Orthogonal factors of operators on the Rosenthal spaces and the Bourgain-Rosenthal-Schechtman spaces

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Abstract

We study factorization properties of bounded linear operators on the Rosenthal spaces $X_{p,w}$ and the Bourgain-Rosenthal-Schechtman spaces $R_{\alpha}^{p,0}$, $1 \leqslant \alpha < \omega_1$. Specifically, we prove that the Rosenthal spaces $X_{p,w}$ and the limit Bourgain-Rosenthal-Schechtman spaces $R_{\alpha}^{p,0}$, equipped with their natural bases, have the factorization property, and that the isomorphic spaces $R_{\omega}^{p,0}$ and $X_{p,w}$ have the primary factorization property.

We establish the factorization property of the spaces $R_{\omega}^{p,0}$ and $X_{p,w}$ with their respective bases separately. For the spaces $X_{p,w}$, the proof relies on the notion of a strategically reproducible basis. In contrast, for $R_{\omega}^{p,0}$, the proof is based on an approximate orthogonal reduction to a diagonal operator: every bounded linear operator on $R_{\omega}^{p,0}$ can, via distributional embedding and up to arbitrary precision, be reduced to a diagonal operator. Moreover, we show that $R_{\omega}^{p,0}$ has the primary factorization property, which immediately implies the same property for $X_{p,w}$. The proof of the aforementioned reduction relies on an approximate orthogonal reduction to a scalar multiple of the identity operator.

In addition, we establish the factorization property for the limit spaces $R_{\alpha}^{p,0}$ with their standard martingale difference sequence bases. Although this proof is derived from an approximate orthogonal reduction to a scalar finite-dimensional decomposition (FDD)-diagonal operator, the construction is significantly more involved than in the case $\alpha = \omega$.

For every $1 \leqslant \alpha < \omega_1$, we construct an explicit unconditional FDD $(X_{\lambda})_{\lambda \in \mathcal{T}_{\alpha}}$ of the Bourgain-Rosenthal-Schechtman space $R_{\alpha}^{p,0}$. This FDD has strong reproducing properties and supports a theory of distributional representations between the spaces $R_{\alpha}^{p,0}$, for $1 \leqslant \alpha < \omega_1$. We use this framework to establish the approximate orthogonal reduction to a scalar FDD-diagonal operator.

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Chapter 1

Introduction

This work is motivated by the study of conditions under which a bounded linear operator T on a Banach space X becomes a factor of the identity operator id on X; that is, there exist bounded linear operators $L, R\colon X \to X$ such that

$$id = LTR.$$

Factors of the identity have played a crucial role in Banach space theory—for instance, in the study of decompositions of classical Banach spaces and in the analysis of closed ideals in the Banach algebras

$$\mathcal{L}(X) = \{T : X \to X\colon T \text{ is a bounded linear operator on } X\}.$$

In this work, we focus on complemented subspaces of L_p for a fixed $1 < p < \infty$. Specifically, we study operators on the Rosenthal spaces $X_{p,w}$ [24] and the Bourgain-Rosenthal-Schechtman spaces R^p_α, $1 \leqslant \alpha < \omega_1$ [4].

The Rosenthal space is the first known complemented subspace of L_p, apart from L_p, ℓ_2, ℓ_p, $\ell_2 \oplus \ell_p$, and $\ell_p(\ell_2)$. It is the closed linear span in L_p of an independent 3-valued symmetric sequence $(f_n)_{n=1}^\infty$ such that, if $p > 2$,

$$\left(\frac{\|f_n\|_2}{\|f_n\|_p}\right)_{n=1}^\infty$$

vanishes at a slow rate (when $p < 2$, we consider its conjugate exponent). To study the properties of these spaces, such as their complementation, for $p > 2$, Rosenthal introduced what is now known as Rosenthal's inequality, which states that for any scalar sequence $(a_n)_{n=1}^\infty$,

$$\Big\| \sum_{n=1}^\infty a_n f_n \Big\|_p \sim_p \max\Big\{ \Big(\sum_{n=1}^\infty |a_n|^p \|f_n\|_p^p \Big)^{1/p}, \Big(\sum_{n=1}^\infty |a_n|^2 \|f_n\|_2^2 \Big)^{1/2} \Big\}.$$

This equivalence makes it possible to study the aforementioned space as a sequence space, $X_{p,w}$, whose norm admits a simple and explicit description.

The Bourgain-Rosenthal-Schechtman spaces R^p_α, $\alpha < \omega_1$, are defined by transfinite recursion.

The first space R_0^p consists of the constant functions. If R_α^p is defined, then $R_{\alpha+1}^p$ is the disjoint sum of two 1/2-compressed, isometric copies of R_α^p. More formally, for $I = [a, b) \subset [0, 1)$, define $T_I : L_p \to L_p$ by setting $(T_I f)(t) = f((t-a)/(b-a))$ for $t \in I$ and $(T_I f)(t) = 0$ otherwise. This T_I is an example of a scaled isometry of a special type, called a θ-compression, where $\theta = b - a$. Then,

$$R_{\alpha+1}^p = T_{[0,1/2)}(R_\alpha^p) + T_{[1/2,1)}(R_\alpha^p).$$

For a limit ordinal number α, the space R_α^p is the independent sum of the spaces R_β^p, $0 \leqslant \beta < \alpha$, that is, there are distributional embeddings $T_\beta^\alpha : R_\beta^p \to L_p$, $0 \leqslant \beta < \alpha$ with independent ranges such that the space R_α^p is the following closed linear span in L_p,

$$R_\alpha^p = [(T_\beta^\alpha(R_\beta^p))_{\beta<\alpha}].$$

By construction, each space R_α^p is spanned by the constant function together with a symmetric three-valued martingale difference sequence, and thus by Burkholder's martingale theorem from [5] the space admits an unconditional basis. Now, an appropriate cofinal subset of

$$\{R_\alpha^p : \alpha < \omega_1\}$$

yields uncountably many pairwise non-isomorphic complemented subspaces of L_p. Specifically, we focus on $R_\alpha^{p,0}$, more accurately, on its isomorphic hyperplanes of all mean-zero random variables. In all aforementioned cases, the complementation in L_p is witnessed via an orthogonal projection that is also well defined on L_p.

The first two results we prove concern the factorization property of the Rosenthal spaces $X_{p,w}$ and the Bourgain-Rosenthal-Schechtman spaces $R_\alpha^{p,0}$. In particular, we establish the following theorems.

Theorem. [11] For $1 < p < \infty$, $X_{p,w}$ and $R_\omega^{p,0}$ with their respective standard bases have the factorization property.

Theorem. [10] For $1 < p < \infty$, the limit $R_\alpha^{p,0}$ spaces with their standard martingale difference sequence bases have the factorization property.

This means the following. If X is one of the above spaces and $(e_n)_{n=1}^\infty$ denotes its standard basis, then every bounded linear operator $T\colon X \to X$ with large diagonal, i.e., such that $\inf_n |e_n^*(T(e_n))| \geqslant \delta > 0$, is a factor of the identity. The factorization property, formally introduced in [13], has driven the study of reproducing properties of bases in Banach spaces and has led to further developments in the theory of L_p and other classical Banach spaces (see, e.g., [15], [17], [18], [19], [25]).

The third result that we prove concerns the primary factorization property of the spaces $X_{p,w}$ and $R_\omega^{p,0}$.

Theorem. [11] For $1 < p < \infty$, the spaces $X_{p,w}$ and $R_\omega^{p,0}$ have the primary factorization property.

The above means that for every bounded linear operator $T\colon X \to X$, where X is one of these spaces, either T or $id - T$ is a factor of the identity. Note that the statement of this theorem is basis-free, and the spaces $X_{p,w}$ and $R^{p,0}_\omega$ are isomorphic. We prove the primary factorization property for $R^{p,0}_\omega$, using its basis; this result then immediately implies the same property for $X_{p,w}$. It has long been known that Banach spaces satisfying the primary factorization property together with Pełczyński's accordion property are primary. Although this is a basis-free concept, it has been long studied for classical Banach spaces, often using their standard bases. Examples of primary spaces derived from this technique can be found in [3], [8], [17] [20] and [22]. Additionally, the aforementioned property is closely related to the study of closed ideals in $\mathcal{L}(X)$, e.g., if the space X has the primary factorization property, then the set

$$\mathcal{M}_X = \{T \in \mathcal{L}(X) : \ id \neq LTR \ \text{ for all } \ L, R \in \mathcal{L}(X)\}$$

is a maximal ideal of $\mathcal{L}(X)$ [7].

In most of the above papers given a bounded linear operator $T\colon X \to X$, it is shown that there are bounded linear operators $L, R\colon X \to X$ and a "simpler" operator $S\colon X \to X$ such that LTR is approximately S. This process is referred to as reducing T to S. After several successive reductions, usually, T is reduced to a scalar operator, i.e., an operator $\lambda \cdot id$, for some $\lambda \in \mathbb{R}$. In this work we develop and study a specific kind of reduction called orthogonal reduction. For this purpose, we utilize the notion of an orthogonally complemented subspace of L_p, that was introduced by Alspach in 1999 [2]. A subspace X of L_p is called orthogonally complemented if the bounded linear projection from L_p onto X is derived from an orthogonal projection. For example, the Rosenthal space and the Bourgain-Rosenthal-Schechtman spaces are of this type. If X is such a subspace of L_p and $T, S : X \to X$ are bounded linear operators, we say that T is an orthogonal factor of S with error $\epsilon > 0$ if there exists a distributional embedding $J : X \to L_p$ with $J(X) \subset X$, such that $\|J^\dagger T J - S\| < \epsilon$. The operator $J^\dagger : L_p \to X$ denotes the formal adjoint operator of J, satisfying $J^\dagger J = id : X \to X$, and $JJ^\dagger : L_p \to L_p$ is the orthogonal projection onto $J(X)$.

The proof of the factorization property of $X_{p,w}$ with its natural basis relies on some machinery developed in [16], namely of strategical reproducibility of Schauder bases. This means that in a certain two-player game, one of the players has a winning strategy allowing them to reproduce the basis and its biorthogonal sequence in the dual. Using Rosenthal's analysis of block sequences of the standard basis of $X_{p,w}$, we deduce that it is strategically reproducible and thus it satisfies the factorization property.

With regards to the factorization property of $R^{p,0}_\omega$ with its standard martingale difference sequence basis, it requires use of probabilistic and combinatorial arguments. For each $n \in \mathbb{N}$, let $\mathcal{D}^n$ denote the set of left-closed, right-open dyadic intervals of length $1/2^n$ in $[0, 1)$, and $\mathcal{D}_n = \cup_{k=0}^n \mathcal{D}^k$. For every $n \in \mathbb{N}$, denote $h^n_I = T^\omega_n(h_I)$, $I \in \mathcal{D}_{n-1}$, where T^ω_n, $n \in \mathbb{N}$ are the distributional embeddings with independent ranges involved in the construction of the limit space R^p_ω. The space $R^{p,0}_\omega$ is the closed linear span of the collection $((h^n_I)_{I\in\mathcal{D}_{n-1}})_{n\in\mathbb{N}}$, where for $n \in \mathbb{N}$, $(h^n_I)_{I\in\mathcal{D}_{n-1}}$ is a distributional copy of the first $(n-1)$-levels of the Haar system and the collection of the σ-algebras $\sigma(h^n_I)_{I\in\mathcal{D}_{n-1}}$,

$n \in \mathbb{N}$, is independent. With an appropriate enumeration, this basis is a martingale difference sequence, and thus, by Burkholder's inequality, it is an unconditional basis of $R^{p,0}_\omega$. Because each part $(h^n_I)_{I\in\mathcal{D}_{n-1}}$ spans a distributional copy of $L^{n,0}_p = \langle h_I : I \in \mathcal{D}_{n-1}\rangle$, the probabilistic techniques from [14] apply to this setting. We refine them to construct, for every $T\colon R^{p,0}_\omega \to R^{p,0}_\omega$, an operator $D\colon R^{p,0}_\omega \to R^{p,0}_\omega$ that is diagonal with respect to the basis, and its eigenvalues are averages of finitely many diagonal entries $((|I|^{-1}\langle h^n_I, Th^n_I\rangle)_{I\in\mathcal{D}_{n-1}})_{n\in\mathbb{N}}$ of T, and show that T is an approximate orthogonal factor of D. With this, the factorization property follows relatively easily from the unconditionality of the basis. The proof of the primary factorization property of $R^{p,0}_\omega$ is based on an orthogonal reduction of an arbitrary diagonal operator to a scalar multiple $\lambda \cdot id$ of the identity, where λ is an average of eigenvalues of T. Because approximate orthogonal factorization is transitive, every $T\colon R^{p,0}_\omega \to R^{p,0}_\omega$ is an approximate orthogonal factor of a scalar operator $\lambda \cdot id$, where λ is in the convex hull of the diagonal entries of T. The primary factorization property then follows easily. The orthogonal reduction of D to $\lambda \cdot id$ is based on a probabilistic future-level stabilization argument from [18] that has its roots in [17], together with some combinatorial arguments. Because $X_{p,w}$ is isomorphic to $R^{p,0}_\omega$, we directly obtain the primary factorization property of $X_{p,w}$.

Although the factorization property of the limit $R^{p,0}_\alpha$ spaces with their standard martingale difference sequence bases is based on an approximate orthogonal reduction to a diagonal operator, the proof is significantly more involved than in the case $\alpha = \omega$. Toward this direction, for every $\alpha < \omega_1$, we construct an explicit unconditional FDD $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ for $R^{p,0}_\alpha$, and we we develop a theory around the martingale difference sequence basis of $R^{p,0}_\alpha$. Moreover, we define and study general distributional embedding schemes that define a class of distributional embeddings between the spaces $R^{p,0}_\alpha$. Utilizing the aforementioned appropriate framework, for every countable limit ordinal number α, we prove that every bounded linear operator on $R^{p,0}_\alpha$ is an approximate orthogonal factor of a scalar FDD-diagonal operator, yielding the factorizational property of $R^{p,0}_\alpha$.

With regards to the explicit unconditional FDD $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ for each space $R^{p,0}_\alpha$, it is constructed recursively, following the recursive structure of the spaces $R^{p,0}_\alpha$ for $1 \leqslant \alpha < \omega_1$. For every $n \in \mathbb{N}$, $R^{p,0}_n = L^{n,0}_p$, and we consider the trivial FDD $L^{n,0}_p$ for this space. For a countable limit ordinal α, assume that for each $R^{p,0}_\beta$, $1 \leqslant \beta < \alpha$, an FDD $(X_\lambda)_{\lambda\in\mathcal{T}_\beta}$ has been defined. We then define $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ by taking a union of the independent FDDs $(T^\alpha_\beta(X_\lambda))_{\lambda\in\mathcal{T}_\beta}$, $1 \leqslant \beta < \alpha$. If α is a limit ordinal, then for every $n \in \mathbb{N}$,

$$R^{p,0}_{\alpha+n} = L^{n,0}_p + \langle T_I(R^{p,0}_\alpha) : I \in \mathcal{D}^n\rangle.$$

We define the FDD $(X_\lambda)_{\lambda\in\mathcal{T}_{\alpha+n}}$ by letting one component be $L^{n,0}_p$ and taking the 2^n disjointly supported $1/2^n$-compressed copies $(T_I(X_\lambda))_{\lambda\in\mathcal{T}_\alpha}$, $I \in \mathcal{D}^n$, of $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$. For $1 \leqslant \alpha < \omega_1$, we formalize the index set $\mathcal{T}_\alpha$ of the FDD as an explicit tree of finite sequences of ordinal numbers and dyadic intervals, in accordance with the above construction. Each X_λ is an isometric copy of some finite-dimensional space $L^{\kappa(\lambda),0}_p$ via a θ_λ-compression operator T_λ determined, e.g., by the entries of λ.

Next, we introduce a class of distributional embeddings between the spaces $R^{p,0}_\alpha$, later used in the factorization of operators. Let $\alpha < \omega_1$. A distributional embedding scheme of $R^{p,0}_\alpha$ into $R^{p,0}_\beta$

is a family $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu\in\mathcal{M}_\lambda})_{\lambda\in\mathcal{T}_\alpha}$, where $(\mathcal{M}_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ is a collection of finite subsets of $\mathcal{T}_\beta$ that respect the tree structure. For instance, for each $\lambda \in \mathcal{T}_\alpha$, the associated compressions $(T_\mu)_{\mu\in\mathcal{M}_\lambda}$ have pairwise disjointly supported ranges and the family $(J_\mu)_{\mu\in\mathcal{M}_\lambda}$ consists of distributional embeddings with common domain a finite-dimensional $L_p^{\kappa(\lambda),0}$ space. A substantial portion of the paper is devoted to proving that the operator $T_\Psi : R_\alpha^{p,0} \to R_\beta^{p,0}$ given by

$$T_\Psi|_{X_\lambda} = \sum_{\mu\in\mathcal{M}_\lambda} T_\mu J_\mu T_\lambda^{-1}$$

is a distributional embedding. While the finite-dimensional components $T_\Psi|_{X_\lambda}$, $\lambda \in \mathcal{T}_\alpha$ are almost trivially distributional embeddings, the main difficulty lies in controlling how these pieces interact with one another. Proceeding by transfinite induction on α, we decompose T_Ψ into component operators determined by distributional embedding schemes on Bourgain-Rosenthal-Schechtman spaces on smaller ordinals. Each piece satisfies the hypothesis, and reassembling the pieces recovers T_Ψ and establishes it as a distributional embedding. The inductive step differs for successor and limit ordinals α.

Utilizing the above framework, we prove the approximate orthogonal reduction to a FDD-diagonal operator: Let α be a limit countable ordinal number. For every bounded linear operator $T : R_\alpha^{p,0} \to R_\alpha^{p,0}$ and $\epsilon > 0$ there exist a distributional embedding scheme Ψ of $R_\alpha^{p,0}$ into itself and a scalar FDD-diagonal bounded linear operator $D : R_\alpha^{p,0} \to R_\alpha^{p,0}$ such that T is an orthogonal factor of D with error ϵ. Furthermore, the entries of D are averages of finitely many diagonal entries of the matrix representation of T with respect to the standard martingale difference sequence basis of the space. As an easy conclusion, we obtain that the limit spaces $R_\alpha^{p,0}$ with their standard martingale difference sequence bases satisfy the factorization property.

Chapter 2

Preliminaries

In this chapter, we introduce the standard terminology and notation used throughout the thesis. We begin with basic notation, basic concepts from Banach space theory, recall the notation for the Haar system and conclude by recalling basic notions from probability theory.

2.1 Basic notation

This section introduces the notation employed throughout the thesis.

We write $\mathbb{N} = \{1, 2, \dots\}$ for the set of positive integers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^{\mathbb{N}}$ the set of real sequences. If r is a real number, we denote by $[r]$ the integer part of r. Moreover, if A is a set, we denote by $\#A$ the cardinality of A. For a subset S of a linear (respectively, normed) space, we denote by $\langle S \rangle$ (respectively, $[S]$) its linear (respectively, closed linear) span. For a normed linear space X we write X^* for the dual space of X. In this thesis, we consider Banach spaces over the real field.

Given a probability space $(\Omega, \mathcal{A}, \mu)$, the characteristic function of a set $A \subset \Omega$ is denoted by $\chi_A : \Omega \to \{0, 1\}$. Whenever we work within a probability space, all set relations are understood in the essential sense; for instance, $A \subset B$ means that $\mu(A \backslash B) = 0$. Given a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$, the conditional expectation of an integrable function $f : \Omega \to \mathbb{R}$ with respect to $\mathcal{B}$ is denoted by $\mathbb{E}(f|\mathcal{B})$. The support of f is the set $\operatorname{supp}(f) = [f \neq 0] := \{\omega \in \Omega : \ f(\omega) \neq 0\}$. If Ω is endowed with a topology, its Borel σ-algebra is denoted $\mathcal{B}(\Omega)$. The Lebesgue measure of a measurable subset A of $\mathbb{R}$ is denoted $|A|$. Throughout the thesis, we fix $1 < p < \infty$, denote by q its conjugate exponent and denote $p^* = \max\{p, q\}$. An L_p space refers to a space of the form $L_p(\Omega, \mathcal{A}, \mu)$ for some probability space $(\Omega, \mathcal{A}, \mu)$; for brevity we write $L_p(\mu)$. When the underlying probability space is the unit interval $[0, 1]$ with its Borel σ-algebra and the Lebesgue measure, we simply write L_p. When multiple such spaces are considered simultaneously, we may denote them, for instance, as $L_p(\mu)$, $L_p(\nu)$, or $L_p(\xi)$. For $f \in L_p(\mu)$ and $g \in L_q(\mu)$, we write $\langle g, f \rangle = \int g f \, d\mu$.

2.2 Concepts from Banach space theory

In this section, we recall fundamental concepts from Banach space theory, including the notions of a Schauder basis and a Schauder decomposition. The following can be found in [1] and [21].

In the following definition we recall the notion of a Schauder basis. If we select a basis in a finite-dimensional vector space, then we are, in effect, selecting a system of coordinates. Bases in infinite-dimensional Banach spaces play the same role.

Definition 2.2.1. Let X be an infinite dimensional Banach space and let $(e_n)_{n=1}^{\infty}$ be a sequence in X. The sequence $(e_n)_{n=1}^{\infty}$ is called a Schauder basis or simply basis if every $x \in X$ admits a unique representation $x = \sum_{n=1}^{\infty} a_n e_n$ for some scalar sequence $(a_n)_{n=1}^{\infty}$.

The convergence is in the norm topology of X, that is we require that the sequence $(\sum_{n=1}^{N} a_n e_n)_{N=1}^{\infty}$ converges to x in the norm topology of X. Notice that a basis consists of linearly independent, and in particular nonzero, vectors. Moreover, if $(e_n)_{n=1}^{\infty}$ is a basis of X, then the closed linear span of $(e_n)_{n=1}^{\infty}$ in X coincides with the space X, that is $X = [(e_n)_{n=1}^{\infty}]$, and thus, X is separable.

Definition 2.2.2. A Schauder basis $(e_n)_{n=1}^{\infty}$ of a Banach space X induces the sequence of bounded linear projections $(S_n)_{n=1}^{\infty}$ defined by $S_n(\sum_{m=1}^{\infty} a_m e_m) = \sum_{m=1}^{n} a_m e_m$. The projections $(S_n)_{n=1}^{\infty}$ are called natural projections associated with the basis $(e_n)_{n=1}^{\infty}$.

It is well known that $K = \sup_n \|S_n\| < \infty$. The number K is called the basis constant. In the optimal case that $K = 1$ the basis $(e_n)_{n=1}^{\infty}$ is said to be monotone.

Definition 2.2.3. A Schauder basis $(e_n)_{n=1}^{\infty}$ of a Banach space X induces the sequence of bounded linear functionals $(e_n^*)_{n=1}^{\infty}$ in X^* defined by $e_n^*(\sum_{m=1}^{\infty} a_m e_m) = a_n$. The sequence $(e_n^*)_{n=1}^{\infty}$ is called the biorthogonal sequence associated with the basis $(e_n)_{n=1}^{\infty}$.

Note that the unique representation of each element of X can be expressed via the biorthogonal sequence, that is $x = \sum_{n=1}^{\infty} e_n^*(x) e_n$.

In the next definition we relax the assumption that a basis must span the entire space.

Definition 2.2.4. Let X be an infinite dimensional Banach space. A sequence $(e_n)_{n=1}^{\infty}$ in X is called a Schauder basic sequence or simply basic sequence if $(e_n)_{n=1}^{\infty}$ is a basis for $[(e_n)_{n=1}^{\infty}]$.

Example 2.2.5.

(a) The sequence of the standard unit vectors $(e_n)_{n=1}^{\infty}$ is a monotone basis for c_0 and ℓ_p, for all $1 \leqslant p < \infty$.

(b) The sequence of the standard unit vectors $(e_n)_{n=1}^{\infty}$ is not a basis for ℓ_∞, but it is a monotone basic sequence.

In the next definition we recall the definition of two equivalent bases or basic sequences. Equivalent bases allow flexibility in choosing bases for computations, expansions, or theoretical work without changing the underlying structure of the space.

Definition 2.2.6. Two bases (or basis sequences) $(e_n)_{n=1}^{\infty}$ and $(e'_n)_{n=1}^{\infty}$ in the respective Banach spaces X and X' are called equivalent, if there exists a constant $C > 0$ such that for all finitely non zero sequences of scalars $(a_n)_{n=1}^{\infty}$ we have

$$\frac{1}{C}\| \sum_{n=1}^{\infty} a_n e'_n\| \leqslant \| \sum_{n=1}^{\infty} a_n e_n\| \leqslant C\| \sum_{n=1}^{\infty} a_n e'_n\|.$$

Or, equivalently, there exists an isomorphism $T : [e_n : n \in \mathbb{N}] \to [e'_n : n \in \mathbb{N}]$ such that $T(e_n) = e'_n$ for each $n \in \mathbb{N}$.

Let X be a Banach space with a basis $(e_n)_{n=1}^{\infty}$. We identify a bounded linear operator $T : X \to X$ with its matrix representation $(e_m^*(Te_n))_{m,n=1}^{\infty}$. Thus, the concept of a diagonal operator with respect to a basis naturally arises.

Definition 2.2.7. Let X be a Banach space with a basis $(e_n)_{n=1}^{\infty}$ and let $T : X \to X$ be a bounded linear operator. The operator T is said to be diagonal with respect to $(e_n)_{n=1}^{\infty}$ (or simply diagonal when the basis is understood) if $e_m^*(Te_n) = 0$ for all $m \neq n$. Equivalently, each e_n is an eigenvector of T corresponding to some eigenvalue λ_n.

We will now take a closer look at particular classes of bases. Namely, unconditional, shrinking and boundedly-complete bases.

Definition 2.2.8. A basis $(e_n)_{n=1}^{\infty}$ is said to be C-unconditional for some $C \geqslant 1$ if, for all scalars $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfying $|a_k| \leqslant |b_k|$ for $1 \leqslant k \leqslant n$,

$$\Big\| \sum_{k=1}^{n} a_k e_k \Big\| \leqslant C \Big\| \sum_{k=1}^{n} b_k e_k \Big\|.$$

A basis that is C-unconditional for some $C \geqslant 1$ is simply called unconditional.

Definition 2.2.9. A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is called shrinking if the biorthogonal sequence $(e_n^*)_{n=1}^{\infty}$ associated with $(e_n)_{n=1}^{\infty}$ is a basis for X^*.

Definition 2.2.10. A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is said to be boundedly-complete if whenever $(a_n)_{n=1}^{\infty}$ is a sequence of scalars such that

$$\sup_N \| \sum_{n=1}^{N} a_n e_n\| < \infty,$$

then the series $\sum_{n=1}^{\infty} a_n e_n$ converges.

In 1951, James proved that if X is a Banach space with a basis $(e_n)_{n=1}^{\infty}$, then X is reflexive if and only if $(e_n)_{n=1}^{\infty}$ is both boundedly-complete and shrinking.

Example 2.2.11.

(a) The standard unit vector basis $(e_n)_{n=1}^\infty$ is an unconditional basis of c_0 and ℓ_p, for all $1 \leqslant p < \infty$. It is shrinking for c_0 and ℓ_p, for all $1 < p < \infty$, and it is boundedly-complete for ℓ_p, for all $1 \leqslant p < \infty$, but it is not boundedly-complete for c_0.

(b) For every $n \in \mathbb{N}$ denote $f_n = \sum_{k=1}^n e_k$, where $(e_n)_{n=1}^\infty$ is the standard unit vector basis of c_0. The sequence $(f_n)_{n=1}^\infty$ is a basis of c_0 with biorthogonal sequence defined by $f_n^* = e_n^* - e_{n+1}^*$ for all $n \in \mathbb{N}$, and it is known as the summing basis. The basis $(f_n)_{n=1}^\infty$ is neither unconditional, nor shrinking, nor boundedly-complete.

A Schauder basis decomposes, in a sense, a Banach space into a sum of one dimensional spaces. It is sometimes useful to consider cruder decompositions where the components into which we decompose a given Banach space are subspaces of dimension larger than 1.

Definition 2.2.12. Let X be a Banach space and let $(E_n)_{n=1}^\infty$ be a sequence of closed subspaces of X.

(a) The sequence $(E_n)_{n=1}^\infty$ is called a Schauder decomposition of X if every $x \in X$ can be uniquely written as $x = \sum_{n=1}^\infty x_n$ with $x_n \in E_n$. If each E_n is finite-dimensional, (E_n) is called a finite-dimensional decomposition (FDD). A Schauder decomposition induces a sequence of bounded linear projections $(P_n)_{n=1}^\infty$ defined by $P_n(\sum_{m=1}^\infty x_m) = x_n$.

(b) If every sequence of non-zero vectors $(x_n)_{n=1}^\infty$ with $x_n \in E_n$ forms an unconditional basis of its closed linear span, the decomposition is called an unconditional Schauder decomposition.

Remark 2.2.13. In [26], the author constructs a Banach space admitting a FDD but not a Schauder basis.

Let $(E_n)_{n=1}^\infty$ be an FDD of a Banach space X with corresponding projections $(P_n)_{n=1}^\infty$ and let $T : X \to X$ be a bounded linear operator. The matrix representation of T with respect to the aforementioned FDD naturally arises and can be visualized as $(P_m T P_n)_{m,n=1}^\infty$.

Definition 2.2.14. Let X be a Banach space that admits an FDD $(E_n)_{n=1}^\infty$ and let $T : X \to X$ be a bounded linear operator. The operator $T : X \to X$ is said to be diagonal with respect to $(E_n)_{n=1}^\infty$ (or FDD-diagonal when the decomposition is understood) if $P_m T P_n = 0$ for all $m \neq n$. If each E_n is an eigenspace of T corresponding to some eigenvalue λ_n, we say that T is scalar diagonal with respect to $(E_n)_{n=1}^\infty$ (or simply scalar FDD-diagonal when the FDD is understood) and we call $(\lambda_n)_{n=1}^\infty$ the entries of T.

2.3 Haar system notation

In this section, we recall basic notation for the standard Haar system. Although this system is not a basis of the $R_\alpha^{p,0}$ spaces, the canonical basis of each such space can be blocked into an FDD whose components are spanned by compressed copies of finite initial segments of the Haar system. For this reason, we will use Haar-related notation extensively.

We enumerate the Haar basis over the dyadic intervals of $[0,1]$, ordered lexicographically.

Notation 2.3.1. We establish the following notation used in handling the Haar system.

(a) We denote by $\mathcal{D}$ the set of all dyadic intervals I of $[0,1]$, i.e., of the form

$$I = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right),$$

where $n \in \mathbb{N}_0$ and $1 \leqslant i \leqslant 2^n$.

(b) For $n \in \mathbb{N}_0$ we denote

$$\mathcal{D}^n = \{I \in \mathcal{D} : |I| = 1/2^n\} = \left\{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) : 1 \leqslant i \leqslant 2^n\right\} \text{ and}$$
$$\mathcal{D}_n = \bigcup_{k=0}^{n} \mathcal{D}^k.$$

(c) We enumerate $\mathcal{D}$ in the lexicographical order as follows. Let $I \in \mathcal{D}^m$ and $J \in \mathcal{D}^n$. Then $I < J$ means either $m < n$, or both $m = n$ and $\min(I) < \min(J)$. That is, $[0,1) < [0,1/2) < [1/2,1) < [0,1/4) < [1/4,1/2) < [1/2,3/4) < [3/4,1) < \ldots$.

(d) Let $n \in \mathbb{N}_0$ and $I = [(i-1)/2^n, i/2^n) \in \mathcal{D}^n$. We denote the left and right half of I in $\mathcal{D}^{n+1}$ by

$$I^+ = \left[\frac{2(i-1)}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\right) \text{ and } I^- = \left[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\right).$$

We call I the immediate predecessor of I^+ and I^- and write

$$\pi(I^+) = I \text{ and } \pi(I^-) = I.$$

In this context, the term "predecessor" refers to an implicit binary tree structure of $\mathcal{D}$, rather than the lexicographical order. We define the signs of I^+ and I^- as

$$\epsilon(I^+) = 1 \text{ and } \epsilon(I^-) = -1.$$

Notation 2.3.2. We recall the standard Haar system and its relation to standard subspaces of L_p.

(a) For $I \in \mathcal{D}$, we define

$$h_I = \chi_{I^+} - \chi_{I^-}.$$

The collection $(\chi_{[0,1]}) \cup (h_I)_{I \in \mathcal{D}}$ is the standard Haar system. Moreover, by the Burkholder inequality, it forms an unconditional basis of L_p, and then,

$$L_p = [(\chi_{[0,1]}) \cup (h_I)_{I \in \mathcal{D}}].$$

(b) We denote
$$L_p^0 = \{f \in L_p : \mathbb{E}(f) = 0\} = [(h_I)_{I\in\mathcal{D}}].$$

(c) For $n \in \mathbb{N}$ denote
$$L_p^n = \langle(\chi_I)_{I\in\mathcal{D}^n}\rangle = \langle(\chi_{[0,1]}) \cup (h_I)_{I\in\mathcal{D}_{n-1}}\rangle$$
and
$$L_p^{n,0} = \{f \in L_p^n : \mathbb{E}(f) = 0\} = \langle(h_I)_{I\in\mathcal{D}_{n-1}}\rangle.$$

2.4 Basic notions from probability theory

In this section, we recall basic notions from probability theory, including the concepts of the distribution of a random variable and a martingale difference sequence.

Definition 2.4.1. Let f be a real-valued random variable on a probability space $(\Omega, \mathcal{A}, \mu)$. The distribution of f, denoted $\operatorname{dist}(f)$, is the probability measure on the Borel sets of $\mathbb{R}$ defined by
$$(\operatorname{dist}(f))(A) = \mu([f \in A]).$$

In the following example we make use of the notions of the Dirac measure and the standard Rademacher functions. Recall that the Dirac measure centered on some fixed point t in some measurable space $(\Omega, \mathcal{A})$, denoted δ_t, is the probability measure on $(\Omega, \mathcal{A})$ defined by
$$\delta_t(A) = \begin{cases} 1 & t \in A \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, recall that for every $n \in \mathbb{N}_0$, the n-th Rademacher function, denoted r_n, is defined by
$$r_n = \sum_{I\in\mathcal{D}^n} h_I$$

Example 2.4.2.

1. For every $n \in \mathbb{N}_0$,
$$\operatorname{dist}(r_n) = \frac{1}{2}(\delta_{-1} + \delta_1)$$

2. For every $I \in \mathcal{D}$,
$$\operatorname{dist}(h_I) = \frac{|I|}{2}(\delta_{-1} + \delta_1) + (1 - |I|)\delta_0$$

Based on the above definition, we recall the notion of a distributional embedding, which is used extensively.

Definition 2.4.3. Let X, Y be subspaces of some L_p-spaces. A linear operator $T : X \to Y$ is called a distributional embedding if, for all $f \in X$, $\mathrm{dist}(Tf) = \mathrm{dist}(f)$. If T is additionally onto then it is called a distributional isomorphism, and we write $X \equiv^{\mathrm{dist}} Y$.

A distributional embedding is an isometry, but it is much more than that. When the domain of a distributional embedding is not closed, its extension to the closure remains a distributional embedding. This can be proved through the characteristic function of random variables.

Next, we present an example of a distributional embedding associated with a measurable, measure-preserving function—known as the Koopman operator—which is used in the definition of the R^p_α spaces.

Example 2.4.4. Let $\mathcal{A}$ be a σ-subalgebra of $\mathcal{B}[0,1]$ and let $\varphi\colon ([0,1],\mathcal{A}) \to ([0,1],\mathcal{B}([0,1]))$ be measurable and measure-preserving, that is $|\varphi^{-1}(A)| = |A|$ for all $A \in \mathcal{B}[0,1]$. Then the Koopman operator of φ, denoted $T_\varphi\colon L_p \to L_p$ and defined by

$$T_\varphi(f) = f \circ \varphi$$

is a distributional embedding.

Each R^p_α space has a basis given by a 3-valued symmetric martingale difference sequence. In the following definition we recall the notions of a symmetric random variable and of a martingale difference sequence.

Definition 2.4.5.

(a) A random variable f in a space $L_p(\mu)$ is called symmetric if for all $A \in \mathcal{B}(\mathbb{R})$ we have

$$\mu\Big([f \in A]\Big) = \mu\Big([-f \in A]\Big).$$

(b) A sequence $(f_n)_{n=1}^\infty$ in a space $L_p(\mu)$ is called a martingale difference sequence if, for all $n \geqslant 2$,

$$\mathbb{E}(f_n | f_1, \ldots, f_{n-1}) = 0.$$

Example 2.4.6. The standard Haar system $(\chi_{[0,1]}) \cup (h_I)_{I\in\mathcal{D}}$ is a martingale difference sequence with the lexicographic order of $\mathcal{D}$. Each function h_I is symmetric.

Remark 2.4.7. Let $(f_n)_{n=1}^\infty$ be a martingale difference sequence in a space $L_p(\mu)$. When all the terms of $(f_n)_{n=1}^\infty$ are non-zero, by the non-expansiveness of the conditional expectation operator, $(f_n)_{n=1}^\infty$ is a monotone Schauder basic sequence. Moreover, by the Burkholder inequality ([5, Theorem 9]), it is unconditional.

Chapter 3

Basic concepts

In this chapter, we recall key notions from the theory of operator factorizations, we then outline the framework of orthogonally complemented subspaces of L_p, and we conclude with the notion of an orthogonal factor.

3.1 Factorization of operators

In this section, we recall the notions of a factor and a projectional factor of an operator, note their distinction, and prepare for the discussion of orthogonal factors in the last subsection. We also recall the factorization property for Banach spaces with a Schauder basis and the primary factorization property for a Banach space.

We begin with the concepts of a factor and a projectional factor of an operator.

Definition 3.1.1. Let X be a Banach space, $T, S : X \to X$ be bounded linear operators, and $K > 0$.

(a) We call T a K-factor of S if there exist bounded linear operators $A, B : X \to X$ such that

$$S = ATB \text{ and } \|A\|\|B\| \leqslant K.$$

When we do not need to be specific about the constant K, then we call T a factor of S.

(b) We call T a K-projectional factor of S is there exist bounded linear operators $A, B : X \to X$ such that

$$S = ATB, \ \|A\|\|B\| \leqslant K, \text{ and } AB = id : X \to X.$$

When we do not need to be specific about the constant K, then we call T a projectional factor of S.

Remark 3.1.2. An operator $T : X \to X$ is a factor of the identity operator $id : X \to X$ if and only if there exist projections $P : X \to Y$ and $Q : X \to Z$, with images isomorphic to X, such that $QT|_Y : Y \to Z$ is an isomorphism. By contrast, T is a projectional factor of the identity operator precisely when there exists a projection $P : X \to Y$, with image isomorphic to X, such that $PT|_Y = id : Y \to Y$. More generally, if T is a projectional factor of S, then there exists a decomposition $X = Y \oplus Z$ such that $X \simeq Y$ and the 2×2 matrix representation of T with respect to this decomposition contains a matrix similar to S in the upper left entry. Thus, S may be viewed as a "simpler" operator than T, with T effectively "reduced" to S. Projectional factorizations enjoy stronger properties and are employed in stepwise reductions that lead to clean factorization results often related to decompositions of Banach spaces (see, e.g., [17, 18]).

In this work we impose additional structure, further tightening the notion of a projectional factor: the decomposition $X = Y \oplus Z$ must be orthogonal, and the spaces X and Y are distributionally isomorphic. These notions are well defined for subspaces of L_p-spaces. This refinement grants additional properties to the factorization, notably the automatic continuity of certain operators. One of our main results is that every bounded linear operator on a limit space $R^{p,0}_{\alpha}$ "orthogonally reduces" to a scalar FDD-diagonal operator. The necessary, though elementary, theory to define orthogonal factorizations is developed in Section 3.2.

To recall the notion of the factorization property, we first review the concept of the large diagonal of an operator defined on a Banach space with a basis. In Chapters 4 and 6 we show that the Rosenthal $X_{p,w}$ space and the Bourgain-Rosenthal-Schechtman $R^{p,0}_{\omega}$ space with their respective natural bases have the factorization property. Additionally, in Chapter 10 we prove that the standard bases of the limit $R^{p,0}_{\alpha}$ spaces have the factorization property.

Notation 3.1.3. Let X be a Banach space with a basis $(e_n)_{n=1}^{\infty}$ and let $T : X \to X$ be a bounded linear operator. Denote by $\mathcal{D}(T)$ the set of the diagonal elements of T with respect to the basis $(e_n)_{n=1}^{\infty}$, that is

$$\mathcal{D}(T) = \{e_n^*(Te_n) : \ n \in \mathbb{N}\}.$$

Additionally, denote by $\mathcal{A}(\mathcal{D}(T))$ the collection of all averages of finitely many elements of $\mathcal{D}(T)$.

Definition 3.1.4. Let X be a Banach space with a basis $(e_n)_{n=1}^{\infty}$. We say that the basis $(e_n)_{n=1}^{\infty}$ has the factorization property if every bounded linear operator $T\colon X \to X$ with large diagonal with respect to the basis $(e_n)_{n=1}^{\infty}$, that is for some $\delta > 0$, we have $\inf\{|d| : \ d \in \mathcal{D}(T)\} \geqslant \delta$, is a factor of the identity operator id on X.

The preceding definition depends on the basis of the Banach space. In contrast, the following definition of the primary factorization property is basis-free.

In Chapter 6, we prove that the Rosenthal $X_{p,w}$ spaces and the Bourgain-Rosenthal-Schechtman $R^{p,0}_{\omega}$ space have the primary factorization property.

Definition 3.1.5. Let X be a Banach space. We say that the space X has the primary factorization property if for every bounded linear operator $T : X \to X$ we have that T or $id - T$ is a factor of the identity operator id on X.

Remark 3.1.6.

(a) In [16, Remark 2.4, page 17], the authors provide an example of a Banach space, namely the James space, that admits a Schauder basis but does not satisfy the factorization property.

(b) There is an example of a Banach space that does not satisfy the primary factorization property. Indeed, let $1 < p \neq q < \infty$, and set

$$X = \ell_p \oplus \ell_q,$$

where P denotes the natural projection onto ℓ_p. If P were a factor of the identity, then ℓ_q would embed isomorphically into ℓ_p, which is absurd. An analogous argument applies to the projection $id - P$ onto ℓ_q. Hence, X does not satisfy the primary factorization property.

(c) The factorization property and the primary factorization property are related and rely on similar techniques, but they are not equivalent. For instance, the standard basis of $\ell_p \oplus \ell_2$, for $1 < p \neq 2 < \infty$, satisfies the factorization property (as a strategically reproducible basis), but fails the primary factorization property (see (b)).

3.2 Orthogonally complemented spaces

A subspace of a space $L_p(\mu)$ is called orthogonally complemented if it is the image of a bounded linear projection stemming from an orthogonal projection on $L_2(\mu)$. To our knowledge, this concept was initially presented in [2] for $p > 2$, but it will be used extensively to create the suitable framework for examining the factorization properties of operators on the Rosenthal space, X_p, and the limit Bourgain-Rosenthal-Schechtman spaces, R^p_α. We introduce the notion of an orthogonally complemented space and we construct and study the formal adjoint operator of a distributional embedding that allows us to show that orthogonal complementability is preserved under distributional isomorphisms.

In the following definition we give a name to the orthogonal projections on a space $L_2(\mu)$ bounded with respect to the $\|\cdot\|_p$-norm, and next, we introduce the notion of an orthogonally complemented subspace of $L_p(\mu)$.

Recall that $1 < p < \infty$, and let q denote the conjugate exponent of p.

Definition 3.2.1. Consider the space $L_p(\mu)$ and let $C \geqslant 1$.

(a) An orthogonal projection $P : L_2(\mu) \to L_2(\mu)$ is called p-C-bounded if, for every $f \in L_2(\mu) \cap L_p(\mu)$, $\|Pf\|_p \leqslant C\|f\|_p$.

