$ of order $k = 1,\ 2,\ 3,\ 4$. The end result is clearly the same, but computing α_1, α_2, β_1 and β_2 is much simpler, as we demonstrate in a moment. With this change, the system of four equations for finding α_1, α_2, β_1 and β_2 looks as follows:

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\ln\left[(1 + z/\beta_1)^{-\alpha_1}(1 + z/\beta_2)^{-\alpha_2}\right]\Big|_{z=1} = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\ln[\phi(z)]\Big|_{z=1}, \quad k \in \{1, 2, 3, 4\}. \tag{5.4}$$

Let us denote $s_k := -(1/k!)\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}\ln[\phi(z)]\Big|_{z=1}$. As we see later, it is easy to compute s_k numerically in each case of interest, thus for now we treat these numbers as known quantities. Computing the derivatives in the left-hand side of (5.4), we obtain a system of four equations

$$\frac{\alpha_1}{(1 + \beta_1)^k} + \frac{\alpha_2}{(1 + \beta_2)^k} + \frac{\alpha_2}{(1 + \beta_2)^k} + \frac{\alpha_2}{(1 + \beta_2)^k} = (-1)^k s_{k-1}, \quad k \in \{1, 2, 3, 4\}. \tag{5.5}$$

At this stage, that is in order to solve the nonlinear equations in (5.5) and find α_1, α_2, β_1 and β_2, we employ the toolbox of *Padé approximations*.

Namely, we introduce yet another function $\psi(z) := -\phi'(z)/\phi(z)$. Then system (5.4) is equivalent to

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left[\frac{\alpha_1}{z + \beta_1} + \frac{\alpha_2}{z + \beta_2}\right]\Big|_{z=1} = \psi^{(k)}(1), \quad k \in \{0, 1, 2, 3\}. \tag{5.6}$$

Note that we can write

$$\frac{\alpha_1}{z + \beta_1} + \frac{\alpha_2}{z + \beta_2} = \frac{A + B(z - 1)}{1 + C(z - 1) + D(z - 1)^2} \tag{5.7}$$

for some constants A, B, C and D. Now we can express system of four equations (5.6) in an equivalent way by saying that the first four terms of the Maclaurin expansion of the rational function

$$\frac{P(w)}{Q(w)} := \frac{A + Bw}{1 + Cw + Dw^2}$$

must match the corresponding terms of the Maclaurin expansion

$$\psi(1 + w) = s_0 + s_1 w + s_2 w^2 + s_3 w^3 + \ldots,$$

in other words we have a single equation

$$\frac{A + Bw}{1 + Cw + Dw^2} = s_0 + s_1 w + s_2 w^2 + s_3 w^3 + O(w^4), \quad w \to 0, \tag{5.8}$$

(note that we have introduced here a new variable $w := z - 1$). Equation (5.8) tells us that the rational function $P(w)/Q(w)$ is a $[1/2]$ Padé approximation to the function $\psi(1 + w)$. In general, a $[p/q]$ Padé approximation to a function f is a rational function $P(w)/Q(w)$ (with $\deg(P) = p$ and $\deg(Q) = q$) that has the same first $p + q + 1$ terms in Maclaurin expansion as the function $f(w)$.

It turns out that single equation (5.8) contains all the information necessary for finding the constants A, B, C and D. By multiplying both sides of equation (5.8) by $Q(w) = 1+Cw+Dw^2$ we obtain

$$A + Bw = (1 + Cw + Dw^2)(s_0 + s_1 w + s_2 w^2 + s_3 w^3 + O(w^4)), \quad w \to 0. \tag{5.9}$$

Identifying the coefficients in front of powers of w in both sides of the above equation we obtain a system of four linear (!) equations

$$\begin{cases} A = s_0, \\ B = Cs_0 + s_1, \\ 0 = Ds_0 + Cs_1 + s_2, \\ 0 = Ds_1 + Cs_2 + s_3. \end{cases} \tag{5.10}$$

Note that this system of four linear equations can be solved very easily by first finding C and D from the third and fourth equations, and then substituting these results into the first and second equations would give us the values of A and B.

Now that we have found A, B, C, and D, and so we can compute β_1 and β_2 by noting that

$$(1 + \beta_1 + w)(1 + \beta_2 + w) = 1 + Cw + Dw^2,$$

and then the constants α_1 and α_2 can be found by the formula $\alpha_i = P'(-1-\beta_i)$. Remarkably, all we have done so far is the partial fraction decomposition for the rational function $P(w)/Q(w)$ as in (5.7). This process, which can be done *by hand*, has provided us with the desired constants α_1, α_2, β_1 and β_2, and so with the approximant LT $\tilde{\phi}_{(2)}$.

5.3.2 Extension to the approximation of any order

To generalize the method described in the previous subsection to arbitrary orders m and arbitrary choice z^*, we follow the same steps and arrive at the problem of finding $[m-1/m]$ Padé approximant to the function $\psi(z^*+w)$, that is, instead of (5.7) and (5.8) we have

$$\begin{aligned}\sum_{i=1}^{m} \frac{\alpha_i}{z^*+\beta_i+w} &= \frac{a_0+a_1w+\cdots+a_{m-1}w^{m-1}}{1+b_1w+b_2w^2+\cdots+b_mw^m} \\ &= s_0+s_1w+s_2w^2+\cdots+s_{2m-1}w^{2m-1}+O(w^{2m}),\end{aligned} \tag{5.11}$$

where $s_k := (1/k!)\psi^{(k)}(z^*)$. This allows us to find the coefficients b_i, a_i by solving a system of linear equations similar to (5.10) and then to obtain the required numbers $\{\alpha_i\}_{1\le i\le m}$ and $\{\beta_i\}_{1\le i\le m}$ by doing the partial fraction decomposition in (5.11).

The main input for this algorithm is the sequence of coefficients $s_k = (1/k!)\psi^{(k)}(z^*)$. These coefficients are typically not available in closed form, but they can be easily computed numerically. Indeed, we first compute the numbers

$$g_k := \phi^{(k)}(z^*) = (-1)^k \int_0^\infty x^k e^{-z^*x}\mathrm{d}F(x), \quad k\ge 0, \tag{5.12}$$

to a high precision by a numerical quadrature (the double-exponential quadrature of [56] is particularly well-suited for such calculations). Next we observe that

$$\frac{d}{dz}\phi(z) = -\psi(z)\phi(z). \tag{5.13}$$

Rewriting this identity in terms of Taylor series centred at z^* gives us

$$\sum_{k\ge 0} g_{k+1}(z-z^*)^k/k! = -\left(\sum_{n\ge 0} s_n(z-z^*)^n\right)\times\left(\sum_{k\ge 0} g_k(z-z^*)^k/k!\right).$$

Comparing the constant term in the Taylor series in the left-hand side and the right-hand side of the above equation we find that $s_0 = -g_1/g_0$, and comparing the coefficients in front of $(z-z^*)^k$ gives us

$$s_k = -\frac{1}{g_0}\left(\frac{g_{k+1}}{k!} + \sum_{i=0}^{k-1} s_i \frac{g_{k-i}}{(k-i)!}\right), \tag{5.14}$$

which allows to compute s_k recursively for all $k\ge 1$.

5.3.3 A simple numerical example and a question arising from it

To illustrate our method on a numerical example of actuarial interest, let us assume that X is distributed Weibull with the probability density function (PDF)

$$f(x) = (3/4)x^{-1/4}\exp(-x^{3/4}),\ x \geq 0.$$

Weibull distribution has been frequently chosen to model insurance risks [47, 39, for general discussions].

We fix $m = 2$ and $z^* = 1$; by computing numerically the integral in (5.12) we calculate

$$g_0 \approx 0.5193711\quad g_1 \approx -0.2123717,\quad , g_2 \approx 0.2179689,\quad g_3 \approx -0.3665409,\quad g_4 \approx 0.8649004.$$

Then we use (5.14) to find

$$s_1 \approx 0.4089017,\quad s_2 \approx -0.0868391,\quad s_3 \approx 0.0630061,\quad s_4 \approx -0.0478247$$

and solving system of linear equations (5.10) we obtain

$$A \approx 0.4089017,\quad B \approx -0.2292057,\quad C \approx 0.7729116,\quad D \approx 0.0100583.$$

The polynomial $Q(w) = 1 + Cw + Dw^2$ has two roots ≈ -75.526356 and ≈ -1.3163588, thus we find $\beta_1 \approx 74.526356$ and $\beta_2 \approx 0.3163588$. Using the formula $\alpha_i = P'(-1-\beta_i)$ we find $\alpha_1 \approx 22.6439903$ and $\alpha_2 \approx 0.1435963$. Thus we have found an approximation to the Weibull distribution of interest. Namely, we have approximated X with the help of $\tilde{X}_{(2)}$, such that the latter RV is equal in distribution to $\Gamma_1 + \Gamma_2$, with $\Gamma_i \sim Gamma(\alpha_i, \beta_i),\ i = 1,\ 2$.

We conclude this section with an important observation, which paves the path to the introduction of the class of GGCs latter on in the next section. Namely, let the RV X be distributed Weibull with the following PDF (the shape parameter is now greater than one)

$$f(x) = (3/2)x^{1/2}\exp(-x^{3/2}),\ x \geq 0.$$

For this set-up, we find that $\alpha_1 \approx -29.421049$ and $\beta_1 \approx -66.009893$, which shows that our algorithm does not always give a legitimate approximation in the form $\Gamma_1 + \Gamma_2$, with $\Gamma_i \sim Gamma(\alpha_i, \beta_i)$. This outcome is rather unfortunate, but similar inconveniences have been encountered in the context of other MMAs [e.g. 22].

This raises an important question:

Does there exist a (rich) class of CDFs for which the Padé approximations described above always yield meaningful results?

In the next section we show that the answer to this question is in affirmative, and we also discuss the convergence of our refined MMA method.

5.4 Generalized Gamma Convolutions

The class of Generalized Gamma Convolutions comprises distributions which are weak limits of the RVs of the form $\sum_{i=1}^m \Gamma_i$, where $\Gamma_i \sim Gamma(\alpha_i, \beta_i)$ and $\alpha_i > 0,\ \beta_i > 0,\ i = 1, \ldots, m$. It seems that GGCs were first used by Thorin in 1977, who employed their properties to prove that the log-normal distribution is infinitely divisible [e.g., 15, for an excellent discussion].

The most convenient way to describe the class of GGCs is via LTs. Note that for a RV $\sum_{i=1}^m \Gamma_i := \tilde{X}_{(m)}$, which is our approximant from Section 5.3, we have

$$\mathbb{E}[\exp(-z\tilde{X}_{(m)})] = \prod_{i=1}^m \mathbb{E}[\exp(-z\Gamma_i)] = \prod_{i=1}^m (1+z/\beta_i)^{-\alpha_i} = \exp\Big(-\int_0^\infty \ln(1+z/t)U(\mathrm{d}t)\Big),$$

where $U(\mathrm{d}t)$ is a discrete measure having support at points β_i with mass $U(\{\beta_i\}) = \alpha_i$. Thus, the following result is not surprising.

Proposition 1 ([61], see also [15])**.** *The distribution on* $[0,\ \infty)$ *of the r.v.* X *is a GGC if and only if its Laplace transform is*

$$\phi(z) := \mathbb{E}[\exp(-zX)] = \exp\Big(-az - \int_0^\infty \ln(1+z/t)U(\mathrm{d}t)\Big) \text{ for } \operatorname{Re}(z) \geq 0, \tag{5.15}$$

where $a \in [0,\ \infty)$ *is a constant, and* $U(\mathrm{d}t)$ *is a positive Radon measure, also called Thorin measure, which must satisfy*

$$\int_0^\infty \min(|\ln(t)|, 1/t)U(dt) < \infty.$$

It turns out that the algorithm outlined in Section 5.3 always produces meaningful results when the RV being approximated has a CDF that belongs to the class of GGCs. By meaningful results we mean that the algorithm produces a set of *positive* numbers α_i and β_i that determine the approximant $\tilde{X}_{(m)} := \sum_{i=1}^m \Gamma_i$, where $\Gamma_i \sim Gamma(\alpha_i, \beta_i)$. Moreover, it can be shown [30] that, as $m \to +\infty$, the LTs $\mathbb{E}[\exp(-z\tilde{X}_{(m)})]$ converge to the LT $\mathbb{E}[\exp(-zX)]$ exponentially fast and uniformly in z on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. One may wonder at this point whether the class of GGCs is rich enough. We note in this respect that it comprises, e.g., such important distributions for actuarial applications as gamma, inverse gamma, inverse Gaussian, Pareto, log-normal, and Weibull with the shape parameter less than one, among a great variety of other distributions.

To summarize the findings of Sections 5.3 and 5.4, we emphasize that our MMA (i) converges very fast, (ii) always provides legitimate outcomes if the risk RV to be approximated has a CDF in the class of GGCs ,(iii) yields unique parameters of the approximant CDF, and (iv) is optimal in the sense that only $2m$ parameters are required to match the first $2m$ moments.

We conclude this section with outlining the cause for the algorithm described in Section 5.3 to be so well-suited for the RVs having CDFs in the class of GGCs. This reason in fact stems from the fact that the function $\psi(z) = -\mathrm{d}/\mathrm{d}z \ln(\mathbb{E}[\exp(-zX)])$ is given by

$$\psi(z) = \int_0^\infty \frac{U(\mathrm{d}t)}{t+z}, \tag{5.16}$$

which can be easily obtained from (5.15). Functions of the form $\int_0^\infty (t+z)^{-1} U(\mathrm{d}t)$ are called *Stieltjes functions*, and they have been shown to enjoy many nice analytical properties. In particular, it has been proved that Padé approximations to such functions always exist and converge exponentially fast and uniformly in z on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ (see [9]). Since the algorithm outlined in Section 5.3 is essentially a Padé approximation method, this explains why RVs with CDFs in the class of GGCs fit perfectly within our approximation scheme.

5.5 Gaver-Stehfest algorithm

At this point of the discussion, we have hopefully convinced the reader that there is a mathematically sound way to approximate the risk RV X having a CDF in the class of GGCs with the help of an approximant RV $\tilde{X}$. The solution, however attractive, involves the corresponding LTs, that is $\tilde{\phi} \approx \phi(z)$. However, our ultimate goal is to find the CDF of the approximant RV $\tilde{X}$, denoted by $\tilde{F}$. Thus, the remaining step is to rediscover this CDF from the LT $\tilde{\phi}$. Doing that is a classical problem in analysis, and a variety of solutions exist. One popular method is to use Bromwich integral, which requires integration over a path in the complex plane. Another method, and this is what we do in the present paper, is via the

Gaver-Stehfest algorithm. The main difference between the Gaver-Stehfest algorithm and the one based on the Bromwich integral, as well as most other Laplace inversion methods, is that it uses only the values of the Laplace transform on the positive real line and does not require any complex numbers. This method was invented in 1970 by Stehfest [54], by improving upon the earlier method of Gaver, and since then it has been successfully used in many areas of Applied Mathematics, including in Probability and Statistics [3, 40], Actuarial Science [7] and Mathematical Finance [52].

The Gaver-Stehfest algorithm is very simple and easy to implement. To introduce it, consider a function f and its Laplace transform

$$\phi(z) := \int_0^\infty e^{-zx} f(x)\mathrm{d}x,$$

where we assume that $\phi(z)$ is finite for all $z > 0$. For all integers $m \geq 1$ we define

$$f_m(x) := \ln(2)x^{-1} \sum_{k=1}^{2m} a_k(n)\phi\left(k\ln(2)x^{-1}\right), \quad x > 0, \tag{5.17}$$

where the coefficients are defined as follows:

$$a_k(m) := \frac{(-1)^{m+k}}{m!} \sum_{j=[(k+1)/2]}^{\min(k,m)} j^{m+1}\binom{n}{j}\binom{2j}{j}\binom{j}{k-j}, \quad m \geq 1,\ 1 \leq k \leq 2m.$$

It is known [41] that the approximations $f_m(x)$ converge to $f(x)$ if f is continuous at x and of bounded variation in a neighbourhood of x. There is also a lot of numerical evidence that the approximations $f_m(x)$ converge to $f(x)$ very fast, provided that f is smooth enough at x [e.g., 3, 23]; in Chapter 6 we provide a rigorous proof of this assertion. When using the Gaver-Stehfest algorithm one should be careful with the loss of significant digits in the sum (6.2). This is due to the fact that the coefficients $a_k(m)$ are very large (for large k and m) and of alternating signs. This problem is readily solved by using any high-precision arithmetic package.

