Module: Probability Basics

Module Outline

- Introduction
- Basic Axioms of probability
- Rules of probability
- Independence and dependence of events

Probability: Introduction

- Uncertainty is ubiquitous: *weather, economy, health, jobs, sports, stocks,........*

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- Lots of related decisions: *insurance, vaccinations, stocks vs bond, which stocks to buy,......*
- How to capture uncertainty/chance more formally?

Example 1: 6-sided Die

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- What is the probability that an even number comes up?
- What is the probability that the number exceeds 4?

Example 2: Office

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- An office has 2 *secretaries*, 4 *interns*, 20 *data analysts*, 2 *accountants and* 2 *managers.*

  One of them is randomly picked to go to a convention.

- What is the probability that one of the interns gets picked? $\frac{4}{30}$

- A family is expecting a child? Boy or girl?

Probability: Terminology

- *Sample space* S : Set of all possible outcomes

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- *Event*: a collection (subset) of outcomes
- Venn diagrams and sets

Example 3: Students and car-ownership

- 100 people were surveyed: 60 own a car

  30 were students, 15 of them own a car

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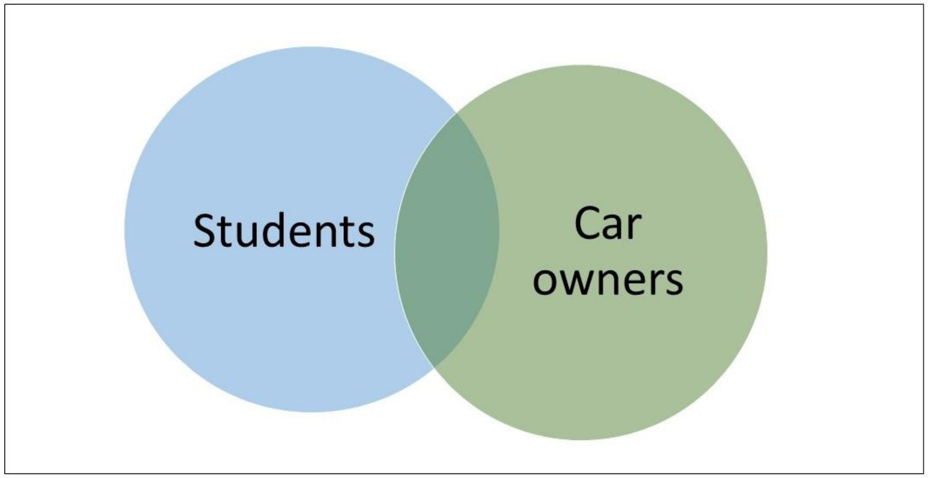

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- $St.$ = set of students $\qquad$ C = set of car owners

- $St \cap C$ = students AND car-owners

  $St\ U\ C$ = students OR car-owners

- In the survey, what is the prob. of picking someone who is a car-owner, BUT not a student?

Example 3 extended: 100 surveyed

- 60 own cars; of them, 20 own 1+ cars

  30 students, 15 of them own cars, 5 own 1+ cars

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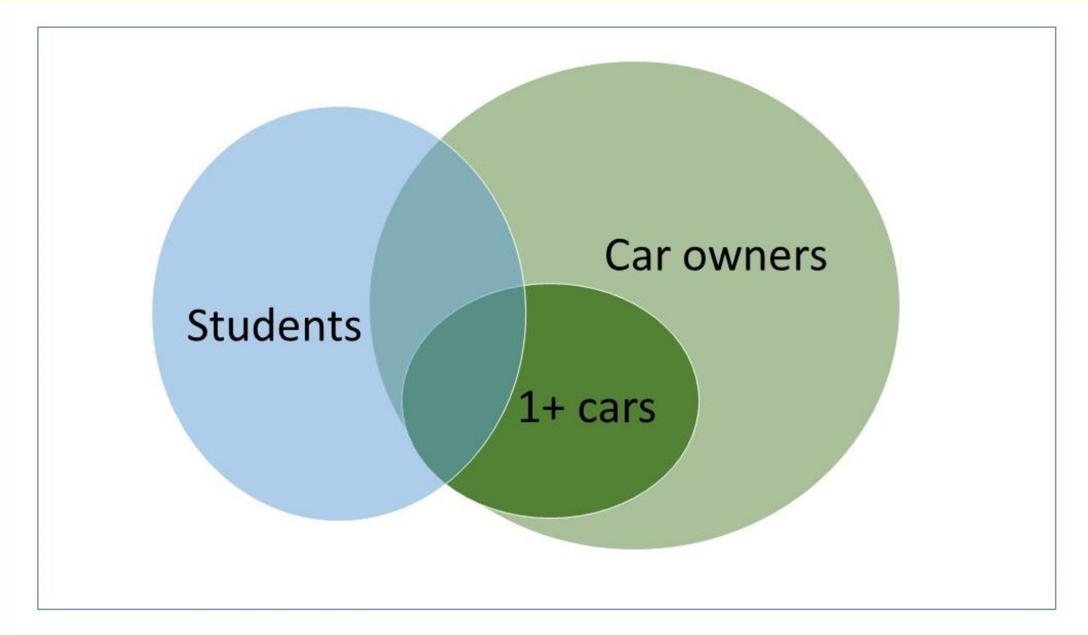