(b) A subspace X of $L_p(\mu)$ is called C-orthogonally complemented if there exists a p-C-bounded orthogonal projection $P : L_2(\mu) \to L_2(\mu)$ such that

$$X = \overline{P(L_2(\mu) \cap L_p(\mu))}^{\|\cdot\|_p}.$$

Notation 3.2.2. Consider the space $L_p(\mu)$, $C \geqslant 1$, let $P : L_2(\mu) \to L_2(\mu)$ be a p-C-bounded orthogonal projection, and denote $X = \overline{P(L_2(\mu) \cap L_p(\mu))}^{\|\cdot\|_p}$.

(a) By the density of $L_2(\mu) \cap L_p(\mu)$ in $L_p(\mu)$, there exists a unique bounded linear projection

$$P_X : L_p(\mu) \to L_p(\mu)$$

such that $P_X|_{L_2(\mu)\cap L_p(\mu)} = P|_{L_2(\mu)\cap L_p(\mu)}$. Furthermore, $P_X(L_p(\mu)) = X$ and $\|P_X\| \leqslant C$.

(b) The self-adjointness of P implies that it is q-C-bounded. We denote

$$X' = \overline{P(L_2(\mu) \cap L_q(\mu))}^{\|\cdot\|_q}$$

and $P_{X'} : L_q(\mu) \to L_q(\mu)$ the projection onto X' stemming from P.

Remark 3.2.3. In the context of the above notation, the following hold.

(a) The projection $P_{X'} : L_q(\mu) \to L_q(\mu)$ is the Banach space adjoint of $P_X : L_p(\mu) \to L_p(\mu)$. Consequently, the space X' is C-isomorphic to X^*.

(b) Because the intersection $X \cap X'$ contains $P(L_\infty(\mu))$, it is dense in both X and X' with respect to the norms $\|\cdot\|_p$ and $\|\cdot\|_q$, respectively.

For every $\alpha < \omega_1$, the space R^p_α admits a basis consisting of a symmetric martingale difference sequence taking values in $\{-1, 0, 1\}$, with all terms nonzero. This naturally leads to the following topic.

Our next goal is to show that the formula of the projection onto an orthogonally complemented $\{-1, 0, 1\}$-valued symmetric martingale difference sequence with non-zero terms in a space $L_p(\mu)$ admits an explicit expression in terms of the martingale difference sequence itself, that is Proposition 3.2.5. The following remark will be used in the proof of the aforementioned proposition.

Remark 3.2.4. Let f be a 3-valued symmetric random variable in a space $L_p(\mu)$. Then $\|f\|_p\|f\|_q = \|f\|_2^2$ (see [24, page 280]).

Proposition 3.2.5. Let $(f_n)_{n=1}^\infty$ be a $\{-1, 0, 1\}$-valued symmetric martingale difference sequence with non-zero terms in a space $L_p(\mu)$ and assume that $X = [(f_n)_{n=1}^\infty]$ is C-orthogonally complemented. Then

(a) $\langle \|f_m\|_q^{-1} f_m, \|f_n\|_p^{-1} f_n\rangle = \delta_{m,n}$ for all $m, n \in \mathbb{N}$,

(b) the biorthogonal sequence associated with $(\|f_n\|_p^{-1} f_n)_{n=1}^\infty$ is equivalent to $(\|f_n\|_q^{-1} f_n)_{n=1}^\infty$, and

(c) the projection P_X has the explicit formula

$$P_X f = \sum_{n=1}^\infty \|f_n\|_q^{-1} \langle f_n, f\rangle \|f_n\|_p^{-1} f_n,$$

where the above series convergences with respect to $\|\cdot\|_p$, and $X' = [(f_n)_{n=1}^\infty]$ in $L_q(\mu)$.

Proof. Remark 3.2.4 yields $\langle \|f_n\|_q^{-1} f_n, \|f_n\|_p^{-1} f_n \rangle = 1$ for all $n \in \mathbb{N}$, and since $(f_n)_{n=1}^\infty$ is a martingale difference sequence from basic properties of the conditional expectation we have $\langle f_m, f_n \rangle = 0$ for all $m \neq n \in \mathbb{N}$, which proves (a). Remark 2.4.7 implies that the sequence $(f_n)_{n=1}^\infty$ is a monotone basis of X. Denote by $(\phi_n)_{n\in\mathbb{N}}$ the biorthogonal sequence associated with $(\|f_n\|_p^{-1} f_n)_{n=1}^\infty$ in X^*, and denote by $(S_n)_{n=1}^\infty$ the natural projections associated with the basis $(\|f_n\|_p^{-1} f_n)_{n=1}^\infty$. We prove that $(\phi_n)_{n\in\mathbb{N}}$ is equivalent to $(\|f_n\|_q^{-1} f_n)_{n=1}^\infty$. Since P_X is derived from an orthogonal projection we obtain that $P_X^*(f_n) = f_n$ for all $n \in \mathbb{N}$. Let $a_1, ..., a_n \in \mathbb{R}$. We observe that

$$\Big\| \sum_{k=1}^n a_k f_k \Big\|_q = \Big\| P_X^* \Big(\sum_{k=1}^n a_k f_k \Big) \Big\|_q = \Big(\sum_{k=1}^n a_k f_k \Big) \Big(P_X g \Big)$$

for some $g \in L_p$ with $\|g\| \leqslant 1$. Note $\Big(\sum_{k=1}^n a_k f_k \Big) \Big(P_X g \Big) = \Big(\sum_{k=1}^n a_k \phi_k \Big) \Big(P_X g \Big)$. Thus, $\Big\| \sum_{k=1}^n a_k f_k \Big\|_q \leqslant C \Big\| \sum_{k=1}^n a_k \phi_k \Big\|$. Now, we have

$$\Big\| \sum_{k=1}^n a_k \phi_k \Big\| = \Big(\sum_{k=1}^n a_k \phi_k \Big)(g)$$

for some $g \in X$ with $\|g\| \leqslant 1$, and observe that $\Big(\sum_{k=1}^n a_k \phi_k \Big)(g) = \Big(\sum_{k=1}^n a_k f_k \Big)(g)$. Thus, $\Big\| \sum_{k=1}^n a_k \phi_k \Big\| \leqslant \Big\| \sum_{k=1}^n a_k f_k \Big\|_q$, which proves (b). Now, we prove the boundedness of $Q : L_p \to L_p$ defined by

$$Qf = \sum_{n=1}^\infty \|f_n\|_q^{-1} \langle f_n, f \rangle \|f_n\|_p^{-1} f_n,$$

which in combination with

$$Q|L_2(\mu) \cap L_p(\mu) = P_X|L_2(\mu) \cap L_p(\mu),$$

and the fact that P_X is derived from an orthogonal projection, we obtain (c). Next we establish the boundedness of Q. Let $f \in L_p$ with $\|f\| \leqslant 1$. Since the basis $(f_n)_{n=1}^\infty$ is boundedly-complete it suffices to show that for all $n \in \mathbb{N}$, $\| \sum_{k=1}^n \|f_k\|_q^{-1} \langle f_k, f \rangle \|f_k\|_p^{-1} f_k \| \leqslant C$. Indeed, for $n \in \mathbb{N}$, we have

$$\Big\| \sum_{k=1}^n \|f_k\|_q^{-1} \langle f_k, f \rangle \|f_k\|_p^{-1} f_k \Big\| = g \Big(\sum_{k=1}^n \|f_k\|_q^{-1} \langle f_k, f \rangle \|f_k\|_p^{-1} f_k \Big),$$

for some $g \in X^*$ with $\|g\| \leqslant 1$, and observe that

$$\Big(\sum_{k=1}^n \langle g, \|f_k\|_p^{-1} f \rangle \|f_k\|_q^{-1} f_k \Big)(f) \leqslant \Big\| \Big(\sum_{k=1}^n \langle g, \|f_k\|_p^{-1} f \rangle \|f_k\|_q^{-1} f_k \Big) \Big\|.$$

From part (b) and the monotonicity of the basis $(f_n)_{n=1}^{\infty}$ we obtain that

$$\Big\|\Big(\sum_{k=1}^{n}\langle g,\|f_k\|_p^{-1}f\rangle\|f_k\|_q^{-1}f_k\Big)\Big\| \leqslant C\Big\|\Big(\sum_{k=1}^{n}\langle g,\|f_k\|_p^{-1}f\rangle\phi_k\Big)\Big\| = C\Big\|S_k^*(g)\Big\| \leqslant C,$$

which proves $\|\sum_{k=1}^{n}\|f_k\|_q^{-1}\langle f_k,f\rangle\|f_k\|_p^{-1}f_k\| \leqslant C$. $\square$

We next construct the formal adjoint operator of a distributional embedding.

Notation 3.2.6. Consider the spaces $L_p(\mu)$, $L_p(\nu)$, a C-orthogonally complemented subspace X of $L_p(\mu)$, and a distributional embedding $T: X \to L_p(\nu)$. We construct the formal adjoint of T,

$$T^{\dagger}: L_p(\nu) \to X,$$

as follows. Initially, we extend $T|_{X\cap X'}$ to a distributional embedding $T: X' \to L_q(\nu)$ (see the comment below Definition 2.4.3), which by slightly abusing notation, we also denote T. We then identify $L_q(\nu)^*$ with $L_p(\nu)$ and X^* with X' and let $T^{\dagger}$ be the Banach-space adjoint of $T: X' \to L_q(\nu)$. Because the identification $X' \simeq X^*$ is not isometric, $\|T^{\dagger}\| \leqslant C$.

Remark 3.2.7. By beginning with the operator $T: X' \to L_q(\nu)$, we analogously construct its formal adjoint $T^{\dagger}: L_q(\nu) \to X'$. By definition, for all $f \in L_p(\nu), g \in X', u \in X, v \in L_q(\nu)$, considering the operators T and $T^{\dagger}$ with appropriate domains,

$$\langle g, T^{\dagger}f\rangle = \langle Tg, f\rangle \text{ and } \langle u, T^{\dagger}v\rangle = \langle Tu, v\rangle.$$

In particular, the two distinct versions of $T^{\dagger}$ coincide on $L_p(\nu)\cap L_q(\nu)$, and thus

$$T^{\dagger}(L_p(\nu)\cap L_q(\nu)) \subset X\cap X'.$$

In the following proposition we prove that C-orthogonal complementability is preserved under distributional isomorphisms.

Proposition 3.2.8. Consider the spaces $L_p(\mu)$ and $L_p(\nu)$. Let X be a C-orthogonally complemented subspace of $L_p(\mu)$ and $T: X \to L_p(\nu)$ be a distributional embedding. Denoting $T^{\dagger}: L_p(\nu) \to X$ its formal adjoint, the following hold.

(a) $T^{\dagger}T = id: X \to X$

(b) $TT^{\dagger}: L_p(\nu) \to L_p(\nu)$ is a linear projection of norm at most C onto $T(X)$ witnessing the C-orthogonal complementation of $T(X)$ in $L_p(\nu)$.

Proof. The first statement is shown as follows: for $f \in X$, $g \in X'$, we have that $\langle g, f\rangle = \langle Tg, Tf\rangle$ since (g,f) and (Tg,Tf) have the same joint distribution. Hence, $\langle g,f\rangle = \langle g, T^{\dagger}Tf\rangle$, and thus, $T^{\dagger}T = id: X \to X$. This yields that $TT^{\dagger}$ is a projection onto $T(X)$. It remains to show that $TT^{\dagger}|_{L_2(\nu)\cap L_p(\nu)}$ has norm one with respect to $\|\cdot\|_2$, because in Hilbert spaces norm-one projections are automatically orthogonal (see, e.g., [6, Chapter 3 §III, Proposition 3.3]). For $f \in L_2(\nu) \cap L_p(\nu) = L_q(\nu) \cap L_p(\nu)$, we have $T^\dagger f \in X \cap X' \subset L_2(\mu)$, and therefore,

$$\|TT^\dagger f\|_2^2 = \|T^\dagger f\|_2^2 = \langle T^\dagger f, T^\dagger f\rangle = \langle f, TT^\dagger f\rangle \leqslant \|f\|_2 \|TT^\dagger f\|_2.$$

We deduce $\|TT^\dagger f\|_2 \leqslant \|f\|_2$. $\square$

From Proposition 3.2.5 (c) and Proposition 3.2.8 (b) we obtain a formula for the formal adjoint of a distributional embedding that acts on an orthogonally complemented $\{-1, 0, 1\}$-valued symmetric martingale difference sequence. This formula will be used in the orthogonal reductions of operators on the limit $R_\alpha^{p,0}$ spaces.

Corollary 3.2.9. Let $(f_n)_{n=1}^\infty$ be a $\{-1, 0, 1\}$-valued symmetric martingale difference sequence with non-zero terms in a space $L_p(\mu)$ and assume that $X = [(f_n)_{n=1}^\infty]$ is C-orthogonally complemented. Let $T : X \to L_p(\nu)$ be a distributional embedding into some space $L_p(\nu)$ and, for $n \in \mathbb{N}$, denote $b_n = Tf_n$. Then, for all $f \in X$ and $g \in L_p(\nu)$,

$$Tf = \sum_{n=1}^{\infty} \|f_n\|_q^{-1} \langle f_n, f\rangle \|f_n\|_p^{-1} b_n \text{ and } T^\dagger g = \sum_{n=1}^{\infty} \|f_n\|_q^{-1} \langle b_n, g\rangle \|f_n\|_p^{-1} f_n.$$

We next note that the formal adjoint acts as an anti-automorphism.

Remark 3.2.10. If X is C-orthogonally complemented in $L_p(\mu)$ and $T : X \to L_p(\nu)$, $S : T[X] \to L_p(\xi)$ are distributional embeddings, we have $(ST)^\dagger = T^\dagger S^\dagger$.

3.3 Orthogonal factors

With the necessary theory in place, we give the definition of approximate orthogonal factors of operators that is used in our main theorems. The key term of orthogonal factors of operators is primarily modeled on projectional factors from [17].

Definition 3.3.1. Let X, Y be subspaces of some spaces $L_p(\mu)$ and $L_p(\nu)$, respectively, and assume that Y is a C-orthogonally complemented. Let $R\colon X \to X$ and $S\colon Y \to Y$ be bounded linear operators. For $\epsilon > 0$, we say that R is an orthogonal factor of S with error ϵ if there exists a distributional embedding $T\colon Y \to L_p(\mu)$ with $T(Y) \subset X$ such that

$$\|T^\dagger R T - S\| \leqslant \epsilon.$$

If the above holds for $\epsilon = 0$, then we call R an orthogonal factor of S.

The following remark highlights the relation between orthogonal factors and projectional factors. In Definition 3.1.1 (b), we introduced the notion of a projectional factor, which is a special case of the more general concept of a projectional factor with error $\varepsilon > 0$. Let $K > 0$, and let $T, S : X \to X$

be bounded linear operators on a Banach space X. We say that T is a K-projectional factor of S with error $\epsilon > 0$ if there exist bounded linear operators $A, B : X \to X$ such that $\|ATB - S\| < \epsilon$ and $AB = id : X \to X$ and $\|A\|\|B\| \leqslant K$.

Remark 3.3.2. Let $C \geqslant 1$ and $\epsilon > 0$. Let X be a C-orthogonally complemented subspace of a space $L_p(\mu)$. Moreover, let $T, S \colon X \to X$ be bounded linear operators. If T is an orthogonal factor of S with error $\epsilon > 0$, then T is a C-projectional factor of S with error $\epsilon > 0$. Note that C is the orthogonal complementation constant of X.

When R is an orthogonal factor of S, this should be understood as a reduction of R to S, which in applications is typically the simpler operator. In the following proposition we prove that being an approximate orthogonal factor is a transitive property. This follows from Remark 3.2.10.

Proposition 3.3.3. Let X, Y, and Z be subspaces of some L_p-spaces, and assume that Y and Z are C and D-orthogonally complemented. Let $R : X \to X$, $S : Y \to Y$, and $U : Z \to Z$ be bounded linear operators, and assume that R is an orthogonal factor of S with error ϵ and S is an orthogonal factor of U with error δ. Then R is an orthogonal factor of U with error $D\epsilon + \delta$.

Proof. From the assumptions we have

$$\|T_1^\dagger R T_1 - S\| \leqslant \epsilon \tag{3.1}$$

and

$$\|T_2^\dagger S T_2 - U\| \leqslant \delta, \tag{3.2}$$

where $T_1 \colon Y \to X$ and $T_2 \colon Z \to Y$ are distributional embeddings. Denote $T = T_1 T_2$. From Remark 3.2.10 , (3.1) and (3.2) we obtain that

$$\begin{aligned}
\|T^\dagger R T - U\| &\leqslant \|T_2^\dagger T_1^\dagger R T_1 T_2 - T_2^\dagger S T_2\| + \|T_2^\dagger S T_2 - U\| \\
&\leqslant \|T_2^\dagger\| \|T_1^\dagger R T_1 - S\| + \|T_2^\dagger S T_2 - U\| \\
&\leqslant D\epsilon + \delta
\end{aligned}$$

Therefore, R is an orthogonal factor of U with error $D\epsilon + \delta$. □

Remark 3.3.4. By Proposition 3.2.8 (a), being an orthogonal factor showcases symmetry around the identity operator, which means the following: Let X, Y be subspaces of some L_p-spaces, and assume that Y is C-orthogonally complemented. Let $R \colon X \to X$ and $S \colon Y \to Y$ be bounded linear operators. If R is an orthogonal factor of S with error ϵ, then $id - R : X \to X$ is an orthogonal factor of $id - S : Y \to Y$ with the same error ϵ.

We state a general result on the factorization property of orthogonally complemented symmetric three-valued martingale difference sequences, formulated so as to apply in our context. In view of

Theorems 6.2.6 and 10.2.1, it yields the factorization property of the standard basis of the limit spaces $R^{p,0}_\alpha$, that is, Theorems 6.2.7 and 10.2.3.

Proposition 3.3.5. Let $X = [(f_n)_{n=1}^\infty] \leqslant L_p(\mu)$, where $(f_n)_{n=1}^\infty$ is a $\{-1,0,1\}$-valued symmetric martingale difference sequence and let $C \geqslant 1$. Assume that

(a) X is C-orthogonally complemented, and

(b) for every $\eta > 0$ and bounded linear operator $T\colon X \to X$, there exists a diagonal operator $R\colon X \to X$ with diagonal entries in $\mathcal{A}(\mathcal{D}(T))$ such that T is an orthogonal factor of R with error η.

Then, for every $0 < \epsilon < 1$, every operator T with δ-large diagonal, i.e.,

$$\inf_{n\in\mathbb{N}} \left| \|f_n\|_2^{-2} \langle f_n, T f_n\rangle \right| \geqslant \delta,$$

is a $C(p^*-1)(1-\epsilon)^{-1}\delta^{-1}$-factor of the identity. In particular, the basis $(f_n)_{n=1}^\infty$ of X has the factorization property.

Proof. Define the multiplication operator $M : X \to X$ given by

$$f_n \mapsto \left(\|f_n\|_2^{-2} \langle f_n, T f_n\rangle \right)^{-1} f_n, \quad n \in \mathbb{N}.$$

Because every martigale difference sequence in L_p is (p^*-1)-unconditional, $\|M\| \leqslant (p^*-1)\delta^{-1}$ and observe that the diagonal entries of TM are all equal to one. From the assumptions, we have that there exists a diagonal operator $R\colon X \to X$ with diagonal entries one, i.e. $R = \mathrm{id}$, such that TM is an orthogonal factor of id with error ϵ. Thus, there exists a distributional embedding $J\colon X \to L_p(\mu)$, with $J(X) \subset X$, such that

$$\|J^\dagger TMJ - id\| < \epsilon.$$

Thus, the operator $J^\dagger TMJ$ is invertible with $\left\| \left(J^\dagger TMJ \right)^{-1} \right\| \leqslant (1-\epsilon)^{-1}$. Now, letting

$$L = \left(J^\dagger TMJ \right)^{-1} J^\dagger \text{ and } R' = MJ$$

we have $LTR' = id$ and

$$\|L\| \cdot \|R'\| \leqslant \frac{C(p^*-1)}{(1-\epsilon)\delta}.$$

□

We state a general result on the primary factorization property of orthogonally complemented spaces. In view of Theorem 6.3.5, it implies that $R^{p,0}_\omega$ has the primary factorization property, that is, Theorem 6.3.6.

Proposition 3.3.6. Let $C \geqslant 1$ and let X be a C-orthogonally complemented subspace of $L_p(\mu)$. Assume that for every $\eta > 0$ and operator $T \colon X \to X$, there exists a scalar λ such that T is an orthogonal factor of $\lambda \cdot id$ with error η. Then, for every $0 < \epsilon < 1$, either T or $id - T$ is a $\frac{2C}{1-\epsilon}$-factor of the identity. In particular, X has the primary factorization property.

Proof. Let $0 < \epsilon < 1$. From the assumptions, we have that there exists a scalar λ such that T is an orthogonal factor of $\lambda \cdot id : X \to X$ with error $\epsilon/2$. Thus, there exists a distributional embedding $J : X \to L_p(\mu)$ with $J(X) \subset X$ such that

$$\|J^\dagger T J - \lambda \cdot id\| < \epsilon/2.$$

If $|\lambda| \geqslant 1/2$, then

$$\|\lambda^{-1} J^\dagger T J - id\| < \epsilon.$$

Hence, the operator $\lambda^{-1} J^\dagger T J$ is invertible with $\left\| \left(\lambda^{-1} J^\dagger T J\right)^{-1} \right\| \leqslant (1-\epsilon)^{-1}$. If

$$L = (\lambda^{-1} J^\dagger T J)^{-1} \lambda^{-1} J^\dagger \quad \text{and} \quad R = J$$

then $LTR = id$ and

$$\|L\|\|R\| \leqslant \frac{2C}{1-\epsilon}$$

Now, if $|\lambda| < 1/2$, then from Remark 3.3.4 we have

$$\|J^\dagger (id - T) J - (1-\lambda) \cdot id\| < \epsilon/2.$$

Hence, we achieve the same conclusion for $id - T$ instead of T. $\square$

Chapter 4

Factorizing in the Rosenthal $X_{p,w}$ spaces

In this chapter, the main goal is to show that the Rosenthal $X_{p,w}$ spaces with their natural bases have the factorization property. We recall the definition of the $X_{p,w}$ spaces and some of their properties that will be used in the main theorem of this chapter. Next, we recall the notion of a strategically reproducible basis, which characterizes the factorization property of a basis, and we conclude by proving that the Rosenthal $X_{p,w}$ spaces have the factorization property.

4.1 Definition of the $X_{p,w}$ spaces

In this section, we recall the definition of the $X_{p,w}$ spaces and we present some of their basic properties. Moreover, we show that the Rosenthal $X_{p,w}$ space is isomorphic to an orthogonally complemented subspace of L_p.

Definition 4.1.1. Let $2 < p < \infty$ and $w = (w_n)_{n=1}^\infty$ be a sequence of positive real numbers. Define

$$X_{p,w} = \{(x_n)_{n=1}^\infty \in \mathbb{R}^{\mathbb{N}} : (x_n)_{n=1}^\infty \in \ell_p \text{ and } (x_n w_n)_{n=1}^\infty \in \ell_2\}.$$

The space $X_{p,w}$ equipped with the norm

$$\|(x_n)_{n=1}^\infty\|_{X_{p,w}} = \max\{\|(x_n)_{n=1}^\infty\|_p, \|(x_n w_n)_{n=1}^\infty\|_2\},$$

admits the sequence $(e_n)_{n=1}^\infty$ as an unconditional basis. Recall that the n-th coordinate of $(e_n)_{n=1}^\infty$ is 1 and all other coordinates are 0.

In the following definition, we assign a name to sequences w that vanish at a slow rate.

Definition 4.1.2. Let $2 < p < \infty$ and $w = (w_n)_{n=1}^\infty$ be a sequence of positive real numbers. We say that w satisfies the $(*)$ property if

$$w_n \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} w_n^{2p/(p-2)} = \infty.$$

We focus on such sequences throughout this chapter. The following result implies that all $X_{p,w}$ spaces, with w as specified above, have the same linear and topological structure. This can be found in [24, Theorem 13].

Theorem 4.1.3. Let $2 < p < \infty$ and w, w' be sequences of positive reals satisfying the $(*)$ property. Then $X_{p,w}$ is isomorphic to $X_{p,w'}$.

Based on the above theorem we recall the following definition.

Definition 4.1.4. Let $2 < p < \infty$. The Rosenthal X_p space is an $X_{p,w}$ space, where w satisfies the $(*)$ property.

It is proved in [24, Corollary, page 282] that for $2 < p < \infty$, the Rosenthal X_p space is isomorphic to a complemented subspace of L_p. Additionally, we have the following.

Theorem 4.1.5. Let $2 < p < \infty$. The Rosenthal X_p space is isomorphic to a $7.35(p/\log(p))$-orthogonally complemented subspace of L_p.

This is not stated in [24], but it can be easily deduced as follows: The Rosenthal X_p space is isomorphic to the closed linear span of a 3-valued symmetric independent sequence $(f_n)_{n=1}^{\infty}$ in L_p and this closed linear span is $7.35(p/\log(p))$-orthogonally complemented subspace of L_p because the restriction of the orthogonal projection $P\colon L_2 \to L_2$, defined by

$$P(f) = \sum_{n=1}^{\infty} \|f_n\|_2^{-2} \langle f_n, f \rangle f_n,$$

to L_p is a projection onto the closed linear span of $(f_n)_{n=1}^{\infty}$ in L_p of norm at most K_p ([24, Theorem 4]). The constant K_p is at most $7.35(p/\log(p))$ from [9, Theorem 4.1].

The following can be found in [24, Lemma 7]. This will be used in the main theorem of this chapter, that is Theorem 4.3.1.

Lemma 4.1.6. Let $2 < p < \infty$ and w be a sequence of positive reals satisfying the $(*)$ property. Let $(E_j)_{j=1}^{\infty}$ be a sequence of disjoint finite subsets of the positive integers. For each j, put

$$f_j = \sum_{n \in E_j} w_n^{2/(p-2)} e_n,$$

$$\beta_j = \left(\sum_{n \in E_j} w_n^{2p/(p-2)} \right)^{(p-2)/2p}$$

and

$$\tilde{f}_j = \|f_j\|_p^{-1} f_j.$$

Let Y denote the closed linear span of $(\tilde{f}_j)_{j=1}^{\infty}$ in $X_{p,w}$. Then $(\tilde{f}_j)_{n=1}^{\infty}$ is an unconditional basis of Y, isometrically equivalent to the natural basis of $X_{p,(\beta_j)}$ and Y is a 1-complemented subspace of $X_{p,w}$ with projection defined by

$$P(x) = \sum_{j=1}^{\infty} \|f_j\|_p \|f_j\|_2^{-2} \langle f_j, x\rangle \tilde{f}_j.$$

4.2 Strategically reproducible bases

In this section, we recall the notion of a strategically reproducible basis, which will be used to deduce the factorization property for a basis of a Banach space.

The following two definitions are from [16]. See Remark 4.2.3 below for an intuitive explanation of them.

Definition 4.2.1. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be basic sequences in (possibly different) Banach spaces and $C \geqslant 1$. We say that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are impartially C-equivalent if for any finite choice of scalars $(a_n)_{n=1}^{\infty} \in c_{00}$ we have

$$\frac{1}{\sqrt{C}} \| \sum_{n=1}^{\infty} a_n y_n \| \leqslant \| \sum_{n=1}^{\infty} a_n x_n \| \leqslant \sqrt{C} \| \sum_{n=1}^{\infty} a_n y_n \|.$$

Definition 4.2.2. Assume that X is a Banach space with a basis $(e_n)_{n=1}^{\infty}$ which is unconditional and shrinking. Let $(e_n^*)_{n=1}^{\infty} \subset X^*$ be the biorthogonal sequence associated with the basis $(e_n)_{n=1}^{\infty}$. We say that $(e_n)_{n=1}^{\infty}$ is strategically reproducible if the following condition is satisfied for some $C \geqslant 1$:

$$\begin{array}{lll}
\forall n_1 \in \mathbb{N} & \exists b_1 \in \langle e_n : n \geqslant n_1 \rangle, & \exists b_1^* \in \langle e_n^* : n \geqslant n_1 \rangle \\
\forall n_2 \in \mathbb{N} & \exists b_2 \in \langle e_n : n \geqslant n_2 \rangle, & \exists b_2^* \in \langle e_n^* : n \geqslant n_2 \rangle \\
\forall n_3 \in \mathbb{N} & \exists b_3 \in \langle e_n : n \geqslant n_3 \rangle, & \exists b_3^* \in \langle e_n^* : n \geqslant n_3 \rangle \\
 & \vdots &
\end{array}$$

such that

(a) $(b_k)_{k=1}^{\infty}$ is impartially C-equivalent to $(e_k)_{k=1}^{\infty}$,

(b) $(b_k^*)_{k=1}^{\infty}$ is impartially C-equivalent to $(e_k^*)_{k=1}^{\infty}$, and

(c) $b_k^*(b_l) = \delta_{k,l} \;\; \forall k, l \in \mathbb{N}$.

Remark 4.2.3. The condition in Definition 4.2.2 can be interpreted as player (II) having a winning strategy in a two-player game:
We fix $C \geqslant 1$. Player (I) chooses $n_1 \in \mathbb{N}$, then player (II) chooses

$$b_1 \in \langle e_n : n \geqslant n_1 \rangle \quad \text{and} \quad b_1^* \in \langle e_n^* : n \geqslant n_1 \rangle.$$

They repeat the moves infinitely many times, obtaining, for every $k \in \mathbb{N}$, numbers n_k, and vectors b_k and b_k^*. Player (II) wins if they were able to choose the sequences $(b_k)_{k=1}^{\infty}$ in X and $(b_k^*)_{k=1}^{\infty}$ in X^* such that (a), (b) and (c) are satisfied. Thus, the basis $(e_n)_{n=1}^{\infty}$ is strategically reproducible if and only if for some $C \geqslant 1$ player (II) has a winning strategy.

The following theorem will be used inside the proof of the main theorem of the following section, Theorem 4.3.1, and it can be found in [16, Theorem 3.12].

Theorem 4.2.4. Let X be a Banach space with an unconditional and shrinking basis $(e_n)_{n=1}^{\infty}$. If the basis $(e_n)_{n=1}^{\infty}$ is strategically reproducible, then the basis has the factorization property.

4.3 The factorization property of the basis of the $X_{p,w}$ space

In this section, we prove the main theorem of this chapter, that is the natural basis of $X_{p,w}$ has the factorization property.

Theorem 4.3.1. Let $2 < p < \infty$. For every w satisfying the $(*)$ property, the basis $(e_n)_{n=1}^{\infty}$ of the $X_{p,w}$ space has the factorization property.

Proof. From Theorem 4.2.4 it is enough to show that the basis $(e_n)_{n=1}^{\infty}$ of the $X_{p,w}$ space is strategically reproducible for some $C \geqslant 1$. In the two-player game of Remark 4.2.3 we assume the role of player (II).

Let $\epsilon > 0$ and let $C = 1 + \epsilon$. In each round, after player (I) chooses $n_k \in \mathbb{N}$, we choose a finite subset E_k of the positive integers such that $\min(E_k) > n_k$, $\max(E_{k-1}) < \min(E_k)$, where $E_0 = \varnothing$, and such that putting

$$\beta_k = \left(\sum_{n \in E_k} w_n^{2p/(p-2)} \right)^{(p-2)/2p}$$

we have

$$w_k \leqslant \beta_k \leqslant \sqrt{1+\epsilon}\, w_k.$$

Now, put

$$b_k = \sum_{n \in E_k} w_n^{2/(p-2)} e_n,$$

$$\tilde{b}_k = \|b_k\|_p^{-1} b_k$$

and

$$b_k^* = \sum_{n \in E_k} \|b_n\|_p \|b_n\|_2^{-2} w_n^{2/(p-2)} e_n^*.$$

We show that this choice satisfies Definition 4.2.2. We observe that $\tilde{b}_k \in \langle e_n : n \geqslant n_k \rangle$ and $b_k^* \in \langle e_n^* : n \geqslant n_k \rangle$. Moreover, put $\beta = (\beta_k)_{k=1}^{\infty}$. From Lemma 4.1.6 we have that $(\tilde{b}_k)_{k=1}^{\infty}$ is isometrically equivalent to the usual basis of the $X_{p,\beta}$ space. Now, we will show that the usual basis of $X_{p,\beta}$ is impartially $(1+\epsilon)$-equivalent to the usual basis for $X_{p,w}$.

Indeed, let $a_1, ..., a_n \in \mathbb{R}$. We have

$$\|\sum_{k=1}^{n} a_k e_k\|_{X_{p,\beta}} = \max\{\|\sum_{k=1}^{n} a_k e_k\|_p, \|\sum_{k=1}^{n} a_k \beta_k e_k\|_2\}$$

and

$$\|\sum_{k=1}^{n} a_k e_k\|_{X_{p,w}} = \max\{\|\sum_{k=1}^{n} a_k e_k\|_p, \|\sum_{k=1}^{n} a_k w_k e_k\|_2\}.$$

Since for every $k \in \mathbb{N}$, $w_k \leqslant \beta_k \leqslant \sqrt{1+\epsilon} w_k$, we obtain

$$\|\sum_{k=1}^{n} a_k \beta_k e_k\|_2 \leqslant \sqrt{1+\epsilon} \|\sum_{k=1}^{n} a_k w_k e_k\|_2$$

and

$$\|\sum_{k=1}^{n} a_k w_k e_k\|_2 \leqslant \|\sum_{k=1}^{n} a_k \beta_k e_k\|_2.$$

Hence,

$$\|\sum_{k=1}^{n} a_k e_k\|_{X_{p,w}} \leqslant \|\sum_{k=1}^{n} a_k e_k\|_{X_{p,\beta}} \leqslant \sqrt{1+\epsilon} \|\sum_{k=1}^{n} a_k e_k\|_{X_{p,w}}.$$

Thus, $(\tilde{b}_k)_{k=1}^{\infty}$ is impartially $(1+\epsilon)$-equivalent to the usual basis for $X_{p,w}$. Let Y be the closed linear span of the sequence $(\tilde{b}_k)_{k=1}^{\infty}$ in $X_{p,w}$. From Lemma 4.1.6 we know that Y is 1-complemented subspace of $X_{p,w}$ with

$$P(x) = \sum_{k=1}^{\infty} \|b_k\|_p \|b_k\|_2^{-2} \langle b_k, x \rangle \tilde{b}_k$$

as the norm-1 projection onto Y. Since $(\tilde{b}_k)_{k=1}^{\infty}$ is impartially $(1+\epsilon)$-equivalent to the usual basis for $X_{p,w}$, there exists an isomorphism $T\colon Y \to X_{p,w}$ with $T(\tilde{b}_k) = e_k$ and $\|T\| \leqslant \sqrt{1+\epsilon}$. We observe that $b_k^* = e_k^* TP$ and $b_k^*(\tilde{b}_l) = \delta_{k,l}$. Now, it remains to prove that $(b_k^*)_{k=1}^\infty$ is impartially $(1+\epsilon)$-equivalent to $(e_k^*)_{k=1}^\infty$.

Indeed, let $a_1, ..., a_n \in \mathbb{R}$. We observe that

$$\|\sum_{k=1}^n a_k b_k^*\| = \|\sum_{k=1}^n a_k e_k^* TP\| = \|(TP)^* \left(\sum_{k=1}^n a_k e_k^*\right)\| \leqslant \sqrt{1+\epsilon}\|\sum_{k=1}^n a_k e_k^*\|.$$

For the other inequality, we have

$$\|\sum_{k=1}^n a_k e_k^*\| = \left(\sum_{k=1}^n a_k e_k^*\right)(x),$$

where x belongs to the unit sphere of $X_{p,w}$ and $x = \sum_{k=1}^\infty \lambda_k e_k$. We have

$$\|\sum_{k=1}^n a_k e_k^*\| = \sum_{k=1}^n a_k \lambda_k = \left(\sum_{k=1}^n a_k b_k^*\right)\left(\sum_{k=1}^n \lambda_k \tilde{b_k}\right) \leqslant \sqrt{1+\epsilon}\|\sum_{k=1}^n a_k b_k^*\|.$$

□

It is easy to see that a reflexive Banach space with a basis (e_n) has the factorization property if and only if the dual space X^* with the biorthogonal sequence (e_n^*) has the factorization property. For $1 < q < 2$, Rosenthal (see [24, page 280]) defines $X_{q,w}$ to be the dual space $X_{p,w}^*$ of $X_{p,w}$. Therefore, we have the following.

Theorem 4.3.2. For $1 < q < 2$ and w satisfying the $(*)$ property with respect to p, the Rosenthal $X_{q,w}$ space with the biorthogonal sequence associated with the natural basis of the $X_{p,w}$ space has the factorization property.

From the paragraph below Theorem 4.1.5, it follows that, for $1 < q < 2$, the Rosenthal $X_{q,w}$ space is isomorphic to the closed linear span in L_q of an independent sequence of 3-valued symmetric random variables $(f_n)_{n=1}^\infty$.

Chapter 5

The Bourgain-Rosenthal-Schechtman R_α^p spaces

In this chapter, we recall the original definition of the Bourgain-Rosenthal-Schechtman spaces R_α^p and we observe that they are orthogonally complemented spaces, giving explicitly the steps even though it was established implicitly in [4]. Next, we introduce an equivalent definition of the R_α^p spaces as subspaces of $L_p[0,1]$, providing clearer intuition and insight into this class of spaces, used throughout the rest of the thesis. We conclude with the description of the spaces $R_\alpha^{p,0}$, that is the spaces of all mean zero random variable of R_α^p, and we observe that they enjoy the orthogonal complementation.

5.1 The R_α^p spaces

In this section, we recall the original definition of the Bourgain-Rosenthal-Schechtman spaces R_α^p and their orthogonal complementation from [4]. Each space R_α^p was originally defined as a subspace of $L_p(\{0,1\}^{T_\alpha})$, for some countable index set T_α depending on α. While we briefly revisit this approach here, in the next section we present an equivalent formulation in which these spaces are realized as subspaces of $L_p = L_p[0,1]$.

In the following definition we recall the notions of the disjoint sum and the independent sum.

Definition 5.1.1.

(a) For a subspace B of $L_p(\Omega, \mathcal{A}, \mu)$, the disjoint sum $(B \oplus B)_p$ of B is the subspace

$$\Big\{ b(\omega,\epsilon) \in L_p(\Omega \times \{0,1\}) : \text{there exists } b_\epsilon \in B \text{ with } b(\omega,\epsilon) = b_\epsilon(\omega) \\ \text{for all } \omega \in \Omega,\ \epsilon = 0 \text{ or } 1 \Big\},$$

of $L_p(\Omega \times \{0,1\})$, where $\{0,1\}$ is equipped with the uniform probability measure.

(b) For every $n \in \mathbb{N}$, let B_n be a subspace of $L_p(\Omega_n, \mathcal{A}_n, \mu_n)$. We define the independent sum of

B_n as follows. For every n, let

$$\overline{B}_n = \left\{ b \in L_p \left(\prod_k \Omega_k \right) : \text{ there exists } f \in B_n \text{ with } b(\omega) = f(\omega_n) \text{ for all } \omega \in \prod_k \Omega_k \right\},$$

in $L_p\left(\prod_n \Omega_n\right)$. The subspace $\overline{B}_n$ is a distributional copy of B_n that depends only on the n-th coordinate. Then the independent sum of $(B_n)_{n=1}^{\infty}$ is defined as

$$\Big(\sum_n B_n\Big)_{\mathrm{Ind},p} = [\bigcup_n \overline{B}_n],$$

in $L_p\left(\prod_n \Omega_n\right)$. Independent sums can be taken over any countable collection of spaces, as their enumeration does not matter.

The spaces R_α^p are defined recursively on the countable ordinals. The first Bourgain-Rosenthal-Schechtman space is the space of constant functions, while the successor and limit steps are carried out using the disjoint and independent sums, respectively.

Definition 5.1.2. Define $R_0^p = \langle \chi_{\{0,1\}} \rangle$ in $L_p\{0,1\}$. Let α be an ordinal with $0 < \alpha < \omega_1$. Suppose that we have defined the R_β^p space for all $\beta < \alpha$.

(a) If $\alpha = \beta + 1$, then $R_\alpha^p = (R_\beta^p \oplus R_\beta^p)_p$.

(b) If α is a limit ordinal, then $R_\alpha^p = (\sum_{\beta<\alpha} R_\beta^p)_{\mathrm{Ind},p}$.

The orthogonal complementation of the R_α^p spaces is established implicitly in [4], with the steps made explicit in [11, Theorem 4.6].

Theorem 5.1.3. Let $\alpha < \omega_1$. The R_α^p space is $(p^* - 1)^2 (p^*/2)^{3/2}$-orthogonally complemented.

Proof. The proof is based on technical details from [4]. We give a description that avoids some of them, but the unfamiliar reader is invited to skip this proof. For $1 < p < \infty$, Bourgain, Rosenthal, and Schechtman define an explicit auxiliary subspace $X_{\mathcal{D}}^p$ of $L_p\{0,1\}^{\mathcal{D}}$, for some countable set $\mathcal{D}$. The specific form of $\mathcal{D}$ and $X_{\mathcal{D}}^p$ is not relevant to this proof, but, for context, we mention it is the closed linear span, with respect to $\|\cdot\|_p$, of a vector space X of measurable functions that is the same for all values of p. For more details on $X_{\mathcal{D}}^p$, the interested reader may refer to [4, Section 3]. According to [4, Theorem 3.1], $X_{\mathcal{D}}^p$ is complemented in $L_p\{0,1\}^{\mathcal{D}}$. More precisely, it is stated in [4, page 222, paragraph above equation 3.8] that the projection P_0 onto $X_{\mathcal{D}}^p$ is in fact orthogonal. A precise formula for P_0 is given, and it is independent of p. The estimate $\|P_0\| \leqslant K_p^2 (p/2)^{3/2}$ (where $K_p = (p-1)$ is the constant in Burkholder's inequality, see, e.g. [23, Theorem 8.6]) is given in [4, bottom of page 222] only for the case $p > 2$. However, the "selfadjointness" of P_0 yields that the same estimate is true for $1 < p < 2$ replacing p with p^*, i.e., $\|P_0\| \leqslant (K_{p*})^2 (p^*/2)^{3/2}$, for $1 < p < \infty$.

Fixing $\alpha < \omega_1$, it is shown in [4, Lemma 3.9] that there exists a distributional copy $X_{T_\alpha}^p$ of R_α^p in $X_{\mathcal{D}}^p$. In [4, line 1 of proof of Lemma 3.6] it is stated that $X_{T_\alpha}^p$ is complemented in $X_{\mathcal{D}}^p$ via a conditional expectation operator $\mathbb{E}_\alpha$. Therefore, $\mathbb{E}_\alpha P_0$ is a p-C-bounded orthogonal projection on $L_p\{0,1\}^{\mathcal{D}}$ onto $X^p_{T_\alpha}$, for $C = (K_{p^*})^2(p^*/2)^{3/2}$. Therefore, R^p_α is $(p^*-1)^2(\frac{p^*}{2})^{3/2}$-orthogonally complemented. □

5.2 The R^p_α spaces as subspaces of $L_p[0,1]$

In this section, we give an equivalent definition of the R^p_α spaces as subspaces of $L_p = L_p[0,1]$. In this framework, disjoint sums are described in terms of compressions with disjointly supported ranges, and independent sums in terms of images under distributional embeddings with independent ranges.

The notion of a θ-compression is central to this work and will be used repeatedly throughout. It is essentially the same as the classical θ-dilation associated with the Boyd indices of a rearrangement-invariant space. However, contrary to standard convention, we adopt the term "compression" since we restrict to values $\theta \in (0,1]$.