5.6 Illustrative examples with log-normal, Pareto and Weibul risks of varying tail thickness

In this section, we demonstrate the usefulness of our new MMA method by applying it to a number of examples of actuarial interest. More specifically, motivated by numerical examples in Bahnemann [8], we consider applications to risks that are distributed log-normally, Pareto of the second kind, a.k.a., Lomax, and Weibull. These distributions have been routinely chosen to model insurance risks [e.g., 38, 47, 39, for a general discussion].

In what follows, we denote the CDFs of the log-normal, Lomax, and Weibull distributions, by, respectively, F_L, F_P, and F_W. All these CDFs have explicit forms given by

$$F_L(x;\mu,\sigma) = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\ln x-\mu}{\sigma\sqrt{2}}\right)\right],\ \ x>0,\ \mu\in\mathbb{R},\ \sigma>0, \tag{5.18}$$

$$F_P(x;\alpha,\beta) = 1-\left(\frac{\beta}{x+\beta}\right)^{\alpha},\ \ x\geq 0,\ \alpha>0,\ \beta>0, \tag{5.19}$$

$$F_W(x;\beta,\delta) = 1-e^{-(x/\beta)^{\delta}},\ \ x\geq 0,\ \beta>0,\ \delta>0. \tag{5.20}$$

In the above, σ, α and δ are shape parameters that determine the tail weight of the corresponding distribution. Namely, in the case of the log-normal distribution, larger values of σ imply heavier right tails, whereas in the case of the Lomax and Weibull distributions, smaller values of α and δ, respectively, suggest heavier right tails. Interestingly, there has been ample practical evidence that these are the parametrizations that correspond to the heavier tails that are of particular interest in non-life insurance applications, yet in the theoretical literature the light-tailed examples prevail [e.g., 22, 57, for recent references]. In the rest of this section, we consider both heavy-tailed and light-tailed parametrizations of the CDFs F_L, F_P, and F_W. Our choices of parameters are inspired by Bahnemann [8].

5.6.1 Stand-alone risks

We begin by approximating CDFs (5.18),(5.19), and (5.20). Genuinely speaking, these approximations are not the ultimate goal of this paper, and hence we remind the reader that: (i) after we have the approximations for the just-mentioned CDFs of the stand-alone risks RVs, the CDFs of the aggregate risk RVs are obtained with the help of equations (5.1) and (5.2), (ii) in the context of the stand-alone risks, we are able to evaluate the accuracy of an approximation by comparing the approximant CDF with the actual CDF, and thus the discussion in this subsection can serve as an important evaluation of the various MMAs tested here.

To explore the effectiveness of the distinct MMAs, we employ the Kolmogorov-Smirnov (KS) metric to measuring how close the distribution of an approximation is to the desired distribution. The KS distance (d_{KS}) of an approximant RV $\tilde{X}$ (with CDF $\tilde{F}$) to the RV X (with CDF F), denoted $d_{KS}(\tilde{F}, F)$, is given by

$$d_{KS}(\tilde{X}, X) = \sup_{x \geq 0} \left| \tilde{F}(x) - F(x) \right|.$$

This metric yields the worst distance and, thus, small values of d_{KS} suggest the approximation is good on the entire domain. However, on a different note, relatively large values of d_{KS} do not necessarily mean the approximation is worthless, as it can turn out very reasonable in some regions of the CDFs domain.

In the examples below, we illustrate the effectiveness of our approximation by comparing it with three simple MMAs: the normal approximation, the normal power approximation (NPA), and the shifted gamma approximation (SGA), when applicable. These three MMAs are referred to as 'the most commonly cited in practice' by Hardy [34]. In addition, we compare our approximation with the mixed Erlang distribution (MED) approach of Cossette et al. [22], when applicable. The MED method uses the first m moments of a risk RV to determine a mixture of Erlang distributions for approximating this RVs CDF. We use the results in Cossette et al. [22] for the log-normal distribution ($\mu = 0,\ \sigma = 0.5$) and apply their algorithm to our additional cases of interest (Pareto and Weibull), when applicable. (When implementing the MED method, we only considered moment-matching of orders $m = 3$ and $m = 4$, as with $m \geq 5$ the method requires a large number of computations.)

In what follows, we denote the CDF of our approximant RV $\tilde{X}_{(m)}$ by $\tilde{F}_{(m)}$, we denote the mixed Erlang CDF by $\tilde{F}_{W_{m,l}}$, and the CDFs of the NA, NPA, and SGA methods by $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$, respectively.

Example 1. To start off, let us consider log-normally distributed risk RVs. The log-normal distribution has been found appropriate for modelling losses originating from a great variety of non-life insurance risks [e.g., 47, 39]. More specifically, [38] mention applications in property, fire, hurricane, and motor insurances, to name just a few [also, e.g., 25, 14, 50]. Furthermore the standard formula of the European Insurance and Occupational Pensions Authority explicitly assumes the log-normality of insurers' losses [28].

Table 5.1 provides the KS distances of several approximations for the parameter sets ($\mu = 0, \sigma = 0.5$), ($\mu = 1.5240, \sigma = 1.2018$), and ($\mu = 5.9809, \sigma = 1.8$). The choice of the larger value of the σ parameters is motivated by Example 5.8 in Bahnemann [8], whereas the choice of the smaller one is due to Cossette et al. [22] and Dufresne [26]. As the parameter

σ gets larger, it becomes increasingly difficult to obtain a valid approximation with any of the methods (this is perhaps the reason why in the examples manifesting in the academic literature, smaller values of the σ parameter are very common). With $\sigma = 1.2018$ and $\sigma = 1.8$, the MED method does not produce potential solutions using the parameters $m = 4$ and $l = 70$, in the former case, and $m = 3$, or 4 and $l = 70$, in the latter case. As suggested in Cossette et al. [22], this can be remedied by increasing the parameter l; however, increasing this parameter too much results in the method being computationally infeasible.

Figure 5.1 provides the plots of the approximating CDFs in Table 5.1 as well as the actual CDF F_L (in blue). Figures 5.1 a), c), and e) depict our approximations (dark green and light green, where light green corresponds to the better approximation) and the MED approximation (in red, when applicable). Figures 5.1 b), d), and f) depict the simple moment-matching methods: NA (orange), NPA (salmon), and SGA (black). Apparently, the NA and NPA approximations are inadequate for the more skewed log-normal cases, that is for the parametrizations ($\mu = 1.5240$, $\sigma = 1.2018$) and ($\mu = 5.9809$, $\sigma = 1.8$). This is not surprising, as it is well-known that the NPA provides fairly accurate results for the skewness values not exceeding one [e.g. 51], whereas the two aforementioned cases of the log-normal distribution lead to the skewness values of 11.23 and 136.38, respectively. In the context of the SGA, higher skewness values imply very small scale parameters, and these in turn result in a nearly vertical rise in the corresponding CDF, thus aggravating the accuracy significantly.

Example 2. The next distribution we consider is Lomax. As in the case of the log-normal distribution discussed earlier, there is ample of evidence that the Lomax distribution is an adequate model to describing non-life insurance risks. We refer to Seal [53] for a list of references, and in particular to Benckert and Sternberg [12], Andersson [5], and Ammeter [4] for fire insurance losses, and Benktander [13] for automobile insurance losses.

Table 5.2 provides the KS distances for two parameter sets: ($\alpha = 2.7163, \beta = 16.8759$), and ($\alpha = 2, \beta = 3000$) [e.g., Example 5.7 in 8]. Recall that the m^{th} moment of the Pareto distribution is finite only when $m < \alpha$. Consequently, most of the existing MMAs are not applicable for these examples; in the first case, only the NA method and MED method with $m = 2$, are applicable. In contrast, our approximation is feasible in both cases and works well.

Figure 5.2 compares the plots of the CDFs in Table 5.2 against the actual CDF F_P (in blue). Figures 5.2 a), and c) depict our approximations (dark green and light green, where

Table 5.1: The KS distances, $d_{KS}(\cdot, X_L)$, of the approximations to risk RVs distributed log-normally

Parameters	$\tilde{F}_{(5)}$	$\tilde{F}_{(10)}$	$\tilde{F}_{W_{5,70}}$	$\tilde{F}_{NA}$	$\tilde{F}_{NP}$	$\tilde{F}_{SG}$
$\mu = 0$, $\sigma = 0.5$	9.029e−4	2.950e−6	1.131e−3	9.834e−2	7.165e−2	3.870e−2
Parameters	$\tilde{F}_{(3)}$	$\tilde{F}_{(16)}$	$\tilde{F}_{W_{3,70}}$	$\tilde{F}_{NA}$	$\tilde{F}_{NP}$	$\tilde{F}_{SG}$
$\mu = 1.5240$, $\sigma = 1.2018$	6.070e−3	2.118e−6	3.400e−2	0.290e0	0.730e0	0.5919e0
Parameters	$\tilde{F}_{(3)}$	$\tilde{F}_{(36)}$	$\tilde{F}_{W_{3,70}}$	$\tilde{F}_{NA}$	$\tilde{F}_{NP}$	$\tilde{F}_{SG}$
$\mu = 5.9809$, $\sigma = 1.8$	7.968e−3	1.113e−6	-	0.420e0	0.835e0	0.8015e0

light green corresponds to the better approximation) and the MED approximation (in red, when applicable). Figure 5.2 b) depicts the normal approximation (orange).

Example 3. The final distribution we consider is Weibull, which is a GGC when $\delta < 1$, and these are the values of interest when modelling non-life insurance risks [8]. Very much like the log-normal and Pareto distributions, the Weibull distribution has been commonly chosen to model insurance data. We refer to Mikosch [47] and Klugman, Panjer, and Willmot [39] for a general discussion, as well as to Hogg and Klugman [35] for an application to huricane loss data.

Table 5.3 provides the KS distances for the parameters $\beta = 220.653$ and $\delta = 0.8$ [e.g., Example 2.13 in 8].

Figure 5.3 compares the plots of the CDFs in Table 5.3 with the actual CDF F_W (in blue). Figure 5.3 a) depicts our approximations (dark green and light green, where light green corresponds to the better approximation) and the MED approximation (red). Figure 5.3 b) depicts the simple moment-matching methods: NA (orange), NPA (salmon), and SGA (black).

In summary, we note with satisfaction that the proposed refined MMA method has

Table 5.2: The KS distances, $d_{KS}(\cdot, X_P)$, of the approximations to risk RVs distributed Pareto

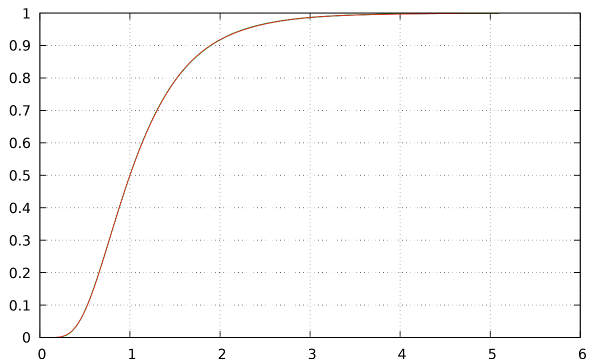
Parameters	$\tilde{F}_{(2)}$	$\tilde{F}_{(10)}$	$\tilde{F}_{W_{2,200}}$	$\tilde{F}_{NA}$	$\tilde{F}_{NP}$	$\tilde{F}_{SG}$
$\alpha = 2.7163$, $\beta = 16.8759$	1.273e−2	4.320e−5	3.477e−2	0.302e0	-	-
Parameters	$\tilde{F}_{(3)}$	$\tilde{F}_{(10)}$	-	$\tilde{F}_{NA}$	$\tilde{F}_{NP}$	$\tilde{F}_{SG}$
$\alpha = 2$, $\beta = 3000$	6.532e−2	9.525e−3	-	-	-	-

Table 5.3: The KS distances, $d_{KS}(\cdot, X_W)$, of the approximations to a risk RV distributed Weibull

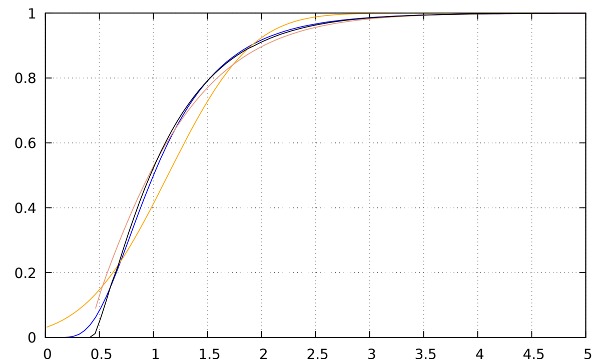
Parameters	$\tilde{F}_{(3)}$	$\tilde{F}_{(10)}$	$\tilde{F}_{W_{3,70}}$	$\tilde{F}_{NA}$	$\tilde{F}_{NP}$	$\tilde{F}_{SG}$
$\beta = 220.653$, $\delta = 0.8$	3.297e−2	5.337e−4	2.075e−2	2.13e−1	3.45e−1	1.374e−1

Table 5.4: Computation times for the CDFs in Tables 5.1, 5.2, and 5.3.

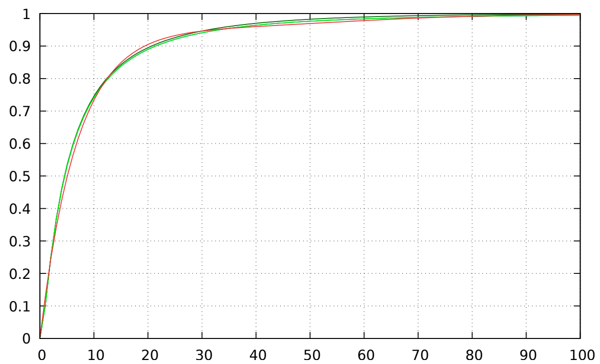
Log-normal	$\tilde{F}_{(5)}$	$\tilde{F}_{(10)}$
$\mu = 0$, $\sigma = 0.5$	19.256	39.120
Log-normal	$\tilde{F}_{(3)}$	$\tilde{F}_{(16)}$
$\mu = 1.5240$, $\sigma = 1.2018$	14.068	71.052
Log-normal	$\tilde{F}_{(3)}$	$\tilde{F}_{(36)}$
$\mu = 5.9809$, $\sigma = 1.8$	15.924	162.052
Pareto	$\tilde{F}_{(2)}$	$\tilde{F}_{(10)}$
$\alpha = 2.7163$, $\beta = 16.8759$	8.876	43.856
Pareto	$\tilde{F}_{(3)}$	$\tilde{F}_{(10)}$
$\alpha = 2$, $\beta = 3000$	13.308	40.912
Weibull	$\tilde{F}_{(3)}$	$\tilde{F}_{(10)}$
$\beta = 220.653$, $\delta = 0.8$	12.528	39.852



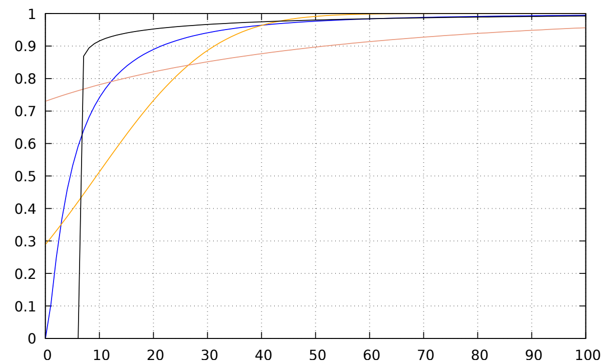
(a) The CDFs F_L, $\tilde{F}_{(5)}$, $\tilde{F}_{(10)}$, and $\tilde{F}_{W_{5,70}}$.