Basic Axioms of Probability

- **[i]** $0 \leq P(A) \leq 1$

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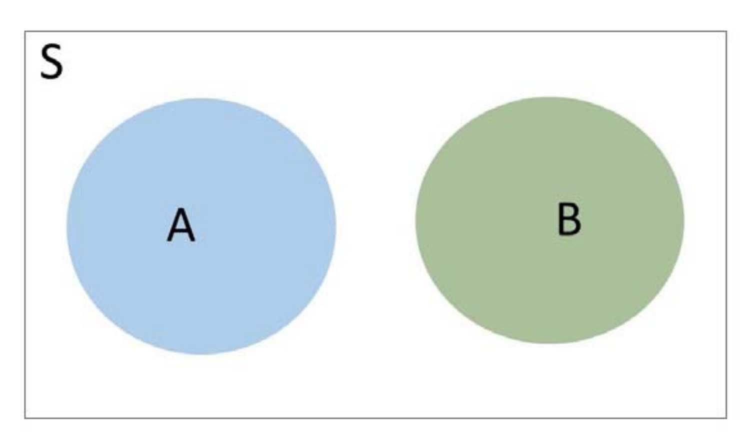
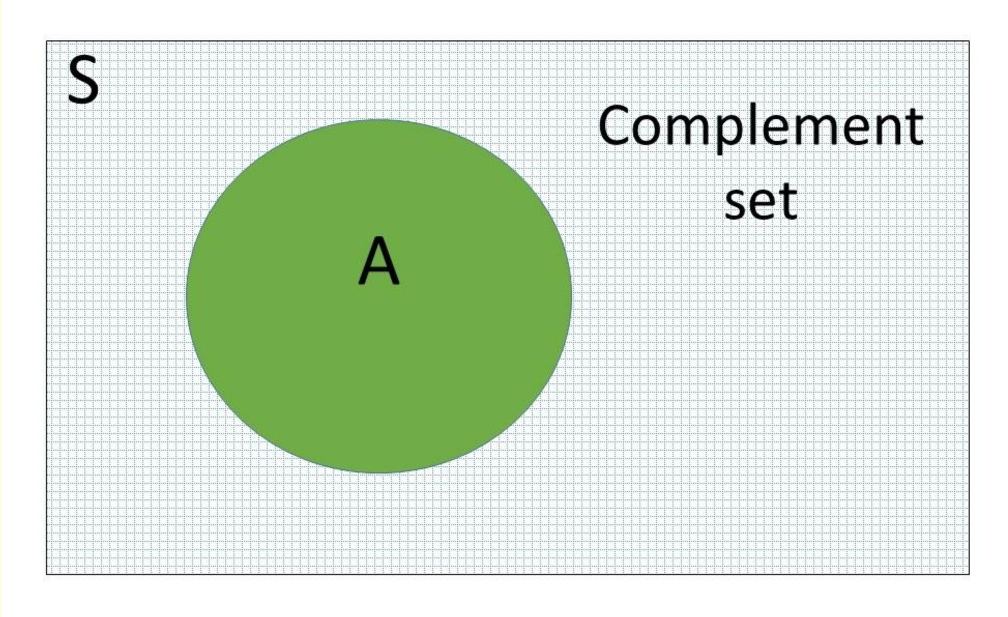
- **[i]** $0 \leq P(A) \leq 1$
- **[ii]** $P(\mathit{Sample\ Space}) = 1$
- **[iii]** If A and B are disjoint (*mutually exclusive*),
  $P(A \cup B) = P(A) + P(B)$