Definition 5.2.1. Let $1 < p < \infty$, $0 < \theta \leqslant 1$ and X, Y be subspaces of L_p. A linear operator $T\colon X \to Y$ is called a θ-compression if for every $f \in X$ and $A \in \mathcal{B}(\mathbb{R})$,

$$|[Tf \in A]| = \theta|[f \in A]| + (1-\theta)\chi_A(0),$$

i.e., $\mathrm{dist}(Tf) = \theta\mathrm{dist}(f) + (1-\theta)\delta_0$.

Remark 5.2.2. Let $0 < \theta \leqslant 1$, $0 < \eta \leqslant 1$, and X, Y, Z be subspaces of L_p.

(a) If $T\colon X \to Y$ is a θ-compression and $S\colon Y \to Z$ is an η-compression, then $TS\colon X \to Z$ is a $\theta\eta$-compression.

(b) If $T\colon X \to Y$ is a θ-compression, then for every $f \in X$ we have

$$\|Tf\|_p = \theta^{1/p}\|f\|_p.$$

(c) If $T\colon X \to Y$ is a 1-compression, then T is a distributional embedding.

The following piece of notation will be used extensively to define and study the Bourgain-Rosenthal-Schechtman spaces as subspace of $L_p = L_p[0,1]$.

Notation 5.2.3. Let $0 \leqslant a < b \leqslant 1$ and $I = [a,b)$. Let $T_I\colon L_p \to L_p$ be defined by

$$T_I(f)(t) = \begin{cases} f(\frac{t-a}{b-a}), & a \leqslant t < b \\ 0, & \text{otherwise.} \end{cases}$$

Evidently, T_I is a $(b-a)$-compression.

Recall that set relations and operations should be interpreted in the essential sense.

Remark 5.2.4. Let X be a subspace of L_p, $0 < \theta \leqslant 1$, and $T : X \to L_p$ be a θ-compression. For $f, g \in X$, we have that

(a) $\operatorname{supp}(g) \subset \operatorname{supp}(f)$ if and only if $\operatorname{supp}(Tg) \subset \operatorname{supp}(Tf)$ and

(b) $\operatorname{supp}(g) \cap \operatorname{supp}(f) = \varnothing$ if and only if $\operatorname{supp}(Tg) \cap \operatorname{supp}(Tf) = \varnothing$.

The first assertion arises from the linearity of T and the characterization

$$\operatorname{supp}(g) \subset \operatorname{supp}(f) \text{ if and only if } \lim_{r\to\infty} \big|[rf + g \neq 0]\big| = |[f \neq 0]|.$$

The second assertion is simpler.

Notation 5.2.5. For every limit ordinal $\alpha < \omega_1$, we fix, for the rest of the thesis, a collection

$$T_\beta^\alpha \colon L_p \to L_p,\ 0 \leqslant \beta < \alpha$$

of distributional embeddings with independent ranges; that is, for any family $(f_\beta)_{\beta<\alpha}$ in L_p, the functions $(T_\beta^\alpha f_\beta)_{\beta<\alpha}$ are independent. For example, this can be achieved by fixing a collection $\varphi_\beta^\alpha \colon ([0,1], \mathcal{B}([0,1])) \to ([0,1], \mathcal{B}([0,1]))$, $\beta < \alpha$, of independent measure-preserving maps, and setting $T_\beta^\alpha(f) = f \circ \varphi_\beta^\alpha$, as in Example 2.4.4.

Using Notation 5.2.3 and 5.2.5, the spaces R_α^p can equivalently be defined as subspaces of L_p as follows.

Definition 5.2.6. Let $R_0^p = \langle \chi_{[0,1]} \rangle$, and, if α is a countable ordinal number such that, for all $0 \leqslant \beta < \alpha$, R_β^p has been defined then

(a) if $\alpha = \beta + 1$ let

$$R_\alpha^p = T_{[0,\frac{1}{2})}(R_\beta^p) + T_{[\frac{1}{2},1)}(R_\beta^p),$$

(b) and if α is a limit ordinal let

$$R_\alpha^p = \left[T_\beta^\alpha(R_\beta^p) : \beta < \alpha \right].$$

Henceforth, we will always work with this version of the Bourgain-Rosenthal-Schechtman spaces.

Remark 5.2.7. The choice of maps at each step does not matter: at successor steps one may use any 1/2-compressions with disjointly supported ranges, and at limit steps any distributional embeddings with independent ranges; the outcome is distributionally isomorphic.

Notation 5.2.8. Let $0 \leqslant \alpha < \omega_1$. Denote $R_\alpha^{p,0} = \{f \in R_\alpha^p : \mathbb{E}(f) = 0\}$.

For the purpose of this work we will mostly work with $R_\alpha^{p,0}$ instead of R_α^p, where $1 \leqslant \alpha < \omega_1$.

Remark 5.2.9. We point out the following recursive relations satisfied by the spaces $R_\alpha^{p,0}$.

(a) For $1 \leqslant \alpha < \omega_1$, for every $n \in \mathbb{N}$, by induction, we have that

$$R_{\alpha+n}^{p,0} = L_p^{n,0} + \langle T_I(R_\alpha^{p,0}) : I \in \mathcal{D}^n \rangle = \langle h_I : I \in \mathcal{D}_{n-1} \rangle + \langle T_I(R_\alpha^{p,0}) : I \in \mathcal{D}^n \rangle.$$

(b) For a limit ordinal α,

$$R_\alpha^{p,0} = [T_\beta^\alpha(R_\beta^{p,0}) : 1 \leqslant \beta < \alpha] \equiv^{\mathrm{dist}} \Big(\sum_{1 \leqslant \beta < \alpha} R_p^{\beta,0} \Big)_{\mathrm{Ind},p}.$$

Example 5.2.10. Let $n \in \mathbb{N}$.

(a) $R_n^{p,0} = L_p^{n,0}$.

(b) $R_\omega^{p,0} = \Big[\bigcup_{k \in \mathbb{N}} T_k^\omega(R_k^{p,0}) \Big]$.

(c) $R_{\omega+n}^{p,0} = \Big[L_p^{n,0} \cup \bigcup_{I \in \mathcal{D}^n} T_I(R_\omega^{p,0}) \Big] = \Big[\langle h_I : I \in \mathcal{D}_{n-1} \rangle \cup \bigcup_{I \in \mathcal{D}^n} T_I(R_\omega^{p,0}) \Big]$.

In the following remark we observe that the $R_\alpha^{p,0}$ space is orthogonally complemented.

Remark 5.2.11. Let $\alpha < \omega_1$. By Theorem 5.1.3, R_α^p, and by Proposition 3.2.8 every distributional copy of it, is $(p^* - 1)^2(\frac{p^*}{2})^{3/2}$-orthogonally complemented. Let P denote the corresponding p-C-orthogonal projection, where $C = (p^* - 1)^2(\frac{p^*}{2})^{3/2}$. Since L_p^0 is 2-orthogonally complemented via $Q(f) = f - \mathbb{E}(f)$, it follows that $R_\alpha^{p,0}$ is $2C$-orthogonally complemented via QP.

Chapter 6

Factorizing in $R_\omega^{p,0}$

In this chapter, we establish the factorization and primary factorization property of $R_\omega^{p,0}$. We begin by introducing the standard martingale difference sequence basis of $R_\omega^{p,0}$. We then show that every bounded linear operator on $R_\omega^{p,0}$ can be orthogonally approximately reduced to a diagonal operator, which yields the factorization property of $R_\omega^{p,0}$ with respect to its natural basis. We conclude by deriving the primary factorization property of $R_\omega^{p,0}$ from the orthogonal reduction of every bounded linear operator on $R_\omega^{p,0}$ to a scalar multiple of the identity operator on $R_\omega^{p,0}$.

6.1 The basis of $R_\omega^{p,0}$

In this section, we present the natural martingale difference sequence basis of $R_\omega^{p,0}$. For the higher order countable ordinals, in Chapter 7, we develop a theory around the bases of the spaces $R_\alpha^{p,0}$ resulting in an explicit description of the martingale difference sequence bases of the $R_\alpha^{p,0}$ spaces. We conclude by establishing the relationship between the spaces $R_\omega^{p,0}$, R_ω^p and X_p.

Notation 6.1.1. From Example 5.2.10 (a) and (b) we have

$$R_\omega^{p,0} = \left[\bigcup_{n \in \mathbb{N}} T_n^\omega (L_p^{n,0}) \right].$$

Recall that $L_p^{n,0} = \langle h_I : I \in \mathcal{D}_{n-1} \rangle$, for all $n \in \mathbb{N}$, and for every $n \in \mathbb{N}$ and $I \in \mathcal{D}_{n-1}$ denote $h_I^n = T_n^\omega(h_I)$. Thus,

$$R_\omega^{p,0} = [h_I^n : I \in \mathcal{D}_{n-1}, n \in \mathbb{N}].$$

We will use the following linearly ordered set to enumerate the basis of $R_\omega^{p,0}$.

Notation 6.1.2. Denote

$$\mathcal{D}_\omega = \{(n, I) : n \in \mathbb{N}, I \in \mathcal{D}_{n-1}\},$$

and define the linear order on $\mathcal{D}_\omega$ as follows: $(m, J) \prec (n, I)$ if and only if $m < n$ or both $m = n$ and $J < I$ in the lexicographic order of $\mathcal{D}_{m-1}$. The set $\mathcal{D}_\omega$ can be visualized as disjoint copies of the sets $\mathcal{D}_{n-1}$, $n \in \mathbb{N}$. Thus,

$$R_\omega^{p,0} = [(h_I^n)_{(n,I)\in\mathcal{D}_\omega}].$$

The subscript ω for the set $\mathcal{D}_\omega$ indicates the space $R_\omega^{p,0}$. We now observe that the sequence $(h_I^n)_{(n,I)\in\mathcal{D}_\omega}$ is an unconditional basis of $R_\omega^{p,0}$.

Remark 6.1.3. The sequence $(h_I^n)_{(n,I)\in\mathcal{D}_\omega}$ described in Notation 6.1.1 is a martingale difference sequence with respect to the linear order of $\mathcal{D}_\omega$. This follows from the fact that $(h_I^n)_{I\in\mathcal{D}_{n-1}}$, $n \in \mathbb{N}$, are independent and for every $n \in \mathbb{N}$ we have that $(h_I^n)_{I\in\mathcal{D}_{n-1}}$ is a distributional copy of the martingale difference sequence basis $(h_I)_{I\in\mathcal{D}_{n-1}}$ of $L_p^{n,0}$. Therefore, $(h_I^n)_{(n,I)\in\mathcal{D}_\omega}$ is an unconditional basis of $R_\omega^{p,0}$ by the Burkholder inequality.

In the reminder of this section we focus on the relation between the spaces R_ω^p, $R_\omega^{p,0}$ and X_p. The following is proved in [2] (see the penultimate paragraph of page 11 until the first seven lines of page 12).

Proposition 6.1.4. The Bourgain-Rosenthal-Schechtman space $R_\omega^{p,0}$ and the Rosenthal space X_p are isomorphic.

Note that R_ω^p is isomorphic to its hyperplanes as an infinite-dimensional complemented subspace of L_p. In particular, $R_\omega^p \simeq R_\omega^{p,0} \simeq X_p$.

6.2 The orthogonal reduction to a diagonal operator

In this section, we approximately orthogonally reduce an arbitrary operator T on $R_\omega^{p,0}$ to a diagonal operator R on $R_\omega^{p,0}$, that is for every small error $\epsilon > 0$ we construct a distributional embedding $J : R_\omega^{p,0} \to L_p$ with $J(R_\omega^{p,0}) \subset R_\omega^{p,0}$ and a diagonal operator R on $R_\omega^{p,0}$ such that $\|J^\dagger T J - R\| < \epsilon$. This is based on probabilistic arguments first used in [14]. This reduction in combination with Proposition 3.3.5 yields the factorization property of the basis of $R_\omega^{p,0}$.

We describe now in broad strokes the general idea of the proof of Theorem 6.2.6, that is the orthogonal reduction to a diagonal operator. Let $T\colon R_\omega^{p,0} \to R_\omega^{p,0}$ be a bounded linear operator. Recall that

$$\mathcal{D}(T) = \{|I|^{-1}\langle h_I^n, T h_I^n\rangle : (n, I) \in \mathcal{D}_\omega\},$$

and $\mathcal{A}(\mathcal{D}(T))$ is the collection of all averages of finitely many elements of $\mathcal{D}(T)$.

Using an induction on the linear order of $\mathcal{D}_\omega$, for every $n \in \mathbb{N}$, we construct finite sets of dyadic intervals $(\mathcal{B}_I^n)_{I\in\mathcal{D}_{n-1}}$ and blocks $(b_I^n)_{I\in\mathcal{D}_{n-1}}$ of the basis $(h_I^n)_{I\in\mathcal{D}_{n-1}}$ of $T_n^\omega(L_p^{n,0})$ in some

$T^{\omega}_{N(n)}(L_p^{N(n),0})$, $N(n) > n$, with $b_I^n = \sum_{K \in \mathcal{B}_I^n} \theta_K h_K^{N(n)}$, where $(\theta_K)_{K \in \mathcal{B}_I^n} \in \{-1,1\}^{\mathcal{B}_I^n}$, such that

$$\begin{aligned}
\operatorname{supp}(b^n_{[0,1]}) &= [0,1] \\
\operatorname{supp}(b^n_{I^+}) &= [b_I^n = 1] \\
\operatorname{supp}(b^n_{I^-}) &= [b_I^n = -1],
\end{aligned}$$

for every $n \in \mathbb{N}$. From this we obtain that $(b_I^n)_{(n,I) \in \mathcal{D}_\omega}$ is a distributional copy of $(h_I^n)_{(n,I) \in \mathcal{D}_\omega}$. We construct the block sequence $(b_I^n)_{(n,I) \in \mathcal{D}_\omega}$ such that

$$|\langle b_J^m, T b_I^n \rangle| < \epsilon_{J,I}^{m,n},$$

for every $(m,J) \neq (n,I) \in \mathcal{D}_\omega$, and

$$|\langle b_I^n, T b_I^n \rangle - \sum_{K \in \mathcal{B}_I^n} \langle h_K^{N(n)}, T h_K^{N(n)} \rangle| < \epsilon_I^n,$$

for every $(n,I) \in \mathcal{D}_\omega$, where $\epsilon_{J,I}^{m,n}$ and ϵ_I^n are small errors. From this we obtain that T is an approximate orthogonal factor of a diagonal operator R, that is

$$\|J^\dagger T J - R\| < \epsilon,$$

where $J : R_\omega^{p,0} \to L_p$ with $J(R_\omega^{p,0}) \subset R_\omega^{p,0}$ is the distributional embedding defined by $J(h_I^n) = b_I^n$, and R is the diagonal operator on $R_\omega^{p,0}$ defined by

$$R(h_I^n) = \left(|I|^{-1} \sum_{K \in \mathcal{B}_I^n} \langle h_K^{N(n)}, T h_K^{N(n)} \rangle \right) h_I^n$$

and has diagonal element in $\mathcal{A}(\mathcal{D}(T))$.

In each inductive step, the choice of the block vector b_I^n is performed probabilistically, and for this purpose, we give the following notation.

Notation 6.2.1. Fix a finite collection of disjointly supported and symmetric $\{-1,0,1\}$-valued measurable functions $\mathcal{B} = (h_k)_{k=1}^m$. Denote

$$H_k = \operatorname{supp}(h_k), 1 \leqslant k \leqslant m, \text{ and } H = \cup_{k=1}^m H_k.$$

Considering the probability space $\Omega = \{-1,1\}^m$ equipped with the uniform probability measure, we define the random vector $b_{\mathcal{B}} : \Omega \to \langle \{h_k : 1 \leqslant k \leqslant m\} \rangle$ given by

$$b_{\mathcal{B}}(\vartheta) = \sum_{k=1}^m \theta_k h_k,$$

where $\vartheta = (\theta_k)_{k=1}^m \in \Omega$. For $g \in L_q$, $f \in L_p$, and a linear operator $T : \langle \{h_k : 1 \leqslant k \leqslant m\} \rangle \to L_p$, we define the random variables $Y_{\mathcal{B},g}, W_{\mathcal{B},f}, Z_{\mathcal{B},T} : \Omega \to \mathbb{R}$ given by

$$Y_{\mathcal{B},g}(\vartheta) = \langle g, b_{\mathcal{B}}(\vartheta)\rangle \text{ and } W_{\mathcal{B},f}(\vartheta) = \langle b_{\mathcal{B}}(\vartheta), f\rangle,$$

and

$$Z_{\mathcal{B},T}(\vartheta) = \langle b_{\mathcal{B}}(\vartheta), Tb_{\mathcal{B}}(\vartheta)\rangle - \sum_{k=1}^{m}\langle h_k, Th_k\rangle.$$

In the above notation, the random vector $b_{\mathcal{B}}$ will be used in the inductive step of the previously described proof to choose an appropriate vector b_I^n. The random variables $Y_{\mathcal{B},g}$, $W_{\mathcal{B},f}$, and $Z_{\mathcal{B},T}$ describe the interaction of this vector with previously constructed vectors via T, and they are employed to obtain the small errors $\epsilon_{J,I}^{m,n}$ and ϵ_I^n.

With regards to this notation, we formulated the statement in a general setting in order to derive estimates for the expectation and variance of more general random variables. We will use specific instances of Notation 6.2.1 in Lemmas 6.2.5 and 10.1.2

In Lemma 6.2.5, the general random variables h_k will be replaced by distributional copies of the Haar functions. We now establish estimates for the variance of the above random variables. These estimates will be used in the proof of Lemma 6.2.5.

Lemma 6.2.2. Following Notation 6.2.1, we have $\mathbb{E}(Y_{\mathcal{B},g}) = 0$ and

$$\mathbb{V}(Y_{\mathcal{B},g}) \leqslant \|g\|_q^2\Big(|H| \max_{1\leqslant k\leqslant m} |H_K|\Big)^{1/p}.$$

Proof. We have

$$Y_{\mathcal{B},g}(\vartheta) = \langle g, b_{\mathcal{B}}(\vartheta)\rangle = \sum_{k=1}^{m}\theta_k\langle g, h_k\rangle.$$

We follow the common convention of denoting by $\theta_k : \Omega \to \{-1, 1\}$ the random variable defined by $(\theta_i)_{i=1}^m \mapsto \theta_k$. Each random variable $\theta_k : \Omega \to \{-1, 1\}$ is of mean zero, and thus, we obtain that

$$\mathbb{E}(Y_{\mathcal{B},g}) = \sum_{k=1}^{m}\mathbb{E}(\theta_k)\langle g, h_k\rangle = 0.$$

Now, observe that

$$\mathbb{V}(Y_{\mathcal{B},g}) = \mathbb{E}(Y_{\mathcal{B},g}^2).$$

We have

$$Y_{\mathcal{B},g}^2(\vartheta) = \Big(\sum_{k=1}^{m}\theta_k\langle g, h_k\rangle\Big)\Big(\sum_{l=1}^{m}\theta_l\langle g, h_l\rangle\Big)$$

$$= \sum_{k,l=1}^{m} \theta_k \theta_l \langle g, h_k \rangle \langle g, h_l \rangle.$$

Now, taking the expectation we observe that

$$\mathbb{E}(Y_{\mathcal{B},g}^2) = \sum_{k,l=1}^{m} \mathbb{E}(\theta_k \theta_l) \langle g, h_k \rangle \langle g, h_l \rangle.$$

From the fact that the random variables $(\theta_k)_{k=1}^m$ are independent, and for every $1 \leqslant k \leqslant m$ we have $\theta_k^2 = 1$, we obtain that $\mathbb{E}(\theta_k \theta_l) = 0$, for all $k \neq l$, and $\mathbb{E}(\theta_k^2) = 1$. Hence,

$$\mathbb{V}(Y_{\mathcal{B},g}) = \sum_{k=1}^{m} \langle g, h_k \rangle^2.$$

Put $a_{g,k} = \langle g, h_k \rangle$ and we observe that $|a_{g,k}| \leqslant \|g\|_q |H_k|^{1/p}$. Therefore,

$$\begin{aligned}
\mathbb{V}(Y_{\mathcal{B},g}) &= \langle g, \sum_{k=1}^{m} a_{g,k} h_k \rangle \leqslant \|g\|_q \| \sum_{k=1}^{m} a_{g,k} h_k \|_p \\
&\leqslant \|g\|_q \max_{1 \leqslant k \leqslant m} |a_{g,k}| \| \sum_{k=1}^{m} h_k \|_p = \|g\|_q \max_{1 \leqslant k \leqslant m} |a_{g,k}| |H|^{1/p} \\
&\leqslant \|g\|_q^2 |H|^{1/p} \max_{1 \leqslant k \leqslant m} |H_k|^{1/p} = \|g\|_q^2 \Big(|H| \max_{1 \leqslant k \leqslant m} |H_k| \Big)^{1/p}.
\end{aligned}$$

□

We skip the proof of the following lemma as it is almost identical to that of Lemma 6.2.2.

Lemma 6.2.3. Following Notation 6.2.1, we have $\mathbb{E}(W_{\mathcal{B},f}) = 0$ and

$$\mathbb{V}(W_{\mathcal{B},f}) \leqslant \|f\|_p^2 \Big(|H| \max_{1 \leqslant k \leqslant m} |H_K| \Big)^{1/q}.$$

Lemma 6.2.4. Following Notation 6.2.1, we have $\mathbb{E}(Z_{\mathcal{B},T}) = 0$ and

$$\mathbb{V}(Z_{\mathcal{B},T}) \leqslant 2\|T\|^2 |H|^{1+1/p} \Big(\max_{1 \leqslant k \leqslant m} |H_k| \Big)^{1/q}.$$

Proof. We observe that

$$Z_{\mathcal{B},T}(\vartheta) = \sum_{\substack{k,l=1 \\ k \neq l}}^{m} \theta_k \theta_l \langle h_k, T h_l \rangle,$$

and thus,

$$\mathbb{E}(Z_{\mathcal{B},T}) = \sum_{\substack{k,l=1\\k\neq l}}^{m} \mathbb{E}(\theta_k\theta_l)\langle h_k, Th_l\rangle = 0.$$

Hence, $\mathbb{V}(Z_{\mathcal{B},T}) = \mathbb{E}(Z^2_{\mathcal{B},T})$. Now, we observe that

$$\begin{aligned} Z^2_{\mathcal{B},T}(\vartheta) &= \left(\sum_{\substack{k_1,l_1=1\\k_1\neq l_1}}^{m} \theta_{k_1}\theta_{l_1}\langle h_{k_1}, Th_{l_1}\rangle\right)\left(\sum_{\substack{k_2,l_2=1\\k_2\neq l_2}}^{m} \theta_{k_2}\theta_{l_2}\langle h_{k_2}, Th_{l_2}\rangle\right) \\ &= \sum_{\substack{k_1,l_1=1,k_2,l_2=1\\k_1\neq l_1,k_2\neq l_2}}^{m} \theta_{k_1}\theta_{l_1}\theta_{k_2}\theta_{l_2}\langle h_{k_1}, Th_{l_1}\rangle\langle h_{k_2}, Th_{l_2}\rangle. \end{aligned}$$

Now, taking the expectation we obtain that

$$\mathbb{E}(Z^2_{\mathcal{B},T}) = \sum_{\substack{k_1,l_1=1,k_2,l_2=1\\k_1\neq l_1,k_2\neq l_2}}^{m} \mathbb{E}(\theta_{k_1}\theta_{l_1}\theta_{k_2}\theta_{l_2})\langle h_{k_1}, Th_{l_1}\rangle\langle h_{k_2}, Th_{l_2}\rangle.$$

We have that $\mathbb{E}(\theta_{k_1}\theta_{l_1}\theta_{k_2}\theta_{l_2}) = 1$ if and only if $k_1 = k_2, l_1 = l_2$ and $k_1 = l_2, k_2 = l_1$, otherwise, $\mathbb{E}(\theta_{k_1}\theta_{l_1}\theta_{k_2}\theta_{l_2}) = 0$. Thus,

$$\mathbb{V}(Z_{\mathcal{B},T}) = \underbrace{\sum_{\substack{k,l=1\\k\neq l}}^{m} \langle h_k, Th_l\rangle\langle h_l, Th_k\rangle}_{=:A} + \underbrace{\sum_{\substack{k,l=1\\k\neq l}}^{m} \langle h_k, Th_l\rangle^2}_{=:B}.$$

We compute bounds for A and B separately:

$$\begin{aligned} A &= \sum_{l=1}^{m} \Big\langle h_l, T\Big(\sum_{\substack{k=1\\k\neq l}}^{m}\langle h_k, Th_l\rangle h_k\Big)\Big\rangle \leqslant \|T\| \sum_{l=1}^{m} \|h_l\|_q \Big\| \sum_{\substack{k=1\\k\neq l}}^{m}\langle h_k, Th_l\rangle h_k\Big\|_p \\ &\leqslant \|T\|^2 \sum_{l=1}^{m} \|h_l\|_q \max_{1\leqslant k\leqslant m} \big(\|h_k\|_q\|h_l\|_p\big)\Big\| \sum_{k=1}^{m} h_k\Big\|_p = \|T\|^2 \sum_{l=1}^{m} |H_l|^{1/q}|H_l|^{1/p}\big(\max_{1\leqslant k\leqslant m}|H_k|^{1/q}\big)|H|^{1/p} \\ &= \|T\|^2\Big(\max_{1\leqslant l\leqslant m}|H_l|\Big)^{1/q}|H|^{1+1/p} \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{k=1}^{m} \Big\langle h_k, T\Big(\sum_{\substack{l=1\\l\neq k}}^{m}\langle h_k, Th_l\rangle h_l\Big)\Big\rangle \leqslant \|T\| \sum_{k=1}^{m} \|h_k\|_q \Big\| \sum_{\substack{l=1\\l\neq k}}^{m}\langle h_k, Th_l\rangle h_l\Big\|_p \\ &\leqslant \|T\|^2 \sum_{k=1}^{m} \|h_k\|_q \max_{1\leqslant l\leqslant m} \big(\|h_k\|_q\|h_l\|_p\big)\Big\| \sum_{l=1}^{m} h_l\Big\|_p = \|T\|^2|H|^{1/p}\Big(\max_{1\leqslant l\leqslant m}|H_l|\Big)^{1/p} \sum_{k=1}^{m} |H_k|^{2/q} \end{aligned}$$

$$\leqslant \|T\|^2|H|^{1/p}\Big(\max_{1\leqslant l\leqslant m}|H_l|\Big)^{1/p}\Big(\max_{1\leqslant k\leqslant m}|H_k|\Big)^{2/q-1}|H| = \|T\|^2\Big(\max_{1\leqslant l\leqslant m}|H_l|\Big)^{1/q}|H|^{1+1/p}.$$

$\square$

In the following lemma, we present a specific instance of Notation 6.2.1 that fits into the proof of the main theorem of this section described above. Using the preceding estimates for the expectation and variance of the aforementioned random variables, we now determine a value for a random vector. This will be used in the inductive step of the proof of Theorem 6.2.6.

Lemma 6.2.5. Let $T\colon R^{p,0}_\omega \to R^{p,0}_\omega$ be a bounded linear operator. Let $(n, I) \in \mathcal{D}_\omega$, $M, N \in \mathbb{N}$ with $M \geqslant N$ and $\mathcal{B} \subseteq \mathcal{D}^N$ with $|\cup \mathcal{B}| = |I|$. Let $f_{(m,J)} \in R^{p,0}_\omega$ and $g_{(m,J)} \in L_q$, where $(m, J) \prec (n, I)$ such that

$$\|f_{(m,J)}\| \leqslant |J|^{1/p}$$

and

$$\|g_{(m,J)}\| \leqslant |J|^{1/q}.$$

For $\vartheta = (\theta_K)_{K\in\mathcal{B}} \in \{-1, 1\}^{\mathcal{B}}$ we define the random vector

$$b_{\mathcal{B},M}(\vartheta) = \sum_{K\in\mathcal{B}} \theta_K h^M_K,$$

and the random variables

$$Z(\vartheta) = \langle b_{\mathcal{B},M}(\vartheta), Tb_{\mathcal{B},M}(\vartheta)\rangle - \sum_{K\in\mathcal{B}} \langle h^M_K, Th^M_K\rangle,$$

$$Y_{(m,J)}(\vartheta) = \langle T^*g_{(m,J)}, b_{\mathcal{B},M}(\vartheta)\rangle, \quad (m, J) \prec (n, I)$$

and

$$W_{(m,J)}(\vartheta) = \langle b_{\mathcal{B},M}(\vartheta), Tf_{(m,J)}\rangle, \quad (m, J) \prec (n, I).$$

Let $\eta > 0$ and $\eta_{(m,J)} > 0$, where $(m, J) \prec (n, I)$. If

$$N > p^*\left(2\log\|T\| + \log\left(\frac{2^{2n+3}}{\eta^2} + \sum_{(m,J)\prec(n,I)} \frac{1}{\eta^2_{(m,J)}}\right)\right) \tag{6.1}$$

then there exists $\vartheta^0 = (\theta^0_K)_{K\in\mathcal{B}}$ such that

$$|Z(\vartheta^0)| < \eta,$$

$$|Y_{(m,J)}(\vartheta^0)| < \eta, \quad (m,J) \prec (n,I)$$

and

$$|W_{(m,J)}(\vartheta^0)| < \eta_{(m,J)}, \quad (m,J) \prec (n,I).$$

Proof. To prove the existence of ϑ^0, we define the following events:

$$\mathcal{Z} = \{\vartheta : |Z(\vartheta)| > \eta\},$$

$$\mathcal{Y}(m,J) = \{\vartheta : |Y_{(m,J)}(\vartheta)| > \eta\}, \qquad (m,J) \prec (n,I)$$

and

$$\mathcal{W}(m,J) = \{\vartheta : |W_{(m,J)}(\vartheta)| > \eta_{(m,J)}\}, \qquad (m,J) \prec (n,I).$$

Denote by $\mathcal{A}$ the union of the aforementioned events, that is

$$\mathcal{A} = \mathcal{Z} \cup \Big(\bigcup_{(m,J) \prec (n,I)} \mathcal{Y}(m,J) \Big) \cup \Big(\bigcup_{(m,J) \prec (n,I)} \mathcal{W}(m,J) \Big).$$

Now, taking the probability and using the Chebyshev inequality, we observe that

$$\mathbb{P}(\mathcal{A}) \leqslant \mathbb{P}(\mathcal{Z}) + \sum_{(m,J) \prec (n,I)} \mathbb{P}(\mathcal{Y}(m,J)) + \sum_{(m,J) \prec (n,I)} \mathbb{P}(\mathcal{W}(m,J))$$

$$\leqslant \frac{1}{\eta^2} \mathbb{V}(Z) + \frac{1}{\eta^2} \sum_{(m,J) \prec (n,I)} \mathbb{V}(Y_{(m,J)}) + \sum_{(m,J) \prec (n,I)} \frac{1}{\eta^2_{(m,J)}} \mathbb{V}(W_{(m,J)}).$$

Using the upper estimates of the variance of the random variables Z, $Y_{(m,J)}$ and $W_{(m,J)}$, that is Lemmas 6.2.4, 6.2.2 and 6.2.3, we obtain

$$\begin{aligned} \mathbb{P}(\mathcal{A}) &\leqslant \frac{2}{\eta^2} \|T\|^2 |I|^{(1/p)+1} 2^{-N/q} + \frac{1}{\eta^2} \sum_{(m,J) \prec (n,I)} \|T\|^2 |J|^{2/q} |I|^{1/p} 2^{-N/p} \\ &+ \sum_{(m,J) \prec (n,I)} \frac{1}{\eta^2_{(m,J)}} \|T\|^2 |J|^{2/p} |I|^{1/q} 2^{-N/q} \\ &\leqslant \left(\frac{2^{2n+3}}{\eta^2} + \sum_{(m,J) \prec (n,I)} \frac{1}{\eta^2_{(m,J)}} \right) \|T\|^2 2^{-N/p^*}. \end{aligned}$$

Now, (6.1) yields $\mathbb{P}(\mathcal{A}) < 1$, that is the existence of ϑ^0. $\square$

This is the main theorem of this section, that is the approximate orthogonal reduction to a diagonal operator. We use the probabilistic Lemma 6.2.5.

Theorem 6.2.6. Let $T\colon R^{p,0}_\omega \to R^{p,0}_\omega$ be a bounded linear operator and $\epsilon > 0$. Then there exist a distributional embedding $S : R^{p,0}_\omega \to L_p$ with $S(R^{p,0}_\omega) \subset R^{p,0}_\omega$ and a diagonal operator $R\colon R^{p,0}_\omega \to R^{p,0}_\omega$ with diagonal entries in $\mathcal{A}(D(T))$ such that

$$\|S^\dagger T S - R\| < \epsilon.$$

In particular, T is an orthogonal factor of R with error ϵ.

Proof. We choose an increasing sequence of positive integers $k(n)$ such that

$$k(n) > p^* \left(12n + 13 + 2\log\left(\frac{\|T\|}{\epsilon}\right)\right)$$

and put $N(n) = k(n) + n$.

By induction on the linear order of $\mathcal{D}_\omega$ we will construct a distributional copy $(b^n_I)_{(n,I)\in\mathcal{D}_\omega}$ of the basis $(h^n_I)_{(n,I)\in\mathcal{D}_\omega}$ of $R^{p,0}_\omega$ consisting of blocks of the basis $(h^n_I)_{(n,I)\in\mathcal{D}_\omega}$, of the form

$$b^n_I = \sum_{K\in\mathcal{B}^n_I} \theta^{(n,I)}_K h^{N(n)}_K, \tag{6.2}$$

where $\mathcal{B}^n_I \subset \mathcal{D}^{k(n)+i}$ with $I \in \mathcal{D}^i$ and $(\theta^{(n,I)}_K)_{K\in\mathcal{B}^n_I} \in \{-1,1\}^{\mathcal{B}^n_I}$. The blocks $(b^n_I)_{(n,I)\in\mathcal{D}_\omega}$ will be chosen to satisfy certain properties, which will be given explicitly in the inductive construction (see (6.4), (6.5) and (6.6)). Using the formulas of the operators S and $S^\dagger$ from Corollary 3.2.9, we will show that (see (6.7))

$$\|(S^\dagger T S - R)(h^n_I)\| < \frac{\epsilon}{2^{2n+1+n/p}}, \tag{6.3}$$

where $R\colon R^{p,0}_\omega \to R^{p,0}_\omega$ is a diagonal operator defined by

$$R(h^n_I) = \left(|I|^{-1} \sum_{K\in\mathcal{B}^n_I} \langle h^{N(n)}_K, T h^{N(n)}_K\rangle\right) h^n_I$$

with diagonal elements in $\mathcal{A}(\mathcal{D}(T))$. Given (6.3), we prove $\|S^\dagger T S - R\| < \epsilon$. Let $f \in R^{p,0}_\omega$ with

$$f = \sum_{n=1}^{\infty} \sum_{I\in\mathcal{D}_{n-1}} |I|^{-1}\langle h^n_I, f\rangle h^n_I.$$

We have

$$\|(S^\dagger T S - R)(f)\| = \|\sum_{n=1}^{\infty} \sum_{I\in\mathcal{D}_{n-1}} |I|^{-1}\langle h^n_I, f\rangle (S^\dagger T S - R)(h^n_I)\|$$

$$\leqslant \sum_{n=1}^{\infty} \sum_{I \in \mathcal{D}_{n-1}} 2^{n/p} \frac{\epsilon}{2^{n/p+2n+1}} \|f\| \leqslant \epsilon \|f\|.$$

We skip the basis step as it is simpler than the induction step. Now, let $(n, I) \in \mathcal{D}_\omega$ be fixed and assume that we have constructed the blocks (b_J^m) for every $(m, J) \prec (n, I)$ as in (6.2). We will only treat the case $I \neq [0,1]$, as the case $I = [0,1]$ is slightly simpler.

We will define a random block

$$b_I^n(\vartheta) = \sum_{K \in \mathcal{B}_I^n} \theta_K^{(n,I)} h_K^{N(n)},$$

with $\mathcal{B}_I^n \subset \mathcal{D}^{k(n)+i}$, to be chosen later, where i is such that $I \in \mathcal{D}^i$, such that for the random variables

$$Z_I^n(\vartheta) = \langle b_I^n(\vartheta), T b_I^n(\vartheta) \rangle - \sum_{K \in \mathcal{B}_I^n} \langle h_K^{N(n)}, T h_K^{N(n)} \rangle,$$

$$Y^{(n,I)}(m, J)(\vartheta) = \langle T^* b_J^m, b_I^n(\vartheta) \rangle \text{ for } (m, J) \prec (n, I)$$

$$W^{(n,I)}(m, J)(\vartheta) = \langle b_I^n(\vartheta), T b_J^m \rangle \text{ for } (m, J) \prec (n, I),$$

where $\vartheta = (\theta_K^{(n,I)})_{K \in \mathcal{B}_I^n} \in \{-1, 1\}^{\mathcal{B}_I^n}$, there exists $\vartheta_0 \in \{-1, 1\}^{\mathcal{B}_I^n}$ such that

$$|Z_I^n(\vartheta_0)| < \frac{\epsilon}{32^{5n+2}}, \tag{6.4}$$

$$|Y^{(n,I)}(m, J)(\vartheta_0)| < \frac{\epsilon}{32^{5n+2}} \text{ for } (m, J) \prec (n, I) \text{ and} \tag{6.5}$$

$$|W^{(n,I)}(m, J)(\vartheta_0)| < \frac{\epsilon}{32^{2n+m+3+n/q+m/p}} \text{ for } (m, J) \prec (n, I). \tag{6.6}$$

The choice of signs will be given by an appropriate application of Lemma 6.2.5.

We now explicitly define the set $\mathcal{B}_I^n$. Let $J = \pi(I)$ (such J exists because we assumed $I \neq [0,1]$) and we assume without loss of generality that $I = J^+$. By the induction hypothesis there exists a block b_J^n with $\mathcal{B}_J^n \subset \mathcal{D}^{k(n)+j}$ and $J \in \mathcal{D}^j$, where $j = i - 1$. Define

$$\mathcal{B}_I^n = \Big\{ K \in \mathcal{D}^{k(n)+i} : \operatorname{supp}(h_K^{N(n)}) \subset [b_J^n = 1] \Big\}.$$

Note that $\mathcal{B}_I^n$ is the subset of $\mathcal{D}^{k(n)+i}$ such that

$$\bigcup_{K \in \mathcal{B}_I^n} \operatorname{supp}(h_K^{N(n)}) = [b_J^n = 1].$$

(If $I = J^-$, then $\mathcal{B}_I^n = \Big\{ K \in \mathcal{D}^{k(n)+i} : \operatorname{supp}(h_K^{N(n)}) \subset [b_J^n = -1] \Big\}$, and the set $\mathcal{B}_I^n$ is the subset of

$\mathcal{D}^{k(n)+i}$ such that $\bigcup_{K\in\mathcal{B}_I^n}\operatorname{supp}(h_K^{N(n)}) = [b_J^n = -1]$). We define the random block

$$b_I^n(\vartheta) = \sum_{K\in\mathcal{B}_I^n} \theta_K^{(n,I)} h_K^{N(n)},$$

where $\vartheta = (\theta_K^{(n,I)})_{K\in\mathcal{B}_I^n} \in \{-1,1\}^{\mathcal{B}_I^n}$. We will show that the assumptions of Lemma 6.2.5 are satisfied, and this will yield the existence of signs $\vartheta_0 = (\theta_K^{0,(n,I)}) \in \{-1,1\}^{\mathcal{B}_I^n}$ satisfying inequalities (6.4), (6.5) and (6.6). By the choice of $k(n)$, $N(n)$ satisfies (6.1), and therefore we may apply Lemma 6.2.5, where now the $g_{(m,J)}$ and $f_{(m,J)}$ of Lemma 6.2.5 are $g_{(m,J)} = b_J^m$ and $f_{(m,J)} = b_J^m$. Lemma 6.2.5 yields the existence of signs $\theta_0 = (\theta_K^{0,(n,I)})$ and we put

$$b_I^n = b_I^n(\vartheta_0) = \sum_{K\in\mathcal{B}_I^n} \theta_K^{0,(n,I)} h_K^{N(n)}.$$

For brevity, we denote

$$\begin{aligned} Z_I^n &= Z_I^n(\vartheta_0)\\ Y^{(n,I)}(m,J) &= Y^{(n,I)}(m,J)(\vartheta_0)\\ W^{(n,I)}(m,J) &= W^{(n,I)}(m,J)(\vartheta_0). \end{aligned}$$

We have constructed a distributional copy $(b_I^n)_{(n,I)\in\mathcal{D}_\omega}$ of $(h_I^n)_{(n,I)\in\mathcal{D}_\omega}$ that satisfies (6.4), (6.5) and (6.6).

It remains to show (6.3), so fix $(n,I)\in\mathcal{D}_\omega$. We have

$$\begin{aligned} &\|(S^\dagger TS - R)(h_I^n)\| =\\ &\quad= \Big\|\sum_{m=1}^{\infty}\sum_{J\in\mathcal{D}_{m-1}} |J|^{-1}\langle b_J^m, Tb_I^n\rangle h_J^m - \Big(|I|^{-1}\sum_{K\in\mathcal{B}_I^n}\langle h_K^{N(n)}, Th_K^{N(n)}\rangle\Big) h_I^n\Big\|\\ &\quad= \Big\|\sum_{(m,J)<(n,I)} |J|^{-1}\underbrace{\langle b_J^m, Tb_I^n\rangle}_{=Y^{(n,I)}(m,J)} h_J^m + \sum_{(m,J)>(n,I)} |J|^{-1}\underbrace{\langle b_J^m, Tb_I^n\rangle}_{=W^{(m,J)}(n,I)} h_J^m +\\ &\quad |I|^{-1}\underbrace{\Big(\langle b_I^n, Tb_I^n\rangle - \sum_{K\in\mathcal{B}_I^n}\langle h_K^{N(n)}, Th_K^{N(n)}\rangle\Big)}_{=Z_I^n} h_I^n\Big\|. \end{aligned}$$

Using inequalities (6.4), (6.5) and (6.6) we obtain

$$\begin{aligned} &\|(S^\dagger TS - R)(h_I^n)\| \\ &\leqslant 2^{n/q}|Z_I^n| + 2^{2n+1+n/q}\max_{(m,J)<(n,I)}|Y^{(n,I)}(m,J)| \end{aligned} \tag{6.7}$$

$$+ \sum_{m \geqslant n} 2^{m+1+m/q} \max_{J \in \mathcal{D}_{m-1}} |W^{(m,J)}(n, I)| < \frac{\epsilon}{2^{2n+1+n/p}}.$$

□

Theorem 6.2.6 and Proposition 3.3.5 imply the factorization property of the natural basis of $R^{p;0}_\omega$.

Theorem 6.2.7. The standard martingale difference sequence basis of $R^{p;0}_\omega$ has the factorization property.

6.3 The orthogonal reduction to a scalar multiple of the identity

In this section, we approximately orthogonally reduce an arbitrary diagonal operator on $R^{p;0}_\omega$ to a scalar multiple of the identity operator on $R^{p;0}_\omega$ using a probabilistic lemma from [18] with origin in [17] (Lemma 6.3.1). This yields the approximate orthogonal reduction of an arbitrary operator on $R^{p;0}_\omega$ to a scalar multiple of the identity operator on $R^{p;0}_\omega$. Then we conclude with the primary factorization property of $R^{p;0}_\omega$.

The proof of the following probabilistic lemma can be found in [18, Lemma 3.9]. We will use it inside the proof of Theorem 6.3.3, that is the finite dimensional stabilization, to construct, by induction on the linear order of $\mathcal{D}_{m-1}$, a distributional copy $(b_I)_{I \in \mathcal{D}_{m-1}}$ of $(h_I)_{I \in \mathcal{D}_{m-1}}$ that stabilizes the eigenvalues of a diagonal operator. As in the proof of Theorem 6.2.6, in each step of the induction, we use a random block $b_I(\vartheta)$, defined using a family of dyadic intervals $\mathcal{B}_I$ that has been fixed in a previous inductive step. Because R is diagonal, the value $\langle b_I, R(b_I)\rangle$ is independent of the outcome θ, but the latter defines two families of dyadic intervals, $\mathcal{B}_{I^+}(\theta)$ and $\mathcal{B}_{I^-}(\theta)$, to be used in a future inductive step. The role of the following Lemma is to establish conditions under which, with positive probability, there is a vector b_I such that the associated families $\mathcal{B}_{I^+}$ and $\mathcal{B}_{I^-}$, define future random blocks $b_{I^+}(\theta), b_{I^-}(\theta)$ with values $\langle b_{I^+}, T(b_{I^+})\rangle$ and $\langle b_{I^-}, T(b_{I^-})\rangle$ approximately equal.