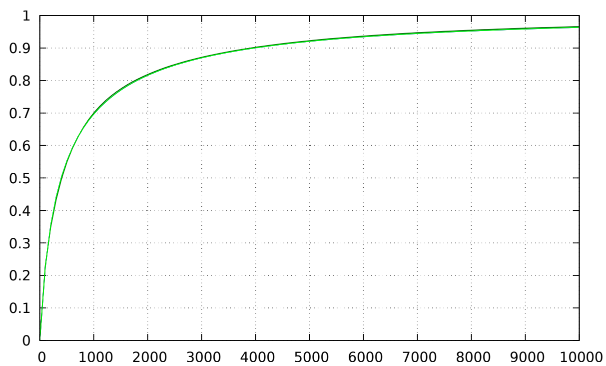
(b) The CDFs F_L, $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$.



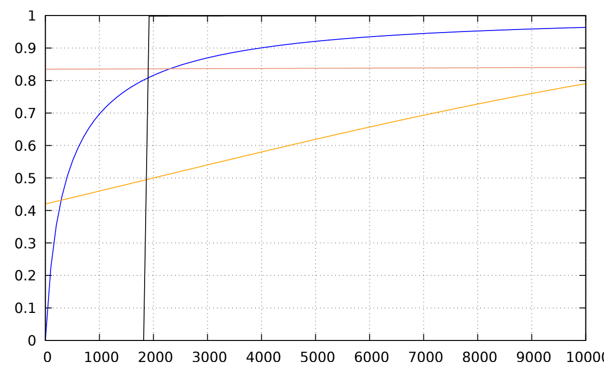
(c) The CDFs F_L, $\tilde{F}_{(3)}$, $\tilde{F}_{(16)}$, and $\tilde{F}_{W_{3,70}}$.



(d) The CDFs F_L, $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$.

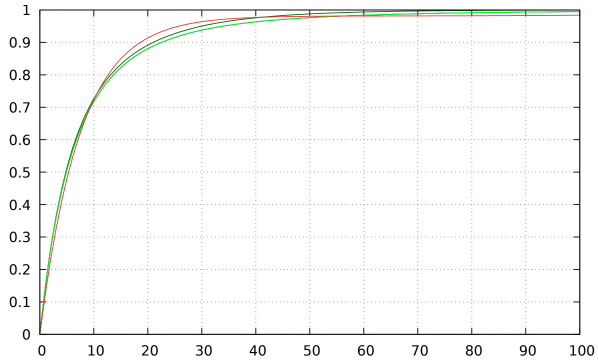


(e) The CDFs F_L, $\tilde{F}_{(3)}$, and $\tilde{F}_{(36)}$.

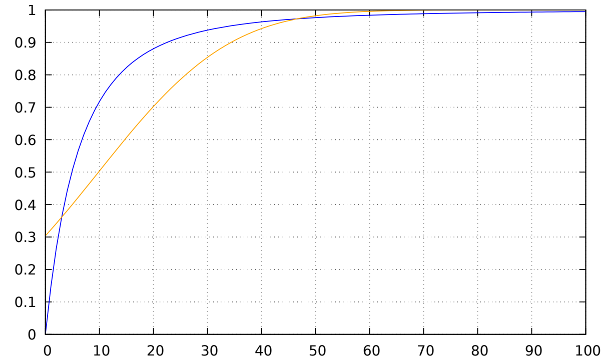


(f) The CDFs F_L, $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$.

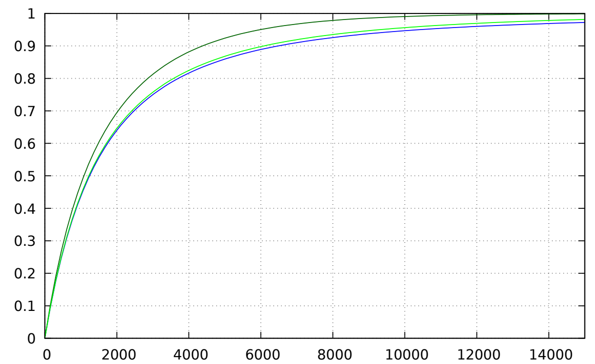
Figure 5.1: The log-normal CDFs and the corresponding approximations as per Table 5.1.



(a) The CDFs F_P, $\tilde{F}_{(2)}$, $\tilde{F}_{(10)}$, and $\tilde{F}_{W_{2,200}}$.

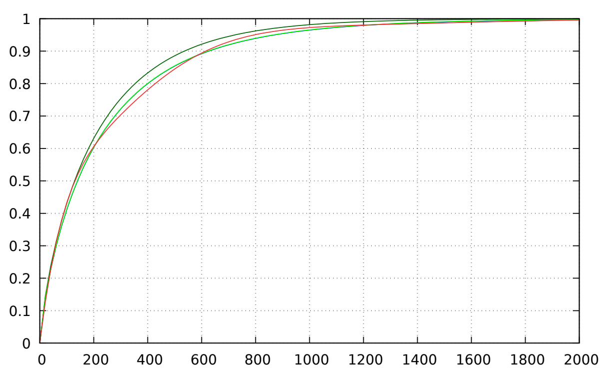


(b) The CDFs F_P, and $\tilde{F}_{NA}$.

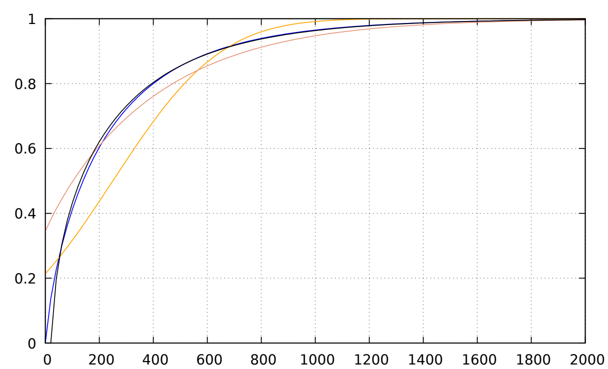


(c) The CDFs F_P, $\tilde{F}_{(3)}$, and $\tilde{F}_{(16)}$.

Figure 5.2: The Pareto CDFs and the corresponding approximations as per Table 5.2.



(a) The CDFs F_W, $\tilde{F}_{(3)}$, $\tilde{F}_{(10)}$, and $\tilde{F}_{W_{3,70}}$.



(b) The CDFs F_W, $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$.

Figure 5.3: The Weibull CDF and the corresponding approximations as per Table 5.3.

performed exactly as expected in all of the three examples discussed above. That is, it has outperformed all other MMAs in the cases when the latter are applicable (e.g., the lighter-tailed log-normal CDFs with $\sigma = 0.5,\ \sigma = 1.2018$ and the Weibull CDF), and it has provided a feasible and accurate alternative, otherwise (e.g., the heavier-tailed log-normal CDF with $\sigma = 1.8$ and the Pareto CDFs).

We conclude this section by addressing a request of a referee to report the times required to compute the CDFs in Tables 5.1, 5.2, and 5.3 using the MMA approach put forward herein. These times are provided in Table 5.4. (All calculations were performed on a laptop computer with 12 GB of memory and an Intel® Core™ i5-5200U CPU.) The numbers represent the time, in seconds, required to perform the Gaver Stehfest algorithm described in Section 5 with $m = 25$ and arithmetic precision to 300 digits and precomputed coefficients of the approximant Laplace transforms. Remarkably, the computation speed can be significantly enhanced (without dramatic drop in accuracy) by reducing the degree of precision and using a smaller value for m. Specifically, we set $m = 10$ and the arithmetic precision to 100 digits. Then in, e.g., the log-normal case with $\mu = 5.9809$ and $\sigma = 1.8$, $\tilde{F}_{(3)}$ and $\tilde{F}_{(36)}$ can be computed in 0.948 and 9.396 seconds, respectively, while achieving the KS distances of 7.968e−3 and 1.123e−6, respectively.

We further turn to the aggregate risk RVs, and we demonstrate that the advantages of our approximation method carry on.

5.6.2 Aggregate risks with full and partial coverage

We begin by approximating the CDF of the RV $S_n = X_1 + \cdots + X_n$ - the individual risk model [e.g., 24], and the CDF of the RV $S_N = X_1 + \cdots + X_N$ - the collective risk model [e.g., 32]. In the former case, we sum n independent RVs with possibly different distributions. In the latter case, we sum a random number N of IID RVs. Furthermore, as ceding insurers often turn to a reinsurer in order to reduce the variability of the underwriting outcomes, we conclude the discussion in this section with the stop-loss reinsurance set-up. We note briefly that due to Borch [17], the variance of the cedent insurer's payouts is the smallest under the stop-loss reinsurance contract. Irrespective of whether coverage modifications of the aggregate risks are allowed or not, we assume that the stand-alone risks have CDFs (5.18),(5.19), and (5.20). As explicit CDFs of the aggregate risks are not available, we use the Monte-Carlo (MC) method as a benchmark.

Example 4. Consider the CRM, $S_N = X_1 + \cdots + X_N$, where N has a Poisson(λ) distribution, and $X_i,\ i = 1, \ldots, N$ has a log-normal distribution with parameters $\mu = 5.9809$, and $\sigma = 1.8$ [8, Example 5.9]. The Laplace transform of the RV S_N is given by

$$\phi_{S_N}(z) = \exp\left[\lambda(\phi_{X_1}(z) - 1)\right].$$

Thus we obtain an approximation for $\phi(z) := \mathbb{E}[\exp(-zS_N)]$ as following

$$\tilde{\phi}(z) = \exp\left[\lambda(\tilde{\phi}_1(z) - 1)\right].$$

Then we evoke the Gaver-Stehfest algorithm to obtain the approximating CDF of the RV S_N.

Figure 5.4 summarizes the results. Namely, Figures 5.4 a), c), and e) show the CDFs of our approximation (dark green; succinctly $\tilde{F}_{S_N,(36)}$) and the CDF produced by means of MC simulation with 10^6 samples (blue; succinctly F_M), for $\lambda = 5$, 10, and 15, respectively. In each one of our approximations, we used $m = 36$ to approximate the underlying log-normal severity distribution. Figures 5.4 b), d), and f) depict the MC CDF (blue), as well as the CDF obtained via NA (orange), NPA (salmon), and SGA (black).

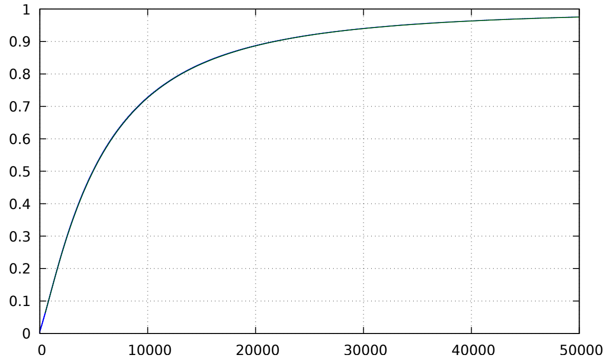
Example 5. Consider the IRM, $S_n = X_1 + \cdots + X_n$, with $n = 15$. We assume X_i, $i = 1, \ldots, 5$, are distributed log-normally with parameters $\mu = 5.9809, \sigma = 1.8$, X_i, $i = 6, ..10$, are distributed Pareto with parameters $\alpha = 2, \beta = 3000$, and X_i, $i = 11, \ldots, 15$ are distributed Weibull with parameters $\beta = 220.653, \delta = 0.8$; all independent. The LT of the aggregate risk RV S_n, denoted by $\phi(z) := \mathbb{E}[\exp(-zS_n)]$, is given by

$$\phi(z) = (\phi_{X_1}(z) \cdot \phi_{X_6}(z) \cdot \phi_{X_{11}}(z))^5.$$

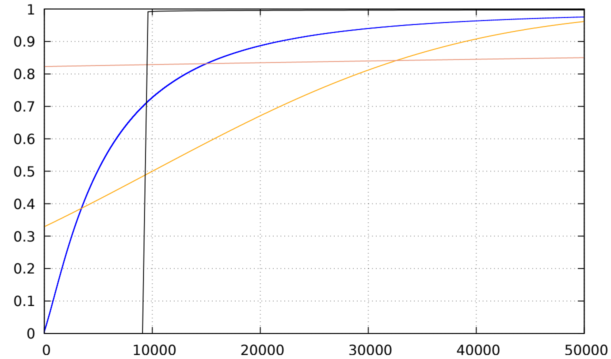
Thus we approximate the CDF of the RV S_n by first approximating the LTs of the severities and then evoking the Gaver-Stehfest method on

$$\tilde{\phi}(z) = (\tilde{\phi}_1(z) \cdot \tilde{\phi}_6(z) \cdot \tilde{\phi}_{11}(z))^5.$$

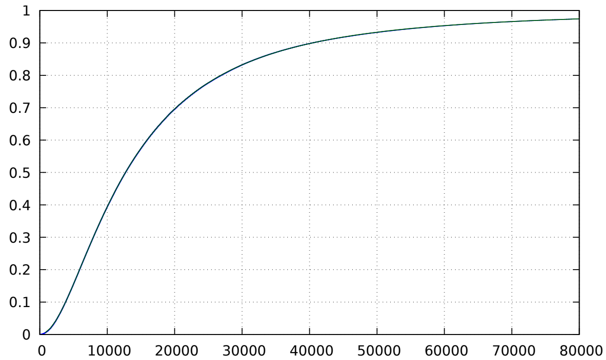
Figure 5.5 shows our approximation of the CDF of S_n (dark green; denoted by $\tilde{F}_{S_n,(40),(36),(30)}$) and the CDF obtained by means of MC simulation (blue). In this case, we used approximation orders $m = 40$, $m = 36$, and $m = 30$ to approximate the underlying log-normal, Pareto, and Weibull CDFs, respectively. Note that the RV S_n has only one finite moment and, hence, other MMAs are not applicable.



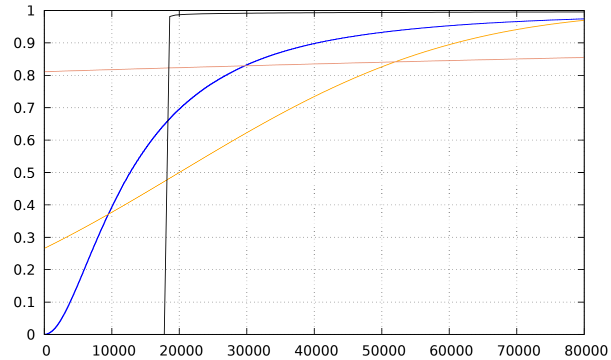
(a) The CDFs $\tilde{F}_{S_N,(36)}$ and F_M ($\lambda = 5$).



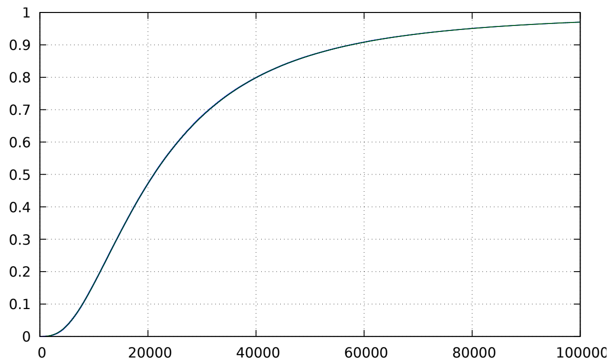
(b) The CDFs F_M, $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$ ($\lambda = 5$).



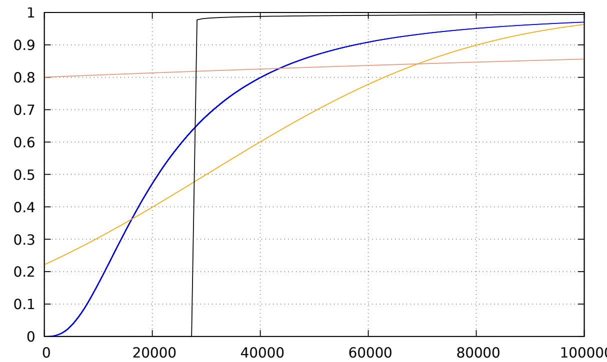
(c) The CDFs $\tilde{F}_{S_N,(36)}$ and F_M ($\lambda = 10$).