Probability rules

- **[Complement rule]** $P(A^c) = 1 - P(A)$



Example 4: Lottery

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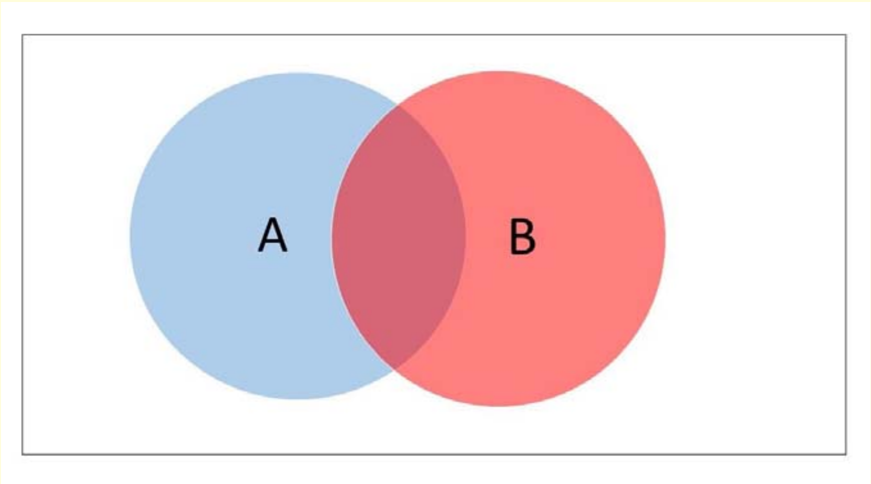
Probability rules: Addition rule

- **[Addition rule]** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

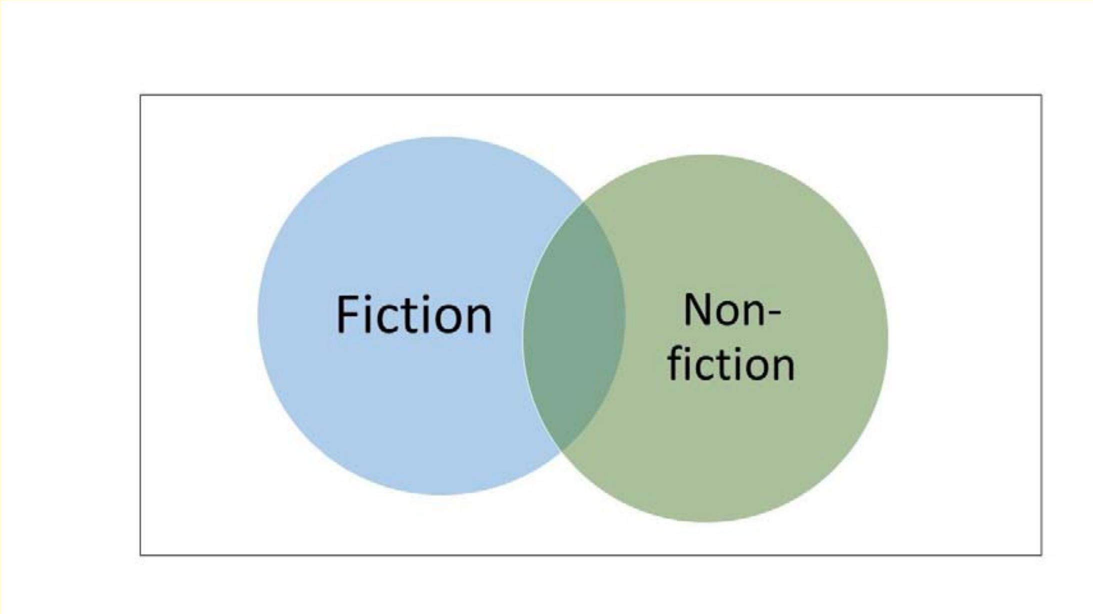


Example 5: Bookstore

- 200 customers visit a bookstore

  75 bought a non-fiction book, 100 bought a fiction

  25 bought both



Example 5: Bookstore

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- What is the prob. of a customer buying a book?

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Example 6: Winterfest in Ottawa

- Organizing a festival in Ottawa on Feb. 5th.

  <u>*Past* 10 *yrs*</u>: $\quad \texttt{snowed:}\ \frac{2}{10}, \quad \texttt{very cold}: \frac{5}{10},$

  $\texttt{snow AND very cold}: \frac{1}{10}$

Example 6: Winterfest in Ottawa

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- Prob. of good weather?

Clicker question 1

- Toronto, Dec. 25th, *past* 10 *yrs*:

  `snowed:` $\frac{2}{10}$, `cold:` $\frac{4}{10}$, `snow AND cold:` $\frac{1}{10}$

  Q: What is the prob. of there being neither snow NOR being cold on Dec. 25th?