Lemma 6.3.1. Let $R\colon L^0_p \to L^0_p$ be a diagonal operator with diagonal elements $d_I = |I|^{-1}\langle h_I, Rh_I\rangle$, where $(h_I)_{I \in \mathcal{D}}$ is the standard Haar system. Let $m, k \in \mathbb{N}$ with $m < k$ and let $\mathcal{B} \subset \mathcal{D}^m$. Consider the probability space $\{-1, 1\}^{\mathcal{B}}$ equipped with the uniform probability measure. We define the random vector

$$b(\vartheta) = \sum_{J \in \mathcal{B}} \theta_J h_J$$

where $\vartheta = (\theta_J)_{J \in \mathcal{B}}$, and denote

$$\mathcal{B}^+(\vartheta) = \{J \in \mathcal{D}^k : \ J \subset [b(\vartheta) = 1]\} \text{ and } \mathcal{B}^-(\vartheta) = \{J \in \mathcal{D}^k : J \subset [b(\vartheta) = -1]\}.$$

We define the random variables

$$\lambda^+ \colon \{-1,1\}^{\mathcal{B}} \to \mathbb{R} \quad \text{and} \quad \lambda^- \colon \{-1,1\}^{\mathcal{B}} \to \mathbb{R}$$

with

$$\lambda^+(\vartheta) = \frac{1}{(\#\mathcal{B}^+(\vartheta))} \sum_{J \in \mathcal{B}^+(\vartheta)} d_J \quad \text{and} \quad \lambda^-(\vartheta) = \frac{1}{(\#\mathcal{B}^-(\vartheta))} \sum_{J \in \mathcal{B}^-(\vartheta)} d_J.$$

Then

$$\mathbb{E}(\lambda^+) = \mathbb{E}(\lambda^-) = \frac{1}{(\#\{J \in \mathcal{D}^k : J \subset \cup\mathcal{B}\})} \sum_{\{J \in \mathcal{D}^k : J \subset \cup\mathcal{B}\}} d_J,$$

and

$$\mathbb{V}(\lambda^+) \leqslant \frac{2^{-m}}{|\cup \mathcal{B}|}\|R\|^2 \quad \text{and} \quad \mathbb{V}(\lambda^-) \leqslant \frac{2^{-m}}{|\cup \mathcal{B}|}\|R\|^2.$$

Remark 6.3.2. Let $m \in \mathbb{N}$ and $S : L_p^{m,0} \to L_p^0$ be a distributional embedding. The orthogonal projection $E \colon L_p^0 \to L_p^0$ onto $S[L_p^{m,0}]$ can be expressed as

$$E(f) = \sum_{I \in \mathcal{D}_{m-1}} |I|^{-1} \langle Sh_I, f \rangle Sh_I.$$

The above projection is of norm 1 as it is equal to the conditional expectation operator with respect to $\sigma(Sh_I : I \in \mathcal{D}_{m-1})$.

The following finite dimensional stabilization plays a key role in the proof of Theorem 6.3.4. The proof of the latter then relatively easily follows from a compactness argument. To prove this finite dimensional stabilization we use a combinatorial argument, based on the pigeonhole principle and a repeated application of the earlier probabilistic future-level stabilization argument (Lemma 6.3.1) to stabilize the eigenvalues of a given diagonal operator.

Lemma 6.3.3. Let $m \in \mathbb{N}$, $\Gamma > 0$ and $\epsilon > 0$. If

$$M > m + 3 + 3\log(m) + 2\log(\frac{\Gamma}{\epsilon}) + m\frac{4(p^*-1)\Gamma}{\epsilon} + 2\log(p^*-1) \tag{6.8}$$

then for every diagonal operator $R \colon L_p^{M,0} \to L_p^{M,0}$ with $\|R\| \leqslant \Gamma$, there exists a distributional copy $(b_I)_{I \in \mathcal{D}_{m-1}}$ of $(h_I)_{I \in \mathcal{D}_{m-1}}$ in $L_p^{M,0}$ satisfying the following:

(i) There are $0 \leqslant n_0 < n_1 < \cdots < n_{m-1} \leqslant M-1$ such that, for $0 \leqslant i \leqslant m-1$ and $I \in \mathcal{D}^i$,

$$b_I = \sum_{K \in \mathcal{B}_I} \theta_K h_K,$$

where $\mathcal{B}_I \subset \mathcal{D}^{n_i}$ and $(\theta_K)_{K\in\mathcal{B}_I} \in \{-1,1\}^{\mathcal{B}_I}$.

(ii) Denote $Y = \langle b_I : I \in \mathcal{D}_{m-1}\rangle$, E the conditional expectation operator onto the space Y as in Remark 6.3.2, and $\lambda_0 = \langle b_{[0,1]}, Rb_{[0,1]}\rangle \in \mathcal{A}(D(R))$. Then

$$\left\| \left(ER - \lambda_0 \cdot id\right)|Y \right\| < \epsilon.$$

Proof. Denote by $(d_I)_{I\in\mathcal{D}_{M-1}}$ the diagonal entries of R with respect to the basis $(h_I)_{I\in\mathcal{D}_{M-1}}$ of L_p^M, that is $d_I = |I|^{-1}\langle h_I, Rh_I\rangle$, for all $I \in \mathcal{D}_{M-1}$. Choose

$$x = [m + 3 + 3\log(m) + 2\log(\frac{\Gamma}{\epsilon}) + 2\log(p^* - 1)] + 1.$$

For every $k \in \{x, x+1, ..., M-1\}$, denote

$$\lambda_k = \frac{1}{(\#\mathcal{D}^k)} \sum_{J\in\mathcal{D}^k} d_J,$$

that is the finite average of the diagonal elements of R over the level $\mathcal{D}^k$. Note $|\lambda_k| \leqslant \Gamma$, for all $k \in \{x, x+1, ..., M-1\}$. Now, (6.8) and the pigeonhole principle yield the existence of levels $(n_j)_{j=0}^{m-1}$ with

$$x \leqslant n_0 < n_2 < ... < n_{m-1} \leqslant M - 1$$

such that

$$|\lambda_{n_k} - \lambda_{n_l}| < \frac{\epsilon}{2(p^* - 1)},$$

for every $0 \leqslant k, l \leqslant m-1$.

By induction on the linear order of $\mathcal{D}_{m-1}$ we will construct families $(\mathcal{B}_I)_{I\in\mathcal{D}_{m-1}}$ of dyadic intervals and signs $((\theta_J)_{J\in\mathcal{B}_I})_{I\in\mathcal{D}_{m-1}}$ with $\mathcal{B}_{[0,1]} = \mathcal{D}^{n_0}$ and if

$$b_I = \sum_{J\in\mathcal{B}_I} \theta_J h_J,$$

where $I \in \mathcal{D}^k$, $0 \leqslant k < m-1$, then

$$\mathcal{B}_{I^+} = \{J \in \mathcal{D}^{n_{k+1}} : J \subset [b_I = 1]\}$$

and

$$\mathcal{B}_{I^-} = \{J \in \mathcal{D}^{n_{k+1}} : J \subset [b_I = -1]\}.$$

The vectors $(b_I)_{I\in\mathcal{D}_{m-1}}$ will form a distributional copy of $(h_I)_{I\in\mathcal{D}_{m-1}}$ and they will be constructed such that (i) and (ii) of the statement are satisfied.

We begin the inductive construction. Let $\mathcal{B}_{[0,1]} = \mathcal{D}^{n_0}$. For signs $\theta = (\theta_J)_{J\in\mathcal{B}_{[0,1]}}$ define the random vector

$$b_{[0,1]}(\theta) = \sum_{J\in\mathcal{B}_{[0,1]}} \theta_J h_J.$$

and denote

$$\mathcal{B}^{n_j}_{[0,\frac{1}{2}]}(\theta) = \{J \in \mathcal{D}^{n_j} : J \subset [b_{[0,1]}(\theta) = 1]\} \text{ and } \mathcal{B}^{n_j}_{[\frac{1}{2},1]}(\theta) = \{J \in \mathcal{D}^{n_j} : J \subset [b_{[0,1]}(\theta) = -1]\},$$

for every $1 \leqslant j \leqslant m-1$. Moreover, we define the random variables

$$\lambda^{n_j}_{[0,\frac{1}{2}]} : \{-1,1\}^{\mathcal{B}_{[0,1]}} \to \mathbb{R} \text{ and } \lambda^{n_j}_{[\frac{1}{2},1]} : \{-1,1\}^{\mathcal{B}_{[0,1]}} \to \mathbb{R}$$

with

$$\lambda^{n_j}_{[0,\frac{1}{2}]}(\theta) = \frac{1}{(\#\mathcal{B}^{n_j}_{[0,\frac{1}{2}]}(\theta))} \sum_{J\in\mathcal{B}^{n_j}_{[0,\frac{1}{2}]}(\theta)} d_J \text{ and } \lambda^{n_j}_{[\frac{1}{2},1]}(\theta) = \frac{1}{(\#\mathcal{B}^{n_k}_{[\frac{1}{2},1]}(\theta))} \sum_{J\in\mathcal{B}^{n_j}_{[\frac{1}{2},1]}(\theta)} d_J,$$

for every $1 \leqslant j \leqslant m-1$. We will choose signs $\theta = (\theta_J)_{J\in\mathcal{B}_{[0,1]}}$ such that

$$|\lambda^{n_j}_{[0,\frac{1}{2}]}(\theta) - \lambda_{n_j}| < \frac{\epsilon}{2m(p^*-1)} \text{ and } |\lambda^{n_j}_{[\frac{1}{2},1]}(\theta) - \lambda_{n_j}| < \frac{\epsilon}{2m(p^*-1)}$$

for every $1 \leqslant j \leqslant m-1$. Lemma 6.3.1 implies that

$$\mathbb{E}(\lambda^{n_j}_{[0,\frac{1}{2}]}) = \mathbb{E}(\lambda^{n_j}_{[\frac{1}{2},1]}) = \lambda_{n_j}$$

and

$$\mathbb{V}(\lambda^{n_j}_{[0,\frac{1}{2}]}) \leqslant \frac{1}{2^{n_0}}\Gamma^2 \text{ and } \mathbb{V}(\lambda^{n_j}_{[\frac{1}{2},1]}) \leqslant \frac{1}{2^{n_0}}\Gamma^2,$$

for every $1 \leqslant j \leqslant m-1$. To choose signs we define the events,

$$\mathcal{Z}^{n_j}_{[0,\frac{1}{2}]} = \{\theta : |\lambda^{n_j}_{[0,\frac{1}{2}]}(\theta) - \lambda_{n_j}| > \frac{\epsilon}{2m(p^*-1)}\}$$

and

$$\mathcal{Z}^{n_j}_{[\frac{1}{2},1]} = \{\theta : |\lambda^{n_j}_{[\frac{1}{2},1]}(\theta) - \lambda_{n_j}| > \frac{\epsilon}{2m(p^*-1)}\},$$

for every $1 \leqslant j \leqslant m-1$. Using the Chebyshev inequality we obtain that

$$\mathbb{P}(\mathcal{Z}^{n_j}_{[0,\frac{1}{2}]}) \leqslant \frac{4m^2(p^*-1)^2}{\epsilon^2}\mathbb{V}(\lambda^{n_j}_{[0,\frac{1}{2}]}) \leqslant 4m^2(p^*-1)^2\frac{\Gamma^2}{\epsilon^2}\frac{1}{2^{n_0}}$$

and

$$\mathbb{P}(\mathcal{Z}^{n_j}_{[\frac{1}{2},1]}) \leqslant \frac{4m^2(p^*-1)^2}{\epsilon^2}\mathbb{V}(\lambda^{n_j}_{[\frac{1}{2},1]}) \leqslant 4m^2(p^*-1)^2\frac{\Gamma^2}{\epsilon^2}\frac{1}{2^{n_0}},$$

for every $1 \leqslant j \leqslant m-1$. Thus

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{1\leqslant j\leqslant m-1}\left(\mathcal{Z}^{n_j}_{[0,\frac{1}{2}]} \cup \mathcal{Z}^{n_j}_{[\frac{1}{2},1]}\right)\right) &\leqslant \sum_{1\leqslant j\leqslant m-1}\left(\mathbb{P}(\mathcal{Z}^{n_j}_{[0,\frac{1}{2}]}) + \mathbb{P}(\mathcal{Z}^{n_j}_{[\frac{1}{2},1]})\right)\\
\leqslant \sum_{1\leqslant j\leqslant m-1} 8m^2(p^*-1)^2\frac{\Gamma^2}{\epsilon^2}\frac{1}{2^{n_0}} &\leqslant 2^3m^3(p^*-1)^2\frac{\Gamma^2}{\epsilon^2}\frac{1}{2^{n_0}} < 1,
\end{aligned}$$

as $x \leqslant n_0$. Hence, there exist signs $\theta = (\theta_J)_{J\in\mathcal{B}_{[0,1]}}$ such that

$$|\lambda^{n_j}_{[0,\frac{1}{2}]}(\theta) - \lambda_{n_j}| < \frac{\epsilon}{2m(p^*-1)} \text{ and } |\lambda^{n_j}_{[\frac{1}{2},1]}(\theta) - \lambda_{n_j}| < \frac{\epsilon}{2m(p^*-1)}$$

for every $1 \leqslant j \leqslant m-1$.

Now, let $I \in \mathcal{D}^k$, $0 \leqslant k < m-1$, and assume that we have found $\mathcal{B}_I \subset \mathcal{D}^{n_k}$. For signs $\theta = (\theta_J)_{J\in\mathcal{B}_I}$ define the random vector

$$b_I(\theta) = \sum_{J\in\mathcal{B}_I}\theta_J h_J.$$

Moreover, denote

$$\mathcal{B}^{n_j}_{I^+}(\theta) = \{J \in \mathcal{D}^{n_j} : J \subset [b_I(\theta) = 1]\} \text{ and } \mathcal{B}^{n_j}_{I^-}(\theta) = \{J \in \mathcal{D}^{n_j} : J \subset [b_I(\theta) = -1]\},$$

and define the random variables

$$\lambda^{n_j}_{I^+} : \{-1,1\}^{\mathcal{B}_I} \to \mathbb{R} \quad\text{and}\quad \lambda^{n_j}_{I^-} : \{-1,1\}^{\mathcal{B}_I} \to \mathbb{R}$$

with

$$\lambda^{n_j}_{I^+}(\theta) = \frac{1}{(\#\mathcal{B}^{n_j}_{I^+}(\theta))}\sum_{J\in\mathcal{B}^{n_j}_{I^+}(\theta)} d_J \text{ and } \lambda^{n_j}_{I^-}(\theta) = \frac{1}{(\#\mathcal{B}^{n_j}_{I^-}(\theta))}\sum_{J\in\mathcal{B}^{n_j}_{I^-}(\theta)} d_J$$

for every $k+1 \leqslant j \leqslant m-1$. We will choose signs $\theta = (\theta_J)_{J\in\mathcal{B}_I}$ such that

$$|\lambda^{n_j}_{I^+}(\theta) - \lambda^{n_j}_I| < \frac{\epsilon}{2m(p^*-1)} \text{ and } |\lambda^{n_j}_{I^-}(\theta) - \lambda^{n_j}_I| < \frac{\epsilon}{2m(p^*-1)}$$

for every $k+1 \leqslant j \leqslant m-1$. To choose signs we define the events

$$\mathcal{Z}_{I^+}^{n_j} = \{\theta : |\lambda_{I^+}^{n_j}(\theta) - \lambda_I^{n_j}| > \frac{\epsilon}{2m(p^*-1)}\}$$

and

$$\mathcal{Z}_{I^-}^{n_j} = \{\theta : |\lambda_{I^-}^{n_j}(\theta) - \lambda_I^{n_j}| > \frac{\epsilon}{2m(p^*-1)}\},$$

for every $k+1 \leqslant j \leqslant m-1$. Lemma 6.3.1 implies that

$$\mathbb{E}(\lambda_{I^+}^{n_j}) = \mathbb{E}(\lambda_{I^-}^{n_j}) = \lambda_I^{n_j},$$

and

$$\mathbb{V}(\lambda_{I^+}^{n_j}) \leqslant \frac{2^k}{2^{n_k}}\Gamma^2 \text{ and } \mathbb{V}(\lambda_{I^-}^{n_j}) \leqslant \frac{2^k}{2^{n_k}}\Gamma^2,$$

for every $k+1 \leqslant j \leqslant m-1$. Using the Chebyshev inequality we obtain that

$$\mathbb{P}(\mathcal{Z}_{I^+}^{n_j}) \leqslant \frac{4m^2(p^*-1)^2}{\epsilon^2}\mathbb{V}(\lambda_{I^+}^{n_j})$$

and

$$\mathbb{P}(\mathcal{Z}_{I^-}^{n_j}) \leqslant \frac{4m^2(p^*-1)^2}{\epsilon^2}\mathbb{V}(\lambda_{I^-}^{n_j}),$$

for every $k+1 \leqslant j \leqslant m-1$. Therefore,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{j=k+1}^{m-1}\left(\mathcal{Z}_{I^+}^{n_j} \cup \mathcal{Z}_{I^-}^{n_j}\right)\right) &\leqslant \sum_{j=k+1}^{m-1}\left(\mathbb{P}(\mathcal{Z}_{I^+}^{n_j}) + \mathbb{P}(\mathcal{Z}_{I^-}^{n_j})\right) \\ &\leqslant \sum_{j=k+1}^{m-1} 2^3\frac{m^2(p^*-1)^2}{\epsilon^2}\frac{2^k}{2^{n_k}}\Gamma^2 < 2^{m+3}m^3(p^*-1)^2\frac{\Gamma^2}{\epsilon^2}\frac{1}{2^{n_k}} < 1,\end{aligned}$$

as $x \leqslant n_0 < n_2 < \ldots < n_{m-1}$. Hence, there exist $\theta = (\theta_J)_{J\in\mathcal{B}_I}$ such that

$$|\lambda_{I^+}^{n_j}(\theta) - \lambda_I^{n_j}| < \frac{\epsilon}{2m(p^*-1)} \text{ and } |\lambda_{I^-}^{n_j}(\theta) - \lambda_I^{n_j}| < \frac{\epsilon}{2m(p^*-1)},$$

for every $k+1 \leqslant j \leqslant m-1$.

Now, choose $\lambda_0 = \lambda_{n_0}$. For $I \in \mathcal{D}^k$ and $k \leqslant j \leqslant m-1$, we observe that

$$|\lambda_I^{n_j} - \lambda_{n_j}| \leqslant \sum_{i=0}^{k-1}|\lambda_{\pi(I)^i}^{n_j} - \lambda_{\pi(I)^{i+1}}^{n_j}| < \frac{\epsilon}{2(p^*-1)},$$

for every $k \leqslant j \leqslant m-1$, where $\lambda^{n_j}_{\pi(I)^0} = \lambda^{n_j}_I$ and $\lambda^{n_j}_{\pi(I)^k} = \lambda_{n_j}$. Thus,

$$|\lambda^{n_j}_I - \lambda_0| \leqslant |\lambda^{n_j}_I - \lambda_{n_j}| + |\lambda_{n_j} - \lambda_0| < \frac{\epsilon}{2(p^*-1)} + \frac{\epsilon}{2(p^*-1)} = \frac{\epsilon}{p^*-1},$$

for every $k \leqslant j \leqslant m-1$. Now, for $I \in \mathcal{D}^k$, $0 \leqslant k \leqslant m-1$, put $\mathcal{B}_I = \mathcal{B}^{n_k}_I$ and $\lambda_I = \lambda^{n_k}_I$. We have $|\lambda_I - \lambda_0| < \epsilon/(p^*-1)$, for every $I \in \mathcal{D}_{m-1}$, and note $\lambda_I = \langle b_I, Rb_I \rangle$. Finally, for an arbitrary $f \in Y$ with $f = \sum_{I \in \mathcal{D}_{m-1}} a_I b_I$, from the Paley-Burkholder inequality, we obtain that

$$\begin{aligned} \|ER - \lambda_0 \cdot id)(f)\| &= \|\sum_{I \in \mathcal{D}_{m-1}} (\lambda_I - \lambda_0) a_I b_I\| \\ &\leqslant (p^*-1)\frac{\epsilon}{(p^*-1)}\|\sum_{I \in \mathcal{D}_{m-1}} a_I b_I\| = \epsilon\|f\|, \end{aligned}$$

finishing the proof. □

The above finite-dimensional stabilization result remains valid with the same estimates and properties if the spaces L^m_p and L^M_p are replaced by any of their distributional copies.

We now orthogonally reduce an arbitrary diagonal operator to a scalar multiple of the identity operator using the above finite dimensional stabilization.

Theorem 6.3.4. Let $R\colon R^{p,0}_\omega \to R^{p,0}_\omega$ be a diagonal operator with diagonal elements $d^n_I = |I|^{-1}\langle h^n_I, Rh^n_I \rangle$, where $(n, I) \in \mathcal{D}_\omega$, and $\epsilon > 0$. Then there exists a distributional embedding $J : R^{p,0}_\omega \to L_p$ with $J(R^{p,0}_\omega) \subset R^{p,0}_\omega$ and a scalar $\lambda_0 \in \mathcal{A}(\mathcal{D}(R))$ such that

$$\|J^\dagger RJ - \lambda_0 \cdot id\| < \epsilon.$$

In particular, R is an orthogonal factor of $\lambda_0 \cdot id\colon R^{p,0}_\omega \to R^{p,0}_\omega$ with error ϵ.

Proof. For the parameters $\frac{\epsilon}{2}$ and $\|R\|$, choose an increasing sequence of non-negative integers $N(m)$ with

$$N(m) > m + 3 + 3\log(m) + 2\log(\frac{2\|R\|}{\epsilon}) + m\frac{8(p^*-1)\|R\|}{\epsilon} + 2\log(p^*-1).$$

Then, for every m, Theorem 6.3.3 implies the existence of a distributional copy $(b^m_I)_{I \in \mathcal{D}_{m-1}}$ of $(h^m_I)_{I \in \mathcal{D}_{m-1}}$ in the space $\langle h^{N(m)}_I : I \in \mathcal{D}_{N(m)-1} \rangle$ satisfying (i) and (ii) of Theorem 6.3.3. From the proof of the aforementioned theorem, for every m, if we put $\lambda^m_0 = \langle b^m_{[0,1]}, Rb^m_{[0,1]} \rangle$, and for every $I \in \mathcal{D}_{m-1}$ we put $\lambda^m_I = \langle b^m_I, Rb^m_I \rangle$ then we have

$$|\lambda^m_I - \lambda^m_0| < \frac{\epsilon}{2(p^*-1)},$$

for every $I \in \mathcal{D}_{m-1}$. Since for every m we have $\lambda^m_0 \in \mathcal{A}(\mathcal{D}(R))$, the sequence $(\lambda^m_0)^\infty_{m=1}$ is bounded.

Thus, there exists a subsequence $(\lambda_0^{m_k})_{k=1}^{\infty}$ and $\lambda_0 \in \mathcal{A}(\mathcal{D}(R))$ such that

$$|\lambda_0^{m_k} - \lambda_0| < \frac{\epsilon}{2(p^* - 1)}$$

for every $k = 1, 2....$ Thus,

$$|\lambda_I^{m_k} - \lambda_0| < \frac{\epsilon}{(p^* - 1)},$$

for every $I \in \mathcal{D}_{m_k - 1}$ and $k = 1, 2....$ Now, put

$$Y = [(b_I^{m_k})_{I \in \mathcal{D}_{k-1}} : k = 1, 2, ...]$$

and define the distributional isomorphism $J\colon R_\omega^{p,0} \to Y$ with $J(h_I^k) = b_I^{m_k}$, for all $I \in \mathcal{D}_{k-1}$. Using the aforementioned distributional embedding $J : R_\omega^{p,0} \to L_p$ with $J(R_\omega^{p,0}) \subset R_\omega^{p,0}$ and the above scalar $\lambda_0 \in \mathcal{A}(\mathcal{D}(R))$ we observe that $J^\dagger R J$ is a diagonal operator with entries $(\lambda_I^{m_k})_{(k,I) \in \mathcal{D}_\omega}$. Therefore,

$$\|J^\dagger R J - \lambda_0 \cdot id\| < \epsilon,$$

completing the proof. □

Recall that Theorem 6.2.6 proves that an arbitrary operator $T : R_\omega^{p,0} \to R_\omega^{p,0}$ is an approximate orthogonal factor of a diagonal operator $R : R_\omega^{p,0} \to R_\omega^{p,0}$, and Theorem 6.3.4 shows that an arbitrary diagonal operator $R : R_\omega^{p,0} \to R_\omega^{p,0}$ is an approximate orthogonal factor of a scalar multiple of the identity $\lambda_0 \cdot id\colon R_\omega^{p,0} \to R_\omega^{p,0}$. The aforementioned theorems and Proposition 3.3.3, that is the transitivity of the orthogonal factors yield the following.

Theorem 6.3.5. Let $T\colon R_\omega^{p,0} \to R_\omega^{p,0}$ be a bounded linear operator and $\epsilon > 0$. Then there exists a distributional embedding $J : R_\omega^{p,0} \to L_p$ with $J(R_\omega^{p,0}) \subset R_\omega^{p,0}$ and $\lambda_0 \in \mathrm{conv}(\mathcal{D}(T))$ such that

$$\|J^\dagger T J - \lambda_0 \cdot id\| < \epsilon.$$

In particular, T is an orthogonal factor of $\lambda_0 \cdot id\colon R_\omega^{p,0} \to R_\omega^{p,0}$ with error ϵ.

Theorem 6.3.5 and Proposition 3.3.6 yield the primary factorization property of $R_\omega^{p,0}$, which in combination with Proposition 6.1.4 we obtain the following.

Theorem 6.3.6. The spaces $R_\omega^{p,0}$ and X_p have the primary factorization property.

Chapter 7

Decompositions of $R_\alpha^{p,0}$

The main goal of this chapter is to construct an unconditional Schauder decomposition of $R_\alpha^{p,0}$ with the required reproducing properties, suitable for defining and studying classes of distributional embeddings of $R_\alpha^{p,0}$ relevant to the orthogonal reduction of operators. For every ordinal $\alpha < \omega_1$, we construct a countable well-founded tree $\mathcal{T}_\alpha$. The elements of each tree $\mathcal{T}_\alpha$ are finite sequences whose entries are ordinal numbers and dyadic intervals, ordered by initial segments. For each $\lambda \in \mathcal{T}_\alpha$, its entries encode information that determine a positive integer $\kappa(\lambda)$, a parameter $\theta_\lambda \in (0,1]$, and a corresponding θ_λ-compression T_λ, which in turn define the space $X_\lambda = T_\lambda(L_p^{\kappa(\lambda),0})$. We then show that, under a suitable enumeration, the collection $((T_\lambda h_I)_{I\in\mathcal{D}_{\kappa(\lambda)-1}})_{\lambda\in\mathcal{T}_\alpha}$ forms a martingale difference sequence spanning $R_\alpha^{p,0}$. Finally, we study the supports of the spaces $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$, a fact needed in subsequent sections where the aforementioned distributional embeddings are introduced and studied.

7.1 The trees $\mathcal{T}_\alpha$

In this section, for every $1 \leqslant \alpha < \omega_1$, we construct a countable well-founded tree $\mathcal{T}_\alpha$. Each member λ of $\mathcal{T}_\alpha$ is a finite sequence $(x_1, \dots, x_l)$, where each x_i is either an ordinal number or a dyadic interval, and this information encodes the position of the subspace X_λ (which will be defined in Section 7.2) inside $R_\alpha^{p,0}$.

Notation 7.1.1. For every ordinal number α we define its integer part $\kappa(\alpha) \in \mathbb{N} \cup \{0\}$ as follows:

- $\kappa(\alpha) = 0$, for every limit ordinal α and $\alpha = 0$.
- $\kappa(\alpha + 1) = \kappa(\alpha) + 1$, for every ordinal α.

For example, $\kappa(0) = 0$, $\kappa(n) = n$ and $\kappa(\omega+n) = n$. More generally, when we write $\alpha = \beta+\kappa(\alpha)$, then β is a limit ordinal number.

Notation 7.1.2. For a finite sequence $\lambda = (x_1, \dots, x_l)$ we denote its length by $|\lambda| = l$ and for $1 \leqslant m \leqslant l$ we denote $\lambda|_m = (x_1, \dots, x_m)$ and $\lambda(m) = x_m$. Let $\mu = (y_1, ..., y_k)$ be another finite sequence.

(a) Denote the concatenation of λ and μ by

$$\lambda^\frown\mu = (x_1, ..., x_l, y_1, ..., y_k).$$

(b) If $x_l = y_1$, then denote the gluing of λ and μ by

$$\lambda \oslash \mu = (x_1, ..., x_l, y_2, ..., y_k).$$

The operation "$^\frown$" is used in the definition of the trees $\mathcal{T}_\alpha$, whereas both "$^\frown$" and "$\oslash$" will be employed in the study of distributional embedding schemes in Chapters 8 and 9.

We define the trees $\mathcal{T}_\alpha$, $1 \leqslant \alpha < \omega_1$ in two equivalent ways. The first is recursive and highlights how $\mathcal{T}_\alpha$ depends on the trees for smaller ordinals; the second is direct and gives an explicit description of the elements of $\mathcal{T}_\alpha$.

The recursive definition of $\mathcal{T}_\alpha$ naturally mirrors the recursive structure of the spaces $R^{p,0}_\alpha$ described in Remark 5.2.9. For practical reasons, however, we postpone this until after we introduce the operators $(T_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ in Section 7.2, which allows a more convenient treatment of these spaces.

Definition 7.1.3. Let $1 \leqslant \alpha < \omega_1$.

(a) If $\alpha = n < \omega$, then

$$\mathcal{T}_n = \{(n)\}.$$

This is based on $R^{p,0}_n = L^{n,0}_p$ and $\mathcal{T}_n = \{(n)\}$ because the FDD will be trivial in this case.

(b) If α is a limit ordinal, then

$$\mathcal{T}_\alpha = \bigcup_{1\leqslant\beta<\alpha} \{(\alpha)^\frown\lambda : \lambda \in \mathcal{T}_\beta\}.$$

Because $R^{p,0}_\alpha \equiv^{\text{dist}} (\sum_{\beta<\omega_1} R^{\beta,0}_p)_{\text{Ind},p}$, the index set $\mathcal{T}_\alpha$ of the FDD of $R^{p,0}_\alpha$ is the disjoint union of the sets $\mathcal{T}_\beta$, $\beta < \alpha$.

(c) If $\alpha = \beta + \kappa(\alpha)$ is a successor, then

$$\mathcal{T}_\alpha = \{(\alpha)\} \cup \bigcup_{I\in\mathcal{D}^{\kappa(\alpha)}} \{(\alpha, I)^\frown\lambda : \lambda \in \mathcal{T}_\beta\}.$$

In this case, $R^{p,0}_\alpha = L^{\kappa(\alpha),0}_p + [T_I(R^{p,0}_\beta) : I \in \mathcal{D}^{\kappa(\alpha)}]$. Therefore the index set $\mathcal{T}_\alpha$ for the FDD of $R^{p,0}_\alpha$ is the disjoint union of $2^{\kappa(\alpha)}$ copies of the set $\mathcal{T}_\beta$ together with (α), the latter corresponding to the space $L^{\kappa(\alpha),0}_p$, viewed as lying "above" the spaces $T_I(R^{p,0}_\beta)$, $I \in \mathcal{D}^{\kappa(\alpha)}$.

The pair $(\mathcal{T}_\alpha, \sqsubseteq)$, where $\sqsubseteq$ is the initial-segment order, is a countable well-founded tree.

We follow standard tree terminology. For $\lambda, \mu \in \mathcal{T}_\alpha$, if $\mu \subsetneq \lambda$, we say that λ is a successor of μ and that μ is a predecessor of λ. If μ is the maximum predecessor of λ, then we say that λ is an immediate successor of μ. A member $\lambda \in \mathcal{T}_\alpha$ with no predecessors is called a root. Denote

$$\mathcal{Root}(\mathcal{T}_\alpha) = \{\lambda \in \mathcal{T}_\alpha : \ \lambda \text{ is a root}\}.$$

The height of a member $\lambda \in \mathcal{T}_\alpha$, denoted $h(\lambda)$, is the minimal path distance from a root of $\mathcal{T}_\alpha$ to λ. There can be indices m with $1 \leqslant m \leqslant |\lambda|$ for which $\lambda|_m \notin \mathcal{T}_\alpha$, hence in general $h(\lambda)$ can be arbitrarily smaller than $|\lambda|$.

Remark 7.1.4. Let $\alpha < \omega_1$.

(a) If α is a limit ordinal, then $\mathcal{Root}(\mathcal{T}_\alpha)$ is countably infinite.

(b) If it is a successor ordinal and $\alpha = \beta + \kappa(\alpha)$, then $\mathcal{Root}(\mathcal{T}_\alpha)$ is the singleton $\{(\beta + \kappa(\alpha))\}$.

Example 7.1.5. The trees $\mathcal{T}_\omega$, $\mathcal{T}_{\omega+n}$, and $\mathcal{T}_{\omega\cdot 2}$ can be easily described as follows:

$$\begin{aligned}
\mathcal{T}_\omega &= \{(\omega, n) : n \in \mathbb{N}\},\\
\mathcal{T}_{\omega+n} &= \{(\omega+n)\} \cup \{(\omega+n, I, \omega, k) : I \in \mathcal{D}^n, k \in \mathbb{N}\}, \text{ where } n \in \mathbb{N} \text{ is fixed, and}\\
\mathcal{T}_{\omega\cdot 2} &= \{(\omega \cdot 2, \omega + n) : n \in \mathbb{N}\} \cup \{(\omega \cdot 2, \omega+n, I, \omega, k) : n \in \mathbb{N}, I \in \mathcal{D}^n, k \in \mathbb{N}\} \cup\\
&\quad\ \{(\omega \cdot 2, \omega, k) : k \in \mathbb{N}\} \cup \{(\omega \cdot 2, k) : k \in \mathbb{N}\}.
\end{aligned}$$

The following definition provides an explicit alternative description of all trees $\mathcal{T}_\alpha$, highlighting the structure of the entries of a sequence $\lambda = (x_1, \ldots, x_l) \in \mathcal{T}_\alpha$ and giving a clearer intuition for the form of these trees. Moreover, the entries of λ carry a more specific meaning, as they can be used directly to construct the operator T_λ and consequently the space X_λ.

Definition 7.1.6. Let $\mathcal{T}$ be the collection of all finite sequences $(x_1, ..., x_l)$ in $[1, \omega_1) \cup \mathcal{D}$ satisfying the following:

- $x_1 \in [1, \omega_1)$.
- For $1 \leqslant i \leqslant l$, if x_i is a limit ordinal, then $i < l$ and $x_{i+1} \in [1, \omega_1)$ with $x_{i+1} < x_i$.
- For $1 \leqslant i \leqslant l$, if x_i is a successor ordinal, then either $i = l$ or $i \leqslant l - 2$. In the latter case $x_{i+1} \in \mathcal{D}^{\kappa(x_i)}$ and $x_{i+2} + \kappa(x_i) = x_i$.

The following remark can be shown easily by induction on the countable ordinal numbers.

Remark 7.1.7. For every $1 \leqslant \alpha < \omega_1$, we have that

$$\mathcal{T}_\alpha = \{(x_1, ..., x_l) \in \mathcal{T} : x_1 = \alpha\}.$$

Notation 7.1.8. Let $\alpha < \omega_1$ and $\lambda = (x_1, ..., x_l) \in \mathcal{T}_\alpha$, Definition 7.1.6 yields that x_l is a successor ordinal number, i.e., there is a limit (or zero) ordinal ξ such that $x_l = \xi + \kappa(x_l)$. Denote

$$b(\lambda) = \xi \text{ and}$$
$$\kappa(\lambda) = \kappa(x_l).$$

For example, for $\lambda = (\omega \cdot 2, \omega + n) \in \mathcal{T}_{\omega\cdot 2}$ we have $\kappa(\lambda) = n$ and $b(\lambda) = \omega$.

Remark 7.1.9. Let $\alpha < \omega_1$ and $\lambda \in \mathcal{T}_\alpha$.

(a) If μ is a predecessor of a λ, then there exist $I \in \mathcal{D}^{\kappa(\mu)}$ and $\nu \in \mathcal{T}_{b(\mu)}$ such that

$$\lambda = \mu^\frown I^\frown \nu.$$

If, furthermore, λ is an immediate successor of μ then ν is a root of $\mathcal{T}_{b(\mu)}$.

(b) If $\nu \in \mathcal{T}_{b(\lambda)+\kappa(\lambda)}$ then $\lambda \oslash \nu \in \mathcal{T}_\alpha$.

(c) λ is a root if and only if $\lambda(i) \in [1, \omega_1)$ for every $1 \leqslant i \leqslant |\lambda|$.

7.2 Operators and spaces indexed over $\mathcal{T}_\alpha$

In this section, following the recursive definition of the trees $\mathcal{T}_\alpha$, we define, for every $\lambda \in \mathcal{T}_\alpha$, the θ_λ-compression $T_\lambda : L_p \to L_p$, which will be used to directly define $X_\lambda = T_\lambda(L_p^{\kappa(\lambda),0})$, but also later to study distributional copies of the FDD $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$.

Definition 7.2.1. Recursively on the countable ordinal numbers α, we define, for every $\lambda \in \mathcal{T}_\alpha$, a number $0 < \theta_\lambda \leqslant 1$ and a θ_λ-compression $T_\lambda\colon L_p \to L_p$ as follows.

(a) Let $\alpha = n$ and $\lambda = (n) \in \mathcal{T}_n$. We let $\theta_\lambda = 1$ and $T_\lambda = id$.

(b) Let α be a limit ordinal and $\lambda \in \mathcal{T}_\alpha$. Then there are $1 \leqslant \beta < \alpha$ and $\mu \in \mathcal{T}_\beta$ such that $\lambda = (\alpha)^\frown\mu$. We let $\theta_\lambda = \theta_\mu$ and $T_\lambda = T_\beta^\alpha \circ T_\mu$, where T_β^α is as in Notation 5.2.5.

(c) Let $\alpha = \beta + \kappa(\alpha)$ be an infinite successor ordinal and $\lambda \in \mathcal{T}_\alpha$. Then

   (i) if $\lambda = (\alpha)$, let $\theta_\lambda = 1$ and $T_\lambda = id$.

   (ii) if $\lambda = (\alpha, I)^\frown\mu$, where $I \in \mathcal{D}^{\kappa(\alpha)}$ and $\mu \in \mathcal{T}_\beta$, let $\theta_\lambda = |I|\theta_\mu$ and $T_\lambda = T_I T_\mu$, where T_I is the $|I|$-compression as in Notation 5.2.3.

Example 7.2.2. Let $\lambda = (\omega \cdot 2, \omega + n, I, \omega, k) \in \mathcal{T}_{\omega\cdot 2}$, where $n \in \mathbb{N}$, $I \in \mathcal{D}^n$ and $k \in \mathbb{N}$. Then,

$$\theta_\lambda = \theta_{(\omega+n,I,\omega,k)} = |I|\theta_{(\omega,k)} = |I|\theta_{(k)} = |I|$$

and

$$T_\lambda = T^{\omega\cdot 2}_{\omega+n} T_{(\omega+n,I,\omega,k)} = T^{\omega\cdot 2}_{\omega+n} T_I T_{(\omega,k)} = T^{\omega\cdot 2}_{\omega+n} T_I T^\omega_k.$$

Remark 7.2.3. Let $\alpha < \omega_1$ and $\lambda \in \mathcal{T}_\alpha$.

(a) If $b(\lambda)$ is infinite, for $I \in \mathcal{D}^{\kappa(\lambda)}$ and $\mu \in \mathcal{T}_{b(\lambda)}$, we have

$$\theta_{\lambda^\frown I^\frown \mu} = \theta_\lambda |I| \theta_\mu \text{ and } T_{\lambda^\frown I^\frown \mu} = T_\lambda T_I T_\mu.$$

(b) For $\mu \in \mathcal{T}_{b(\lambda)+\kappa(\lambda)}$, we have

$$\theta_{\lambda \oslash \mu} = \theta_\lambda \theta_\mu \text{ and } T_{\lambda \oslash \mu} = T_\lambda T_\mu.$$

In the next remark, we provide an explicit definition of θ_λ and T_λ, although it will not be used directly.

Remark 7.2.4. Let $\alpha < \omega_1$ and $\lambda = (x_1, \dots, x_l) \in \mathcal{T}_\alpha$.

(a) Following the convention $\prod_\varnothing = 1$,

$$\theta_\lambda = \prod_{\substack{1 \leqslant i \leqslant l \\ x_i \in \mathcal{D}}} |x_i| = \prod_{\substack{1 \leqslant i < l \\ x_i \in [1, \omega_1)}} 2^{-\kappa(x_i)}.$$

(b) For $1 \leqslant j \leqslant l$ define

$$T_j = \begin{cases} id & : j = 1 \text{ or } x_{j-1} \in \mathcal{D}, \\ T_{x_j}^{x_{j-1}} & : x_{j-1}, x_j \in [1, \omega_1), \\ T_J & : x_j = J \in \mathcal{D}. \end{cases}$$

Then, $T_\lambda = T_1 T_2 \cdots T_l$.

For example, if $\lambda = (\omega \cdot 3, \omega + n, I, \omega, k) \in \mathcal{T}_{\omega \cdot 3}$ then $\theta_\lambda = |I| = 2^{-n}$ and $T_\lambda = T_{\omega+n}^{\omega \cdot 3} T_I T_k^\omega$.

Definition 7.2.5. Let $\alpha < \omega_1$ and $\lambda \in \mathcal{T}_\alpha$.

(a) For $I \in \mathcal{D}_{\kappa(\lambda)-1}$, let

$$h_I^\lambda = T_\lambda(h_I),$$

(b) and let

$$\begin{aligned} X_\lambda &= T_\lambda(L_p^{\kappa(\lambda),0}) = \langle h_I^\lambda : I \in \mathcal{D}_{\kappa(\lambda)-1} \rangle \text{ and} \\ Y_\lambda &= [X_\mu : \mu \sqsupseteq \lambda]. \end{aligned}$$

Remark 7.2.6. Let $1 \leqslant \alpha < \omega_1$. Definitions 7.2.1 and 7.2.5 immediately yield the following recursive structure for the spaces $(X_\lambda)_{\lambda \in \mathcal{T}_\alpha}$, reflecting the recursive construction of the space $R_\alpha^{p,0}$. The remarks given inside Definition 7.1.3 apply here as well.

(a) If $\alpha = n$, then $\theta_n = 1$ and $X_\lambda = L_p^{n,0}$.

(b) If α is a limit ordinal and $\lambda \in \mathcal{T}_\alpha$, then $\lambda = (\alpha)^\frown\mu$ for some $\mu \in \mathcal{T}_\beta$ with $\beta < \alpha$, and $\theta_\lambda = \theta_\mu$ and $X_\lambda = T_\beta^\alpha(X_\mu)$.

(c) If $\alpha = \beta + \kappa(\alpha)$ is a successor and $\lambda \in \mathcal{T}_\alpha$, there are two possibilities for X_λ:

(a) If $\lambda = (\alpha)$, then $\theta_\lambda = 1$ and $X_\lambda = L_p^{\kappa(\alpha),0}$.