(d) The CDFs F_M, $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$ ($\lambda = 10$).



(e) The CDFs $\tilde{F}_{S_N,(36)}$ and F_M ($\lambda = 15$).



(f) The CDFs F_M, $\tilde{F}_{NA}$, $\tilde{F}_{NP}$, and $\tilde{F}_{SG}$ ($\lambda = 15$).

Figure 5.4: The collective risk model: $S_N = X_1 + \cdots + X_N$, $N \sim \text{Poisson}(\lambda)$, and the X_i's are IID log-normally with the parameters ($\mu = 5.9809, \sigma = 1.8$).

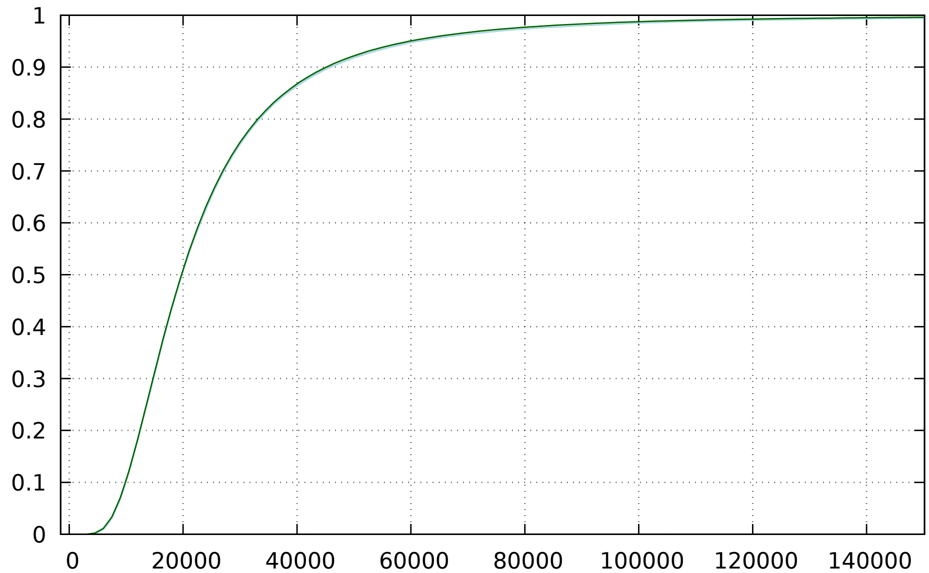


Figure 5.5: The CDFs $\tilde{F}_{S_n,(40),(36),(30)}$ and F_M for the individual risk model S_n, with $n = 15$.

Example 6. In the previous examples we considered aggregate risks with full coverages, that is there were no deductibles, policy limits, or other policy modifications applied by the insurer to reduce the payouts of the benefits. However, this is not always the case. Consequently, in this example, we explorer the so-called stop-loss r.v. $S_{r,l}$, which is given, for positive retention r and limit l, by

$$S_{r,l} = \begin{cases} 0, & X < r \\ S - r, & r \leq X < r + l \\ l, & r + l \leq X < \infty \end{cases}$$

where the RV S is the aggregate risk RV within either the IRM or CRM. It is a standard exercise to show that the CDF of the RV $S_{r,l}$ is given by

$$F_{r,l}(s) = \begin{cases} \mathbb{P}[S \leq s + r), & s < l \\ 1, & l \leq s \end{cases}.$$

Consequently, the CDF of the risk RV $S_{r,l}$ is approximated similarly to the case in Example 4 (if the RV S refers to the aggregate risk within CRM), and similarly to the case in Example 5 (if the RV S refers to the aggregate risk within IRM). To demonstrate, we consider the following CRM: $S := S_N = X_1 + \cdots + X_N$, $N \sim$ Poisson(15), and the X_i's are IID with log-normal distribution and the parameters $\mu = 5.9809, \sigma = 1.8$. For the sake of the

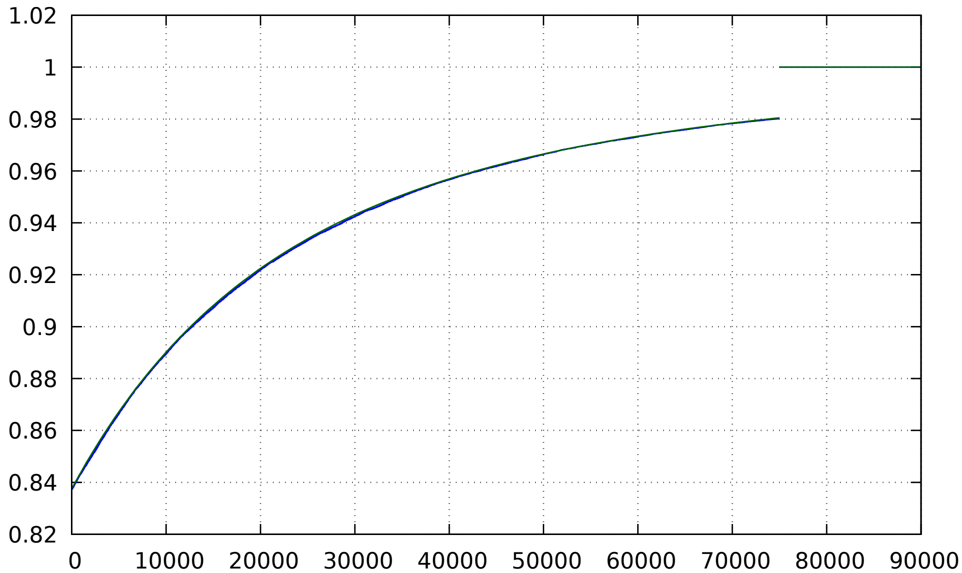


Figure 5.6: The CDFs $\tilde{F}_{r,l}$ and F_M for the stop-loss contract with retention $r = 45000$ and limit $l = 75000$

demonstration, we set $r = \mathbb{E}[N] \cdot 3000 = 45000$, and $l = \mathbb{E}[N] \cdot 5000 = 75000$. Figure 5.6 depicts the approximating CDF $\tilde{F}_{r,l}$ of the risk RV $S_{r,l}$ (dark green) and the CDF obtained by means of the MC simulation F_M (blue).

Chapter 6

On the rate of convergence of the Gaver-Stehfest algorithm

The Gaver-Stehfest algorithm is widely used for numerical inversion of Laplace transform. In this chapter we provide the first rigorous study of the rate of convergence of the Gaver-Stehfest algorithm. We prove that Gaver-Stehfest approximations converge exponentially fast if the target function is analytic in a neighbourhood of a point and they converge at a rate $o(n^{-k})$ if the target function is $(2k+3)$-times differentiable at a point.

6.1 Introduction and main results

The Gaver-Stehfest algorithm for numerical inversion of Laplace transform has a long history. In 1966 Gaver [31] has introduced simple (but rather slowly convergent) approximations for the inverse Laplace transform, and in 1970 Stehfest [54, 55] has applied convergence acceleration to Gaver's approximation and thus the Gaver-Stehfest algorithm was born. The algorithm turned out to be very popular with practitioners due to a number of desirable properties: it is linear, it is exact for constant functions, all the coefficients can be computed explicitly and, most importantly, the algorithm does not require the use of complex numbers, as it needs the values of the Laplace transform only on the positive real line. The price one has to pay for this latter feature is that the algorithm requires high-precision arithmetic for its implementation.

Let us present the Gaver-Stehfest algorithm. We start with a locally integrable function

$f : (0, \infty) \mapsto \mathbb{R}$, such that its Laplace transform

$$F(z) := \int_0^\infty e^{-zx} f(x) \mathrm{d}x \tag{6.1}$$

is finite for all $z > 0$. We want to solve the following inverse problem: given the values of $F(z)$ for $z > 0$, compute the value of $f(x)$ at a given point $x > 0$. Gaver-Stehfest approximations are given by

$$f_n(x) := \ln(2) x^{-1} \sum_{k=1}^{2n} a_k(n) F\left(k \ln(2) x^{-1}\right), \quad n \geq 1, \ x > 0, \tag{6.2}$$

where

$$a_k(n) := \frac{(-1)^{n+k}}{n!} \sum_{j=[(k+1)/2]}^{\min(k,n)} j^{n+1} \binom{n}{j} \binom{2j}{j} \binom{j}{k-j}, \quad 1 \leq k \leq 2n.$$

In [41] several conditions for convergence of $f_n(x_0)$ were established. It was proved that if f has bounded variation or is Hölder continuous in a neighbourhood of $x_0 > 0$, then $f_n(x_0)$ converge to $(f(x_0+) + f(x_0-))/2$ as $n \to \infty$. The question of the rate of convergence was left open, and until now there were no rigorous results about the rate of convergence of the Gaver-Stehfest algorithm (although there were many numerical studies of the convergence of the algorithm – see [2, 23, 37, 46, 64] and the references therein). It is the goal of this paper to provide the first rigorous treatment of the rate of convergence of the Gaver-Stehfest algorithm. We establish the following two results:

Theorem 8. *Assume that f is analytic in a neighborhood of $x_0 > 0$. Then there exists $c > 0$ such that*

$$f_n(x_0) = f(x_0) + O(e^{-cn}), \quad n \to +\infty. \tag{6.3}$$

Theorem 9. *Assume that $m \geq 5$ and f is m-times differentiable at $x_0 > 0$. Set $k = [(m-3)/2]$. Then*

$$f_n(x_0) = f(x_0) + o(n^{-k}), \quad n \to +\infty. \tag{6.4}$$

The above two theorems lead to two natural problems: determine the largest values of c and k in (6.3) and (6.4). The first problem, that is trying to determine the largest value of c in (6.3) is likely to be very hard and we do not have any intuition as to what the answer may be. For the second problem we do have the following conjecture, supported by a number of numerical experiments

Conjecture: *If f is m-times differentiable at $x_0 > 0$ then*

$$f_n(x_0) = f(x_0) + O(n^{-m}), \quad n \to +\infty.$$

We arrived at this conjecture by investigating the rate of convergence of the Gaver-Stehfest approximations to functions of the form

$$f(x) = (x-1)^{\alpha} e^{-\beta x} \times \mathbf{1}_{\{x>1\}}, \tag{6.5}$$

where $\alpha > 0$ and $\beta \in \mathbb{C}$ with $\text{Re}(\beta) \geq 0$. This function clearly satisfies $f(1) = 0$ and is m-times differentiable at $x = 1$ for any integer $m < \alpha$. The corresponding Laplace transform is easily computed explicitly

$$F(z) = \int_0^{\infty} f(x) e^{-zx} \mathrm{d}x = \int_1^{\infty} (x-1)^{\alpha} e^{-(\beta+z)x} \mathrm{d}x = \Gamma(\alpha+1)(\beta+z)^{-\alpha-1} e^{-\beta-z}, \quad z > 0.$$

To find the optimal value of k in (6.4) we computed Gaver-Stehfest approximations $f_n(1)$ for $1 \leq n \leq 300$ (using high-precision arithmetic) and then we used linear regression to compute k that provides the best fit for $\ln |f_n(1)| \sim C - k \ln(n)$, $1 \leq n \leq 300$. This procedure was repeated many times with different values of parameters α and β and the above conjecture seems to hold true for all functions of the form (6.5).

This chapter is organized as follows: In Section 6.2 we state and prove Theorem 10, which is the foundation of our approach. In Section 6.3 we prove Theorem 8 and in Section 6.4 we prove Theorem 9.

6.2 Preliminary results

Let us review some properties of the Lambert W-function, which will be needed later. The principal branch of the Lambert W-function, denoted by $W(z)$, is an analytic function in the neighborhood of $z = 0$ that satisfies $W(z)\exp(W(z)) = z$. It is well-known [21] that W is analytic in $\mathbb{C} \setminus (-\infty, -e^{-1}]$, and it has the following Taylor series at $z = 0$ (see formula (3.1) in [21])

$$W(z) = \sum_{n \geq 1} (-n)^{n-1} \frac{z^n}{n!}, \quad |z| < 1/e, \tag{6.6}$$

and a branching singularity at $z = -1/e$

$$W(z) = -1 + p - \frac{p^2}{3} + \frac{11}{72}p^3 - \frac{43}{540}p^4 + \frac{769}{17280}p^5 + ... = \sum_{n\geq 0} \mu_n p^n, \tag{6.7}$$

where $p = \sqrt{2(1+ez)}$ and the series converges for $|p| < \sqrt{2}$ (see formula (4.22) in [21]). The coefficients μ_n are certain rational numbers that can be computed recursively (see formulas (4.23) and (4.24) in [21]).

We define

$$H(z) := -\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 W(z) = -\frac{W(z)}{(1+W(z))^3}. \tag{6.8}$$

The second equality follows from the identity $zW'(z) = W(z)/(1+W(z))$, which can be easily derived from the functional equation $W(z)\exp(W(z)) = z$. Since W is analytic in $\mathbb{C} \setminus (-\infty, -e^{-1}]$ and satisfies $W(0) = 0$, it is clear from (6.8) that H is also analytic in $\mathbb{C} \setminus (-\infty, -e^{-1}]$ and satisfies $H(0) = 0$.

From (6.7) and (6.8) we derive the series representation

$$H(z) = p^{-3} - \frac{11}{24}p^{-1} - \frac{4}{135} - \frac{1}{1152}p - \frac{31}{405}p^2 - \cdots = p^{-3} - \frac{11}{24}p^{-1} + \sum_{n\geq 0} c_n p^n, \tag{6.9}$$

where, as above, $p = \sqrt{2(1+ez)}$ and the series converges for $|p| < \sqrt{2}$. The coefficients c_n in (6.9) are certain rational numbers that can be computed recursively using values of μ_n. We define the following two functions in terms of coefficients c_n:

$$A(u) := \frac{1}{2\sqrt{2}} - \frac{11}{24\sqrt{2}}(1+u) + \sum_{n\geq 0} c_{2n+1} 2^{n+1/2}(1+u)^{n+2}, \tag{6.10}$$

$$B(u) := \sum_{n\geq 0} c_{2n} 2^n (1+u)^n. \tag{6.11}$$

Since the series in (6.9) converges for $|p| < \sqrt{2}$, we conclude that the series (6.10) and (6.11) converge for $|1+u| < 1$, thus functions A and B are analytic in the disk $D_1(-1)$: here and everywhere else in this chapter we will denote

$$D_r(a) := \{z \in \mathbb{C} \ : \ |z-a| < r\},$$

for $r > 0$ and $a \in \mathbb{C}$. By construction we have an identity

$$H(z/e) = (1+z)^{-3/2} A(z) + B(z), \tag{6.12}$$

which is valid for $z \in D_1(-1) \setminus (-\infty, -1]$.

Next, given a function f and $x_0 > 0$ we define

$$\tilde{f}(v) := \frac{f(x_0 \log_{1/2}((1+v)/2))}{1+v} + \frac{f(x_0 \log_{1/2}((1-v)/2))}{1-v}, \quad -1 < v < 1, \tag{6.13}$$

and

$$\phi(x) := \frac{1}{\pi\sqrt{1-x}} \int_0^{\frac{\pi}{2}} \tilde{f}(\sqrt{x}\sin(y))\mathrm{d}y, \quad 0 \leq x < 1. \tag{6.14}$$

We also define $w(z) := ze^{z+1}$ and

$$\Lambda(w) = \Lambda(w;\sigma) := \int_0^{\sigma} (1 + w(1-x))^{-\frac{3}{2}} A(w(1-x))\phi(x)\mathrm{d}x, \tag{6.15}$$

where $\sigma \in (0,1)$ and A is defined in (6.10). For every $\sigma \in (0,1)$ the function $w \mapsto \Lambda(w;\sigma)$ is well-defined for $w \in D_\delta(-1) \setminus (-\infty, -1]$ for some $\delta = \delta(\sigma) > 0$ small enough. Finally, for $\epsilon > 0$ we denote

$$\mathcal{D}_\epsilon := D_{1+\epsilon}(0) \setminus D_{\epsilon^{1/4}}(-1) = \{z \in \mathbb{C} \ : \ |z| < 1 + \epsilon \ \text{ and } \ |1+z| > \epsilon^{1/4}\}. \tag{6.16}$$

The main goal of this section is to establish the following result.