  (a) $\frac{2}{10}$ **(b)** $\frac{4}{10}$

  (c) $\frac{5}{10}$ **(d)** $\frac{6}{10}$

Solution to Clicker question 1

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Addition rule: Application

- *Example 7*: Candn. immigrants in 2017:

  Total: $300,000$; *Economic class:* $160,000$

  Ontario: $110,000$; *Economic class in ON:* $50,000$

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  Total: $300,000$; *Economic class:* $160,000$

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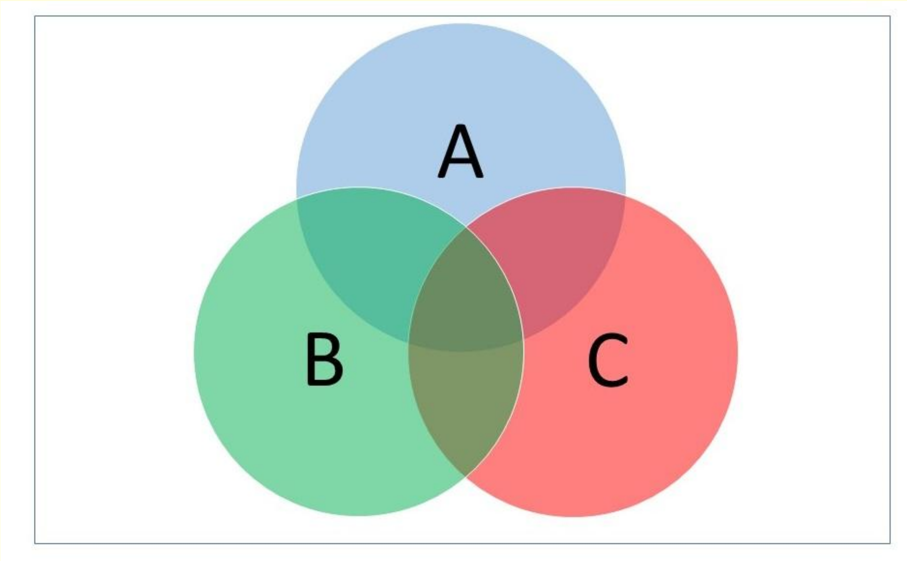
- What is the prob. that a randomly selected immigrant is

  (a) *of economic class,* **(b)** *of economic class or in ON?*

Probability rules

- **[Addition rule extension]**

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\ &\quad - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \end{aligned}$$



Addition Rule Extension: Application

- *Example 8*: 100 students in a class did projects $A/B/C$

A	B	C	A and B	B and C	A and C	all 3
30	40	20	10	12	15	5

What is the prob. that a random student did at least one of the assignments?

Addition Rule Extension: Application

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Clicker question 2

- In the above example, what is the prob. that a random student did not submit any of the projects?

  (a) 0.10 **(b)** 0.42

  (c) 0.55 **(d)** 0.62

Solution to Clicker question 2

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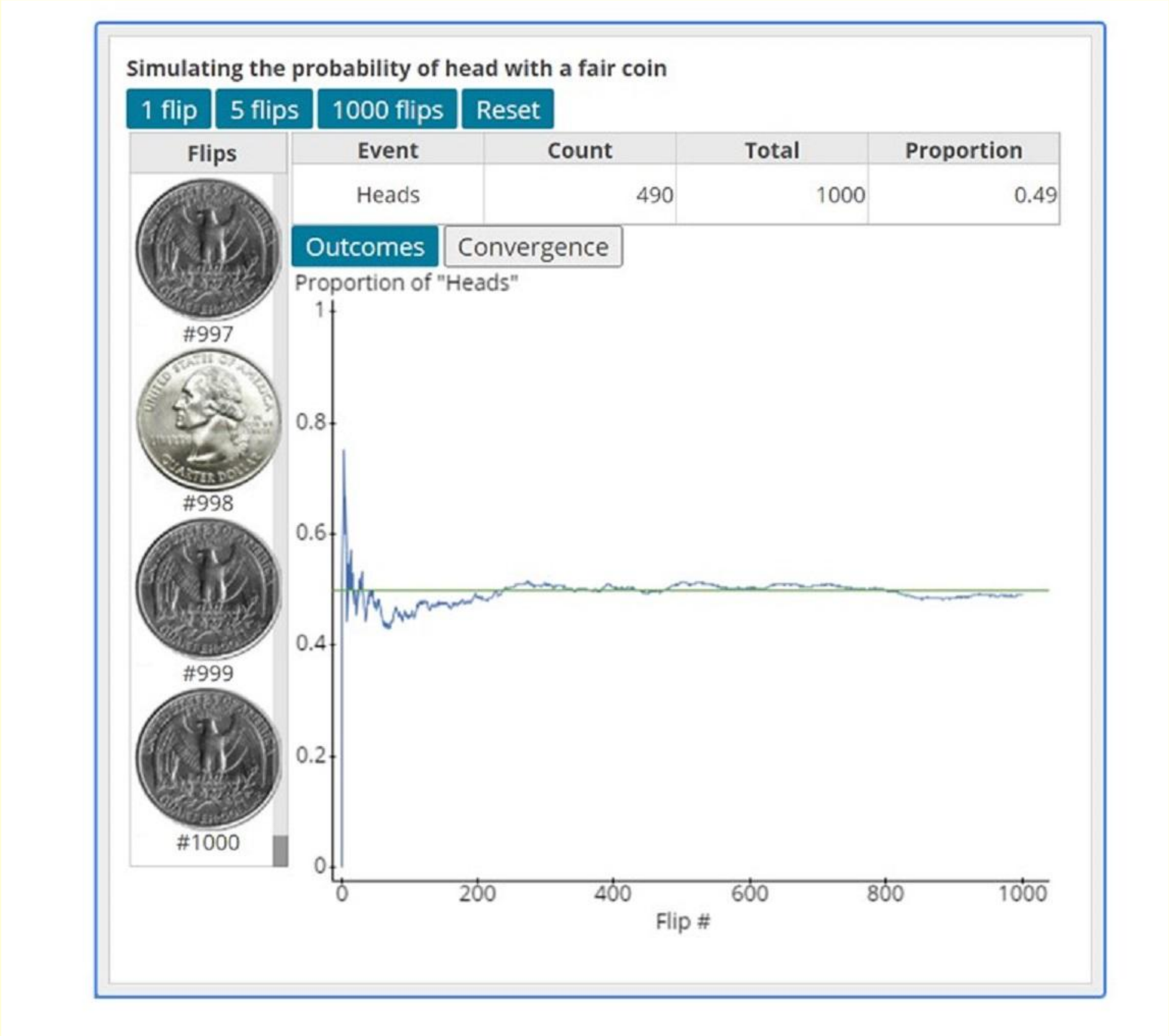
Interpretation of probability