(b) If $\lambda = (\alpha, I)^\frown\mu$ with $I \in \mathcal{D}^{\kappa(\alpha)}$ and $\mu \in \mathcal{T}_\beta$, then $\theta_\lambda = |I|\theta_\mu$ and $X_\lambda = T_I(X_\mu)$.

7.3 The finite-dimensional decomposition of $R_\alpha^{p,0}$

In this section, we show that, under a suitable enumeration, the family $((h_I^\lambda)_{I\in\mathcal{D}_{\kappa(\lambda)-1}})_{\lambda\in\mathcal{T}_\alpha}$ forms a martingale difference sequence spanning $R_\alpha^{p,0}$. This is essentially [4, Proposition 2.8], adapted to our notation and framework. In particular, $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ is an unconditional FDD of $R_\alpha^{p,0}$.

We first show that $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ spans $R_\alpha^{p,0}$.

Proposition 7.3.1. Let $\alpha < \omega_1$. Then,

$$R_\alpha^{p,0} = [(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}] = [(Y_\lambda)_{\lambda\in\mathcal{R}oot(\mathcal{T}_\alpha)}].$$

Proof. The second equality is immediate. For the first, we argue by induction on α. Let $\alpha = n \in \mathbb{N}$. By Definition 7.1.3(a), $\mathcal{T}_n = \{(n)\}$. For $\lambda = (n)$, by Remark 7.2.6, $X_\lambda = L_p^{n,0} = R_n^{p,0}$.

Let α be a limit ordinal. By Definition 7.1.3 (b), every $\lambda \in \mathcal{T}_\alpha$ has the form $(\alpha)^\frown\mu$ with $\mu \in \mathcal{T}_\beta$ for some $\beta < \alpha$. By the induction hypothesis and Remark 7.2.6,

$$[(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}] = \Big[\bigcup_{\beta<\alpha} T_\beta^\alpha\big(\bigcup_{\mu\in\mathcal{T}_\beta} X_\mu\big)\Big] = \Big[\bigcup_{\beta<\alpha} T_\beta^\alpha(R_\beta^{p,0})\Big] = R_\alpha^{p,0}.$$

Let $\alpha = \beta + \kappa(\alpha)$. By the induction hypothesis and Remark 7.2.6, we conclude:

$$\begin{aligned}[(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}] &= \Big[X_{(\alpha)} \bigcup \bigcup_{I\in\mathcal{D}^{\kappa(\alpha)}} \bigcup_{\mu\in\mathcal{T}_\beta} X_{(\alpha,I)^\frown\mu}\Big] = L_p^{\kappa(\alpha),0} + \Big[\bigcup_{I\in\mathcal{D}^{\kappa(\alpha)}} T_I\Big(\bigcup_{\mu\in\mathcal{T}_\beta} X_\mu\Big)\Big] \\ &= L_p^{\kappa(\alpha),0} + \Big[\bigcup_{I\in\mathcal{D}^{\kappa(\alpha)}} T_I(R_\beta^{p,0})\Big] = R_\alpha^{p,0}.\end{aligned}$$

□

The following is the main result of this section. Before proceeding to the proof, we introduce the necessary notation. Recall that, by Notation 7.1.8, for every $\alpha < \omega_1$ and every $\lambda \in \mathcal{T}_\alpha$, we have $\kappa(\lambda) - 1 \geqslant 0$.

Theorem 7.3.2. Let $\alpha < \omega_1$ and $(\lambda_n, I_n)_{n=1}^\infty$ be an enumeration of

$$\{(\lambda, I) : \lambda \in \mathcal{T}_\alpha \text{ and } I \in \mathcal{D}_{\kappa(\lambda)-1}\}$$

such that for all $n < m$ one of the following holds:

(a) λ_n and λ_m are incomparable, or

(b) $\lambda_n \subsetneq \lambda_m$, or

(c) $\lambda_n = \lambda_m$ and $I_n < I_m$ in the lexicographical order of $\mathcal{D}_{\kappa(\lambda_n)-1}$.

Then, $(h_{I_n}^{\lambda_n})_{n=1}^{\infty}$ is a martingale difference sequence, and by the Burkholder inequality, it is an unconditional basis of $R_\alpha^{p,0}$, and $(X_\lambda)_{\lambda\in\mathcal{T}_\alpha}$ is an unconditional FDD of the same space.

Notation 7.3.3. For $0 \leqslant a < b \leqslant 1$ and $I = [a,b)$ denote $q_I : [0,1] \to [0,1]$ given by $q_I(t) = a + (b-a)t$.

Lemma 7.3.4. For $0 \leqslant a < b \leqslant 1$, $I = [a,b)$, and $f \in L_p$ the following hold:

(a) for every $A \subset \mathbb{R}\backslash\{0\}$,

$$\big[T_I f \in A\big] = q_I\big([f \in A]\big) \text{ and}$$

(b) for every $A \subset [0,1]$,

$$\int_A (T_I f)(t)dt = |I| \int_{q_I^{-1}(A)} f(t)dt.$$

Proof. The first assertion is proved with an elementary double inclusion, of which we showcase one direction. Let $t \in [0,1]$ such that $(T_I f)(t) \in A$. Since $(T_I f)(t) \neq 0$, it follows that $t \in I$, and $(T_I f)(t) = f((b-a)^{-1}(t-a))$. Defining $s = (b-a)^{-1}(t-a)$ leads to $s \in [f \in A]$, resulting in $t = q_I(s)$.

We establish the second assertion for $f = \chi_B$, where B is a measurable subset of $[0,1]$. We will use $T_I\chi_B = \chi_{a+(b-a)B}$, where $a + (b-a)B \subset I$, and $q_I^{-1}(A) = (b-a)^{-1}[(A \cap I) - a]$. Indeed, by the homogeneity of the Lebesgue measure:

$$\begin{aligned}
\int_A (T_I\chi_B)(t)dt &= \Big|A \cap \big(a + (b-a)B\big)\Big| = \Big|(A \cap I) \cap \big(a + (b-a)B\big)\Big| \\
&= \Big|[(A \cap I) - a] \cap \big((b-a)B\big)\Big| \\
&= (b-a)\Big|[(b-a)^{-1}(A \cap I) - a] \cap B\Big| = |I| \int_{q^{-1}(A)} \chi_B(t)dt.
\end{aligned}$$

□

Proposition 7.3.5. Let $n \in \mathbb{N}$ and, for $I \in \mathcal{D}^n$, $\mathcal{A}_I$ be a σ-subalgebra of $\mathcal{B}[0,1]$. Denote

$$\mathcal{A} = \bigcap_{I\in\mathcal{D}^n} \Big\{A \subset [0,1] : \; q_I^{-1}(A) \in \mathcal{A}_I\Big\}.$$

Then, $\mathcal{A}$ is a σ-subalgebra of $\mathcal{B}[0,1]$ and the following hold.

(a) For every $I \in \mathcal{D}^n$ and $\mathcal{A}_I$-measurable function f, the function $T_I f$ is $\mathcal{A}$-measurable.

(b) For every $f \in L_p$,

$$\mathbb{E}\big(T_I f|\mathcal{A}\big) = T_I \mathbb{E}\big(f|\mathcal{A}_I\big).$$

Proof. To prove (a), let $I \in \mathcal{D}^n$, f be a $\mathcal{A}_I$-measurable function and $A \in \mathcal{B}(\mathbb{R})$. If $A \subset \mathbb{R}\backslash\{0\}$, then from Lemma 7.3.4 (a) we have that $\big[T_I f \in A\big] = q_I\big([f \in A]\big) \in \mathcal{A}$. Similarly, if $A = \{0\}$, then we have that $\big[T_I f \in A\big] = q_I\big([f = 0]\big) \cup [0,1]\backslash I \in \mathcal{A}$, because $I \in \mathcal{A}$. To prove (b), let $f \in L_p$. By (a), $T_I \mathbb{E}\big(f|\mathcal{A}_I\big)$ is $\mathcal{A}$-measurable. Therefore, it is enough to prove that, for $A \in \mathcal{A}$,

$$\int_A \Big(T_I \mathbb{E}\big(f|\mathcal{A}_I\big)\Big)(t)dt = \int_A (T_I f)(t)dt.$$

By Lemma 7.3.4 (b) we have that

$$\int_A \Big(T_I \mathbb{E}\big(f|\mathcal{A}_I\big)\Big)(t)dt = |I| \int_{q_I^{-1}(A)} \Big(\mathbb{E}\big(f|\mathcal{A}_I\big)\Big)(t)dt = |I| \int_{q_I^{-1}(A)} f(t)dt = \int_A (T_I f)(t)dt.$$

$\square$

Proof of Theorem 7.3.2. We proceed by induction on α. For $\alpha = n$, we have $R_n^{p,0} = L_p^{n,0}$, and, as is well known, the basis $(h_I)_{I \in \mathcal{D}_{n-1}}$ of $L_p^{n,0}$ forms a martingale difference sequence in the lexicographic order of $\mathcal{D}_{n-1}$.

Let α be a limit ordinal. For every $n \in \mathbb{N}$, there exist $\beta < \alpha$ and $\mu_n \in \mathcal{T}_\beta$ such that $h_{I_n}^{\lambda_n} = T_\beta^\alpha(h_{I_n}^{\mu_n})$. For each $\beta < \alpha$, define $F(\beta) = \{n \in \mathbb{N} : \mu_n \in \mathcal{T}_\beta\}$. The sets $F(\beta)$, $\beta < \alpha$, form a partition of $\mathbb{N}$. By the induction hypothesis, for every $\beta < \alpha$, the sequence $(h_{I_n}^{\mu_n})_{n \in F(\beta)}$ is a martingale difference sequence with respect to the increasing enumeration of $F(\beta)$. Because $(T_\beta^\alpha)_{\beta<\alpha}$ are distributional embeddings with independent ranges, $(h_{I_n}^{\lambda_n})_{n \in \mathbb{N}} = ((h_{I_n}^{\mu_n})_{n \in F(\beta)})_{\beta<\alpha}$ is a martingale difference sequence.

Let $\alpha = \beta + \kappa(\alpha)$, where β is a limit ordinal, and let $n \in \mathbb{N}$ be arbitrary. Denote

$$\sigma = \sigma(h_{I_k}^{\lambda_k} : 1 \leqslant k \leqslant n-1).$$

We will show that $\mathbb{E}(h_{I_n}^{\lambda_n} \mid \sigma) = 0$. We distinguish two cases for λ_n: either $\lambda_n = (\beta + \kappa(\alpha))$, or $\lambda_n \sqsupseteq (\beta + \kappa(\alpha))^\frown J_n$ for some $J_n \in \mathcal{D}^{\kappa(\alpha)}$. In the first case, by the enumeration of $\mathcal{T}_\alpha$, $\lambda_1 = \lambda_2 = \cdots = \lambda_n = (\beta + \kappa(\alpha))$. Since $(h_I)_{I \in \mathcal{D}_{\kappa(\alpha)-1}}$ is a martingale difference sequence in the lexicographic order of $\mathcal{D}_{\kappa(\alpha)-1}$, it follows that $\mathbb{E}(h_{I_n}^{\lambda_n} \mid \sigma) = 0$. In the second case, we have $h_{I_n}^{\lambda_n} = T_{J_n}(h_{I_n}^{\mu_n})$ for some $\mu_n \in \mathcal{T}_\beta$. For every $J \in \mathcal{D}^{\kappa(\alpha)}$, define

$$F(J) = \{1 \leqslant k \leqslant n-1 : \lambda_k \sqsupseteq (\beta + \kappa(\alpha))^\frown J\}.$$

If $k \in F(J)$, then $h_{I_k}^{\lambda_k} = T_J(h_{I_k}^{\mu_k})$ for some $\mu_k \in \mathcal{T}_\beta$. For each $J \in \mathcal{D}^{\kappa(\alpha)}$, let

$$\mathcal{A}_J = \sigma(h_{I_k}^{\mu_k} : k \in F(J)).$$

Observe $\sigma \subset \mathcal{A}$, where $\mathcal{A}$ is as in Proposition 7.3.5. By the induction hypothesis, $(h_{I_k}^{\mu_k})_{k\in F(J_n)} \cup \{h_{I_n}^{\mu_n}\}$ is a martingale difference sequence in the increasing enumeration of $F(J_n) \cup \{n\}$. Combining this with Proposition 7.3.5 (b), we obtain that

$$\mathbb{E}(h_{I_n}^{\lambda_n} \mid \mathcal{A}) = \mathbb{E}(T_{J_n} h_{I_n}^{\mu_n} \mid \mathcal{A}) = T_{J_n}\mathbb{E}(h_{I_n}^{\mu_n} \mid \mathcal{A}_{J_n}) = 0.$$

Therefore, $\mathbb{E}(h_{I_n}^{\lambda_n} \mid \sigma) = 0$. □

Remark 7.3.6. Let $\alpha < \omega_1$ and $(\lambda_n, I_n)_{n=1}^{\infty}$ be an enumeration of $\{(\lambda, I) : \lambda \in \mathcal{T}_\alpha,\ I \in \mathcal{D}_{\kappa(\lambda)-1}\}$ as in Theorem 7.3.2. By Example 3.2.9, if $(b_{I_n}^{\lambda_n})_{n=1}^{\infty}$ is a distributional copy of $(h_{I_n}^{\lambda_n})_{n=1}^{\infty}$, then the orthogonal projection P onto $[(b_{I_n}^{\lambda_n})_{n=1}^{\infty}]$ is given by

$$P(f) = \sum_{n=1}^{\infty} \langle \|b_{I_n}^{\lambda_n}\|_q^{-1} b_{I_n}^{\lambda_n}, f\rangle \|b_{I_n}^{\lambda_n}\|_p^{-1} b_{I_n}^{\lambda_n}.$$

7.4 Independent decompositions of $R_\alpha^{p,0}$

In this section, we show that for a limit ordinal number α, the space $R_\alpha^{p,0}$ is the independent sum of the successor ordinal spaces $R_{b(\lambda)+\kappa(\lambda)}^{p,0}$, $\lambda \in \mathcal{R}oot(\mathcal{T}_\alpha)$. This is derived from Proposition 7.3.1, which states that the $R_\alpha^{p,0}$ space is the closed linear span of the spaces $(Y_\lambda)_{\lambda\in\mathcal{R}oot(\mathcal{T}_\alpha)}$, and from Proposition 7.4.1, which states that the spaces $(Y_\lambda)_{\lambda\in\mathcal{R}oot(\mathcal{T}_\alpha)}$ are independent and for every $\lambda \in \mathcal{R}oot(\mathcal{T}_\alpha)$ the spaces $R_{b(\lambda)+\kappa(\lambda)}^{p,0}$ and Y_λ are distributionally isomorphic.

While the following statement holds in general, the second part is non-trivial only for limit ordinals α, that is, when $\mathcal{R}oot(\mathcal{T}_\alpha)$ is not a singleton. The second part, the independence of the spaces $(Y_\lambda)_{\lambda\in\mathcal{R}oot(\mathcal{T}_\alpha)}$, is used in Chapter 9, where we prove that distributional embedding schemes induce distributional embeddings.

Proposition 7.4.1. For $\alpha < \omega_1$, the following hold.

(a) For every $\lambda \in \mathcal{T}_\alpha$,

$$T_\lambda\big(R_{b(\lambda)+\kappa(\lambda)}^{p,0}\big) = Y_\lambda.$$

(b) The spaces $(Y_\lambda)_{\lambda\in\mathcal{R}oot(\mathcal{T}_\alpha)}$ are independent, and in particular,

$$R_\alpha^{p,0} \equiv^{\mathrm{dist}} \Big(\sum_{\lambda\in\mathcal{R}oot(\mathcal{T}_\alpha)} R_{b(\lambda)+\kappa(\lambda)}^{p,0}\Big)_{\mathrm{Ind},p}.$$

Proof. Firstly, we prove (a) by induction on α. If $\alpha = n \in \mathbb{N}$, $\mathcal{T}_\alpha = \{(n)\}$. For $\lambda = (n)$ we have that $T_\lambda = id$, and thus,

$$T_\lambda(R_{b(\lambda)+\kappa(\lambda)}^{p,0}) = R_n^{p,0} = X_{(n)} = Y_{(n)}.$$

Let α be a limit ordinal. Let $\lambda \in \mathcal{T}_\alpha$. Then $\lambda = (\alpha)^\frown\mu$, where $\mu \in \mathcal{T}_\beta$, for some $\beta < \alpha$. By the induction hypothesis, we observe that

$$\begin{aligned} T_\lambda\big(R^{p,0}_{b(\lambda)+\kappa(\lambda)}\big) &= T^\alpha_\beta T_\mu\big(R^{p,0}_{b(\mu)+\kappa(\mu)}\big) = T^\alpha_\beta Y_\mu = \big[T^\alpha_\beta X_{\mu'} : \mu \sqsubseteq \mu'\big] \\ &= \big[T^\alpha_\beta T_{\mu'} L_p^{\kappa(\mu'),0} : \mu \sqsubseteq \mu'\big] = \big[T_{\lambda'} L_p^{\kappa(\lambda'),0} : \lambda \sqsubseteq \lambda'\big] = [X_{\lambda'} : \lambda \sqsubseteq \lambda'] = Y_\lambda. \end{aligned}$$

Now, let $\alpha = \beta + \kappa(\alpha)$. Let $\lambda \in \mathcal{T}_\alpha$. Then $\lambda = (\beta + \kappa(\alpha))$ or $\lambda = (\beta+\kappa(\alpha))^\frown I^\frown\mu$, where $I \in \mathcal{D}^{\kappa(\alpha)}$ and $\mu \in \mathcal{T}_\beta$. In the first case, $T_\lambda = id$, consequently,

$$T_\lambda\big(R^{p,0}_{b(\lambda)+\kappa(\lambda)}\big) = R^{p,0}_\alpha = Y_\lambda.$$

For the latter case, by the induction hypothesis, we observe that

$$\begin{aligned} T_\lambda\big(R^{p,0}_{b(\lambda)+\kappa(\lambda)}\big) &= T_I T_\mu\big(R^{p,0}_{b(\mu)+\kappa(\mu)}\big) = T_I Y_\mu = \big[T_I X_{\mu'} : \mu \sqsubseteq \mu'\big] \\ &= \big[T_I T_{\mu'} L_p^{\kappa(\mu'),0} : \mu \sqsubseteq \mu'\big] = \big[T_{\lambda'} L_p^{\kappa(\lambda'),0} : \lambda \sqsubseteq \lambda'\big] = [X_{\lambda'} : \lambda \sqsubseteq \lambda'] = Y_\lambda. \end{aligned}$$

Now, we prove (b) by induction on α. From Remark 7.1.4 (b) the finite and successor cases trivially hold. Let α be a limit ordinal and note $\mathcal{Root}(\mathcal{T}_\alpha) = \{(\alpha)^\frown\mu : \mu \in \mathcal{Root}(\mathcal{T}_\beta), \beta < \alpha\}$. For every $\beta < \alpha$, by the induction hypothesis and because T^α_β is a distributional embedding, the spaces $(Y_{(\alpha)^\frown\mu})_{\mu\in\mathcal{T}_\beta} = (T^\alpha_\beta(Y_\mu))_{\mu\in\mathcal{T}_\beta}$ are independent. Because the operators $(T^\alpha_\beta)_{\beta<\alpha}$ have independent ranges, the spaces $((Y_{(\alpha)^\frown\mu})_{\mu\in\mathcal{T}_\beta})_{\beta<\alpha}$ are independent. Proposition 7.3.1 and the above assertion (a) and (b) imply

$$R^{p,0}_\alpha \equiv^{\text{dist}} \Big(\sum_{\lambda\in\mathcal{Root}(\mathcal{T}_\alpha)} R^{p,0}_{b(\lambda)+\kappa(\lambda)}\Big)_{\text{Ind},p}.$$

□

7.5 The supports of the spaces X_λ, $\lambda \in \mathcal{T}_\alpha$

In this section, we show that, for two incomparable $\lambda, \mu \in \mathcal{T}_\alpha$, the spaces X_λ and X_μ have disjoint supports if and only if the sequences λ and μ first split at a position where their entries are dyadic intervals. This fact is used in Chapters 8 and 9, where distributional embedding schemes are studied.

Recall that all set-theoretic relations are understood in the essential sense.

Definition 7.5.1. Let X be a (not necessarily closed) subspace of L_p and A be a measurable subset of $[0,1]$.

(a) We write $\text{supp}(X) \subset A$ if, for every $f \in X$, $\text{supp}(f) \subset A$.

(b) If $\text{supp}(X) \subset A$ and, additionally, for every measurable subset B of $[0,1]$ such that $\text{supp}(X) \subset B$, we have $A \subset B$, then we write $\text{supp}(X) = A$.

It is standard that there exists a set A such that $\operatorname{supp}(X) = A$, and that A is essentially unique.

Proposition 7.5.2. For $\alpha < \omega_1$ and $\lambda \in \mathcal{T}_\alpha$, the following hold.

(a) We have

$$\operatorname{supp}(X_\lambda) = \operatorname{supp}(T_\lambda \chi_{[0,1]}),$$

and in particular $\theta_\lambda = |\operatorname{supp}(X_\lambda)|$.

(b) If μ is a successor of λ, then

$$\operatorname{supp}(X_\mu) \subset \operatorname{supp}(X_\lambda).$$

More precisely, if $\mu = \lambda^\frown I^\frown \nu$, where $I \in \mathcal{D}^{\kappa(\lambda)}$ and $\nu \in \mathcal{T}_{b(\lambda)}$, then

$$\operatorname{supp}(X_\mu) \subset [T_\lambda \chi_I = 1].$$

Proof. Both assertions are based on Remark 5.2.4. For $f \in X_\lambda$ there is $g \in L_p^{\kappa(\lambda),0}$ such that $f = T_\lambda g$. Recalling that $T_\lambda h_{[0,1]} \in X_\lambda$, we obtain

$$\operatorname{supp}(f) \subset \operatorname{supp}(T_\lambda \chi_{[0,1]}) = \operatorname{supp}(T_\lambda h_{[0,1]}) \subset \operatorname{supp}(X_\lambda).$$

In other words, $\operatorname{supp}(X_\lambda) \subset [T_\lambda \chi_{[0,1]} = 1] \subset \operatorname{supp}(X_\lambda)$. Also,

$$|\operatorname{supp}(T_\lambda \chi_{[0,1]})| = |[T_\lambda \chi_{[0,1]} = 1]| = \theta_\lambda |[\chi_{[0,1]} = 1]| = \theta_\lambda.$$

Next, for $\mu = \lambda^\frown I^\frown \nu$,

$$\operatorname{supp}(T_\mu \chi_{[0,1]}) = \operatorname{supp}(T_\lambda T_I T_\nu \chi_{[0,1]}) \subset \operatorname{supp}(T_\lambda T_I \chi_{[0,1]}) = \operatorname{supp}(T_\lambda \chi_I).$$

∎

Remark 7.5.3. By Proposition 7.5.2 (b), for $\alpha < \omega_1$ and $\lambda \in \mathcal{T}_\alpha$, we have $\operatorname{supp}(Y_\lambda) = \operatorname{supp}(X_\lambda)$.

Definition 7.5.4. Let $\alpha < \omega_1$ and $\lambda, \mu \in \mathcal{T}_\alpha$. We say that λ and μ are disjointly supported if

$$\operatorname{supp}(X_\lambda) \cap \operatorname{supp}(X_\mu) = \varnothing.$$

Proposition 7.5.5. Let $\alpha < \omega_1$. Let $\lambda, \mu \in \mathcal{T}_\alpha$ be incomparable and denote

$$i_0 = \min\{i : \lambda(i) \neq \mu(i)\}.$$

(a) The entries $\lambda(i_0)$ and $\mu(i_0)$ are either both ordinals or both intervals.

(b) λ and μ are disjointly supported if and only if $\lambda(i_0)$ and $\mu(i_0)$ are both intervals.

Proof. We prove the second assertion, since the first follows directly from Definition 7.1.6. Assume that $\lambda(i_0)$ and $\mu(i_0)$ are both intervals, say $\lambda(i_0) = I$ and $\mu(i_0) = J$. Then $\lambda = \nu^\frown I^\frown\tilde{\lambda}$ and $\mu = \nu^\frown J^\frown\tilde{\mu}$, where $I \neq J \in \mathcal{D}^{\kappa(\nu)}$ and $\tilde{\lambda}, \tilde{\mu} \in \mathcal{T}_{b(\nu)}$. By Proposition 7.5.2 (b), we have $\mathrm{supp}(X_\lambda) \subset [T_\nu\chi_I = 1]$ and $\mathrm{supp}(X_\mu) \subset [T_\nu\chi_J = 1]$. Remark 5.2.4, together with the disjointness of I and J, implies that $\mathrm{supp}(X_\lambda)$ and $\mathrm{supp}(X_\mu)$ are disjoint.

We prove the other direction by induction on α, i.e., in the α inductive step we assume that for every $\beta < \alpha$, any λ, μ in $\mathcal{T}_\beta$ that split at an ordinal are not disjointly supported and we will show this also holds in $\mathcal{T}_\alpha$. The finite case $\alpha = n$ trivially holds, because $\mathcal{T}_n$ is a singleton.

Let α be a limit ordinal, and assume that $\lambda, \mu \in \mathcal{T}_\alpha$ split at an ordinal. We have that $\lambda = \nu^\frown\lambda(i_0)^\frown\tilde{\lambda}$ and $\mu = \nu^\frown\mu(i_0)^\frown\tilde{\mu}$. From Definition 7.1.6 we observe that $\lambda(i_0 - 1) = \mu(i_0 - 1)$ is a limit ordinal. Thus, either $\lambda(i_0 - 1) = \alpha$ or $\lambda(i_0 - 1) < \alpha$. For the first case, we have that $\lambda = (\alpha)^\frown\lambda(i_0)^\frown\tilde{\lambda}$ and $\mu = (\alpha)^\frown\mu(i_0)^\frown\tilde{\mu}$. Thus, $X_\lambda = T^\alpha_{\lambda(i_0)}(X_{(\lambda(i_0))^\frown\tilde{\lambda}})$ and $X_\mu = T^\alpha_{\mu(i_0)}(X_{(\mu(i_0))^\frown\tilde{\mu}})$. Recall that $T^\alpha_{\lambda(i_0)}$ and $T^\alpha_{\mu(i_0)}$ have independent ranges. Because $\mathrm{supp}(X_\lambda) = [T_\lambda h_{[0,1]} \neq 0]$ and $\mathrm{supp}(X_\mu) = [T_\mu h_{[0,1]} \neq 0]$, they are independent. From Proposition 7.5.2 (a) we have that $\mathrm{supp}(X_\lambda)$ and $\mathrm{supp}(X_\mu)$ are of positive measure. Hence, λ and μ are not disjointly supported. For the latter case, we have that $\lambda = \nu^\frown\lambda(i_0)^\frown\tilde{\lambda}$ and $\mu = \nu^\frown\mu(i_0)^\frown\tilde{\mu}$, where $\lambda(i_0 - 1) = \mu(i_0 - 1) < \alpha$. We have that $\lambda(i_0 - 1)^\frown\lambda(i_0)^\frown\tilde{\lambda}$ and $\mu(i_0 - 1)^\frown\mu(i_0)^\frown\tilde{\mu}$ belong to $\mathcal{T}_{\lambda(i_0-1)}$ and split at an ordinal. By the induction hypothesis, we have that $\lambda(i_0 - 1)^\frown\lambda(i_0)^\frown\tilde{\lambda}$ and $\mu(i_0 - 1)^\frown\mu(i_0)^\frown\tilde{\mu}$ are not disjointly supported. Combining this with Remark 5.2.4, we obtain that λ and μ are not disjointly supported.

Now, let $\alpha = \beta + \kappa(\alpha)$ and assume that $\lambda, \mu \in \mathcal{T}_\alpha$ split at an ordinal. By the definition of $\mathcal{T}_\alpha$, there exist $I \in \mathcal{D}^{\kappa(\alpha)}$ and $\tilde{\lambda}, \tilde{\mu} \in \mathcal{T}_\beta$ such that $\lambda = (\beta + \kappa(\alpha))^\frown I^\frown\tilde{\lambda}$ and $\mu = (\beta + \kappa(\alpha))^\frown I^\frown\tilde{\mu}$ and $\tilde{\lambda}$ and $\tilde{\mu}$ split at an ordinal. By the induction hypothesis we have that $\tilde{\lambda}$ and $\tilde{\mu}$ are not disjointly supported. Thus, Remark 5.2.4 implies that λ and μ are not disjointly supported. $\square$

Chapter 8

Distributional embedding schemes

In this chapter, we present distributional embedding schemes, a fundamental concept in our paper. For ordinal numbers $\alpha, \beta < \omega_1$, a distributional embedding scheme from $R^{p,0}_{\alpha}$ to $R^{p,0}_{\beta}$ is an object Ψ comprising a collection of finite subsets of $\mathcal{T}_\beta$ with a tree structure resembling $\mathcal{T}_\alpha$ and an accompanying collection of distributional embeddings of finite-dimensional L_p-spaces. The object Ψ is designed so that it induces a distributional embedding $T_\Psi : R^{p,0}_{\alpha} \to R^{p,0}_{\beta}$. In Chapter 10, this tool will facilitate the stabilization of operators. Although T_Ψ is easily defined as a linear operator, establishing that it preserves the distribution is an extensive process that will be concluded in Chapter 9. Here, we will deliver essential estimates related to Ψ and introduce specific sub-distributional embedding schemes of Ψ, which are utilized in Chapter 9.

We define distributional embedding schemes in a slightly broader setting than that described in the preceding paragraph. Specifically, we work with subtrees of $\mathcal{T}_\alpha$ and with the corresponding subspaces of $R^{p,0}_{\alpha}$ that they determine. This degree of generality is unnecessary for the reduction to diagonal operators but becomes essential for the scalar reduction (see Section 10.3). Although this latter step is not included in the thesis, the additional generality does not alter the proof, apart from minor adjustments in notation.

Definition 8.0.1. Let $\alpha < \omega_1$, and let $\mathcal{E}$ be a subset of $\mathcal{T}_\alpha$.

(a) We call $\mathcal{E}$ a subtree of $\mathcal{T}_\alpha$ if, for every $\lambda_0 \in \mathcal{E}$, $\{\lambda \in \mathcal{T}_\alpha : \lambda \sqsubseteq \lambda_0\} \subset \mathcal{E}$.

(b) We call $\mathcal{E}$ an interval-branching subtree of $\mathcal{T}_\alpha$ if it is a subtree and, for every $\lambda_0 \in \mathcal{E}$ that is non-maximal in $\mathcal{E}$ and $I \in \mathcal{D}^{\kappa(\lambda_0)}$, there exists $\lambda \in \mathcal{E}$ such that $\lambda_0{}^\frown I \sqsubseteq \lambda$.

(c) We call $\mathcal{E}$ a perfectly branching subtree of $\mathcal{T}_\alpha$ if it is a subtree and, for every $\lambda_0 \in \mathcal{E}$ that is non-maximal in $\mathcal{T}_\alpha$ and for every $I \in \mathcal{D}^{\kappa(\lambda_0)}$,

$$\sup\Big\{b(\lambda) + \kappa(\lambda) : \lambda \in \mathcal{E} \text{ and } \lambda_0{}^\frown I \sqsubseteq \lambda\Big\} = b(\lambda_0).$$

Given a subtree $\mathcal{E}$ of $\mathcal{T}_\alpha$, we define the subspace $R_{\mathcal{E}} = [(X_\lambda)_{\lambda \in \mathcal{E}}]$ of $R^{p,0}_{\alpha}$.

Remark 8.0.2. A subtree $\mathcal{E}$ is a downward-closed subset of $\mathcal{T}_\alpha$. It is termed interval-branching if every element λ either has no extensions or can be extended along all intervals $I \in \mathcal{D}^{\kappa(\lambda)}$. A subtree is perfectly branching if the complexity of its extensions matches that of $\mathcal{T}_\alpha$.

We will prove the main result concerning distributional embedding schemes, namely Theorem 9.0.1, in the setting of interval-branching subtrees. For the purposes of our main Theorem 10.2.1 this additional generality is not required, since it would have been sufficient to consider the full trees $\mathcal{E} = \mathcal{T}_\alpha$. Our reason for working in this broader setting is to establish a framework that may later serve in proving the primary factorization property for the spaces $R^{p,0}_{\omega^\alpha}$, $1 \leqslant \alpha < \omega_1$, (see Section 10.3). Perfectly branching subtrees were introduced for the same purpose; although they play no role in the present argument and will not be mentioned again, we retain them to maintain continuity with a prospective extension of this work.

Definition 8.0.3. Let $\alpha, \beta < \omega_1$ and consider the subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$. Let $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu\in\mathcal{M}_\lambda})_{\lambda\in\mathcal{E}}$ be a collection such that, for every $\lambda \in \mathcal{E}$, $\mathcal{M}_\lambda$ is a finite subset of $\mathcal{F}$ and, for every $\mu \in \mathcal{M}_\lambda$, $J_\mu : L_p^{\kappa(\lambda),0} \to L_p^{\kappa(\mu),0}$ is a distributional embedding. We call Ψ a distributional embedding scheme of $R_\mathcal{E}$ in $R_\mathcal{F}$ if the following hold.

(a) For $\lambda \in \mathcal{E}$,

  (i) the members of $\mathcal{M}_\lambda$ are pairwise disjointly supported and

  (ii) $\sum\limits_{\mu\in\mathcal{M}_\lambda} \theta_\mu = \theta_\lambda$.

(b) If $\lambda_1, \lambda_2 \in \mathcal{E}$ are incomparable and $\mu_1 \in \mathcal{M}_{\lambda_1}$, $\mu_2 \in \mathcal{M}_{\lambda_2}$, then μ_1, μ_2 are incomparable.

(c) Let $\lambda \in \mathcal{E}$ be a non-root, λ_0 be the immediate predecessor of λ, and let $I \in \mathcal{D}^{\kappa(\lambda_0)}$ such that $\lambda_0{}^\frown I \sqsubseteq \lambda$. Then, for every $\mu \in \mathcal{M}_\lambda$, there exist $\mu_0 \in \mathcal{M}_{\lambda_0}$ and $P \in \mathcal{D}^{\kappa(\mu_0)}$ such that

$$\mu_0{}^\frown P \sqsubseteq \mu \text{ and } P \subset [J_{\mu_0} h_{\pi(I)} = \epsilon(I)].$$

For the definitions of $\pi(I)$ and $\epsilon(I)$, see Notation 2.3.1.

Remark 8.0.4. A brief explanation of the definition of Ψ is necessary.

(i) When focusing solely on the tree structure of the subtrees $\mathcal{E}$ and $\mathcal{F}$, then $(\mathcal{M}_\lambda)_{\lambda\in\mathcal{E}}$ is a disjoint collection of non-empty finite antichains of $\mathcal{F}$, indexed over $\mathcal{E}$, such that, for every $\lambda_1, \lambda_2 \in \mathcal{E}$, $\lambda_1 \sqsubsetneq \lambda_2$ if and only if there exists $\mu_1 \in \mathcal{M}_{\lambda_1}$, $\mu_2 \in \mathcal{M}_{\lambda_2}$ such that $\mu_1 \sqsubsetneq \mu_2$. While this aspect helps establish some properties of Ψ, Ψ encodes significantly more information required for representing $R_\mathcal{E}$ in $R_\mathcal{F}$.

(ii) Each set $\mathcal{M}_\lambda$ will be used to define a distributional copy of X_λ in $\langle X_\mu : \mu \in \mathcal{M}_\lambda\rangle$ via $(J_\mu)_{\mu\in\mathcal{M}_\lambda}$. Item (a) asserts that the spaces $(X_\mu)_{\mu\in\mathcal{M}_\lambda}$ are disjointly supported and the measure of the support of their union aligns with the measure of the support of X_λ.

(iii) While (b) appears to be an innocent condition related only to the tree structure, when considered alongside the other items, it guarantees the preservation of independence.

(iv) Item (c) indicates that each $\mu \in \mathcal{M}_\lambda$ is the (not necessarily immediate) successor of some $\mu_0 \in \mathcal{M}_{\lambda_0}$ along an interval P. Although it may seem logical to choose P so that $\epsilon(P) = \epsilon(I)$, this perspective is overly simplistic because of the involvement of J_{μ_0}. Therefore, P is restricted by a condition linked to J_{μ_0} and $\pi(I)$ and $\epsilon(I)$.

We provide the next piece of notation for context, but we will not address it yet.

Notation 8.0.5. Fix $\alpha, \beta < \omega_1$, subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$, and a distributional embedding scheme $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu\in\mathcal{M}_\lambda})_{\lambda\in\mathcal{E}}$ of $R_\mathcal{E}$ in $R_\mathcal{F}$. We denote by $T_\Psi : \langle (X_\lambda)_{\lambda\in\mathcal{E}}\rangle \to \langle (X_\mu)_{\mu\in\mathcal{F}}\rangle$ the linear map given by

$$T_\Psi|_{X_\lambda} = \sum_{\mu\in\mathcal{M}_\lambda} T_\mu J_\mu T_\lambda^{-1}|_{X_\lambda},$$

for $\lambda \in \mathcal{E}$. That is, for $\lambda \in \mathcal{E}$ and $I \in \mathcal{D}_{\kappa(\lambda)-1}$,

$$T_\Psi h_I^\lambda = \sum_{\mu\in\mathcal{M}_\lambda} T_\mu J_\mu h_I.$$

Remark 8.0.6. In Chapter 9, we will show that if $\mathcal{E}$ and $\mathcal{F}$ are interval-branching subtrees, then T_Ψ is a distributional embedding. The same result also holds for non-interval-branching subtrees, with an identical proof apart from heavier notation needed to distinguish the intervals along which a given $\lambda \in \mathcal{E}$ may extend. Since the proof is already long and no application of the more general statement is known to us, we state and prove the result under the interval-branching assumption.

8.1 Estimates on distributional embedding schemes

We will provide precise estimates for quantities of the form $\sum_{\nu\in\Gamma} \theta_\nu$, for subsets Γ of $\mathcal{M}_\lambda$ of a special type stemming from the tree structure of Ψ.

Notation 8.1.1. For the rest of this subsection, we fix $\alpha, \beta < \omega_1$, interval-branching subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$, and a distributional embedding scheme $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu\in\mathcal{M}_\lambda})_{\lambda\in\mathcal{E}}$ of $R_\mathcal{E}$ in $R_\mathcal{F}$. The partial relation "$\sqsubseteq$" should be understood as referring to $\mathcal{E}$ or $\mathcal{F}$, rather than $\mathcal{T}_\alpha$ or $\mathcal{T}_\beta$.

To facilitate the main statement of this subsection, we introduce some notation, which implicitly depends on Ψ.

Notation 8.1.2. Let $\lambda \in \mathcal{E}$.

(a) For $\mu \in \mathcal{F}$ denote

$$\Gamma_\lambda^\mu = \Big\{\nu \in \mathcal{M}_\lambda : \mu \sqsubseteq \nu\Big\},$$

and, for $J \in \mathcal{D}^{\kappa(\mu)}$, denote

$$\Gamma_\lambda^{\mu\frown J} = \Big\{\nu \in \mathcal{M}_\lambda : \mu\frown J \sqsubseteq \nu\Big\}.$$

(b) For $I \in \mathcal{D}^{\kappa(\lambda)}$ and $\mu \in \mathcal{M}_\lambda$ denote

$$\mathcal{D}_I^{\kappa(\mu)} = \Big\{P \in \mathcal{D}^{\kappa(\mu)} : P \subset [J_\mu h_{\pi(I)} = \epsilon(I)]\Big\}.$$

The above will be used extensively in the sequel. The reader may realize that $\mathcal{D}_I^{\kappa(\mu)}$ depends on μ, rather than only $\kappa(\mu)$. However, no confusion should arise from this.

Remark 8.1.3. Based on the above notation, the following hold.

(a) For $\lambda \in \mathcal{E}$, $I \in \mathcal{D}^{\kappa(\lambda)}$, and $\mu \in \mathcal{M}_\lambda$ we have

$$\sum_{P \in \mathcal{D}_I^{\kappa(\mu)}} |P| = |I|,$$

because $J_\mu h_{\pi(I)}$ is a distributional copy of $h_{\pi(I)}$ in $L_p^{\kappa(\mu),0}$.

(b) Definition 8.0.3 (c) is reformulated as follows. Let $\lambda_0 \in \mathcal{E}$, $I \in \mathcal{D}^{\kappa(\lambda_0)}$, and λ be an immediate successor of λ_0 with $\lambda_0{}^\frown I \sqsubseteq \lambda$. Then the collections of sets

$$\Big\{\Gamma_\lambda^{\mu^\frown P} : \mu \in \mathcal{M}_{\lambda_0} \text{ and } P \in \mathcal{D}_I^{\kappa(\mu)}\Big\}$$

forms a partition of $\mathcal{M}_\lambda$. By iteration, the same applies if we substitute "immediate successor" with "successor".

(c) There exist variations of (b), such as the following. Let $\lambda_0 \in \mathcal{E}$, $I \in \mathcal{D}^{\kappa(\lambda_0)}$, and λ be a successor of λ_0 with $\lambda_0{}^\frown I \sqsubseteq \lambda$. Let $\mu_0 \in \mathcal{F}$ and $J \in \mathcal{D}^{\kappa(\mu_0)}$ such that $\Gamma_{\lambda_0}^{\mu_0{}^\frown J} \neq \varnothing$. Then the collections of sets

$$\Big\{\Gamma_\lambda^{\mu^\frown P} : \mu \in \Gamma_{\lambda_0}^{\mu_0{}^\frown J} \text{ and } P \in \mathcal{D}_I^{\kappa(\mu)}\Big\}$$

forms a partition of $\Gamma_\lambda^{\mu_0{}^\frown J}$.

The following is the main initial result about distributional embedding schemes, focusing on specific estimates. It will be used in the next subsection, where we define sub-distributional embedding schemes of Ψ.

Proposition 8.1.4. Fix $\mu_0 \in \mathcal{F}$ and let $\lambda_0 \in \mathcal{E}$ be minimal such that $\Gamma_{\lambda_0}^{\mu_0} \neq \varnothing$, assuming it exists. The following hold.

(a) For every $\lambda \sqsupseteq \lambda_0$,

$$\sum_{\nu \in \Gamma_\lambda^{\mu_0}} \theta_\nu = \theta_{\mu_0} \frac{\theta_\lambda}{\theta_{\lambda_0}}.$$

(b) If $\mu_0 \notin \mathcal{M}_{\lambda_0}$ then, for every $J \in \mathcal{D}^{\kappa(\mu_0)}$ and $\lambda \sqsupseteq \lambda_0$,

$$\sum_{\nu \in \Gamma_\lambda^{\mu_0{}^\frown J}} \theta_\nu = \theta_{\mu_0} |J| \frac{\theta_\lambda}{\theta_{\lambda_0}}.$$

(c) If $\mu_0 \in \mathcal{M}_{\lambda_0}$ then, for every $I \in \mathcal{D}^{\kappa(\lambda_0)}$ and $J \in \mathcal{D}_I^{\kappa(\mu_0)}$, and for every $\lambda \sqsupseteq \lambda_0{}^\frown I$,

$$\sum_{\nu \in \Gamma_\lambda^{\mu_0{}^\frown J}} \theta_\nu = \theta_{\mu_0} \frac{|J|}{|I|} \frac{\theta_\lambda}{\theta_{\lambda_0}}.$$

We will prove Proposition 8.1.4 in steps, by first showing (c), then (b), and finally we will conclude (a). We start with a lemma.

Lemma 8.1.5. Let Γ be a set of disjointly supported members of $\mathcal{F}$.