Theorem 10. *Assume that $f(x_0) = 0$.*

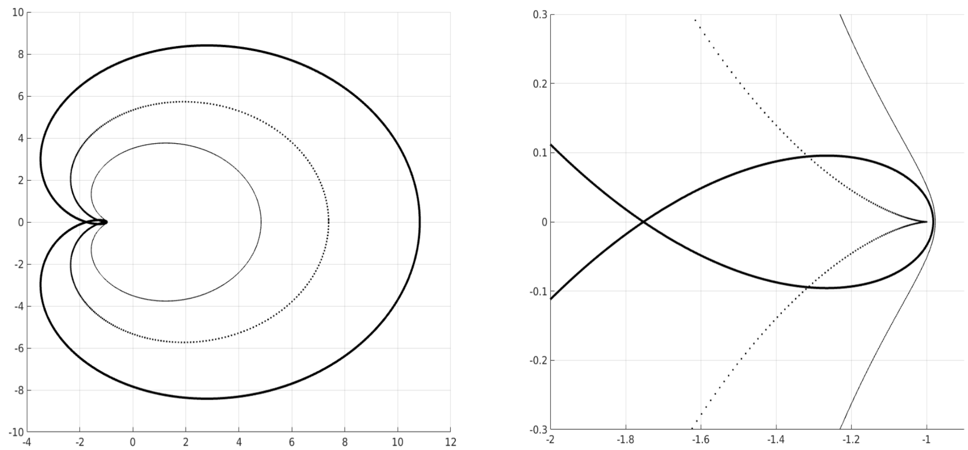
(i) The function

$$\Delta(z) := \sum_{n \geq 1} f_n(x_0)(-1)^n z^n, \tag{6.17}$$

is analytic in $\mathcal{D}_\epsilon$ for $\epsilon < 1/100$.

(ii) For any $\sigma \in (0,1)$ the function $\Delta(z) - \Lambda(w(z);\sigma)$ is analytic in $D_\delta(-1)$ for some $\delta > 0$ small enough.

Theorem 10 will be our main tool in proving Theorems 8 and 9. We will apply it as follows: suppose we can show that for some $\sigma \in (0,1)$ the function $z \mapsto \Lambda(w(z);\sigma)$ is analytic in $D_\delta(-1)$ for some $\delta > 0$. Then Theorem 10 would imply that $\Delta(z)$ is analytic in $D_R(0)$ for some $R > 1$. This latter fact combined with (6.17) would prove that the sequence $\{f_n(x_0)\}_{n \geq 1}$ converges to zero exponentially fast. Alternatively, if the function $\Lambda(w(z))$ is not analytic in $D_\delta(-1)$ for any $\delta > 0$, it must have a singularity at $z = -1$, and then the behavior of $\Lambda(w(z))$ at this singularity (for example, the number of times $\Lambda(w(z))$ is differentiable at $z = -1$)



(a) (b)

Figure 6.1: The images of three circles $|z| = 0.8$ (thin line), $|z| = 1$ (dotted line) and $|z| = 1.2$ (thick line) under the map $z \mapsto w = ze^{z+1}$. Figure (b) magnifies the area near $w = -1$ of the figure (a).

would give us information about the singularity of $\Delta(z)$ at $z = -1$, and this informatoin coupled with (6.17) would again lead to estimates on the rate of convergence of the sequence $\{f_n(x_0)\}_{n\geq 1}$ to zero.

Before we prove Theorem 10, we need to establish a number of preliminary results. The next technical result collects some properties of the map $z \mapsto w = ze^{z+1}$ (see Figure 6.1).

Lemma 2. *Let $w(z) = ze^{z+1}$.*

(i) For any $c \in (0, 1)$ there exists $R > 1$ such that the function $c \times w(z)$ maps $D_R(0)$ into $\mathbb{C} \setminus (-\infty, -1]$.

(ii) For $\epsilon \in (0, 1/100]$ the function $w = w(z)$ maps the domain $\mathcal{D}_\epsilon$ (defined in (6.16)) into $\mathbb{C} \setminus (-\infty, -1]$.

Proof. First we will establish the following

Fact: If $\epsilon \in (0, 1/100]$ and for some $y \in \mathbb{R}$ we have $z = -y\cot(y) + \mathrm{i}y \in D(0; 1 + \epsilon)$, then necessarily $z \in D_{\epsilon^{1/4}}(-1)$.

To prove this fact we will need the following two inequalities

$$0 < 1 - y\cot(y) < y^2/2 \quad \text{for all } \ y \in (-1/4, 1/4), \tag{6.18}$$

$$1 + y^2/6 < y\csc(y) \quad \text{for all } \ y \in (-\pi, \pi). \tag{6.19}$$

These inequalities can be easily established by examining MacLaurin series of $y\cot(y)$ and $y\sec(y)$. Alternatively, these inequalities follow at once from inequalities (17) and (19) in [19].

Now, if $\epsilon \in (0, 1/100]$ and $z = -y\cot(y) + \mathrm{i}y \in D_{1+\epsilon}(0)$, then

$$|z|^2 = y^2\cot(y)^2 + y^2 < (1+\epsilon)^2,$$

thus $|y| < 1 + \epsilon$ and from (6.19) we find

$$(1 + y^2/6)^2 < y^2\csc(y)^2 = y^2\cot(y)^2 + y^2 < (1+\epsilon)^2,$$

which implies that $|y| < \sqrt{6\epsilon} < 1/4$. Then applying (6.18) we estimate

$$|z+1|^2 = (1 - y\cot(y))^2 + y^2 < y^4/4 + y^2 < 9\epsilon^2 + 6\epsilon = \epsilon^{1/2} \times \epsilon^{1/2}(9\epsilon + 6) < \epsilon^{1/2},$$

and this implies $z \in D_{\epsilon^{1/4}}(-1)$. This ends the proof of the Fact above.

Let us now prove part (i) of Lemma 2. Since $w(-1) = -1$ and w is an entire (and thus, continuous) function, there exists $\epsilon \in (0, 1/100]$ small enough such that $|z + 1| < \epsilon^{1/4}$ implies $|w(z) + 1| < 1/c - 1$. Take $R = 1 + \epsilon$ and let $z \in D_R(0)$. If $z = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$, then

$$\mathrm{Im}(w(z)) = e^{x+1}(y\cos(y) + x\sin(y)).$$

Thus $w(z) \in \mathbb{R}$ if $y = 0$ or $x = -y\cot(y)$. For $y = 0$ it is easy to see that $w(z) = w(x) \geq -1$, thus $-1 < c \times w(z)$. If $y \neq 0$ and $w(z) \in \mathbb{R}$ then $z = -y\cot(y) + \mathrm{i}y$. Since $z \in D_{1+\epsilon}(0)$, by the Fact above we conclude that $z \in D_{\epsilon^{1/4}}(-1)$, thus $|w(z) + 1| < 1/c - 1$, which implies that $-1 < c \times w(z)$. Thus if $z \in D_{1+\epsilon}(0)$ and $w(z)$ is real, then necessarily $-1 < c \times w(z)$. In other words, the function $c \times w(z)$ maps $D_{1+\epsilon}(0)$ into $\mathbb{C} \setminus (-\infty, 1]$.

It remains to prove part (ii) of Lemma 2. Let $\epsilon \in (0, 1/100]$ and $z = x + \mathrm{i}y \in \mathcal{D}_\epsilon$ for $x, y, \in \mathbb{R}$. As we argued above, if $w(z) \in \mathbb{R}$ then either $y = 0$ or $x = -y\cot(y)$. In the former case z is real and the minimum of $w(z)$ over real $z \in \mathcal{D}_\epsilon$ is strictly greater than -1 (the minimum of $w(x)$ over $x \in \mathbb{R}$ is -1 and is achieved at $z = -1$, and $z = -1 \notin \mathcal{D}_\epsilon$). In the latter case, we use the Fact above and conclude that $z \in D_{\epsilon^{1/4}}(-1)$, which is impossible since by definition $\mathcal{D}_\epsilon$ does not contain points from $D_{\epsilon^{1/4}}(-1)$. Therefore, the function $w(z)$ maps $\mathcal{D}_\epsilon$ into $\mathbb{C} \setminus (-\infty, 1]$. $\square$

Lemma 3. *Let Ω be a compact set in $\mathbb{R}^n$ and $a(\mathbf{x})$ be a continuous function and $b(\mathbf{x})$ an integrable function of $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \Omega$. Assume that $g(z)$ is analytic in the domain $G \subset \mathbb{C}$ and $za(\mathbf{x}) \in G$ for all $z \in G$ and $\mathbf{x} \in \Omega$. Then the function*

$$\Phi(z) := \int\limits_{\Omega} g(za(\mathbf{x}))b(\mathbf{x})\mathrm{d}x_1\mathrm{d}x_2 \dots \mathrm{d}x_n \tag{6.20}$$

is also analytic in G.

Proof. For each $\mathbf{x} \in \Omega$, the function $z \mapsto g(za(\mathbf{x}))$ is analytic in G. By Cauchy's integral theorem, for each $\mathbf{x} \in \Omega$ and for any triangle T contained in G we have

$$\int_T g(za(\mathbf{x}))\mathrm{d}z = 0.$$

Since T and Ω are compact and $g(za(\mathbf{x}))$ is continuous on $T \times \Omega$, we have

$$\sup_{(z,x) \in T \times \Omega} |g(za(\mathbf{x}))| < \infty.$$

Using this fact and the assumption that b is integrable on Ω, we can apply Fubini's theorem and conclude that

$$\int_T \Phi(z)\mathrm{d}z = \int_\Omega \left[\int_T g(za(\mathbf{x}))\mathrm{d}z\right] g(\mathbf{x})\mathrm{d}x_1\mathrm{d}x_2 \dots \mathrm{d}x_n = 0$$

for any every triangle T contained G. Morera's Theorem tells us that Φ is analytic in G. $\square$

Next, we define

$$G(z) := \frac{2}{\pi} \int\limits_0^{\frac{\pi}{2}} H(z \sin(t)^2)\mathrm{d}t, \tag{6.21}$$

where H was defined in (6.8). As we discussed on page 75, H is analytic in $\mathbb{C} \setminus (-\infty, -e^{-1}]$ and satisfies $H(0) = 0$. This fact and Lemma 3 applied to the integral in (6.21) implies that G is also analytic in $\mathbb{C} \setminus (-\infty, -e^{-1}]$ and satisfies $G(0) = 0$.

Lemma 4. *The function* $\Delta(z)$ *defined by* (6.17) *has integral representation*

$$\Delta(z) = \int_0^{\infty} G(4e^{-1-u}(1-e^{-u})w(z))f(x_0u/\ln(2))\mathrm{d}u \tag{6.22}$$

and it is analytic in $\mathcal{D}_\epsilon$ *for* $\epsilon < 1/100$.

Proof. From [41] we know that Gaver-Stehfest approximants are given by an integral representation

$$f_n(x) = \int_0^{\infty} q_n\left(4e^{-u}(1-e^{-u})\right) f(xu/\ln(2))\mathrm{d}u. \tag{6.23}$$

where

$$q_n(v) := \sum_{k=1}^{n} \frac{k^{n+1}(\frac{1}{2})_k}{(n-k)!(k!)^2}(-1)^{n+k}v^k, \quad n \geq 1. \tag{6.24}$$

Also, from Proposition 2.2 in [41] we find that for $0 \leq v \leq 1$ and $|z| < 1/(2e)$

$$G\left(vze^{z}\right) = \sum_{n\geq 1} q_n(v)(-1)^n z^n. \tag{6.25}$$

Also, from (6.24) we find (using the Binomial Theorem and the trivial estimates $(\frac{1}{2})_k < k!$ and $k^{n+1} \leq n^{n+1}$) that

$$|q_n(v)| \leq vn^{n+1}\sum_{k=1}^{n}\frac{1}{(n-k)!(k!)} < v\frac{n^{n+1}2^n}{n!}, \quad \text{for all } \ 0 \leq v \leq 1. \tag{6.26}$$

Thus for every $|z| < 1/(4e)$ we have the bound

$$\sum_{n\geq 1} |q_n(v)| \times |z|^n < C \times v, \quad 0 \leq v \leq 1,$$

for some $C > 0$, so that we can apply the Dominated Convergence Theorem to conclude that

$$\begin{aligned}\Delta(z) = \sum_{n\geq 1} f_n(x_0)(-1)^n z^n &= \int_0^{\infty}\left[\sum_{n\geq 1} q_n\left(4e^{-u}(1-e^{-u})\right)(-1)^n z^n\right] f(x_0u/\ln(2))\mathrm{d}u \\ &= \int_0^{\infty} G(4ze^{z-u}(1-e^{-u}))f(x_0u/\ln(2))\mathrm{d}u.\end{aligned} \tag{6.27}$$

Thus we have established (6.22) for $|z| < 1/(4e)$. The fact that $\Delta(z)$ can be extended to an analytic function in $\mathcal{D}_\epsilon$ follows from (6.22), Lemma 2(ii), Lemma 3 and the fact that $G(z)$ is an analytic function in $\mathbb{C} \setminus (-\infty, -e^{-1}]$. $\square$

Next, we define

$$\Delta_1(z) = \Delta_1(z;\sigma) := \int_0^\sigma G(e^{-1}(1-v^2)w(z))\tilde{f}(v)\mathrm{d}v, \tag{6.28}$$

where $\sigma \in (0,1)$ and $\tilde{f}$ is defined in (6.13).

Lemma 5. *For any $\sigma \in (0,1)$ there exists $R > 1$ such that the function $z \mapsto \Delta(z) - \Delta_1(z;\sigma)$ is analytic in $D_R(0)$.*

Proof. First we compute

$$\begin{aligned}\Delta(z) &= \int_0^{\ln(2)} G(4e^{-1-u}(1-e^{-u})w(z))f(x_0 u/\ln(2))\mathrm{d}u \\ &\quad + \int_{\ln(2)}^{\infty} G(4e^{-1-u}(1-e^{-u})w(z))f(x_0 u/\ln(2))\mathrm{d}u \\ &= \int_0^1 G(e^{-1}(1-v^2)w(z))\left[\frac{f(x_0\log_{1/2}((1+v)/2))}{1+v} + \frac{f(x_0\log_{1/2}((1-v)/2))}{1-v}\right]\mathrm{d}v.\end{aligned} \tag{6.29}$$

Here we changed variables $u = -\ln((1+v)/2)$ in the integral over $u \in (0,\ln(2))$ and $u = -\ln((1-v)/2)$ in the integral over $u \in (\ln(2),\infty)$.

Next, we define $g(z) := G(z/e)/z$. As we pointed out on page 79, the function G is analytic in $\mathbb{C} \setminus (-\infty,-e^{-1}]$ and satisfies $G(0) = 0$, thus the function g is analytic in $\mathbb{C} \setminus (-\infty,-1]$. From (6.28) and (6.29) we obtain

$$\Delta(z) - \Delta_1(z) = w(z)\int_\sigma^1 g(w(z)(1-v^2))b(v)\mathrm{d}v$$

where

$$b(v) := (1-v^2)\tilde{f}(v) = (1-v)f(x_0\log_{1/2}((1+v)/2)) + (1+v)f(x_0\log_{1/2}((1-v)/2)).$$

According to Lemma 2(i), there exists $R > 1$ such that the function $z \mapsto (1-\sigma^2) \times w(z)$ maps $D_R(0)$ into $\mathbb{C} \setminus (-\infty,-1]$. Then for every $v \in (\sigma,1]$ we have

$$(1-v^2) \times w(z) \in \mathbb{C} \setminus (-\infty,-1], \quad \text{for all } \; z \in D_R(0).$$

Note also that the function $b(v)$ is integrable over $v \in (\sigma,1]$. Applying Lemma 3, we conclude that $\Delta(z) - \Delta_1(z)$ is analytic in $D_R(0)$. □

Next we define

$$\Delta_2(z) = \Delta_2(z;\sigma) := \frac{1}{\pi}\int_0^\sigma \int_{v^2}^{\sigma^2} \frac{H(e^{-1}(1-x)w(z))}{\sqrt{(1-x)(x-v^2)}} \mathrm{d}x \tilde{f}(v)\mathrm{d}v \tag{6.30}$$

where $\sigma \in (0,1)$, H was defined in (6.8) and $\tilde{f}$ was defined in (6.13).