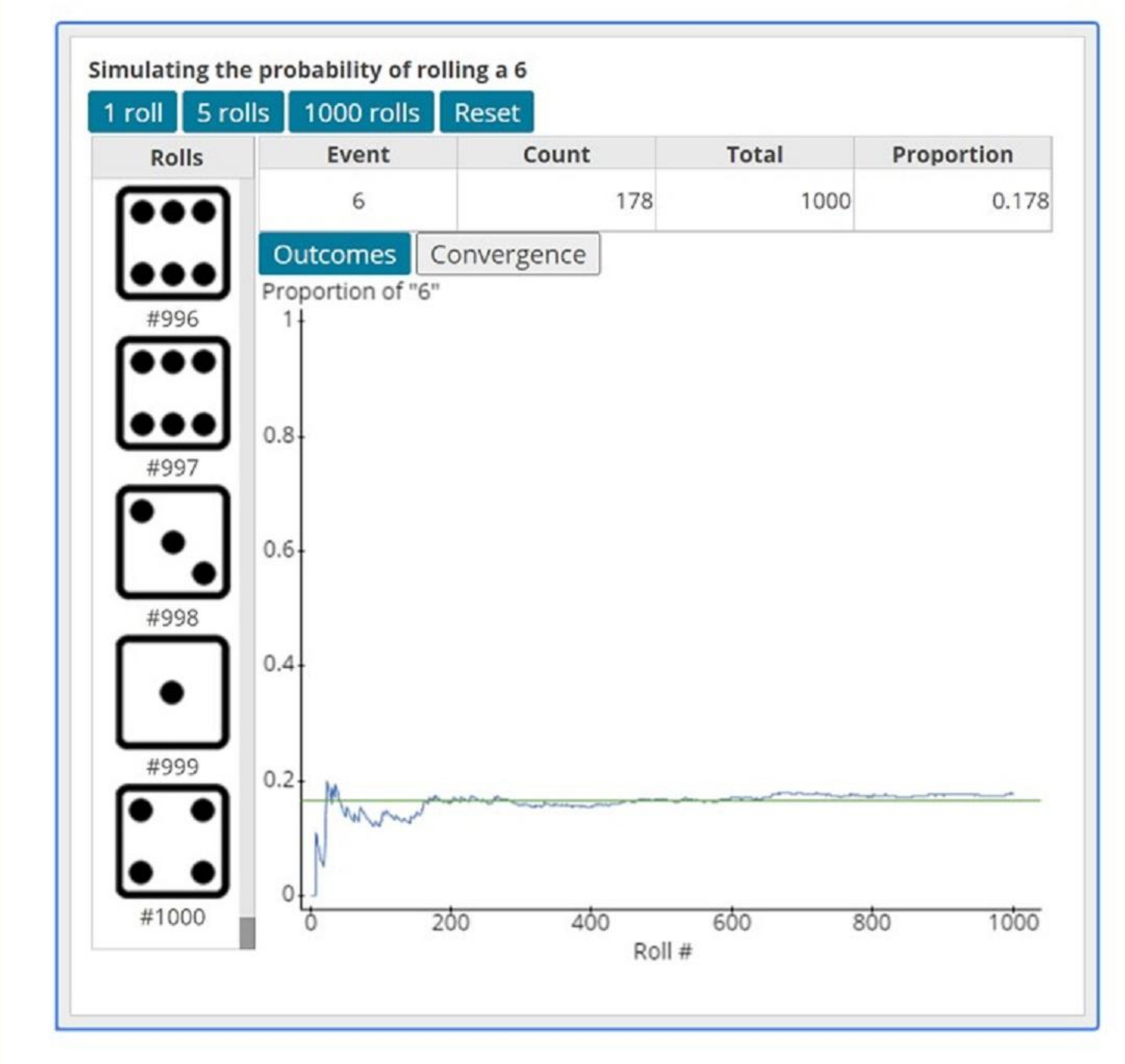
- *Frequentist*:

  If you repeat the experiment many, many times, the fraction will go to probability

Frequentist: Tossing a coin 1000 times (statcrunch.com)



Frequentist: Rolling a die 1000 times (statcrunch.com)



Interpretation of probability

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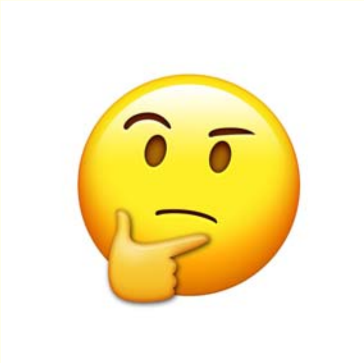
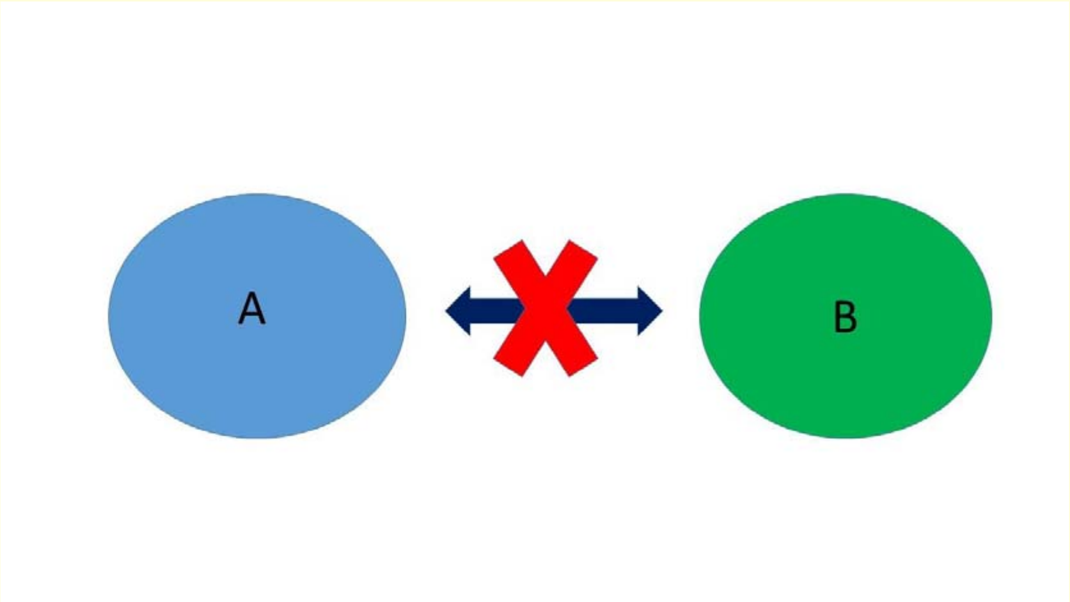
- *Personal prob.*:

  Beliefs around rare events (e.g. will I do well in the course?, will Trump get re-elected?.....)



Relationship between events

- *Independence*: Two events A and B are independent if the occurrence of one **does not influence** the prob. of the other occurring.



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- *Example*: draw a card from a deck of 52 cards

  1^{st} one is *King* of *Hearts*: **Replace** it back or **not**

  Effect on 2^{nd} card drawn?