(a) Assume that there is $\mu \in \mathcal{F}$ such that, for all $\nu \in \Gamma$, $\mu \sqsubseteq \nu$. Then,

$$\sum_{\nu \in \Gamma} \theta_\nu \leqslant \theta_\mu.$$

(b) Assume that there are $\mu \in \mathcal{F}$ and $I \in \mathcal{D}^{\kappa(\mu)}$ such that, for all $\nu \in \Gamma$, $\mu^\frown J \sqsubseteq \nu$. Then,

$$\sum_{\nu \in \Gamma} \theta_\nu \leqslant \theta_\mu |J|.$$

Proof. The first statement directly follows from Proposition 7.5.2 (b), according to which, $\operatorname{supp}(X_\nu)$, $\nu \in \Gamma$ are disjoint subsets of $\operatorname{supp}(X_\mu)$, and thus,

$$\sum_{\nu \in \Gamma} \theta_\nu = \sum_{\nu \in \Gamma} |\operatorname{supp}(X_\nu)| = \Big| \bigcup_{\nu \in \Gamma} \operatorname{supp}(X_\nu) \Big| \leqslant |\operatorname{supp}(X_\mu)| = \theta_\mu.$$

The proof of the second statement is a variation of the above, also involving Proposition 7.5.2 (b) stating, for $\nu \in \Gamma$, $\operatorname{supp}(X_\nu) \subset [T_\mu \chi_J = 1]$. Therefore,

$$\sum_{\nu \in \Gamma} \theta_\nu = \sum_{\nu \in \Gamma} |\operatorname{supp}(X_\nu)| = \Big| \bigcup_{\nu \in \Gamma} \operatorname{supp}(X_\nu) \Big| \leqslant |[T_\mu \chi_J = 1]| = \theta_\mu |J|,$$

because T_μ is a θ_μ-compression. $\square$

Proof of Proposition 8.1.4 (c). We proceed by induction on $n = h(\lambda) - h(\lambda_0)$, i.e., the path distance from λ_0 to λ in the tree $\mathcal{E}$. To elaborate further, the inductive hypothesis in the n'th step will state that the conclusion holds for all μ_0, λ_0, and λ, as outlined, for which $h(\lambda) - h(\lambda_0) < n$. We initiate with the base case, $n = 1$. Let $\lambda_0 \in \mathcal{E}$, $\mu_0 \in \mathcal{M}_{\lambda_0}$, $I \in \mathcal{D}^{\kappa(\lambda_0)}$ and $J \in \mathcal{D}_I^{\kappa(\mu_0)}$, and let λ be an immediate successor of λ_0 with $\lambda_0{}^\frown I \sqsubseteq \lambda$. Note that, in particular, $\theta_\lambda = \theta_{\lambda_0}|I|$. We will prove

$$\sum_{\nu \in \Gamma_\lambda^{\mu_0{}^\frown J}} \theta_\nu = \theta_{\mu_0}|J| = \theta_{\mu_0} \frac{|J|}{|I|} \frac{\theta_\lambda}{\theta_{\lambda_0}}.$$

By Lemma 8.1.5 (b), for any $\mu \in \mathcal{M}_{\lambda_0}$ and $P \in \mathcal{D}_I^{\kappa(\mu)}$, we have

$$\sum_{\nu \in \Gamma_\lambda^{\mu \frown P}} \theta_\nu \leqslant \theta_\mu |P|.$$

If we assume, towards a contradiction, that $\sum_{\nu \in \Gamma_\lambda^{\mu_0 \frown J}} \theta_\nu < \theta_{\mu_0}|J|$, then

$$\begin{aligned}
\theta_\lambda &= \sum_{\nu \in \mathcal{M}_\lambda} \theta_\nu \text{ (by Definition 8.0.3 (a))} \\
&= \sum_{\mu \in \mathcal{M}_{\lambda_0}} \sum_{P \in \mathcal{D}_I^{\kappa(\mu)}} \sum_{\nu \in \Gamma_{\lambda_0}^{\mu \frown P}} \theta_\nu \text{ (by Remark 8.1.3 (b))} \\
&< \sum_{\mu \in \mathcal{M}_{\lambda_0}} \theta_\mu \sum_{P \in \mathcal{D}_I^{\kappa(\mu)}} |P| = \sum_{\mu \in \mathcal{M}_{\lambda_0}} \theta_\mu |I| \text{ (by Remark 8.1.3 (a))} \\
&= \theta_{\lambda_0}|I| = \theta_\lambda.
\end{aligned}$$

We now perform the n'th inductive step. Let $\lambda_0 \in \mathcal{T}_\alpha$, $\mu_0 \in \mathcal{M}_{\lambda_0}$, $I \in \mathcal{D}^{\kappa(\lambda_0)}$ and $J \in \mathcal{D}_I^{\kappa(\mu_0)}$, and let $\lambda \sqsupseteq \lambda_0 \frown I$ with $h(\lambda) - h(\lambda_0) = n$. Let λ_{n-1} denote the immediate predecessor of λ and let $I_n \in \mathcal{D}^{\kappa(\lambda_{n-1})}$ such that $\lambda_{n-1} \frown I_n \sqsubseteq \lambda$. By Remark 8.1.3 (c), the sets

$$\left\{ \Gamma_\lambda^{\mu \frown P} : \mu \in \Gamma_{\lambda_{n-1}}^{\mu_0 \frown J} \text{ and } P \in \mathcal{D}_{I_n}^{\kappa(\mu)} \right\}$$

form a partition of $\Gamma_\lambda^{\mu_0 \frown J}$, and thus,

$$\sum_{\nu \in \Gamma_\lambda^{\mu_0 \frown J}} \theta_\nu = \sum_{\mu \in \Gamma_{\lambda_{n-1}}^{\mu_0 \frown J}} \sum_{P \in \mathcal{D}_{I_n}^{\kappa(\mu)}} \sum_{\nu \in \Gamma_\lambda^{\mu \frown P}} \theta_\nu.$$

By the base case, we have that, for each $\mu \in \Gamma_{\lambda_{n-1}}^{\mu_0 \frown J}$ and $P \in \mathcal{D}_{I_n}^{\kappa(\mu)}$,

$$\sum_{\nu \in \Gamma_\lambda^{\mu \frown P}} \theta_\nu = \theta_\mu |P|,$$

and therefore

$$\begin{aligned}
\sum_{\nu \in \Gamma_\lambda^{\mu_0 \frown J}} \theta_\nu &= \sum_{\mu \in \Gamma_{\lambda_{n-1}}^{\mu_0 \frown J}} \theta_\mu \sum_{P \in \mathcal{D}_{I_n}^{\kappa(\mu)}} |P| = \sum_{\mu \in \Gamma_{\lambda_{n-1}}^{\mu_0 \frown J}} \theta_\mu |I_n| \text{ (by Remark 8.1.3 (a))} \\
&= \left(\theta_{\mu_0} \frac{|J|}{|I|} \frac{\theta_{\lambda_{n-1}}}{\theta_{\lambda_0}} \right) |I_n| \text{ (by the case } n-1) \\
&= \theta_{\mu_0} \frac{|J|}{|I|} \frac{\theta_{\lambda_n}}{\theta_{\lambda_0}}.
\end{aligned}$$

$\square$

Before proving Proposition 8.1.4 (b), we establish a lemma that will also be used frequently later, in Chapter 9.

Lemma 8.1.6. Let $\lambda \in \mathcal{E}$.

(a) There exists a unique root $\mu \in \mathcal{F}$ such that $\Gamma_\lambda^\mu = \mathcal{M}_\lambda$.

(b) Let $\mu \in \mathcal{F}$ and $J \in \mathcal{D}^{\kappa(\mu)}$ such that $\Gamma_\lambda^{\mu^\frown J} \neq \varnothing$. Then there exists a unique root $\xi \in \mathcal{T}_{b(\mu)}$ such that $\mu^\frown J^\frown \xi \in \mathcal{F}$ and $\Gamma_\lambda^{\mu^\frown J} = \Gamma_\lambda^{\mu^\frown J^\frown \xi}$.

Proof. The two statements are shown with virtually the same argument. We start with statement (a). We fix an arbitrary $\nu \in \mathcal{M}_\lambda$ and let μ be the unique root such that $\mu \sqsubseteq \nu$. By Remark 7.1.9 (c) all the entries of μ are ordinal numbers. For $\nu' \in \mathcal{M}_\lambda \backslash \{\nu\}$, we will show $\nu' \in \Gamma_\lambda^\mu$. Because ν, ν' are disjointly supported, by Proposition 7.5.5, for the minimum entry i for which $\nu(i) \neq \nu'(i)$, $\nu(i)$ is some interval I, and since $\mu \sqsubseteq \nu$, $i > |\mu|$. In conclusion, $\nu'|_{|\mu|} = \nu|_{|\mu|} = \mu$.

We now demonstrate (b). Fix an arbitrary $\nu \in \Gamma_\lambda^{\mu^\frown J}$. Because $\mu^\frown J \sqsubseteq \nu$, there exists a unique root $\xi \in \mathcal{T}_{b(\mu)}$ such that $\mu^\frown J^\frown \xi \sqsubseteq \nu$. In particular, for $1 \leqslant i \leqslant |\xi|$, $\xi(i)$ is an ordinal number. Clearly, $\Gamma_\lambda^{\mu^\frown J^\frown \xi} \subset \Gamma_\lambda^{\mu^\frown J}$. For $\nu' \in \Gamma_\lambda^{\mu^\frown J} \backslash \{\nu\}$, we will show $\nu' \in \Gamma_\lambda^{\mu^\frown J^\frown \xi}$. Let i denote the minimum entry for which $\nu(i) \neq \nu'(i)$. Because $\nu, \nu' \in \Gamma_\lambda^{\mu^\frown J}$, $i \notin \{1, \ldots, |\mu| + 1\}$. Since ν, ν' are disjointly supported, for the minimum entry i for which $\nu(i) \neq \nu'(i)$, $\nu(i)$ is not an ordinal number, and thus, $i \notin \{|\mu| + 2, \ldots, |\mu| + 1 + |\xi|\}$. Therefore, $\nu'|_{|\mu|+1+|\xi|} = \nu|_{|\mu|+1+|\xi|} = \mu^\frown J^\frown \xi$. □

Proof of Proposition 8.1.4 (b). We will first prove the conclusion only in the case $\lambda = \lambda_0$. More precisely, we will prove that for every $\mu_0 \in \mathcal{F}$, for which there is a minimal $\lambda_0 \in \mathcal{T}_\alpha$ with $\Gamma_{\lambda_0}^{\mu_0} \neq \varnothing$, if $\mu_0 \notin \mathcal{M}_{\lambda_0}$ then, for every $J \in \mathcal{D}^{\kappa(\mu_0)}$, we have

$$\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0{}^\frown J}} \theta_\nu = \theta_{\mu_0} |J|. \tag{8.1}$$

This will be proved by induction on $h(\mu_0)$, i.e., the height of μ_0 in the tree $\mathcal{F}$. In the base case, μ_0 is a root. By Remark 8.0.4 (i), λ_0 is a root, and by Lemma 8.1.6 (a), $\Gamma_{\lambda_0}^{\mu_0} = \mathcal{M}_{\lambda_0}$. We may assume $\mu_0 \notin \mathcal{M}_{\lambda_0}$, as, otherwise, the conclusion vacuously holds. This, in particular, suggests that the collection of sets

$$\left\{\Gamma_{\lambda_0}^{\mu_0{}^\frown P} : P \in \mathcal{D}^{\kappa(\mu_0)}\right\}$$

form a partition of $\Gamma_{\lambda_0}^{\mu_0} = \mathcal{M}_{\lambda_0}$. By Lemma 8.1.5, for any $J \in \mathcal{D}^{\kappa(\mu_0)}$,

$$\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0{}^\frown J}} \theta_\nu \leqslant \theta_{\mu_0} |J| = |J|.$$

If, for some $J \in \mathcal{D}^{\kappa(\mu_0)}$, we had $\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0 \frown J}} \theta_\nu < |J|$, then,

$$1 = \theta_{\lambda_0} = \sum_{\nu \in \mathcal{M}_{\lambda_0}} \theta_\nu = \sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0}} \theta_\nu = \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} \sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0 \frown J}} \theta_\nu < \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} |J| = 1.$$

This contradiction concludes the base case. Assume now that μ_0 is a non-root with immediate predecessor μ, for which the conclusion is assumed to hold. Let $P_0 \in \mathcal{D}^{\kappa(\mu)}$ such that $\mu^\frown P_0 \sqsubseteq \mu_0$. By Lemma 8.1.6 (b), we have

$$\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0}} \theta_\nu = \sum_{\nu \in \Gamma_{\lambda_0}^{\mu \frown P_0}} \theta_\nu. \tag{8.2}$$

Claim:

$$\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0}} \theta_\nu = \theta_{\mu_0}.$$

To establish the claim, we distinguish two cases regarding μ, the immediate predecessor of μ_0. In the first case, λ_0 is minimal such that $\Gamma_{\lambda_0}^{\mu} \neq \varnothing$. Then, by the inductive hypothesis,

$$\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0}} \theta_\nu = \sum_{\nu \in \Gamma_{\lambda_0}^{\mu \frown P_0}} \theta_\nu = \theta_\mu |P_0| = |\theta_{\mu_0}|.$$

In the second case, the immediate predecessor λ_p of λ_0 is minimal such that $\Gamma_{\lambda_p}^{\mu} \neq \varnothing$, and if this is the case, $\mu \in \Gamma_{\lambda_p}^{\mu}$. Let $I_0 \in \mathcal{D}^{\kappa(\lambda_p)}$ such that $\lambda_p{}^\frown I_0 \sqsubseteq \lambda_0$. By Proposition 8.1.4 (c),

$$\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0}} \theta_\nu = \sum_{\nu \in \Gamma_{\lambda_0}^{\mu \frown P_0}} \theta_\nu = \theta_\mu |P_0| \frac{\theta_{\lambda_0}}{|I_0| \theta_{\lambda_p}} = |\theta_{\mu_0}|.$$

The proof of the claim is complete.

We use the claim as follows. By Lemma 8.1.5, for any $J \in \mathcal{D}^{\kappa(\mu_0)}$,

$$\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0 \frown J}} \theta_\nu \leqslant \theta_{\mu_0} |J|.$$

If, for some $J \in \mathcal{D}^{\kappa(\mu_0)}$, we had $\sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0 \frown J}} \theta_\nu < |J| \theta_{\mu_0}$, then,

$$\theta_{\mu_0} = \sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0}} \theta_\nu = \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} \sum_{\nu \in \Gamma_{\lambda_0}^{\mu_0 \frown J}} \theta_\nu < \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} |J| \theta_{\mu_0}| = \theta_{\mu_0}.$$

This contradiction concludes the proof of the case $\lambda = \lambda_0$.

We finally prove the general case. Let $\mu_0 \in \mathcal{F}$ and let $\lambda_0 \in \mathcal{E}$ be minimal such that $\Gamma_{\lambda_0}^{\mu_0} \neq \varnothing$, assuming it exists, and assume $\mu_0 \notin \mathcal{M}_{\lambda_0}$, as, otherwise, the conclusion vacuously holds. Let

$J \in \mathcal{D}^{\kappa(\mu_0)}$ and $\lambda \sqsupsetneq \lambda_0$. Let $I \in \mathcal{D}^{\kappa(\lambda_0)}$ such that $\lambda \sqsupseteq \lambda_0{}^\frown I$, and note that, by Remark 8.1.3 (c), the sets

$$\left\{\Gamma_\lambda^{\mu^\frown P} : \mu \in \Gamma_{\lambda_0}^{\mu_0{}^\frown J} \text{ and } P \in \mathcal{D}_I^{\kappa(\mu)}\right\}$$

form a partition of $\Gamma_\lambda^{\mu_0{}^\frown J}$. We now compute

$$\begin{aligned}
\sum_{\nu\in\Gamma_\lambda^{\mu_0{}^\frown J}} \theta_\nu &= \sum_{\mu\in\Gamma_{\lambda_0}^{\mu_0{}^\frown J}} \sum_{P\in\mathcal{D}_I^{\kappa(\mu)}} \sum_{\nu\in\Gamma_\lambda^{\mu^\frown P}} \theta_\nu = \sum_{\mu\in\Gamma_{\lambda_0}^{\mu_0{}^\frown J}} \sum_{P\in\mathcal{D}_I^{\kappa(\mu)}} \theta_\mu \frac{|P|}{|I|}\frac{|\lambda|}{|\lambda_0|} \text{ (by Proposition 8.1.4 (c))}\\
&= \sum_{\mu\in\Gamma_{\lambda_0}^{\mu_0{}^\frown J}} \theta_\mu \frac{|I|}{|I|}\frac{|\lambda|}{|\lambda_0|} \text{ (by Remark 8.1.3 (a))}\\
&= \theta_{\mu_0}|J|\frac{|\lambda|}{|\lambda_0|} \text{ (by the case } \lambda = \lambda_0).
\end{aligned}$$

□

Remark 8.1.7. Proposition 8.1.4 (a) follows from the argument in the Claim inside the proof of Proposition 8.1.4 (b).

8.2 Sub-distributional embedding schemes associated with Ψ

We construct three new types of distributional embedding schemes by isolating substructures of Ψ emanating from nodes $\lambda_0 \in \mathcal{E}$ and $\mu_0 \in \mathcal{F}$. These new schemes have lower ordinal numbers associated with them. This reduction in complexity is important, and it is used in Chapter 9 to prove, by induction on α, that T_Ψ is a distributional embedding.

Notation 8.2.1. Let $\alpha < \omega_1$ and $\mathcal{E}$ be an interval-branching subtree of $\mathcal{T}_\alpha$. For $\lambda_0 \in \mathcal{E}$ and $I \in \mathcal{D}^{\kappa(\lambda_0)}$, we define the interval-branching subtrees

$$\mathcal{E}^{\lambda_0{}^\frown I} = \left\{\lambda \in \mathcal{T}_{b(\lambda_0)} : \lambda_0^\frown I^\frown \lambda \in \mathcal{E}\right\} \text{ and } \mathcal{E}^{\lambda_0} = \left\{\lambda \in \mathcal{T}_{b(\lambda_0)+\kappa(\lambda_0)} : \lambda_0 \oslash \lambda \in \mathcal{E}\right\}$$

of $\mathcal{T}_{b(\lambda_0)}$ and $\mathcal{T}_{b(\lambda_0)+\kappa(\lambda_0)}$, respectively.

Notation 8.2.2. For the remainder of this subsection, we fix $\alpha, \beta < \omega_1$, interval-branching subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$, and a distributional embedding scheme $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu\in\mathcal{M}_\lambda})_{\lambda\in\mathcal{E}}$ of $R_\mathcal{E}$ in $R_\mathcal{F}$.

In the following proposition, we construct the first type of a sub-distributional embedding scheme of Ψ, which is used in the successor case of the inductive step in Section 9.

Proposition 8.2.3. Let $\lambda_0 \in \mathcal{E}$, $\mu_0 \in \mathcal{M}_{\lambda_0}$, $I \in \mathcal{D}^{\kappa(\lambda_0)}$, and $J \in \mathcal{D}_I^{\kappa(\mu_0)}$. For every $\lambda \in \mathcal{E}^{\lambda_0{}^\frown I}$, denote

$$\mathcal{M}_\lambda^{(\lambda_0{}^\frown I, \mu_0{}^\frown J)} = \{\mu \in \mathcal{F}^{\mu_0{}^\frown J} : \mu_0^\frown J^\frown \mu \in \mathcal{M}_{\lambda_0^\frown I^\frown \lambda}\}$$

and, for every $\mu \in \mathcal{M}_\lambda^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)}$, denote

$$J_\mu^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)} = J_{\mu_0{}^\frown J^\frown \mu}.$$

Then the collection

$$\Psi_{\lambda_0{}^\frown I}^{\mu_0{}^\frown J} = \big(\mathcal{M}_\lambda^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)}, (J_\mu^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)})_{\mu\in\mathcal{M}_\lambda^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)}}\big)_{\lambda\in\mathcal{E}^{\lambda_0{}^\frown I}}$$

is a distributional embedding scheme of $R_{\mathcal{E}^{\lambda_0{}^\frown I}}$ in $R_{\mathcal{F}^{\mu_0{}^\frown J}}$.

To aid the proof of Proposition 8.2.3, we note the following observation, which is evident from the relevant definitions.

Remark 8.2.4. For every $\lambda \in \mathcal{E}^{\lambda_0{}^\frown I}$, considering the set $\Gamma_\lambda = \Gamma_{\lambda_0{}^\frown I^\frown\lambda}^{\mu_0{}^\frown J}$, the map

$$\phi_\lambda : \mathcal{M}_\lambda^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)} \to \Gamma_\lambda$$

given by $\phi_\lambda(\mu) = \mu_0{}^\frown J^\frown\mu$ is a bijection and, for all $\mu \in \mathcal{M}_\lambda^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)}$,

$$\theta_{\phi_\lambda(\mu)} = \theta_{\mu_0}|J|\theta_\mu.$$

Proof of Proposition 8.2.3. For the rest of the poof, for $\lambda \in \mathcal{E}^{\lambda_0{}^\frown I}$, we denote $\mathcal{M}_\lambda^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)} = \mathcal{M}'_\lambda$ and, for $\mu \in \mathcal{M}'_\lambda$, we denote $J_\mu^{(\lambda_0{}^\frown I,\mu_0{}^\frown J)} = J'_\mu$. For $\lambda \in \mathcal{E}^{\lambda_0{}^\frown I}$, have that $\mathcal{M}'_\lambda$ is finite and $\mathcal{M}'_\lambda \subset \mathcal{F}^{\mu_0{}^\frown J}$. Clearly, for every $\mu \in \mathcal{M}'\lambda$ we have that $J'_\mu : L_p^{\kappa(\lambda),0} \to L_p^{\kappa(\mu),0}$ is a distributional embedding. We need to verify Definition 8.0.3 (a) (i) and (ii), (b), and (c).

First, we establish Definition 8.0.3 (a) (i), so, let $\lambda \in \mathcal{E}^{\lambda_0{}^\frown I}$ and $\mu, \mu' \in \mathcal{M}'_\lambda$ with $\mu \neq \mu'$. Hence, we have that $\mu_0^\frown J^\frown\mu$ and $\mu_0^\frown J^\frown\mu'$ belong to $\mathcal{M}_{\lambda_0^\frown I^\frown\lambda}$ and $\mu_0^\frown J^\frown\mu \neq \mu_0^\frown J^\frown\mu'$. Thus, $\mu_0^\frown J^\frown\mu$ and $\mu_0^\frown J^\frown\mu'$ are disjointly supported. Applying Proposition 7.5.5 (b) we easily obtain that μ and μ' are disjointly supported. The property in Definition 8.0.3 (b) is a clear consequence of the tree structure of the distributional embedding scheme described in Remark 8.0.4 (i). We use Remark 8.2.4 and Proposition 8.1.4 (c) to verify Definition 8.0.3 (a) (ii) as follows. For $\lambda \in \mathcal{E}^{\lambda_0{}^\frown I}$,

$$\sum_{\mu\in\mathcal{M}'_\lambda} \theta_\mu = \theta_{\mu_0}^{-1}|J|^{-1}\sum_{\nu\in\Gamma_\lambda}\theta_\nu = \theta_{\mu_0}^{-1}|J|^{-1}\theta_{\mu_0}\frac{|J|}{|I|}\frac{\theta_{\lambda_0{}^\frown I^\frown\lambda}}{\theta_{\lambda_0}} = \theta_\lambda.$$

Finally, we prove that Definition 8.0.3 (c) is satisfied. Let $\lambda \in \mathcal{E}^{\lambda_0{}^\frown I}$ be an immediate successor of $\pi(\lambda)$ and $J' \in \mathcal{D}^{\kappa(\pi(\lambda))}$ such that $\pi(\lambda)^\frown J' \sqsubseteq \lambda$ (here, $\pi(\lambda)$ denotes the immediate predecessor of λ; this notation should not be confused with the notation $\pi(I)$ used for dyadic intervals). We will show that, for $\mu \in \mathcal{M}'_\lambda$, there exist $\nu \in \mathcal{M}'_{\pi(\lambda)}$ and $Q \in \mathcal{D}_J^{\kappa(\nu)}$ such that $\nu^\frown Q \sqsubseteq \mu$. It is clear that the immediate predecessor of $\bar\lambda = \lambda_0{}^\frown I^\frown\lambda$ is $\pi(\bar\lambda) = \lambda_0{}^\frown I^\frown\pi(\lambda)$ and $\pi(\bar\lambda)^\frown J' \sqsubseteq \bar\lambda$. By Remark

8.1.3 (c), the collection of sets

$$\left\{\Gamma_{\bar{\lambda}}^{\nu^\frown Q} : \nu \in \Gamma_{\pi(\bar{\lambda})}^{\mu_0{}^\frown J} \text{ and } Q \in \mathcal{D}_J^{\kappa(\nu)}\right\}$$

form a partition of $\Gamma_{\bar{\lambda}}^{\mu_0{}^\frown J}$. Therefore, for every $\mu \in \mathcal{M}'_\lambda$, there exist $\nu \in \Gamma_{\pi(\bar{\lambda})}^{\mu_0{}^\frown J}$ and $Q \in \mathcal{D}_J^{\kappa(\nu)}$ such that $\phi_{\pi(\lambda)}(\nu)^\frown Q \sqsubseteq \phi_\lambda(\mu)$, and thus, $\nu^\frown Q \sqsubseteq \mu$. □

The last two types of distributional embedding schemes from Ψ are utilized in the limit case of the inductive step in Chapter 9. The proofs closely resemble that of Proposition 8.2.3. Although we highlight some differences, we choose to omit specific details to avoid repetition.

Proposition 8.2.5. Let $\lambda_0 \in \mathcal{E}$ be a root, $\mu_0 \in \mathcal{F}$, and $J \in \mathcal{D}^{\kappa(\mu_0)}$ such that $\Gamma_{\lambda_0}^{\mu_0{}^\frown J} \neq \varnothing$. For every $\lambda \in \mathcal{E}^{\lambda_0}$, denote

$$\mathcal{M}_\lambda^{(\lambda_0,\mu_0{}^\frown J)} = \{\mu \in \mathcal{F}^{\mu_0{}^\frown J} : \mu_0{}^\frown J^\frown \mu \in \mathcal{M}_{\lambda_0 \oslash \lambda}\}$$

and, for every $\mu \in \mathcal{M}_\lambda^{(\lambda_0,\mu_0{}^\frown J)}$, denote

$$J_\mu^{(\lambda_0,\mu_0{}^\frown J)} = J_{\mu_0{}^\frown J^\frown \mu}.$$

Then the collection

$$\Psi_{\lambda_0}^{\mu_0{}^\frown J} = \left(\mathcal{M}_\lambda^{(\lambda_0,\mu_0{}^\frown J)}, (J_\mu^{(\lambda_0,\mu_0{}^\frown J)})_{\mu \in \mathcal{M}_\lambda^{(\lambda_0,\mu_0{}^\frown J)}}\right)_{\lambda \in \mathcal{E}^{\lambda_0}}$$

is a distributional embedding scheme of $R_{\mathcal{E}^{\lambda_0}}$ in $R_{\mathcal{F}^{\mu_0{}^\frown J}}$.

The key distinction between the proofs of Proposition 8.2.5 and Proposition 8.2.3 lies in its dependence on Proposition 8.1.4 (b) instead of (c). We point out an observation that is analogous to Remark 8.2.4.

Remark 8.2.6. For every $\lambda \in \mathcal{E}^{\lambda_0}$, considering the set $\Gamma_\lambda = \Gamma_{\lambda_0 \oslash \lambda}^{\mu_0{}^\frown J}$, the map

$$\phi_\lambda : \mathcal{M}_\lambda^{(\lambda_0,\mu_0{}^\frown J)} \to \Gamma_\lambda$$

given by $\phi_\lambda(\mu) = \mu_0{}^\frown J^\frown \mu$ is a bijection and, for all $\mu \in \mathcal{M}_\lambda^{(\lambda_0,\mu_0{}^\frown J)}$,

$$\theta_{\phi_\lambda(\mu)} = \theta_{\mu_0} |J| \theta_\mu.$$

Proof of Proposition 8.2.5. We will only verify Definition 8.0.3 (b), as the rest are shown in the exact same way as in the proof of Proposition 8.2.3. By Remark 8.2.6 and Proposition 8.1.4 (b), for $\lambda \in \mathcal{E}^{\lambda_0}$,

$$\sum_{\mu \in \mathcal{M}_\lambda^{(\lambda_0,\mu_0{}^\frown J)}} \theta_\mu = \theta_{\mu_0}^{-1} |J|^{-1} \sum_{\nu \in \Gamma_\lambda} \theta_\nu = \theta_{\mu_0}^{-1} |J|^{-1} \theta_{\mu_0} |J| \frac{\theta_{\lambda_0 \oslash \lambda}}{\theta_{\lambda_0}} = \theta_\lambda.$$

$\square$

The proof of the final proposition is based on a remark analogous to Remarks 8.2.4 and 8.2.6, as well as Proposition 8.1.4 (a). The process is identical to the previous two propositions; therefore, we will not repeat it.

Proposition 8.2.7. Let $\lambda_0 \in \mathcal{E}$ and $\mu_0 \in \mathcal{F}$ such that $\Gamma_{\lambda_0}^{\mu_0} \neq \varnothing$. For every $\lambda \in \mathcal{E}^{\lambda_0}$, denote

$$\mathcal{M}_\lambda^{(\lambda_0,\mu_0)} = \{\mu \in \mathcal{F}^{\mu_0} : \mu_0 \oslash \mu \in \mathcal{M}_{\lambda_0 \oslash \lambda}\}$$

and, for every $\mu \in \mathcal{M}_\lambda^{(\lambda_0,\mu_0)}$, denote

$$J_\mu^{(\lambda_0,\mu_0)} = J_{\mu_0 \oslash \mu}.$$

Then the collection

$$\Psi_{\lambda_0}^{\mu_0} = \left(\mathcal{M}_\lambda^{(\lambda_0,\mu_0)}, (J_\mu^{(\lambda_0,\mu_0)})_{\mu \in \mathcal{M}_\lambda^{(\lambda_0,\mu_0)}}\right)_{\lambda \in \mathcal{E}^{\lambda_0}}$$

is a distributional embedding scheme of $R_{\mathcal{E}^{\lambda_0}}$ in $R_{\mathcal{F}^{\mu_0}}$.

Chapter 9

Distributional representation of $R_\alpha^{p,0}$ in $R_\beta^{p,0}$

In this chapter, the goal is to establish the following theorem about the map T_Ψ given in Notation 8.0.5.

Theorem 9.0.1. Let $\alpha, \beta < \omega_1$. For interval-branching subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$ and a distributional embedding scheme $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu \in \mathcal{M}_\lambda})_{\lambda \in \mathcal{E}}$ of $R_\mathcal{E}$ in $R_\mathcal{F}$, we define the linear map $T_\Psi : \langle (X_\lambda)_{\lambda \in \mathcal{E}} \rangle \to \langle (X_\mu)_{\mu \in \mathcal{F}} \rangle$ by

$$T_\Psi|_{X_\lambda} = \sum_{\mu \in \mathcal{M}_\lambda} T_\mu J_\mu T_\lambda^{-1},$$

where $\lambda \in \mathcal{E}$. Then, T_Ψ extends to a distributional embedding $T_\Psi \colon R_\mathcal{E} \to R_\mathcal{F}$.

The proof is carried out by transfinite induction on α, and to aid the process, we introduce the following notation.

Notation 9.0.2. For $\alpha < \omega_1$, we say that property $P(\alpha)$ is true if the conclusion of Theorem 9.0.1 holds for the given α and all $\beta < \omega_1$.

We have structured the proof by breaking it down into three parts. The first part, Section 9.1, addresses the general theory of distributional embeddings and compressions, and the ways they can be assembled to create new such operators. Its results will enable an increase in the complexity of distributional embeddings in the inductive step of the proof of Theorem 9.0.1. In the second part, Section 9.2, we prove the base case and develop the specialized tools required to carry out the successor inductive step. In the third, and lengthiest, part, Section 9.3, we prove a series of operator reductions and use them to carry out the limit inductive step, thus completing the proof of Theorem 9.0.1.

9.1 Assembling abstract θ-compressions

Here, we establish the necessary elementary tools used to assemble compressions and distributional embeddings, creating new operators of the same type. We initially recall that distributional embeddings with independent domains and independent ranges can define an ambient distributional embedding. We then define disjointly supported compressions and demonstrate how they can be utilized to create distributional embeddings.

The following is a standard result proved using the characteristic function of a random variable, which characterizes its distribution. If f is a random variable, then its characteristic function is defined by

$$\Phi_f(t) = \int_0^1 \mathrm{e}^{itf(x)}\,\mathrm{d}x.$$

For a linear operator T, we denote its range by $\mathrm{R}(T)$.

Proposition 9.1.1. Let $(X_n)_{n=1}^\infty$ be a sequence of independent subspaces of L_p and $(T_n : X_n \to L_p)_{n=1}^\infty$, be a sequence distributional embeddings such that the spaces $(\mathrm{R}(T_n))_{n=1}^\infty$ are independent. Then, there exists a distributional embedding

$$T : [(X_n)_{n=1}^\infty] \to L_p$$

such that, for all $n \in \mathbb{N}$,

$$T|_{X_n} = T_n.$$

The notion of the support of a linear subspace of L_p was given in Definition 7.5.1. This leads to the natural notion of compressions with disjointly supported ranges. We will prove that the sum of such compressions with a common domain is still a compression, a result which is visually rather obvious. At the end of this section, we formulate and prove another intuitive statement of how to define a distributional embedding on a disjoint sum $\big(\oplus_{I\in\mathcal{D}^n} X_I\big) = \langle\{T_I[X_I] : I \in \mathcal{D}^n\}\rangle$, for a collection of subspaces $(X_I)_{I\in\mathcal{D}^n}$ of L_p.

Definition 9.1.2. Let $n \in \mathbb{N}$, $X_1, \dots, X_n$ be subspaces of L_p, and $T_i : X_i \to L_p$, $1 \leqslant i \leqslant n$, be bounded linear operators. We say that $(T_i)_{i=1}^n$ have disjointly supported ranges if the sets $(\mathrm{supp}(\mathrm{R}(T_i))_{i=1}^n$ are essentially disjoint.

Example 9.1.3. These are disjointly supported compressions that will be used later.

(a) For $0 \leqslant a < b \leqslant 1$, $\mathrm{supp}(\mathrm{R}(T_{[a,b]})) = [a,b]$ and, for $n \in \mathbb{N}$, the $1/2^n$-compressions $(T_I)_{I\in\mathcal{D}^n}$ have disjointly supported ranges.

(b) For $\alpha < \omega_1$ and $\mu \in \mathcal{T}_\alpha$, $\mathrm{supp}(\mathrm{R}(T_\mu)) = \mathrm{supp}(X_\mu)$. Therefore, if $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu\in\mathcal{M}_\lambda})_{\lambda\in\mathcal{E}}$ is a distributional embedding scheme, then, for every $\lambda \in \mathcal{T}_\alpha$, the compressions $(T_\mu)_{\mu\in\mathcal{M}_\lambda}$ have disjointly supported ranges.

Lemma 9.1.4. The following identities hold.

(a) Let X be a subspace of L_p and $T : X \to L_p$ be a θ-compression. Then, for every bounded measurable function $\phi : \mathbb{R} \to \mathbb{R}$ and $f \in X$,

$$\int \phi\big((Tf)(t)\big)dt = \theta \int \phi\big(f(t)\big)dt + (1-\theta)\phi(0).$$

(b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Let $X_1, \ldots, X_n$ be subspaces of L_p, and, for $1 \leqslant i \leqslant n$, $T_i : X \to L_p$ be a θ_i-compression such that $(T_i)_{i=1}^n$ have disjointly supported ranges. Then, for $f_1 \in X_1, \ldots, f_n \in X_n$,

$$\int \phi\Big(\sum_{i=1}^n (T_i f_i)(t)\Big)dt = \sum_{i=1}^n \theta_i \int \phi\big(f_i(t)\big)dt + \big(1 - \sum_{i=1}^n \theta_i\big)\phi(0).$$

Proof. Let us promptly prove the first assertion. Assuming that T is a θ-compression,

$$\begin{aligned}\int \phi\big((Tf)(t)\big)dt &= \int \phi(x)\, d\big(\mathrm{dist}(Tf)\big)(x) = \theta \int \phi(x) d\big(\mathrm{dist}(f)\big)(x) + (1-\theta)\int \phi(x) d\delta_0(x)\\ &= \theta \int \phi\big(f(t)\big)dt + (1-\theta)\phi(0).\end{aligned}$$

In order to deduce the second assertion, observe

$$\begin{aligned}\int \phi\Big(\sum_{i=1}^n (T_i f_i)(t)\Big)dt &= \sum_{j=1}^n \int_{\mathrm{supp}(\mathrm{R}(T_j))} \phi\Big(\sum_{i=1}^n (T_i f_i)(t)\Big)dt = \sum_{i=1}^n \int_{\mathrm{supp}(\mathrm{R}(T_i))} \phi\big((T_i f_i)(t)\big)dt\\ &= \sum_{i=1}^n \int \phi\big((T_i f_i)(t)\big)dt.\end{aligned}$$

Apply the first statement to conclude the proof. □

Proposition 9.1.5. Let X be a subspace of L_p, $n \in \mathbb{N}$, and, for $1 \leqslant i \leqslant n$, $T_i : X \to L_p$ be a θ_i-compression such that $(T_i)_{i=1}^n$ have disjointly supported ranges. Then,

$$T = \sum_{i=1}^n T_i : X \to L_p$$

is a $(\sum_{i=1}^n \theta_i)$-compression.

Proof. Let A be a measurable subset of $\mathbb{R}$ and $f \in X$. We apply Lemma 9.1.4 (b) for $X_1 = \cdots = X_n = X$, $f_1 = \cdots = f_n = f$, and $\phi = \chi_A$, to obtain

$$\Big|\Big[\sum_{i=1}^n T_i f \in A\Big]\Big| = \sum_{i=1}^n \theta_i |[f_i \in A]| + (1 - \sum_{i=1}^n \theta_i)\chi_A(0).$$

□

Notation 9.1.6. For $0 \leqslant a < b \leqslant 1$ and $I = [a, b)$ we define the operator $Q_I : L_p \to L_p$ given by

$$\big(Q_I f\big)(t) = f\big(q_I(t)\big) = f\big(a + t(b - a)\big).$$

Remark 9.1.7. For $n \in \mathbb{N}$ and $I, J \in \mathcal{D}^n$, $Q_I T_J f = \delta_{I,J} f$.

Proposition 9.1.8. Let $n \in \mathbb{N}$ and, for $I \in \mathcal{D}^n$, let X_I be a subspace of L_p and $S_I : X_I \to L_p$ be a $(1/2^n)$-compression such that $(S_I)_{I \in \mathcal{D}^n}$ have disjointly supported ranges. Then,

$$S = \sum_{I \in \mathcal{D}^n} S_I Q_I : \langle (T_I[X_I])_{I \in \mathcal{D}^n} \rangle \to L_p$$

is a distributional embedding.

Proof. Let $f \in \langle (T_I[X_I])_{I \in \mathcal{D}^n} \rangle$, i.e., for $I \in \mathcal{D}^n$, there is $f_I \in X_I$ such that $f = \sum_{I \in \mathcal{D}^n} T_I f_I$. By Remark 9.1.7, $Sf = \sum_{I \in \mathcal{D}^n} S_I f_I$. Let now A be a measurable subset of $\mathbb{R}$, and apply Lemma 9.1.4 (b) twice for $\phi = \chi_A$ to obtain

$$|[Sf \in A]| = \Big|\Big[\sum_{I \in \mathcal{D}^n} S_I f_I \in A\Big]\Big| = \sum_{I \in \mathcal{D}^n} \frac{1}{2^n} |[f_I \in A]| = \Big|\Big[\sum_{I \in \mathcal{D}^n} T_I f_I \in A\Big]\Big| = |[f \in A]|,$$

i.e., $\mathrm{dist}(Sf) = \mathrm{dist}(f)$. $\square$

9.2 The base case and the inductive step for successor ordinals

We initially establish the base case, i.e., when α is a positive integer, and then we prepare to prove the successor case of the inductive step in the proof of Theorem 9.0.1. If $\alpha = \alpha_0 + n$, where α_0 is a non-zero limit ordinal number and $n \in \mathbb{N}$, and $\mathcal{E}$ is a interval-branching subtree of $\mathcal{T}_\alpha$, we analyze T_Ψ as a sum of operators that includes sub-distributional embedding schemes with domains over interval-branching subtrees of $\mathcal{T}_{\alpha_0}$, meaning schemes of lesser complexity. We combine this with the inductive hypothesis $P(\alpha_0)$ and Proposition 9.1.8 to achieve the desired conclusion.

Proposition 9.2.1. Let $n \in \mathbb{N}$, $\beta < \omega_1$, $\mathcal{E}$ be an interval-branching subtree of $\mathcal{T}_n$, $\mathcal{F}$ be an interval-branching subtree of $\mathcal{T}_\beta$, and Ψ be a distributional embedding scheme from $R_\mathcal{E}$ to $R_\mathcal{F}$. Then T_Ψ is a distributional embedding.

Proof. In this case, $\mathcal{T}_n = \{(n)\}$, and thus $\mathcal{E} = \{(n)\}$ and $R_\mathcal{E} = L_p^{n,0}$. Because $T_{(n)} = id$,

$$T_\Psi = \sum_{\mu \in \mathcal{M}_{(n)}} T_\mu J_\mu : R_\mathcal{E} \to R_\mathcal{F},$$

and thus, by Proposition 9.1.5, it is a distributional embedding. $\square$

Remark 9.2.2. The same argument shows that, for any $\alpha < \omega_1$ and trivial subtree $\mathcal{E} = \{\lambda_0\}$ of $\mathcal{T}_\alpha$, any distributional embedding scheme Ψ from $R_\mathcal{E}$ to some space $R_\mathcal{F}$, the operator T_Ψ is a distributional embedding.

We augment the operator T_Ψ by incorporating the constant functions into its domain. This adjustment simplifies the proof of the successor case in the inductive step, making it more intuitive and eliminating some technicalities that would arise otherwise.

Notation 9.2.3. We set the following notation.

(a) For $\alpha < \omega_1$ and an interval-branching subtree $\mathcal{E}$ of $\mathcal{T}_\alpha$ we denote $\widetilde{R}_\mathcal{E} = \langle \{\chi_{[0,1]}\} \cup R_\mathcal{E} \rangle$.

(b) For $\alpha, \beta < \omega_1$, interval-branching subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$, and a distributional embedding scheme Ψ of $R_\mathcal{E}$ in $R_\mathcal{F}$, we denote by $\widetilde{T}_\Psi : \widetilde{R}_\mathcal{E} \to \widetilde{R}_\mathcal{F}$ the extension of T_Ψ given by letting $\widetilde{T}_\Psi \chi_{[0,1]} = \chi_{[0,1]}$.

Clearly, T_Ψ is a distributional embedding if and only if $\widetilde{T}_\Psi$ is.

Notation 9.2.4. For this section, fix $\alpha, \beta < \omega_1$, with $\alpha = \alpha_0 + n$, where α_0 is a non-zero limit ordinal number and $n \in \mathbb{N}$, interval-branching subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$ and a distributional embedding scheme $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu \in \mathcal{M}_\lambda})_{\lambda \in \mathcal{E}}$ of $R_\mathcal{E}$ in $R_\mathcal{F}$.

Proposition 9.2.5. If $P(\alpha_0)$ is true, then T_Ψ is a distributional embedding.

Before proving this, we will disassemble the operator $\widetilde{T}_\Psi$. It is worth noting that the ensuing formula is not valid in its current form if we do not use the augmented operators.