Lemma 6. *For any $\sigma \in (0,1)$ there exists $R > 1$ such that the function $z \mapsto \Delta_1(z;\sigma) - \Delta_2(z;\sigma)$ is analytic in $D_R(0)$.*

Proof. We define two sets

$$\Omega_1 := \left\{(t,v) \in \mathbb{R}^2 \ : \ 0 \le v \le \sigma,\ 0 \le t < \arcsin\left(\sqrt{\frac{1-\sigma^2}{1-v^2}}\right)\right\},$$
$$\Omega_2 := \left\{(t,v) \in \mathbb{R}^2 \ : \ 0 \le v \le \sigma,\ \arcsin\left(\sqrt{\frac{1-\sigma^2}{1-v^2}}\right) \le t \le \frac{\pi}{2}\right\}.$$

Using formulas (6.21) and (6.28) we write

$$\begin{aligned}\Delta_1(z) &= \frac{2}{\pi}\int_0^\sigma \int_0^{\frac{\pi}{2}} H(e^{-1}(1-v^2)\sin(t)^2 w(z))\mathrm{d}t\tilde{f}(v)\mathrm{d}v \\ &= \frac{2}{\pi}\iint_{\Omega_1} H(e^{-1}(1-v^2)\sin(t)^2 w(z))\mathrm{d}t\tilde{f}(v)\mathrm{d}v \\ &\quad + \frac{2}{\pi}\iint_{\Omega_2} H(e^{-1}(1-v^2)\sin(t)^2 w(z))\mathrm{d}t\tilde{f}(v)\mathrm{d}v.\end{aligned}$$

We change the variable of integration $t \mapsto x = 1-(1-v^2)\sin(t)^2$ so that

$$\mathrm{d}t = -1/(2\sqrt{(1-x)(x-v^2)})\mathrm{d}x$$

and we obtain

$$\frac{2}{\pi}\iint_{\Omega_2} H(e^{-1}(1-v^2)\sin(t)^2 w(z))\mathrm{d}t\tilde{f}(v)\mathrm{d}v = \Delta_2(z),$$

which implies

$$\Delta_1(z) - \Delta_2(z) = \frac{2}{\pi}\iint_{\Omega_1} H(e^{-1}(1-v^2)\sin(t)^2 w(z))\mathrm{d}t\tilde{f}(v)\mathrm{d}v. \tag{6.31}$$

Note that on the set Ω_1 we have $(1-v^2)\sin(t)^2 \le 1-\sigma^2$, thus we can use the fact that $H(z/e)$ is an analytic function in $\mathbb{C} \setminus (-\infty,-1]$ and apply Lemma 2(i) and Lemma 3 to conclude that the integral in the right-hand side of (6.31) is an analytic function of z in the disk $D_R(0)$ for some $R > 1$. $\square$

Proof of Theorem 10: Part (i) of Theorem 10 was established in Lemma 4. To prove part (ii), it is enough to show that the function $z \mapsto \Delta_2(z;\sqrt{\sigma}) - \Lambda(w(z);\sigma)$ is analytic in $D_\delta(-1)$ for some $\delta > 0$, since

$$\Delta(z) - \Lambda(w(z);\sigma) = [\Delta(z) - \Delta_1(z;\sqrt{\sigma})] + [\Delta_1(z;\sqrt{\sigma}) - \Delta_2(z;\sqrt{\sigma})] + \Delta_2(z;\sqrt{\sigma}) - \Lambda(w(z);\sigma),$$

and both terms in square brackets are analytic in $D_R(0)$ for some $R > 1$ (by Lemmas 5 and 6), thus they are analytic in $D_\delta(-1)$ for any $\delta \in (0, R-1]$.

We apply Fubini's Theorem to the double integral (6.30) and interchange the order of integration to obtain

$$\begin{aligned}
\Delta_2(z;\sqrt{\sigma}) &= \frac{1}{\pi}\int_0^\sigma \int_0^{\sqrt{x}} \frac{H(e^{-1}(1-x)w(z))}{\sqrt{(1-x)(x-v^2)}} \tilde{f}(v)\mathrm{d}v\mathrm{d}x \\
&= \int_0^\sigma H(e^{-1}(1-x)w(z))\left[\frac{1}{\pi\sqrt{1-x}}\int_0^{\sqrt{x}} \frac{\tilde{f}(v)}{\sqrt{x-v^2}}\mathrm{d}v\right]\mathrm{d}x \\
&= \int_0^\sigma H(e^{-1}(1-x)w(z))\left[\frac{1}{\pi\sqrt{1-x}}\int_0^{\frac{\pi}{2}} \tilde{f}(\sqrt{x}\sin(y))\mathrm{d}y\right]\mathrm{d}x \\
&= \int_0^\sigma H(e^{-1}(1-x)w(z))\phi(x)\mathrm{d}x.
\end{aligned}$$

In deriving this formula we have changed variable of integration $v = \sqrt{x}\sin(y)$ and used (6.14). Next, we apply (6.12) to the above identity and obtain

$$\begin{aligned}
\Delta_2(z;\sqrt{\sigma}) &= \int_0^\sigma H(e^{-1}(1-x)w(z))\phi(x)\mathrm{d}x \\
&= \int_0^\sigma (1 + w(z)(1-x))^{-3/2} A(w(z)(1-x))\phi(x)\mathrm{d}x + \int_0^\sigma B(w(z)(1-x))\phi(x)\mathrm{d}x,
\end{aligned}$$

which is equivalent to

$$\Delta_2(z;\sqrt{\sigma}) - \Lambda(w(z);\sigma) = \int_0^\sigma B(w(z)(1-x))\phi(x)\mathrm{d}x.$$

According to the discussion on page 75, the function B is analytic in $D_1(-1)$. The function $w(z)$ is entire and satisfies $w(-1) = -1$, thus there exists $\delta > 0$ small enough such that $w(z)(1-x) \in D_1(-1)$ for all $x \in (0,\sigma)$ and $z \in D_\delta(-1)$. Applying Lemma 3 we conclude that the function

$$z \mapsto \int_0^\sigma B(w(z)(1-x))\phi(x)\mathrm{d}x = \Delta_2(z;\sqrt{\sigma}) - \Lambda(w(z);\sigma)$$

is analytic in $D_\delta(-1)$. $\square$

6.3 Proof of Theorem 8

We are working under assumption that f is analytic in a neighbourhood of $x_0 > 0$. We can also assume, without loss of generality, that $f(x_0) = 0$, since Gaver-Stehfest approximations are linear in f and they are exact for constant functions.

Our goal is to show that for some $\sigma \in (0,1)$ and $\delta > 0$ the function $\Lambda(w(z);\sigma)$ is analytic in $D_\delta(-1)$. Once this is established, Theorem 10 would imply that the function $\Delta(z)$ is analytic in $D_R(0)$ for some $R > 1$ and then Cauchy estimates for derivatives of analytic function would give us the desired result: for every $r \in (1;R)$ we have $|f_n(x_0)| = O(r^{-n})$ as $n \to +\infty$.

We recall that $\phi(x)$ is defined by

$$\phi(x) = \frac{1}{\pi\sqrt{1-x}} \int_0^{\frac{\pi}{2}} \tilde{f}(\sqrt{x}\sin(y))\mathrm{d}y.$$

Since f is analytic in the neighbourhood of x_0 and satisfies $f(x_0) = 0$, the function $\tilde{f}$ (defined by (6.13)) is even and analytic in a neighbourhood of $x = 0$ and also satisfies $\tilde{f}(0) = 0$, which implies that the function $x \mapsto \tilde{f}(\sqrt{x}\sin(y))$ is analytic in a neighbourhood of $x = 0$. Applying Lemma 3 we conclude that the function $\varphi(x) := \phi(x)/x$ is analytic in a neighbourhood of $x = 0$.

Our problem is now reduced to the following one: *given that $A(u)$ is analytic in $D_1(-1)$ and $\varphi(x)$ is analytic in a neighbourhood of $x = 0$, prove that there exist $\sigma \in (0,1)$ and $\delta > 0$ such that the function*

$$\Lambda(w(z)) = \int_0^{\sigma} (1 + w(z)(1-x))^{-\frac{3}{2}} A(w(z)(1-x)) x\varphi(x)\mathrm{d}x, \tag{6.32}$$

is analytic in $D_\delta(-1)$.

Since $\varphi(x)$ is analytic in a neighbourhood of $x = 0$, there exists $\epsilon \in (0,1)$ small enough such that the function of two variables

$$(w,u) \mapsto \varphi((1 + w - u^2)/w)$$

is analytic in $(w,u) \in D_\epsilon(-1) \times D_\epsilon(0)$. We set $\sigma = \epsilon^2/4$. Recall that $w(z) = ze^{z+1}$ and it an entire function that satisfies $w(-1) = -1$. Therefore we can find $\delta \in (0,1)$ small enough such that the following two conditions hold

(i) $w(z) \in D_\sigma(-1)$ for $z \in D_\delta(-1)$;

(ii) $1 + w(z) = 0$ for $z \in D_\delta(-1)$ only if $z = -1$.

Note that $w(-1) = -1$, $w'(-1) = 0$ and $w''(-1) = 1$, thus $1 + w(z) = (z+1)^2\tilde{w}(z)$ for some function $\tilde{w}(z)$ with $\tilde{w}(-1) = 1/2$. According to condition (ii) above, $\tilde{w}(z) \neq 0$ for $z \in D_\delta(-1)$. Thus we conclude that the function $\eta_1(z) := \sqrt{1 + w(z)} = (z+1)\sqrt{\tilde{w}(z)}$ is analytic in $D_\delta(-1)$. It is also clear that $\eta_1(z) \in D_\epsilon(0)$ for $z \in D_\delta(-1)$.

From condition (i) above we find that

$$1 + w(z)(1-\sigma) \in D_{(1-\sigma)\sigma}(\sigma) \quad \text{for } \ z \in D_\delta(-1), \tag{6.33}$$

The fact that $0 \notin D_{(1-\sigma)\sigma}(\sigma)$ implies $1 + w(z)(1-\sigma) \neq 0$ for $z \in D_\delta(-1)$, so that the function $\eta_2(z) := \sqrt{1 + w(z)(1-\sigma)}$ is analytic and nonzero in $D_\delta(-1)$. From (6.33) we also conclude that

$$|\eta_2(z)| \leq \sqrt{\sigma + (1-\sigma)\sigma} < \sqrt{2\sigma} = \frac{\epsilon}{\sqrt{2}} < \epsilon \quad \text{for } \ z \in D_\delta(-1), \tag{6.34}$$

thus $\eta_2(z) \in D_\epsilon(0)$ for $z \in D_\delta(-1)$.

Assume now that $z \in (-1, -1+\delta)$, so that $1 + w(z) \in (0, \sigma)$ and $1 + w(z)(1-\sigma) \in (\sigma, \sigma + (1-\sigma)\sigma)$. We change the variable of integration $x \mapsto u = \sqrt{1 + w(1-x)}$ in (6.32), so that $x = (1 + w - u^2)/w$ and obtain

$$\begin{aligned} \Lambda(w(z);\sigma) &= \int_0^\sigma (1 + w(z)(1-x))^{-\frac{3}{2}} A(w(z)(1-x)) x\varphi(x)\mathrm{d}x \\ &= \frac{2}{w(z)^2} \int_{\eta_2(z)}^{\eta_1(z)} \Big(\frac{1+w(z)}{u^2} - 1\Big) K(w(z), u)\mathrm{d}u \end{aligned} \tag{6.35}$$

where we defined

$$K(w,u) := A(u^2 - 1)\varphi((1 + w - u^2)/w).$$

Since A is analytic in $D_1(-1)$ and $\varphi((1 + w - u^2)/w)$ is analytic in $(w,u) \in D_\epsilon(-1) \times D_\epsilon(0)$, we conclude that the function $K(w,u)$ is analytic in $(w,u) \in D_\epsilon(-1) \times D_\epsilon(0)$.

Next, we define

$$L(w,u) := \frac{1+w}{u^2}\Big[K(w,u) - K(w,0)\Big] - K(w,u).$$

Since the function $u \mapsto K(w,u)$ is even and analytic in $(w,u) \in D_\epsilon(-1) \times D_\epsilon(0)$ we conclude that the function $u \mapsto (K(w,u) - K(w,0))/u^2$ is analytic in $u \in D_\epsilon(-1)$ for each $w \in D_\epsilon(0)$, thus the function $L(w,u)$ is analytic in $(w,u) \in D_\epsilon(-1) \times D_\epsilon(0)$. Therefore, there exists a function $M(w,u)$ that is analytic in $(w,u) \in D_\epsilon(-1) \times D_\epsilon(0)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}u} M(w,u) = L(w,u), \quad \text{for } (w,u) \in D_\epsilon(-1) \times D_\epsilon(0).$$

With these definitions of L and M we can rewrite the integrand in the right-hand side of (6.35) as follows

$$\left(\frac{1+w}{u^2} - 1\right) K(w,u) = \frac{1+w}{u^2} K(w,0) + L(w,u) = \frac{1+w}{u^2} K(w,0) + \frac{\mathrm{d}}{\mathrm{d}u} M(w,u).$$

Now we can evaluate the integral in (6.35):

$$\begin{aligned}
&\int_{\eta_2(z)}^{\eta_1(z)} \left[\frac{1+w}{u^2} K(w,0) + \frac{\mathrm{d}}{\mathrm{d}u} M(w,u)\right] \mathrm{d}u \\
&= \left[-\frac{1+w}{u} K(w,0) + M(w,u) \right] \Bigg|_{u=\eta_2(z)}^{u=\eta_1(z)} \\
&= \left(-\frac{1+w}{\eta_1(z)} + \frac{1+w}{\eta_2(z)} \right) K(w,0) + M(w,\eta_1(z)) - M(w,\eta_2(z)),
\end{aligned}$$

so that we finally obtain (using the fact that $1 + w(z) = \eta_1^2(z)$)

$$\Lambda(w(z);\sigma) = \frac{2}{w(z)^2} \times \left[\left(-\eta_1(z) + \frac{1+w(z)}{\eta_2(z)} \right) K(w(z),0) + M(w(z),\eta_1(z)) - M(w(z),\eta_2(z)) \right]. \tag{6.36}$$

So far we have established (6.36) for $z \in (-1, -1+\delta)$. However, due to our choice of σ and δ, the right-hand side in (6.36) is an analytic function of $z \in D_\delta(-1)$, which proves that the function $\Lambda(w(z);\sigma)$ can be extended to an analytic function in $z \in D_\delta(-1)$. This ends the proof of Theorem 8. $\square$

6.4 Proof of Theorem 9

We are working under assumption that f is m times differentiable at $x_0 > 0$ and $f(x_0) = 0$. We can also assume, without loss of generality, that $f^{(j)}(x_0) = 0$ for $j = 1, \ldots, m$. Indeed, the Taylor expansion of f at x_0 gives us

$$f(x) = \sum_{k=1}^{m} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + h_m(x)(x - x_0)^m = P(x) + R(x),$$

where $h_m(x) \to 0$ as $x \to x_0$. Since Gaver-Stehfest approximations are linear, we have

$$f_n(x) = P_n(x) + R_n(x),$$

where $P_n(x)$, and $R_n(x)$ are the n-th Gaver-Stehfest approximations of $P(x)$ and $R(x)$, respectively. The function P is a polynomial, in particular it is analytic and thus Theorem 8 implies that $P_n(x_0)$ converge to $0 = P(x_0)$ exponentially fast as $n \to +\infty$. Therefore, $f_n(x_0) = o(n^{-k})$ as $n \to +\infty$ if and only if $R_n(x_0) = o(n^{-k})$.