Independent and Dependent events

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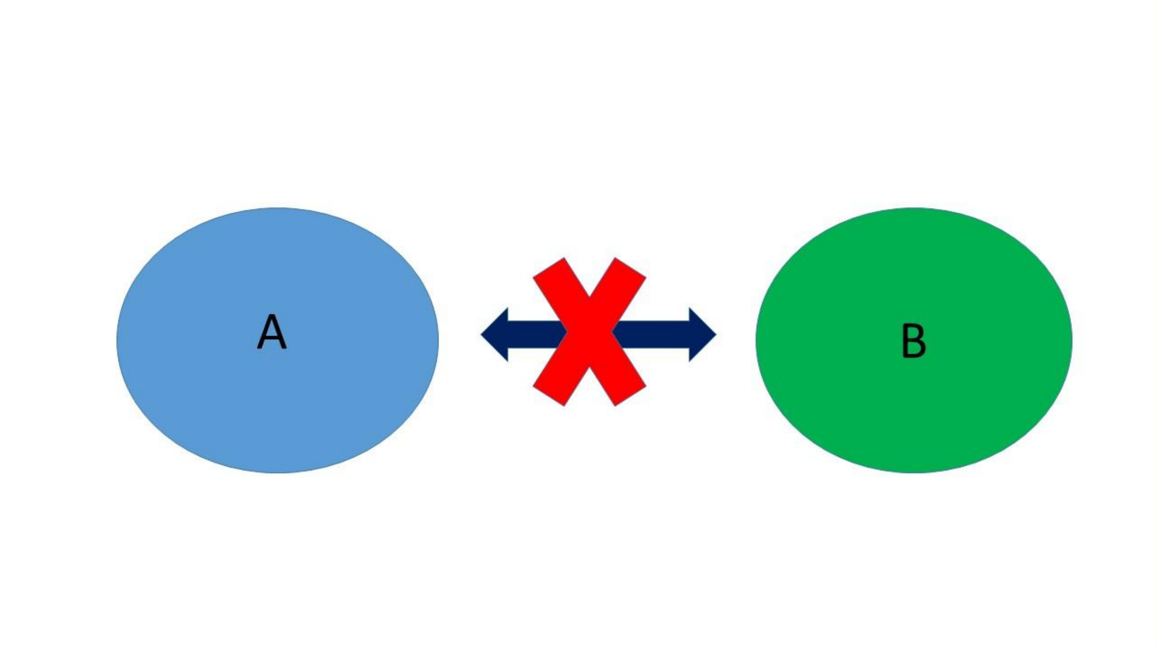

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- *Random walk theory* of stock prices

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- Without replacement: Prob. that first two are *Hearts*?

Independence Rule: Extension

- If events $A_1, A_2, ..., A_n$ are independent:

$$P(A_1 \cap A_2 \cap .... \cap A_n) = P(A_1)P(A_2)....P(A_n)$$

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- $P(\text{all boys}) = P(BBB)$
- $P(BBB) = \left(\frac{1}{2}\right) * \left(\frac{1}{2}\right) * \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

Example 10 continued

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- $1 - P(\text{all boys}) = 1 - P(BBB) =$
- $P(\text{at least one girl}) = 1 - \left(\frac{1}{8}\right) = 0.875$

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- **Q:** A family has 3 children. What is the prob. that exactly two of them are girls?

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- GGB, GBG, BGG...... Total prob. $= \frac{3}{8}$

Clicker question 3

- Which of the following events are independent?

  (a) *Parking illegally* and *getting a ticket*

  (b) *Owning a phone* and *being a hockey fan*

  (c) *A dry summer* and *forest fires*

  (d) *Buying more lottery tickets* and *winning the lottery*

Solution to Clicker question 3

- Which of the following events are independent?

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Independence: Implications

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- Random sampling

Independence: Application

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- (b) exactly one of them is a smoker?
- (c) at least two of them are smokers

Clicker question 4

- A family has 3 children. What is the prob. that exactly one of them is a boy?

(a) $\frac{1}{8}$ **(b)** $\frac{1}{2}$

(c) $\frac{3}{8}$ **(d)** $\frac{1}{3}$

Solution to Clicker question 4

- A family has 3 children. What is the prob. that exactly one of them is a boy?

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Example of dependent event

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- **if** 1^{st} **successful**: success-rate on 2^{nd} : 79%

  if 1^{st} **unsuccessful**: success-rate on 2^{nd} : 72%

Checking for independence

- **Example 12**: In a university sample, 70% of the students reported living off-campus. 40% reported belonging to some club, 20% reported living off-campus and belonging to a club.

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  $O \equiv$ lives off-campus

  Q: Are C and O independent?