Proposition 9.2.6. Let $\lambda_0 = (\alpha_0 + n)$ denote the unique root of $\mathcal{E}$. Then,

$$\widetilde{T}_\Psi = \sum_{\mu \in \mathcal{M}_{\lambda_0}} \sum_{I \in \mathcal{D}^{\kappa(\lambda_0)}} \sum_{J \in \mathcal{D}_I^{\kappa(\mu)}} T_\mu T_J \widetilde{T}_{\Psi^{\mu \frown J}_{\lambda_0 \frown I}} Q_I.$$

Proof. We denote by S the map on the right of the above equation. We initially show that, for $f \in \langle \{\chi_{[0,1]}\} \cup X_{\lambda_0} \rangle = L_p^{\kappa(\lambda_0)}$, $Sf = \widetilde{T}_\Psi f$. It suffices to verify this for $f = \chi_{I_0}$, $I_0 \in \mathcal{D}^{\kappa(\lambda_0)}$. By a standard identity already known to Haar,

$$\chi_{I_0} = \chi_{[h_{\pi(I_0)} = \epsilon(I_0)]} = 2^{-\kappa(\lambda_0)} \chi_{[0,1]} + \sum_{r=1}^{\kappa(\lambda_0)} 2^{-\kappa(\lambda_0)+r-1} \epsilon(\pi^{r-1}(I_0)) h_{\pi^r(I_0)}.$$

This is established by a straightforward induction argument using that, for all $I \in \mathcal{D} \setminus \mathcal{D}^0$, $\chi_I = (1/2)(\chi_{\pi(I)} + \epsilon(I) h_{\pi(I)})$. The same identity holds if we replace $(h_I)_{I \in \mathcal{D}_{\kappa(\lambda_0)-1}}$ with a distributional copy of it. We now compute:

$$\begin{aligned} S\chi_{I_0} &= \sum_{\mu \in \mathcal{M}_{\lambda_0}} \sum_{I \in \mathcal{D}^{\kappa(\lambda_0)}} \sum_{J \in \mathcal{D}_I^{\kappa(\mu)}} T_\mu T_J \widetilde{T}_{\Psi^{\mu \frown J}_{\lambda_0 \frown I}} Q_I \chi_{I_0} = \sum_{\mu \in \mathcal{M}_{\lambda_0}} \sum_{J \in \mathcal{D}_{I_0}^{\kappa(\mu)}} T_\mu T_J \widetilde{T}_{\Psi^{\mu \frown J}_{\lambda_0 \frown I}} \chi_{[0,1]} \\ &= \sum_{\mu \in \mathcal{M}_{\lambda_0}} \sum_{J \in \mathcal{D}_{I_0}^{\kappa(\mu)}} T_\mu T_J \chi_{[0,1]} = \sum_{\mu \in \mathcal{M}_{\lambda_0}} T_\mu \sum_{J \in \mathcal{D}_{I_0}^{\kappa(\mu)}} \chi_J = \sum_{\mu \in \mathcal{M}_{\lambda_0}} T_\mu \chi_{[J_\mu h_{\pi(I_0)} = \epsilon(I_0)]} \\ &= \sum_{\mu \in \mathcal{M}_{\lambda_0}} T_\mu \Big(2^{-\kappa(\lambda_0)} \chi_{[0,1]} + \sum_{r=1}^{\kappa(\lambda_0)} 2^{-\kappa(\lambda_0)+r-1} \epsilon(\pi^{r-1}(I_0)) J_\mu h_{\pi(I_0)} \Big) \end{aligned}$$

$$= \Big(\sum_{\mu\in\mathcal{M}_{\lambda_0}} T_\mu\Big)2^{-\kappa(\lambda_0)}\chi_{[0,1]} + \sum_{r=1}^{\kappa(\lambda_0)} 2^{-\kappa(\lambda_0)+r-1}\epsilon(\pi^{r-1}(I_0))\Big(\sum_{\mu\in\mathcal{M}_{\lambda_0}} T_\mu J_\mu h_{\pi(I_0)}\Big)$$

$$= \widetilde{T}_\Psi 2^{-\kappa(\lambda_0)}\chi_{[0,1]} + \sum_{r=1}^{\kappa(\lambda_0)} 2^{-\kappa(\lambda_0)+r-1}\epsilon(\pi^{r-1}(I_0))\widetilde{T}_\Psi h^{\lambda_0}_{\pi^r(I_0)} = \widetilde{T}_\Psi(\chi_{I_0}).$$

We fix $\lambda \supsetneqq \lambda_0$, i.e., there are $I_0 \in \mathcal{D}^{\kappa(\lambda_0)}$ and $\xi \in \mathcal{E}^{\lambda_0{}^\frown I_0}$ such that $\lambda = \lambda_0{}^\frown I_0{}^\frown\xi$. We will show $S|_{X_\lambda} = T_\Psi|_{X_\lambda}$, or, equivalently, $ST_\lambda|_{L_p^{\kappa(\lambda),0}} = T_\Psi T_\lambda|_{L_p^{\kappa(\lambda),0}}$. Fix $f \in L_p^{\kappa(\lambda),0}$. By Remark 8.1.3 (b), the sets

$$\Big\{\Gamma_\lambda^{\mu^\frown P} : \mu \in \mathcal{M}_{\lambda_0} \text{ and } P \in \mathcal{D}_{I_0}^{\kappa(\mu)}\Big\}$$

form a partition of $\mathcal{M}_\lambda$. We thus calculate

$$\begin{aligned}
ST_\lambda f &= \sum_{I\in\mathcal{D}^{\kappa(\lambda_0)}}\Big(\sum_{\mu\in\mathcal{M}_{\lambda_0}}\sum_{J\in\mathcal{D}_I^{\kappa(\mu)}} T_\mu T_J \widetilde{T}_{\Psi^{\mu^\frown J}_{\lambda_0{}^\frown I}}\Big)Q_I(T_{I_0}T_\xi f) = \Big(\sum_{\mu\in\mathcal{M}_{\lambda_0}}\sum_{J\in\mathcal{D}_{I_0}^{\kappa(\mu)}} T_\mu T_J \widetilde{T}_{\Psi^{\mu^\frown J}_{\lambda_0{}^\frown I_0}}\Big)T_\xi f\\
&= \Big(\sum_{\mu\in\mathcal{M}_{\lambda_0}}\sum_{J\in\mathcal{D}_{I_0}^{\kappa(\mu)}} T_\mu T_J\Big)\sum_{\nu\in\mathcal{M}_\xi^{(\lambda_0{}^\frown I_0,\mu^\frown J)}} T_\nu J_\nu^{(\lambda_0{}^\frown I_0,\mu^\frown J)} f \text{ (definition of } T_{\Psi^{\mu^\frown J}_{\lambda_0{}^\frown I_0}})\\
&= \sum_{\mu\in\mathcal{M}_{\lambda_0}}\sum_{J\in\mathcal{D}_{I_0}^{\kappa(\mu)}}\sum_{\nu\in\mathcal{M}_\xi^{(\lambda_0{}^\frown I_0,\mu^\frown J)}} T_{\mu^\frown J^\frown \nu}J_{\mu^\frown J^\frown\nu} f \text{ (definition of } J_\nu^{(\lambda_0{}^\frown I_0,\mu^\frown J)})\\
&= \sum_{\mu\in\mathcal{M}_{\lambda_0}}\sum_{J\in\mathcal{D}_{I_0}^{\kappa(\mu)}}\sum_{\nu\in\Gamma_\lambda^{\mu^\frown J}} T_\nu J_\nu f = \sum_{\nu\in\mathcal{M}_\lambda} T_\nu J_\nu f = T_\Psi T_\lambda f.
\end{aligned}$$

□

Lemma 9.2.7. We have

$$\widetilde{R}_{\mathcal{E}} = \langle\{T_I\big(\widetilde{R}_{\mathcal{E}^{\lambda_0{}^\frown I}}\big) : I \in \mathcal{D}^n\}\rangle.$$

Proof. Noting, $\langle\{\chi_{[0,1]}\} \cup X_{\lambda_0}\rangle = L_p^{\kappa(\lambda_0)}$, we have

$$\begin{aligned}
\widetilde{R}_{\mathcal{E}} &= \big[L_p^{\kappa(\lambda_0)} \cup \bigcup_{I\in\mathcal{D}^{\kappa(\lambda_0)}}\bigcup_{\mu\in\mathcal{E}^{\lambda_0{}^\frown I}} X_{\lambda_0{}^\frown I^\frown\mu}\big] = \big[(T_I\chi_{[0,1]})_{I\in\mathcal{D}^{\kappa(\lambda_0)}} \cup \bigcup_{I\in\mathcal{D}^{\kappa(\lambda_0)}} T_I\big([(X_\mu)_{\mu\in\mathcal{E}^{\lambda_0{}^\frown I}}]\big)\big]\\
&= \big[\bigcup_{I\in\mathcal{D}^{\kappa(\lambda_0)}} T_I\big([\{\chi_{[0,1]}\}\cup R_{\mathcal{E}^{\lambda_0{}^\frown I}}]\big)\big] = \big\langle\bigcup_{I\in\mathcal{D}^{\kappa(\lambda_0)}} T_I\big(\widetilde{R}_{\mathcal{E}^{\lambda_0{}^\frown I}}\big)\big\rangle.
\end{aligned}$$

□

Proof of Proposition 9.2.5. We will prove that $\widetilde{T}_\Psi$ is a distributional embedding, which is equivalent to the conclusion. Using Proposition 9.2.6, write

$$\widetilde{T}_\Psi = \sum_{\mu\in\mathcal{M}_{\lambda_0}} T_\mu\Big[\sum_{I\in\mathcal{D}^{\kappa(\lambda_0)}}\underbrace{\Big(\sum_{J\in\mathcal{D}_I^{\kappa(\mu)}} T_J\widetilde{T}_{\Psi^{\mu^\frown J}_{\lambda_0{}^\frown I}}\Big)}_{=:S_I^\mu}Q_I.\Big]$$

For $\mu \in \mathcal{M}_{\lambda_0}$, $I \in \mathcal{D}^{\kappa(\lambda_0)}$, and $J \in \mathcal{D}_I^{\kappa(\mu_0)}$, because $P(\alpha_0)$ is true, $\widetilde{T}_{\Psi^{\mu\frown J}_{\lambda_0\frown I}}$ is a distributional embedding with domain $\widetilde{R}_{\mathcal{E}^{\lambda_0\frown I}}$. It follows that $(T_J \widetilde{T}_{\Psi^{\mu\frown J}_{\lambda_0\frown I}})_{J\in\mathcal{D}_I^{\kappa(\mu)}}$ are $2^{-\kappa(\mu)}$-compressions with disjointly supported ranges. By Proposition 9.1.5, for $\mu \in \mathcal{M}_{\lambda_0}$ and $I \in \mathcal{D}^{\kappa(\lambda_0)}$,

$$S_I^\mu = \sum_{J\in\mathcal{D}_I^{\kappa(\mu)}} T_J \widetilde{T}_{\Psi^{\mu\frown J}_{\lambda_0\frown I}}$$

is a $|I|$-compression with domain $\widetilde{R}_{\mathcal{E}^{\lambda_0\frown I}}$ and range supported in $\cup\mathcal{D}_I^\mu$. By Proposition 9.1.8,

$$S_\mu = \sum_{I\in\mathcal{D}^{\kappa(\lambda_0)}} S_I^\mu Q_I$$

is a distributional embedding with domain $\langle\{T_I[\widetilde{R}_{\mathcal{E}^{\lambda_0\frown I}}] : I \in \mathcal{D}^n\}\rangle = \widetilde{R}_{\mathcal{E}}$, by Lemma 9.2.7. By Proposition 9.1.5, $\widetilde{T}_\Psi = \sum_{\mu\in\mathcal{M}_{\lambda_0}} T_\mu S_\mu$ is a distributional embedding. $\square$

9.3 The inductive step for limit ordinals

The limit ordinal case of the inductive step is more involved than the successor case. We first establish some notation.

Notation 9.3.1. For $\alpha < \omega_1$ and an interval-branching subtree $\mathcal{E}$ of $\mathcal{T}_\alpha$, we denote

(a) $\mathcal{R}oot(\mathcal{E}) = \{\lambda \text{ is a root of } \mathcal{E}\}$ and,

(b) for $\lambda \in \mathcal{R}oot(\mathcal{E})$, $Y_\lambda^{\mathcal{E}} = \big[(X_{\lambda'})_{\lambda'\in\mathcal{E},\lambda'\sqsupseteq\lambda}\big]$.

Remark 9.3.2. By definition, we have $R_{\mathcal{E}} = [(Y_\lambda^{\mathcal{E}})_{\lambda\in\mathcal{R}oot(\mathcal{E})}]$. Furthermore, for every $\lambda \in \mathcal{R}oot(\mathcal{E})$, $Y_\lambda^{\mathcal{E}} \subset Y_\lambda$, and therefore by Proposition 7.4.1, the spaces $(Y_\lambda^{\mathcal{E}})_{\lambda\in\mathcal{R}oot(\mathcal{E})}$ are independent.

Remark 9.3.3. For $\lambda \in \mathcal{R}oot(\mathcal{E})$, there exists a relation between the spaces $R_{\mathcal{E}^\lambda}$ and $Y_\lambda^{\mathcal{E}}$:

$$T_\lambda[R_{\mathcal{E}^\lambda}] = Y_\lambda^{\mathcal{E}}.$$

Therefore,

$$R_{\mathcal{E}} \equiv^{\text{dist}} \Big(\sum_{\lambda\in\mathcal{R}oot(\mathcal{E})} R_{\mathcal{E}^\lambda}\Big)_{\text{Ind},p}.$$

Indeed, for $\mu \in \mathcal{E}^\lambda$,

$$T_\lambda[X_\mu] = T_\lambda T_\mu[L_p^{\kappa(\mu),0}] = T_\lambda T_\mu[L_p^{\kappa(\lambda\oslash\mu),0}] = T_{\lambda\oslash\mu}[L_p^{\kappa(\lambda\oslash\mu),0}] = X_{\lambda\oslash\mu}.$$

Because $\mathcal{E}^\lambda = \{\mu \in \mathcal{T}_{b(\lambda)+\kappa(\lambda)} : \lambda \oslash \mu \in \mathcal{E}\}$, the map $\mu \mapsto \lambda \oslash \mu$ defines a bijection from $\mathcal{E}^\lambda$ to $\{\lambda' \in \mathcal{E} : \lambda \sqsubseteq \lambda'\}$. We conclude

$$T_\lambda[R_{\mathcal{E}^\lambda}] = [(T_\lambda[X_\mu])_{\mu\in\mathcal{E}^\lambda}] = [(X_{\lambda\oslash\mu})_{\mu\in\mathcal{E}^\lambda}] = [(X_{\lambda'})_{\lambda'\in\mathcal{E} \text{ and } \lambda\sqsubseteq\lambda'}] = Y_\lambda^{\mathcal{E}}.$$

Notation 9.3.4. For this section, fix $\alpha, \beta < \omega_1$, where α is a limit ordinal, interval-branching subtrees $\mathcal{E}$ of $\mathcal{T}_\alpha$ and $\mathcal{F}$ of $\mathcal{T}_\beta$ and a distributional embedding scheme $\Psi = (\mathcal{M}_\lambda, (J_\mu)_{\mu \in \mathcal{M}_\lambda})_{\lambda \in \mathcal{E}}$ of $R_\mathcal{E}$ in $R_\mathcal{F}$.

We will prove the following.

Proposition 9.3.5. If, for all successor ordinals $\gamma < \alpha$, $P(\gamma)$ is true then T_Ψ is a distributional embedding.

We begin by outlining a sequence of disassemblies of operators associated with sub-distributional embedding schemes derived from Ψ.

Lemma 9.3.6. Let $\lambda_0 \in \mathcal{R}oot(\mathcal{E})$ and let μ_0 be the unique root of $\mathcal{F}$ such that $\Gamma_{\lambda_0}^{\mu_0} = \mathcal{M}_{\lambda_0}$ (which exists by Lemma 8.1.6 (a)). Then

$$T_\Psi|_{Y_{\lambda_0}^{\mathcal{E}}} = T_{\mu_0} T_{\Psi_{\lambda_0}^{\mu_0}} T_{\lambda_0}^{-1}|_{Y_{\lambda_0}^{\mathcal{E}}}.$$

Proof. The tree structure of $(\mathcal{M}_\lambda)_{\lambda \in \mathcal{E}}$ (see Remark 8.0.4 (i)) yields that, for $\lambda \sqsupsetneq \lambda_0$, $\Gamma_\lambda^{\mu_0} = \mathcal{M}_\lambda$. In particular, if $\mu \in \mathcal{E}^{\lambda_0}$ is such that $\lambda = \lambda_0 \oslash \mu$ then, by the definition of $\mathcal{M}_\mu^{(\lambda_0, \mu_0)}$,

$$\mathcal{M}_\lambda = \left\{ \mu_0 \oslash \nu : \nu \in \mathcal{M}_\mu^{(\lambda_0, \mu_0)} \right\}.$$

Let us fix $f \in X_\lambda$, i.e., there is $g \in L_p^{\kappa(\lambda),0}$ such that $f = T_\lambda g = T_{\lambda_0 \oslash \mu} g = T_{\lambda_0} T_\mu g$. Then

$$\begin{aligned} T_{\mu_0} T_{\Psi_{\lambda_0}^{\mu_0}} T_{\lambda_0}^{-1} f = T_{\mu_0} T_{\Psi_{\lambda_0}^{\mu_0}} T_\mu g &= \sum_{\nu \in \mathcal{M}_\mu^{(\lambda_0, \mu_0)}} T_{\mu_0} T_\nu J_\nu^{(\lambda_0, \mu_0)} T_\mu^{-1} T_\mu g \\ &= \sum_{\nu \in \mathcal{M}_\mu^{(\lambda_0, \mu_0)}} T_{\mu_0 \oslash \nu} J_{\mu_0 \oslash \nu} g = \sum_{\nu \in \mathcal{M}_\lambda} T_\nu J_\nu g = T_\Psi f. \end{aligned}$$

□

Lemma 9.3.7. Let $\lambda_0 \in \mathcal{R}oot(\mathcal{E})$ and $\mu_0 \in \mathcal{F} \backslash \mathcal{M}_{\lambda_0}$ such that $\Gamma_{\lambda_0}^{\mu_0} \neq \varnothing$. Then

$$T_{\Psi_{\lambda_0}^{\mu_0}} = \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} T_J T_{\Psi_{\lambda_0}^{\mu_0 \frown J}}.$$

Proof. Because $\Gamma_{\lambda_0}^{\mu_0} \neq \varnothing$ and $\mu_0 \notin \mathcal{M}_{\lambda_0}$, we have, for all $\lambda \sqsupseteq \lambda_0$, the sets

$$\left\{ \Gamma_\lambda^{\mu_0 \frown J} : J \in \mathcal{D}^{\kappa(\mu_0)} \right\} \tag{9.1}$$

form a partition of $\Gamma_\lambda^{\mu_0}$. We fix $\lambda \in \mathcal{E}^{\lambda_0}$ and $f \in X_\lambda$, i.e., $f = T_\lambda g$, for some $g \in L_p^{\kappa(\lambda),0}$. By definition,

$$T_{\Psi_{\lambda_0}^{\mu_0}} f = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0)}} T_\nu J_\nu^{(\lambda_0, \mu_0)} T_\lambda^{-1} T_\lambda g = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0)}} T_\nu J_{\mu_0 \oslash \nu} g$$

and, for $J \in \mathcal{D}^{\kappa(\mu_0)}$

$$T_{\Psi_{\lambda_0}^{\mu_0 {}^\frown J}} f = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 {}^\frown J)}} T_\nu J_\nu^{(\lambda_0, \mu_0 {}^\frown J)} T_\lambda^{-1} T_\lambda g = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 {}^\frown J)}} T_\nu J_{\mu_0 {}^\frown J {}^\frown \nu} g.$$

We associate the index sets in the rightmost above sums as follows:

$$\mathcal{M}_\lambda^{(\lambda_0, \mu_0)} = \bigcup_{J \in \mathcal{D}^{\kappa(\mu_0)}} \Big\{ (b(\mu_0) + \kappa(\mu_0)) {}^\frown J {}^\frown \nu : \nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 {}^\frown J)} \Big\}. \tag{9.2}$$

Indeed, by Remark 8.2.6, for $J \in \mathcal{D}^{\kappa(\mu_0)}$, the map $\phi_\lambda^J : \mathcal{M}_\lambda^{(\lambda_0, \mu_0 {}^\frown J)} \to \Gamma_{\lambda_0 \oslash \lambda}^{\mu_0 {}^\frown J}$ given by

$$\phi_\lambda^J(\nu) = \mu_0 {}^\frown J {}^\frown \nu$$

is a bijection. Similarly, the map $\psi_\lambda : \mathcal{M}_\lambda^{(\lambda_0, \mu_0)} \to \Gamma_{\lambda_0 \oslash \lambda}^{\mu_0}$ given by

$$\psi_\lambda(\nu) = \mu_0 \oslash \nu$$

is a bijection. Therefore, by (9.1), for $J \in \mathcal{D}^{\kappa(\mu_0)}$ the map

$$\psi_\lambda^{-1} \phi_\lambda^J(\nu) = (b(\mu_0) + \kappa(\mu_0)) {}^\frown J {}^\frown \nu$$

defines an injection from $\mathcal{M}_\lambda^{(\lambda_0, \mu_0 {}^\frown J)}$ to $\mathcal{M}_\lambda^{(\lambda_0, \mu_0)}$, and by (9.1), the images of $(\psi_\lambda^{-1} \phi_\lambda^J)_{J \in \mathcal{D}^{\kappa(\mu_0)}}$ partition $\mathcal{M}_\lambda^{(\lambda_0, \mu_0)}$, which establishes (9.2). We conclude the proof as follows:

$$\begin{aligned} T_{\Psi_{\lambda_0}^{\mu_0}} f &= \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 {}^\frown J)}} T_{(b(\mu_0) + \kappa(\mu_0)) {}^\frown J {}^\frown \nu} J_{\mu_0 {}^\frown J {}^\frown \nu} g \\ &= \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 {}^\frown J)}} T_J T_\nu J_{\mu_0 {}^\frown J {}^\frown \nu} g = \sum_{J \in \mathcal{D}^{\kappa(\mu_0)}} T_J T_{\Psi_{\lambda_0}^{\mu_0 {}^\frown J}} f. \end{aligned}$$

□

Lemma 9.3.8. Let $\lambda_0 \in \mathcal{R}oot(\mathcal{E})$, $\mu_0 \in \mathcal{F}$, and $J \in \mathcal{D}^{\kappa(\mu_0)}$ such that $\Gamma_{\lambda_0}^{\mu_0 {}^\frown J} \neq \varnothing$. Let μ be the unique root of $\mathcal{F}^{\mu_0 {}^\frown J}$ such that $\Gamma_{\lambda_0}^{\mu_0 {}^\frown J} = \Gamma_{\lambda_0}^{\mu_0 {}^\frown J {}^\frown \mu}$ (which exists by Lemma 8.1.6 (b)). Then

$$T_{\Psi_{\lambda_0}^{\mu_0 {}^\frown J}} = T_\mu T_{\Psi_{\lambda_0}^{\mu_0 {}^\frown J {}^\frown \mu}}.$$

Proof. The ensuing argument is along the lines of the proof of Lemma 9.3.7, but we include it for completeness. Similarly to Remark 8.1.3 (c), for $\lambda \supsetneq \lambda_0$,

$$\Gamma_\lambda^{\mu_0 {}^\frown J} = \bigcup_{\mu \in \Gamma_{\lambda_0}^{\mu_0 {}^\frown J}} \Gamma_\lambda^\mu = \bigcup_{\mu \in \Gamma_{\lambda_0}^{\mu_0 {}^\frown J {}^\frown \mu}} \Gamma_\lambda^\mu = \Gamma_\lambda^{\mu_0 {}^\frown J {}^\frown \mu}. \tag{9.3}$$

We fix $\lambda \in \mathcal{E}^{\lambda_0}$ and $f \in X_\lambda$, i.e., $f = T_\lambda g$, for some $g \in L_p^{\kappa(\lambda),0}$. By definition,

$$T_{\Psi_{\lambda_0}^{\mu_0 ^\frown J}} f = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J)}} T_\nu J_\nu^{(\lambda_0,\mu_0 ^\frown J)} T_\lambda^{-1} T_\lambda g = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J)}} T_\nu J_{\mu_0 ^\frown J ^\frown \nu} g$$

and

$$T_{\Psi_{\lambda_0}^{\mu_0 ^\frown J ^\frown \mu}} f = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J ^\frown \mu)}} T_\nu J_\nu^{(\lambda_0,\mu_0 ^\frown J ^\frown \mu)} T_\lambda^{-1} T_\lambda g = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J ^\frown \mu)}} T_\nu J_{\mu_0 ^\frown J ^\frown \mu \oslash \nu} g.$$

We associate the index sets of the rightmost above sums as follows:

$$\mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J)} = \Big\{ \mu \oslash \nu : \nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J ^\frown \mu)} \Big\}. \tag{9.4}$$

Indeed, by Remark 8.2.6, the map $\phi_\lambda : \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J)} \to \Gamma_{\lambda_0 \oslash \lambda}^{\mu_0 ^\frown J}$ given by

$$\phi_\lambda(\nu) = \mu_0 {}^\frown J ^\frown \nu$$

is a bijection. Similarly, the map $\psi_\lambda : \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J ^\frown \mu)} \to \Gamma_{\lambda_0 \oslash \lambda}^{\mu_0 ^\frown J ^\frown \mu}$ given by

$$\psi_\lambda(\nu) = \mu_0 {}^\frown J ^\frown \mu \oslash \nu$$

is a bijection. Given $\Gamma_{\lambda_0 \oslash \lambda}^{\mu_0 ^\frown J} = \Gamma_{\lambda_0 \oslash \lambda}^{\mu_0 ^\frown J ^\frown \mu}$, the map $\phi_\lambda^{-1} \psi_\lambda(\nu) = \mu \oslash \nu$ is a bijection witnessing (9.4). We conclude as follows:

$$T_{\Psi_{\lambda_0}^{\mu_0 ^\frown J}} f = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J)}} T_\nu J_{\mu_0 ^\frown J ^\frown \nu} g = \sum_{\nu \in \mathcal{M}_\lambda^{(\lambda_0, \mu_0 ^\frown J ^\frown \mu)}} T_{\mu \oslash \nu} J_{\mu_0 ^\frown J ^\frown \mu \oslash \nu} g = T_\mu T_{\Psi_{\lambda_0}^{\mu_0 ^\frown J ^\frown \mu}} f.$$

□

We now perform the inductive step. To reach the intended conclusion, we carry out an induction on a specific sub-tree $\mathcal{S}$ of $\mathcal{F}$, associated with Ψ. As we progress from the leaves to the roots of $\mathcal{S}$, the intention is to show that for each $\mu \in \mathcal{S}$, a certain operator S_μ associated with μ and Ψ sufficiently preserves independence.

Proof of Proposition 9.3.5. For every $\mu \in \mathcal{F}$ (whether a root or otherwise) denote

$$\mathcal{R}oot(\mathcal{E}, \mu) = \{\lambda \in \mathcal{R}oot(\mathcal{E}) : \Gamma_\lambda^\mu \neq \varnothing\}$$

and $Z_\mu = [(Y_\lambda^{\mathcal{E}})_{\lambda \in \mathcal{R}oot(\mathcal{E},\mu)}]$. Note that, by Lemma 8.1.6 (b), the sets $\big(\mathcal{R}oot(\mathcal{E},\mu)\big)_{\mu \in \mathcal{R}oot(\mathcal{F})}$ form a partition of the roots of $\mathcal{R}oot(\mathcal{E})$. By Remark 9.3.2, $R_{\mathcal{E}} = [(Z_\mu)_{\mu \in \mathcal{R}oot(\mathcal{F})}]$, and the spaces $(Z_\mu)_{\mu \in \mathcal{R}oot(\mathcal{F})}$ are independent.

We formulate the following claim.

Claim. For all $\mu \in \mathcal{F}$ for which $\mathcal{R}oot(\mathcal{E}, \mu) \neq \varnothing$, there exists a distributional embedding

$$S_\mu : Z_\mu \to R_{\mathcal{F}^\mu}$$

such that, for all $\lambda \in \mathcal{R}oot(\mathcal{E}, \mu)$,

$$S_\mu|_{Y^{\mathcal{E}}_\lambda} = T_{\Psi^\mu_\lambda} T_\lambda^{-1}|_{Y^{\mathcal{E}}_\lambda}.$$

Assuming the above is true, we will first conclude the proof by applying the claim. For $\mu \in \mathcal{R}oot(\mathcal{F})$, because T_μ is a distributional embedding, we have that $T_\mu S_\mu : Z_\mu \to R^{p,0}_\beta$ is a distributional embedding with range inside Y_μ, and thus, by Proposition 7.4.1, $(T_\mu S_\mu)_{\mu \in \mathcal{R}oot(\mathcal{F})}$ have independent ranges. By Proposition 9.1.1, the operator $S : R_{\mathcal{E}} = [(Z_\mu)_{\mu \in \mathcal{R}oot(\mathcal{F})}] \to R^{p,0}_\beta$ given by $S|_{Z_\mu} = T_\mu S_\mu$, $\mu \in \mathcal{R}oot(\mathcal{F})$, is a distributional embedding. We will use Lemma 9.3.6 to show that $S = T_\Psi$. Indeed, for $\lambda \in \mathcal{R}oot(\mathcal{E})$ and $\mu \in \mathcal{R}oot(\mathcal{F})$ such that $\lambda \in \mathcal{R}oot(\mathcal{E}, \mu)$,

$$S|_{Y^{\mathcal{E}}_\lambda} = T_\mu S_\mu|_{Y^{\mathcal{E}}_\lambda} = T_\mu T_{\Psi^\mu_\lambda} T_\lambda^{-1}|_{Y^{\mathcal{E}}_\lambda} = T_\Psi|_{Y^{\mathcal{E}}_\lambda}.$$

We proceed to prove the claim by induction on the sub-tree $\mathcal{S} = \{\mu \in \mathcal{F} : \mathcal{R}oot(\mathcal{E}, \mu) \neq \varnothing\}$ of $\mathcal{F}$ with maximal nodes the members of $\cup_{\lambda \in \mathcal{R}oot(\mathcal{E})} \mathcal{M}_\lambda$. The inductive step will essentially be a refinement of the preceding argument.

In the basis step, let $\mu \in \mathcal{S}$ be a terminal node. Then $\mathcal{R}oot(\mathcal{E}, \mu) = \{\lambda\}$, for some $\lambda \in \mathcal{R}oot(\mathcal{E})$. The claim can be rephrased as follows: $T_{\Psi^\mu_\lambda} T_\lambda^{-1} : Y^{\mathcal{E}}_\lambda \to R_{\mathcal{F}^\mu}$ is a distributional embedding, which we will now verify. Because $P(b(\lambda) + \kappa(\lambda))$ is true, $T_{\Psi^\mu_\lambda} : R_{\mathcal{E}^\lambda} \to R_{\mathcal{F}^\mu}$ is a distributional embedding. By Remark 9.3.3, $T_\lambda^{-1} : Y^{\mathcal{E}}_\lambda \to R_{\mathcal{E}^\lambda}$ is a distributional isomorphism. We conclude that $T_{\Psi^\mu_\lambda} T_\lambda^{-1} : Y^{\mathcal{E}}_\lambda \to R_{\mathcal{F}^\mu}$ is a distributional embedding.

Let μ be non-maximal with the property $\mathcal{R}oot(\mathcal{E}, \mu) \neq \varnothing$ and assume that all successors of μ satisfy the claim. We will first prove that, for $J \in \mathcal{D}^{\kappa(\mu)}$,

$$\mathcal{R}oot(\mathcal{E}, \mu) = \bigcup_{\nu \in \mathcal{R}oot(\mathcal{F}^{\mu^\frown J})} \mathcal{R}oot(\mathcal{E}, \mu^\frown J^\frown \nu). \tag{9.5}$$

The "$\supset$" inclusion is trivial. Let $\lambda \in \mathcal{R}oot(\mathcal{E}, \mu)$. Because μ is non-maximal in $\mathcal{S}$, $\mu \notin \mathcal{M}_\lambda$, and thus, there is $P \in \mathcal{D}^{\kappa(\mu)}$ such that $\Gamma^{\mu^\frown P}_\lambda \neq \varnothing$. By Proposition 8.1.4 (b), $\sum_{\xi \in \Gamma^{\mu^\frown J}_\lambda} \theta_\xi = \sum_{\xi \in \Gamma^{\mu^\frown P}_\lambda} \theta_\xi$, and thus $\Gamma^{\mu^\frown J}_\lambda \neq \varnothing$. We deduce that there exists $\nu \in \mathcal{R}oot(\mathcal{F}^{\mu^\frown J})$ such that $\Gamma^{\mu^\frown J^\frown \nu}_\lambda \neq \varnothing$, i.e., $\lambda \in \mathcal{R}oot(\mathcal{E}, \mu^\frown J^\frown \nu)$.

Fix momentarily $J \in \mathcal{D}^{\kappa(\mu)}$. By the inductive hypothesis, for every $\nu \in \mathcal{R}oot(\mathcal{F}^{\mu^\frown J})$, there exists a distributional embedding

$$S_{\mu^\frown J^\frown \nu} : Z_{\mu^\frown J^\frown \nu} \to R_{\mathcal{F}^{\mu^\frown J^\frown \nu}}$$

such that, for all $\lambda \in \mathcal{R}oot(\mathcal{E}, \mu^\frown J^\frown \nu)$,

$$S_{\mu^\frown J^\frown \nu}|_{Y^{\mathcal{E}}_\lambda} = T_{\Psi^{\mu^\frown J^\frown \nu}_\lambda} T_\lambda^{-1}|_{Y^{\mathcal{E}}_\lambda}.$$

Therefore, for $\nu \in \mathcal{R}oot(\mathcal{F}^{\mu^\frown J})$, the operator $T_\nu S_{\mu^\frown J^\frown \nu} : Z_{\mu^\frown J^\frown \nu} \to R^{p,0}_{b(\mu)}$ is a distributional embedding with image in Y_ν. By Proposition 9.1.1, there exists a distributional embedding

$$S_J : [(Z_{\mu^\frown J^\frown \nu})_{\nu \in \mathcal{R}oot(\mathcal{F}^{\mu^\frown J})}] = [\big((Y^{\mathcal{E}}_\lambda)_{\lambda \in \mathcal{R}oot(\mathcal{E}, \mu^\frown J^\frown \nu)}\big)_{\nu \in \mathcal{R}oot(\mathcal{F}^{\mu^\frown J})}] \overset{(9.5)}{=} Z_\mu \to R^{p,0}_{b(\mu)}$$

such that, for all $\nu \in \mathcal{R}oot(\mathcal{F}^{\mu^\frown J})$,

$$S_J|_{Z_{\mu^\frown J^\frown \nu}} = T_\nu S_{\mu^\frown J^\frown \nu}.$$

By using Proposition 9.1.5, we deduce that the operator $S = \sum_{J \in \mathcal{D}^\kappa(\mu)} T_J S_J : Z_\mu \to R^{p,0}_{b(\mu)}$ is a distributional embedding. To conclude the proof, it remains to be verified that, for $\lambda \in \mathcal{R}oot(\mathcal{E}, \mu)$, $S|_{Y^{\mathcal{E}}_\lambda} = T_{\Psi^\mu_\lambda} T_\lambda^{-1}|_{Y^{\mathcal{E}}_\lambda}$. For $J \in \mathcal{D}^{\kappa(\mu)}$, let ν_J be the unique root of $\mathcal{F}^{\mu^\frown J}$ such that $\lambda \in \mathcal{R}oot(\mathcal{E}, \mu^\frown J^\frown \nu_J)$. Then,

$$\begin{aligned}
S|_{Y^{\mathcal{E}}_\lambda} &= \sum_{J \in \mathcal{D}^\kappa(\mu)} T_J S_J|_{Y^{\mathcal{E}}_\lambda} = \sum_{J \in \mathcal{D}^\kappa(\mu)} T_J T_{\nu_J} S_{\mu^\frown J^\frown \nu_J}|_{Y^{\mathcal{E}}_\lambda} = \sum_{J \in \mathcal{D}^\kappa(\mu)} T_J T_{\nu_J} T_{\Psi^{\mu^\frown J^\frown \nu_J}_\lambda} T_\lambda^{-1}|_{Y^{\mathcal{E}}_\lambda} \\
&= \sum_{J \in \mathcal{D}^\kappa(\mu)} T_J T_{\Psi^{\mu^\frown J}_\lambda} T_\lambda^{-1}|_{Y^{\mathcal{E}}_\lambda} \text{ (by Lemma 9.3.8)} \\
&= T_{\Psi^\mu_\lambda} T_\lambda^{-1}|_{Y^{\mathcal{E}}_\lambda} \text{ (by Lemma 9.3.7).}
\end{aligned}$$

$\square$

Chapter 10

Factorizing in the limit $R^{p,0}_\alpha$ spaces

In this chapter, we show that for any bounded linear operator $T : R^{p,0}_\alpha \to R^{p,0}_\alpha$, with α a countable limit ordinal, and any $\epsilon > 0$, there exist a distributional embedding scheme Ψ of $R^{p,0}_\alpha$ into itself and a scalar FDD-diagonal operator $R : R^{p,0}_\alpha \to R^{p,0}_\alpha$ such that $\|T^\dagger_\Psi T T_\Psi - R\| < \epsilon$. The entries of R are averages of the diagonal entries of T, and hence, using Proposition 3.3.5, we obtain the factorization property of the standard martingale difference sequence basis of $R^{p,0}_\alpha$. The construction of Ψ is inductive. Enumerate $\mathcal{T}_\alpha$ as $(\lambda_n)_{n=1}^\infty$, compatible with the tree order. At each step we select $(\mathcal{M}_{\lambda_n}, (J_\mu)_{\mu\in\mathcal{M}_{\lambda_n}})$ and a scalar r_n such that, letting $S_n = \sum_{\mu\in\mathcal{M}_{\lambda_n}} T_\mu J_\mu : L_p^{\kappa(\lambda_n),0} \to L^0_p$, Y_n its image, and E_n the orthogonal projection onto Y_n, we have

$$\left\|(E_nT - r_n\cdot id)|_{Y_n}\right\| < \frac{\epsilon}{3\cdot 2^n}, \quad \left\|\Big(\sum_{m=1}^{n-1} E_m\Big)TE_n\right\| < \frac{\epsilon}{3\cdot 2^n}, \quad \left\|E_nT\Big(\sum_{m=1}^{n-1} E_m\Big)\right\| < \frac{\epsilon}{3\cdot 2^n}.$$

Provided Definition 8.0.3 is respected, this yields $\|T^\dagger_\Psi T T_\Psi - R\| < \epsilon$, where $R|_{X_{\lambda_n}} = r_n\cdot id|_{X_{\lambda_n}}$. The inductive choice of $(\mathcal{M}_{\lambda_{n+1}}, (J_\mu)_{\mu\in\mathcal{M}_{\lambda_{n+1}}})$ rests on a finite-dimensional stabilization argument depending on the parameters $\dim(X_{\lambda_{n+1}})$, $\sum_{m=1}^n \dim(X_{\lambda_m})$, $\|T\|$, and the error $\epsilon/2^{n+1}$. These determine a large integer κ_{n+1}, permitting the choice of $\mathcal{M}_{\lambda_{n+1}}$, respecting $(\mathcal{M}_{\lambda_m}, (J_\mu)_{\mu\in\mathcal{M}_{\lambda_m}})_{m=1}^n$, with $\kappa(\mu) = \kappa_{n+1}$ for all $\mu\in\mathcal{M}_{\lambda_{n+1}}$. This provides enough room to carry out a probabilistic choice of the family $(J_\mu)_{\mu\in\mathcal{M}_{\lambda_{n+1}}}$ and define S_{n+1}. The Chapter is divided into two parts: the finite-dimensional stabilization and the construction of the distributional embedding scheme, the latter completing the proof.

10.1 A finite-dimensional stabilization

We now outline the main result of this subsection and its proof. Given $n, m \in \mathbb{N}$, $\Gamma > 0$, and $0 < \epsilon < 1$, we show the existence of a large integer k_0'' with the following property: for every operator T of norm at most Γ, operators Q_1, Q_2 of norm at most Γ and rank at most m, every $k \geqslant k_0''$, and compressions $S_l : L_p^{k,0} \to L_p^0$, $1 \leqslant l \leqslant N$, with pairwise disjointly supported ranges (k_0'' does not depend on N), there exist distributional embeddings $J_l : L_p^{n,0} \to L_p^{k,0}$, $1 \leqslant l \leqslant N$, such that, letting $S = \sum_{l=1}^{N} S_l J_l : L_p^{n,0} \to L_p^0$, with image Y and E the orthogonal projection onto Y, the operator $ET|_Y$ is ϵ-close to $r \cdot id|_Y$, with r given by an explicit formula, and EQ_1 and Q_2E have norm less than ϵ. This result is directly applicable in the context outlined above.

The proof proceeds in two stages: first, a reduction where $ET|_Y$ is shown to be close to a diagonal operator $R : Y \to Y$, and then a further reduction to a scalar multiple of the identity, each step incurring a dimensional penalty. The first step is proved here in full. The construction proceeds by enumerating $\mathcal{D}_{n-1}$ lexicographically and, assuming $(J_l h_J)_{l=1}^N$ has been defined for $J < I$, simultaneously defining $(J_l h_I)_{l=1}^N$ by means of a random family $(J_l h_I(\vartheta))_{l=1}^N$, where ϑ ranges over a suitable probability space. One then shows that, with high probability, sufficient criteria are satisfied. The second step is not proved from scratch: we first dilate an appropriate compressed finite-dimensional L_p^0-space to a non-compressed one, apply the stabilization of Lemma 6.3.3, and finally compress back to the original setting.

Remark 10.1.1. Let $m \in \mathbb{N}$, $\theta \in (0,1]$, and $S : L_p^{m,0} \to L_p^0$ be a θ-compression. The orthogonal projection $E\colon L_p^0 \to S[L_p^{m,0}]$ can be expressed as

$$E(f) = \sum_{I \in \mathcal{D}_{m-1}} \theta^{-1} |I|^{-1} \langle Sh_I, f \rangle Sh_I = \mathbb{E}(f\chi_A | \mathcal{A}),$$

where $\mathcal{A} = \sigma(Sh_I : I \in \mathcal{D}_{m-1})$ and $A = \cup_{I \in \mathcal{D}^{m-1}} \operatorname{supp}(Sh_I)$ (the above equality of formulas holds in L_p^0 but not in L_p). We will mainly rely on the former, while the latter reveals that $\|E\| = 1$.

This first step of the reduction is carried out in full and yields a stabilization where $ET|_Y$ is close to a diagonal operator R on a compressed copy Y of $L_p^{n,0}$, obtained from pieces of disjoint compressed copies of $L_p^{k,0}$ for sufficiently large k.

Lemma 10.1.2. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, $\Gamma > 0$, and $0 < \epsilon < 1$. There exists $k_0 := k_0(n, m, \Gamma, \epsilon) \in \mathbb{N}$ such that for every $k \geqslant k_0 + n$ and

(a) operator $T\colon X \to L_p^0$ of norm at most Γ, where X is a subspace of L_p^0,

(b) operators $Q_1 : Z \to L_p^0, Q_2 : X \to W$ of norm at most Γ and rank at most m, where Z, W are normed spaces, and

(c) operators $S_1, \ldots, S_N : L_p^{k,0} \to X$ with disjointly supported ranges such that, for $1 \leqslant l \leqslant N$, S_l is a θ_l-compression, for some $\theta_l \in (0,1]$,

there exist distributional embeddings $J_l : L_p^{n,0} \to L_p^{k,0}$, $1 \leqslant l \leqslant N$, satisfying the following:

(i) For each $1 \leqslant l \leqslant N$, $0 \leqslant i \leqslant n-1$ and $I \in \mathcal{D}^i$, there are $\mathcal{B}_I^l \subset \mathcal{D}^{k_0+i}$ and $(\theta_K^l)_{K \in \mathcal{B}_I^l} \in \{-1,1\}^{\mathcal{B}_I^l}$ such that

$$J_l h_I = \sum_{K \in \mathcal{B}_I^l} \theta_K^l h_K.$$

(ii) Denote $S = \sum_{l=1}^N S_l J_l : L_p^{n,0} \to X$, which is a θ-compression with $\theta = \sum_{l=1}^N \theta_l$, $b_I = Sh_I$, $I \in \mathcal{D}_{n-1}$, $Y = [(b_I)_{I\in\mathcal{D}_{n-1}}]$, $E : L_p^0 \to Y$ the orthogonal projection, and $R : Y \to Y$ the diagonal operator given by $Rb_I = d_I b_I$, where

$$d_I = \frac{1}{|I|\theta} \sum_{l=1}^{N} \sum_{K\in\mathcal{B}_I^l} \langle S_l h_K, TS_l h_K\rangle,\ I \in \mathcal{D}_{n-1}.$$

Then

$$\Big\|\Big(ET - R\Big)|Y\Big\| < \epsilon.$$

(iii) We have $\|EQ_1\| < \epsilon$ and $\|Q_2E\| < \epsilon$.