Next, we argue that Theorem 9 will be established if we can show that for some $\sigma > 0$ and $\delta > 0$ the function $\frac{\mathrm{d}^k}{\mathrm{d}z^k}\Lambda(w(z);\sigma)$ is bounded in $D_\delta(-1) \cap D_1(0)$. Assuming this result, Theorem 10 implies that the function $\Delta^{(k)}(z)$ is continuous on $\overline{D_1(0)} \setminus \{-1\}$ and bounded in $\overline{D_1(0)}$. From (6.17) we find

$$\Delta^{(k)}(z) = \sum_{n \geq k} n(n-1)\cdots(n-k+1) f_n(x_0)(-1)^n z^{n-k}, \quad |z| < 1.$$

Thus, for any $0 < r < 1$ and $n \geq k$, we have

$$\begin{aligned} n(n-1)\cdots(n-k+1) f_n(x_0)(-1)^n &= \frac{1}{2\pi i}\int_{|z|=r} z^{-(n-k)-1}\Delta^{(k)}(z)\mathrm{d}z \\ &= r^{-(n-k)}\int_0^1 e^{-2\pi i(n-k)t}\Delta^{(k)}(re^{2\pi i t})\mathrm{d}t. \end{aligned}$$

Taking the limit as $r \uparrow 1$ (and using the Dominated Convergence Theorem) we conclude that

$$n(n-1)\cdots(n-k+1) f_n(x_0)(-1)^{n-k} = \int_0^1 e^{-2\pi i(n-k)t}\Delta^{(k)}(e^{2\pi i t})\mathrm{d}t.$$

Since $\Delta^{(k)}(e^{2\pi i t})$ is continuous and bounded on $(0, 1/2) \cup (1/2, 1)$, it follows from the Riemann-Lebesgue lemma that

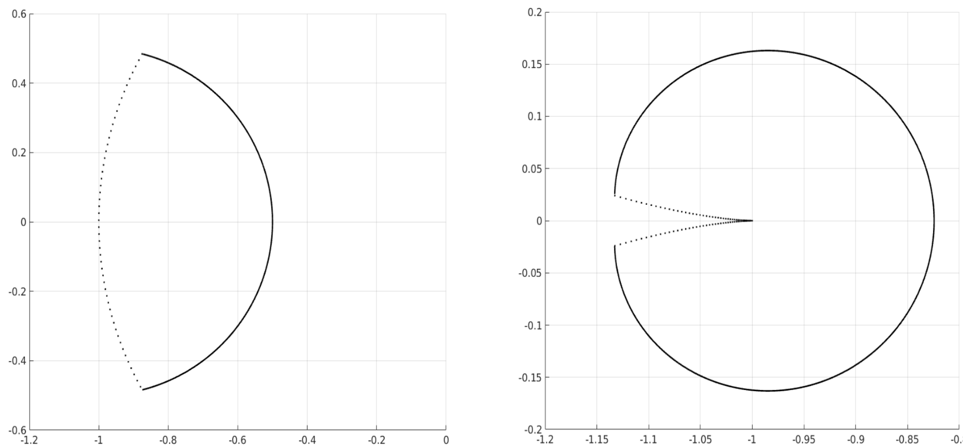
$$n(n-1)\cdots(n-k+1) f_n(x_0) \to 0, \quad n \to +\infty,$$

which is equivalent to $f_n(x_0) = o(n^{-k})$.

Next, we recall that $\phi(x)$ is defined via (6.14). Since $f^{(j)}(x_0) = 0$ for $j = 0, 1, \ldots, m$, we also have $\tilde{f}^{(j)}(x_0) = 0$ for $j = 0, 1, \ldots, m$ (see (6.13)), thus $\tilde{f}(x) = O(x^m)$ as $x \to 0$ and therefore $\phi(x) = O(x^{m/2})$ as $x \downarrow 0$.

Our problem is now reduced to the following one: *given that $m = 2k+3$, $A(u)$ is analytic in $D_1(-1)$ and $\phi(x)$ is an integrable function on $(0, 1-\epsilon)$ (for any $\epsilon > 0$) that satisfies $\phi(x) = O(x^{m/2})$ as $x \downarrow 0$, prove that there exist $\sigma \in (0,1)$ and $\delta > 0$ such that the function*

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\Lambda(w(z);\sigma) = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\int_0^\sigma (1 + w(z)(1-x))^{-\frac{3}{2}} A(w(z)(1-x))\phi(x)\mathrm{d}x, \qquad (6.37)$$



(a) (b)

Figure 6.2: (a) Domain $D_{\frac{1}{2}}(-1) \cap D_1(0)$ and (b) its image $\Omega_{\frac{1}{2}}$ under the map $z \mapsto w = ze^{z+1}$

is bounded in $D_\delta(-1) \cap D_1(0)$.

For $\delta \in (0,1)$ we define

$$\Omega_\delta := \{w \in \mathbb{C} \ : \ w = ze^{z+1}, \ z \in D_\delta(-1) \cap D_1(0)\}.$$

On Figure 6.2 we plot the domains $D_\delta(-1) \cap D_1(0)$ and Ω_δ for $\delta = 1/2$.

Before we can proceed with the proof of Theorem 9, we need to establish three auxiliary results.

Lemma 7.

(i) As $z \to -1$ *we have* $w(z) = -1 + \frac{1}{2}(z+1)^2 + \frac{1}{3}(z+1)^3 + O((z+1)^4)$.

(ii) Let $w \in \Omega_\delta$ *and* $1 + w = a + \mathrm{i}b$ *for real* a *and* b*. If* $a < 0$ *then* $b^2 > C|a|^3$ *for some positive constant* $C = C(\delta)$.

(iii) Let $w \in \Omega_\delta$ *and* $(1+w)/(-w) = a + \mathrm{i}b$ *for real* a *and* b*. If* $a < 0$ *then* $b^2 > C|a|^3$ *for some positive constant* $C = C(\delta)$.

Proof. Part (i) follows by Taylor expansion of $w(z) = ze^{z+1}$. To prove part (ii), we parametrize the circle $|z| = 1$ as $z(u) = -\cos(u) - \mathrm{i}\sin(u)$, so that $u = 0$ corresponds to $z = -1$. Writing Taylor series near $u = 0$ we see that

$$z(u) = -1 + \frac{u^2}{2} - \mathrm{i}u + O(u^3),$$

and using the result in item (i) we compute

$$1+w(z(u))=\frac{1}{2}\Big(\frac{u^2}{2}-\mathrm{i}u\Big)^2+\frac{1}{3}\Big(\frac{u^2}{2}-\mathrm{i}u\Big)^3+O(u^4)=-\frac{1}{2}u^2-\mathrm{i}\frac{1}{6}u^3+O(u^4).$$

Thus we see that the boundary of the domain Ω_δ near $w=-1$ (that is represented by the dotted line on figure 6.2b) is paramaterized by the curve $\gamma(u)=-1-\frac{1}{2}u^2-\mathrm{i}\frac{1}{6}u^3+O(u^4)$ near $u=0$. Equivalently, if $\gamma(u)=-1+a+\mathrm{i}b$, then we have parametrization $b^2=\frac{2}{9}|a|^3+o(|a|^3)$ near the point $w=-1$. Thus if we take $C>0$ small enough, then the entire curve $\{z=-1+x+\mathrm{i}y\ :\ x<0,\ y\in\mathbb{R},\ y^2=C|x|^3\}$ will lie outside of the domain Ω_δ. This ends the proof of item (ii).

Item (iii) follows from (i) and (ii). □

Next, we define

$$Q_{\alpha,\beta}(w):=\int_0^\sigma |1+w(1-x)|^{-\alpha}x^\beta \mathrm{d}x,$$

where $\alpha>0$, $\beta>0$, $\sigma\in(0,1)$ and $w\in\mathbb{C}\setminus(-\infty,-1]$.

Lemma 8. *Assume that $\beta>0$ and $0\le\gamma<\alpha$. The function $w\mapsto|1+w|^\gamma Q_{\alpha,\beta}(w)$ is bounded in Ω_δ if $\beta+1\ge\max(\alpha,3\alpha/2-\gamma)$.*

Proof. First we need to bound from below the value of $|1+w(1-x)|$, for $x\in(0,\sigma)$ and $w\in\Omega_\delta$. For s and t ranging over some subsets of $(0,\infty)$ we will write $s\approx t$ if for some positive constants C_1 and C_2 we have $C_1t<s<C_2t$ for all s and t. Thus, for $w\in\Omega_\delta$ we have $|w|\approx 1$ and

$$|1+w(1-x)|=|w|\times|(1+w)/(-w)+x|\approx|(1+w)/(-w)+x|.$$

Let $(1+w)/(-w)=a+\mathrm{i}b$ for real a and b. It is clear that $a=O(1)$ and $b=O(1)$ when $w\in\Omega_\delta$. If $a>-x/2$ then $x+a>x/2$ and we have an inequality

$$|(1+w)/(-w)+x|^2=|(x+a)+\mathrm{i}b|^2=(x+a)^2+b^2>x^2/4+b^2.$$

If $a\le -x/2$ (so that $a<0$ and $x\le 2|a|$) we have

$$|(1+w)/(-w)+x|^2=|(x+a)+\mathrm{i}b|^2=(x+a)^2+b^2\ge b^2.$$

Thus, there exists a constant $C > 0$ such that

$$Q_{\alpha,\beta}(w) = \int_0^\sigma |1 + w(1-x)|^{-\alpha} x^\beta \mathrm{d}x < C \int_0^\sigma |(1+w)/(-w) + x|^{-\alpha} x^\beta \mathrm{d}x$$
$$< C \times \left[\mathbf{1}_{\{a<0\}} \int_0^{2|a|} |b|^{-\alpha} x^\beta \mathrm{d}x + \int_0^\sigma (x^2/4 + b^2)^{-\alpha/2} x^\beta \mathrm{d}x \right].$$

We have

$$\int_0^{2|a|} |b|^{-\alpha} x^\beta \mathrm{d}x = O(|a|^{\beta+1}|b|^{-\alpha}).$$

Performing change of variables $x = 2|b|y$ we compute

$$I := \int_0^\sigma (x^2/4 + b^2)^{-\alpha/2} x^\beta \mathrm{d}x = 2^{\beta+1}|b|^{\beta+1-\alpha} \int_0^{\sigma/|b|} (1+y^2)^{-\alpha/2} y^\beta \mathrm{d}y.$$

If $\sigma/|b| \le 1$ the integral in the right-hand side of the above equation is $O(1)$, and since $|b| = O(1)$ and $\beta + 1 \ge \alpha$ we conclude that in this case $I = O(1)$. If $\sigma/|b| > 1$, we write

$$\int_0^{\sigma/|b|} (1+y^2)^{-\alpha/2} y^\beta \mathrm{d}y = \int_0^1 (1+y^2)^{-\alpha/2} y^\beta \mathrm{d}y + \int_1^{\sigma/|b|} (1+y^2)^{-\alpha/2} y^\beta \mathrm{d}y.$$

The first integral is a constant (depending only on α and β). In the second integral, the integrand can be bounded from above and below by a constant multiple of $y^{\beta-\alpha}$. Thus, when $\sigma/|b| > 1$, the second integral can be estimated as

$$\int_1^{\sigma/|b|} (1+y^2)^{-\alpha/2} y^\beta \mathrm{d}y \approx \int_1^{\sigma/|b|} y^{\beta-\alpha} \mathrm{d}y = O(1) + O(|b|^{\alpha-\beta-1}).$$

Combining these results we obtain an estimate (in the case $\sigma/|b| > 1$)

$$I = \int_0^\sigma (x^2/4 + b^2)^{-\alpha/2} x^\beta \mathrm{d}x = |b|^{\beta+1-\alpha} \times (O(1) + O(|b|^{\alpha-\beta-1})) = O(|b|^{\beta+1-\alpha}) + O(1) = O(1),$$

where in the last step we again used the fact that $|b| = O(1)$ and $\beta + 1 \ge \alpha$.

It is clear that $|1+w|^\gamma = O(1)$ in Ω_δ. Thus, combining the above estimates, we conclude

$$|1+w|^\gamma Q_{\alpha,\beta}(w) = O(1) + \mathbf{1}_{\{a<0\}} O(|1+w|^\gamma |a|^{\beta+1} |b|^{-\alpha}). \tag{6.38}$$

For $w \in \Omega_\delta$ we have

$$|1+w| = O(|(1+w)/(-w)|) = O((a^2+b^2)^{1/2}) = O(|b|(1+(a/b)^2)^{1/2}).$$

When $a < 0$ we have $|b|^{-1} = O(|a|^{-3/2})$ (see Lemma 7(iii)), thus we obtain

$$|1+w| = O(|b|(1+|a|^{-1})^{1/2}) = O(|b| \times |a|^{-1/2}). \tag{6.39}$$

From (6.38) and (6.39) (and using $|b|^{-1} = O(|a|^{-3/2})$) we estimate for $a < 0$

$$|1+w|^{\gamma}|a|^{\beta+1}|b|^{-\alpha} = |a|^{\beta+1-\gamma/2}|b|^{-\alpha+\gamma} = |a|^{\beta+1-\gamma/2}|a|^{-3/2(\alpha-\gamma)} = |a|^{\beta+\gamma+1-3\alpha/2}$$

and this latter quantity is bounded since $|a| = O(1)$ and $\beta + 1 \geq 3\alpha/2 - \gamma$. $\square$

We leave to the reader the proof of the next result: it can be done by induction or using Faa di Bruno's formula.

Lemma 9. *For every $k \in \mathbb{N}$ there exist polynomials $\{P_{k,j}(x_1, \ldots, x_k)\}_{1\leq j\leq k}$ such that for any smooth functions g and h*

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} g(h(z)) = \sum_{j=1}^{k} (h'(z))^{\max(2j-k,0)} \times g^{(j)}(h(z)) \times P_{k,j}(h'(z), \ldots, h^{(k)}(z)). \tag{6.40}$$

Now we are ready to complete the proof of Theorem 9. We recall that all that is left to do is to establish the fact stated (in italic font) on page 87. To simplify notation, we define $\psi(w) := (1+w)^{-3/2}A(w)$. With this notation we have

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\Lambda(w(z);\sigma) = \int_0^{\sigma} \frac{\mathrm{d}^k}{\mathrm{d}z^k}\psi(w(z)(1-x))\phi(x)\mathrm{d}x.$$

Invoking Lemma 9, we have

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}z^k}\Lambda(w(z);\sigma) &= \sum_{j=1}^{k} (w'(z))^{\max(2j-k,0)} \\ &\times \int_0^{\sigma} \psi^{(j)}(w(z)(1-x))(1-x)^{\max(2j-k,0)} P_{k,j}(w'(z)(1-x), \ldots, w^{(k)}(z)(1-x))\phi(x)\mathrm{d}x. \end{aligned} \tag{6.41}$$

The function $A(u)$ is analytic in $D_1(-1)$. We choose $\sigma > 0$ and $\delta > 0$ small enough so that $w(1-x) \in D_{\frac{1}{2}}(-1)$ for $w \in \Omega_\delta$ and $x \in (0,\sigma)$ and $|\phi(x)| < C_1 x^{m/2}$ for some $C_1 > 0$ and all $x \in (0,\sigma)$. We compute

$$\psi^{(j)}(w(1-x)) = \sum_{l=0}^{j} \binom{j}{l} \times \left[\prod_{i=0}^{l-1}(-3/2-i)\right](1+w(1-x))^{-3/2-l}A^{(j-l)}(w(1-x)).$$

The terms $A^{(j-l)}(w(1-x))$ are bounded for $w \in \Omega_\delta$ and $x \in (0,\sigma)$. Thus

$$|\psi^{(j)}(w(1-x))| = O(|1+w(1-x)|^{-3/2-j}), \quad w \in \Omega_\delta, \quad x \in (0,\sigma).$$

The functions $P_{k,j}(w'(z)(1-x),\dots,w^{(k)}(z)(1-x))$ are bounded for $z \in D_\delta(-1) \cap D_1(0)$ and $x \in (0,\sigma)$, since $P_{k,j}$ is a polynomial and w an entire function. We observe that $w'(z) = (z+1)e^{z+1} = w(1+z)/z$. This fact coupled with the result $1+w(z) = \frac{1}{2}(z+1)^2 + O((z+1)^3)$ (that was proved earlier in Lemma 7) implies that $|w'(z)| = O(|1+w(z)|^{1/2})$ in $D_\delta(-1)\cap D_1(0)$. Combining all these observations and using (6.41) we conclude that there exists $C_2 > 0$ such that for all $z \in D_\delta(-1) \cap D_1(0)$

$$\begin{aligned}\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}\Lambda(w(z);\sigma)\right| &< C_2 \sum_{j=1}^{k} |1+w(z)|^{\max(j-k/2,0)} \int_0^\sigma |1+w(z)(1-x)|^{-3/2-j}\phi(x)\mathrm{d}x \qquad (6.42)\\ &= C_1 \times C_2 \sum_{j=1}^{k} |1+w(z)|^{\max(j-k/2,0)} Q_{3/2+j,m/2}(w(z)).\end{aligned}$$

We leave it to the reader to check that if $m = 2k+3$ then for all $j = 1,2,\dots,k$

$$m/2+1 \geq 3/2+j \qquad \text{and} \qquad m/2+1+\max(j-k/2,0) \geq (3/2)\times(3/2+j).$$

According to Lemma 8, each term $|1+w|^{\max(j-k/2,0)}Q_{3/2+j,m/2}(w)$ in (6.42) is bounded when $w \in \Omega_\delta$, thus $\frac{\mathrm{d}^k}{\mathrm{d}z^k}\Lambda(w(z);\sigma)$ is bounded in $D_\delta(-1)\cap D_1(0)$.