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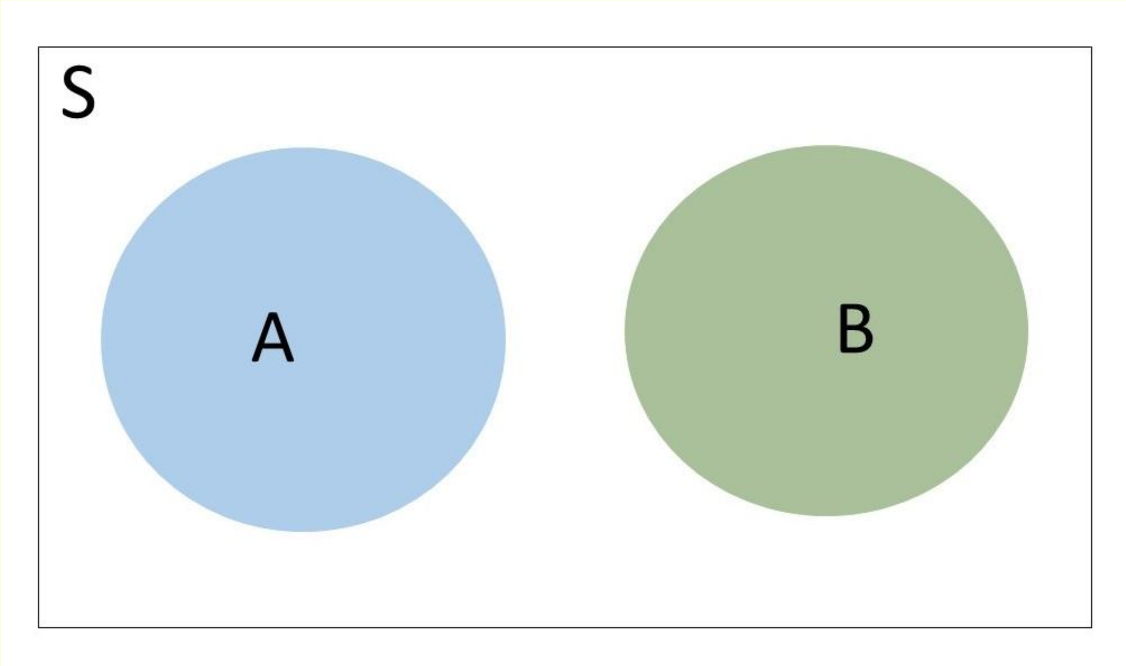
  Q: Are C and O independent?

- If C and O are independent:

$$P(C \cap O) = P(C)P(O)$$

Independence vs Mutually Exclusive

- *Mutually exclusive*: two events cannot occur at the same time



Independence vs Mutually Exclusive

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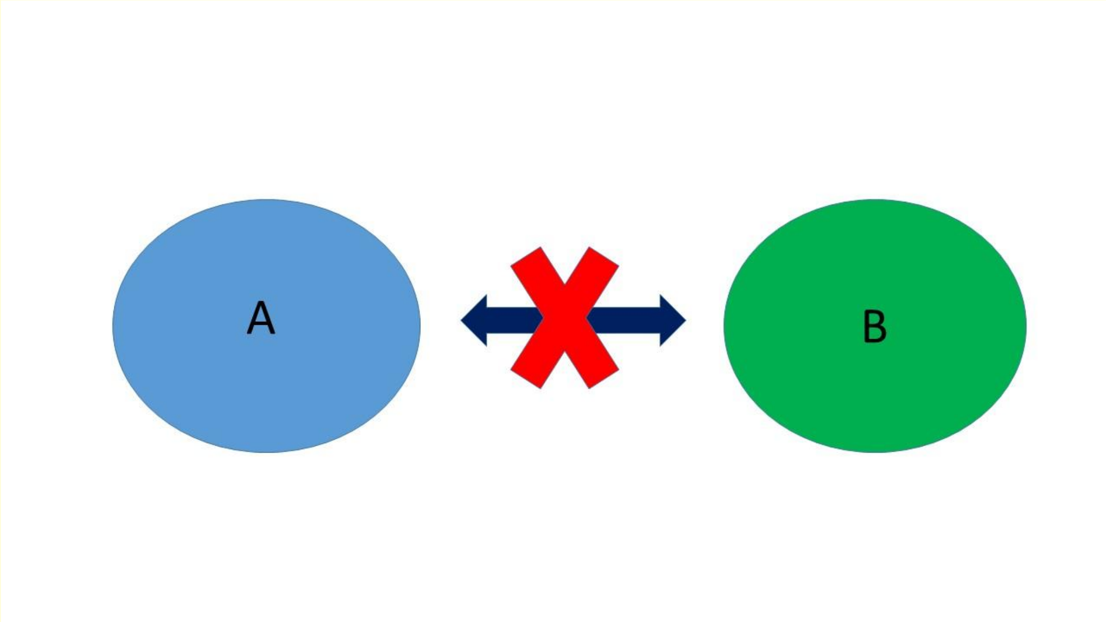
- *Independence*: one event does not influence the other



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  $F \equiv$ Jim fails the course
- P and F are disjoint, but not independent