Proof. Let $M = 5^m - 1$, which is sufficiently large to ensure that the unit sphere of every m-dimensional normed space contains a $(1/2)$-dense set of cardinality at most M. Let

$$k_0(n, m, \Gamma, \epsilon) = \Big\lfloor p^*\Big(5n + 3 + 2\log(\Gamma) + 3m + 2\log(\epsilon^{-1})\Big)\Big\rfloor + 1, \tag{10.1}$$

and fix $k \geqslant k_0 + n$ and items such as in (a), (b) and (c). Fix $(1/2)$-dense subsets $(x_i)_{i=1}^M$ of the unit sphere of $\mathrm{R}(Q_1)$ and $(y_i^*)_{i=1}^M$ of the unit sphere of the dual of $\mathrm{R}(Q_2)$.

For $1 \leqslant l \leqslant N$, we will define $J_l : L_p^{n,0} \to L_p^k$ by choosing a distributional copy $(c_I^l)_{I\in\mathcal{D}_{n-1}}$ of $(h_I)_{I\in\mathcal{D}_{n-1}}$ in $L_p^{k,0}$, and letting $J_l h_I = c_I^l$, $I \in \mathcal{D}_{n-1}$. Following the notation of the statement, the construction will satisfy (i) and

$$\Big|\Big\langle b_I, Tb_J\Big\rangle\Big| < \frac{\epsilon\theta|I|^{1/q}|J|^{1/p}}{2^{2n+1}},\ \ I \neq J \in \mathcal{D}_{n-1}, \tag{10.2}$$

$$\Big|\theta^{-1}|I|^{-1}\Big\langle b_I, Tb_I\Big\rangle - d_I\Big| < \frac{\epsilon}{2^{n+1}},\ \ I \in \mathcal{D}_{n-1}, \tag{10.3}$$

$$|\langle b_I, x_i\rangle| \leqslant \frac{\epsilon(\theta|I|)^{1/q}}{\Gamma 2^{n+1}} \text{ and } \big|Q_2^* x_i^*(b_I)\big| \leqslant \frac{\epsilon(\theta|I|)^{1/p}}{2^{n+1}},\ I \in \mathcal{D}_{n-1},\ \ 1 \leqslant i \leqslant M. \tag{10.4}$$

We note that (10.2) and (10.3) yield (ii) as follows: for $y \in Y$, with $\|y\| \leqslant 1$,

$$\begin{aligned}
\Big\|ETy - Ry\Big\| &= \Big\| \sum_{I\in\mathcal{D}_{n-1}} \Big(\sum_{J\in\mathcal{D}_{n-1}} \theta^{-1}|I|^{-1}\theta^{-1}|J|^{-1}\langle b_I, Tb_J\rangle\langle b_J, y\rangle\Big) b_I - \sum_{I\in\mathcal{D}_{n-1}} \theta^{-1}|I|^{-1} d_I\langle b_I, y\rangle b_I\Big\| \\
&\leqslant \Big\| \sum_{I\in\mathcal{D}_{n-1}} \theta^{-1}|I|^{-1}\langle b_I, y\rangle\Big(\theta^{-1}|I|^{-1}\langle b_I, Tb_I\rangle - d_I\Big) b_I\Big\| \\
&\quad + \Big\| \sum_{I\in\mathcal{D}_{n-1}} \Big(\sum_{\substack{J\in\mathcal{D}_{n-1}\\ J\neq I}} \theta^{-2}|I|^{-1}|J|^{-1}\langle b_I, Tb_J\rangle\langle b_J, y\rangle\Big) b_I\Big\| \\
&< \sum_{I\in\mathcal{D}_{n-1}} \theta^{-1}|I|^{-1}\theta^{1/q}|I|^{1/q}\theta^{1/p}|I|^{1/p}\frac{\epsilon}{2^{n+1}}
\end{aligned}$$

$$+\sum_{I\in\mathcal{D}_{n-1}}\sum_{\substack{J\in\mathcal{D}_{n-1}\\ J\neq I}}\theta^{-2}|I|^{-1}|J|^{-1}\theta^{1/q}|J|^{1/q}\theta^{1/p}|I|^{1/p}\frac{\epsilon\theta|I|^{1/q}|J|^{1/p}}{2^{2n+1}}$$
$$\leqslant \big(\#\mathcal{D}_{n-1}\big)\frac{\epsilon}{2^{n+1}}+\big(\#\mathcal{D}_{n-1}\big)^2\frac{\epsilon}{2^{2n+1}}<\epsilon.$$

Furthermore, (10.4) yields (iii) as follows: for $1\leqslant i\leqslant M$,

$$\|Ex_i\|=\Big\|\sum_{I\in\mathcal{D}_{n-1}}\theta^{-1}|I|^{-1}\langle b_I,x_i\rangle b_I\Big\|\leqslant\sum_{I\in\mathcal{D}_{n-1}}\theta^{-1}|I|^{-1}\frac{\epsilon(\theta|I|)^{1/q}}{\Gamma 2^{n+1}}(\theta|I|)^{1/p}<\frac{\epsilon}{2\Gamma},$$

and $\|EQ_1\|\leqslant 2\Gamma\max_{1\leqslant i\leqslant M}\|Ex_i\|<\epsilon$. An argument of a similar nature yields $\|Q_2E\|<\epsilon$.

We return to the main objective of the proof, namely, the construction of the distributional copies $(c_I^l)_{I\in\mathcal{D}_{n-1}}$ of $(h_I)_{I\in\mathcal{D}_{n-1}}$ in $L_p^{k,0}$, $1\leqslant l\leqslant N$, such that item (i) and (10.2) to (10.4) are satisfied. We proceed by induction on the lexicographical order of $\mathcal{D}_{n-1}$, and for each $I\in\mathcal{D}_{n-1}$, if $I\in\mathcal{D}^i$, we will choose, simultaneously for all $1\leqslant l\leqslant N$, collections $\mathcal{B}_I^l\subset\mathcal{D}^{k_0+i}$, with $\cup\mathcal{B}_I^l=[c_{\pi(I)}^l=\epsilon(I)]$ (or $\mathcal{B}_{[0,1)}=\mathcal{D}^{k_0}$, for $I=[0,1)$), and $(\theta_K^l)_{K\in\mathcal{B}_I}\in\{-1,1\}^{\mathcal{B}_I^l}$, and let

$$c_I^l=\sum_{K\in\mathcal{B}_I^l}\theta_K^l h_I.$$

We skip the basis step because the argument is analogous to the general one. Let $I\in\mathcal{D}^i$, with $1\leqslant i\leqslant n-1$, and assume that for every J preceding I in the lexicographical order, we have chosen the collection $\mathcal{B}_J^l$ and the signs $(\theta_K^l)_{K\in\mathcal{B}_J^l}$, $1\leqslant l\leqslant N$, satisfying the inductive hypothesis so far. For $1\leqslant l\leqslant N$, define

$$\mathcal{B}_I^l=\Big\{K\in\mathcal{D}^{k_0+i}:K\subset[c_{\pi(I)}^l=\epsilon(I)]\Big\},$$

i.e., $\mathcal{B}_I^l$ satisfies $\cup\mathcal{B}_I^l=[c_{\pi(I)}^l=\epsilon(I)]$. We consider the randomized vector

$$b_I(\vartheta)=\sum_{l=1}^N\sum_{K\in\mathcal{B}_I^l}\theta_K^l S_l h_K$$

where

$$\vartheta=\big((\theta_K^l)_{K\in\mathcal{B}_I^l}\big)_{l=1}^N\in\Omega:=\prod_{l=1}^N\{-1,1\}^{\mathcal{B}_I^l},$$

equipped with the uniform measure. Note that $\big((S_lh_K)_{K\in\mathcal{B}_I^l}\big)_{l=1}^N$ are disjointly supported and symmetric $\{-1,0,1\}$-valued vectors. We introduce the following random variables based on Notation 6.2.1. For $\vartheta\in\Omega$, we let

$$Y_J(\vartheta):=Y_{\mathcal{B}_I,T^*b_J}(\vartheta)=\Big\langle b_J,Tb_I(\vartheta)\Big\rangle,\quad J<I\in\mathcal{D}_{n-1},$$
$$W_J(\vartheta):=W_{\mathcal{B}_I,Tb_J}(\vartheta)=\Big\langle b_I(\vartheta),Tb_J\Big\rangle,\quad J<I\in\mathcal{D}_{n-1},$$

$$Z_I(\vartheta) := Z_{\mathcal{B}_I,T}(\vartheta) = \Big\langle b_I(\vartheta), Tb_I(\vartheta)\Big\rangle - \sum_{l=1}^{N}\sum_{K\in\mathcal{B}_I^l}\Big\langle S_l h_K, TS_l h_K\Big\rangle, \text{ and}$$

$$W_j(\vartheta) := W_{\mathcal{B}_I,x_j}(\vartheta) = \langle b_I(\vartheta), x_j\rangle \text{ and } Y_j(\vartheta) := Y_{\mathcal{B}_I,Q_2^* x_j^*}(\vartheta) = Q_2^* x_j^*\big(b_I(\vartheta)\big),\ \ 1 \leqslant j \leqslant M.$$

We relate (10.2) to (10.4) to these random variables as follows: there exists $\vartheta \in \Omega$ such that

$$\begin{aligned}
\Big|Y_J(\vartheta)\Big| < \frac{\epsilon\theta|J|^{1/q}|I|^{1/p}}{2^{2n+1}} \text{ and } \Big|W_J(\vartheta)\Big| < \frac{\epsilon\theta|I|^{1/q}|J|^{1/p}}{2^{2n+1}},\ \ J < I \in \mathcal{D}_{n-1},\\
\Big|Z_I(\vartheta)\Big| < \frac{\epsilon\theta|I|}{2^{n+1}}, \text{ and}\\
|W_j(\vartheta)| < \frac{\epsilon(\theta|I|)^{1/q}}{\Gamma 2^{n+1}} \text{ and } |Y_j(\vartheta)| < \frac{\epsilon(\theta|I|)^{1/p}}{2^{n+1}},\ \ 1 \leqslant j \leqslant M.
\end{aligned}\tag{10.5}$$

Once the existence of such ϑ has been established, we can let, for $1 \leqslant l \leqslant N$, $c_I^l = \sum_{K\in\mathcal{B}_I^l}\theta_K^l h_K$, and the proof will be complete. We proceed by examining the following events in Ω:

$$\begin{aligned}
\mathcal{Y}_J = \Big[\big|Y_J(\vartheta)\big| \geqslant \frac{\epsilon\theta|J|^{1/q}|I|^{1/p}}{2^{2n+1}}\Big] \text{ and } \mathcal{W}_J = \Big[\big|W_J(\vartheta)\big| \geqslant \frac{\epsilon\theta|I|^{1/q}|J|^{1/p}}{2^{2n+1}}\Big],\ \ J < I \in \mathcal{D}_{n-1},\\
\mathcal{Z}_I = \Big[\big|Z_I(\vartheta)\big| \geqslant \frac{\epsilon\theta|I|}{2^{n+1}}\Big], \text{ and}\\
\mathcal{W}_j = \Big[\big|W_j(\vartheta)\big| \geqslant \frac{\epsilon(\theta|I|)^{1/q}}{\Gamma 2^{n+1}}\Big] \text{ and } \mathcal{Y}_j = \Big[\big|Y_j(\vartheta)\big| \geqslant \frac{\epsilon(\theta|I|)^{1/p}}{2^{n+1}}\Big],\ \ 1 \leqslant j \leqslant M.
\end{aligned}$$

Denoting $\mathcal{E}$ the union of the above events, it suffices to show $\mathbb{P}(\mathcal{E}) < 1$. Taking the probability of $\mathcal{E}$ and applying Chebyshev's inequality, we obtain:

$$\begin{aligned}
\mathbb{P}(\mathcal{E}) &\leqslant \sum_{J<I\in\mathcal{D}_{n-1}}\mathbb{P}(\mathcal{Y}_J) + \sum_{J<I\in\mathcal{D}_{n-1}}\mathbb{P}(\mathcal{W}_J) + \mathbb{P}(\mathcal{Z}_I) + \sum_{j=1}^{M}\mathbb{P}(\mathcal{W}_j) + \sum_{j=1}^{M}\mathbb{P}(\mathcal{Y}_j)\\
&\leqslant \sum_{J<I\in\mathcal{D}_{n-1}}\frac{2^{4n+2}}{\epsilon^2\theta^2|J|^{2/q}|I|^{2/p}}\mathbb{V}(Y_J) + \sum_{J<I\in\mathcal{D}_{n-1}}\frac{2^{4n+2}}{\epsilon^2\theta^2|I|^{2/q}|J|^{2/p}}\mathbb{V}(W_J) + \frac{2^{2n+2}}{\epsilon^2\theta^2|I|^2}\mathbb{V}(Z_I)\\
&+ \sum_{j=1}^{M}\frac{\Gamma^2 2^{2n+2}}{\epsilon^2(\theta|I|)^{2/q}}\mathbb{V}(W_j) + \sum_{j=1}^{M}\frac{2^{2n+2}}{\epsilon^2(\theta|I|)^{2/p}}\mathbb{V}(Y_j).
\end{aligned}$$

Lemmas 6.2.2, 6.2.3 and 6.2.4 yield the following estimates for the above variances:

$$\mathbb{V}(Y_J) \leqslant \|T\|^2\big(\theta|J|\big)^{2/q}\Big(\theta\frac{1}{2^i}\big(\max_{1\leqslant l\leqslant N}\theta_l\big)\frac{1}{2^{k_0+i}}\Big)^{1/p} \leqslant \frac{\theta^2\Gamma^2|J|^{2/q}}{2^{(k_0+2i)/p}},\ \ J < I \in \mathcal{D}_{n-1},$$

$$\mathbb{V}(W_J) \leqslant \|T\|^2\big(\theta|J|\big)^{2/p}\Big(\theta\frac{1}{2^i}\big(\max_{1\leqslant l\leqslant N}\theta_l\big)\frac{1}{2^{k_0+i}}\Big)^{1/q} \leqslant \frac{\theta^2\Gamma^2|J|^{2/p}}{2^{(k_0+2i)/q}},\ \ J < I \in \mathcal{D}_{n-1},$$

$$\mathbb{V}(Z_I) \leqslant 2\|T\|^2\Big(\theta\frac{1}{2^i}\Big)^{1+1/p}\Big(\max_{1\leqslant l\leqslant N}\theta_l\frac{1}{2^{k_0+i}}\Big)^{1/q} \leqslant \frac{2\theta^2\Gamma^2}{2^{2i+k_0/q}}, \text{ and}$$

$$\mathbb{V}(W_j) \leqslant \Big(\theta\frac{1}{2^i}\big(\max_{1\leqslant l\leqslant N}\theta_l\big)\frac{1}{2^{k_0+i}}\Big)^{1/q} \leqslant \frac{\theta^{2/q}}{2^{(k_0+2i)/q}} \text{ and}$$

$$\mathbb{V}(Y_j) \leqslant \Gamma^2\Big(\theta\frac{1}{2^i}\big(\max_{1\leqslant l\leqslant N}\theta_l\big)\frac{1}{2^{k_0+i}}\Big)^{1/p} \leqslant \frac{\Gamma^2\theta^{2/p}}{2^{(k_0+2i)/p}},\ \ 1\leqslant j\leqslant M.$$

By combining the upper bound for $\mathbb{P}(\mathcal{E})$ with the variance bounds we conclude:

$$\begin{aligned}
\mathbb{P}(\mathcal{E}) \leqslant{}& \sum_{J<I\in\mathcal{D}_{n-1}}\frac{2^{4n+2}}{\epsilon^2\theta^2|J|^{2/q}|I|^{2/p}}\frac{\theta^2\Gamma^2|J|^{2/q}}{2^{(k_0+2i)/p}} + \sum_{J<I\in\mathcal{D}_{n-1}}\frac{2^{4n+2}}{\epsilon^2\theta^2|I|^{2/q}|J|^{2/p}}\frac{2\theta^2\Gamma^2|J|^{2/p}}{2^{(k_0+2i)/q}}\\
&+\frac{2^{2n+2}}{\epsilon^2\theta^2|I|^2}\frac{2\theta^2\Gamma^2}{2^{2i+k_0/q}} + M\frac{\Gamma^2 2^{2n+2}}{\epsilon^2(\theta|I|)^{2/q}}\frac{\theta^{2/q}}{2^{(k_0+2i)/q}} + M\frac{2^{2n+2}}{\epsilon^2(\theta|I|)^{2/p}}\frac{\Gamma^2\theta^{2/p}}{2^{(k_0+2i)/p}}\\
\leqslant{}& 2^i\frac{2^{4n+2}\Gamma^2}{\epsilon^2 2^{k_0/p}} + 2^i\frac{2^{4n+2}\Gamma^2}{\epsilon^2 2^{k_0/q}} + \frac{2^{2n+2}2\Gamma^2}{\epsilon^2 2^{k_0/q}} + \frac{2^{2n+3}M\Gamma^2}{\epsilon^2 2^{k_0/p^*}} \leqslant \frac{2^{5n+3}\Gamma^2(1+M)}{\epsilon^2 2^{k_0/p^*}}.
\end{aligned}$$

From (10.1), $\mathbb{P}(\mathcal{E}) < 1$, finishing the proof. □

The following reduction is the main result of this section.

Proposition 10.1.3. Let $n\in\mathbb{N}$, $m\in\mathbb{N}$, $\Gamma>0$, and $0<\epsilon<1$. There exists $k_0'' := k_0''(n,m,\Gamma,\epsilon)\in\mathbb{N}$ such that for every $k\geqslant k_0''$ and

(a) operator $T\colon X\to L_p$, where X is a subspace of L_p, with $\|T\|\leqslant\Gamma$,

(b) operators $Q_1 : Z\to L_p^0, Q_2 : X\to W$ of norm at most Γ and rank at most m, where Z and W are normed spaces, and

(c) operators $S_1,\ldots,S_N : L_p^{k,0}\to X$ with disjointly supported ranges such that, for $1\leqslant l\leqslant N$, S_l is a θ_l-compression, for some $\theta_l\in(0,1]$,

there exist distributional embeddings $J_l : L_p^{n,0}\to L_p^{k,0}$, $1\leqslant l\leqslant N$ such that the following hold.

(i) Denote $S=\sum_{l=1}^N S_lJ_l : L_p^{n,0}\to X$, which is a θ-compression with $\theta=\sum_{l=1}^N\theta_l$, $b_I = Sh_I$, $I\in\mathcal{D}_{n-1}$, $Y=[(b_I)_{I\in\mathcal{D}_{n-1}}]$, and $E : L_p^0\to Y$ the orthogonal projection. Then, for some $0\leqslant i\leqslant k-1$ and
$$r = \theta^{-1}\sum_{l=1}^N\sum_{K\in\mathcal{D}^i}\langle S_lh_K, TS_lh_K\rangle,$$
we have
$$\Big\|\Big(ET - r\cdot id\Big)|Y\Big\| < \epsilon.$$

(ii) We have $\|EQ_1\|<\epsilon$ and $\|Q_2E\|<\epsilon$.

Proof. Let

$$k_1 = k_0'(n, \Gamma + \epsilon/2, \epsilon/2) \text{ and } k_0'' = k_0(k_1, m, \Gamma, \epsilon/2) + k_1,$$

where k_0 is as in Lemma 10.1.2 and k_0' is as in Lemma 6.3.3 (that is it plays the role of M in (6.8)). Fix $k \geqslant k_0''$ and items such as in (a), (b) and (c). Applying Lemma 10.1.2 to these parameters yields distributional embeddings $\hat{J}_1, \ldots, \hat{J}_N : L_p^{k_1,0} \to L_p^{k,0}$ satisfying its conclusion (here, k_1 has replaced n in Lemma 10.1.2). For clarity we adopt the hat notation for all objects associated with them. Specifically, for each $1 \leqslant l \leqslant N$, $0 \leqslant i \leqslant k_1 - 1$, and $I \in \mathcal{D}^i$ there are $\hat{\mathcal{B}}_I^l \subset \mathcal{D}^{k_0+i}$ and signs $(\hat{\theta}_K^l)_{K \in \hat{\mathcal{B}}_I^l}$ such that

$$\hat{J}_l h_I = \sum_{K \in \hat{\mathcal{B}}_I^l} \hat{\theta}_K^l h_K.$$

Defining the θ-compression $\hat{S} = \sum_{l=1}^N S_l \hat{J}_l : L_p^{k,0} \to X$, $\hat{b}_I = \hat{S} h_I$,

$$\hat{d}_I = (|I|\theta)^{-1} \sum_{l=1}^N \sum_{K \in \hat{\mathcal{B}}_K^l} \langle S_l h_K, T S_l h_K \rangle,\ I \in \mathcal{D}_{k_1-1},$$

$\hat{Y} = [(\hat{b}_I)_{I \in \mathcal{D}_{k_1-1}}]$, $\hat{E} : L_p^0 \to \hat{Y}$ the orthogonal projection, and $\hat{R} : \hat{Y} \to \hat{Y}$ given by $\hat{R}\hat{b}_I = \hat{d}_I \hat{b}_I$, we have that Lemma 10.1.2 (ii) and (iii) hold for $\epsilon/2$.

Define

$$R = (\hat{S})^{-1} \hat{R} \hat{S} : L_p^{k_1,0} \to L_p^{k_1,0}$$

and note that $R h_I = \hat{d}_I h_I$, for $I \in \mathcal{D}_{k_1-1}$, i.e., R is a diagonal operator, and $\|R\| = \|\hat{R}\| \leqslant \Gamma + \epsilon/2$. We may therefore apply Lemma 6.3.3 to R to obtain a distributional embedding $J' : L_p^{n,0} \to L_p^{k_1,0}$ satisfying its conclusion. For clarity we adopt the prime notation for all objects associated with it. More precisely, there exist $0 \leqslant m_0 < m_1 < \cdots < m_{n-1} \leqslant k-1$ such that for $0 \leqslant i \leqslant n-1$ and $I \in \mathcal{D}^i$, there are $\mathcal{B}_I' \subset \mathcal{D}^{m_i}$ and signs $(\theta_K')_{K \in \mathcal{B}_I'}$ such that

$$J' h_I = \sum_{K \in \mathcal{B}_I'} \theta_K' h_K.$$

Denote $Y' = J'(L_p^{n,0})$, E' the conditional expectation operator onto Y', and

$$\begin{aligned} r = \langle J' h_{[0,1]}, R J' h_{[0,1]} \rangle &= \Big\langle \sum_{L \in \mathcal{D}^{m_0}} \theta_L' h_L, \sum_{L \in \mathcal{D}^{m_0}} \hat{d}_L \theta_L' h_L \Big\rangle = \frac{1}{2^{m_0}} \sum_{L \in \mathcal{D}^{m_0}} \hat{d}_L \\ &= \frac{1}{2^{m_0}} \sum_{L \in \mathcal{D}^{m_0}} \frac{2^{m_0}}{\theta} \sum_{l=1}^N \sum_{K \in \hat{\mathcal{B}}_L^l} \langle S_l h_K, T S_l h_K \rangle = \frac{1}{\theta} \sum_{l=1}^N \sum_{K \in \mathcal{D}^{k_0+m_0}} \langle S_l h_K, T S_l h_K \rangle. \end{aligned}$$

Then,

$$\Big\| \Big(E' R - r \cdot id \Big)\Big|_{Y'} \Big\| < \frac{\epsilon}{2}. \tag{10.6}$$

For $1 \leqslant l \leqslant N$, we define $J_l = \hat{J}_l J'$, and we will show that the desired conclusion is satisfied. Denote $S = \sum_{l=1}^N S_l J_l = \hat{S}J' : L_p^{n,0} \to X$, which is a θ-compression with $\theta = \sum_{l=1}^N \theta_l$, $b_I = Sh_I$, $I \in \mathcal{D}_{n-1}$, $Y = [(b_I)_{I\in\mathcal{D}_{n-1}}]$, and $E : L_p^0 \to Y$ the orthogonal projection. Because $\hat{S}$ is a θ-compression, $\mathrm{dist}(\hat{S}f, \hat{S}g) = \theta \mathrm{dist}(f,g) + (1-\theta)\delta_{(0,0)}$, for all f,g in its domain. We obtain that for every $I \in \mathcal{D}_{n-1}$, $f \in L_p^{k_1,0}$,

$$\langle \hat{S}J'h_I, \hat{S}f\rangle = \theta\langle J'h_I, f\rangle.$$

We deduce that, for $f \in \hat{Y}$,

$$\begin{aligned} E'(\hat{S})^{-1}f &= \sum_{I\in\mathcal{D}_{n-1}} |I|^{-1}\langle J'h_I, (\hat{S})^{-1}f\rangle J'h_I = \sum_{I\in\mathcal{D}_{n-1}} \theta^{-1}|I|^{-1}\langle \hat{S}J'h_I, f\rangle J'h_I \\ &= \sum_{I\in\mathcal{D}_{n-1}} \theta^{-1}|I|^{-1}\langle Sh_I, f\rangle (\hat{S})^{-1}Sh_I = (\hat{S})^{-1}E(f). \end{aligned}$$

Therefore, $\hat{S}E'(\hat{S})^{-1}|_{\hat{Y}} = E$. We use this in conjunction with (10.6):

$$\begin{aligned} \frac{\epsilon}{2} &> \Big\|\Big(E'(\hat{S})^{-1}\hat{R}\hat{S} - r\cdot id\Big)\Big|_{Y'}\Big\| = \Big\|\Big(\hat{S}E'(\hat{S})^{-1}\hat{R}\hat{S} - r\cdot \hat{S}\Big)\Big|_{Y'}\Big\| = \Big\|\Big(E\hat{R} - r\cdot id\Big)\Big|_{\hat{S}(Y')}\Big\| \\ &= \Big\|\Big(E\hat{R} - r\cdot id\Big)\Big|_{Y}\Big\|. \end{aligned}$$

We combine this with the conclusion of Lemma 10.1.2 (ii) for T, $\hat{Y}$, and $\hat{E}$, which states

$$\frac{\epsilon}{2} > \Big\|\Big(\hat{E}T - \hat{R}\Big)\Big|_{\hat{Y}}\Big\| \geqslant \Big\|\Big(E\hat{E}T - E\hat{E}\hat{R}\Big)\Big|_{Y}\Big\| = \Big\|\Big(ET - E\hat{R}\Big)\Big|_{Y}\Big\|,$$

and therefore, $\|(ET - r\cdot id)|_Y\| < \epsilon$.

It remains only to verify (ii), which follows at once from $\|\hat{E}Q_i\| < \epsilon/2$, $\|Q_2\hat{E}\| < \epsilon/2$, and $E\hat{E} = \hat{E}E = E$. $\square$

10.2 The orthogonal reduction to a scalar FDD-diagonal operator

We now address the reduction of an arbitrary bounded linear operator $T : R_\alpha^{p,0} \to R_\alpha^{p,0}$ to a scalar FDD-diagonal operator R whose entries are averages of the diagonal entries of T. With Proposition 10.1.3 in place, we turn to the inductive construction of a distributional embedding scheme Ψ, as described at the beginning of Chapter 10.

The following theorem involves only limit ordinal numbers and yields the factorization property of the standard martingale difference sequence bases of the limit $R_\alpha^{p,0}$ spaces (see Theorem 10.2.3). The factorization property of the standard martingale difference sequence bases of the infinite successor $R_\alpha^{p,0}$ spaces is addressed in Remark 10.2.4.

Theorem 10.2.1. Let α be a limit ordinal number, $T\colon R_\alpha^{p,0} \to R_\alpha^{p,0}$ be a bounded linear operator and $\epsilon > 0$. Then there exist a distributional embedding scheme Ψ from $R_\alpha^{p,0}$ to itself and a scalar

FDD-diagonal operator $R\colon R_\alpha^{p,0} \to R_\alpha^{p,0}$ with diagonal entries in $\mathcal{A}(D(T))$ such that

$$\|T_\Psi^\dagger T T_\Psi - R\| < \epsilon.$$

In particular, T is an orthogonal factor of R with error ϵ.

Proof. Let $(\lambda_n)_{n=1}^\infty$ be an enumeration of the elements of $\mathcal{T}_\alpha$ compatible with the order of the initial segments, that is $\lambda_n \subsetneq \lambda_m$ implies $n < m$. We will construct a distributional embedding scheme Ψ of $R_\alpha^{p,0}$ in $R_\alpha^{p,0}$ and a scalar FDD-diagonal operator R on $R_\alpha^{p,0}$ such that $\|T_\Psi^\dagger T T_\Psi - R\| < \epsilon$, and furthermore, for every $n \in \mathbb{N}$, there is an r_n in $\mathcal{A}(D(T))$ such that $R|_{X_{\lambda_n}} = r_n \cdot id$. The construction of $\Psi = (\mathcal{M}_{\lambda_n}, (J_\mu)_{\mu\in\mathcal{M}_{\lambda_n}})_{n=1}^\infty$ and choice of $(r_n)_{n=1}^\infty$ will be carried out inductively on $n \in \mathbb{N}$ such that after n steps, the partial family $(\mathcal{M}_{\lambda_k}, (J_\mu)_{\mu\in\mathcal{M}_{\lambda_k}})_{k=1}^n$ satisfies Definition 8.0.3, up to that point (the set $\{\lambda_1, \ldots, \lambda_n\}$ is a backward closed subtree of $\mathcal{T}_\alpha$), and the following bounds hold: for every $n \in \mathbb{N}$, denote $S_n = \sum_{\mu\in\mathcal{M}_{\lambda_n}} T_\mu J_\mu$, which is a θ_{λ_n}-compression,

$$Y_n = S_n(L_p^{\kappa(\lambda_n),0}) = T_\Psi(X_{\lambda_n}),$$

and $E_n : R_\alpha^{p,0} \to Y_n$ the orthogonal projection. Then, for all $n \in \mathbb{N}$,

$$\left\|\Big(E_nT - r_n \cdot id\Big)\Big|_{Y_n}\right\| < \frac{\epsilon}{3\cdot 2^n}, \tag{10.7}$$

$$\Big\|\Big(\sum_{m=1}^{n-1} E_m\Big)TE_n\Big\| < \frac{\epsilon}{3\cdot 2^n}, \text{ and} \tag{10.8}$$

$$\Big\|E_nT\Big(\sum_{m=1}^{n-1} E_m\Big)\Big\| < \frac{\epsilon}{3\cdot 2^n}. \tag{10.9}$$

Before initiating the construction, we demonstrate how the above bounds yield the conclusion. Consider the space $Y = T_\Psi[R_\alpha^{p,0}]$ that admits the Schauder decomposition $(Y_n)_{n=1}^\infty$ with corresponding coordinate projections $(E_n)_{n=1}^\infty$, which have norm one. Denote the projection

$$P_\Psi = T_\Psi T_\Psi^\dagger : R_\alpha^{p,0} \to Y, \qquad P_\Psi f = \sum_{n=1}^\infty E_n f,$$

as in Remark 7.3.6 (see also Example 3.2.5 and Proposition 3.2.8), recalling also $T_\Psi^\dagger T_\Psi = id : R_\alpha^{p,0} \to R_\alpha^{p,0}$. Consider the operator $\tilde{R} = T_\Psi R T_\Psi^{-1}$, which is diagonal with respect to the aforementioned decomposition, and more precisely, for all $n \in \mathbb{N}$, $\tilde{R}E_n = r_nE_n$. Then,

$$\begin{aligned}
\Big\|T_\Psi^\dagger T T_\Psi - R\Big\| &= \Big\|T_\Psi\big(T_\Psi^\dagger T T_\Psi - R\big)T_\Psi^{-1}\Big\| = \Big\|P_\Psi T|_Y - \tilde{R}\Big\| \\
&= \Big\|\mathrm{SOT} - \sum_{n=1}^\infty \Big(\Big(\sum_{m=1}^{n-1} E_m\Big)TE_n + E_nT\Big(\sum_{m=1}^{n-1} E_m\Big) + E_nTE_n - E_n\tilde{R}E_n\Big)\Big\| \\
&\leqslant \sum_{n=1}^\infty \Big(\Big\|\Big(\sum_{m=1}^{n-1} E_m\Big)TE_n\Big\| + \Big\|E_nT\Big(\sum_{m=1}^{n-1} E_m\Big)\Big\| + \Big\|\Big(E_nT - r_n\cdot id\Big)\Big|_{Y_n}\Big\|\Big) < \epsilon,
\end{aligned}$$

where the above notation $\mathrm{SOT} - \sum$ means that the convergence of the series is taken with respect to the strong operator topology. We return to the inductive construction of the family $(\mathcal{M}_{\lambda_n}, (J_\mu)_{\mu \in \mathcal{M}_{\lambda_n}})_{n=1}^\infty$ and the scalars $(r_n)_{n=1}^\infty$. The inductive hypothesis up to n consists of

(i) the partial family $(\mathcal{M}_{\lambda_m}, (J_\mu)_{\mu \in \mathcal{M}_{\lambda_m}})_{m=1}^n$ satisfies Definition 8.0.3,

(ii) (10.7) to (10.9) hold up to n,

(iii) there exist $\kappa_n \in \mathbb{N}$ and $\theta_n \in (0, 1]$ such that for all $\mu \in \mathcal{M}_{\lambda_n}$,

$$\kappa(\mu) = \kappa_n, \ \theta_\mu = \theta_n, \ \text{and } b(\mu) = b(\lambda_n), \ \text{and}$$

(iv) if λ_n is a root then $\mathcal{M}_{\lambda_n} = \{\lambda'_n\}$, for some root λ'_n, and otherwise, for every $\mu \in \mathcal{M}_{\lambda_n}$, $\pi(\mu) \in \mathcal{M}_{\pi(\lambda_n)}$.

We omit the basis step, since the same method applies. Assume the construction is complete up to n, and set $\lambda = \lambda_{n+1}$. There are two cases: λ is a root, or a successor. The root case is analogous to the successor case, so we only record that then $\mathcal{M}_\lambda = \{\lambda'\}$, where λ' is the root of $\mathcal{T}_\alpha$ obtained from λ by repeating all entries except that the last, $b(\lambda) + \kappa(\lambda)$, is replaced with $b(\lambda) + \kappa_{n+1}$ for some sufficiently large κ_{n+1}. While this is analogous to the successor case, this is the only point where the fact that α is a limit ordinal matters; otherwise $\mathcal{T}_\alpha$ would have a unique root. We now turn to the successor case, where

$$\lambda = \pi(\lambda)^\frown O^\frown \nu,$$

with $\pi(\lambda) = \lambda_{n_0}$, for some $1 \leqslant n_0 \leqslant n$, $O \in \mathcal{D}^{\kappa(\pi(\lambda))}$ and $\nu \in \mathcal{T}_{b(\pi(\lambda))}$ is a root. We apply Proposition 10.1.3 to obtain an integer

$$k''_0 = k''_0\Big(\kappa(\lambda), \sum_{m=1}^n 2^{\kappa(\lambda_m)}, \|T\| \cdot \Big\| \sum_{m=1}^n E_m \Big\|, \frac{\epsilon}{3 \cdot 2^{n+1}}\Big)$$

satisfying its conclusion, and we define

$$\kappa_{n+1} = k''_0 \vee \kappa(\nu) \vee \Big(\max_{1 \leqslant m \leqslant n} \big(\kappa_m + 1\big)\Big) \text{ and } \theta_{n+1} = \frac{\theta_{n_0}}{2^{\kappa_{n_0}}}.$$

We define the root ν' of $\mathcal{T}_{b(\pi(\lambda))}$ by taking the entries of ν unchanged, except that the last entry, $b(\nu) + \kappa(\nu)$, is replaced with $b(\nu) + \kappa_{n+1}$. Letting $\mathcal{D}_O^{\kappa(\mu)} = \{P \in \mathcal{D}^{\kappa(\mu)} : P \subset [J_\mu h_{\pi(O)} = \epsilon(O)]\}$, $\mu \in \mathcal{M}_{\pi(\lambda)}$, as in Notation 8.1.2, we define

$$\mathcal{M}_\lambda = \Big\{\mu^\frown P^\frown \nu' : \mu \in \mathcal{M}_{\pi(\lambda)} \text{ and } P \in \mathcal{D}_O^{\kappa(\mu)}\Big\}.$$

It is clear that, for $\mu \in \mathcal{M}_\lambda$, $b(\mu) = b(\lambda)$, $\kappa(\mu) = \kappa_{n+1}$ and $\theta_\mu = \theta_{n+1}$. By the inductive hypothesis, the members of $\mathcal{M}_{\pi(\mu)}$ are pairwise disjointly supported; hence, by Proposition 7.5.2, so are the members of $\mathcal{M}_\lambda$ and

$$\sum_{\mu\in\mathcal{M}_\lambda}\theta_\mu = \sum_{\mu\in\mathcal{M}_{\pi(\lambda)}}\sum_{P\in\mathcal{D}_O^{\kappa(\mu)}}\theta_\mu|P| = \sum_{\mu\in\mathcal{M}_{\pi(\lambda)}}\frac{\#\mathcal{D}_O^{\kappa(\mu)}}{2^{\kappa_{n_0}}}\theta_\mu = \sum_{\mu\in\mathcal{M}_{\pi(\lambda)}}\frac{|I|2^{\kappa_{n_0}}}{2^{\kappa_{n_0}}}\theta_\mu = |I|\theta_{\pi(\lambda)} = \theta_\lambda.$$

Therefore, Definition 8.0.3 (a) is satisfied. Clearly, Definition 8.0.3 (c) holds by design. The verification of Definition 8.0.3(b) relies on the tree structure of $(\mathcal{M}_{\lambda_k})_{k=1}^n$, on (iv), which holds by construction, and on the fact that, by the choice of κ_{n+1}, $\mathcal{M}_\lambda$ is disjoint from $\mathcal{M}_{\lambda_1},\ldots,\mathcal{M}_{\lambda_n}$. It remains to choose distributional embeddings $J_\mu : L_p^{\kappa(\lambda),0}\to L_p^{\kappa(\mu),0}$, $\mu\in\mathcal{M}_\lambda$, and a scalar $r_{n+1}\in\mathcal{A}(D(T))$ satisfying (ii). We now note that the numerical restrictions tied to the integer k_0'' are satisfied by the operators T, $Q_1 = T\Big(\sum_{m=1}^n E_m\Big)$, $Q_2 = (\sum_{m=1}^n E_m)T$, and by the θ_{n+1}-compressions $(T_\mu)_{\mu\in\mathcal{M}_\lambda}$, whose ranges are disjointly supported. Proposition 10.1.3 yields distributional embeddings $J_\mu : L_p^{\kappa(\lambda),0}\to L_p^{\kappa(\mu),0}$, $\mu\in\mathcal{M}_\lambda$, such that letting $S_{n+1}=\sum_{\mu\in\mathcal{M}_\lambda}T_\mu J_\mu$, $Y_{n+1}=S_{n+1}[L_p^{\kappa(\lambda),0}]$, and E_{n+1} the orthogonal projection onto Y_{n+1}, then, for some $0\leqslant i\leqslant\kappa(\lambda)-1$ and

$$\begin{aligned} r_{n+1} &= \theta_\lambda^{-1}\sum_{\mu\in\mathcal{M}_\lambda}\sum_{K\in\mathcal{D}^i}\langle T_\mu h_K, TT_\mu h_K\rangle = 2^{-i}\frac{\theta_{n+1}}{\theta_\lambda}\sum_{\mu\in\mathcal{M}_\lambda}\sum_{K\in\mathcal{D}^i}|K|^{-1}\theta_\mu^{-1}\langle h_K^\mu, Th_K^\mu\rangle \\ &= \frac{1}{(\#\mathcal{M}_\lambda)\cdot(\#\mathcal{D}^i)}\sum_{\mu\in\mathcal{M}_\lambda}\sum_{K\in\mathcal{D}^i}|K|^{-1}\theta_\mu^{-1}\langle h_K^\mu, Th_K^\mu\rangle\in\mathcal{A}(D(T)) \end{aligned}$$

we have

$$\Big\|\Big(E_{n+1}T - r_{n+1}\cdot id\Big)|Y_{n+1}\Big\| < \frac{\epsilon}{3\cdot 2^{n+1}},$$
$$\Big\|\big(\sum_{m=1}^n E_m\big)TE_{n+1}\Big\| < \frac{\epsilon}{3\cdot 2^{n+1}},\ \text{and}\ \Big\|E_{n+1}T\big(\sum_{m=1}^n E_m\big)\Big\| < \frac{\epsilon}{3\cdot 2^{n+1}}.$$

□

Remark 10.2.2. As the proof of Theorem 10.2.1 suggests, the "error" operator $T_\Psi^\dagger TT_\Psi - R$ is compact.

Theorem 10.2.1 and Proposition 3.3.5 yield the factorization property of $R_\alpha^{p,0}$ for a countable limit ordinal number α.

Theorem 10.2.3. The standard martingale difference sequence basis of $R_\alpha^{p,0}$ has the factorization property, whenever α is a countable limit ordinal number.

Remark 10.2.4. Let α be a countable limit ordinal and $n\in\mathbb{N}$. Every bounded linear operator $T: R_{\alpha+n}^{p,0}\to R_{\alpha+n}^{p,0}$ is a factor of a bounded linear operator $\tilde{T}: R_\alpha^{p,0}\to R_\alpha^{p,0}$ whose diagonal entries are a subcollection of the diagonal entries of T with respect to the standard martingale difference sequence basis. This is achieved, e.g., by fixing $I\in\mathcal{D}^n$ and taking $A=(id-\mathbb{E})Q_I : R_{\alpha+n}^{p,0}\to R_\alpha^{p,0}$ and $B=T_I : R_\alpha^{p,0}\to R_{\alpha+n}^{p,0}$ and letting $\tilde{T}=ATB$, where $\mathbb{E}(f)$ denotes the expectation of the random variable f, that is $\mathbb{E}(f)=\int_0^1 f(x)\mathrm{d}x$. Together with Theorem 10.2.3 and the isomorphism

$R_\alpha^{p,0} \simeq R_{\alpha+n}^{p,0}$ from [2], this yields the factorization property for the standard martingale difference sequence bases of all spaces $R_\alpha^{p,0}$, $\omega \leqslant \alpha < \omega_1$. Note that the operator-norm bounds for the factoring operators deteriorate as $n \to \infty$.

10.3 Open questions

From the previous chapters, the problem of the primary factorization property of the spaces $R_{\omega^\alpha}^{p,0}$, $\alpha < \omega_1$, naturally arises.

Question. Let $\alpha < \omega_1$. Does $R_{\omega^\alpha}^{p,0}$ have the primary factorization property?

Towards this direction, the main step is the following open question.

Question. Let $\alpha < \omega_1$. Let $D\colon R_{\omega^\alpha}^{p,0} \to R_{\omega^\alpha}^{p,0}$ be a scalar FDD-diagonal operator and $\epsilon > 0$. Does there exist a distributional embedding scheme Ψ from $R_{\omega^\alpha}^{p,0}$ to itself and a scalar λ such that

$$\|T_\Psi^\dagger D T_\Psi - \lambda \cdot id\| < \epsilon?$$

In particular, is D an orthogonal factor of $\lambda \cdot id$ with error ϵ?

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