□

Bibliography

[1] K.K. Aase. “Pooling in insurance”. In: *Encyclopaedia of Actuarial Science*. Ed. by J. L. Teugels and B. Sundt. Vol 3. Chichester, England: Wiley, 2004, pp. 1308–1311.

[2] J. Abate and W. Whitt. “A unified framework for numerically inverting Laplace transforms”. In: *INFORMS Journal on Computing* 18 (2006), pp. 408–421.

[3] Joseph Abate and Ward Whitt. “The Fourier-series method for inverting transforms of probability distributions”. In: *Queueing Systems* 10.1-2 (1992), pp. 5–87.

[4] H. Ammeter. “Grosstschaden-Verteilungen und ihre Anwendungen, Mitt. Verein. Schweiz”. In: *Versich.-Mathr* 71 (1971), pp. 35–62.

[5] H. Andersson. “An Analysis of the Development of the Fire Losses in the Northern Countries after the Second World War”. In: *ASTIN Bulletin: the Journal of the International Actuarial Association* 6 (1971), pp. 25–30.

[6] Søren Asmussen, Jens Ledet Jensen, and Leonardo Rojas-Nandayapa. “On the Laplace Transform of the Lognormal Distribution”. In: *Methodology and Computing in Applied Probability* 18.2 (June 2016), pp. 441–458. ISSN: 1573-7713. DOI: 10.1007/s11009-014-9430-7. URL: https://doi.org/10.1007/s11009-014-9430-7.

[7] Andrei Badescu et al. “Risk processes analyzed as fluid queues”. In: *Scandinavian Actuarial Journal* 2005.2 (2005), pp. 127–141.

[8] D. Bahnemann. *Distributions for actuaries*. Casualty Actuarial Society, 2015.

[9] S. G. Baker and P. Graves-Morris. *Padé Approximants*. 2nd ed. Vol. 1. Cambridge University Press, Cambridge–New York, 1996.

[10] Richard Barakat. “Sums of independent lognormally distributed random variables”. In: *J. Opt. Soc. Am.* 66.3 (Mar. 1976), pp. 211–216. DOI: 10.1364/JOSA.66.000211. URL: http://www.osapublishing.org/abstract.cfm?URI=josa-66-3-211.

[11] E. Barouch and Gordon M. Kaufman. *On sums of lognormal random variables.* Working Paper. Alfred P. Sloan School of Management, MIT., Feb. 1976.

[12] L.G. Benckert and I. Sternberg. “An Attempt to find an Expression for the Distribution of Fire Damage Amount”. In: *Transactions of the XV International Congress of Actuaries* 2 (1957), pp. 288–294.

[13] G. Benktander. “Notes sur la Distribution Conditionné du Montant d’un Sinistre par Rapport a l’Hypothese qu’il y a eu un Sinistre dans l’Assurance Automobile”. In: *ASTIN Bulletin: the Journal of the International Actuarial Association* 2 (1962), pp. 24–29.

[14] David R. Bickerstaff. “Automobile collision deductibles and repair cost groups: the log-normal model”. In: *Proceedings of the Casualty Actuarial Society* LIX (1972), p. 68.

[15] Lennart Bondesson. *Generalized gamma convolutions and related classes of distributions and densities.* Lecture No. New-York: Springer-Verlag, 1992.

[16] Lennart Bondesson. “On the Lévy Measure of the Lognormal and the LogCauchy Distributions”. In: *Methodology And Computing In Applied Probability* 4.3 (Aug. 2002), pp. 243–256. ISSN: 1573-7713. DOI: 10.1023/A:1022533817579. URL: https://doi.org/10.1023/A:1022533817579.

[17] K. Borch. “The optimal reinsurance treaty”. In: *ASTIN Bulletin: the Journal of the International Actuarial Association* 5 (1969), pp. 293–297.

[18] P.L. Brockett. “On the misuse of the Central Limit Theorem in some risk calculations”. In: *Journal of Risk and Insurance* 50.4 (1983), pp. 727–731.

[19] Chao-Ping Chen and Feng Qi. “Inequalities of some trigonometric functions”. In: *Publikacije Elektrotehnickog fakulteta. Serija Matematika* 15 (2004), pp. 72–79.

[20] Davis K. Cope. “Convergence of Piessens’ Method for Numerical Inversion of the Laplace Transform on the Real Line”. In: *SIAM Journal on Numerical Analysis* 27.5 (1990), pp. 1345–1354. DOI: 10.1137/0727077. eprint: https://doi.org/10.1137/0727077. URL: https://doi.org/10.1137/0727077.

[21] R. M. Corless et al. “On the Lambert W function”. In: *Advances in Computational Mathematics*. 1996, pp. 329–359.

[22] H. Cossette et al. “Moment-based approximation with mixed Erlang distributions”. In: *Variance* 10.1 (2016), pp. 161–182.

[23] Brian Davies and Brian Martin. “Numerical inversion of the Laplace transform: a survey and comparison of methods”. In: *Journal of Computational Physics* 33.1 (1979), pp. 1–32.

[24] J. Dhaene and D. Vincke. “Individual risk models”. In: *Encyclopaedia of Actuarial Science*. Ed. by J. L. Teugels and B. Sundt. Vol 2. Chichester, England: Wiley, 2004, pp. 871–875.

[25] Leset B. Dropkin. “Size of loss distributions in workmens compensation insurance”. In: *Proceedings of the Casualty Actuarial Society* LI (1964), p. 198.

[26] D. Dufresne. “Fitting Combinations of Exponential to Probability Distributions,””. In: *Applied Stochastic Models in Business and Industry* 23 (2007), pp. 23–48.

[27] Daniel Dufresne. “Sums of Log-normals”. In: *Actuarial Research Conference Proceedings*. 2008.

[28] EIOPA-14-322. *The underlying assumptions in the standard formula for the Solvency Capital Requirement calculation*. Tech. rep. European Insurance and Occupational Pensions Authority, 2014.

[29] L. N. G. Filon. “On a Quadrature Formula for Trigonometric Integrals”. In: *Proceedings of the Royal Society of Edinburgh* 49 (1928), pp. 38–47. DOI: 10.1017/S0370164600026262.

[30] E. Furman, D. Hackmann, and A. Kuznetsov. “On log-normal convolutions: An analytical-numerical method with applications to economic capital determination”. In: *Insurance: Mathematics and Economics* (2018). under review.

[31] D. P. Gaver, Jr. “Observing Stochastic Processes, and Approximate Transform Inversion”. In: *Operations Research* 14.3 (1966), pp. 444–459.

[32] M. Goovaerts. “Collective risk models”. In: *Encyclopaedia of Actuarial Science*. Ed. by J. L. Teugels and B. Sundt. Chichester, England: Wiley, 2004.

[33] John A. Gubner. “A New Formula for Lognormal Characteristic Functions”. In: *Vehicular Technology, IEEE Transactions on* 55.5 (Oct. 2006), pp. 1668–1671.

[34] M. R. Hardy. “Approximating the aggregate claims distributions”. In: *Encyclopaedia of Actuarial Science*. Ed. by J. L. Teugels and B. Sundt. Vol 1. Chichester, England: Wiley, 2004, pp. 78–86.

[35] R.V. Hogg and S.A. Klugman. “ON THE ESTIMATION OF LONG TAILED SKEWED DISTRIBUTIONS WITH ACTUARIAL APPLICATIONS”. In: *Journal of Econometrics* 23 (1983), pp. 91–102.

[36] Philip Holgate. “The log-normal characteristic function.” In: *Communications in Statistics - Theory and Methods* 18.12 (1989), pp. 4539–4548.

[37] R. G. Jacquot, J. W. Steadman, and C. N. Rhodine. “The Gaver-Stehfest algorithm for approximate inversion of Laplace transforms”. In: *Circuits Systems Magazine* 5.1 (1983), pp. 4–8.

[38] Christian Kleiber and Samuel Kotz. *Statistical Size Distributions in Economics and Actuarial Sciences*. John Wiley & Sons, Inc., Hoboken, New Jersey. Canada. 2003, p. 355. ISBN: 9780471457169. DOI: 10.1002/0471457175. arXiv: arXiv:1011.1669v3. URL: http://books.google.ru/books?id=7wLGjyB128IC.

[39] Stuatrt A. Klugman, Harry H. Panjer, and Gordon E. Willmot. *Loss models: from data to decisions, 3rd ed.* Society of Actuaries. 2012. ISBN: 9781118315323.

[40] S. G. Kou and Hui Wang. “First Passage Times of a Jump Diffusion Process”. In: *Advances in Applied Probability* 35.2 (2003), pp. 504–531.

[41] A. Kuznetsov. “On the convergence of the Gaver-Stehfest algorithm”. In: *SIAM Journal on Numerical Analysis* 6.51 (2013), pp. 2984–2998.

[42] Alexey Kuznetsov, Andreas E. Kyprianou, and Victor Rivero. “The Theory of Scale Functions for Spectrally Negative Lévy Processes”. eng. In: *Lévy Matters II*. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 97–186. ISBN: 3642314066.

[43] Alexey Kuznetsov and Justin Miles. “On the rate of convergence of the Gaver–Stehfest algorithm”. In: *IMA Journal of Numerical Analysis* (Apr. 2021). drab015. ISSN: 0272-4979. DOI: 10.1093/imanum/drab015. eprint: https://academic.oup.com/imajna/advance-article-pdf/doi/10.1093/imanum/drab015/37248831/drab015.pdf. URL: https://doi.org/10.1093/imanum/drab015.

[44] Patrick J. Laub et al. “Approximating the Laplace transform of the sum of dependent lognormals”. In: *Advances in Applied Probability* 48.A (2016), pp. 203–215. DOI: 10.1017/apr.2016.50.

[45] Roy B. Leipnik. “On lognormal random variables: I-the characteristic function”. In: *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics* 32.3 (1991), pp. 327–347. DOI: 10.1017/S0334270000006901.

[46] Viktoriya Masol and Jef L. Teugels. “Numerical accuracy of real inversion formulas for the Laplace transform”. In: *Journal of Computational and Applied Mathematics* 233.10 (2010), pp. 2521–2533.

[47] Thomas Mikosch. *Non-Life Insurance Mathematics: An Introduction with Poisson Process*. Vol. 32. 1. SIAM Review. 2009, p. 184. ISBN: 978-3-540-88232-9. DOI: 10.1007/978-3-540-88233-6. arXiv: arXiv:1011.1669v3. URL: http://dx.doi.org/10.1007/978-3-540-88233-6.

[48] Justin Miles. “On the Laplace transform of the lognormal distribution: Analytic continuation and series approximations”. In: *Journal of Computational and Applied Mathematics* 404 (2022), p. 113904. ISSN: 0377-0427. DOI: https://doi.org/10.1016/j.cam.2021.113904. URL: https://www.sciencedirect.com/science/article/pii/S0377042721005276.

[49] Justin Miles, Edward Furman, and Alexey Kuznetsov. “Risk aggregation: A general approach via the class of Generalized Gamma Convolutions; In press in the Variance”. In: Variance (Dec. 2018).

[50] Brian O’Neill and W. T. Wells. “Some Recent Results in Lognormal Parameter Estimation Using Grouped and Ungrouped Data”. In: *Journal of the American Statistical Association* 67.337 (1972), p. 76. ISSN: 01621459. DOI: 10.2307/2284705. URL: http://www.jstor.org/stable/2284705?origin=crossref.

[51] C.M. Ramsay. “A note of the normal power approximation”. In: *ASTIN Bulletin: the Journal of the International Actuarial Association* 21 (1991), pp. 147–150.

[52] Wim Schoutens and Geert Van Damme. “The β-variance gamma model”. In: *Review of Derivatives Research* 14.3 (2011), pp. 263–282. ISSN: 1380-6645.

[53] H.L. Seal. “Survival probabilities based on pareto claim distributions”. In: *ASTIN Bulletin: the Journal of the International Actuarial Association* 11 (1980), pp. 61–71.

[54] H. Stehfest. “Algorithm 368: Numerical inversion of Laplace transforms”. In: *Commun. ACM* 13.1 (1970), pp. 47–49. ISSN: 0001-0782.

[55] H. Stehfest. “Remark on algorithm 368: Numerical inversion of Laplace transforms”. In: *Commun. ACM* 13.10 (1970), p. 624.

[56] H. Takahasi and M. Mori. “Double exponential formulas for numerical integration”. In: *Publ. RIMS, Kyoto Univ.* 9 (1974), pp. 721–741.

[57] J. Tao, S.B. Provost, and J. Ren. “Moment-based density approximations for aggregate losses”. In: *Scandinavian Actuarial Journal* 2016.3 (2016), pp. 216–245.

[58] C. Tellambura and D. Senaratne. “Accurate Computation of the MGF of the Lognormal Distribution and Its Application to Sum of Lognormals”. In: *Trans. Comm.* 58.5 (May 2010), pp. 1568–1577. ISSN: 0090-6778. DOI: 10.1109/TCOMM.2010.05.080640. URL: http://dx.doi.org/10.1109/TCOMM.2010.05.080640.

[59] Olof Thorin. “An extension of the notion of a generalized -convolution”. In: *Scandinavian Actuarial Journal* 1978.3 (1978), pp. 141–149. DOI: 10.1080/03461238.1978.10432021. eprint: https://doi.org/10.1080/03461238.1978.10432021. URL: https://doi.org/10.1080/03461238.1978.10432021.

[60] Olof Thorin. “On the infinite divisbility of the Pareto distribution”. In: *Scandinavian Actuarial Journal* 1977 (1977), pp. 31–40. URL: https://api.semanticscholar.org/CorpusID:119591466.

[61] Olof Thorin. “On the infinite divisibility of the lognormal distribution”. In: *Scandinavian Actuarial Journal* 1977.3 (1977), pp. 121–148. ISSN: 0346-1238. DOI: 10.1080/03461238.1977.10405635. URL: http://dx.doi.org/10.1080/03461238.1977.10405635.

[62] Olof Thorin. “Proof of a conjecture of L. Bondesson concerning infinite divisibility of powers of a gamma variable”. In: *Scandinavian Actuarial Journal* 1978.3 (1978), pp. 151–164. DOI: 10.1080/03461238.1978.10432022. eprint: https://doi.org/10.1080/03461238.1978.10432022. URL: https://doi.org/10.1080/03461238.1978.10432022.

[63] E. C. Titchmarsh. *Introduction to the Theory of Fourier Integrals*. 2nd. Oxford University Press, 1948.

[64] P.P. Valkó and J. Abate. “Comparison of sequence accelerators for the Gaver method of numerical Laplace transform inversion”. In: *Computers & Mathematics with Applications* 48.3–4 (2004), pp. 629–636.

[65] D. V. (David Vernon) Widder. *The Laplace transform.* eng. Princeton mathematical series ; 6. Princeton, N.J: Princeton University Press.